Nonparametric probabilistic approach of uncertainties with correlated mass and stiffness random matrices

Anas Batou*[c], Airton Nabarrete[d]

[a]*Department of Mechanical, Materials and Aerospace Engineering, School of Engineering, University of Liverpool, Liverpool, L69 7ZF, United Kingdom.*
[b]*Aerospace Engineering Division, Instituto Tecnológico de Aeronáutica (ITA), Pça. Mal Eduardo Gomes, 50 - São José dos Campos, 12228-900 Brazil*

---

*Corresponding author

Nonparametric probabilistic approach of uncertainties with correlated mass and stiffness random matrices

Anas Batou*[c], Airton Nabarrete[d]

[c]*Department of Mechanical, Materials and Aerospace Engineering, School of Engineering, University of Liverpool, Liverpool, L69 7ZF, United Kingdom.*
[d]*Aerospace Engineering Division, Instituto Tecnológico de Aeronáutica (ITA), Pça. Mal Eduardo Gomes, 50 - São José dos Campos, 12228-900 Brazil*

Abstract

This paper concerns the probabilistic modeling of uncertainties in structural dynamics. For real complex structures, the accurate modeling and identification of uncertainties is challenging due to the large number of involved uncertain parameters. In this context, the nonparametric probabilistic approach which consists in modeling globally the uncertainties by replacing the mass, stiffness and damping reduced matrices by random matrices is attractive since it yields a stochastic modeling for which the level of uncertainties is controlled by a small number of dispersion parameters. In its classical version, these random matrices are assumed to be independent. This assumption is valid (and proven) in absence of information concerning the dependence structure of these random matrices. In some situation, such as the presence of geometry uncertainties, this assumption is not valid any more and may yield an overestimation of the output levels of fluctuation. In this context, the present paper presents an extension of the classical nonparametric probabilistic to take into account a dependence between the random mass and stiffness matrices. This new modeling is illustrated on a beam structure for which the diameter presents spatial random fluctuations along the longitudinal direction.

Keywords: random matrix theory, nonparametric probabilistic approach, structural dynamics, mass/stiffness correlation

1. Introduction

For so long time engineers have performed structural analyses using deterministic mathematical models. In especial for many structures the increase of the safety margin was usually employed in the design to support the uncertainties of the physical construction or the variations of the loading or material resistance. In the last years, the development of stochastic models has enhanced the reliability of the dynamic analyses of these structures.

*Corresponding author

Regardless the chosen strategy uncertainties shall be always considered in structural analyses. There are several methods for including the uncertainties into computational models. Considering probabilistic methods, the classical parametric approach takes into account the uncertainties in the parameters of the computational model [1, 2, 3]. This method consists in replacing the uncertain parameters by random variables and then propagate the uncertainty using devoted methods such as the Monte Carlo simulation method. This approach is quite simple to implement and use if there are uncertainty sources with known probability distributions. In cases of real complex structures, this approach becomes inadequate because the sources of uncertainties are not completely known, or they are known but modeled with a large number of hyperparameters that need to be identified.

The nonparametric probabilistic approach [4, 5] is an alternative to overcome the limitations of the parametric one. In this approach, the uncertainties happen at the operator level by modeling the reduced-order generalized mass, stiffness and damping matrices of a structural system as random matrices. There is a small number of dispersion parameters that control the constructed stochastic model and have their experimental identification feasible in an industrial context [6, 7, 8, 9, 10, 11]. Furthermore, it is not necessary to know the probability density functions of the model input data. An explicit generator for independent realizations of the random matrices is directly available [4], making its use very practical in the context of a Monte Carlo simulation [12]. The randomness applied directly to the reduced-order generalized matrices allows the solution of the dynamical equations for a chosen amount of vibration modes, which results in a significant reduction of the required computational cost when compared to parametric models.

In its classical form, the nonparametric probabilistic approach assumes that the random mass, stiffness and damping matrices are independent. This assumption completely make sense in case of uncertainties related to the material properties for instance. In absence of information about the dependence of these random matrices, this assumption is correct considering the Information Theory [4]. This assumption yields a conservative stochastic model that will encompass a possible dependence if exists.

In a situation of a structure whose distributed mass has significant influence in its dynamic behaviour, it is clear that uncertainties in chosen geometry parameters promote correlated mass and stiffness uncertainties and consequently, dependent ones. For instance, in most industries (automotive vehicles, aircraft ...), manufacturing tolerances yield uncertainties in the geometry of the structure that will affect the mass and stiffness matrix in a dependent way. In this case, not taking into account the existing dependence structure may overestimate the level of fluctuations of the quantities of interest and in the case of an inverse identification of the dispersion parameters, those later may be underestimated to compensate the non identifiability of the stochastic model.

The objective of this paper is to extend the classical nonparametric probabilistic approach to take into account a correlation/dependency between the mass and stiffness random matrices. To achieve this objective, a global correlation parameter is introduced in addition to the mass and stiffness dispersion parameters. The generator of independent realizations of these pair of random matrices is then constructed. For the proposed construction only one additional parameter is introduced keeping the advantage of the nonparametric probabilistic approach and the classical nonparametric probabilistic approach can be a special case of this new model when setting the correlation parameter to zero.

In this paper, Section 2 presents the nominal model and its related equation of motion. Then Section 3 presents the new extended nonparametric probabilistic approach. Finally, in Section 3, the proposed approach is illustrated and validated through a numerical application consisting of a beam structure with geometry spatial fluctuations.

2. Nominal model

The computational model is constructed using the finite element method in the frequency domain. The structure is analyzed on the frequency range of $0 \leq \omega \leq \omega_{max}$ and the vector $\boldsymbol{y}(\omega)$ is the $n \times 1$ vector of frequency-dependent displacement amplitudes, where n is the number of degrees of freedom. The equation of motion for the nominal model is

$$(-\omega^2 \mathtt{M} + i\omega \mathtt{D} + \mathtt{K})\, \boldsymbol{y}(\omega)\, e^{i\omega t} = \mathtt{F}(\omega)\, e^{i\omega t} \tag{1}$$

where $\mathtt{M}$, $\mathtt{D}$ and $\mathtt{K}$ are the $n \times n$ mass, damping and stiffness matrices, and $\mathtt{F}(\omega)$ is the $n \times 1$ vector of time-independent force amplitudes.

The reduced nominal model is obtained by projecting the nominal model on the subspace spanned by the m first mode shapes. Considering the subscript $k = 1, 2, \ldots, m$ we write the generalized eigenvalue problem as follows

$$\mathtt{K}\, \boldsymbol{\phi}_k = \lambda_k \mathtt{M}\, \boldsymbol{\phi}_k \tag{2}$$

The m eigenvalues are indexed as $0 < \lambda_1 \leq \ldots \leq \lambda_m$ and associated with the corresponding mode shapes $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_m$. Here, only deformation modes are considered for the structure.

We introduce the approximation $\boldsymbol{y}^{(m)}(\omega)$ of $\boldsymbol{y}(\omega)$ written as

$$\boldsymbol{y}^{(m)}(\omega) = \boldsymbol{\Phi}\, \boldsymbol{q}(\omega) \tag{3}$$

where $\boldsymbol{q}(\omega)$ is the vector of the m generalized coordinates and $\boldsymbol{\Phi}$ is the $n \times m$ modal matrix for the mass-normalized mode shapes.

The reduced nominal model is written as

$$(-\omega^2 \mathcal{M} + i\omega \mathcal{D} + \mathcal{K})\, \boldsymbol{q}(\omega) = \mathcal{F}(\omega)\,, \tag{4}$$

In Eq. 4, $\mathcal{M} = \boldsymbol{\Phi}^T\, \mathtt{M}\, \boldsymbol{\Phi} = I_m$, $\mathcal{D} = \boldsymbol{\Phi}^T\, \mathtt{D}\, \boldsymbol{\Phi}$ and $\mathcal{K} = \boldsymbol{\Phi}^T\, \mathtt{K}\, \boldsymbol{\Phi} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ are the $m \times m$ matrices of generalized mass, damping and stiffness, and $\mathcal{F}(\omega) = \boldsymbol{\Phi}^T\, \mathtt{F}(\omega)$ is the $m \times 1$ vector of generalized forces.

3. Stochastic model

It is now assumed that the nominal computational model presented in Section 2 has some sources of uncertainties related to its parameters such as geometry or material properties. As explained in the introduction, all the sources of uncertainties are modeled globally using a nonparametric probabilistic approach for uncertainties. In this section, an extended version of the classical nonparametric probabilistic approach of uncertainties is presented allowing the correlation between the random mass and stiffness matrices to be taken into account.

3.1. Nonparametric probabilistic approach of uncertainties.

The nonparametric probabilistic approach of uncertainties consists in modeling the uncertainties directly and globally at the reduced-order level by replacing the mass and the stiffness matrices of the nominal computational by random matrices [4]. The probability distribution for each of these matrices is constructed using the Maximum Entropy Principle [13] yielding a probabilistic model controlled by two dispersion parameters. In the classical nonparametric probabilistic approach, the obtained mass and stiffness random matrices are shown to be independent. If necessary this procedure can be also applied to the damping matrix. The damping random matrix would be constructed as independent too and a third dispersion parameter would control of the probabilistic model. In the following subsections, at first, we describe the classical nonparametric probabilistic approach for mass and stiffness matrices and then, we present an extended approach introducing a correlation, and consequently the dependence, between the mass and stiffness random matrices.

3.1.1. Classical nonparametric approach of uncertainties: independent mass and stiffness random matrices

In the classical nonparametric approach, the matrices of generalized mass and stiffness are replaced by independent random matrices.

The Cholesky factorization of the generalized mass and stiffness matrices are $\mathcal{M} = \mathcal{L}_M^T\,\mathcal{L}_M$ and $\mathcal{K} = \mathcal{L}_K^T\,\mathcal{L}_K$, respectively. If the subscript i denotes M and K, we can write $\boldsymbol{A}_i$ for representing the random matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ in the form

$$\boldsymbol{A}_i = \mathcal{L}_i^T \mathbf{G}_i \mathcal{L}_i \tag{5}$$

where $\mathbf{G}_i$ is the normalized random matrix whose probability distribution is constructed using the Maximum Entropy principle with the constraints: (a) it must be positive definite almost surely; (b) it has a mean value which is equal to the identity matrix; and (c) it must verify the inequality $|E\{\log[\det(\mathbf{G}_i)]\}| < +\infty$, which means that the stochastic response of the stochastic computational dynamic model is a second-order random stochastic process.

The random mass and stiffness matrices are then independent and the probability distribution is defined by the probability density function $p_{\mathbf{G}_i}(\mathbf{G}_i)$ as

$$p_{\mathbf{G}_i}(\mathbf{G}_i) = \mathbb{1}_{\mathbb{M}_m^+(\mathbb{R})}(\mathbf{G}_i) \times C_i \times \left[\det(\mathbf{G}_i)\right]^{\frac{m+1}{2}[(\delta_i)^{-2}-1]} \times e^{-\frac{m+1}{2}(\delta_i)^{-2}\,\mathrm{tr}[\mathbf{G}_i]}\,, \tag{6}$$

where the positive constant C_i is given by

$$C_i = (2\pi)^{-\frac{m(m-1)}{4}} \left(\frac{m+1}{2\delta_i^2}\right)^{\frac{m(m+1)}{2}(\delta_i)^{-2}} \left\{\Pi_{j=1}^m \Gamma\left(\frac{m+1}{2\delta_i^2} + \frac{1-j}{2}\right)\right\}^{-1}, \tag{7}$$

and the gamma function defined for $z > 0$ is $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$.

The dispersion parameter δ_i in Equations 6 and 7 controls the global statistical fluctuations of the matrix $\mathbf{G}_i$ and is evaluated as

$$\delta_i = \sqrt{\frac{E\{\|\mathbf{G}_i - I_m\|_F^2\}}{E\{\|I_m\|_F^2\}}} = \sqrt{\frac{E\{\|\mathbf{G}_i - I_m\|_F^2\}}{m}} \tag{8}$$

where $\|.\|_F$ is the Frobenius norm.

The dispersion parameter must be real valued and chosen in the interval $0 < \delta_i < \sqrt{(m+1)/(m+5)}$ to guarantee the constraint (c) of the Maximum Entropy principle.

The independent realizations of the random mass and stiffness matrices can be easily performed within the Monte Carlo numerical simulation. To do this, the normalized random matrix is constructed as $\mathbf{G}_i = \mathbf{L}_i^T \mathbf{L}_i$, in which $\mathbf{L}_i$ is an upper triangular random matrix with values in $\mathbb{M}_m(\mathbb{R})$ such that:

(1) random variables $[\mathbf{L}_i]_{jj'}$ for $j \leq j'$ are independent,

(2) for $j < j'$, real-valued random variables $[\mathbf{L}_i]_{jj'}$ can be written as $[\mathbf{L}_i]_{jj'} = \sigma_m U_{jj'}$ in which $\sigma_m = \delta_i/\sqrt{m+1}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1,

(3) for $j = j'$, positive-valued random variables $[\mathbf{L}_i]_{jj}$ can be written as $[\mathbf{L}_i]_{jj} = \sigma_m \sqrt{2V_j}$ in which V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma(\frac{m+1}{2}(\delta_i)^{-2} + \frac{1-j}{2})}\; v^{\frac{m+1}{2}(\delta_i)^{-2} - \frac{1+j}{2}} e^{-v} \tag{9}$$

3.1.2. Extended nonparametric approach of uncertainties: dependent mass and stiffness random matrices

In the former nonparametric approach, there is no information concerning the correlation between the normalized mass and stiffness random matrices. In this section, we propose to add a correlation, and then a dependence between both. With this in mind the global correlation between these normalized random matrices can be measured as

$$\eta = \frac{E\{\mathrm{tr}\,[(\mathbf{G}_M - I_m)^T(\mathbf{G}_K - I_m)]\}}{\sqrt{E\{\|\mathbf{G}_M - I_m\|_F^2\}\, E\{\|\mathbf{G}_K - I_m\|_F^2\}}} = \frac{E\{\mathrm{tr}\,[(\mathbf{G}_M - I_m)^T(\mathbf{G}_K - I_m)]\}}{m\,\delta_M\,\delta_K} \tag{10}$$

where η is the measured global correlation, δ_M and δ_K are the dispersion parameters for the normalized mass and stiffness random matrices, constructed as $\mathbf{G}_M = \mathbf{L}_M^T \mathbf{L}_M$ and $\mathbf{G}_K = \mathbf{L}_K^T \mathbf{L}_K$, respectively.

It would be also possible to construct the joint probability distribution of the normalized random matrices directly by taking into account the measured correlation as described by Eq. 10 and using the Maximum Entropy principle. Although this strategy yields the joint probability distribution, it would be difficult to construct the explicit generator of independent realizations such as the one obtained in the classical nonparametric probabilistic approach. Instead, we propose a correlation coefficient which previously introduces a correlation between entries of the upper triangular random matrices $\mathbf{L}_M$ and $\mathbf{L}_K$ using the Nataf transform [14] of correlated Gaussian random matrices. This kind of procedure was previously employed by Soize [15] to compute the tensor-valued random fields for the anisotropic elastic material with spatial fluctuations. *A posteriori* we verify if the measured global correlation between the matrices $\mathbf{G}_M$ and $\mathbf{G}_K$ is close to the introduced correlation between entries.

Let $\mathbf{L}_1^G$ and $\mathbf{L}_2^G$ be two upper triangular random matrices for which all the non-zero entries are independent and normalized Gaussian random variables. The correlation coefficient is defined as $0 \leq \gamma \leq 1$, and different from the one introduced in Eq. 10. The matrices $\mathbf{L}_M^G$ and $\mathbf{L}_K^G$ are constructed as follows

$$\begin{aligned} \mathbf{L}_M^G &= \mathbf{L}_1^G \\ \mathbf{L}_K^G &= \gamma\, \mathbf{L}_1^G + \sqrt{1-\gamma^2}\, \mathbf{L}_2^G \end{aligned} \tag{11}$$

By setting $\gamma = 0$, the matrices $\mathbf{L}_M^G$ and $\mathbf{L}_K^G$ become uncorrelated, and consequently independent. On the contrary, setting $\gamma = 1$, both matrices become equal. Recalling the subscript i as denoting M and K, the upper random matrix $\mathbf{L}_i$ is now constructed as a transformation of the random matrices $\mathbf{L}_i^G$ such that

(1) for $j < j'$, $[\mathbf{L}_i]_{jj'} = \sigma_m [\mathbf{L}_i^G]_{jj'}$ in which $\sigma_m = \delta_i / \sqrt{m+1}$;
(2) for $j = j'$, $[\mathbf{L}_i]_{jj} = \sigma_m \sqrt{2 F_{V_j}^{-1}\{\Phi([\mathbf{L}_i^G]_{jj})\}}$.

where the function $u \mapsto \Phi(u)$ is the cumulative distribution function of the normalized Gaussian random variable and the function $v \mapsto F_{V_j}^{-1}(v)$ is the reciprocal function of the cumulative distribution of the random variable V_j with probability distribution given by Eq. 9.

In this approach, the normalized random matrices $\mathbf{G}_M$ and $\mathbf{G}_K$ are now dependent and the parameters δ_M, δ_K and γ control the probabilistic model of uncertainties. Finally, the mass and stiffness random matrices are constructed as

$$\begin{aligned} \boldsymbol{M} &= \mathcal{L}_M^T \mathbf{G}_M \mathcal{L}_M \\ \boldsymbol{K} &= \mathcal{L}_K^T \mathbf{G}_K \mathcal{L}_K \end{aligned} \tag{12}$$

This extended nonparametric approach represents a generalization of the classical nonparametric approach. By selecting $\gamma = 0$ we have independent random matrices which correspond to the most conservative case and yields the most variability to the outputs. With the dependency of the random matrices the output variability decreases which may be interesting for several industrial applications.

The three parameters $\delta_{_M}$, $\delta_{_K}$ and γ can be identified using experimental data or random data resulting from the use of a fully parametric model of uncertainties, as demonstrated in Section 4. The correlation parameter η that measures the correlation between the two random matrices and the correlation coefficient γ used as input parameter for the construction of the random matrices are slightly different. In order to demonstrate how this difference is negligible, Table 1 shows estimated values, considering the Monte Carlo simulation with 10 000 realizations. The correlation parameter η is evaluated for different values of the correlation coefficient γ, considering different sizes of the matrix and for $\delta_m = \delta_k = 0.2$.

	$n = 2$	$n = 10$	$n = 40$
$\gamma = 0.1$	0.1008	0.1011	0.0990
$\gamma = 0.5$	0.4971	0.4957	0.4967
$\gamma = 0.9$	0.9014	0.9004	0.8989

Table 1: Estimated value of η.

It can be seen in this table a very good correspondence between γ and η, which validated the herein before construction.

3.2. Random modal data

Mode shapes of the nonparametric model may present substantial differences in comparison with the nominal elastic modes calculated in section 2. In order to evaluate this comparison, we project the nonparametric model on the subspace spanned by the m first random mode shapes. Considering $k = 1, 2, \ldots, m$ we write the random eigenvalue problem as follows

$$\boldsymbol{K}\,\boldsymbol{\psi}_k = \Lambda_k\,\boldsymbol{M}\,\boldsymbol{\psi}_k \tag{13}$$

The random eigenfrequencies are calculated as $\Omega_k = \sqrt{\Lambda_k}$. Similarly to the nominal model, the eigenvalues are indexed as $0 < \Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_m$ and associated with the corresponding random eigenvectors $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_m$. These random eigenvectors are mass-normalised and indexed to compose the $m \times m$ matrix of the mass-normalised random elastic modes $\widetilde{\boldsymbol{\Phi}} = [\widetilde{\phi}_1, \widetilde{\phi}_2, \ldots, \widetilde{\phi}_m]$.

The random matrices constructed with the extended nonparametric approach are diagonalized as follows

$$\begin{aligned} \widetilde{\mathbf{M}} &= \widetilde{\boldsymbol{\Phi}}^T \boldsymbol{M} \widetilde{\boldsymbol{\Phi}} \\ \widetilde{\mathbf{K}} &= \widetilde{\boldsymbol{\Phi}}^T \boldsymbol{K} \widetilde{\boldsymbol{\Phi}} \end{aligned} \tag{14}$$

The diagonalized mass and stiffness random matrices are almost surely evaluated as $\widetilde{\mathbf{M}} = I_m$ and $\widetilde{\mathbf{K}} = \mathrm{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_\mathrm{m})$.

3.3. Random FRFs

For all ω in the range $0 \leq \omega \leq \omega_{max}$, the random response $\mathbf{Y}(\omega)$ of the stochastic reduced-order computational model, is written as

$$\mathbf{Y}(\omega) = \widetilde{\boldsymbol{\Phi}}\, \mathbf{Q}(\omega) \tag{15}$$

in which the random vector $\mathbf{Q}(\omega)$ of the random generalized coordinates, is the solution of the following random reduced-order matrix equation

$$(-\omega^2 \widetilde{\mathbf{M}} + i\omega \widetilde{\mathbf{D}} + \widetilde{\mathbf{K}})\, \mathbf{Q}(\omega) = \widetilde{\mathbf{F}}(\omega) \tag{16}$$

in which $\widetilde{\mathbf{D}} = \widetilde{\boldsymbol{\Phi}}^T \mathbb{D} \widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Psi}^T \mathcal{D} \boldsymbol{\Psi}$ with $\boldsymbol{\Psi} = [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_m]$ and where $\widetilde{\mathbf{F}}(\omega) = \widetilde{\boldsymbol{\Phi}}^T\, \mathbb{f}(\omega) = \boldsymbol{\Psi}^T\, \mathcal{F}(\omega)$.

4. Application

A clamped-supported beam is investigated in the present study as depicted in Fig. 1. The beam made of isotropic material has two different circular sections named as Beam 1 and Beam 2. The dimensions and properties of the beam are described in Table 2.

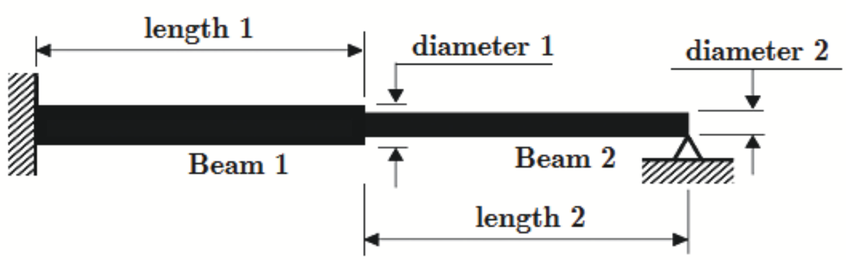

Figure 1: Clamped-supported beam.

	Beam 1	Beam 2
length [m]	0.12	0.18
diameter [m]	7.5×10^{-3}	5×10^{-3}
Young Modulus [GPa]	70	
Poisson's coefficient	0.29	
mass density [kg/m^3]	2 800	

Table 2: Beam characteristics.

The stiffness and mass matrices have been obtained using Euler-Bernoulli theory. Each beam is discretized with 100 elements. The reduced model is then constructed using $m = 8$ elastic modes ranging from 227 Hz to 8 632 Hz. The damping generalized matrix is constructed as diagonal considering the value of 2% for the damping ratio. In Fig. 2, the nominal solution is depicted for a frequency range of $[0, 4\,000]$ Hz. A unit frequency dependent force is applied laterally to the beam at a distance 0.104 m from the left end. The magnitude of the velocity is plotted for the observed degree of freedom located at 0.098 m from the left end.

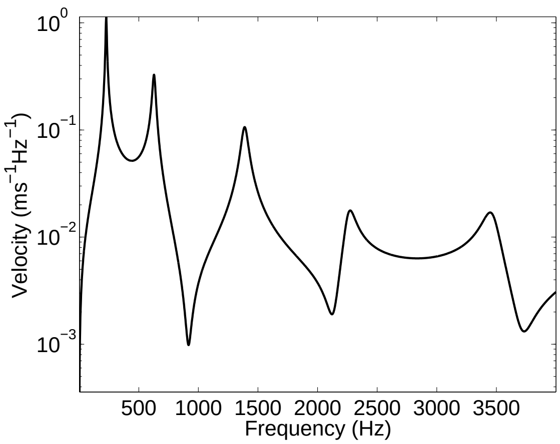

Figure 2: Response of the nominal model.

4.1. Construction of the reference response using a parametric stochastic model

In this application, the diameter of the beam section is assumed to be a random field that has different values along the beam length. Thus it is a uniform random variable with support $[7.3\ 7.7] \times 10^{-3}$ m for Beam 1 and $[4.8\ 5.2] \times 10^{-3}$ m for Beam 2. The correlation along the beam length is introduced through a Nataf transform of the stationary Gaussian random field with the correlation length of 0.01 m. Figure 3 shows two realizations of the random beam diameters.

The parametric reduced stochastic model is constructed by projecting the random dynamical equations of the beam on the basis constituted by the 8 elastic modes of the nominal model studied in the previous section. Therefore the reduced mass and stiffness matrices, $\widehat{\mathbf{K}}$ and $\widehat{\mathbf{M}}$, are full and random. Then the random modal data of the parametric random beam can be obtained by resolving the eigenvalue problem as follows

$$\widehat{\mathbf{K}}\,\boldsymbol{\psi}_k^{par} = \Lambda_k^{par}\,\widehat{\mathbf{M}}\,\boldsymbol{\psi}_k^{par} \tag{17}$$

The random eigenfrequencies are calculated as $\Omega_k^{par} = \sqrt{\Lambda_k^{par}}$ and indexed as $0 < \Omega_1^{par} \leq \Omega_2^{par} \leq \ldots \leq \Omega_m^{par}$. These eigenfrequencies are associated with the corresponding mass-normalised random elastic modes $\widetilde{\boldsymbol{\Phi}}^{par} = [\widetilde{\phi}_1^{par}, \widetilde{\phi}_2^{par}, \ldots, \widetilde{\phi}_m^{par}]$.

The statistics, estimated using the Monte Carlo Method with 1000 realizations, of the 8 first eigenfrequencies are reported in Table 3. The coefficient of variation is

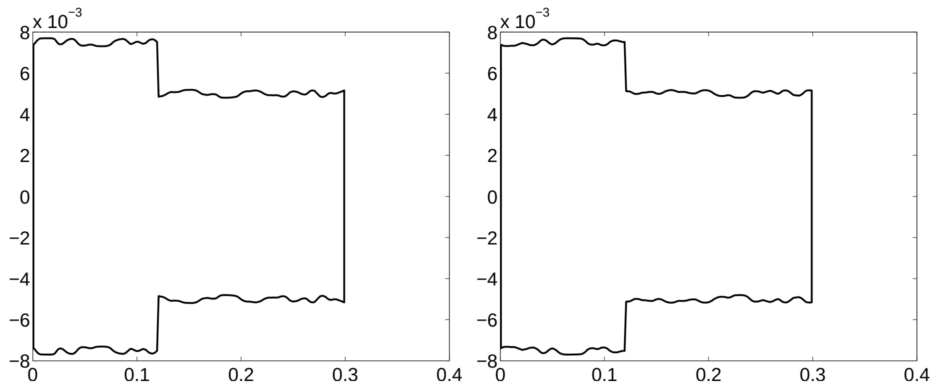

Figure 3: Two realizations of the random beam diameters.

the ratio standard deviation/mean value. The random velocity response is plotted as magnitude in Fig. 4.

Mode	1	2	3	4	5	6	7	8
Mean value (Hz)	226.2	627.4	1386	2258	3460	5053	6588	8623
Coef. Var. ($\times 10^{-3}$)	7.9	7.8	5.6	5.6	6.2	5.3	5.4	5.6

Table 3: Parametric model - statistics for the 8 first eigenfrequencies.

Let $E\{\widehat{\mathbf{M}}\} = \widehat{\mathbf{L}}_M^T \widehat{\mathbf{L}}_M$ and $E\{\widehat{\mathbf{K}}\} = \widehat{\mathbf{L}}_K^T \widehat{\mathbf{L}}_K$ be the Cholesky factorization for the mean mass and stiffness matrices, respectively, considering the parametric random beam. For this case, the normalized random mass and stiffness matrices are defined by

$$\widehat{\mathbf{G}}_M = \widehat{\mathbf{L}}_M^{T^{-1}} \widehat{\mathbf{M}} \, \widehat{\mathbf{L}}_M^{-1} \quad \text{and} \quad \widehat{\mathbf{G}}_K = \widehat{\mathbf{L}}_K^{T^{-1}} \widehat{\mathbf{K}} \, \widehat{\mathbf{L}}_K^{-1} \tag{18}$$

In this case, the dispersion parameters and the correlation parameter for the parametric model are defined by

$$\delta_M^{par} = \sqrt{\frac{E\{\|\widehat{\mathbf{G}}_M - I_m\|_F^2\}}{m}} \quad \text{and} \quad \delta_K^{par} = \sqrt{\frac{E\{\|\widehat{\mathbf{G}}_K - I_m\|_F^2\}}{m}} \tag{19}$$

$$\eta^{par} = \frac{\operatorname{tr} E\{(\widehat{\mathbf{G}}_M - I_m)^T (\widehat{\mathbf{G}}_K - I_m)\}}{m \, \delta_M^{par} \, \delta_K^{par}} \tag{20}$$

The estimated values for this parameters are $\delta_M^{par} = 0.026$, $\delta_K^{par} = 0.049$ and $\eta^{par} = 0.86$. It can then be concluded that for the parametric random beam with random field diameter the correlation between the mass and stiffness matrix is very large. These

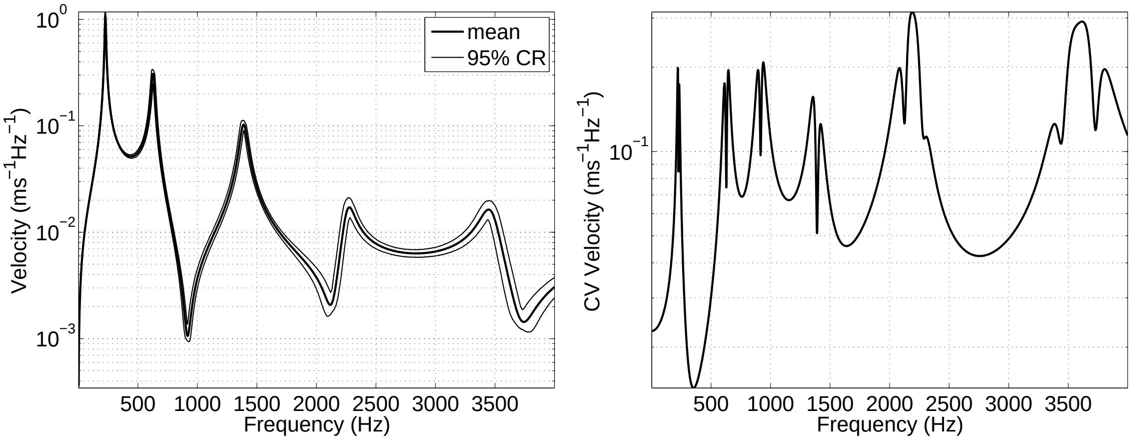

Figure 4: Random response for the parametric random beam. On the left: Mean response and confidence region at 95 %. On the right: Coefficient of variation (ratio standard deviation/mean value).

reference estimated values for δ_M^{par}, δ_K^{par} and η^{par} will now be used as the values δ_M, δ_K and γ of the nonparametric correlated stochastic model.

4.2. Nonparametric correlated stochastic model

In this section, the nonparametric correlated stochastic model presented in Section 3 is analyzed using the values of the parameters identified for the parametric stochastic model, i.e, $\delta_M = \delta_M^{par} = 0.026$, $\delta_K = \delta_K^{par} = 0.049$ and $\gamma = \eta^{par} = 0.86$. The statistics of the 8 first eigenfrequencies are reported in Table 4.

Mode	1	2	3	4	5	6	7	8
Mean value (Hz)	226.3	628.3	1387	2260	3463	5059	6596	8632
Coef. Var. ($\times 10^{-3}$)	7.1	7.1	7.1	7.3	7.0	7.0	7.0	7.0

Table 4: Nonparametric correlated model - statistics for the 8 first eigenfrequencies.

The probability distribution of the 4 first eigenfrequencies are shown in Fig 6. Compared to the values obtained for the parametric stochastic model (see Table 3), the mean values are in good agreement and the coefficients of variation are in quite good agreement. It should be noted that the level of fluctuation using the nonparametric approach is defined globally through the two dispersion parameters. Therefore it cannot be fitted frequencies by frequencies. The random velocity response and its comparison with the one obtained using the parametric stochastic model are plotted as magnitude in Figures 6 and 7. The levels of fluctuation of the response is comparable to the ones obtained for the parametric stochastic model.

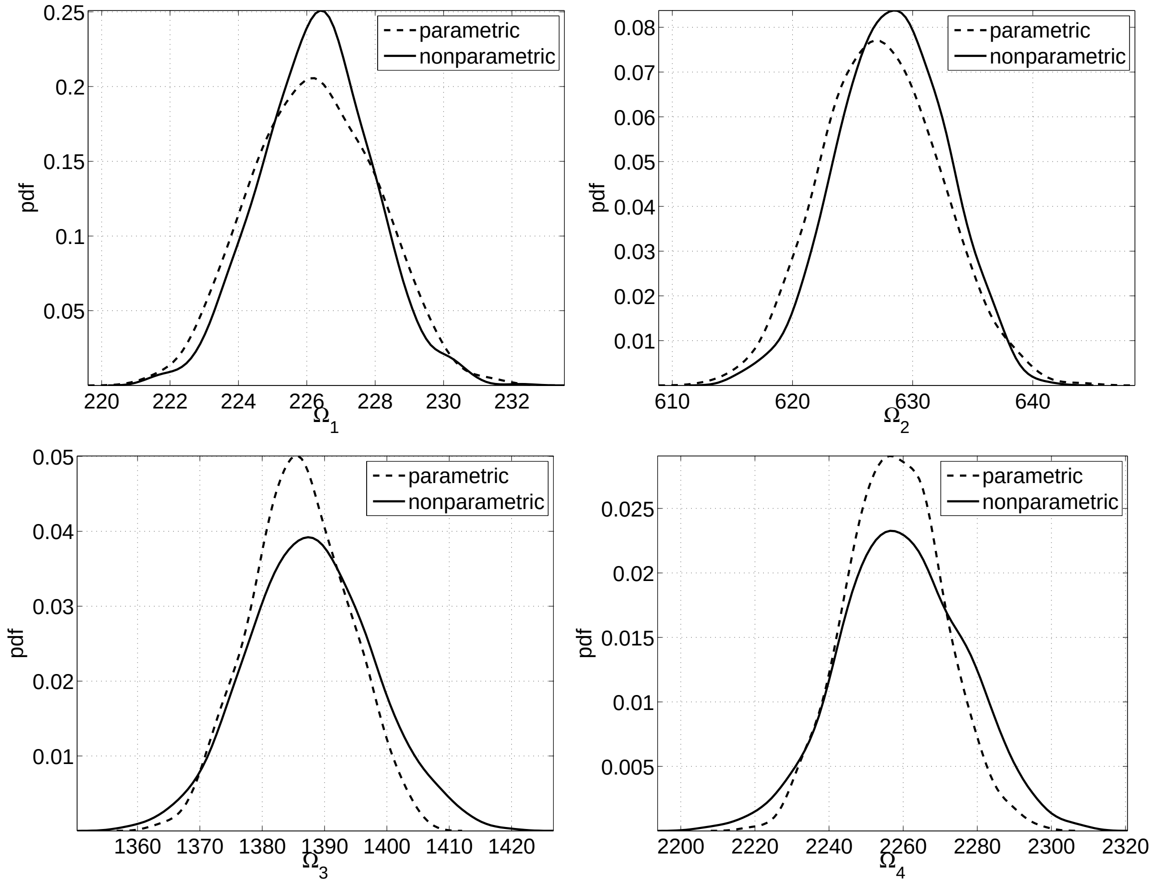

Figure 5: Correlated nonparametric model - Probability distribution of the 4 first eigenfrequencies.

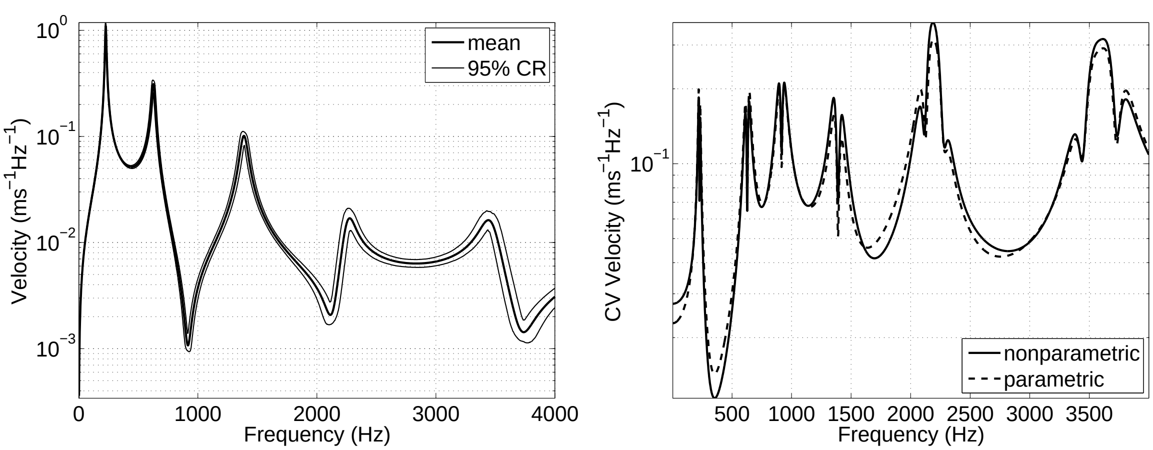

Figure 6: Random response for the correlated nonparametric stochastic model. On the left: Mean response and confidence region at 95 %. On the right: Coefficient of variation (ratio standard deviation/mean value).

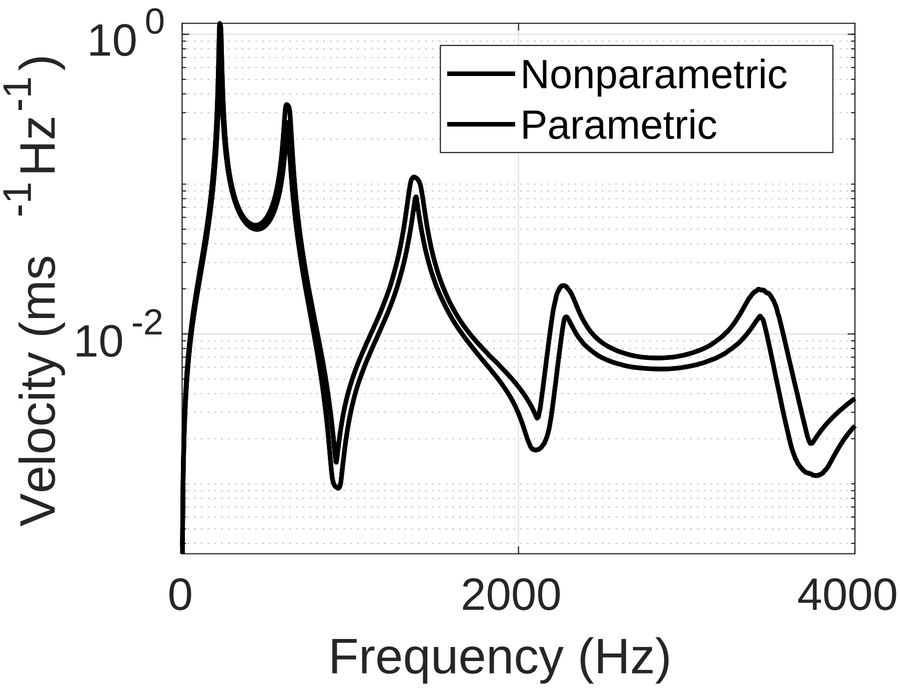

Figure 7: Random response: Comparison between the correlated nonparametric stochastic model and the parametric stochastic model (confidence region at 95 %, the plots are almost superimposed).

4.3. Nonparametric uncorrelated stochastic model (classical approach)

The results of the previous section are now compared with the ones obtained using the classical non parametric correlated stochastic model. Therefore, we now have $\delta_M = \delta_M^{par} = 0.026$, $\delta_K = \delta_K^{par} = 0.049$ and $\gamma = 0$. The statistics of the 8 first eigenfrequencies are reported in Table 5. The probability distribution of the 4 first eigen-

Mode	1	2	3	4	5	6	7	8
Mean value (Hz)	226.5	627.7	1387	2261	3463	5056	6597	8643
Coef. Var. ($\times 10^{-3}$)	13.3	13.3	13.6	13.6	13.5	12.9	13.0	12.4

Table 5: Nonparametric uncorrelated model - statistics for the 8 first eigenfrequencies.

frequencies are shown in Fig 6. Compared to the values obtained for the parametric stochastic model (see Table 3), the mean values are in good agreement. Concerning the coefficients of variation, the values are overestimated. The random velocity response and its comparison with the one obtained using the parametric stochastic model are plotted as magnitude in Figures 9 and 10. As expected, the levels of fluctuation of the response are larger than the ones obtained for the parametric stochastic model. To obtain an output level of fluctuation comparable to the parametric case, the dispersion parameters δ_M, δ_K could be changed to lower values but these new values would

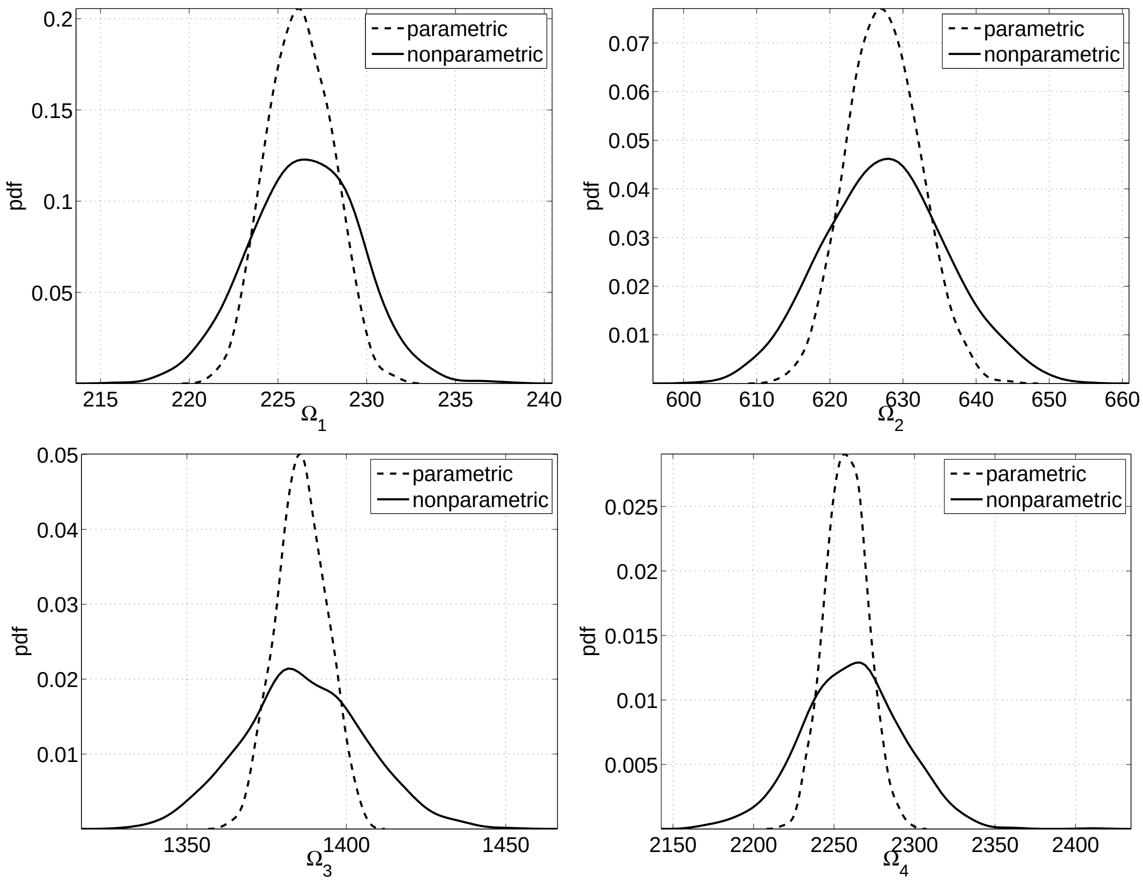

Figure 8: Uncorrelated nonparametric model - Probability distribution of the 4 first eigenfrequencies.

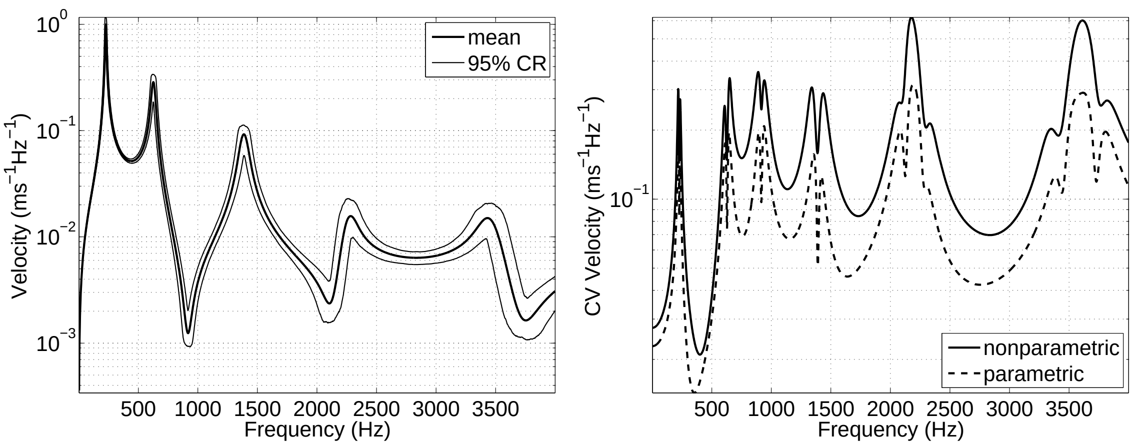

Figure 9: Random response for the nonparametric uncorrelated stochastic model. On the left: Mean response and confidence region at 95 %. On the right: Coefficient of variation (ratio standard deviation/mean value).

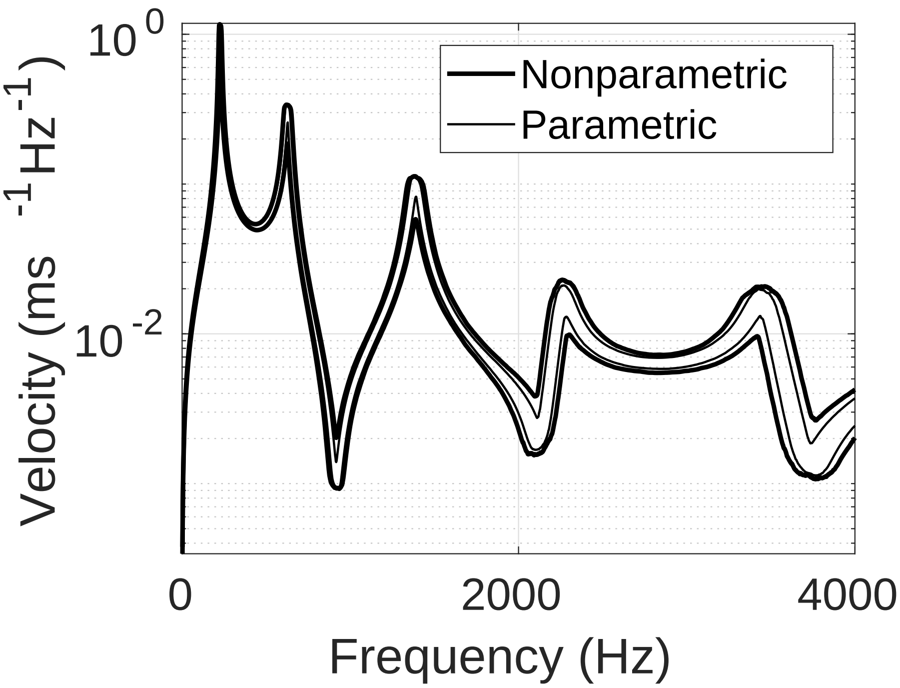

Figure 10: Random response: Comparison between the uncorrelated nonparametric stochastic model and the parametric stochastic model (confidence region at 95 %).

not correspond to the correct values of dispersion for the random mass and stiffness matrices. It can then be concluded (through this example which is representative of many industrial applications, this is not a general proof) that, if exists, the correlation of between the mass and the stiffness random generalized matrices have to be taken into account in order to not overestimate the level of fluctuation of the quantities of interest.

5. Conclusion

In this paper, an extended nonparametric probabilistic approach of uncertainties has been presented. A global correlation between the mass and the stiffness generalized random matrices has been introduced. The advantages of the classical nonparametric probabilistic approach are kept: (1) the probabilistic model is controlled by a small number of parameters, only one correlation coefficient has been introduced in addition to the mass and stiffness dispersion parameters, (2) an explicit generator of independent realizations has been constructed, (3) the probabilistic model is constructed at the reduced-order level. The classical nonparametric probabilistic approach of uncertainties is a particular case ($\gamma = 0$) of this extended new probabilistic model. This new model has been validated through a numerical application. The results clearly show that for this example, as soon as a correlation between the mass and the stiffness exists, it has to be taken into account by setting the correlation coefficient at the good value. Otherwise, the output variability is overestimated.

[1] Ghanem, R.G. and Spanos, P.D. (1991) Stochastic Finite Elements: A Spectral Approach. Dover Publications Inc.

[2] Schueller, G.I. (2006) Developments in stochastic structural mechanics. Archive of Applied Mechanics Vol: 75 (10-12) pp. 755-773.

[3] Schueller, G.I. (2007) On the treatment of uncertainties in structural mechanics and analysis. Computers and Structures, Vol: 85(5-6) pp. 235-243.

[4] Soize, C. (2000) A nonparametric model of random uncertainties for reduced matrix models in structural dynamics. Probabilistic Engineering Mechanics, Vol: 15(3) pp. 277-294.

[5] Soize, C. (2005) Random matrix theory for modeling random uncertainties in computational mechanics. Computer Methods in Applied Mechanics and Engineering, 194, pp. 13331366.

[6] Batou, A., Soize, C. and Corus, M. (2011) Experimental identification of an uncertain computational dynamical model representing a family of structures. Computer and Structures, Vol: 89(13-14), pp. 1440-1448.

[7] Chebli, H. and Soize, C. (2004) Experimental validation of a nonparametric probabilistic model of non homogeneous uncertainties for dynamical systems. Journal of the Acoustical Society of America, Vol: 115(2) pp. 697-705.

[8] Chen, C., Duhamel, D., and Soize, C. (2006) Probabilistic approach for model and data uncertainties and its experimental identification in structural dynamics: case of composite sandwich panels. Journal of Sound and Vibration, Vol: 294(1-2) pp. 64-81.

[9] Duchereau, J. and Soize, C. (2006) Transient dynamics in structures with nonhomogeneous uncertainties induced by complex joints. Mechanical Systems and Signal Processing, Vol: 20(4) pp. 854-867.

[10] Durand, J-F., Soize, C., and Gagliardini, L. (2008) Structural-acoustic modeling of automotive vehicles in presence of uncertainties and experimental identification and validation. Journal of the Acoustical Society of America, Vol: 124 (3) pp. 1513-1525.

[11] Fernandez, C., Soize, C. and Gagliardini, L. (2010) Sound-insulation layer modelling in car computational vibroacoustics in the medium-frequency range. Acta Acustica united with Acustica (AAUWA), Vol: 96(3) pp. 437-444.

[12] Rubinstein, R.Y. (2007) Simulation and the Monte Carlo Method. Series in Probability and Statistics. John Wiley and Sons, New Jersey, USA, 2nd edition.

[13] Jaynes, E.T. (2003) Probability Theory: The Logic of Science. Cambridge University Press.

[14] Nataf, A. (1962) Détermination des distributions de probabilités dont les marges sont données, Comptes Rendus de lAcadémie des Sciences, 225, pp. 42-43.

[15] Soize, C. (2008) Tensor-valued random fields for meso-scale stochastic model of anisotropic elastic microstructure and probabilistic analysis of representative volume element size. Probabilistic Engineering Mechanics, 23(2-3), pp. 307-323.