The Price of Stability of Weighted Congestion Games[*]

George Christodoulou[†] Martin Gairing[†] Yiannis Giannakopoulos[‡]
Paul Spirakis[§]

April 16, 2018

Abstract

We give exponential lower bounds on the Price of Stability (PoS) of weighted congestion games with polynomial cost functions. In particular, for any positive integer d we construct rather simple games with cost functions of degree at most d which have a PoS of at least $\Omega(\Phi_d)^{d+1}$, where $\Phi_d \sim d/\ln d$ is the unique positive root of equation $x^{d+1} = (x+1)^d$. This asymptotically closes the huge gap between $\Theta(d)$ and Φ_d^{d+1} and matches the Price of Anarchy upper bound. We further show that the PoS remains exponential even for singleton games. More generally, we also provide a lower bound of $\Omega((1+1/\alpha)^d/d)$ on the PoS of α-approximate Nash equilibria. All our lower bounds extend to network congestion games, and hold for mixed and correlated equilibria as well.

On the positive side, we give a general upper bound on the PoS of approximate Nash equilibria, which is sensitive to the range W of the player weights. We do this by explicitly constructing a novel approximate potential function, based on Faulhaber's formula, that generalizes Rosenthal's potential in a continuous, analytic way. From the general theorem, we deduce two interesting corollaries. First, we derive the existence of an approximate pure Nash equilibrium with PoS at most $(d+3)/2$; the equilibrium's approximation parameter ranges from $\Theta(1)$ to $d+1$ in a smooth way with respect to W. Secondly, we show that for unweighted congestion games, the PoS of α-approximate Nash equilibria is at most $(d+1)/\alpha$.

1 Introduction

In the last 20 years, a central strand of research within Algorithmic Game Theory has focused on understanding and quantifying the inefficiency of equilibria compared to centralized, optimal solutions. There are two standard concepts that measure this inefficiency. The Price of Anarchy (PoA) [32] which takes the worst-case perspective, compares the worst-case equilibrium with the system optimum. It is a very robust measure of performance. On the other hand, the Price of Stability (PoS) [44, 5], which is also the focus of this work, takes an optimistic perspective, and uses the best-case equilibrium for this comparison. The PoS is an appropriate concept to analyse the ideal solution that we would like our protocols to produce.

The initial set of problems that arose from the Price of Anarchy theory have now been resolved. The most rich and well-studied among these models are, arguably, the atomic and non-atomic variants of congestion games (see [37, Ch. 18] for a detailed discussion). This class

[*]Supported by the Alexander von Humboldt Foundation with funds from the German Federal Ministry of Education and Research (BMBF), and by EPSRC grant EP/M008118/1.

[†]Department of Computer Science, University of Liverpool. Emails: G.Christodoulou@liverpool.ac.uk, gairing@liverpool.ac.uk

[‡]Department of Mathematics, TU Munich. Email: Yiannis.Giannakopoulos@tum.de

[§]Department of Computer Science, University of Liverpool, and Computer Engineering and Informatics Department, University of Patras. Email: P.Spirakis@liverpool.ac.uk

of games is very descriptive and captures a large variety of scenarios where users compete for resources, most prominently routing games. The seminal work of Roughgarden and Tardos [42, 43] gave the answer for the non-atomic variant, where each player controls a negligible amount of traffic. Awerbuch et al. [6], Christodoulou and Koutsoupias [15] resolved the Price of Anarchy for atomic congestion games with affine latencies, generalized by Aland et al. [3] to polynomials; this led to the development of Roughgarden's Smoothness Framework [41] which extended the bounds to general cost functions, but also distilled and formulated previous ideas to bound the Price of Anarchy in an elegant, unified framework. At the computational complexity front, we know that even for simple congestion games, finding a (pure) Nash equilibrium is a PLS-complete problem [20, 2].

Allowing the players to have different loads, gives rise to the class of *weighted* congestion games [40]; this is a natural and very important generalization of congestion games, with numerous applications in routing and scheduling. Unfortunately though, an immediate dichotomy between weighted and unweighted congestion games occurs: the former may *not* even have pure Nash equilibria [35, 23, 25, 28]; as a matter of fact, it is a strongly NP-hard problem to even determine if that's the case [19]. Moreover, in such games there does not, in general, exist a potential function [36, 29], which is the main tool for proving equilibrium existence in the unweighted case.

As a result, a sharp contrast with respect to our understanding of the two aforementioned inefficiency notions arises. The Price of Anarchy has been studied in depth and general techniques for providing tight bounds are known. Moreover, the asymptotic behaviour of weighted and unweighted congestion games with respect to the Price of Anarchy is identical; it is $\Theta(d/\log d)^d$ for both classes when latencies are polynomials of degree at most d [3].

The situation for the Price of Stability though, is completely different. For unweighted games we have a good understanding[1] and the values are much lower than the Price of Anarchy values, and also *tight*; approximately 1.577 for affine functions [16, 11], and $\Theta(d)$ [14] for polynomials. For weighted games though there is a huge gap; the current state of the art lower bound is $\Theta(d)$ and the upper bound is $\Theta(d/\ln d)^d$. These previous results are summarized at the left of Table 1.

The main focus of this work is precisely to deal with this lack of understanding, and to determine the Price of Stability of weighted congestion games. What makes this problem challenging is that the only general known technique for showing upper bounds for the Price of Stability is the potential method, which is applicable only to potential games. In a nutshell, the idea of this method is to use the global minimizer of Rosenthal's potential [39] as an equilibrium refinement. This equilibrium is also a pure Nash equilibrium and can serve as an upper bound of the Price of Stability. Interestingly, it turns out that, for several classes of potential games, this technique actually provides the tight answer (see for example [5, 16, 11, 14]). However, as already mentioned above, unlike their unweighted counterparts, weighted congestion games are not potential games;[2] so, a completely fresh approach is required.

One way to override the aforementioned limitations of non-existence of pure Nash equilibria, but also their computational hardness, is to consider *approximate* equilibria. In this direction,

[1]Much work has been done on the PoS for network design games, which is though not so closely related to our work. This problem was first studied by Anshelevich et al. [5] who showed a tight bound of H_n, the harmonic number of the number of players n, for directed networks. Finding tight bounds on undirected networks is still a long-standing open problem (see, e.g., [22, 9, 33]). Recently, Bilò et al. [10] (asymptotically) resolved the question for broadcast networks. For the weighted variant of this problem, Albers [4] showed a lower bound of $\Omega(\log W/\log\log W)$, where W is the sum of the players' weights. See [10] and references therein for a thorough discussion of those results.

[2]For the special case of weighted congestion games with linear latency functions, a potential does exist [23] and this was used by [8] to provide a PoS upper bound of 2.

	PoA	PoS
unweighted	$\lfloor\Phi_d\rfloor^{d+1}$ [3]	$\Theta(d)$ [14]
weighted	Φ_d^{d+1} [3]	$[\Theta(d), \Phi_d^{d+1}]$

	PoS lower bound
general	$\Omega(\Phi_d)^{d+1}$
singleton	$\Omega(2^d/d)$
α-approximate equilibria	$\Omega((1+1/\alpha)^d/d)$

Table 1: Previous results (*left table*) regarding the Price of Anarchy and Stability for unweighted and weighted congestion games, with polynomial latency functions of maximum degree d. Φ_d is the unique positive solution of $(x+1)^d = x^{d+1}$ and $\Phi_d = \Theta(d/\log d)$. Tight answers were known for all settings, except for the Price of Stability of the weighted case were only trivial bounds existed. In this paper (*right table*) we (asymptotically) close this gap by showing a lower bound of $\Omega(\Phi_d)^{d+1}$ (Theorem 1), even for network games, which is exponential even for singleton games Theorem 2.

Hansknecht et al. [27] have shown that $(d+1)$–approximate pure Nash equilibria always exist in weighted congestion games with polynomial latencies of maximum degree d, while, in the negative side, there exist games that do not have 1.153–approximate pure Nash equilibria. Notice here, that these results do not take into account computational complexity considerations; if we insist in polynomial-time algorithms for actually finding those equilibria, then the currently best approximation parameter becomes $d^{O(d)}$ [12, 13, 21].

1.1 Our Results

We provide lower and upper bounds on the Price of Stability for the class of weighted congestion games with polynomial latencies with nonnegative coefficients. We consider both exact and approximate equilibria. Our lower bounds are summarized at the right of Table 1.

Lower Bound for Weighted Congestion Games. In our main result in Theorem 1, we resolve a long-standing open problem by providing asymptotically tight bounds for the Price of Stability of weighted congestion games with polynomial latency functions. We construct an instance having a Price of Stability of $\Omega(\Phi_d)^{d+1}$, where d is the maximum degree of the latencies and $\Phi_d \sim \frac{d}{\ln d}$ is the unique positive solution of equation $(x+1)^d = x^{d+1}$.

This bound essentially closes the previously huge gap between $\Theta(d)$ and Φ_d^{d+1} for the PoS of weighted congestion games. The previously best lower and upper bounds were rather trivial: the lower bound corresponds to the PoS results of Christodoulou and Gairing [14] for the unweighted case (and thus, it is also a valid lower bound for the general weighted case as well) and the upper bound comes from the Price of Anarchy results of Aland et al. [3] (PoA, by definition, upper-bounds PoS).

We stress that, although as mentioned before, weighted congestion games do not always possess pure equilibria, our lower bound construction involves a *unique* equilibrium occurring by iteratively eliminating strongly dominated strategies. As a result, this lower bound holds not only for pure, but mixed and correlated equilibria as well.

Singleton Games. Next we switch to the class of singleton congestion games, where a pure strategy for each player is a *single* facility. This class is very well-studied as, on one hand, it abstracts scheduling environments, and on the other, it has very attractive equilibrium properties; unlike general weighted congestion games, there exists an (ordinal) lexicographic potential [24, 30], thus implying the existence of *pure* Nash equilibria. It is important to note that, the tight lower bounds for the Price of Anarchy of general weighted congestion games, hold also for the class of singleton games [11, 7].

Nevertheless, even for this special class, we show in Theorem 2 an exponential lower bound of $\Omega(2^d/d)$. The previous best upper and lower bounds were the same as those of the general case, namely $\Theta(d)$ and Φ_d^{d+1}, respectively. As a matter of fact, this new lower bound comes as a corollary of a much more general result that we show in Theorem 2, that extends to approximate equilibria and gives a lower bound of $\Omega((1+1/\alpha)^d/d)$ on the PoS of α-approximate equilibria, for any (multiplicative) approximation parameter $\alpha \in [1, d)$. Setting $\alpha = 1$ we recover the special case of exact equilibria and the aforementioned exponential lower bound on the standard, exact notion of the PoS. Notice here, that as we show in Theorem 4, the optimal solution (which, in general, is not an equilibrium) itself constitutes a $(d+1)$–approximate equilibrium with a (trivially) optimal PoS of 1.

Positive Results for Approximate Equilibria. In light of the above results, in Section 4, we turn our attention to identifying environments with more structure or flexibility with respect to the underlying solution concept, for which we can hope for improved quality of equilibria. Both our lower bound constructions discussed above use players' weights that form a geometric sequence. In particular the ratio W of the largest over the smallest weight is equal to w^n, which grows very large as the number of players $n \to \infty$. On the other hand, for games where the players have equal weights, i.e. $W = 1$, we know that the PoS is at most $d + 1$. It is therefore natural to ask how the performance of the good equilibria captured by the notion of PoS varies with respect to W. In Theorem 3, we are able to give a general upper bound for approximate equilibria which is sensitive to this parameter W. This general theorem has two immediate, interesting corollaries.

Firstly (Corollary 1), by fixing ξ and allowing the ratio W to range in $[1, \infty)$, we derive the existence of an approximate pure Nash equilibrium with PoS at most $(d+3)/2$; the equilibrium's approximation parameter ranges from $\Theta(1)$ to $d+1$ in a smooth way with respect to W. This is of particular importance in settings where player weights are not very far away from each other (that is, W is small). Secondly (Corollary 2), by setting $W = 1$ and allowing α to range, we get an upper bound of $\frac{d+1}{\alpha}$ for the α-approximate PoS of *unweighted* congestion games which, to the best of our knowledge, was not known before, degrading gracefully from $d + 1$ (which is the actual PoS of exact equilibria in the unweighted case [14]) down to the optimal value of 1 if we allow $(d+1)$-approximate equilibria (which in fact can be achieved by the optimum solution itself; see Theorem 4).

Our Techniques. An advantage of our main lower bound (Theorem 1) is the simplicity of the underlying construction, as well as its straightforward adaptation to network games (see Section 3.1.1)). However, fine-tuning the parameters of the game (player weights and latency functions), to ensure uniqueness of the equilibrium at the "bad" instance, was a technically involved task. This was in part due to the fact that, in order to guarantee uniqueness (via iteratively dominant strategies), each player interacts with a window of μ other players. This μ depends on d in a delicate way (see Fig. 1 and Lemma 1); it has to be an integer but, at the same time, needs also to balance nicely with the algebraic properties of Φ_d (see, e.g., (1), (3) and (4)). Moreover we needed to provide deeper insights on the asymptotic, analytic behaviour of Φ_d, and to explore some new algebraic characteristics of Φ_d (see, e.g., Lemma 7).

In order to derive our upper bounds, we need to define a novel *approximate potential function* [17, 27]. First, we identify (Lemma 2) clear algebraic sufficient conditions for the existence of approximate equilibria with good social-cost guarantees, and then explicitly define (see the (14) and (19) in the proof of Theorem 3) a function that satisfies them. This continuous function, which is defined in the entire space of positive reals, essentially generalizes that of

Rosenthal's in a smooth way: by setting $W = \alpha = 1$, we recover exactly the first significant terms of the well known Rosenthal potential [39] polynomial, with which one can demonstrate the usual PoS results for the unweighted case (see, e.g. [16]). The simple, analytic way in which this function is defined, is the very reason why we can handle both the approximation parameter α of the equilibrium and the ratio W of the weights in a smooth manner while at the same time providing good PoS guarantees.

It is important to stress that, by the purely analytical way in which our approximate potential function is defined, in principle it can also incorporate more general cost functions than polynomials; so, we believe that this technique may be of independent interest. We point towards that direction in Appendix C.

2 Model and Notation

Let $\mathbb{R} = (-\infty, \infty)$ denote the set of real numbers, and in the natural way define $\mathbb{R}_{\geq 0} = [0, \infty)$ and $\mathbb{R}_{>0} = (0, \infty)$.

Weighted Congestion Games. A *weighted congestion game* consists of a finite, nonempty set of players N and resources (or facilities) E. Each player $i \in N$ has a *weight* $w_i \in \mathbb{R}_{>0}$ and a *strategy set* $S_i \subseteq 2^E$. Associated with each resource $e \in E$ is a *cost* (or *latency*) *function* $c_e : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0}$. In this paper we mainly focus on polynomial cost functions with maximum degree $d \geq 0$ and nonnegative coefficients; that is, every cost function is of the form $c_e(x) = \sum_{j=0}^{d} a_{e,j} \cdot x^j$, with $a_{e,j} \geq 0$ for all j. In the following, whenever we refer to polynomial cost functions we mean cost functions of this particular form.

A *pure strategy profile* is a choice of strategies $\mathbf{s} = (s_1, s_2, ...s_n) \in S = S_1 \times \cdots \times S_n$ by the players. We use the standard game-theoretic notation $\mathbf{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$, $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$, such that $\mathbf{s} = (s_i, \mathbf{s}_{-i})$. Given a pure strategy profile $\mathbf{s}$, we define the *load* $x_e(\mathbf{s})$ of resource $e \in E$ as the total weight of players that use resource e on $\mathbf{s}$, i.e., $x_e(\mathbf{s}) = \sum_{i \in N : e \in s_i} w_i$. The *cost* player i is defined by $C_i(\mathbf{s}) = \sum_{e \in s_i} c_e(x_e(\mathbf{s}))$.

A *singleton* weighted congestion game is a special form of congestion games where the strategies of all players consist only of single resources; that is, for all players $i \in N$, $|s_i| = 1$ for all $s_i \in S_i$. In a weighted *network* congestion games the resources E are given as the edge set of some directed graph $G = (V, E)$, and each player $i \in N$ has a source $o_i \in V$ and destination $t_i \in V$ node; then, the strategy set S_i of each player is implicitly given as the edge sets of all directed $o_i \to t_i$ paths in G.

Nash Equilibria. A pure strategy profile $\mathbf{s}$ is a pure *Nash equilibrium* if and only if for every player $i \in N$ and for all $s'_i \in S_i$, we have $C_i(\mathbf{s}) \leq C_i(s'_i, \mathbf{s}_{-i})$. Similarly a strategy profile is an *α-approximate pure Nash equilibrium*, for $\alpha \geq 1$, if $C_i(\mathbf{s}) \leq \alpha \cdot C_i(s'_i, \mathbf{s}_{-i})$ for all players $i \in N$ and $s'_i \in S_i$. As discussed in the introduction, weighted congestion games do not always admit pure Nash equilibria. However, by Nash's theorem they have mixed Nash equilibria. A tuple $\sigma = (\sigma_1, \cdots, \sigma_N)$ of independent probability distributions over players' strategy sets is a *mixed Nash equilibrium* if

$$\mathop{\mathbb{E}}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s})] \leq \mathop{\mathbb{E}}_{\mathbf{s}_{-i} \sim \sigma_{-i}}[C_i(s'_i, \mathbf{s}_{-i})]$$

holds for every $i \in N$ and $s'_i \in S_i$. Here σ_{-i} is a product distribution of all σ_j's with $j \neq i$, and $\mathbf{s}_{-i}$ denotes a strategy profile drawn from this distribution. We use $\mathrm{NE}(G)$ to denote the set of all mixed Nash equilibria of a game G.

Social Cost and Price of Stability. Fix a weighted congestion game G. The *social cost* of a pure strategy profile $\mathbf{s}$ is the weighted sum of the players' costs

$$C(\mathbf{s}) = \sum_{i \in N} w_i \cdot C_i(\mathbf{s}) = \sum_{e \in E} x_e(\mathbf{s}) \cdot c_e(x_e(\mathbf{s})).$$

Denote by $\mathrm{OPT}(G) = \min_{\mathbf{s} \in S} C(\mathbf{s})$ the *optimum social cost* over all strategy profiles $\mathbf{s} \in S$. Then, the *Price of Stability (PoS)* of G is the social cost of the best-case Nash equilibrium over the optimum social cost:

$$\mathrm{PoS}(G) = \min_{\sigma \in \mathrm{NE}(G)} \frac{\mathbb{E}_{\mathbf{s} \sim \sigma}[C(\mathbf{s})]}{\mathrm{OPT}(G)}.$$

The Price of Stability of α-approximate Nash equilibria is defined accordingly. The PoS for a class $\mathcal{G}$ of games is the worst (i.e., largest) PoS among all games in the class, that is, $\mathrm{PoS}(\mathcal{G}) = \sup_{G \in \mathcal{G}} \mathrm{PoS}(G)$. For example, our focus in this paper is determining the Price of Stability for the class $\mathcal{G}$ of weighted congestion games with polynomial cost functions.

Finally, notice that, by using a straightforward scaling argument, it is without loss with respect to the PoS metric to analyse games with player weights in $[1, \infty)$; if not, divide all w_i's with $\min_i w_i$ and scale cost functions accordingly.

3 Lower Bounds

In this section, we present our lower bound constructions. In Section 3.1 we present the general lower bound and then in Section 3.2 the lower bound for singleton games.

3.1 General Congestion Games

The next theorem presents our main negative result on the Price of Stability of weighted congestion games with polynomial latencies of degree d, that asymptotically matches the Price of Anarchy upper bound of Φ_d^{d+1} from Aland et al. [3]. Our result, shows a strong separation of the Price of Stability of weighted and unweighted congestion games, where the Price of Stability is at most $d+1$ [14]. This is in sharp contrast to the Price of Anarchy of these two classes, where the respective bounds are essentially the same.

We will need to introduce some notation. Let $\Phi_d = \Theta\left(\frac{d}{\ln d}\right)$ be the unique positive root of equation $(x+1)^d = x^{d+1}$ and let β_d be a parameter such that $\beta_d \geq 0.38$ for any d, $\lim_{d \to \infty} \beta_d = \frac{1}{2}$. A plot of its values can be seen in Fig. 1.

Theorem 1. *The Price of Stability of weighted congestion games with polynomial latency functions of degree at most $d \geq 9$ is at least $(\beta_d \Phi_d)^{d+1}$.*

We will first need the following technical lemma. Its proof can be found in Appendix A.2.

Lemma 1. *For any positive integer d define*

$$c_d = \frac{1}{d}\left\lfloor d \frac{\ln(\Phi_d^{1+2/d} - \Phi_d) - \ln(\Phi_d^{1+2/d} - \Phi_d - 1)}{\ln \Phi_d} \right\rfloor \tag{1}$$

and

$$\beta_d = 1 - \Phi_d^{-c_d}, \tag{2}$$

Then

$$\Phi_d^{d+2} \leq \left(\Phi_d + \frac{1}{\beta_d}\right)^d, \tag{3}$$

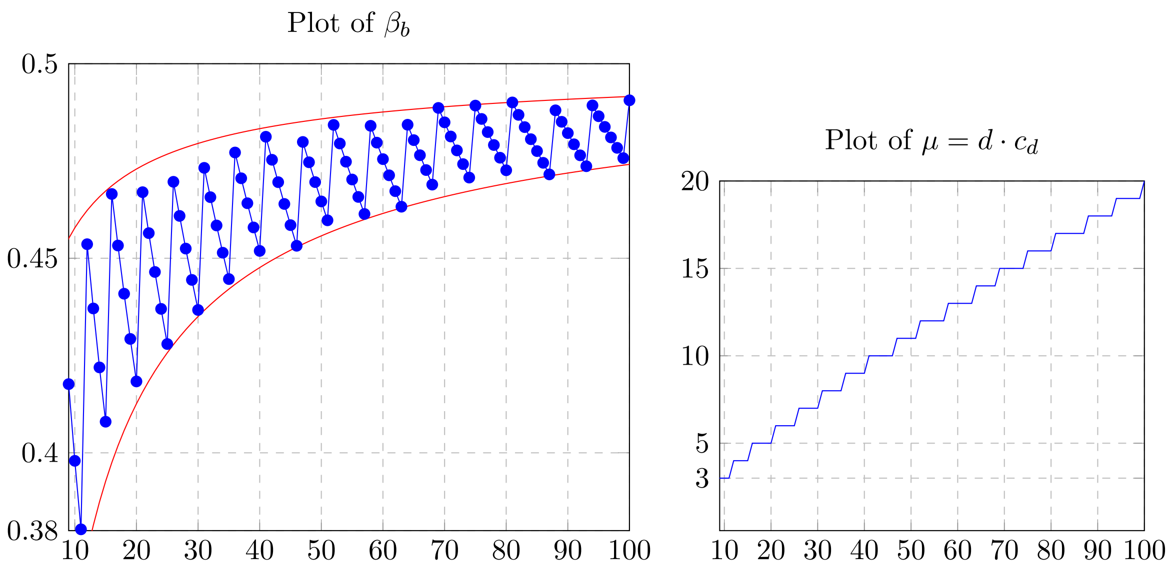

Figure 1: The values of parameters β_b and c_d in Lemma 1 and Theorem 1, for $d = 9, 10, \ldots, 100$

and for all $d \geq 9$,

$$d \cdot c_d \geq 3, \qquad 0.38 \leq \beta_d \leq \frac{1}{2} \qquad \textit{and} \qquad \lim_{d\to\infty} \beta_d = \frac{1}{2}. \tag{4}$$

Plots of parameters c_d and β_d can be found in Fig. 1.

Proof of Theorem 1. We now move on to the description of our congestion game instance. Fix some integer[3] $d \geq 9$. Our instance consists of $n + \mu$ players and $n + \mu + 1$ facilities, where $\mu \equiv c \cdot d$ for some real $c \geq \frac{3}{d}$ (to be specifically determined later on, see (1)) such that $\mu \geq 3$ is an integer. You can think of n as a very large integer, since at the end we will take $n \to \infty$. Every player $i = 1, 2, \ldots, n + \mu$ has a weight of $w_i = w^i$, where $w = 1 + \frac{1}{\Phi_d}$.

It will be useful for subsequent computations to notice that

$$w^d = \left(1 + \frac{1}{\Phi_d}\right)^d = \frac{(\Phi_d + 1)^d}{\Phi_d^d} = \frac{\Phi_d^{d+1}}{\Phi_d^d} = \Phi_d,$$

$$w^{d+1} = w^d \cdot w = \Phi_d \left(1 + \frac{1}{\Phi_d}\right) = \Phi_d + 1.$$

Let also define

$$\alpha = \alpha(\mu) \equiv \sum_{j=1}^{\mu} w^{-j} = \frac{1 - w^{-\mu}}{w - 1} = \frac{1 - (w^d)^{-c}}{w - 1} = \frac{1 - \Phi_d^{-c}}{1 + \frac{1}{\Phi_d} - 1} = \Phi_d \left(1 - \Phi_d^{-c}\right) = \beta_d \Phi_d,$$

where

$$\beta_d \equiv 1 - \Phi_d^{-c} \qquad\qquad \in (0, 1).$$

Observe that

$$w^{-\mu} = 1 - \beta_d \Phi_d (w - 1) = 1 - \beta_d \Phi_d \left(1 + \frac{1}{\Phi_d} - 1\right) = 1 - \beta_d$$

[3] For polynomial latencies of smaller degrees $d \leq 8$ we can instead apply the simpler lower-bound instance for singleton games given in Section 3.2.

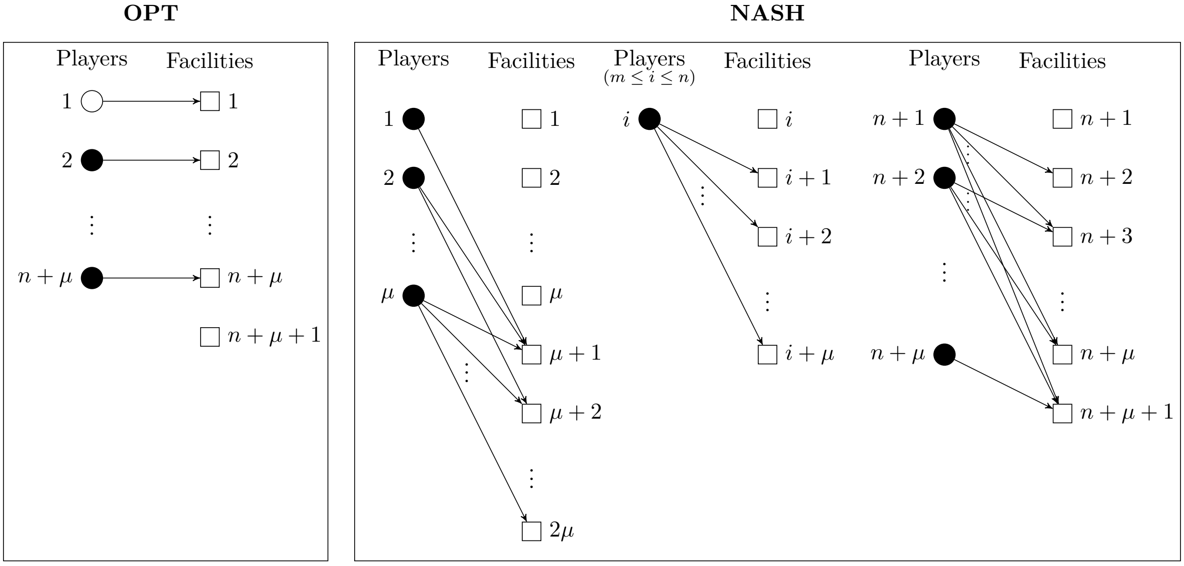

Figure 2: The social optimum $\mathbf{s}^*$ and the unique Nash equilibrium $\tilde{\mathbf{s}}$ in the lower bound construction of Theorem 1 for general weighted congestion games.

and furthermore, for every $i \geq \mu + 1$

$$\sum_{j=i-\mu}^{i-1} w_j = \sum_{j=1}^{\mu} w^{i-j} = \alpha \cdot w^i \qquad \text{and} \qquad \sum_{j=i-\mu}^{i} w_j = (\alpha+1) \cdot w^i,$$

and

$$\sum_{\ell=1}^{\infty} w^{-\ell} = \frac{1}{w-1} = \frac{1}{1+\frac{1}{\Phi_d}-1} = \Phi_d.$$

The facilities have latency functions

$$\begin{aligned} c_j(t) &= (\alpha+1)^d, && \text{if } j = 1, \dots, \mu, \\ c_j(t) &= w^{-j(d+1)} t^d, && \text{if } j = \mu+1, \dots, \mu+n, \\ c_{n+\mu+1}(t) &= 0, \end{aligned}$$

where for simplicity we use j instead e_j to refer to the j-th facility.

Every player i has two available strategies, s_i^* and $\tilde{s}_i$. Eventually we will show that the profile $\mathbf{s}^*$ corresponds to the optimal solution, while $\tilde{\mathbf{s}}$ corresponds to the *unique* Nash equilibrium of the game. Informally, at the former the player chooses to stay at her "own" i-th facility, while at the latter she chooses to deviate and play the μ following facilities $i+1, \dots, i+\mu$. However, special care shall be taken for the boundary cases of the first μ and last μ players, so for any player i we formally define $S_i = \{s_i^*, \tilde{s}_i\}$ where $s_i^* = \{i\}$ and

$$\tilde{s}_i = \begin{cases} \{\mu+1, \dots, \mu+i\}, & \text{if } i = 1, \dots, \mu, \\ \{i+1, \dots, i+\mu\}, & \text{if } i = \mu+1, \dots, n, \\ \{i+1, \dots, n+\mu+1\}, & \text{if } i = n+1, \dots, n+\mu. \end{cases}$$

These two outcomes, $\mathbf{s}^*$ and $\tilde{\mathbf{s}}$, are shown in Fig. 2.

Notice here that any facility j cannot get a load greater than the sum of the weights of the previous μ players plus the weight of the j-th player. So, for and any strategy profile $\mathbf{s}$:

$$x_j(\mathbf{s}) \leq \sum_{\ell=j-\mu}^{j} w_\ell = (\alpha+1) w^j \qquad \text{for all } j \geq \mu+1 \tag{5}$$

Next we will show that the strategy profile $\tilde{\mathbf{s}} = (\tilde{s}_1, \dots, \tilde{s}_{n+\mu})$ is the *unique* Nash equilibrium of our congestion game. We do that by proving that

1. It is a strongly *dominant* strategy for any player $i = 1, \dots, \mu$ to play $\tilde{s}_i$.
2. For any $i = \mu + 1, \dots, n + \mu$, given that every player $k < i$ has chosen to play $\tilde{s}_k$, then it is a strongly *dominant* strategy for player i to deviate to $\tilde{s}_i$ as well.

For the first condition, fix some player $i \leq \mu$ and a strategy profile $\mathbf{s}_{-i}$ for the other players and observe that by choosing $\tilde{s}_i$, player i incurs a cost of at most

$$\begin{aligned}
C_i(\tilde{s}_i, \mathbf{s}_{-i}) &= \sum_{j \in \tilde{s}_i} c_j(x_j(\tilde{s}_i)) \leq \sum_{\ell=\mu+1}^{\mu+i} c_\ell\left((\alpha+1)w^\ell\right) \\
&= \sum_{\ell=d+1}^{d+i} w^{-\ell(d+1)}(\alpha+1)^d w^{\ell d} = (\alpha+1)^d \sum_{\ell=d+1}^{d+i} w^{-\ell} \\
&< (\alpha+1)^d w^{-d} \sum_{\ell=1}^{\infty} w^{-\ell} = (\alpha+1)^d \frac{1}{\Phi_d}\Phi_d = (\alpha+1)^d \\
&= C_i(s_i^*, \mathbf{s}_{-i}),
\end{aligned}$$

where in the first inequality we used the bound from (5).

For the second condition, we will consider the deviations of the remaining players.[4] Fix now some $i = \mu + 1, \dots, n$ and assume a strategy profile $\mathbf{s}_{-i} = (\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_{i+1}, \dots, s_{n+\mu})$ for the remaining players. If player i chooses strategy s_i^* she will experience a cost of

$$C_i(s_i^*, \mathbf{s}_{-i}) = c_i\left(\sum_{\ell=i-\mu}^{i} w_\ell\right) = c_i\left((\alpha+1)w^i\right) = w^{-i(d+1)}(\alpha+1)^d w^{id} = (\alpha+1)^d w^{-i}.$$

It remains to show that

$$C_i(\tilde{s}_i, \mathbf{s}_{-i}) < C_i(s_i^*, \mathbf{s}_{-i}) = (\alpha+1)^d w^{-i}. \tag{6}$$

The cost $C_i(\tilde{s}_i, \mathbf{s}_{-i})$ is complicated to bound immediately, for any profile $\mathbf{s}_{-i}$. Instead, we will resort to the following claim which characterizes the profile $\mathbf{s}_{-i}$ where this cost is maximized, as shown in Fig. 3. Its proof can be found in Appendix A.3.

Claim 1. *There exists a profile $\mathbf{s}'$ with*

1. $s'_j = s_j$ *for all* $j \leq i$ *and* $i > i + \mu$
2. $s'_{i+\mu} = s^*_{i+\mu}$
3. *there exists some* $k \in \{i+1, \dots, i+\mu-1\}$ *such that*

$$s'_j = \tilde{s}_j \qquad \textit{for all } j \in \{i+1, \dots, i+\mu-1\} \setminus \{k\},$$

that dominates $\mathbf{s}$, *i.e.*

$$C_i(\tilde{s}_i, \mathbf{s}_{-i}) \leq C_i(\tilde{s}_i, \mathbf{s}'_{-i}). \tag{7}$$

[4]For the remaining last μ players $i = n+1, \dots, n+\mu$ the proof is similar to the text, and as a matter of fact easier, since when these players deviate to $\tilde{s}_i$ they also use the final "dummy" facility $n + \mu + 1$ that has zero cost.

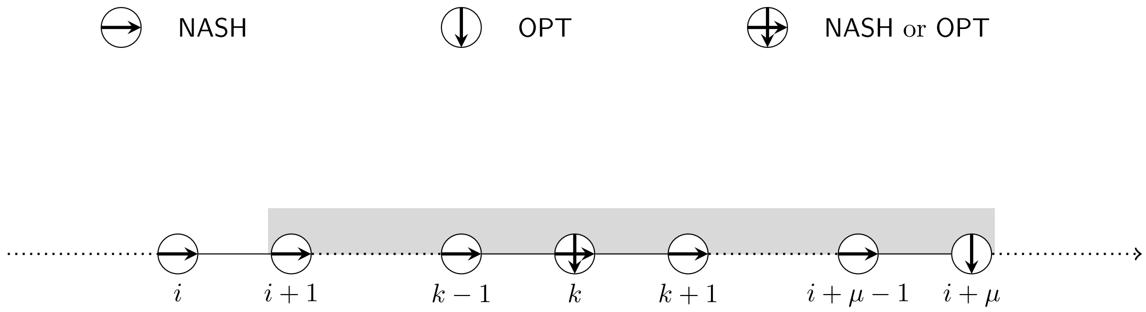

Figure 3: The format of profile $\mathbf{s}'$ described in Claim 1 and returned as output from Procedure $\textsc{Dominate}(\mathbf{s}, i)$ (see Appendix A.3). All players $i+1, \ldots, i+\mu$ (i.e., the those who lie within the window of interest of player i, depicted in grey) play according to the Nash equilibrium $\tilde{\mathbf{s}}$, except the last player $i+\mu$ (that plays according to the optimal profile $\mathbf{s}^*$) and at most one other k (that may play either $\tilde{s}_k$ or s_k^*).

By use of Claim 1, it remains to show

$$C_i(\tilde{s}_i, \mathbf{s}'_{-i}) < (\alpha+1)^d w^{-i}, \tag{8}$$

just for the special case of profiles $\mathbf{s}'$ that are described in Claim 1 and also shown in Fig. 3. We do this in Appendix A.4.

Summarizing, we proved that indeed $\tilde{\mathbf{s}}$ is the *unique* Nash equilibrium of our congestion game. Finally, to conclude with lower-bounding the Price of Stability, let us compute the social cost on profiles $\tilde{\mathbf{s}}$ and $\mathbf{s}^*$. On $\mathbf{s}^*$, any facility j (except the last one) gets a load equal to the weight of player j, so

$$\begin{aligned}
C(\mathbf{s}^*) &= \sum_{j=1}^{n+\mu} w_j c_j(w_j) \\
&= \sum_{j=1}^{\mu} w^j (\alpha+1)^d + \sum_{j=\mu+1}^{n+\mu} w^j w^{-j(d+1)} (w^j)^d \\
&= (\alpha+1)^d \sum_{j=1}^{\mu} w^j + \sum_{j=\mu+1}^{\mu+n} 1 \\
&= (\alpha+1)^d w \frac{w^\mu - 1}{w-1} + n \\
&= n + (\beta\Phi_d + 1)^d \left(1 + \frac{1}{\Phi_d}\right) \frac{\frac{1}{1-\beta} - 1}{1 + \frac{1}{\Phi_d} - 1} \\
&= n + (\beta\Phi_d + 1)^d (\Phi_d + 1) \frac{\beta}{1-\beta} \\
&\leq n + \frac{\beta}{1-\beta} (\Phi_d + 1)^{d+1}.
\end{aligned}$$

On the other hand, at the unique Nash equilibrium $\tilde{\mathbf{s}}$ each facility $j \geq \mu+1$ receives a load equal to the sum of the weights of the previous μ players, i.e.

$$x_j(\tilde{\mathbf{s}}) = \sum_{\ell=j-\mu}^{j-1} w_\ell = \alpha w^j$$

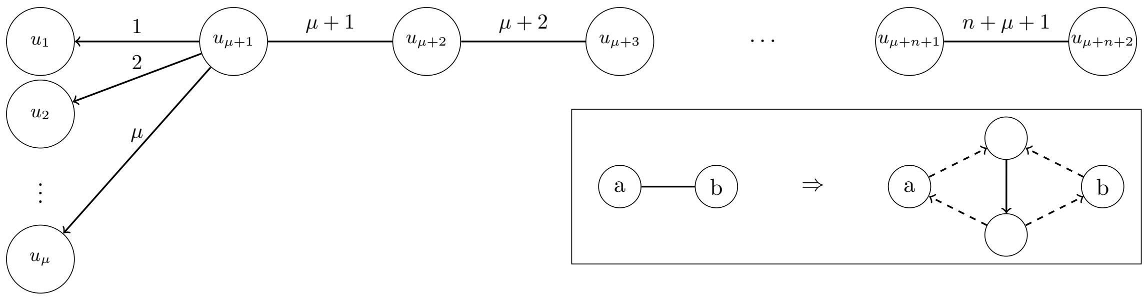

Figure 4: Transformation of the lower bound instance of Theorem 1 for general weighted congestion games to a network game, as described in Proposition 1.

so

$$C(\tilde{\mathbf{s}}) \geq \sum_{j=\mu+1}^{n+\mu} x_j(\tilde{\mathbf{s}})c_j(x_j(\tilde{\mathbf{s}})) = \sum_{j=\mu+1}^{n+\mu} w^{-j(d+1)} \left(\alpha w^j\right)^{d+1} = \alpha^{d+1} \sum_{j=\mu+1}^{\mu+n} 1 = \alpha^{d+1} n.$$

By taking n arbitrarily large we get a lower bound on the Price of Stability of

$$\lim_{n\to\infty} \frac{C(\tilde{\mathbf{s}})}{C(\mathbf{s}^*)} \geq \lim_{n\to\infty} \frac{\alpha^{d+1}n}{n + \frac{\beta}{1-\beta}(\Phi_d+1)^{d+1}} = \alpha^{d+1} = (\beta\Phi_d)^{d+1},$$

where from Lemma 1 we know that $\frac{1}{3} \leq \beta = \frac{1}{2} - o(1)$. $\square$

3.1.1 Network Games

Due to the rather simple structure of the players' strategy sets in the lower bound construction of Theorem 1, it can be readily extended to network games as well:

Proposition 1. *Theorem 1 applies also to* network *weighted congestion games.*

Proof. We arrange the resources from the proof of Theorem 1 as edges in a graph as depicted in Fig. 4. All undirected edges should be replaced by the gadget shown on the bottom, where the solid edge gets the cost function of the original edge and dashed edges are zero cost. Each player $i \in [1, n+\mu]$ has to route its traffic from o_i to t_i, where

$$o_i = \begin{cases} u_{\mu+1}, & \text{if } \ i = 1, \dots, \mu, \\ u_{i+1}, & \text{if } \ i = \mu+1, \dots, n+\mu, \end{cases}$$

and t_i is not shown in the figure but connected with *zero cost* edges as follows:

- For each $i \in [1, n+\mu]$ there is a directed zero cost edge from u_i to t_i.
- For each $i \in [1, n]$ there is a directed zero cost edge from $u_{\mu+1+i}$ to t_i.
- For each $i \in [n+1, n+\mu]$ there is a directed zero cost edge from $u_{\mu+n+2}$ to t_i.

By construction, each player i has two available $o_i \to t_i$ paths, which correspond directly to strategy sets s_i^* and $\tilde{s}_i$ used in the proof of Theorem 1. $\square$

3.2 Singleton Games

In this section we give an exponential lower bound for *singleton* weighted congestion games with polynomial latency functions. The following theorem handles also approximate equilibria and provides a lower bound on the Price of Stability in a very strong sense; even if one allows for the best approximate equilibrium, with approximation factor $\alpha = o\left(\frac{d}{\ln d}\right)$, then its cost is lower-bounded by $\omega(d)$ times the optimal cost.[5] In other words, in order to achieve linear guarantees on the Price of Stability, one has to consider almost $\Omega(d)$-approximate equilibria; this shows, that our positive result in Corollary 1 of the following Section 4.3 is essentially tight. This is complemented by Theorem 4, where we show that the socially optimum profile is a $(d+1)$–approximate equilibrium, achieving an optimal Price of Stability of 1.

Theorem 2. *For any positive integer d and any real $\alpha \in [1, d)$, the α-approximate (mixed) Price of Stability of weighted (singleton) congestion games with polynomial latencies of degree at most d is at least*

$$\frac{1}{e(d+1)}\left(1+\frac{1}{\alpha}\right)^{d+1}. \tag{9}$$

In particular, for the special case of $\alpha = 1$, we derive that the Price of Stability of exact *equilibria is $\Omega(2^d/d) = (2 - o(1))^{d+1}$.*

Proof. Fix a positive integer d and the desired approximation parameter $\alpha \in [1, d)$. Also, let $\gamma \in (\alpha, d)$ be a parameter arbitrarily close to α. Our instance consists of n players with weights $w_i = w^i$, $i = 1, 2, \dots, n$, where we set

$$w = \gamma\frac{d+1}{d-\gamma} > \gamma, \tag{10}$$

the inequality holding due to the fact that $d + 1 > d - \gamma > 0$. At the end of our construction we will take $n \to \infty$, so you can think of n as a very large integer. There are $n + 1$ facilities $e_1, e_2, \dots, e_{n+1}$, with latency functions

$$\begin{aligned} c_{e_1}(t) &= \gamma w^d (w+1)^d, & \\ c_{e_j}(t) &= (\gamma w^d)^{2-j} \cdot t^d, & j = 2, \dots, n, \\ c_{e_{n+1}}(t) &= \gamma^{1-n} w^d (w+1)^d. & \end{aligned}$$

Any player i has exactly two strategies, $s_i^* = \{e_i\}$ and $\tilde{s}_i = \{e_{i+1}\}$ i.e., $S_i = \{\{e_i\}, \{e_{i+1}\}\}$ for all $i = 1, \dots, n$. Let $\mathbf{s}^*, \tilde{\mathbf{s}}$ be the strategy profiles where every player i plays $s_i^*, \tilde{s}_i$ respectively. These two outcomes, $\mathbf{s}^*$ and $\tilde{\mathbf{s}}$ are depicted in Fig. 5. One should think of $\mathbf{s}^*$ as the socially optimal profile. We will show that $\tilde{\mathbf{s}}$ is the *unique* α-approximate Nash equilibrium of our game. To ensure this, it is enough to require the following, which corresponds to eliminating all other possible strictly dominated α-approximate equilibria:

1. It is a strictly α-dominant strategy for player 1 to use facility e_2, i.e., $\alpha C_1(\tilde{s}_1, \mathbf{s}_{-i}) < C_1(\mathbf{s})$ for any profile $\mathbf{s}$.

2. For any $i = 2, \dots, n$, if every player $k < i$ has chosen facility e_{k+1} then it is a strictly α-dominant strategy for player i to chose facility e_{i+1}, i.e., $\alpha C_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, \tilde{s}_i, s_{i+1}, \dots, s_n) < C_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_i, s_{i+1}, \dots, s_n)$ for any strategies $(s_i, s_{i+1}, \dots, s_n) \in S_i \times \cdots \times S_n$.

[5] To see this, just take any upper bound of $\frac{d+1}{c\ln(d+1)}$ on α, for a constant $c > 2$. Then, the lower bound in (9) becomes $\Omega(d^{c-1})$.

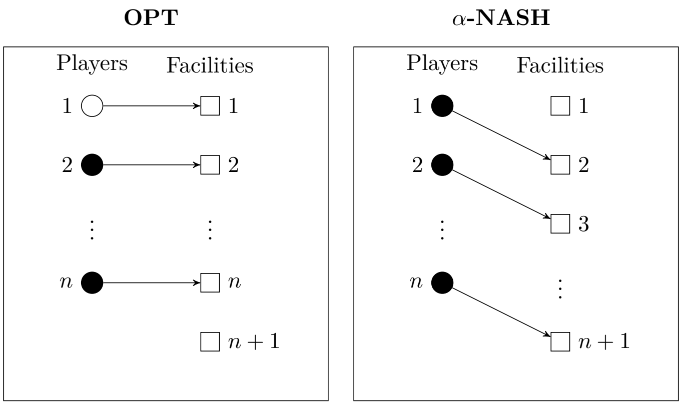

Figure 5: The social optimum $\mathbf{s}^*$ and the unique α-approximate equilibrium $\tilde{\mathbf{s}}$ in the lower bound construction of Theorem 2 for singleton weighted congestion games.

For the first condition, since facility e_2 can be used by at most players 1 and 2, and $\gamma > \alpha$, it is enough to show that $\gamma c_{e_2}(w_1 + w_2) \leq c_{e_1}(w_1)$. Indeed

$$\gamma c_{e_2}(w_1 + w_2) = \gamma(\gamma w^d)^{2-2}(w + w^2)^d = \gamma w^d(1 + w)^d = c_{e_1}(w_1).$$

Similarly, for the second condition, it is enough to show that $\gamma c_{e_{i+1}}(w_i + w_{i+1}) \leq c_{e_i}(w_{i-1} + w_i)$ for $i = 2, \ldots, n-1$, and $\gamma c_{e_{n+1}}(w_n) \leq c_{e_n}(w_{n-1} + w_n)$ for the special case of $i = n$. This is because, facility e_{i+1} can be used by at most players i and $i + 1$, while facility e_i is already being used by player $i - 1$. Indeed, for any $i = 2, \ldots, n-1$ we see that:

$$\gamma c_{e_{i+1}}(w_i + w_{i+1}) = \gamma(\gamma w^d)^{2-(i+1)}(w^i + w^{i+1})^d \quad = (\gamma w^d)^{2-i}(w^{i-1} + w^i)^d = c_{e_i}(w_{i-1} + w_i),$$

while for $i = n$,

$$\begin{aligned} c_{e_n}(w_{n-1} + w_n) &= (\gamma w^d)^{2-n}(w^{n-1} + w^n)^d \\ &= \gamma^{2-n} w^{d(2-n)+d(n-1)}(w + 1)^d \\ &= \gamma \cdot \gamma^{1-n} w^d (w + 1)d \\ &= \gamma c_{e_{n+1}}(w_n). \end{aligned}$$

The social cost at equilibrium $\tilde{\mathbf{s}}$ is at least the cost of player n at $\tilde{\mathbf{s}}$, that is,

$$C(\tilde{\mathbf{s}}) \geq w_n c_{e_{n+1}}(w_n) = w^n \cdot \gamma^{1-n} w^d (1 + w)^d = \left(\frac{w}{\gamma}\right)^n \gamma \cdot w^d (1 + w)^d$$

On the other hand, consider the strategy profile $\mathbf{s}^*$ where every player i chooses facility e_i:

$$\begin{aligned}
C(\mathbf{s}^*) &= w_1 c_{e_1}(w_1) + \sum_{i=2}^{n} w_i c_{e_i}(w_i) \\
&= \gamma w^{d+1}(1+w)^d + \sum_{i=2}^{n} w^i (\gamma w^d)^{2-i} w^{id} \\
&= \gamma w^{d+1}(1+w)^d + \gamma^2 w^{2d} \sum_{i=2}^{n} \left(\frac{w}{\gamma}\right)^i \\
&= \gamma w^{d+1}(1+w)^d + \gamma^2 w^{2d} \cdot \left(\frac{w}{\gamma}\right)^2 \frac{\left(\frac{w}{\gamma}\right)^{n-1} - 1}{\frac{w}{\gamma} - 1} \\
&\leq \gamma w^{d+1}(1+w)^d + \gamma^2 w^{2d} \cdot \left(\frac{w}{\gamma}\right)^2 \frac{\left(\frac{w}{\gamma}\right)^{n-1}}{\frac{w}{\gamma} - 1} \\
&= \left(\frac{w}{\gamma}\right)^n \gamma \cdot \left[\left(\frac{w}{\gamma}\right)^{-n} \cdot w^{d+1}(w+1)^d + \frac{w^{2d+1}}{\frac{w}{\gamma} - 1}\right]
\end{aligned}$$

Recall now that, from (10), $\frac{w}{\gamma} > 1$, and thus $\lim_{n\to\infty} \left(\frac{w}{\gamma}\right)^{-n} = 0$. So, as the number of players n grows large we get the following lower bound on the Price of Stability:

$$\lim_{n\to\infty} \frac{C(\tilde{\mathbf{s}})}{C(\mathbf{s}^*)} \geq \lim_{n\to\infty} \frac{w^d(1+w)^d}{\left(\frac{w}{\gamma}\right)^{-n} \cdot w^{d+1}(w+1)^d + \frac{w^{2d+1}}{\frac{w}{\gamma}-1}} = \left(\frac{w}{\gamma} - 1\right) \frac{(1+w)^d}{w^{d+1}}.$$

Since γ is chosen arbitrarily close to α, deploying (10) to substitute w, the above lower bound can be written as

$$\begin{aligned}
\lim_{n\to\infty} \frac{C(\tilde{\mathbf{s}})}{C(\mathbf{s}^*)} &\geq \left(\frac{d+1}{d-\alpha} - 1\right) \left[1 + \frac{\alpha(d+1)}{d-\alpha}\right]^d \left[\frac{\alpha(d+1)}{d-\alpha}\right]^{d+1} \\
&= \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d \left(1 + \frac{1}{\alpha}\right)^{d+1} \\
&\geq \frac{1}{e} \frac{1}{d+1} \left(1 + \frac{1}{\alpha}\right)^{d+1}.
\end{aligned}$$

□

4 Upper Bounds

The negative results of the previous sections, involve constructions where the ratio W of the largest to smallest weight can be exponential in d. In the main theorem (Theorem 3) of this section we present an analysis which is sensitive to this parameter W, and identify conditions under which the performance of approximate equilibria can be significantly improved.

Our upper bound approach is based on the design of a suitable approximate potential function and has three main steps. First, in Section 4.1, we setup a framework for the definition of this function by identifying conditions that, on the one hand, certify the existence of an approximate equilibrium and, on the other, provide guarantees about its efficiency. Then, in Section 4.2, by use of the Euler-Maclaurin summation formula we present a general form of an approximate potential function, which extends Rosenthal's potential for weighted congestion games (see also Appendix C). Finally, in Section 4.3, we deploy this potential for polynomial latencies. Due to its analytic description, our potential differs from other extensions of the Rosenthal's potential that have appeared in previous work, and we believe that this contribution might be of independent interest, and applied to other classes of latency functions.

4.1 The Potential Method

In the next lemma we lay the ground for the design and analysis of approximate potential functions, by supplying conditions that not only provide guarantees for the existence of approximate equilibria, but also for their performance with respect to the social optimum. In the premises of the lemma, we give conditions on the resource functions ϕ_e, having in mind that $\Phi(\mathbf{s}) = \sum_{e\in E} \phi_e(x_e(\mathbf{s}))$ will eventually serve as the "approximate" potential function.

Lemma 2. *Consider a weighted congestion game with latency functions c_e, for each facility $e \in E$, and player weights w_i, for each player $i \in N$. If there exist functions $\phi_e : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ and parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that for any facility e and player weight $w \in \{w_1, \dots, w_n\}$*

$$\alpha_1 \leq \frac{\phi_e(x+w) - \phi_e(x)}{w \cdot c_e(x+w)} \leq \alpha_2, \qquad \text{for all } x \geq 0, \tag{11}$$

and

$$\beta_1 \leq \frac{\phi_e(x)}{x \cdot c_e(x)} \leq \beta_2, \qquad \text{for all } x \geq \min_n w_n, \tag{12}$$

then our game has an $\frac{\alpha_2}{\alpha_1}$-approximate pure Nash equilibrium which, furthermore, has Price of Stability at most $\frac{\beta_2}{\beta_1}$.

Proof. Denote $\alpha = \frac{\alpha_2}{\alpha_1}$, $\beta = \frac{\beta_2}{\beta_1}$. First we will show that the function $\Phi(\mathbf{s}) = \sum_{e\in E} \phi_e(x_e(\mathbf{s}))$ (defined over all feasible outcomes $\mathbf{s}$) is an α-approximate potential, i.e. for any profile $\mathbf{s}$, any player i and strategy $s_i' \in S_i$,

$$C_i(s_i', \mathbf{s}_{-i}) < \frac{1}{\alpha} C_i(\mathbf{s}) \quad \Longrightarrow \quad \Phi(s_i', \mathbf{s}_{-i}) < \Phi(\mathbf{s}).$$

This would be enough to establish the existence of a pure α-approximate equilibrium, since any (local) minimizer of Φ will do. So, it is enough to prove that

$$\Phi(s_i', \mathbf{s}_{-i}) - \Phi(\mathbf{s}) \leq w_i \alpha_1 \left[\alpha \cdot C_i(s_i', \mathbf{s}_{-i}) - C_i(\mathbf{s})\right].$$

Indeed, if for simplicity we denote $x_e = x_e(\mathbf{s})$ and $x_e' = (s_i', \mathbf{s}_{-i})$ for all facilities e, we can compute

$$\begin{aligned}
\Phi(s_i', \mathbf{s}_{-i}) - \Phi(\mathbf{s}) &= \sum_{e\in E} \left[\phi_e(x_e') - \phi_e(x_e)\right] \\
&= \sum_{e\in s_i' \setminus s_i} \left[\phi_e(x_e + w_i) - \phi_e(x_e)\right] + \sum_{e\in s_i \setminus s_i'} \left[\phi_e(x_e - w_i) - \phi_e(x_e)\right] \\
&\leq \alpha_2 \sum_{e\in s_i' \setminus s_i} w_i c_e(x_e + w_i) - \alpha_1 \sum_{e\in s_i \setminus s_i'} w_i c_e(x_e), \\
&\leq w_i \alpha_2 \left(\sum_{e\in s_i' \setminus s_i} c_e(x_e + w_i) + \sum_{e\in s_i' \cap s_i} c_e(x_e) \right) - w_i \alpha_1 \left(\sum_{e\in s_i \setminus s_i'} c_e(x_e) + \sum_{e\in s_i' \cap s_i} c_e(x_e) \right), \\
&= w_i \alpha_1 \left[\alpha \left(\sum_{e\in s_i' \setminus s_i} c_e(x_e + w_i) + \sum_{e\in s_i' \cap s_i} c_e(x_e) \right) - \left(\sum_{e\in s_i \setminus s_i'} c_e(x_e) + \sum_{e\in s_i' \cap s_i} c_e(x_e) \right) \right] \\
&= w_i \alpha_1 \left[\alpha C_i(s_i', \mathbf{s}_{-i}) - C_i(\mathbf{s})\right].
\end{aligned}$$

where the first inequality holds due to (11) and the second one because $\alpha_2 \geq \alpha_1$.

Next, for the upper bound of β on the Price of Stability, it is enough to show that for any profiles $\mathbf{s}$, $\mathbf{s}'$,

$$\Phi(\mathbf{s}) \leq \Phi(\mathbf{s}') \quad \Longrightarrow \quad C(\mathbf{s}) \leq \beta \cdot C(\mathbf{s}'),$$

because then, if $\mathbf{s}^* \in \operatorname{argmin}_{\mathbf{s}} C(\mathbf{s})$ is an optimal-cost profile and $\tilde{\mathbf{s}} \in \operatorname{argmin}_{\mathbf{s}} \Phi(\mathbf{s})$ is a *global* minimizer of Φ, then $C(\tilde{\mathbf{s}}) \leq \beta C(\mathbf{s}^*)$ (and furthermore, as a minimizer of Φ, $\tilde{\mathbf{s}}$ is clearly an α-approximate equilibrium as well; see the first part of the current proof). Indeed, denoting $x_e = x_e(\mathbf{s})$, $x'_e = x_e(\mathbf{s}')$ for simplicity, we have:

$$\begin{aligned}
\Phi(\mathbf{s}') - \Phi(\mathbf{s}) &= \sum_{e \in E} \phi_e(x_e) - \sum_{e \in E} \phi_e(x'_e) \\
&\leq \beta_2 \sum_{e \in E} x'_e c_e(x'_e) - \beta_1 \sum_{e \in E} x_e c_e(x_e) \\
&= \beta_2 C(\mathbf{s}') - \beta_1 C(\mathbf{s}) \\
&= \beta_1 \left(\beta C(\mathbf{s}') - C(\mathbf{s}) \right).
\end{aligned}$$

$\square$

4.2 Faulhaber's Potential

In this section we propose an approximate potential function, which is based on the following classic number-theoretic result, known as Faulhaber's formula[6], which states that for any positive integers n, m,

$$\begin{aligned}
\sum_{k=1}^{n} k^m &= \frac{1}{m+1} \sum_{j=0}^{m} (-1)^j \binom{m+1}{j} B_j n^{m+1-j} \\
&= \frac{1}{m+1} n^{m+1} + \frac{1}{2} n^m + \frac{1}{m+1} \sum_{j=2}^{m} \binom{m+1}{j} B_j n^{m+1-j}, \qquad (13)
\end{aligned}$$

where the coefficients B_j are the usual Bernoulli numbers.[7] In particular, this shows that the sum of the first n powers of m can be expressed as a polynomial of n with degree $m+1$. Furthermore, this sum corresponds to the well-known potential of Rosenthal [39] for *unweighted* congestion games when the latency function is the monomial $x \mapsto x^m$.

Based on the above observation, we go beyond just integer values of n, and generalize this idea to all positive reals; in that way, we design a "potential" function that can handle different player weights and, furthermore, incorporate in a more powerful, analytically smooth way, approximation factors with respect to both the Price of Stability, as well as the approximation parameter of the equilibrium (in the spirit of Lemma 2). A natural way to do that is to directly generalize (13) and simply define, for any real $x \geq 0$ and positive integer m,

$$S_m(x) \equiv \frac{1}{m+1} x^{m+1} + \frac{1}{2} x^m, \qquad (14)$$

keeping just the first two significant terms.[8] For the special case of $m = 0$ we set $S_0(y) = y$.

[6] See, e.g., [31, p. 287] or [18, p. 106]).

[7] See, e.g., [26, Chapter 6.5] or [1, Chapter 23]. The first Bernoulli numbers are: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$. Also, we know that $B_j = 0$ for all *odd* integers $j \geq 3$.

[8] See Section 4.4 for further discussion on this choice.

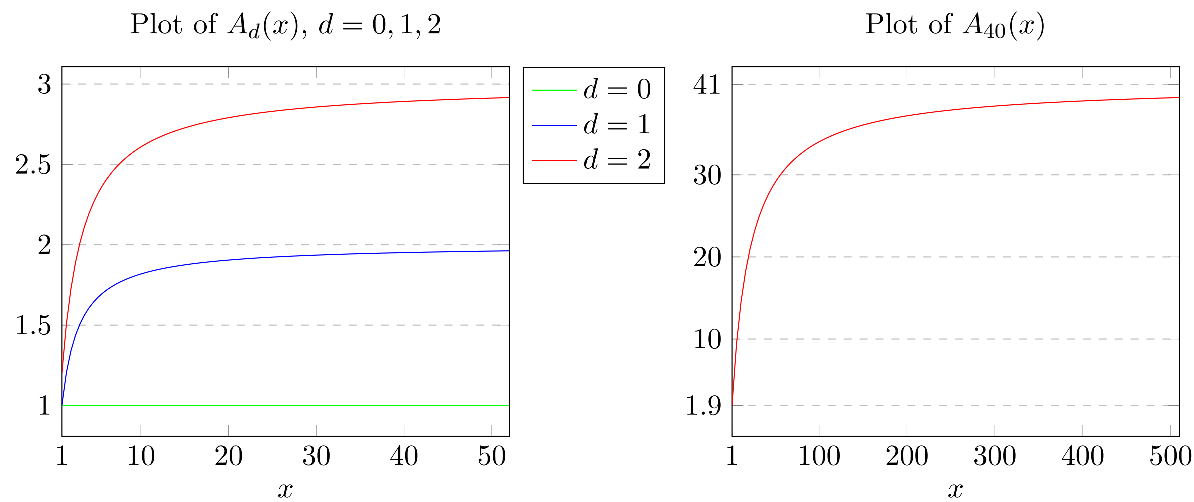

Figure 6: Plots of functions A_d for $d = 0, 1, 2$ (left) and $d = 40$ (right). For $d \geq 1$ they are strictly increasing, starting at $A_d(1) = \frac{2(d+1)}{d+3} \in [1, 2)$ and going up to $d + 1$ at the limit. Here, $A_0(1) = 1$, $A_1(1) = 1$, $A_2(1) = 6/5 = 1.2$ and $A_{40}(1) = 82/43 \approx 1.907$.

For any nonnegative integer m we define the function $A_m : [1, \infty) \longrightarrow \mathbb{R}_{>0}$ with

$$A_m(x) \equiv \left[\frac{S_m(x)}{x^{m+1}}\right]^{-1} = \left(\frac{1}{m+1} + \frac{1}{2x}\right)^{-1} = \frac{2(m+1)x}{2x + m + 1}. \tag{15}$$

Observe that A_m is strictly increasing (in x) for all $m \geq 1$,

$$A_m(1) = \frac{2(m+1)}{m+3} \in [1, 2), \qquad \text{and} \qquad \lim_{x \to \infty} A_m(x) = m + 1. \tag{16}$$

For the special case of $m = 0$ we simply have $A_0(x) = 1$ for all $x \geq 0$. Figure 6 shows a graph of these functions. Since A_m is strictly increasing for $m \geq 1$, its inverse function, $A_m^{-1} : [2\frac{m+1}{m+3}, m + 1] \longrightarrow [1, \infty)$, is well-defined and also strictly increasing for all $m \geq 1$.

The following two lemmas (whose proofs can be found in Appendices B.2 and B.3) describe some useful properties regarding the algebraic behaviour, and the relation among, functions A_m and S_m:

Lemma 3. *Fix any reals $y \geq x \geq 1$. Then the sequences $\frac{A_m(x)}{m+1}$ and $\frac{A_m(x)}{A_m(y)}$ are decreasing, and sequence $A_m(x)$ is increasing (with respect to m).*

Lemma 4. *Fix any integer $m \geq 0$ and reals $\gamma, w \geq 1$. Then*

$$\frac{\gamma^{m+1}}{A_m(\gamma w)} \leq \frac{S_m(\gamma(x + w)) - S_m(\gamma x)}{w(x + w)^m} \leq \gamma^{m+1}, \qquad \textit{for all } x \geq 0, \tag{17}$$

and

$$\frac{\gamma^{m+1}}{m + 1} \leq \frac{S_m(\gamma x)}{x^{m+1}} \leq \frac{\gamma^{m+1}}{A_m(\gamma)}, \qquad \textit{for all } x \geq 1. \tag{18}$$

4.3 The Upper Bound

Now we are ready to state our main positive result:

Theorem 3. *At any congestion game with polynomial latency functions of degree at most $d \geq 1$ and player-weights belonging in the range $[1, W]$, for any $2\frac{d+1}{d+3} \leq \xi \leq d+1$ there exists an $A_d(\bar{\xi} W)$-approximate pure Nash equilibrium, which furthermore has Price of Stability at most $\frac{d+1}{\xi}$, where A_d is a strictly increasing function*[9] *taking values within $[2\frac{d+1}{d+3}, d+1]$ defined in (15), and $\bar{\xi}$ is the inverse image $\bar{\xi} = A_d^{-1}(\xi)$.*

In particular, Theorem 3 has two interesting corollaries, one for $\xi = A_d(1) = \frac{2(d+1)}{d+3}$ and one for $W = 1$ (unweighted games):

Corollary 1. *At any congestion game with polynomial latencies of degree at most $d \geq 1$ where player-weights lie within the range $[1, W]$, there is a $A_d(W)$-approximate pure Nash equilibrium with Price of Stability at most $\frac{d+3}{2}$.*

Notice how, in light of Theorem 2, the above result of Corollary 1 is asymptotically tight as far as the Price of Stability is concerned.

Corollary 2. *At any* unweighted *congestion game with polynomial latencies of degree at most $d \geq 1$, the Price of Stability of α-approximate equilibria is at most $\frac{d+1}{\alpha}$, for any $2\frac{d+1}{d+3} \leq \alpha \leq d+1$.*

Proof of Theorem 3. From now on assume that our latency functions are polynomials of degree at most d with nonnegative coefficients, i.e. for each facility $e \in E$ there exist constants $a_{e,0}, a_{e,1}, \ldots, a_{e,d} \geq 0$ such that

$$c_e(x) = \sum_{j=0}^{d} a_{e,j} x^j.$$

Then, in order to utilize Lemma 4, we choose functions

$$\phi_e(x) = \sum_{j=0}^{d} a_{e,j} \frac{S_j(\gamma x)}{S_j(\gamma)} = \sum_{j=0}^{d} a_{e,j} \frac{A_j(\gamma)}{\gamma^{j+1}} S_j(\gamma x), \tag{19}$$

with parameter $\gamma \geq 1$ selected such that $A_d(\gamma) = \xi$, where $\xi \in [2\frac{d+1}{d+3}, d+1]$ has any value we desire according to the statement of our Theorem 3; notice that γ is well-defined, due to the analytic properties of function A_d (see (16)). Now, (17) gives us that for any $x \geq 0$ and $w \in [1, W]$,

$$\begin{aligned}
\phi_e(x+w) - \phi_e(x) &= \sum_{j=0}^{d} \frac{a_{e,j} A_j(\gamma)}{\gamma^{j+1}} \left(S_j(\gamma(x+w)) - S_j(\gamma x)\right) \\
&\leq \sum_{j=0}^{d} \frac{a_{e,j} A_j(\gamma)}{\gamma^{j+1}} \cdot \gamma^{j+1} w (x+w)^j \\
&= \max_{j=0,\ldots,d} A_j(\gamma) \cdot w \sum_{j=0}^{d} a_{e,j} (x+w)^j \\
&= A_d(\gamma) \cdot w c_e(x+w), \qquad \text{from Lemma 3,}
\end{aligned}$$

[9] See Fig. 6.

and similarly, bounding in the other direction,

$$\begin{aligned}
\phi_e(x+w)-\phi_e(x) &\geq \sum_{j=0}^{d} \frac{a_{e,j}A_j(\gamma)}{\gamma^{j+1}} \cdot \frac{\gamma^{j+1}}{A_j(\gamma w)} w(x+w)^j \\
&= w \sum_{j=0}^{d} \frac{A_j(\gamma)}{A_j(\gamma w)} \cdot a_{e,j}(x+w)^j \\
&\geq \min_{j=0,\dots,d} \frac{A_j(\gamma)}{A_j(\gamma w)} \cdot wc_e(x+w) \\
&= \frac{A_d(\gamma)}{A_d(\gamma w)} \cdot wc_e(x+w), && \text{from Lemma 3 and } \gamma w \geq \gamma \geq 1, \\
&\geq \frac{A_d(\gamma)}{A_d(\gamma W)} \cdot wc_e(x+w), && \text{since } A_d \text{ is monotonically increasing.}
\end{aligned}$$

Furthermore, (18) of Lemma 4 would give us that

$$\phi_e(x) = \sum_{j=0}^{d} \frac{a_{e,j}A_j(\gamma)}{\gamma^{j+1}} S_j(\gamma x) \leq \sum_{j=0}^{d} \frac{a_{e,j}A_j(\gamma)}{\gamma^{j+1}} \cdot \frac{\gamma^{j+1}}{A_j(\gamma)} x^{j+1} = xc_e(x)$$

and

$$\phi_e(x) \geq \sum_{j=0}^{d} \frac{a_{e,j}A_j(\gamma)}{\gamma^{j+1}} \cdot \frac{\gamma^{j+1}}{j+1} x^{j+1} = x \cdot \sum_{j=0}^{d} \frac{A_j(\gamma)}{j+1} \cdot a_{e,j}x^j \geq \frac{A_d(\gamma)}{d+1} \cdot xc_e(x),$$

where the last inequality holds due to Lemma 3.

The above analysis shows us that the functions ϕ_e we defined in (19) satisfy the requirements of Lemma 2 with

$$\alpha_1 = \frac{A_d(\gamma)}{A_d(\gamma W)}, \quad \alpha_2 = A_d(\gamma) \qquad \text{and} \qquad \beta_1 = \frac{A_d(\gamma)}{d+1}, \quad \beta_2 = 1.$$

Thus, we deduce that there exists $A_d(\gamma W)$–approximate pure Nash equilibrium with price of Stability at most $\frac{d+1}{A_d(\gamma)}$, which concludes the proof since γ has been chosen so that $\gamma = A_d^{-1}(\xi)$. □

4.4 Small vs Large Degree Polynomials

One can argue that our choice to keep only the first two terms in Faulhaber's formula (13), when defining our approximate potential in (14), is suboptimal. To some extent, this is correct; it is exactly the reason why this seemingly "unnatural" lower bound of $2\frac{d+1}{d+3}$ for parameter ξ appears in our main result of this section, Theorem 3. It would be nicer if ξ could simply start from 1 instead. Indeed, this can be achieved for small values of d, as described below.

Considering the entire right-hand side expression in (13), one can take the full, exact version of Faulhaber's formula, that can be written[10] in a very elegant way as

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}\left[B_{m+1}(n+1) - B_{m+1}\right], \tag{20}$$

where

$$B_m(y) = \sum_{k=0}^{m} \binom{m}{k} B_k y^{m-k}, \qquad y \geq 0,$$

[10] See, e.g., [31, p. 288] or [1, Eq. 23.1.4].

are the Bernoulli polynomials, and coefficients $B_k = B_k(0)$ are the standard Bernoulli numbers we used before. Now we can use (20) to define a more fine-tuned version for S_m, that is, for $m \geq 1$ set $\hat{S}_m(x) = \frac{1}{m+1}\left[B_{m+1}(x+1) - B_{m+1}\right]$ instead of (14). For example, for degrees up to $m \leq 4$ these new polynomials are:

$$\hat{S}_0(x) = x, \qquad \hat{S}_1(x) = \frac{1}{2}x(x+1), \qquad \hat{S}_2(x) = \frac{1}{6}x(2x^2+3x+1)$$
$$\hat{S}_3(x) = \frac{1}{4}x^2(x+1)^2, \qquad \hat{S}_4(x) = \frac{1}{30}x(6x^4+15x^3+10x^2-1)$$

Using these value, one can verify that for up to $m \leq 4$, all our critical technical requirements for the proof of Theorem 3 are satisfied: most notably Lemmas 3 and 4, and the monotonicity of $\hat{A}_m(x) = \frac{x^{m+1}}{\hat{S}_m(x)}$ (with respect to $x \geq 1$). In particular, now we have that $\hat{A}_m(1) = \frac{1^{m+1}}{\hat{S}_m(1)} = 1$, which is exactly what we wanted. Thus,

> *Theorem 3 can be rewritten for $d \leq 4$, with parameter ξ taking values in the entire range of $\xi \in [1, d+1]$.*

However, there is a catch, that does not allow us to do that in general; as m grows large, the Bernoulli polynomials, that now play a critical role in our definition of functions $\hat{S}_m$ (see (20)), start to behave in a rather erratic, non-smooth way within the interior of the real intervals between consecutive integer values. For example, one can check that, for $d = 14$ function $\hat{A}_{14}$ is *not* monotonically increasing within $[1, 2]$. Even more disastrously, for $d = 20, 21$ functions $\hat{S}_d$ take *negative* values in $[1, 2]$!

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York, 9th printing, 10th gpo printing edition, 1964. ISBN 0486612724. URL http://people.maths.ox.ac.uk/~macdonald/aands/index.html.

[2] H. Ackermann, H. Röglin, and B. Vöcking. On the impact of combinatorial structure on congestion games. *Journal of the ACM*, 55(6):1–22, dec 2008. doi: 10.1145/1455248.1455249.

[3] S. Aland, D. Dumrauf, M. Gairing, B. Monien, and F. Schoppmann. Exact price of anarchy for polynomial congestion games. *SIAM Journal on Computing*, 40(5):1211–1233, jan 2011. doi: 10.1137/090748986.

[4] S. Albers. On the value of coordination in network design. *SIAM Journal on Computing*, 38(6):2273–2302, 2009.

[5] E. Anshelevich, A. Dasgupta, J. Kleinberg, É. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. *SIAM Journal on Computing*, 38(4):1602–1623, jan 2008. doi: 10.1137/070680096.

[6] B. Awerbuch, Y. Azar, and A. Epstein. The price of routing unsplittable flow. *SIAM Journal on Computing*, 42(1):160–177, jan 2013. doi: 10.1137/070702370.

[7] K. Bhawalkar, M. Gairing, and T. Roughgarden. Weighted congestion games: The price of anarchy, universal worst-case examples, and tightness. *ACM Trans. Econ. Comput.*, 2 (4):141–1423, oct 2014. doi: 10.1145/2629666.

[8] V. Bilò. A unifying tool for bounding the quality of non-cooperative solutions in weighted congestion games. *Theory of Computing Systems*, dec 2017. doi: 10.1007/s00224-017-9826-1.

[9] V. Bilò, I. Caragiannis, A. Fanelli, and G. Monaco. Improved Lower Bounds on the Price of Stability of Undirected Network Design Games. *Theory of Computing Systems*, 52(4): 668–686, may 2013. doi: 10.1007/s00224-012-9411-6.

[10] V. Bilò, M. Flammini, and L. Moscardelli. The price of stability for undirected broadcast network design with fair cost allocation is constant. *Games and Economic Behavior*, 2014.

[11] I. Caragiannis, M. Flammini, C. Kaklamanis, P. Kanellopoulos, and L. Moscardelli. Tight bounds for selfish and greedy load balancing. *Algorithmica*, 61(3):606–637, 2010. doi: 10.1007/s00453-010-9427-8.

[12] I. Caragiannis, A. Fanelli, N. Gravin, and A. Skopalik. Efficient computation of approximate pure Nash equilibria in congestion games. In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*, oct 2011. doi: 10.1109/focs.2011.50.

[13] I. Caragiannis, A. Fanelli, N. Gravin, and A. Skopalik. Approximate pure Nash equilibria in weighted congestion games: Existence, efficient computation, and structure. *ACM Trans. Econ. Comput.*, 3(1):2:1–2:32, Mar. 2015. doi: 10.1145/2614687.

[14] G. Christodoulou and M. Gairing. Price of stability in polynomial congestion games. *ACM Transactions on Economics and Computation*, 4(2):1–17, dec 2015. doi: 10.1145/2841229.

[15] G. Christodoulou and E. Koutsoupias. The price of anarchy of finite congestion games. In *STOC '05: Proceedings of the 37th annual ACM symposium on Theory of computing*, pages 67–73, New York, NY, USA, 2005. ACM. doi: 10.1145/1060590.1060600.

[16] G. Christodoulou and E. Koutsoupias. On the price of anarchy and stability of correlated equilibria of linear congestion games. In *Algorithms - ESA 2005, 13th Annual European Symposium*, pages 59–70, 2005.

[17] G. Christodoulou, E. Koutsoupias, and P. G. Spirakis. On the performance of approximate equilibria in congestion games. *Algorithmica*, 61(1):116–140, 2011. doi: 10.1007/s00453-010-9449-2.

[18] J. H. Conway and R. K. Guy. *The Book of Numbers*. Springer New York, 1996. doi: 10.1007/978-1-4612-4072-3.

[19] J. Dunkel and A. S. Schulz. On the complexity of pure-strategy Nash equilibria in congestion and local-effect games. *Mathematics of Operations Research*, 33(4):851–868, 2008. ISSN 0364765X, 15265471.

[20] A. Fabrikant, C. Papadimitriou, and K. Talwar. The complexity of pure Nash equilibria. In *Proceedings of the Thirty-sixth Annual ACM Symposium on Theory of Computing*, STOC '04, pages 604–612, New York, NY, USA, 2004. doi: 10.1145/1007352.1007445.

[21] M. Feldotto, M. Gairing, G. Kotsialou, and A. Skopalik. Computing approximate pure Nash equilibria in Shapley value weighted congestion games. In *Web and Internet Economics*, pages 191–204. Springer International Publishing, 2017. doi: 10.1007/978-3-319-71924-5_14.

[22] A. Fiat, H. Kaplan, M. Levy, S. Olonetsky, and R. Shabo. On the price of stability for designing undirected networks with fair cost allocations. In M. Bugliesi, B. Preneel, V. Sassone, and I. Wegener, editors, *Automata, Languages and Programming: 33rd International Colloquium (ICALP)*, pages 608–618, 2006. doi: 10.1007/11786986_53.

[23] D. Fotakis, S. Kontogiannis, and P. Spirakis. Selfish unsplittable flows. *Theoretical Computer Science*, 348(2):226–239, 2005.

[24] D. Fotakis, S. Kontogiannis, E. Koutsoupias, M. Mavronicolas, and P. Spirakis. The structure and complexity of Nash equilibria for a selfish routing game. *Theoretical Computer Science*, 410(36):3305–3326, 2009. doi: 10.1016/j.tcs.2008.01.004.

[25] M. Goemans, V. Mirrokni, and A. Vetta. Sink equilibria and convergence. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05)*, pages 142–151, Oct 2005. doi: 10.1109/SFCS.2005.68.

[26] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1989.

[27] C. Hansknecht, M. Klimm, and A. Skopalik. Approximate pure Nash equilibria in weighted congestion games. In K. Jansen, J. D. P. Rolim, N. R. Devanur, and C. Moore, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2014)*, volume 28 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 242–257. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2014. doi: 10.4230/LIPIcs.APPROX-RANDOM.2014.242.

[28] T. Harks and M. Klimm. On the existence of pure Nash equilibria in weighted congestion games. *Mathematics of Operations Research*, 37(3):419–436, 2012. doi: 10.1287/moor.1120.0543.

[29] T. Harks, M. Klimm, and R. H. Möhring. Characterizing the existence of potential functions in weighted congestion games. *Theory Comput Syst*, 49:46–70, 2011. doi: 10.1007/s00224-011-9315-x.

[30] T. Harks, M. Klimm, and R. H. Möhring. Strong equilibria in games with the lexicographical improvement property. *International Journal of Game Theory*, 42(2):461–482, 2012. doi: 10.1007/s00182-012-0322-1.

[31] D. E. Knuth. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61 (203):277–277, sep 1993. doi: 10.1090/s0025-5718-1993-1197512-7.

[32] E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. In *16th Annual Symposium on Theoretical Aspects of Computer Science*, STACS '99, pages 404–413, 1999.

[33] E. Lee and K. Ligett. Improved bounds on the price of stability in network cost sharing games. In *Proceedings of the 14th ACM Conference on Electronic Commerce*, EC '13, pages 607–620. ACM, 2013. doi: 10.1145/2482540.2482562.

[34] D. H. Lehmer. On the maxima and minima of Bernoulli polynomials. *The American Mathematical Monthly*, 47(8):533, oct 1940. doi: 10.2307/2303833.

[35] L. Libman and A. Orda. Atomic resource sharing in noncooperative networks. *Telecommunication Systems*, 17(4):385–409, Aug 2001. doi: 10.1023/A:1016770831869.

[36] D. Monderer and L. S. Shapley. Potential games. *Games and economic behavior*, 14(1): 124–143, 1996.

[37] N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani, editors. *Algorithmic Game Theory.* Cambridge University Press, 2007.

[38] P. N. Panagopoulou and P. G. Spirakis. Algorithms for pure Nash equilibria in weighted congestion games. *Journal of Experimental Algorithmics*, 11:27, feb 2007. doi: 10.1145/1187436.1216584.

[39] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

[40] R. W. Rosenthal. The network equilibrium problem in integers. *Networks*, 3(1):53–59, 1973.

[41] T. Roughgarden. Intrinsic robustness of the price of anarchy. *J. ACM*, 62(5):32:1–32:42, Nov. 2015. doi: 10.1145/2806883.

[42] T. Roughgarden and É. Tardos. How bad is selfish routing? *J. ACM*, 49(2):236–259, mar 2002. doi: 10.1145/506147.506153.

[43] T. Roughgarden and É. Tardos. Bounding the inefficiency of equilibria in nonatomic congestion games. *Games and Economic Behavior*, 47(2):389–403, may 2004. doi: 10.1016/j.geb.2003.06.004.

[44] A. S. Schulz and N. S. Moses. On the performance of user equilibria in traffic networks. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '03, pages 86–87, Philadelphia, PA, USA, 2003. Society for Industrial and Applied Mathematics.

A Lower Bound Proofs

A.1 Technical Lemmas

Lemma 5. *For any integer $d \geq 1$ define the function $g : (1, \infty) \longrightarrow (0, \infty)$ with*

$$g(x) = x^{1+2/d} - x.$$

Then g is a monotonically increasing function and for any real constant $\gamma > 0$

$$\lim_{d\to\infty} g\left(\frac{\gamma d}{\ln d}\right) = 2\gamma \qquad \text{and} \qquad \lim_{d\to\infty} \sqrt[d]{\frac{\gamma d}{\ln d}} = 1.$$

Proof. A straightforward application of L'Hospital's rule suffices. □

Lemma 6. *For any integer $d \geq 2$, the function $f : (0, \infty)^2 \longrightarrow (0, \infty)$ defined by*

$$f(x, y) = \frac{(y + x + 1)^d - (y + x)^d}{(y + 1)^d - y^d}$$

is monotonically decreasing with respect to y. Furthermore, for $d \geq 9$,

$$\zeta^{d+1} \leq f\left((\beta_d \Phi_d + 1)(\zeta - 1), \beta_d \Phi_d - (1 - \beta_d)\zeta\right) \qquad \text{for all } \zeta \in [1, 2], \tag{21}$$

where β_d is defined in Lemma 1.

Proof. First let's define function $h : (0,\infty) \longrightarrow (0,\infty)$ with

$$h(t) = \frac{(t+1)^d - t^d}{(t+1)^{d-1} - t^{d-1}}. \tag{22}$$

We will show that h is increasing, which will suffice to prove the desired monotonicity of f since its derivative is

$$\begin{aligned}\frac{\partial f(x,y)}{\partial y} &= \frac{d\left[(x+y+1)^{d-1} - (x+y)^{d-1}\right]}{(y+1)^d - y^d} - \frac{d\left[(y+1)^{d-1} - y^{d-1}\right]\left[(x+y+1)^d - (x+y)^d\right]}{\left[(y+1)^d - y^d\right]^2} \\ &= \frac{d\left[(y+1)^{d-1} - y^{d-1}\right]\left[(x+y+1)^{d-1} - (x+y)^{d-1}\right]}{\left[(y+1)^d - y^d\right]^2}\left[h(y) - h(x+y)\right],\end{aligned}$$

which is negative due to the monotonicity of h. To prove that h is indeed increasing, we will instead show something stronger that we'll need also in the proof of Lemma 1 below, namely that function $\bar{h} : (1,\infty) \longrightarrow (0,\infty)$ with

$$\bar{h}(t) = \frac{t^d - (t-1)^d}{t^d - t(t-1)^{d-1}} \tag{23}$$

is increasing. This will be enough to demonstrate that h is increasing as well, since $h(t) = (t+1)\cdot\bar{h}(t+1)$. Taking its derivative we see that

$$\frac{\partial \bar{h}(t)}{\partial t} = \frac{\left[(t-1)^d - t^d + dt^{d-1}\right](t-1)^d}{\left[t^{d+1} - t^d - t(t-1)^d\right]^2} > 0$$

since from the convexity of function $t \mapsto t^d$ we know that $t^d - (t-1)^d < dt^{d-1}$.

Now let's prove the remaining part of our lemma, that is (21). Observe that if we set $\zeta \leftarrow 1$ to (21) it is satisfied, since $f(0,y) = 1$ for any $y > 0$. So, it is enough if we prove that

$$\zeta^{-(d+1)} f\left((\beta_d\Phi_d + 1)(\zeta - 1), \beta_d\Phi_d - (1-\beta_d)\zeta\right) = \zeta^{-(d+1)}\frac{\left[(\alpha+\beta)\zeta\right]^d - \left[(\alpha+\beta)\zeta - 1\right]^d}{\left[\alpha + 1 - (1-\beta)\zeta\right]^d - \left[\alpha - (1-\beta)\zeta\right]^d}$$

is increasing with respect to $\zeta \in [1,2]$, where here we are using $\alpha = \beta\Phi_d$. So, if we define

$$\begin{aligned} f_1(\zeta) &= \left[(\alpha+\beta)\zeta\right]^d - \left[(\alpha+\beta)\zeta - 1\right]^d \\ f_2(\zeta) &= \left[\alpha + 1 - (1-\beta)\zeta\right]^d - \left[\alpha - (1-\beta)\zeta\right]^d \end{aligned}$$

and we compute the derivative $\frac{\partial}{\partial\zeta}\left(\zeta^{-(d+1)}\frac{f_1(\zeta)}{f_2(\zeta)}\right)$ of the above expression, we need to show that

$$\zeta\left[\frac{f_1'(\zeta)}{f_1(\zeta)} - \frac{f_2'(\zeta)}{f_2(\zeta)}\right] \geq d + 1. \tag{24}$$

Now notice that

$$\zeta\frac{f_1'(\zeta)}{f_1(\zeta)} = d(\alpha+\beta)\zeta\frac{\left[(\alpha+\beta)\zeta\right]^{d-1} - \left[(\alpha+\beta)\zeta - 1\right]^{d-1}}{\left[(\alpha+\beta)\zeta\right]^d - \left[(\alpha+\beta)\zeta - 1\right]^d} = \frac{d}{\bar{h}\left((\alpha+\beta)\zeta\right)},$$

where $\bar{h}$ is the increasing function defined in (23) at the proof of Lemma 6, so taking into consideration that

$$(\alpha+\beta)\zeta = \beta(\Phi_d + 1)\zeta \leq \frac{1}{2}(\Phi_d + 1)2 \leq \Phi_d + 1,$$

we can get that

$$\zeta\frac{f_1'(\zeta)}{f_1(\zeta)} \geq \frac{d}{\bar{h}(\Phi_d+1)} = d(\Phi_d+1)\frac{(\Phi_d+1)^{d-1}-\Phi_d^{d-1}}{(\Phi_d+1)^d-\Phi_d^d} = d\frac{\Phi_d^{d+1}-\Phi_d^d-\Phi_d^{d-1}}{\Phi_d^{d+1}-\Phi_d^d} = d - \frac{d}{\Phi_d^2-\Phi_d}.$$

Similarly, we can see that

$$-\zeta\frac{f_2'(\zeta)}{f_2(\zeta)} = \frac{d(1-\beta)\zeta}{h(\alpha-(1-\beta)\zeta)},$$

where h is the increasing function defined in (22) in the proof of Lemma 6, so taking into consideration that

$$\alpha-(1-\beta)\zeta \leq \beta\Phi_d-(1-\beta) \leq \frac{\Phi_d-1}{2} \qquad \text{and} \qquad (1-\beta)\zeta \geq \frac{1}{2},$$

we get that

$$-\zeta\frac{f_2'(\zeta)}{f_2(\zeta)} \geq \frac{d/2}{h\left((\Phi_d-1)/2\right)} = d\frac{(\Phi_d+1)^{d-1}-(\Phi_d-1)^{d-1}}{(\Phi_d+1)^d-(\Phi_d-1)^d}.$$

Putting everything together, in order to prove the desired (24), it is now enough to show that

$$d\frac{(\Phi_d+1)^{d-1}-(\Phi_d-1)^{d-1}}{(\Phi_d+1)^d-(\Phi_d-1)^d} - \frac{d}{\Phi_d^2-\Phi_d} \geq 1,$$

which we know holds from (27) of Lemma 7. □

Lemma 7. *For any positive integer d,*

$$(\Phi_d+2)^d \leq \Phi_d^{d+2} \tag{25}$$

and

$$\Phi_d \leq \gamma_d\frac{d}{\ln d} \qquad \textit{with} \quad \gamma_d \equiv \frac{\ln d}{\mathcal{W}(d)} \leq 1.368 \quad \textit{and} \quad \lim_{d\to\infty}\gamma_d = 1, \tag{26}$$

where $\mathcal{W}(\cdot)$ denotes the (principal branch of the) Lambert–W function[11]*. Furthermore, for any $d \geq 9$,*

$$\frac{(\Phi_d+1)^{d-1}-(\Phi_d-1)^{d-1}}{(\Phi_d+1)^d-(\Phi_d-1)^d} - \frac{1}{\Phi_d^2-\Phi_d} \geq \frac{1}{d}. \tag{27}$$

Proof. Using the identity $(\Phi_d+1)^d = \Phi_d^{d+1}$, we have

$$(\Phi_d+2)^d = (\Phi_d+1)^d\left(\frac{\Phi_d+2}{\Phi_d+1}\right)^d \leq (\Phi_d+1)^d\left(\frac{\Phi_d+1}{\Phi_d}\right)^d = \Phi_d^{d+1}\frac{\Phi_d^{d+1}}{\Phi_d^d} = \Phi_d^{d+2},$$

proving (25).

Now we move on to upper-bound the values of Φ_d and prove (26). Here we will make use of the following property, which can be readily deduced from the proof of Lemma 5.4 of Aland et al. [3], and in particular where they are upper-bounding Φ_d:

$$d^{1/\gamma} \leq \frac{\gamma\cdot d}{\ln d} \quad\Longrightarrow\quad \Phi_d < \gamma\frac{d}{\ln d}, \tag{28}$$

for any positive real γ. Using $\gamma_d = \frac{\ln d}{\mathcal{W}(d)}$ as defined in the statement of our lemma, we compute:

$$d^{1/\gamma_d} = d^{\frac{\mathcal{W}(d)}{\ln d}} = e^{\mathcal{W}(d)}$$

[11] That is, for any positive real x, $\mathcal{W}(x) = z$ gives the unique positive real solution z to the equation $x = z\cdot e^z$.

and

$$\gamma_d \frac{d}{\ln d} = \frac{\ln d}{\mathcal{W}(d)} \frac{d}{\ln d} = \frac{d}{\mathcal{W}(d)}.$$

Thus, since from the definition of function $\mathcal{W}$ we know that

$$\mathcal{W}(d) e^{\mathcal{W}(d)} = d, \tag{29}$$

we deduce that $\gamma \leftarrow \gamma_d$ indeed satisfies the left hand side of (28), giving us the desired upper bound for Φ_d.

For the asymptotic behaviour of γ_d when d grows large, observe that by taking logarithms in (29) we get

$$\mathcal{W}(d) + \ln \mathcal{W}(d) = \ln d$$

and so

$$\lim_{d\to\infty} \gamma_d = \lim_{d\to\infty} \frac{\ln d}{\mathcal{W}(d)} = \lim_{\gamma\to\infty} \left[\frac{\ln \mathcal{W}(d)}{\mathcal{W}(d)} + 1\right] = 1 + \lim_{z\to\infty} \frac{\ln z}{z} = 1,$$

since it is easy to see that $\lim_{d\to\infty} \mathcal{W}(d) = \infty$.

Finally, let's now prove (27). First, one can simply numerically verify that it indeed holds for all integers $d = 9, 10, \ldots, 14$, so let's just focus on the case when $d \geq 15$. For simplicity, in the remainder of the proof we denote $y = \Phi_d$. It is easy to see[12] then that

$$y \geq \Phi_{15} \approx 7.141 > 7.$$

Performing some elementary algebraic manipulations in (27), we can equivalently write it as

$$(y+1)^{d-1} \left[-y^3 + dy^2 - (2d-1)y - d\right] \geq (y-1)^{d+1}(d-y).$$

Using the fact that $(y+1)^{d-1} = \frac{y^{d+1}}{y+1}$, and then that $y^{d+1} \geq (y-1)^{d+1}$, we can see that it is enough to show that

$$-y^3 + dy^2 - (2d-1)y - d \geq (d-y)(y+1).$$

Since $y > \frac{1}{2}(\sqrt{17}+3) \approx 3.562$, we know that $y^2 - 3y - 2 > 0$, and so the above can be rewritten as

$$\frac{y(y+1)(y-2)}{y^2 - 3y - 2} \leq d.$$

It is a matter of simple calculus to verify that the left-hand side of the above expression is an increasing function of y for $y \geq 7$. Thus, using an upper bound of $y \leq \frac{2d}{\ln d}$ on $y = \Phi_d$ (see [3, Lemma 5.4] or our Lemma 7), it is enough to prove that

$$-4d^2 + (2d^2 + 2d - 1)\ln d + (2 - 3d)\ln^2 d \geq 0,$$

which holds since the function on the left hand side of the inequality is increasing for $d \geq 10$ and for $d = 15$ we can compute its value

$$-43 \ln^2(15) + 479 \ln(15) - 900 \approx 81.814 > 0.$$

□

[12]Here we are silently using the fact that Φ_n is an increasing function of the integer n. One can formally prove this by, e.g., combining Lemmas 5.1 and 5.2 of Aland et al. [3].

A.2 Proof of Lemma 1

To decongest notation a bit in the proof, we will drop the d subscripts from c_d and β_d whenever this is causing no confusion. Starting with (3), if we solve with respect to β we get

$$\Phi_d + \frac{1}{\beta} \geq \Phi_d^{1+\frac{2}{d}} \iff \beta \leq \frac{1}{\Phi_d^{1+2/d} - \Phi_d}, \tag{30}$$

and substituting (2),

$$\begin{aligned} 1 - \Phi_d^{-c} \leq \frac{1}{\Phi_d^{1+2/d} - \Phi_d} &\iff \Phi_d^{-c} \geq \frac{\Phi_d^{1+2/d} - \Phi_d - 1}{\Phi_d^{1+2/d} - \Phi_d} \\ &\iff c \leq \frac{\ln(\Phi_d^{1+2/d} - \Phi_d) - \ln(\Phi_d^{1+2/d} - \Phi_d - 1)}{\ln \Phi_d}, \end{aligned}$$

which holds by the very definition of c in (1) if we relax the floor operator.

Let's now move to (4), and in particular lower-bound the values of parameter β, as d grows large. Due to the floor operator in (1), parameter c can be lower bounded by

$$c \geq \frac{\ln(\Phi_d^{1+2/d} - \Phi_d) - \ln(\Phi_d^{1+2/d} - \Phi_d - 1)}{\ln \Phi_d} - \frac{1}{d} = \log_{\Phi_d} \frac{\Phi_d^{1+2/d} - \Phi_d}{\Phi_d^{1+2/d} - \Phi_d - 1} - \frac{1}{d}$$

and since β is increasing with respect to c,

$$\beta = 1 - \Phi_d^{-c} \geq 1 - \frac{\Phi_d^{1+2/d} - \Phi_d - 1}{\Phi_d^{1+2/d} - \Phi_d} \Phi_d^{1/d} = 1 - \left[1 - \frac{1}{g(\Phi_d)}\right] \Phi_d^{1/d}, \tag{31}$$

where g is the function defined in Lemma 5. Using the upper bound for Φ_d from (26) in Lemma 7, and deploying the monotonicity of function g we can deduce that

$$\beta \geq 1 - \sqrt[d]{\frac{\gamma_d}{\ln d}} \left[1 - \frac{1}{g\left(\gamma_d \frac{d}{\ln d}\right)}\right].$$

From Lemma 7 we also know that $\lim_{d\to\infty} \gamma_d = 1$, so for any arbitrarily small $\varepsilon > 0$ we can make sure that $\gamma_d \leq 1 + \varepsilon$ if we consider sufficiently large d's. Thus, taking limits in the above inequality and using Lemma 5, we finally get the desired

$$\lim_{d\to\infty} \beta \geq 1 - 1 \cdot \left(1 - \frac{1}{2(1+\varepsilon)}\right) = \frac{1}{2(1+\varepsilon)}.$$

For the upper bound of $\beta \leq \frac{1}{2}$, due to (30) it is enough to show that

$$\frac{1}{\Phi_d^{1+2/d} - \Phi_d} \leq \frac{1}{2} \iff \Phi_d + 2 \leq \Phi_d^{1+2/d} \iff (\Phi_d + 2)^d \leq \Phi_d^{d+2},$$

which holds due to (25) of Lemma 7.

For small values of d, and in particular in order to prove that $\beta \geq 0.38$, one can numerically compute the values for β directly from (1). For example, for $d = 9, \ldots, 100$, these values are shown in Fig. 1. The lower and upper red lines in Fig. 1 correspond to the relaxation of the floor operator we used in the lower and upper bounds for β in (31) and (30), respectively. The actual values of β lie between these two lines. Using these values and the resulting monotonicity for β, one can also prove the lower bound of 3 for dc, by observing that by setting $d = 9$ in (1) we have that for any $d \geq 9$

$$d \cdot c \geq \left\lfloor 9 \frac{\ln(\Phi_9^{11/9} - \Phi_9) - \ln(\Phi_9^{11/9} - \Phi_9 - 1)}{\ln \Phi_9} \right\rfloor \approx \lfloor 3.368 \rfloor = 3.$$

A.3 Proof of Claim 1

Since we have fixed that $s_j = \tilde{s}_j$ for all players $j \leq i$ and also the strategies of players $j > i + \mu$ have no effect on the cost of player i (and in particular in (3)), it is safe if we briefly abuse notation and for now assume that $\mathbf{s}_{-i} = (s_{i+1}, \ldots, s_{i+\mu})$.

We generate the above dominating profile $\mathbf{s}'$ inductively, by running Procedure DOMINATE$(\mathbf{s}, i)$ described formally below, scanning and modifying profile $\mathbf{s}$ from right to left.

Procedure DOMINATE$(\mathbf{s}, i)$

Input: Profile $\mathbf{s}_{-i} = (s_{i+1}, \ldots, s_{i+\mu})$; Player $i \in \{\mu + 1, \ldots, n\}$

Output: Profile $\mathbf{s}'_{-i} = (s'_{i+1}, \ldots, s'_{i+\mu})$ of the form described in Items 1 to 3 of page 9, that satisfies Eq. (7)

1 $\mathbf{s}'_{-i} \leftarrow \mathbf{s}_{-i}$;

2 $s'_{i+\mu} \leftarrow s^*_{i+\mu}$;

3 $k \leftarrow i + \mu - 1$;

4 **while** *exists* $j \in \{i+1, \ldots, k-1\}$ *such that* $s'_j = s^*_j$ **do**

5 $\quad s'_k \leftarrow \tilde{s}_k$;

6 $\quad k \leftarrow k - 1$;

end

First, it is not difficult to see that the output profile $\mathbf{s}'$ of DOMINATE$(\mathbf{s}, i)$ indeed has the desired format described in Items 1 to 3 of page 9. In particular, after any execution of the while-loop in lines 4–6 of Procedure DOMINATE, $s'_j = \tilde{s}_j$ for any $j = k+1, \ldots, i + \mu - 1$. Furthermore, it is also easy to see that switching player's $i + \mu$ strategy to $s'_{i+\mu} = s^*_{i+\mu}$ can only increase player's i cost, i.e. (7) is satisfied after line 2 of DOMINATE: if player $i + \mu$ chooses $\tilde{s}_{i+\mu}$ instead, she contributes nothing to the cost of player i, since she does not put her weight in any of the facilities $i+1, \ldots, i+\mu$ played by player i.

So, it remains to be shown that after every iteration of the while-loop, condition (7) is maintained. Since in any such loop only the strategy of player k is possibly switched from s^*_k to $\tilde{s}_k$, it is enough if we show that $\bar{C}_i(\tilde{s}_k, \mathbf{s}'_{-k}) \geq \bar{C}_i(s^*_k, \mathbf{s}'_{-k})$ or, since for any facility $j < k$ it is $x_j(\tilde{s}_k, \mathbf{s}'_{-k}) = x_j(s^*_k, \mathbf{s}'_{-k})$, equivalently

$$\sum_{j=k}^{i+\mu} c_j(x_j(\tilde{s}_k, \mathbf{s}'_{-k})) \geq \sum_{j=k}^{i+\mu} c_j(x_j(s^*_k, \mathbf{s}'_{-k})).$$

If we let z_j, for any $j \geq k$, denote the load on facility j induced by every player *except* from player k, that is formally

$$z_j = \sum \left\{ w_\ell \mid \ell \in \{j-d, \ldots, j\} \setminus \{k\} \quad \wedge \quad j \in s_\ell \right\},$$

the above can be written as

$$\sum_{j=k+1}^{i+\mu} c_j(z_j + w_k) - c_j(z_j) \geq c_k(z_k + w_k) - c_k(z_k).$$

Thus, it is enough if we only take $j = i + \mu$ in the above sum and just prove that

$$c_{i+\mu}(z_{i+\mu} + w_k) - c_{i+\mu}(z_{i+\mu}) \geq c_k(z_k + w_k) - c_k(z_k),$$

which is equivalent to

$$w^{-(i+\mu-k)(d+1)} \frac{(z_{i+\mu} + w_k)^d - z^d_{i+\mu}}{(z_k + w_k)^d - z^d_k} \geq 1. \tag{32}$$

If we define

$$A = \left\{ j \in \{i+1, \dots, k-1\} \;\middle|\; s_j = s_j^* \right\},$$

that is, A is the set of players below k (and above i) that do *not* contribute their weight to the cost of players k and $i+\mu$, we have that

$$z_k = \sum_{\substack{j=k-\mu \\ j \notin A}}^{k-1} w_j = \alpha w_k - \sum_{j \in A} w_j$$

and

$$z_{i+\mu} = \sum_{\substack{j=i \\ j \notin A \cup \{k\}}}^{i+\mu} w_j = (\alpha+1) w_{i+\mu} - w_k - \sum_{j \in A} w_j,$$

since by our inductive process we know that $s'_j = \tilde{s}_j$ for all $j = k+1, \dots, i+\mu-1$. Thus

$$z_{i+\mu} = z_k + (\alpha+1)(w_{i+\mu} - w_k).$$

Now we can rewrite the left hand side of (32) as

$$w^{-(i+\mu-k)(d+1)} \frac{[z_k + (\alpha+1) w_{i+\mu} - \alpha w_k)]^d - [z_k + (\alpha+1) w_{i+\mu} - (\alpha+1) w_k]^d}{(z_k + w_k)^d - z_k^d},$$

and, if we additionally define for simplicity

$$\zeta \equiv w^\lambda, \qquad \text{where} \qquad \lambda \equiv \mu - k + i \in \{1, \dots, \mu-2\}$$

and

$$y_k \equiv \frac{z_k}{w_k},$$

(32) can be written as

$$\zeta^{-(d+1)} \frac{[y_k + (\alpha+1)\zeta - \alpha)]^d - [y_k + (\alpha+1)\zeta - \alpha - 1)]^d}{(y_k+1)^d - y_k^d} \geq 1,$$

or more simply,

$$\zeta^{-(d+1)} f(x,y) \geq 1$$

if we use function f from Lemma 6 with values

$$x = (\alpha+1) w^\lambda - \alpha - 1 = (\alpha+1)(\zeta - 1) > 0 \qquad \text{and} \qquad y = y_k > 0.$$

Deploying the monotonicity of f from Lemma 6 and using that

$$y = \frac{z_k}{w_k} = \alpha - \frac{1}{w_k} \sum_{j \in A} w_j \leq \alpha - \frac{1}{w_k} w_{i+1} = \alpha - w^{i+1-k} = \alpha - w^{\lambda-\mu+1} \leq \alpha - (1-\beta)\zeta,$$

where the first inequality holds due to the fact that from the while-loop test in line 4 of Procedure DOMINATE we know that $A \neq \emptyset$, we finally get that it is enough if we show that

$$\zeta^{d+1} \leq f\left((\alpha+1)(\zeta-1), a - (1-\beta)\zeta\right) \qquad \text{for all } \; \zeta \in \left[1, \frac{1}{\beta - 1}\right],$$

since $\zeta \geq w^1 \geq 1$ and $\zeta \leq w^{\mu-2} \leq w^\mu = (1-\beta)^{-1}$. Thus we can make sure that the above is satisfied if we pick our constant c as in Lemma 1, due to (21) of Lemma 6.

A.4 Proof of Eq. (8)

In any such profile, player $i+\mu$ plays $s^*_{i+\mu}$ and

- Either all other players $j = i+1, \dots, i+\mu-1$ play $\tilde{s}_j$, in which case

$$\begin{aligned}\bar{C}_i(\tilde{s}_i, \mathbf{s}'_{-i}) &= c_{i+\mu}\left(\sum_{\ell=i}^{i+\mu} w_\ell\right) + \sum_{j=i+1}^{i+\mu-1} c_j\left(\sum_{\ell=j-\mu}^{j-1} w_\ell\right)\\ &= c_{i+\mu}\left((\alpha+1)w^{i+\mu}\right) + \sum_{j=i+1}^{i+\mu-1} c_j\left(\alpha w^j\right)\\ &= w^{-(i+\mu)(d+1)}(\alpha+1)^d w^{d(i+\mu)} + \sum_{j=i+1}^{i+\mu-1} w^{-j(d+1)}\alpha^d w^{dj}\\ &= w^{-(i+\mu)}(\alpha+1)^d + \sum_{j=i+1}^{i+\mu-1} w^{-j}\alpha^d\\ &= w^{-i}\left[(\alpha+1)^d w^{-\mu} + \alpha^d \sum_{j=1}^{\mu-1} w^{-j}\right]\\ &= w^{-i}\left[(\alpha+1)^d w^{-\mu} + \alpha^d(\alpha - w^{-\mu})\right]\end{aligned}$$

- Or there might exist a *single* player $k \in \{i+1, \dots, i+\mu-1\}$ that plays s^*_k (instead of $\tilde{s}_k$ which corresponds exactly to the previous case), in which case

$$\begin{aligned}\bar{C}_i(\tilde{s}_i, \mathbf{s}'_{-i}) &\leq c_k\left(\sum_{\ell=k-\mu}^{k} w_\ell\right) + c_{i+\mu}\left(\sum_{\ell=i}^{i+\mu} w_\ell - w_k\right) + \sum_{\substack{j=i+1\\ j\neq k, i+\mu}}^{i+\mu} c_j\left(\sum_{\ell=j-\mu}^{j-1} w_\ell\right)\\ &= c_k\left((\alpha+1)w^k\right) + c_{i+\mu}\left((\alpha+1)w^{i+\mu} - w^k\right) + \sum_{\substack{j=i+1\\ j\neq k}}^{i+\mu-1} c_j\left(\alpha w^j\right)\\ &= w^{-i}\left[(\alpha+1)^d w^{-(k-i)} + (\alpha+1-w^{k-i-\mu})^d w^{-\mu} + \alpha^d(\alpha - w^{-\mu} - w^{-(k-i)})\right],\end{aligned}$$

  which is decreasing with respect to k, so taking the smallest possible value $k = i+1$ we have that

$$\bar{C}_i(\tilde{s}_i, \mathbf{s}'_{-i}) \leq w^{-i}\left[(\alpha+1)^d w^{-1} + (\alpha+1-w^{1-\mu})^d w^{-\mu} + \alpha^d(\alpha - w^{-\mu} - w^{-1})\right].$$

Considering both the above possible scenarios, in order to prove (8) it is thus enough to make sure that

$$\alpha^d(\alpha - w^{-\mu}) < (\alpha+1)^d(1 - w^{-\mu}) \tag{33}$$

and

$$(\alpha+1-w^{1-\mu})^d w^{-\mu} + \alpha^d(\alpha - w^{-\mu} - w^{-1}) < (\alpha+1)^d(1-w^{-1}). \tag{34}$$

For (33), its left-hand side can be written as

$$(\beta\Phi_d)^d\left(\beta\Phi_d - (1-\beta)\right) < (\beta\Phi_d)^d\left(\beta\Phi_d\right) = \beta^{d+1}\Phi_d^{d+1} = \beta^{d+1}(\Phi_d+1)^d,$$

the first inequality holding because $\beta < 1$, while the right-hand side is

$$(\beta\Phi_d+1)^d(1-(1-\beta)) = \beta^{d+1}\left(\Phi_d + \frac{1}{\beta}\right)^d.$$

Thus it is enough to prove that

$$(\Phi_d + 1)^d \leq \left(\Phi_d + \frac{1}{\beta}\right)^d,$$

which holds since $\beta \in (0, 1)$.

For (34), the left-hand side is written as

$$\begin{aligned}
&\left(\beta\Phi_d + 1 - \left(1 + \frac{1}{\Phi_d}\right)(1-\beta)\right)^d (1-\beta) + (\beta\Phi_d)^d \left(\beta\Phi_d - (1-\beta) - \left(1 + \frac{1}{\Phi_d}\right)^{-1}\right) \\
=&\left(\beta\Phi_d + \beta - \frac{1-\beta}{\Phi_d}\right)^d (1-\beta) + \beta^d\Phi_d^d \left(\beta\Phi_d - (1-\beta) - \frac{\Phi_d}{\Phi_d+1}\right) \\
<&\,(\beta\Phi_d + \beta)^d (1-\beta) + \beta^d\Phi_d^d \left(\beta\Phi_d - \frac{\Phi_d}{\Phi_d+1}\right) \\
=&\beta^d (\Phi_d + 1)^d (1-\beta) + \beta^d\Phi_d^d \left(\beta\Phi_d - \frac{\Phi_d}{\Phi_d+1}\right) \\
=&\beta^d (\Phi_d)^{d+1} (1-\beta) + \beta^d\Phi_d^d \left(\beta\Phi_d - \frac{\Phi_d}{\Phi_d+1}\right) \\
=&\beta^d\Phi_d^d \left(\Phi_d - \beta\Phi_d + \beta\Phi_d - \frac{\Phi_d}{\Phi_d+1}\right) \\
=&\beta^d \frac{\Phi_d^{d+2}}{\Phi_d+1}
\end{aligned}$$

and the right-hand side

$$(\beta\Phi_d + 1)^d \left(1 - \left(1 + \frac{1}{\Phi_d}\right)^{-1}\right) = \beta^d \left(\Phi_d + \frac{1}{\beta}\right)^d \frac{1}{\Phi_d + 1}.$$

Thus it is enough to prove that

$$\Phi_d^{d+2} \leq \left(\Phi_d + \frac{1}{\beta}\right)^d,$$

which holds due to (3) since we have already selected parameter c as in (1).

B Upper Bound Proofs

B.1 Technical Lemmas

Lemma 8. *For any positive integer m and real $x > 0$,*

$$\left(1 + \frac{1}{x}\right)^m \geq 1 + \frac{m+1}{2x}$$

Proof. Expanding the power in the left hand side we get

$$\left(1 + \frac{1}{x}\right)^m = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{x^j} \geq \sum_{j=0}^{1} \binom{m}{j} \frac{1}{x^j} = 1 + \frac{m}{x} \geq 1 + \frac{m+1}{2x},$$

since

$$\frac{m}{x} \geq \frac{m+1}{2x} \iff m \geq \frac{m+1}{2} \iff m \geq 1.$$

$\square$

Lemma 9. *For any integer $m \geq 0$ and real $x \geq 0$,*

$$(x+1)^{m+1} - x^{m+1} \leq \frac{m+1}{2}\left[(x+1)^m + x^m\right].$$

Proof. Expanding the powers, our inequality can be rewritten equivalently as:

$$\sum_{j=0}^{m+1} \binom{m+1}{j} x^j - x^{m+1} \leq \frac{m+1}{2}\left[\sum_{j=0}^{m} \binom{m}{j} x^j + x^m\right]$$

$$\sum_{j=0}^{m-1} \binom{m+1}{j} x^j + (m+1)x^m \leq \frac{m+1}{2}\left[\sum_{j=0}^{m-1} \binom{m}{j} x^j + 2x^m\right].$$

$$\sum_{j=0}^{m-1} \binom{m+1}{j} x^j \leq \frac{m+1}{2} \sum_{j=0}^{m-1} \binom{m}{j} x^j.$$

Now, we can see that the above holds by bounding each term; for any $j = 0, 1, \ldots, m-1$:

$$\binom{m+1}{j} = \frac{(m+1)!}{(m+1-j)!j!} = \frac{m+1}{m+1-j}\frac{m!}{(m-j)!j!} = \frac{m+1}{m+1-j}\binom{m}{j} \leq \frac{m+1}{2}\binom{m}{j}.$$

□

B.2 Proof of Lemma 3

Observe that from the definition of A_m in (15),

$$(A_m(x))^{-1} = \frac{1}{m+1} + \frac{1}{2x}$$

which is decreasing with respect to m, and

$$\left(\frac{A_m(x)}{m+1}\right)^{-1} = 1 + \frac{m+1}{2x},$$

which is increasing with respect to m. For the remaining sequence, observe that for any integer $m \geq 0$ and reals $y \geq x \geq 1$,

$$\frac{A_m(x)}{A_m(y)} \geq \frac{A_{m+1}(x)}{A_{m+1}(y)} \iff \frac{A_{m+1}(y)}{A_m(y)} \geq \frac{A_{m+1}(x)}{A_m(x)},$$

so it is enough to show that function $\frac{A_{m+1}(x)}{A_m(x)}$ is monotonically increasing with respect to $x \geq 0$. Indeed,

$$\frac{A_{m+1}(x)}{A_m(x)} = \frac{\frac{1}{m+1} + \frac{1}{2x}}{\frac{1}{m+2} + \frac{1}{2x}} = \frac{2x(m+2) + (m+1)(m+2)}{2x(m+1) + (m+1)(m+2)} = 1 + \frac{1}{m+1}\left(1 + \frac{m+2}{2x}\right)^{-1}.$$

B.3 Proof of Lemma 4

First for (18), notice that it can be rewritten equivalently as

$$\frac{1}{m+1} \leq \frac{S_m(\gamma x)}{(\gamma x)^{m+1}} = \frac{1}{A_m(\gamma x)} \leq \frac{1}{A_m(\gamma)},$$

which holds, as an immediate consequence of the monotonicity of function A_m (see (16)), given that $\gamma x \geq x \geq 1$. For (17), it is enough to prove just the special case when $w = 1$, i.e.,

$$\frac{\gamma^{m+1}}{A_m(\gamma)} = S_m(\gamma) \leq \frac{S_m(\gamma x + \gamma) - S_m(\gamma x)}{(x+1)^m} \leq \gamma^{m+1}, \tag{35}$$

since then it is not difficult to check that we can recover the more general case in (17) by simply substituting $\gamma \leftarrow \gamma w$ and $x \leftarrow \frac{x}{w}$ in (35).

It is not difficult to check that (35) holds for $m = 0$, recalling that $S_0(x) = x$ for all $x \geq 0$. Next, assume for the remainder of the proof that $m \geq 1$.

For the left-hand inequality of (35) first, it can be equivalently rewritten as:

$$(x+1)^m \left(\frac{1}{m+1}\gamma^{m+1} + \frac{1}{2}\gamma^m\right) \leq \frac{1}{m+1}\gamma^{m+1}\left[(x+1)^{m+1} - x^{m+1}\right] + \frac{1}{2}\gamma^m\left[(x+1)^m - x^m\right]$$

$$\frac{1}{2}x^m \leq \frac{\gamma}{m+1}\left[(x+1)^{m+1} - (x+1)^m - x^{m+1}\right],$$

and since $\gamma \geq 1$, it is enough to show that

$$(m+1)x^m \leq 2\left[(x+1)^{m+1} - (x+1)^m - x^{m+1}\right]$$

$$(m+1)x^m \leq 2x\left[(x+1)^m - x^m\right].$$

Now observe that the above trivially holds if $x = 0$, while for $x > 0$ it can be equivalently written as

$$\frac{m+1}{2x} \leq \left(1 + \frac{1}{x}\right)^m - 1,$$

which holds due to Lemma 8.

For the right-hand inequality of (35), it can be equivalently written as:

$$\frac{1}{m+1}\gamma\left[(x+1)^{m+1} - x^{m+1}\right] + \frac{1}{2}\left[(x+1)^m - x^m\right] \leq \gamma(x+1)^m$$

$$2\gamma\left[(x+1)^{m+1} - x^{m+1}\right] \leq (m+1)\left[(2\gamma - 1)(x+1)^m + x^m\right]$$

$$(x+1)^{m+1} - x^{m+1} \leq (m+1)\left[\left(1 - \frac{1}{2\gamma}\right)(x+1)^m + \frac{1}{2\gamma}x^m\right].$$

Since $\gamma \geq 1$, we know that $\frac{1}{2\gamma} \in [0, \frac{1}{2}]$. Thus, taking into consideration that $(x+1)^m > x^m \geq 0$, the linear combination on the right-hand side of the above inequality is minimized for $\frac{1}{2\gamma} = \frac{1}{2}$. So, it is enough to show that

$$(x+1)^{m+1} - x^{m+1} \leq \frac{m+1}{2}\left[(x+1)^m + x^m\right],$$

which holds due to Lemma 9.

C Beyond Polynomial Latencies: Euler-Maclaurin

Our definition of the approximate potential function in Sections 4.2 and 4.4 was based in Faulhaber's formula (13) for the sum of powers of positive integers. This approach can be generalized further, by considering the Euler-Macluarin summation formula[13] :

$$\sum_{j=0}^{n} f(j) = \int_0^n f(t)\,dt + \frac{1}{2}[f(n) + f(0)] + \sum_{j=2}^{m} \frac{B_j}{j!}[f^{(j-1)}(n) - f^{(j-1)}(0)] + \mathcal{R}_m, \tag{36}$$

[13] See, e.g., [26, Section 9.5] and [34].

for any infinitely differentiable function $f : [0, \infty) \longrightarrow (0, \infty)$ (with $f^{(j)}$ denoting the j-th order derivative of f) and integers $n, m \geq 1$, where B_j denotes the Bernoulli numbers we have already used in Section 4.2 and the *error-term* $\mathcal{R}_m$ can by bounded by

$$|\mathcal{R}_m| \leq \frac{2\zeta(m)}{(2\pi)^m} \int_0^n \left| f^{(m)}(t) \right| dt, \tag{37}$$

where $\zeta(m) = \sum_{j=1}^{\infty} \frac{1}{n^j}$ is Riemann's zeta function. Thus, if function f is such that the quantity in the right-hand side of (37) eventually vanishes, i.e. for any real $x \geq 0$,

$$\lim_{m\to\infty} \frac{\zeta(m)}{(2\pi)^m} \int_0^x \left| f^{(m)}(t) \right| dt = 0, \tag{38}$$

then we can define our approximate-potential candidate function on any real $x \geq 0$ by generalizing (36):

$$S(x) = S_f(x) = \int_0^x f(t)\, dt + \frac{1}{2}[f(x) + f(0)] + \sum_{j=2}^{\infty} \frac{B_j}{j!}[f^{(j-1)}(x) - f^{(j-1)}(0)]. \tag{39}$$

For example, it is not difficult to see that, for any monomial $f(x) = x^d$ of degree $d \geq 1$, condition (38) is indeed satisfied (since $f^{(m)} = 0$ for all $m \geq d + 1$) and, because also $f^{(m)}(0) = 0$ and $f^{(m)}(x) = \frac{d!}{(d-m)!}x^{d-m}$, one recovers exactly (13) from (39) above.

Let's now demonstrate this general approach for latency functions f that are not polynomials. For the remaining of this section let $f(x) = e^x$ be an exponential delay function. Then, for any $y \geq 0$,

$$\lim_{m\to\infty} \frac{\zeta(m)}{(2\pi)^m} \int_0^y \left| f^{(m)}(t) \right| dt = (e^y - 1) \lim_{m\to\infty} \frac{\zeta(m)}{(2\pi)^m} = 0,$$

since $\lim_{m\to\infty} \zeta(m) = 1$ and $\lim_{m\to\infty}(2\pi)^m = \infty$. Thus, condition (38) is satisfied, and we can define from (39)

$$\begin{aligned} S(x) &= \int_0^x e^t\, dt + \frac{1}{2}[e^x + e^0] + \sum_{j=2}^{\infty} \frac{B_j}{j!}[e^x - e^0] \\ &= (e^x - 1) - \frac{1}{2}(e^x - 1) + \sum_{j=2}^{\infty} \frac{B_j}{j!}(e^x - 1) + e^x \\ &= (e^x - 1)\sum_{j=0}^{\infty} \frac{B_j}{j!} + e^x. \end{aligned}$$

But since for the integer value $x = 1$ we know that $S(1) = \sum_{j=0}^{1} f(j) = 1 + e$, it must be that

$$e + 1 = (e^x - 1)\sum_{j=0}^{\infty} \frac{B_j}{j!} + e \quad \Longleftrightarrow \quad \sum_{j=0}^{\infty} \frac{B_j}{j!} = \frac{1}{e-1}.$$

So, we finally have that

$$S(x) = (e^x - 1)\frac{1}{e-1} + e^x = \frac{e^{x+1} - 1}{e - 1}.$$

From this, for all reals $x \geq 0$, $w > 0$ we compute:

$$\frac{S(x+w) - S(x)}{wf(x+w)} = \frac{1}{e-1}\frac{e^{x+w+1} - e^{x+1}}{we^{x+w}} = \frac{e}{e-1}\frac{1 - e^{-w}}{w}, \tag{40}$$

which does not depend on x. Thus, from (11) in Lemma 2 we deduce that *exact* pure Nash equilibria always exist for weighted congestion games with exponential latencies. The function S we defined in (40) essentially serves as a weighted potential [36]; its global minimum is a pure Nash equilibrium. Notice here that these results regarding exponential latency functions were already known by the work of Panagopoulou and Spirakis [38].

D Social Optimum is a $(d+1)$–Approximate Equilibrium

Theorem 4. *Consider any weighted congestion game with polynomial latency functions of maximum degree d and let $\mathbf{s}^*$ be a strategy profile that minimizes social cost. Then $\mathbf{s}^*$ is a $(d+1)$–approximate pure Nash equilibrium. As an immediate consequence, the Price of Stability of $(d+1)$–approximate Nash equilibria is 1.*

Proof. Let c be an arbitrary cost function of maximum degree d with non-negative coefficients, i.e., $c(x) = \sum_{j=0}^{d} a_j x^j$, with $a_j \geq 0$ for all j. We will first show that for all $w > 0$ and $x \geq 0$:

$$w \cdot c(x+w) \leq (x+w) \cdot c(x+w) - x \cdot c(x) \leq (d+1) \cdot w \cdot c(x+w). \tag{41}$$

To this end, with $z = \frac{x}{w}$, we get

$$\begin{aligned}
(x+w) \cdot c(x+w) - x \cdot c(x) &= \sum_{j=0}^{d} a_j \cdot \left[(x+w)^{j+1} - x^{j+1}\right] \\
&= \sum_{j=0}^{d} a_j \cdot w^{j+1} \left[(1+z)^{j+1} - z^{j+1}\right] \\
&= \sum_{j=0}^{d} a_j \cdot w^{j+1} \left[\sum_{k=0}^{j} \binom{j+1}{k} z^k\right],
\end{aligned}$$

and

$$\begin{aligned}
w \cdot c(x+w) &= \sum_{j=0}^{d} a_j \cdot w(x+w)^{j+1} \\
&= \sum_{j=0}^{d} a_j \cdot w^{j+1} \left[\sum_{k=0}^{j} \binom{j}{k} z^k\right].
\end{aligned}$$

Clearly, $\binom{j+1}{k} \geq \binom{j}{k}$ for all integer $j \in [0,d], k \in [0,j]$, which immediately implies the first inequality in (41). To see the second inequality, observe that

$$\frac{\binom{j+1}{k}}{\binom{j}{k}} = \frac{j+1}{j+1-k} \leq j+1 \leq d+1.$$

Since $\mathbf{s}^*$ minimizes social cost, for all players $i \in [n]$ and strategies $s_i \in S_i$,

$$C(\mathbf{s}^*) \leq C(s_i, \mathbf{s}^*_{-i}).$$

Denoting $y_e = \sum_{j \in [n] \setminus \{i\} : s \in s_j^*} w_j$, from (41), we get

$$
\begin{aligned}
0 &\leq C(s_i, \mathbf{s}_{-i}^*) - C(\mathbf{s}^*) \\
&= \sum_{e \in s_i \setminus s_i^*} [(y_e + w_i) c_e(y_e + w_i) - y_e c_e(y_e)] - \sum_{e \in s_i^* \setminus s_i} [(y_e + w_i) c_e(y_e + w_i) - y_e c_e(y_e)] \\
&= \sum_{e \in s_i} [(y_e + w_i) c_e(y_e + w_i) - y_e c_e(y_e)] - \sum_{e \in s_i^*} [(y_e + w_i) c_e(y_e + w_i) - y_e c_e(y_e)] \\
&\leq (d+1) \sum_{e \in s_i} w_i \cdot c_e(y_e + w_i) - \sum_{e \in s_i^*} w_i \cdot c_e(y_e + w_i) \\
&= (d+1) C_i(s_i, \mathbf{s}_{-i}^*) - C_i(\mathbf{s}^*),
\end{aligned}
$$

or equivalently $C_i(\mathbf{s}^*) \leq (d+1) C_i(s_i, \mathbf{s}_{-i}^*)$. So $\mathbf{s}^*$ is a $(d+1)$-approximate Nash equilibrium. $\square$