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Partial Pole Assignment with Time Delays for Asymmetric Systems

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Abstract Considering both single and multiple time delays, partial pole assignment for stabilising asymmetric systems is exemplified by friction-induced vibration and aerodynamic flutter. The control strategy is a single-input state feedback including constant time delays in the feedback loop. An unobservability condition is considered to assign some poles while keeping others unchanged. The receptance method is applied to avoid modelling errors from evaluating mass, damping and stiffness matrices by the finite element method. The solution is formulated in linear equations which allow determination of control gains. The stability of the closed-loop system is analysed by evaluating the first few dominant poles and determining a critical time delay. The numerical study shows that the proposed method is capable of making partial pole assignment with time delays. Since many structures and systems with non-conservative forces can be represented by asymmetric systems, this approach is widely applicable for vibration control of engineering structures.

Keywords Partial pole assignment · single and multiple time delays · asymmetric system · friction-induced vibration · aerodynamic flutter · receptance method · unobservability condition · single-input state feedback

1 Introduction

Undesirable vibration generated from machines and natural sources may lead to degradation of machine performances, failure of structures, and health deterioration of humans. It can be reduced in many ways. One of them, pole assignment, is to shift natural frequencies away from the excitation frequencies for avoiding resonances and/or to add damping for preventing excessive vibration. Active pole assignment was firstly formulated in the first-order differential equation [1, 11, 35] and recently it has been implemented in the second-order differential equation [4, 5, 10]. Although the first-order differential equation is a general approach and mostly used in general control theory, the second-order differential equation is the natural formulation for vibration problems and maintains some good properties of the second-order equation of motion such as the symmetry of the structural matrices.

The use of pole assignment to a structural system containing an infinite number of degrees of freedom does not guarantee stability due to unassigned (residue) poles. They may get unintentionally

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shifted from the left-hand side to the right-hand side of the complex plane leading to instability. To overcome this so-called spill-over problem, Datta et al. [6] proposed partial pole assignment by using single-input control which could shift some poles and keep others unchanged. Ram [23] applied this algorithm to control vibration of a rod. Datta and Sarkissian [8], and Ram and Elhay [24] extended the single-input control to the multi-input control. Xu and Qian [37] studied robust partial pole assignment using state feedback control. It can not only assign partial poles but also provide robustness (insensitivity to perturbations). Datta et al. [7] introduced partial eigenstructure assignment which allowed both eigenvalues and eigenvectors to be assigned.

In practice, time delay always exists in active vibration control systems due to sensing and actuating in the feedback loop. It may degrade the control performance and destabilise a system. To deal with this problem, partial pole assignment with time delay using single-input and multiple-input control was developed by Pratt et al. [22] and Bai et al. [2]. However, these works did not investigate the stability. Singh and Ouyang [29] demonstrated various methods, i.e., Taylor series expansion, Newton's eigenvalue iteration method [30] and graphical root-finding algorithm [34], to determine the first few dominant poles of the closed-loop system (the closed-loop system with time delay has an infinite number of poles) and implemented a frequency-sweeping method [9] to determine the critical time delay. Moreover, Singh et al. [28] used traceDDE toolbox in MATLAB [3] to evaluate the first twenty closed-loop poles and applied Cluster Treatment of Characteristic Roots (CTCR) technique [18] to determine the critical time delay of a system with 3 degrees-of-freedom.

Active pole assignment in a model based approach requires the knowledge of mass, damping and stiffness matrices. The matrices can be evaluated from the finite element method (FEM). Mottershead and Ram [15] pointed out that finite element models always had errors because several assumptions must be made when these models were constructed, for example, damping was neglected (or proportional damping was assumed) and model reduction methods were applied. To deal with this problem, the receptance method for a single-input control was proposed [15]. The idea of this method is to measure receptance from experiment rather than evaluating mass, damping and stiffness matrices from the FEM. So modelling errors from FEM can be avoided. The receptance method was extended to pole-zero assignment [16], pole assignment with time delay [26, 27], robust pole assignment [17, 32] and eigenstructure assignment [14]. It was also applied to partial pole assignment to shift some poles and keep others unchanged by using the uncontrollability condition [31] and unobservability condition [25]. Furthermore, Ram and Mottershead [25] extended partial pole assignment to a multi-input receptance-based method.

The aforementioned works studied pole assignment for symmetric systems. Nonetheless, some structures and machines are asymmetric systems due to non-conservative forces such as friction and aerodynamic forces. They are defined by asymmetric damping and/or stiffness matrices and prone to instability reflected by some positive real parts of poles. Friction-induced vibration being an example of an asymmetric system was stabilised by pole assignment using the receptance method [20, 21]. Many researchers extended the previous works to pole assignment with time delay [29], robust pole assignment [13], partial pole assignment by using uncontrollability condition [33] and unobservability condition [12]. Nonetheless, partial pole assignment with time delay for asymmetric systems has not been done yet.

In this paper, partial pole assignment with time delays for asymmetric systems by using the single-input state-feedback control is proposed. The receptance method is implemented which requires no knowledge of mass, damping and stiffness matrices. The unobservability condition is also applied to keep some poles unchanged. The solution is determined by using Sherman-Morrison formula and formulated in linear equations which can be solved directly. The stability is analysed by using traceDDE toolbox in MATLAB [3] to compute the first few dominant poles of the closed-loop system and the frequency-sweeping test [9] is carried out to determine the critical time delay. Two numerical examples of friction-induced vibration and aerodynamic flutter show that the partial pole assignment with the time delays as proposed in this paper can assign the required closed-loop poles precisely without changing the other poles. The dominant closed-loop poles and the critical time delay are determined to ensure that the closed-loop system is stable.

2 Partial pole assignment with time delays

The dynamic equation of asymmetric systems including velocity and displacement feedback control with time delays presented in [29] can be written as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{b}u(t - \tau) + \mathbf{p}(t), \quad (1)$$

$$\mathbf{C} = \mathbf{C}_s + \mathbf{C}_{as}, \quad (2)$$

$$\mathbf{K} = \mathbf{K}_s + \mathbf{K}_{as}, \quad (3)$$

$$u(t - \tau) = -\mathbf{f}^T \dot{\mathbf{x}}(t - \tau_f) - \mathbf{g}^T \mathbf{x}(t - \tau_g), \quad (4)$$

where \mathbf{M} , \mathbf{C}_s and $\mathbf{K}_s \in \mathbb{R}^{n \times n}$ are respectively real symmetric mass, damping and stiffness matrices, \mathbf{C}_{as} and $\mathbf{K}_{as} \in \mathbb{R}^{n \times n}$ are respectively real asymmetric damping and stiffness matrices generated by non-conservative forces such as friction and aerodynamic forces; \mathbf{x} , \mathbf{p} , \mathbf{b} , \mathbf{f} and \mathbf{g} are respectively displacement, excitation force, actuator distribution, velocity gain and displacement gain vectors; u is a control force, τ_f and τ_g are time delays associated with the velocity and displacement state feedback respectively, which either occur naturally in control equipment or are intentionally introduced; and superscript T denotes matrix transposition.

Laplace transform of Eq.(1) gives

$$(s^2\mathbf{M} + s(\mathbf{C}_s + \mathbf{C}_{as}) + \mathbf{K}_s + \mathbf{K}_{as})\mathbf{x}(s)e^{st} = -\mathbf{b}(se^{-s\tau_f}\mathbf{f}^T + e^{-s\tau_g}\mathbf{g}^T)\mathbf{x}(s)e^{st} + \mathbf{p}(s)e^{st}, \quad (5)$$

$$(s^2\mathbf{M} + s(\mathbf{C}_s + \mathbf{C}_{as}) + \mathbf{K}_s + \mathbf{K}_{as} + \mathbf{b}(se^{-s\tau_f}\mathbf{f}^T + e^{-s\tau_g}\mathbf{g}^T))\mathbf{x}(s) = \mathbf{p}(s). \quad (6)$$

Define

$$\mathbf{H}_{as}(s) = (s^2\mathbf{M} + s(\mathbf{C}_s + \mathbf{C}_{as}) + \mathbf{K}_s + \mathbf{K}_{as})^{-1}. \quad (7)$$

$$\widehat{\mathbf{H}}_{as}(s) = (\mathbf{H}_{as}^{-1}(s) + \mathbf{b}(se^{-s\tau_f}\mathbf{f}^T + e^{-s\tau_g}\mathbf{g}^T))^{-1}. \quad (8)$$

Apparently \mathbf{H}_{as} and $\widehat{\mathbf{H}}_{as}$ are (asymmetric) receptance matrices of the open-loop and closed-loop systems. It should be pointed out [33] that \mathbf{H}_{as} is difficult to measure in practice, for example, measuring \mathbf{H}_{as} in a disc brake requires a torque to rotate the disc while the brake is engaged. So, \mathbf{H}_{as} should be rearranged and expressed in term of the open-loop symmetric receptance matrix \mathbf{H}_s , which is easier to measure.

$$\mathbf{H}_{as}(s) = (\mathbf{I} + \mathbf{H}_s(s)(s\mathbf{C}_{as} + \mathbf{K}_{as}))^{-1}\mathbf{H}_s(s). \quad (9)$$

$$\mathbf{H}_s(s) = (s^2\mathbf{M} + s\mathbf{C}_s + \mathbf{K}_s)^{-1}. \quad (10)$$

Using Sherman-Morrison formula, Eq.(8) can be expressed as

$$\widehat{\mathbf{H}}_{as}(s) = \mathbf{H}_{as}(s) - \frac{\mathbf{H}_{as}(s)\mathbf{b}(se^{-s\tau_f}\mathbf{f}^T + e^{-s\tau_g}\mathbf{g}^T)\mathbf{H}_{as}(s)}{1 + (se^{-s\tau_f}\mathbf{f}^T + e^{-s\tau_g}\mathbf{g}^T)\mathbf{H}_{as}(s)\mathbf{b}}. \quad (11)$$

It can be seen that the poles of the closed-loop system (that are different from the poles of the open-loop system) satisfy the following characteristic equation:

$$(\mu_i e^{-\mu_i \tau_f} \mathbf{f}^T + e^{-\mu_i \tau_g} \mathbf{g}^T) \mathbf{H}_{as}(\mu_i) \mathbf{b} = -1. \quad (12)$$

where μ_i is a required closed-loop pole.

According to Eq.(11), a particular pole λ_i can make the closed-loop receptance matrix equal to the open-loop receptance matrix ($\widehat{\mathbf{H}}_{as} = \mathbf{H}_{as}$) when

$$(\lambda_i e^{-\lambda_i \tau_f} \mathbf{f}^T + e^{-\lambda_i \tau_g} \mathbf{g}^T) \mathbf{H}_{as}(\lambda_i) = 0, \quad (13)$$

where λ_i is an unchanged pole. Post-multiplying Eq.(13) by \mathbf{b} , it implies that

$$(\lambda_i e^{-\lambda_i \tau_f} \mathbf{f}^T + e^{-\lambda_i \tau_g} \mathbf{g}^T) \mathbf{H}_{as}(\lambda_i) \mathbf{b} = 0. \quad (14)$$

Combining Eq.(12) and Eq.(14), the velocity gain and displacement gain vectors for the partial pole assignment with the time delays by using the unobservability condition are derived as

$$\begin{bmatrix} e^{-\mu_1 \tau_g} \mathbf{r}_1^T & \mu_1 e^{-\mu_1 \tau_f} \mathbf{r}_1^T \\ \vdots & \vdots \\ e^{-\mu_p \tau_g} \mathbf{r}_p^T & \mu_p e^{-\mu_p \tau_f} \mathbf{r}_p^T \\ e^{-\lambda_{p+1} \tau_g} \mathbf{r}_{p+1}^T & \lambda_{p+1} e^{-\lambda_{p+1} \tau_f} \mathbf{r}_{p+1}^T \\ \vdots & \vdots \\ e^{-\lambda_{2n} \tau_g} \mathbf{r}_{2n}^T & \lambda_{2n} e^{-\lambda_{2n} \tau_f} \mathbf{r}_{2n}^T \end{bmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (15)$$

$$\mathbf{r} = \mathbf{H}_{as} \mathbf{b}. \quad (16)$$

To validate the result, the required closed-loop poles, the unchanged poles, the control gain vectors, the actuator distribution vector and the time delays are substituted back into Eq.(17). If $|D|$ is close to zero, the partial poles are successfully assigned by using the unobservability condition.

$$D = \det(s^2 \mathbf{M} + s(\mathbf{C} + e^{-s\tau_f} \mathbf{b}\mathbf{f}^T) + \mathbf{K} + e^{-s\tau_g} \mathbf{b}\mathbf{g}^T). \quad (17)$$

It should be noted that time delay can vary with time or state. When it varies with time, an analytical mathematical expression of Laplace transform of $u(t - \tau)$, such as in Eq.(5), cannot be obtained. This means that pole assignment as a frequency-domain method cannot be used. When time delay varies with state, Eq.(1) becomes a nonlinear dynamic problem. Again pole assignment cannot be used. Therefore, in this paper, time delay is assumed to be a constant.

3 Stability Analysis

3.1 Root-finding method

The stability must be investigated after applying the feedback control. The time delays in the feedback loop may destabilize the system. Without the time delay $\tau = 0$, the characteristic equation of the closed-loop system,

$$\mathbf{P}(s) = s^2 \mathbf{M} + s(\mathbf{C} + \mathbf{b}\mathbf{f}^T) + \mathbf{K} + \mathbf{b}\mathbf{g}^T, \quad (18)$$

is generally formulated in a polynomial which has $2n$ poles for the n -dimensional system. The stability is defined by pole locations. All negative real parts of the poles indicate a stable system and any positive real part of the pole indicates an unstable system. Nonetheless, the characteristic equation of the closed-loop system with time delays

$$\mathbf{Q}(s) = s^2 \mathbf{M} + s(\mathbf{C} + e^{-s\tau_f} \mathbf{b}\mathbf{f}^T) + \mathbf{K} + e^{-s\tau_g} \mathbf{b}\mathbf{g}^T, \quad (19)$$

has an infinite number of poles. Various methods were proposed, i.e., Taylor series expansion, Newton's eigenvalue iteration method [30] and graphical method [34] to approximate the first few dominant poles (closest to the imaginary axis of the complex plane) of the closed-loop system. In this paper, traceDDE toolbox in MATLAB [3] is applied to evaluate the dominant closed-loop poles.

3.2 Frequency-sweeping test

Although root-finding methods can determine the first few dominant closed-loop poles, they cannot guarantee stability because the remaining poles may have positive real parts. To ensure stability, the frequency-sweeping test [9] is implemented. The closed-loop system with time delays is classified by delay-independent stability, delay-dependent stability and instability. Delay-independent stability means the system is stable for any time delay: $\tau = [0, \infty]$. Delay-dependent stability indicates the system is stable within a limited range of time delay: $\tau = [0, \bar{\tau}]$ where $\bar{\tau}$ is the critical time delay. Instability means an unstable system for any time delay.

In this paper, only commensurate delays are considered. The closed-loop system

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = -\mathbf{b}\mathbf{f}^T\dot{\mathbf{x}}(t - \tau_f) - \mathbf{b}\mathbf{g}^T\mathbf{x}(t - \tau_g), \quad (20)$$

can be studied in cases of two specific delays:

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_0\mathbf{z}(t) + \mathbf{A}_1\mathbf{z}(t - \tau), \text{ for } \tau_f = \tau_g = \tau \geq 0, \\ \mathbf{z} &= \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}, \mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b}\mathbf{g}^T & -\mathbf{M}^{-1}\mathbf{b}\mathbf{f}^T \end{bmatrix}, \end{aligned} \quad (21)$$

or

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_0\mathbf{z}(t) + \mathbf{A}_1\mathbf{z}(t - \tau) + \mathbf{A}_2\mathbf{z}(t - 2\tau), \text{ for } \tau_f = \tau \text{ and } \tau_g = 2\tau \geq 0, \\ \mathbf{z} &= \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}, \mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}^{-1}\mathbf{b}\mathbf{f}^T \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b}\mathbf{g}^T & \mathbf{0} \end{bmatrix}. \end{aligned} \quad (22)$$

For a single time delay $\tau_f = \tau_g = \tau$, Gu et al. [9] stated that the closed-loop system is delay-independent stable if and only if

$$\Re(\lambda_i(\mathbf{A}_0)) < 0 \quad (\mathbf{A}_0 \text{ is stable}), \quad (23)$$

$$\Re(\lambda_i(\mathbf{A}_0 + \mathbf{A}_1)) < 0 \quad (\mathbf{A}_0 + \mathbf{A}_1 \text{ is stable}), \quad (24)$$

$$\rho(i\omega\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_1 < 1, \quad \forall \omega > 0, \quad (25)$$

where ρ is spectral radius: $\rho(\mathbf{A}) = \max_{1 \leq i \leq 2n} |\lambda_i(\mathbf{A})|$, $\lambda_i(\mathbf{A})$ denotes eigenvalues of matrix \mathbf{A} . If one of them is not satisfied, the closed-loop system is not delay-independent stable (may be either delay-dependent stable or unstable). For the delay-dependent stable system, if the closed-loop system without time delay is stable (Eq.(24) is satisfied), the critical time delay is determined by

$$\bar{\tau} := \min_{1 \leq i \leq q} \bar{\tau}_i, \quad \text{for } q = \text{rank}(\mathbf{A}_1), \quad (26)$$

$$\bar{\tau}_i = \min_{1 \leq k \leq 2n} \frac{\theta_k^i}{\omega_k^i} \quad \text{if } \lambda_i((i\omega_k^i\mathbf{I} - \mathbf{A}_0), \mathbf{A}_1) = e^{-i\theta_k^i}, \quad \text{for some } \omega_k^i \in (0, \infty), \theta_k^i \in [0, 2\pi]. \quad (27)$$

For multiple commensurate time delays (for example, $\tau_f = \tau$ and $\tau_g = 2\tau$), Gu et al. [9] stated that the closed-loop system is delay-independent stable if and only if

$$\Re(\lambda_i(\mathbf{A}_0)) < 0 \quad (\mathbf{A}_0 \text{ is stable}), \quad (28)$$

$$\Re(\lambda_i(\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2)) < 0 \quad (\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2 \text{ is stable}), \quad (29)$$

$$\rho\left(\begin{bmatrix} (s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_1 & (s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\right) < 1, \quad \forall \omega > 0. \quad (30)$$

If one of them is not satisfied, the closed-loop system is not delay-independent stable (may be either delay-dependent stable or unstable). For the delay-dependent stable system, if the closed-loop system without time delay is stable (Eq.(29) is satisfied), the critical time delay is determined by

$$\bar{\tau} := \min_{1 \leq i \leq q+2n} \bar{\tau}_i, \text{ for } q = \text{rank}(\mathbf{A}_2), \quad (31)$$

$$\bar{\tau}_i = \min_{1 \leq k \leq 2n} \frac{\theta_k^i}{\omega_k^i} \text{ if } \lambda_i \left(\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(s\mathbf{I} - \mathbf{A}_0) & \mathbf{A}_1 \end{bmatrix}, \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_2 \end{bmatrix} \right) = e^{-i\theta_k^i}, \text{ for some } \omega_k^i \in (0, \infty), \theta_k^i \in [0, 2\pi]. \quad (32)$$

Consequently, the closed-loop system with time delays is stable if $\tau < \bar{\tau}$ but it becomes unstable if $\tau > \bar{\tau}$.

4 Numerical examples

4.1 Friction-induced vibration

A friction-induced vibration problem modelled by a mass-spring-damper system on a conveyor belt as illustrated in Fig.1 is the asymmetric system under the present study. It is the same model studied in [19] (which made structural modifications). The system consists of four masses with m_1 having a degree-of-freedom in the horizontal direction, m_4 having a degree-of-freedom in the vertical direction, m_2 and m_3 having degrees-of-freedom in both directions. When the belt is moving, friction forces are generated to produce an asymmetric stiffness matrix. To simplify the problem, Coulomb friction is considered and stick-slip phenomena is avoided. \mathbf{M} , \mathbf{C}_s , \mathbf{K}_s and \mathbf{K}_{as} corresponding to the displacement vector, $\mathbf{x} = \{x_1, y_4, x_2, x_3, y_2, y_3\}^T$ are given by,

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_3 \end{bmatrix}, \mathbf{C}_s = \begin{bmatrix} c_1 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & c_1 + c_2 & -c_2 & 0 & 0 \\ 0 & 0 & -c_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 \end{bmatrix},$$

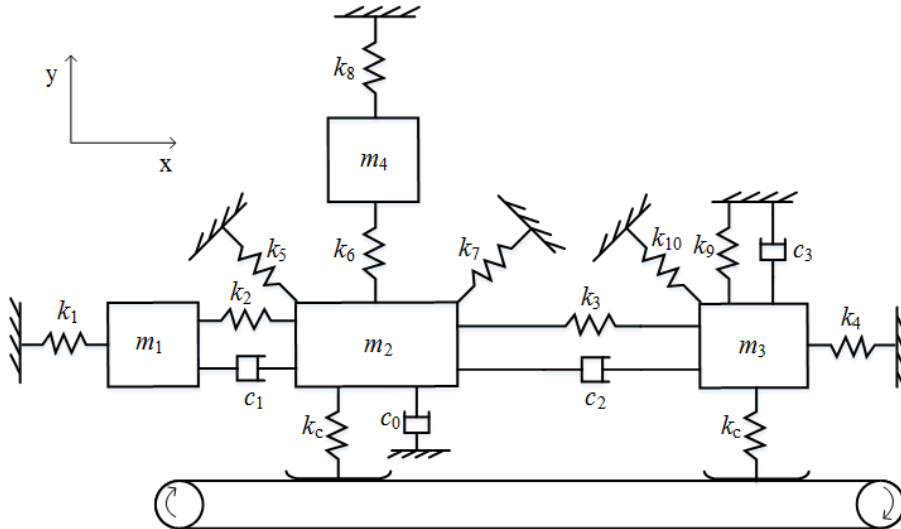


Fig. 1 An asymmetric system of friction-induced vibration

$$\mathbf{K}_s = \begin{bmatrix} k_{11} & 0 & k_{13} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & k_{25} & 0 \\ k_{31} & 0 & k_{33} & k_{34} & k_{35} & 0 \\ 0 & 0 & k_{43} & k_{44} & 0 & k_{46} \\ 0 & k_{52} & k_{53} & 0 & k_{55} & 0 \\ 0 & 0 & 0 & k_{64} & 0 & k_{66} \end{bmatrix}, \mathbf{K}_{as} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_c k_c & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_c k_c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where $k_{11} = k_1 + k_2$, $k_{13} = k_{31} = -k_2$, $k_{22} = k_6 + k_8$, $k_{25} = k_{52} = -k_6$, $k_{33} = k_2 + k_3 + 0.5(k_5 + k_7)$, $k_{34} = k_{43} = -k_3$, $k_{35} = k_{53} = 0.5(k_7 - k_5)$, $k_{44} = k_3 + k_4 + 0.5k_{10}$, $k_{46} = k_{64} = -0.5k_{10}$, $k_{55} = k_c + k_6 + 0.5(k_5 + k_7)$, $k_{66} = k_c + k_9 + 0.5k_{10}$.

Assuming mass; $m_i = 1$ kg ($i = 1, 2, 3, 4$), damping; $c_i = 0.5$ Ns/m ($i = 0, 1, 2, 3$), stiffness; $k_i = 100$ N/m ($i = 1, 2, 3, 4, 5, 6, 8, 9, 10$), $k_7 = 50$ N/m, contact stiffness; $k_c = 110$ N/m and friction coefficient; $\mu_c = 0.5$, the open-loop poles are determined by using polyeig function in MATLAB (see in Table 1). Obviously, the open-loop poles indicate that the system is unstable because the first pair of poles is located on the right-hand side of the complex plane. To stabilise the system, they must be shifted to the left-hand side of the complex plane.

Three cases of required closed-loop poles are shown in Table 1. Partial pole assignment with different time delays (including single and multiple time delays) are demonstrated by using the unobservability condition. The actuator distribution vector; $\mathbf{b} = \{0, 0, 1, 1, 1, 1\}^T$ is assumed. Velocity and displacement gain vectors for assigning the required closed-loop poles including the time delays are determined by solving the linear equations expressed in Eq.(15). The results are shown in Table 2. They are then substituted back into Eq.(17). It is found that $|D|$ is close to zero for all cases. Hence, the algorithm of partial pole assignment with time delays by using the unobservability condition is successful.

The stability can be investigated by examining the poles of the closed-loop system. By using traceDDE toolbox in MATLAB [3], the first twenty poles of the closed-loop system are determined (see Table 3). As can be seen, for all cases, the required closed-loop poles are placed precisely and the other poles have negative real parts. Although the closed-loop poles have negative real parts, they do not guarantee the stability due to unidentified residue poles.

Alternatively, the stability is analysed by using the frequency-sweeping test [9] described in section 3.2. It is clear that the closed-loop systems for all cases are not delay-independent stable because of positive real parts of the open-loop poles (\mathbf{A}_0 is unstable). For delay-dependent stability, the critical time delay can be determined if the closed-loop system without time delay is stable ($\mathbf{A}_0 + \sum_{i=1}^m \mathbf{A}_i$ is stable). It is obtained by using Eq.(26) and Eq.(31) for the single time delay and the multiple time delays respectively. The results are shown in Table 4. Obviously, for case III with the single time delay; $\tau_f = \tau_g = 0.10$ and multiple time delays; $\tau_f = 0.05, \tau_g = 0.10$, the critical time delay cannot be determined because $\mathbf{A}_0 + \sum_{i=1}^m \mathbf{A}_i$ is unstable. For other cases, the closed-loop systems are stable because the time delays are smaller than the critical ones.

In order to validate the critical time delays as shown in Table 4, displacement responses are plotted by using dde23 function in MATLAB. The initial conditions used to test this are taken to be zero displacement and zero velocity for all degrees-of-freedom except $x_3 = 0.01$. For example case II with $\tau_f = \tau_g = 0.05$, the critical time delay at $\bar{\tau} = 0.1406$ is validated by applying time delays $\tau_1 = 0.13$ ($\tau_1 < \bar{\tau}$) and $\tau_2 = 0.15$ ($\tau_1 > \bar{\tau}$). It is clear from Fig.2 that the closed-loop system is stable for τ_1 but it is unstable for τ_2 . By using the same method, the closed-loop systems for case III with $\tau_f = \tau_g = 0.1$ and $\tau_f = 0.05, \tau_g = 0.1$ are stable.

Table 1 Open-loop poles and required closed-loop poles

| Open-loop poles | Closed-loop poles | | |
|--------------------|--------------------|--------------------|--------------------|
| | Case I | Case II | Case III |
| 0.0069 ± 10.3843i | -1.0000 ± 10.5000i | -1.0000 ± 10.5000i | -1.0000 ± 10.5000i |
| -0.0903 ± 11.4497i | -0.0903 ± 11.4497i | -1.0000 ± 11.5000i | -1.0000 ± 11.5000i |
| -0.2517 ± 15.2078i | -0.2517 ± 15.2078i | -1.0000 ± 15.0000i | -1.0000 ± 15.0000i |
| -0.2465 ± 15.9791i | -0.2465 ± 15.9791i | -0.2465 ± 15.9791i | -1.0000 ± 16.0000i |
| -0.0838 ± 18.8646i | -0.0838 ± 18.8646i | -0.0838 ± 18.8646i | -1.0000 ± 18.5000i |
| -0.8346 ± 19.6958i | -0.8346 ± 19.6958i | -0.8346 ± 19.6958i | -0.8346 ± 19.6958i |

Table 2 Control gains obtained by using the unobservability condition

| Case | $\tau_f = \tau_g = 0.05$ | | $\tau_f = \tau_g = 0.10$ | | $\tau_f = 0.05, \tau_g = 0.10$ | |
|------|--------------------------|----------|--------------------------|----------|--------------------------------|----------|
| | g | f | g | f | g | f |
| I | -5.5461 | 1.5305 | -12.6243 | 0.9381 | -6.5012 | 1.2013 |
| | 1.9129 | -0.6826 | 5.1805 | -0.4476 | 2.3218 | -0.5650 |
| | -5.1010 | 1.4081 | -11.6325 | 0.8582 | -5.7976 | 1.1113 |
| | -4.0290 | 0.9744 | -8.3916 | 0.5852 | -4.8354 | -0.7341 |
| | 1.4587 | -0.6289 | 4.5753 | -0.4118 | 1.5600 | -0.5436 |
| | 0.0973 | -0.0324 | 0.2529 | -0.2010 | 0.1035 | -0.0268 |
| II | -15.4097 | -1.4658 | 2.2849 | -1.7048 | -14.1786 | -2.1774 |
| | 19.1193 | 4.8153 | -19.3657 | 4.3470 | 18.1490 | 5.5873 |
| | -22.2576 | 0.0028 | -16.2764 | -1.1700 | -25.1700 | -1.2760 |
| | -24.0482 | 0.5830 | -23.4735 | -0.9000 | -29.8556 | -0.9018 |
| | 18.3706 | 3.1316 | -7.8771 | 3.0537 | 20.1041 | 4.0326 |
| | 2.1532 | -0.1466 | 3.0501 | 0.0231 | 2.9517 | 0.0005 |
| III | 8.8756 | -5.1007 | 61.2222 | -2.4915 | 35.0940 | -3.0658 |
| | 13.6525 | 4.0540 | -15.7040 | 3.6150 | 14.1803 | 4.2553 |
| | -21.9749 | 2.0511 | -45.0928 | -0.1745 | -32.2067 | 0.1430 |
| | -41.4140 | 3.4034 | -68.3566 | -0.1186 | -65.6609 | -0.2696 |
| | -12.0039 | 3.7648 | -38.9014 | 1.8174 | -34.8242 | 1.7928 |
| | 14.9200 | -4.8223 | 67.4949 | -1.9479 | 39.1440 | -2.5662 |

Table 3 First twenty poles of the closed-loop system (sorted by the real part)

| Case | $\tau_f = \tau_g = 0.05$ | $\tau_f = \tau_g = 0.10$ | $\tau_f = 0.05, \tau_g = 0.10$ |
|------|--------------------------|--------------------------|--------------------------------|
| I | $-0.0838 \pm 18.8646i$ | $-0.0838 \pm 18.8646i$ | $-0.0838 \pm 18.8646i$ |
| | $-0.0903 \pm 11.4497i$ | $-0.0903 \pm 11.4497i$ | $-0.0903 \pm 11.4497i$ |
| | $-0.2465 \pm 15.9791i$ | $-0.2465 \pm 15.9791i$ | $-0.2465 \pm 15.9791i$ |
| | $-0.2517 \pm 15.2078i$ | $-0.2517 \pm 15.2078i$ | $-0.2517 \pm 15.2078i$ |
| | $-0.8346 \pm 19.6958i$ | $-0.8346 \pm 19.6958i$ | $-0.8346 \pm 19.6958i$ |
| | $-1.0000 \pm 10.5000i$ | $-1.0000 \pm 10.5000i$ | $-1.0000 \pm 10.5000i$ |
| | -74.707 | -31.615 | $-43.019 \pm 385.400i$ |
| | $-91.877 \pm 146.197i$ | $-43.467 \pm 74.4832i$ | $-43.183 \pm 385.281i$ |
| | $-102.72 \pm 275.863i$ | $-49.405 \pm 138.832i$ | -54.184 |
| | $-105.63 \pm 2288.83i$ | $-52.554 \pm 1144.42i$ | $-56.805 \pm 311.847i$ |
| II | $-0.0838 \pm 18.8646i$ | $-0.0838 \pm 18.8646i$ | $-0.0838 \pm 18.8646i$ |
| | $-0.2465 \pm 15.9791i$ | $-0.2465 \pm 15.9791i$ | $-0.2465 \pm 15.9791i$ |
| | $-0.8346 \pm 19.6958i$ | $-0.8346 \pm 19.6958i$ | $-0.8346 \pm 19.6958i$ |
| | $-1.0000 \pm 10.5000i$ | $-1.0000 \pm 10.5000i$ | $-1.0000 \pm 10.5000i$ |
| | $-1.0000 \pm 11.5000i$ | $-1.0000 \pm 11.5000i$ | $-1.0000 \pm 11.5000i$ |
| | $-1.0000 \pm 15.0000i$ | $-1.0000 \pm 15.0000i$ | $-1.0000 \pm 15.0000i$ |
| | -51.760 | -22.566 | -33.794 |
| | $-76.375 \pm 148.296i$ | $-41.664 \pm 76.9644i$ | $-42.963 \pm 385.497i$ |
| | $-87.790 \pm 277.096i$ | $-48.599 \pm 140.603i$ | $-43.183 \pm 385.281i$ |
| | $-95.013 \pm 404.147i$ | $-52.557 \pm 1144.46i$ | $-56.805 \pm 311.847i$ |
| III | $-0.8346 \pm 19.6958i$ | $-0.8346 \pm 19.6958i$ | $-0.8346 \pm 19.6958i$ |
| | $-1.0000 \pm 10.5000i$ | $-1.0000 \pm 10.5000i$ | $-1.0000 \pm 10.5000i$ |
| | $-1.0000 \pm 11.5000i$ | $-1.0000 \pm 11.5000i$ | $-1.0000 \pm 11.5000i$ |
| | $-1.0000 \pm 15.0000i$ | $-1.0000 \pm 15.0000i$ | $-1.0000 \pm 15.0000i$ |
| | $-1.0000 \pm 16.0000i$ | $-1.0000 \pm 16.0000i$ | $-1.0000 \pm 16.0000i$ |
| | $-1.0000 \pm 18.5000i$ | $-1.0000 \pm 18.5000i$ | $-1.0000 \pm 18.5000i$ |
| | -41.253 | -22.519 | -18.998 |
| | $-71.699 \pm 149.518i$ | $-49.087 \pm 89.2780i$ | $-43.183 \pm 385.281i$ |
| | $-83.433 \pm 277.818i$ | $-54.454 \pm 1144.60i$ | $-43.389 \pm 385.432i$ |
| | $-90.746 \pm 404.658i$ | $-55.351 \pm 1144.93i$ | $-44.683 \pm 84.2841i$ |

Table 4 Critical time delay

| Case | $\tau_f = \tau_g = 0.05$ | $\tau_f = \tau_g = 0.10$ | $\tau_f = 0.05, \tau_g = 0.10$ |
|------|--------------------------|--------------------------|--------------------------------|
| I | 0.1688 | 0.2205 | 0.1334 |
| II | 0.1406 | 0.2015 | 0.1047 |
| III | 0.1118 | - | - |

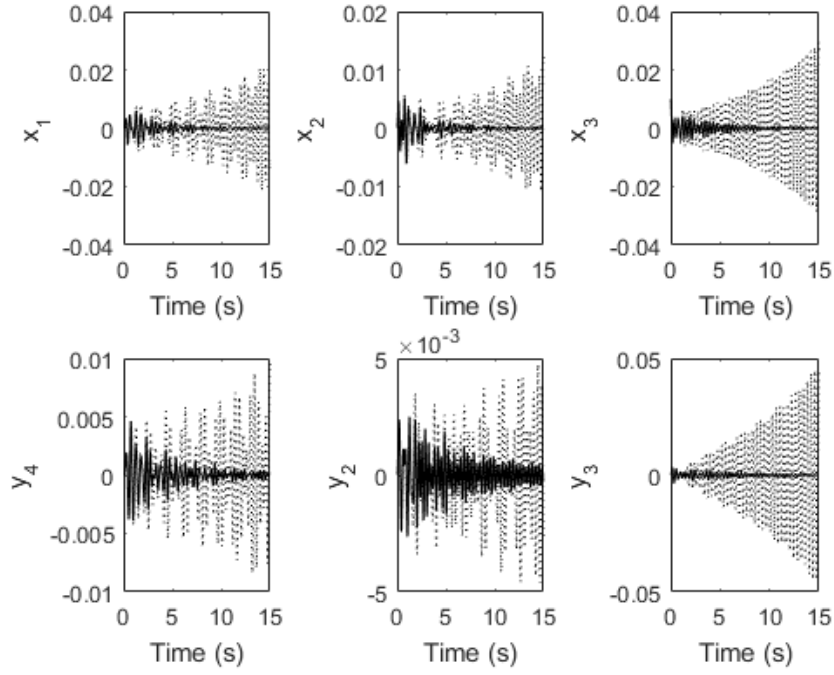


Fig. 2 Displacement responses of the closed-loop system for case II with $\tau_f = \tau_g = 0.05$: $\tau_1 = 0.13$ (solid line) and $\tau_2 = 0.15$ (dotted line)

4.2 Aerodynamic flutter

The aeroelastic equation of a binary bending-torsion rectangular cantilevered wing (see Fig.3) is given by [36]

$$\mathbf{M}\ddot{\mathbf{q}} + (\rho V \mathbf{C}_{as} + \mathbf{C}_s)\dot{\mathbf{q}} + (\rho V^2 \mathbf{K}_{as} + \mathbf{K}_s)\mathbf{q} = 0, \quad (33)$$

$$\mathbf{M} = \begin{bmatrix} \frac{m s c}{5} & \frac{m s}{4} (e^2 - c x_f) \\ \frac{m s}{4} (c^2 - c x_f) & \frac{m s}{3} (\frac{c^3}{3} - c^2 x_f + c x_f^2) \end{bmatrix}, \quad \mathbf{C}_{as} = \begin{bmatrix} \frac{c s a_w}{10} & 0 \\ -\frac{c^2 s e a_w}{8} & -\frac{c^3 s M_{\dot{\theta}}}{24} \end{bmatrix},$$

$$\mathbf{K}_{as} = \begin{bmatrix} 0 & \frac{c s a_w}{8} \\ 0 & -\frac{c^2 s e a_w}{6} \end{bmatrix}, \quad \mathbf{K}_s = \begin{bmatrix} \frac{4 E I}{s^3} & 0 \\ 0 & \frac{G J}{s} \end{bmatrix}, \quad \mathbf{q} = \begin{pmatrix} q_b \\ q_t \end{pmatrix},$$

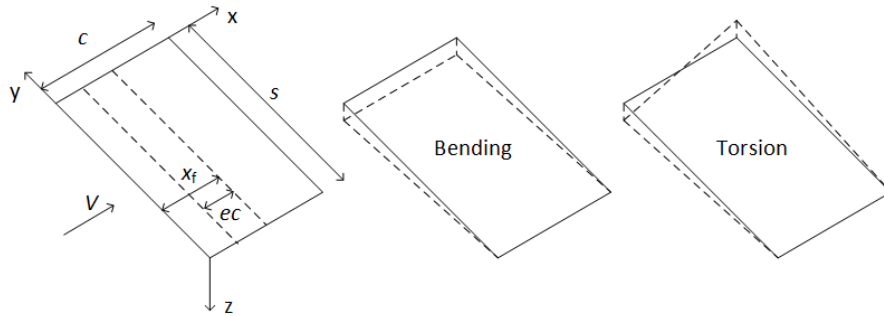


Fig. 3 A rectangular wing showing bending and torsion modes

where \mathbf{M} , \mathbf{C}_s and \mathbf{K}_s are respectively the structural inertia, damping and stiffness (symmetric) matrices, \mathbf{C}_{as} and \mathbf{K}_{as} are the aerodynamic damping and stiffness (asymmetric) matrices and \mathbf{q} is a generalised coordinate vector for bending q_b and torsion q_t .

Parameters used are taken from [36]: mass per unit area of the wing $m = 200 \text{ kg/m}^2$, wing span $s = 7.5 \text{ m}$, chord $c = 2 \text{ m}$, elastic axis location $x_f = 0.48c$, bending rigidity $EI = 2 \times 10^7 \text{ Nm}^2$, torsional rigidity $GJ = 2 \times 10^6 \text{ Nm}^2$, lift curve slope $a_w = 2\pi$, non-dimensional pitch damping derivative $M_{\dot{\theta}} = -1.2$, eccentricity between flexural axis and aero centre $e = x_f/c - 0.25$, air density $\rho = 1.225 \text{ kg/m}^3$, and air speed $V = 150 \text{ m/s}$. The damping matrix \mathbf{C}_s is ignored. The poles of the open-loop system are

$$\{\lambda\}_1^4 = \left\{ \begin{array}{l} 2.8961 \pm 18.7011i \\ -5.2470 \pm 12.2944i \end{array} \right\}.$$

Obviously, the open-loop system is unstable because the first pair of poles have positive real parts. To stabilise the system, partial pole assignment with multiple time delays $\tau_f = 0.03$ and $\tau_g = 0.06$ is considered to assign $\{\mu\}_1^2 = -1 \pm 20i$ and keep $\{\mu\}_3^4$ unchanged. The actuator distribution vector $\mathbf{b} = \{1, 1\}^T$ is assumed. By applying Eq.(15), velocity and displacement gain vectors are obtained as

$$\mathbf{g} = \begin{pmatrix} 4.8126 \\ -2.9705 \end{pmatrix} \times 10^3, \quad \mathbf{f} = \begin{pmatrix} 1.0364 \\ 1.9497 \end{pmatrix} \times 10^3.$$

They are substituted back into Eq.(17). It is found that $|D|$ is close to zero. Hence, the algorithm of partial pole assignment with the given time delays by using unobservability condition is successful. By using Eq.(31), the critical time delay, $\bar{\tau} = 0.0460$, is determined. This means that the closed-loop system with the given time delays is stable. The critical time delay is validated by plotting responses with the given initial conditions; $\mathbf{q} = \{0, 0.01\}^T$, $\dot{\mathbf{q}} = \{0, 0\}^T$. It is clear from Fig.4 that the closed-loop system with $\tau_1 = 0.04$ is stable but the closed-loop system with $\tau_2 = 0.05$ is unstable.

In addition, robustness is an important and interesting topic. The third author has studied robust full pole assignment (see [13]). The authors are studying robust partial pole assignment now but this topic is beyond the scope of this paper and thus it will not be reported here.

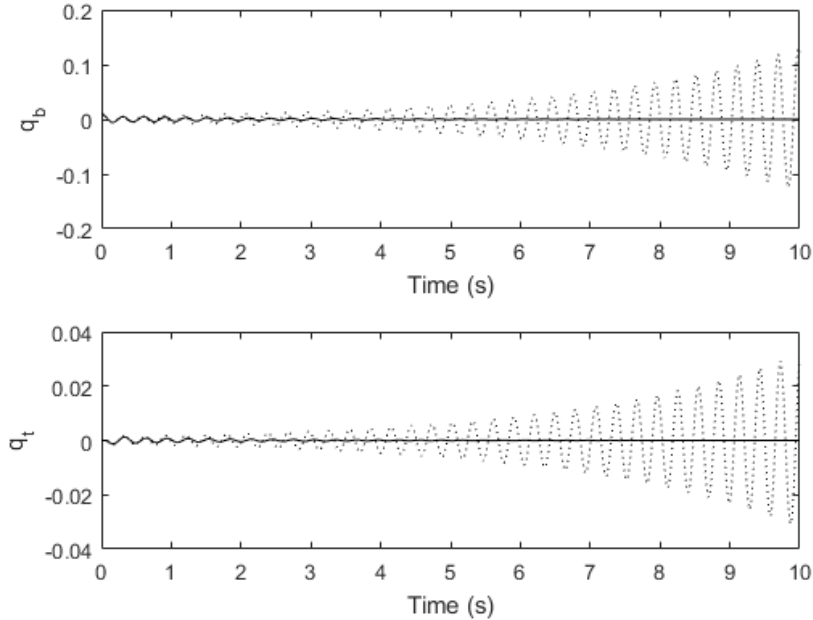


Fig. 4 Responses of the closed-loop system with $\tau_1 = 0.04$ (solid line) and $\tau_2 = 0.05$ (dotted line)

5 Conclusions

In this paper, active partial pole assignment with time delays to asymmetric systems using single-input state feedback control is developed. The unobservability condition is applied to keep some poles unchanged. The receptance method is used to avoid modelling errors from the finite element method and the Sherman-Morrison formula is used to formulate the partial pole assignment problem in linear equations which can be solved directly. Both single time delay and multiple commensurate time delays are tested to ensure that the partial pole assignment as proposed in this paper can assign the required closed-loop poles precisely without spillover.

Stability is investigated by using traceDDE toolbox in MATLAB to determine the dominant closed-loop poles. It is clear that the number of closed-loop poles is higher than $2n$ poles due time delays. Some closed-loop poles calculated by traceDDE are exactly the same as the required ones and others are located in the left-hand side of the complex plane. However, they cannot guarantee the stability since the locations of the remaining poles are not completely known yet. To ensure stability, the frequency-sweeping test is applied to determine the critical time delay.

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