Integrability of exponential process and its application to backward stochastic differential equations

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Abstract

We consider the integrability problem of an exponential process with *unbounded* coefficients. The integrability is established under weaker conditions of Kazamaki type, which complements the results of Yong obtained under a Novikov type condition. As applications, we consider the solvability of linear BSDEs and market completeness, the solvability of a Riccati BSDE and optimal investment, all in the setting of unbounded coefficients.

Keywords: Exponential process; Unbounded coefficients; Linear and Riccati BSDEs; Market completeness; Optimal investment.

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a given complete probability space on which a d-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. The *exponential supermartingale* $S(\cdot)$ is defined as:

$$S(t) := e^{-\frac{1}{2}\int_0^t |\theta(s)|^2 ds - \int_0^t \theta'(s)dW(s)}, \quad t \in [0, T],$$

where $\theta(\cdot)$ is some suitable $\mathcal{F}_t$-adapted process. As its name implies, $S(\cdot)$ satisfies the following inequality (see, for example, Karatzas and Shreve (1988)):

$$\sup_{t\in[0,T]} \mathbb{E}[S(t)] \leq 1. \tag{1.1}$$

In Yong (2006)[1], the following generalisation of $S(\cdot)$ is considered:

$$M(t) := M(t\,; r(\cdot), \theta(\cdot)) := e^{-\int_0^t [r(s)+\frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)' dW(s)}, \quad t \in [0,T], \tag{1.2}$$

for some suitable $\mathcal{F}_t$-adapted process $r(\cdot)$, and named the *exponential process*. Yong's main concern was with the *integrability* problem of this process, i.e. the problem of finding the range of values of $q > 0$ under which these *expectations* are finite:

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,; r(\cdot), \theta(\cdot))^q\right], \tag{1.3}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,; r(\cdot), \theta(\cdot))^q\right], \tag{1.4}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,; r(\cdot), \theta(\cdot))^{-q}\right]. \tag{1.5}$$

The motivation was the solvability of *linear* backward stochastic differential equations (BSDEs) and market completeness with *unbounded* coefficients. Several estimates of (1.3), (1.4), and (1.5) were obtained under different assumptions on $r(\cdot)$. The assumption on $\theta(\cdot)$ was the following:

$$G_\beta := \mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2 ds}\right] < \infty, \tag{1.6}$$

for some $\beta > 0$, which is a *Novikov* type condition.

In this paper, we also consider the integrability problem of $M(\cdot)$ through the finiteness of (1.3), (1.4), and (1.5). Our assumptions on $r(\cdot)$ are mostly the *same* as those of Yong. However, instead of condition (1.6), we assume that $\theta(\cdot)$ satisfies the following *Kazamaki* type condition:

$$H_{\pm\beta} := \sup_{t\in[0,T]} \mathbb{E}\left[e^{\pm\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right] < \infty. \tag{1.7}$$

[1]Throughout the paper, by Yong (2006) we mean the paper: J. Yong, Completeness of security markets and solvability of linear backward stochastic differential equations, *J. Math. Analy. Appl.*, 319 (2006), 333-356.

The Kazamaki condition is *weaker* than the Novikov condition in the sense that (see the Remark in Kazamaki (1978)):

$$G_\beta < \infty \quad \Rightarrow \quad H_{\pm\sqrt{\beta}} < \infty. \tag{1.8}$$

It is to be expected that if we work under weaker conditions, then we should obtain *weaker* integrability. This is indeed the case for our first estimate of (1.3) as given by Theorem 1 below. However, for all our other estimates, we obtain the *same* integrability as Yong while making the *weaker* assumption (1.7). These are *unexpected* conclusions, and complement well those of Yong.

As already mentioned, the main motivation of Yong in considering the integrability problem was the solvability of linear BSDEs. These are equations of the form:

$$\begin{cases} dY(t) = [r(t)Y(t) + \theta'(t)Z(t)]dt + Z'(t)dW(t), \quad t \in [0,T], \\ \\ Y(T) = \xi, \quad a.s., \end{cases} \tag{1.9}$$

where ξ is a given $\mathcal{F}_T$-measurable random variable. The problem of *solvability* for (1.9) is the problem of existence of a *solution pair* $(Y(\cdot), Z(\cdot))$ of adapted processes such that (1.9) holds. Linear BSDEs were introduced by Bismut (1976), whereas the nonlinear BSDEs are introduced by Pardaoux & Peng (1990). The BSDEs are studied extensively since then and have found wide applications in areas such as mathematical finance, stochastic control, and stochastic controllability; see, for example, Lü *et al.* (2012), Lü & Zhang (2014), Tang & Zhang (2004), Tang & Zhang (2009), Gashi & Pantelous (2013), Gashi & Pantelous (2015), Gashi (2015), Karoui *et al.* (1997), Ma & Yong (1999), Mao (2011), Peng (1994), Wang (2013), Yong & Zhou (1999), and the references therein. An assumption in *virtually all* of the literature on BSDEs is that the coefficients of the equation are *bounded*.

The BSDEs with possibly *unbounded* coefficients are important in mathematical finance. When the interest rate is modeled as a solution to a stochastic differential equation (see, for example, Bingham & Kiesel (2004), Date & Gashi (2013), Yong (2006)), which in general is an unbounded process, gives rise to various problems in a market with unbounded coefficients. One such a problem is the *market completeness* (see, for example, Bingham &

Kiesel (2004)). This has motivated Delbaen & Tang (2010), Gashi & Li, Karoui & Huang (1997) and Yong (2006), to consider the problem of solvability of BSDEs with unbounded coefficients. In Gashi & Li and Karoui & Huang (1997), *general* BSDEs are considered, and solution pairs are shown to exist in certain *weighted* spaces. Different from these papers, Yong (2006) considered *linear* BSDEs with unbounded coefficients and under different assumptions on the terminal value ξ. His approach is based on establishing the *integrability* of exponential processes and the reduction of the linear BSDE to a more basic form. The solution pairs in this case belong to *non-weighted* spaces.

In this paper, as the first application of our integrability results, we consider the problem of existence of the solution pair $(Y(\cdot), Z(\cdot))$ for BSDE (1.9) under weaker conditions on the unbounded coefficients as compared to Yong (2006) (see Theorem 4.1 below). One motivation for this is that it almost immediately solves two basic problems of mathematical finance: the *market completeness* and the problem of *pricing and hedging* (see, for example, Karatzas & Shreve (1998), Duffie (1992)). If a market is complete, then any contingent claim (from a certain set) can be hedged, i.e. there exists a self-financing portfolio that exactly replicates the value of the contingent claim. This is equivalent to the solvability of (1.9). On the other hand, as a by product of the solvability problem, we also solve the pricing and hedging problem since $Y(0)$ is the price at time zero of the contingent claim ξ, whereas the process $Z(t)$ is the heading strategy.

Another basic problem of mathematical finance if that of *optimal investment*, i.e. of *financial asset management*. Here we have an investor with wealth given by equation (1.9), but instead of the terminal value ξ being specified, it is the initial value $Y(0)$ that is specified to be equal to the investor's initial capital Y_0. Such an investor is interested to invest in a certain *optimal* way. One popular criterion for optimal investment is the *expected utility* of terminal wealth, i.e. the investor wishes to choose the *trading strategy* $Z(t)$ so that the cost function $\mathbb{E}[U(Y(T))]$ is maximized for some suitable *utility function* U. If the market coefficients are *bounded*, then this problem is essentially solved in Pliska (1986), Cox & Huang (1989), Karatzas *et al.*(1987) (see also Karatzas & Shreve (1998), Korn (1997), Duffie (1992), for a textbook account). This problem has been studied expensively since then (see, for example, Li, Chen and Liu (2016), Li, Rong, and Zhao (2016), Shen and Siu (2017), Stefano and Daniele (2016), Zhao and Rong (2017), for some recent results on the asset management problem). Explicit solutions are known only in special cases, and these include the mean-variance (linear-quadratic utility) Lim & Zhou (2002), exponential and power utility Ferland & Waiter (2008). The problem of optimal investment with *unbounded* coefficients is known to a much lesser extent (see, for example, Bielecki & Pliska (2004), Bieleki *et al.* (2004), Korn & Kraft (2001), Kraft (2005)). The unboundedness of the coefficients is due to the modeling of coefficients by stochastic differential equations, and the *Markovian* structure of the model is exploited in finding the solution. The recent paper Shen (2015) considers the mean-variance optimal investment problem in a market with unbounded coefficients.

In this paper, as our second application, we consider the problem of optimal investment in a market with possibly unbounded coefficients. Using our results on the integrability of the exponential process, as well as our results on the solvability of the linear BSDE (1.9), we first give conditions on the existence of a solution pair for a *Riccati* BSDE (see Lemma 4.2 below), which is a *nonlinear* equation. This is then used to solve in an explicit closed-form the optimal investment problem for the power utility $U(x) = x^\gamma$, where $\gamma \in (0, 1)$ (see Theorem 4.2 below). We remark that in Yong (2006) neither of these two applications are considered. Finally, let us emphasize that an insufficient appreciation of the role of integrability of certain processes when solving the asset management problem in a market with unbounded coefficients has lead to several incomplete or even erroneous results (see, for example, Kraft (2009) and the second of Yong's 2006 papers[2]

The rest of the paper is organized as follows. The next subsection gives the notation for various spaces. In section 2, we give our integrability results and compare them with those of Yong, whereas the proofs follow in section 3. In section 4, which accounts for *half* of the paper, we give the applications mentioned earlier.

[2]Here we mean the paper J. Yong, Remarks on some short rate term structure models, *J. Indu. Man. Opt.*, 2 (2006), 119-134.

1.1. Notation for some spaces

Let E denote a finite dimensional Euclidian space with norm $|\cdot|$.

- $\mathcal{L}^0_{\mathcal{F}_T}(\Omega; E)$ is the set of all $\mathcal{F}_T$-measurable E-valued random variables.

- $\mathcal{L}^p_{\mathcal{F}_T}(\Omega; E)$ is the set of all random variables $\xi \in \mathcal{L}^0_{\mathcal{F}_T}(\Omega; E)$ which for some $p \in (0, \infty)$ satisfy the condition. $\mathbb{E}|\xi|^p < \infty$.

- $\mathcal{L}^0_{\mathcal{F}}(0, T; E)$ is the set of all $\mathcal{F}_t$-adapted processes $\psi : [0, T] \times \Omega \to E$.

- $\mathcal{L}^p_{\mathcal{F}}(\Omega; \mathcal{L}^q(0, T; E))$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0, T; E)$ which for some $p, q \in (0, \infty)$ satisfy the condition

$$\mathbb{E}\left[\int_0^T |\psi(t)|^q dt\right]^{p/q} < \infty.$$

- $\mathcal{L}^p(0, T; E) := \mathcal{L}^p_{\mathcal{F}}(\Omega; \mathcal{L}^p(0, T; E))$, for some $p \in (0, \infty)$.

- $\mathcal{L}^0_{\mathcal{F}}(\Omega; C([0, T]; E)$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0, T; E)$ with almost all paths continuous.

- $\mathcal{L}^p_{\mathcal{F}}(\Omega; C([0, T]; E)$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(\Omega; C([0, T]; E)$ which for some $p > 0$ satisfy the condition

$$\mathbb{E}\left[\sup_{t \in [0,T]} |\psi(t)|^p\right] < \infty.$$

- $\mathcal{L}^q_{\mathcal{F}}(0, T; \mathcal{L}^p(\Omega; E))$ is the set of all processes $\psi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0, T; E)$ which for some $p, q \in (0, \infty)$ satisfy the condition

$$\int_0^T (\mathbb{E}|\psi(t)|^p)^{q/p}\, dt < \infty.$$

- $\mathcal{L}^{q+}_{\mathcal{F}_T}(\Omega; E) := \bigcup_{p \in (q, \infty]} \mathcal{L}^p_{\mathcal{F}_T}(\Omega; E)$ for some $q > 0$.

- $\mathcal{L}^{q-}_{\mathcal{F}_T}(\Omega; E) := \bigcap_{p \in (0, q)} \mathcal{L}^p_{\mathcal{F}_T}(\Omega; E)$ for some $q > 0$.

- $\mathcal{L}^{p\pm}_{\mathcal{F}}(\Omega; \mathcal{L}^{q\pm}(0, T; E))$ and $\mathcal{L}^{p\pm}_{\mathcal{F}}(\Omega; C([0, T]; E))$ are defined in a similar way to the above.

2. Results

In this section we give our integrability results, i.e. we give sufficient conditions under which the expectations (1.3), (1.4), and (1.5) are finite. The motivation for considering these *different* expectations comes from our applications (see §4 below). Our first two results (Theorem 2.1 and Theorem 2.2 below) consider the finiteness of (1.3). They differ on their assumptions on the process $r(\cdot)$. In order to ensure that the process $M(\cdot)$ is well defined, we make the following *standing* assumptions:

$$r(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^1(0,T;\mathbb{R})), \quad \text{and} \quad \theta(\cdot) \in \mathcal{L}^1_{\mathcal{F}}(\Omega; \mathcal{L}^2(0,T;\mathbb{R}^d)).$$

Theorem 1. *If for some $\alpha_0 > 0$, $\beta > 1$, we have*

$$A_{-\alpha_0} := \sup_{t\in[0,T]} \mathbb{E}\left[e^{-\alpha_0 \int_0^t r(s)ds}\right] < \infty, \tag{2.1}$$

$$H_{-\beta} = \sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right] < \infty, \tag{2.2}$$

then

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,; r(\cdot), \theta(\cdot))^{p_1}\right] \le \kappa_1, \tag{2.3}$$

where $p_1 := \alpha_0\beta/(\beta + 2\alpha_0)$ and $\kappa_1 := (A^{\beta}_{-\alpha_0} H^{2\alpha_0}_{-\beta})^{1/(\beta+2\alpha_0)}$.

Theorem 2. *Let (2.2) hold for some $\beta > 1$. If*

$$\inf_{t\in[0,T]} \int_0^t r(s)ds \ge -\kappa, \quad a.s., \tag{2.4}$$

for some $\kappa \in \mathbb{R}$, then

$$\sup_{t\in[0,T]} \mathbb{E}\left[M(t\,; r(\cdot), \theta(\cdot))^{p_2}\right] \le \kappa_2, \tag{2.5}$$

where $p_2 := \beta^2/(2\beta - 1)$ and $\kappa_2 := e^{\frac{\kappa\beta^2}{2\beta-1}} H_{-\beta}^{\frac{2\beta-2}{2\beta-1}}$.

Let us compare the above two results with Theorem 3.2 (ii)-(iii) of Yong (2006), respectively. Our assumptions on $r(\cdot)$ are the same as those in Yong (2006). But for $\theta(\cdot)$, instead of assuming $G_\beta < \infty$ for $\beta > 1$ as in Yong (2006), we make the weaker assumption $H_{-\beta} < \infty$. Now we compare the *degree* of integrability, i.e. the coefficients p_1 and p_2 with those in Yong (2006), which for convenience we denote as $\bar{p}_1$ and $\bar{p}_2$, respectively. Note that due to (1.8), when *comparing* our results with Yong (2006), we should use $\sqrt{\beta}$ instead of β in our results, otherwise we obtain an erroneous comparison. Therefore, our degree of integrability and that of Yong (2006) for the above two results are:

$$p_1 = \frac{\alpha_0\sqrt{\beta}}{\sqrt{\beta}+2\alpha_0}, \qquad \bar{p}_1 = \frac{\alpha_0\sqrt{\beta}}{\sqrt{\beta}+2\alpha_0-(\alpha_0/\sqrt{\beta})}, \tag{2.6}$$

$$p_2 = \frac{\beta}{2\sqrt{\beta}-1}, \qquad \bar{p}_2 = \frac{\beta}{2\sqrt{\beta}-1}. \tag{2.7}$$

From (2.6), it is clear that $p_1 < \bar{p}_1$, which is to be expected since we are assuming less. Also note that as $\beta \to \infty$ we have $p_1 = \bar{p}_1 = \alpha_0$, i.e. *asymptotically* we have the same degree of integrability as in Yong (2006). However, from (2.7) it is clear that we *always* have the same degree of integrability as in Yong (2006), although we assume less, which is unexpected.

Our next three results (Theorems 2.3-2.5 below) consider the problem of finiteness of expectation (1.4) for different ranges of the Kazamaki condition and different assumptions on the process $r(\cdot)$.

Theorem 3. *Let the following condition hold for some* $\alpha_0 > 0$*:*

$$B_{-\alpha_0} := \mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha_0\int_0^t r(s)ds}\right] < \infty. \tag{2.8}$$

If (2.2) *holds for some* $\beta > 1$*, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_3}\right] \le \kappa_3,$$

where $p_3 := \frac{\alpha_0\beta^2}{\beta^2+2\alpha_0\beta-\alpha_0}$, $\kappa_3 := c_1 B_{-\alpha_0}^{\frac{\beta^2}{\beta^2+2\alpha_0\beta-\alpha_0}} H_{-\beta}^{\frac{2\alpha_0(\beta-1)}{\beta^2+2\alpha_0\beta-\alpha_0}}$ *with* $c_1 := \left(\frac{\alpha_0\beta^2}{\alpha_0\beta^2-2\beta^2-2\alpha_0\beta+\alpha_0}\right)^{\frac{\alpha_0\beta^2}{\beta^2+2\alpha_0\beta-\alpha_0}}$.

Theorem 4. *Let* (2.8) *and* (2.2) *hold for some* $\alpha_0 > 0$ *and* $\beta \in (0,1]$*, respectively. Then for any* $p_4 \in \left(\frac{\alpha_0\beta}{2\alpha_0+\beta}, \frac{\alpha_0\beta}{\alpha_0+\beta}\right)$*, we have:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_4}\right] \le \kappa_4,$$

where $\kappa_4 := c_2 B_{-\alpha_0}^{\frac{p_4}{\alpha_0}} H_{-\beta}^{\frac{2\alpha_0\beta - 2p_4(\alpha_0+\beta)}{\alpha_0\beta}}$ with $c_2 := \left(\frac{\alpha_0^2 p_4^2}{\alpha_0^2 p_4^2 - \beta(\alpha_0 - p_4)[(\beta+2\alpha_0)p_4 - \alpha_0\beta]}\right)^{\frac{\alpha_0 p_4^2}{\beta[(2\alpha_0+\beta)p_4 - \alpha_0\beta]}}$.

Theorem 5. *(a) If* (2.2) *and* (2.4) *hold for some $\beta > 1$ and $\kappa \in \mathbb{R}$, respectively, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_5}\right] \le \kappa_5,$$

where $p_5 := \frac{\beta^2}{2\beta-1}$, $\kappa_5 := c_3 H_{-\beta}^{\frac{2\beta-2}{2\beta-1}}$ *with* $c_3 := e^{\frac{\kappa\beta^2}{2\beta-1}}\left[\frac{\beta^2}{(\beta-1)^2}\right]^{\frac{\beta^2}{2\beta-1}}$.

(b) If (2.2) *and* (2.4) *hold for some $\beta \in (0,1]$ and $\kappa \in \mathbb{R}$, respectively, then for any $p_6 \in \left(\frac{\beta}{2}, \beta\right)$, we have:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_6}\right] \le \kappa_6,$$

where $\kappa_6 := c_4 H_{-\beta}^{\frac{2(\beta-p_6)}{\beta}}$ *with* $c_4 := e^{\kappa p_6}\left[\frac{p_6^2}{p_6^2 - \beta(2p_6-\beta)}\right]^{\frac{p_6^2}{\beta(2p_6-\beta)}}$.

Let us now compare the above three results with Theorem 3.4 (i)–(iii) of Yong (2006), respectively. Our assumptions on $r(\cdot)$ are the same as Yong (2006), whereas for $\theta(\cdot)$ we make the weaker assumption $H_{-\beta} < \infty$ instead of $G_\beta < \infty$. It can be easily checked that our degree of integrability is the *same* as that in Yong (2006), i.e. the parameters p_3, p_4, p_5, and p_6 are the same with the corresponding ones of Yong (once we use $\sqrt{\beta}$ instead of β due to (1.8)), despite making weaker assumptions.

Our next two results (Theorem 2.6 and Theorem 2.7 below) consider the finiteness of the expectation (1.5) for different ranges of the Kazamaki condition. Note that different from the previous two results, here we consider the integrability of the *inverse* of the exponential process, and our previous results are used in the proofs (see §3 below).

Theorem 6. *Let $\alpha > 0$ and $\beta_0 > 1$ be given. If*

$$C_\alpha := \mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t [r(s)+|\theta(s)|^2]ds}\right] < \infty, \tag{2.9}$$

$$H_{\beta_0} = \sup_{t\in[0,T]} \mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t \theta'(s)dW(s)}\right] < \infty, \tag{2.10}$$

then

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p_7}\right] \le \kappa_7,$$

where $p_7 := \frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}$, $\kappa_7 := c_5 C_\alpha^{\frac{\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}} H_{\beta_0}^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}}$ *with* $c_5 := \left(\frac{\alpha\beta_0^2}{\alpha\beta_0^2-2\beta_0^2-2\alpha\beta_0+\alpha}\right)^{\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}$.

Theorem 7. *If* (2.9) *and* (2.10) *hold for some* $\alpha > 0$ *and* $\beta_0 \in (0,1]$, *respectively, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p_8}\right] \le \kappa_8,$$

where $p_8 \in \left(\frac{\alpha\beta_0}{2\alpha+\beta_0}, \frac{\alpha\beta_0}{\alpha+\beta_0}\right)$, *and* $\kappa_8 := c_6 C_\alpha^{\frac{p_8}{\alpha}} H_{\beta_0}^{\frac{2\alpha\beta_0-2p_8(\alpha+\beta_0)}{\alpha\beta_0}}$ *with* $c_6 := \left(\frac{\alpha^2 p_8^2}{\alpha^2 p_8^2-\beta_0(\alpha-p_8)[(\beta_0+2\alpha)p_8-\alpha\beta_0]}\right)^{\frac{\alpha p_8^2}{\beta_0[(2\alpha+\beta_0)p_8-\alpha\beta_0]}}$.

Let us now compare these two results with Theorem 3.5 (i) and Theorem 3.5 (ii) of Yong (2006), respectively. In this case, both of our assumptions on $r(\cdot)$ and $\theta(\cdot)$ are different as compared to Yong (2006). Recall first that Yong (2006) obtains his results under assumptions:

$$B_{\alpha_0} < \infty, \quad \text{for some } \alpha_0 > 0, \tag{2.11}$$
$$G_\beta < \infty, \quad \text{for some } \beta > 1. \tag{2.12}$$

However, (2.11) and (2.12) together imply $C_\alpha < \infty$ with $\alpha = \alpha_0\beta/(2\alpha_0+\beta)$ (see the end of page 347 of Yong (2006)). Therefore, for such an α our assumptions are weaker than in Yong (2006). Let us now compare the degree of integrability. Note that in this case our p_7 becomes (with $\alpha_0\beta/(2\alpha_0+\beta)$ instead of α and $\sqrt{\beta}$ instead of β_0):

$$p_7 = \frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)},$$

which is the *same* as the degree of integrability in Theorem 3.5 (i) of Yong (2006). Similarly, we can conclude that even p_8 is the same as the corresponding degree of integrability as in Theorem 3.5 (ii) of Yong (2006).

Finally, let us point out that Yong obtained some other integrability results which either did not make any assumptions on $\theta(\cdot)$ or we could not improve them further.

3. Proofs

We first give two useful lemmas; their proofs are elementary, but we include them for completeness.

Lemma 1. *Let $\alpha_0 > 0$ and $\beta > 1$ be given. For any $\gamma > 1$ there exists a solution pair $(p(\gamma), q(\gamma))$ to the following system of equations and inequalities:*

$$\begin{cases} \alpha_0 = \frac{pq\gamma}{(\gamma-1)(q-1)}, \\ \beta = \frac{2q\gamma}{\gamma-1}(\frac{\sqrt{p\gamma}}{\gamma} + p), \\ p > 0, q > 1. \end{cases} \tag{3.1}$$

Moreover, the global maximum of $p(\gamma)$ is $p^* = \frac{\alpha_0 \beta}{\beta + 2\alpha_0}$.

Proof. The requirement of $q > 1$ implies (from the second equation in (3.1)):

$$q = \frac{\beta(\gamma-1)}{2(p\gamma + \sqrt{p\gamma})} > 1 \Rightarrow \frac{\beta(\gamma-1)}{2} > p\gamma + \sqrt{p\gamma}.$$

If $z := \sqrt{p\gamma}$, then $z > 0$ and

$$z^2 + z - \frac{\beta(\gamma-1)}{2} < 0. \tag{3.2}$$

If

$$z_{1,2} := \frac{-1 \pm \sqrt{1 + 2\beta(\gamma-1)}}{2},$$

then we can write (3.2) as $(z - z_1)(z - z_2) < 0$. For this inequality to hold, it is necessary to have

$$z_2 < z < z_1 = \frac{-1 + \sqrt{1 + 2\beta(\gamma-1)}}{2},$$

i.e.

$$z = \sqrt{p\gamma} < \frac{-1 + \sqrt{1 + 2\beta(\gamma-1)}}{2}. \tag{3.3}$$

Thus, in order to have $q > 1$, (3.3) should be satisfied. Substituting z into (3.1), and knowing that $z > 0$, we have:

$$z = \frac{-1 + \sqrt{1 + 2\beta(\gamma - 1) + \frac{\beta^2}{\alpha_0}(\gamma - 1)}}{2 + \frac{\beta}{\alpha_0}}. \tag{3.4}$$

Since $1 + 2\beta(\gamma - 1) + \frac{\beta^2}{\alpha_0}(\gamma - 1) > 0$, we can write (3.3) as:

$$\frac{-1 + \sqrt{1 + 2\beta(\gamma - 1) + \frac{\beta^2}{\alpha_0}(\gamma - 1)}}{2 + \frac{\beta}{\alpha_0}} < \frac{-1 + \sqrt{1 + 2\beta(\gamma - 1)}}{2}. \tag{3.5}$$

Let $\tilde{A} := 2\beta(\gamma - 1) > 0$ and $\tilde{B} := \frac{\beta}{2\alpha_0} + 1 > 1$. Then (3.5) becomes:

$$\frac{-1 + \sqrt{1 + \tilde{A}\tilde{B}}}{2\tilde{B}} < \frac{-1 + \sqrt{1 + \tilde{A}}}{2}$$

$$\Rightarrow \sqrt{1 + \tilde{A}\tilde{B}} < -\frac{\beta}{2\alpha_0} + \sqrt{\frac{\beta^2}{4\alpha_0^2} + \frac{\beta}{\alpha_0} + 1 + \tilde{A}\tilde{B}^2}.$$

The above inequality holds for any $\gamma > 1$, and so does $q > 1$.

Now let us consider the range of values of p. By (3.4), we have:

$$p = \frac{\left[-1 + \sqrt{1 + 2\beta(\gamma - 1) + \frac{\beta^2}{\alpha_0}(\gamma - 1)}\right]^2}{\left(2 + \frac{\beta}{\alpha_0}\right)^2 \gamma}. \tag{3.6}$$

Therefore, for $\gamma \in (1, \infty)$, we have:

$$p(1) = \lim_{\gamma \to 1^+} p(\gamma) = \frac{-1 + 1}{(2 + \frac{\beta}{\alpha_0})^2} = 0.$$

$$p(\infty) = \lim_{\gamma \to \infty} p(\gamma) = \frac{\alpha_0 \beta}{\beta + 2\alpha_0}.$$

Setting $\frac{dp}{d\gamma} = 0$, we have:

$$\gamma^* = \frac{-\alpha_0 + 2\beta\alpha_0 + \beta^2}{\alpha_0}. \tag{3.7}$$

Note that if $\beta > 1$, then $\gamma^* > 1$ always holds. Hence,

$$\begin{aligned} p(\gamma^*) &= \frac{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)^2\alpha_0}{(-\alpha_0 + 2\beta\alpha_0 + \beta^2)(2\alpha_0 + \beta)^2} < \frac{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)^2\alpha_0}{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)(2\alpha_0 + \beta)^2} \\ &= \frac{(-2\alpha_0 + 2\beta\alpha_0 + \beta^2)\alpha_0}{(2\alpha_0 + \beta)^2} = \frac{\frac{\beta^2}{\alpha_0} + 2\beta - 2}{\left(\frac{\beta}{\alpha_0} + 2\right)^2} \\ &< \frac{\frac{\beta^2}{\alpha_0} + 2\beta}{\left(\frac{\beta}{\alpha_0} + 2\right)^2} = p(\infty). \end{aligned}$$

Therefore, when $\gamma \to \infty$, $p^* := \frac{\alpha_0\beta}{\beta + 2\alpha_0}$ is the global maximum of $p(\gamma)$. □

Lemma 2. *Let $\beta > 1$ be given. For any $\gamma > 1$ there exists a solution $p(\gamma)$ to the following system of an equation and inequality:*

$$\begin{cases} \frac{\beta}{2} = \frac{\gamma}{\gamma - 1}\left(\frac{\sqrt{p\gamma}}{\gamma} + p\right), \\ \\ p > 0. \end{cases} \tag{3.8}$$

Moreover, $p^* := p(2\beta - 1) = \frac{\beta^2}{2\beta - 1}$ is the global maximum of $p(\gamma)$.

Proof. From system (3.8) we have

$$p_{1,2} = \frac{\beta(\gamma - 1) + 1 \pm \sqrt{2\beta(\gamma - 1) + 1}}{2\gamma} \tag{3.9}$$

Therefore for $\gamma \in (1, \infty)$, we have

$$p_1(1) = \lim_{\gamma \to 1^+} p_1(\gamma) = \frac{1 + 1}{2} = 1.$$

$$p_1(\infty) = \lim_{\gamma \to \infty} p_1(\gamma) = \frac{\beta}{2}.$$

$$p_2(1) = \lim_{\gamma \to 1^+} p_2(\gamma) = \frac{1 - 1}{2} = 0.$$

$$p_2(\infty) = \lim_{\gamma\to\infty} p_2(\gamma) = \frac{\beta}{2}.$$

Setting $\frac{dp_1}{d\gamma} = 0$, we have $\gamma^* = 2\beta - 1$. So if $\beta > 1$, then

$$p_1(\gamma^*) = \frac{\beta^2}{2\beta - 1}.$$

Hence we have $p_1(\gamma^*) > p_1(\infty) = \frac{\beta^2}{2\beta}$. Similarly, we obtain $p_2(\gamma^*) = \frac{(\beta-1)^2}{2\beta-1} < p_1(\gamma^*)$. □

Now we present the proofs of previous theorems.

Proof of Theorem 1. Let $p_1 > 0$, $q > 1$ and $\gamma > 1$. By the Hölder's inequality and (1.1), we obtain

$$\begin{aligned}
&\mathbb{E}[M(t\,;r(\cdot),\theta(\cdot))^{p_1}]\\
&=\mathbb{E}\left[e^{-p_1\int_0^t\left[r(s)+\frac{1}{2}\theta^2(s)\right]ds-p_1\int_0^t\theta'(s)dW(s)}\right]\\
&=\mathbb{E}\left[e^{-p_1\int_0^t r(s)ds-(\frac{\sqrt{p_1\gamma}}{\gamma}+p_1)\int_0^t\theta'(s)dW(s)}\cdot e^{\frac{1}{\gamma}\int_0^t\frac{1}{2}(\sqrt{p_1\gamma}\theta(s))^2ds-\frac{1}{\gamma}\int_0^t\sqrt{p_1\gamma}(-\theta'(s))dW(s)}\right]\\
&=\mathbb{E}\left[e^{-p_1\int_0^t r(s)ds-(\frac{\sqrt{p_1\gamma}}{\gamma}+p_1)\int_0^t\theta'(s)dW(s)}\cdot M(t\,;0,-\sqrt{p_1\gamma}\theta(\cdot))^{\frac{1}{\gamma}}\right]\\
&\le\left\{\mathbb{E}\left[e^{-\frac{p_1\gamma}{\gamma-1}\int_0^t r(s)ds-\frac{\gamma}{\gamma-1}(\frac{\sqrt{p_1\gamma}}{\gamma}+p_1)\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{\gamma}}\left\{\mathbb{E}\left[M(t\,;0,-\sqrt{p_1\gamma}\theta(\cdot))^{\frac{1}{\gamma}\cdot\gamma}\right]\right\}^{\frac{1}{\gamma}}\\
&\le\left\{\mathbb{E}\left[e^{-\frac{p_1\gamma}{\gamma-1}\frac{q}{q-1}\int_0^t r(s)ds}\right]\right\}^{\frac{(q-1)(\gamma-1)}{q\gamma}}\left\{\mathbb{E}\left[e^{-\frac{q\gamma}{\gamma-1}(\frac{\sqrt{p_1\gamma}}{\gamma}+p_1)\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{q\gamma}}\\
&=\left\{\mathbb{E}\left[e^{-\alpha_0\int_0^t r(s)ds}\right]\right\}^{\frac{(q-1)(\gamma-1)}{q\gamma}}\cdot\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{q\gamma}},
\end{aligned}$$

where $\alpha_0 = \frac{p_1 q\gamma}{(\gamma-1)(q-1)}$ and $\beta = \frac{2q\gamma}{\gamma-1}(\frac{\sqrt{p_1\gamma}}{\gamma} + p_1)$. Therefore by Lemma 1, when $\gamma \to \infty$, $p_1 = \frac{\alpha_0\beta}{\beta+2\alpha_0}$ is the global maximum of $p_1(\gamma)$. Furthermore we have $\frac{(q-1)(\gamma-1)}{q\gamma} = \frac{\beta}{\beta+2\alpha_0}$ and $\frac{\gamma-1}{q\gamma} = \frac{2\alpha_0}{\beta+2\alpha_0}$. □

Proof of Theorem 2. From Theorem 1 and (2.4), we obtain

$$\begin{aligned}
\mathbb{E}[M(t\,;r(\cdot),\theta(\cdot))^{p_2}] &\le \left\{\mathbb{E}\left[e^{-\frac{p_2\gamma}{\gamma-1}\int_0^t r(s)ds-\frac{\gamma}{\gamma-1}(\frac{\sqrt{p_2\gamma}}{\gamma}+p_2)\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{\gamma}}\\
&\le e^{\kappa p_2}\left\{\mathbb{E}\left[e^{-\frac{\gamma}{\gamma-1}(\frac{\sqrt{p_2\gamma}}{\gamma}+p_2)\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{\gamma-1}{\gamma}}.
\end{aligned}$$

By Lemma 2, we know when $\gamma = 2\beta - 1$, $p_2 = \frac{\beta^2}{2\beta-1}$ is the global maximum of $p(\gamma)$ and thus (2.5) follows. □

Proof of Theorem 3. Note that $M(t\,;0,\theta(\cdot))$ is a martingale when (2.2) holds for $\beta > 1$. By the Hölder's inequality and the Doob's martingale inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_3}\right] \\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^{p_3}\cdot e^{-p_3\int_0^t r(s)ds}\right] \\
\leq&\left\{\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^{\frac{p_3\alpha_0}{\alpha_0-p_3}}\right]\right\}^{\frac{\alpha_0-p_3}{\alpha_0}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha_0\int_0^t r(s)ds}\right]\right\}^{\frac{p_3}{\alpha_0}} \\
\leq&\left(\frac{\frac{p_3\alpha_0}{\alpha_0-p_3}}{\frac{p_3\alpha_0}{\alpha_0-p_3}-1}\right)^{\frac{p_3\alpha_0}{\alpha_0-p_3}\cdot\frac{\alpha_0-p_3}{\alpha_0}}\left\{\mathbb{E}\left[M(T\,;0,\theta(\cdot))^{\frac{p_3\alpha_0}{\alpha_0-p_3}}\right]\right\}^{\frac{\alpha_0-p_3}{\alpha_0}} \\
&\cdot\left\{\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha_0\int_0^t r(s)ds}\right]\right\}^{\frac{p_3}{\alpha_0}}
\end{aligned}
$$

From (2.5), when $r(\cdot) = 0$, we obtain

$$
\sup_{t\in[0,T]}\mathbb{E}\left[M(t\,;0,\theta(\cdot))^{\frac{\beta^2}{2\beta-1}}\right] \leq \sup_{t\in[0,T]}\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{2\beta-2}{2\beta-1}}. \tag{3.10}
$$

Then

$$
\frac{p_3\alpha_0}{\alpha_0-p_3} = \frac{\beta^2}{2\beta-1} \implies p_3 = \frac{\alpha_0\beta^2}{\beta^2+2\alpha_0\beta-\alpha_0} \quad \textit{and} \quad \frac{p_3}{\alpha_0} = \frac{\beta^2}{\beta^2+2\alpha_0\beta-\alpha_0}.
$$

and

$$
\frac{2\beta-2}{2\beta-1}\cdot\frac{\alpha_0-p_3}{\alpha_0} = \frac{2\beta-2}{2\beta-1}\cdot\frac{(2\beta-1)p_3}{\beta^2} = \frac{2\alpha_0(\beta-1)}{\beta^2+2\alpha_0\beta-\alpha_0}.
$$

□

Proof of Theorem 4. Let us denote $\beta' = \frac{\beta}{p_4\gamma} > 1$. By (3.10), we have

$$
\frac{\alpha_0}{\gamma(\alpha_0-p_4)} = \frac{\beta'^2}{2\beta'-1} \implies \gamma = \frac{\beta[(2\alpha_0+\beta)p_4-\alpha_0\beta]}{\alpha_0 p_4^2} > 0.
$$

So it is necessary to have $(2\alpha_0+\beta)p_4-\alpha_0\beta>0$, i.e. $p_4>\frac{\alpha_0\beta}{2\alpha_0+\beta}$. On the other hand, again by (3.10), we have

$$\frac{2\beta'-2}{2\beta'-1}\cdot\frac{\alpha_0-p_4}{\alpha_0}=\frac{2\alpha_0\beta-2p_4(\alpha_0+\beta)}{\alpha_0+\beta}>0,\quad\Longrightarrow\quad p_4<\frac{\alpha_0\beta}{\alpha_0+\beta}.$$

Hence we have $p_4\in(\frac{\alpha_0\beta}{2\alpha_0+\beta},\frac{\alpha_0\beta}{\alpha_0+\beta})$ and

$$0<p_4\gamma=\frac{\beta[(2\alpha_0+\beta)p_4-\alpha_0\beta]}{\alpha_0p_4}<\beta\le 1. \tag{3.11}$$

Note that

$$\sup_{t\in[0,T]}\mathbb{E}\left[e^{-\frac{\beta'}{2}\int_0^t p_4\gamma\theta'(s)dW(s)}\right]=\sup_{t\in[0,T]}\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]<\infty.$$

By Kazamaki (1978), we know $M(t\,;0,p_4\gamma\theta(\cdot))$ is a martingale if $\beta'>1$. By (3.10) and (3.11), the Hölder's inequality and the Doob's martingale inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}M(t\,;r(\cdot),\theta(\cdot))^{p_4}\right]\\
=&\mathbb{E}\left[\sup_{t\in[0,T]}\left(e^{-\int_0^t\frac{1}{2}p_4^2\gamma^2|\theta(s)|^2ds}\right)^{\frac{1}{p_4\gamma}\frac{1}{\gamma}}\cdot e^{-p_4\int_0^t r(s)ds}\cdot\left(e^{-\int_0^t p_4\gamma\theta'(s)dW(s)}\right)^{\frac{1}{\gamma}}\right]\\
\le&\mathbb{E}\left[\sup_{t\in[0,T]}\left(e^{-\int_0^t\frac{1}{2}p_4^2\gamma^2|\theta(s)|^2ds}\right)^{\frac{1}{\gamma}}\cdot e^{-p_4\int_0^t r(s)ds}\cdot\left(e^{-\int_0^t p_4\gamma\theta'(s)dW(s)}\right)^{\frac{1}{\gamma}}\right]\\
=&\mathbb{E}\left[\sup_{t\in[0,T]}M(t\,;0,p_4\gamma\theta(\cdot))^{\frac{1}{\gamma}}\cdot e^{-p_4\int_0^t r(s)ds}\right]\\
\le&\left\{\mathbb{E}\left[\sup_{t\in[0,T]}M(t\,;0,p_4\gamma\theta(\cdot))^{\frac{\alpha_0}{\gamma(\alpha_0-p_4)}}\right]\right\}^{\frac{\alpha_0-p_4}{\alpha_0}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}e^{-\alpha_0\int_0^t r(s)ds}\right]\right\}^{\frac{p_4}{\alpha_0}}\\
\le&\left[\frac{\frac{\alpha_0}{\gamma(\alpha_0-p_4)}}{\frac{\alpha_0}{\gamma(\alpha_0-p_4)}-1}\right]^{\frac{\alpha_0}{\gamma(\alpha_0-p_4)}\cdot\frac{\alpha_0-p_4}{\alpha_0}}\left\{\mathbb{E}\left[M(T\,;0,p_4\gamma\theta(\cdot))^{\frac{\alpha_0}{\gamma(\alpha_0-p_4)}}\right]\right\}^{\frac{\alpha_0-p_4}{\alpha_0}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}e^{-\alpha_0\int_0^t r(s)ds}\right]\right\}^{\frac{p_4}{\alpha_0}}
\end{aligned}$$

$$\leq c_2 \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha_0 \int_0^t r(s)ds}\right]\right)^{\frac{p_4}{\alpha_0}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta'}{2}\int_0^t p_4\gamma\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha_0\beta-2p_4(\alpha_0+\beta)}{\alpha_0\beta}}$$

$$= c_2 \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\alpha_0 \int_0^t r(s)ds}\right]\right)^{\frac{p_4}{\alpha_0}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha_0\beta-2p_4(\alpha_0+\beta)}{\alpha_0\beta}}.$$

□

Proof of Theorem 5. (a) Note that $M(t\,;0,\theta(\cdot))$ is a martingale if $\beta > 1$. By (2.5) and (3.10) and the Doob's martingale inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_5}\right] \\
&= \mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^{p_5}\cdot e^{-p_5\int_0^t r(s)ds}\right] \\
&\leq e^{\kappa p_5}\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,\theta(\cdot))^{p_5}\right] \\
&\leq e^{\kappa p_5}\left(\frac{p_5}{p_5-1}\right)^{p_5}\mathbb{E}[M(T\,;0,\theta(\cdot))^{p_5}] \\
&\leq e^{\frac{\kappa\beta^2}{2\beta-1}}\left[\frac{\beta^2}{(\beta-1)^2}\right]^{\frac{\beta^2}{2\beta-1}}\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{2\beta-2}{2\beta-1}}.
\end{aligned}$$

(b) Let us denote $\beta' = \frac{\beta}{p\gamma} > 1$. By (3.10), we have

$$\frac{1}{\gamma} = \frac{\beta'^2}{2\beta'-1} \implies \gamma = \frac{\beta(2p_6-\beta)}{p_6^2} > 0.$$

So it is necessary to have $2p_6-\beta > 0$, i.e. $p_6 > \frac{\beta}{2}$. On the other hand, again by (3.10), we have

$$\frac{2\beta'-2}{2\beta'-1} = \frac{2(\beta-p_6)}{\beta} > 0, \quad \implies \quad p_6 < \beta.$$

Hence we have $p_6 \in \left(\frac{\beta}{2},\beta\right)$ and

$$0 < \gamma = \frac{\beta(2p_6-\beta)}{p_6^2} < 1.$$

So

$$0 < p_6\gamma = \frac{\beta(2p_6 - \beta)}{p_6} < \beta \leq 1.$$

Again $M(t\,;0, p_6\gamma\theta(\cdot))$ is a martingale if $\beta' > 1$. By the Doob's martingale inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{p_6}\right] \\
=&\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,p_6\gamma\theta(\cdot))^{p_6}\cdot e^{-p_6\int_0^t r(s)ds}\right] \\
\leq&\, e^{\kappa p_6}\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;0,p\gamma\theta(\cdot))^{p_6}\right] \\
\leq&\, e^{\kappa p_6}\left(\frac{\frac{1}{\gamma}}{\frac{1}{\gamma}-1}\right)^{\frac{1}{\gamma}}\mathbb{E}\left[M(T\,;0,p_6\gamma\theta(\cdot))^{\frac{1}{\gamma}}\right] \\
\leq&\, e^{\kappa p_6}\left[\frac{p_6^2}{p_6^2-\beta(2p_6-\beta)}\right]^{\frac{p_6^2}{\beta(2p_6-\beta)}}\left\{\mathbb{E}\left[e^{-\frac{\beta}{2}\int_0^t\theta'(s)dW(s)}\right]\right\}^{\frac{2(\beta-p_6)}{\beta}}.
\end{aligned}$$

□

Proof of Theorem 6. By Theorem 3, when $\beta_0 > 1$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;r(\cdot),\theta(\cdot))^{-p_7}\right] \\
=&\,\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,;-r(\cdot)-|\theta(\cdot)|^2,-\theta(\cdot))^{p_7}\right] \\
\leq&\, c_5\left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t[r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t\theta'(s)dW(s)}\right]\right)^{\frac{2\alpha(\beta_0-1)}{\beta_0^2+2\alpha\beta_0-\alpha}},
\end{aligned}$$

where $c_5 = \left(\frac{\alpha\beta_0^2}{\alpha\beta_0^2-2\beta_0^2-2\alpha\beta_0+\alpha}\right)^{\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}$. □

Proof of Theorem 7. By Theorem 4, for any $p_8 \in \left(\frac{\alpha\beta_0}{2\alpha+\beta_0}, \frac{\alpha\beta_0}{\alpha+\beta_0}\right)$ when $\beta_0 \in (0,1]$ we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,; r(\cdot), \theta(\cdot))^{-p_8}\right] \\
=&\mathbb{E}\left[\sup_{t\in[0,T]} M(t\,; -r(\cdot) - |\theta(\cdot)|^2, -\theta(\cdot))^{p_8}\right] \\
\leq& c_6 \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha\int_0^t [r(s)+|\theta(s)|^2]ds}\right]\right)^{\frac{p_8}{\alpha}} \left(\sup_{t\in[0,T]} \mathbb{E}\left[e^{\frac{\beta_0}{2}\int_0^t \theta'(s)dW(s)}\right]\right)^{\frac{2\alpha\beta_0-2p_8(\alpha+\beta_0)}{\alpha\beta_0}},
\end{aligned}
$$

where $c_6 = \left(\frac{\alpha^2 p_8^2}{\alpha^2 p_8^2 - \beta_0(\alpha-p_8)[(\beta_0+2\alpha)p_8-\alpha\beta_0]}\right)^{\frac{\alpha p_8^2}{\beta_0[(2\alpha+\beta_0)p_8-\alpha\beta_0]}}$. ☐

4. Applications

We give the application of *some* of our results to the solvability of linear and Riccati BSDEs with unbounded coefficients, which are then used to solve the problems of market completeness and optimal investment. The applications to linear BSDE and market completeness are similar to those of Yong (2006) who considers these problems under the Novikov condition, whereas the applications to Riccati BSDE and optimal investment are new.

4.1. Linear BSDE and market completeness

In this section, our aim is to use our integrability results and the approach of Yong (2006) to show the solvability of (1.9), which we repeat here for convenience:

$$
\begin{cases}
dY(t) = [r(t)Y(t) + \theta'(t)Z(t)]dt + Z'(t)dW(t), \quad t \in [0,T], \\
Y(T) = \xi, \quad a.s.,
\end{cases}
\tag{4.1}
$$

As already mentioned in §1, Yong's approach is to reduce (4.1) into a more basic form as follows. The differential of $M(\cdot)Y(\cdot)$ is:

$$
d[M(t)Y(t)] = M(t)[Z(t) - Y(t)\theta(t)]'dW(t), \quad t \in [0,T].
$$

If we define

$$\begin{cases} \widetilde{Y}(t) := & M(t)Y(t), \quad t \in [0,T], \\ \\ \widetilde{Z}(t) := & M(t)[Z(t) - Y(t)\theta(t)], \end{cases} \tag{4.2}$$

then

$$\begin{cases} d\widetilde{Y}(t) = & \widetilde{Z}'(t)dW(t), \quad t \in [0,T], \\ \\ \widetilde{Y}(T) = & \tilde{\xi} := M(T)\xi. \end{cases} \tag{4.3}$$

Note that (4.3) admits a unique adapted solution $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot))$ if $\tilde{\xi}$ has sufficient integrability. In this case, the solution pair of (4.1) is:

$$\begin{cases} Y(t) = & M(t)^{-1}\widetilde{Y}(t), \quad t \in [0,T], \\ \\ Z(t) = & M(t)^{-1}[\widetilde{Z}(t) + \widetilde{Y}(t)\theta(t)]. \end{cases} \tag{4.4}$$

Theorem 8. *Suppose that* (2.1), (2.2), (2.9) *and* (2.10) *hold with positive constants* α_0, α, β *and* β_0 *satisfying:*

$$\begin{cases} \alpha_0 > \frac{\alpha\beta\beta_0^2}{\alpha\beta\beta_0^2 + \alpha\beta - 2\alpha\beta\beta_0 - 2\alpha\beta_0^2 - \beta\beta_0^2}, \\ \\ \alpha > \frac{\beta_0^2}{(\beta_0 - 1)^2}, \\ \\ \beta_0 > 1, \beta > 2. \end{cases} \tag{4.5}$$

and $\theta(\cdot) \in \mathcal{L}_{\mathcal{F}}^{\frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\beta\beta_0^2 + 2\alpha\alpha_0\beta\beta_0 - \alpha_0\alpha\beta}}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)$. *Then for any* $\xi \in \mathcal{L}_{\mathcal{F}_T}^{p+}(\Omega; \mathbb{R})$ *with*

$$p = \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0 - 2\alpha_0\alpha\beta_0^2 - \alpha\beta\beta_0^2} > 1, \tag{4.6}$$

the BSDE (4.1) *admits a unique solution pair* $(Y(\cdot), Z(\cdot))$ *such that:*

$$Y(\cdot) \in \mathcal{L}_{\mathcal{F}}^{1+}(\Omega; C([0,T]; \mathbb{R})), \quad Z(\cdot) \in \mathcal{L}_{\mathcal{F}}^{1+}(\Omega; \mathcal{L}^2([0,T]; \mathbb{R}^d)).$$

Proof. By Theorem 1, we know that if

$$\alpha_0 > \frac{\beta}{\beta - 2}, \tag{4.7}$$

and $\beta > 2$, then

$$M(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_1}(\Omega; C([0,T];\mathbb{R})), \tag{4.8}$$

where

$$p_1 = \frac{\alpha_0 \beta}{2\alpha_0 + \beta} > 1. \tag{4.9}$$

By Hölder inequality, for any $\xi \in \mathcal{L}_{\mathcal{F}_T}^{p_2}(\Omega)$, we have

$$\tilde{\xi} = M(T)\xi \in \mathcal{L}_{\mathcal{F}_T}^{p_3}(\Omega; \mathbb{R}),$$

where

$$p_3 = \frac{p_1 p_2}{p_1 + p_2} = \frac{\alpha_0 \beta p_2}{\alpha_0 \beta + (2\alpha_0 + \beta) p_2}. \tag{4.10}$$

Suppose $p_3 > 1$, $\beta > 2$ and (4.7), then we have

$$p_2 > \frac{\alpha_0 \beta}{\alpha_0 \beta - 2\alpha_0 - \beta} > 1. \tag{4.11}$$

By Theorem 6, if

$$\alpha > \frac{\beta_0^2}{(\beta_0 - 1)^2}, \tag{4.12}$$

and $\beta_0 > 1$, we have

$$M(\cdot)^{-1} \in \mathcal{L}_{\mathcal{F}}^{p_4}(\Omega; C([0,T];\mathbb{R})), \tag{4.13}$$

where

$$p_4 = \frac{\alpha \beta_0^2}{\beta_0^2 + 2\alpha\beta_0 - \alpha} > 1.$$

By BSDE (4.3), Theorem 1, Theorem 6, the Hölder's inequality and the Doob's martingale inequality with $p_1, p_2, p_3, p_4, q_1, q_2 > 1$, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|Y(t)|^{p_5}\right] &= \mathbb{E}\left[\sup_{t\in[0,T]}\left|M(t)^{-1}\widetilde{Y}(t)\right|^{p_5}\right]\\
&\le \left\{\mathbb{E}\left[\sup_{t\in[0,T]}|M(t)|^{-q_1\cdot p_5}\right]\right\}^{\frac{1}{q_1}}\cdot\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{Y}(t)\right|^{\frac{q_1}{q_1-1}p_5}\right]\right\}^{\frac{q_1-1}{q_1}}\\
&\le \left\{\mathbb{E}\left[\sup_{t\in[0,T]}|M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}}\cdot\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{Y}(t)\right|^{p_3}\right]\right\}^{\frac{p_5}{p_3}}\\
&\le C_{p_3,p_5}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}|M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}}\cdot\{\mathbb{E}\,|M(T)|^{q_2\cdot p_3}\}^{\frac{1}{q_2}\frac{p_5}{p_3}}\cdot\left\{\mathbb{E}\,|\xi|^{\frac{q_2}{q_2-1}p_3}\right\}^{\frac{q_2-1}{q_2}\frac{p_5}{p_3}}\\
&\le C_{p_3,p_5}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}|M(t)|^{-p_4}\right]\right\}^{\frac{p_5}{p_4}}\cdot\{\mathbb{E}\,|M(T)|^{p_1}\}^{\frac{p_5}{p_1}}\cdot\{\mathbb{E}\,|\xi|^{p_2}\}^{\frac{p_5}{p_2}}<\infty,
\end{aligned}
$$

where

$$
p_5=\frac{p_3p_4}{p_3+p_4}=\frac{\frac{\alpha_0\beta p_2}{\alpha_0\beta+(2\alpha_0+\beta)p_2}\cdot\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}{\frac{\alpha_0\beta p_2}{\alpha_0\beta+(2\alpha_0+\beta)p_2}+\frac{\alpha\beta_0^2}{\beta_0^2+2\alpha\beta_0-\alpha}}
$$

$$
=\frac{\alpha_0\alpha\beta\beta_0^2p_2}{\alpha_0\beta p_2(\beta_0^2+2\alpha\beta_0-\alpha)+\alpha\beta_0^2[\alpha_0\beta+(2\alpha_0+\beta)p_2]}.
$$

So in order to ensure $p_5>1$, it is necessary to have

$$
(\alpha_0\alpha\beta\beta_0^2+\alpha_0\alpha\beta-\alpha_0\beta\beta_0^2-2\alpha_0\alpha\beta\beta_0-2\alpha_0\alpha\beta_0^2-\alpha\beta\beta_0^2)p_2>\alpha_0\alpha\beta\beta_0^2.
$$

Since $p_2>1$, in other words it is necessary to have

$$
\alpha_0\alpha\beta\beta_0^2+\alpha_0\alpha\beta-\alpha_0\beta\beta_0^2-2\alpha_0\alpha\beta\beta_0-2\alpha_0\alpha\beta_0^2-\alpha\beta\beta_0^2>0,
$$

i.e. if $\alpha>\frac{\beta_0^2}{(\beta_0-1)^2}$ holds, then

$$
\alpha_0>\frac{\alpha\beta\beta_0^2}{(\alpha\beta\beta_0^2+\alpha\beta-2\alpha\beta\beta_0-2\alpha\beta_0^2-\beta\beta_0^2)},\tag{4.14}
$$

and thus

$$
p_2>\frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2+\alpha_0\alpha\beta-\alpha_0\beta\beta_0^2-2\alpha_0\alpha\beta\beta_0-2\alpha_0\alpha\beta_0^2-\alpha\beta\beta_0^2}.\tag{4.15}
$$

It is easy to see that if $\beta_0 > 1$, then (4.14) is bigger than (4.7) and (4.15) is also bigger than (4.11). Hence when (4.12) and (4.14) are satisfied, we always have (4.15). Substituting (4.15) into (4.10), if $\beta_0 > 1$ and (4.12) hold, then

$$p_3 = \frac{\alpha_0\beta p_2}{\alpha_0\beta + (2\alpha_0 + \beta)p_2} > \frac{\alpha_0\alpha\beta\beta_0^2}{\alpha_0\alpha\beta\beta_0^2 + \alpha_0\alpha\beta - \alpha_0\beta\beta_0^2 - 2\alpha_0\alpha\beta\beta_0} > 1.$$

By Theorem 5.1 in Karoui *et al.*(1997), we know that BSDE (4.3) admits a unique solution

$$\widetilde{Y}(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_3}(\Omega; C([0,T];\mathbb{R})), \quad \widetilde{Z}(\cdot) \in \mathcal{L}_{\mathcal{F}}^{p_3}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d)),$$

and the following estimate holds

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{Y}(t)|^{p_3} + \left[\int_0^T |\widetilde{Z}(t)|^2 dt\right]^{\frac{p_3}{2}}\right] = C_{p_3}\,\mathbb{E}\left|\widetilde{\xi}\right|^{p_3} = C_{p_3}\,\mathbb{E}\,|M(T)\xi|^{p_3} \\
&\leq C_{p_3}\left\{\mathbb{E}\,|M(T)|^{q_3\cdot p_3}\right\}^{\frac{1}{q_3}} \cdot \left\{\mathbb{E}\,|\xi|^{\frac{q_3}{q_3-1}p_3}\right\}^{\frac{q_3-1}{q_3}} \leq C_{p_3}\left\{\mathbb{E}\,|M(T)|^{p_1}\right\}^{\frac{p_3}{p_1}} \cdot \left\{\mathbb{E}\,|\xi|^{p_2}\right\}^{\frac{p_3}{p_2}} < \infty,
\end{aligned}$$

where $p_1, p_2, p_3, q_3 > 1$ and C_{p_3} is some constant. Note that

$$p_5 = \frac{p_3 p_4}{p_3 + p_4} > \frac{\alpha_0^2\alpha^2\beta_2\beta_0^4}{\alpha_0^2\alpha\beta^2\beta_0^2(\beta_0^2 + 2\alpha\beta_0 - \alpha) + \alpha_0^2\alpha\beta^2\beta_0^2(\alpha\beta_0^2 + \alpha - \beta_0^2 - 2\alpha\beta_0)} = 1.$$

Finally, taking a constant $\varepsilon \in (0, p_5 - 1)$, using the Hölder's inequality and Minkowski's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |Z(t)|^2 dt\right]^{\frac{p_5-\varepsilon}{2}} \\
=\;& \mathbb{E}\left[\int_0^T \left|M(t)^{-1}\left[\widetilde{Z}(t) + \widetilde{Y}(t)\theta(t)\right]\right|^2 dt\right]^{\frac{p_5-\varepsilon}{2}} \\
\leq\;& \mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-1}\right|^2 \int_0^T \left[\widetilde{Z}(t) + \widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_5-\varepsilon}{2}} \\
=\;& \mathbb{E}\left[\sup_{t\in 0,T} \left|M(t)^{-1}\right|^{p_5-\varepsilon}\left(\int_0^T \left[\widetilde{Z}(t) + \widetilde{Y}(t)\theta(t)\right]^2 dt\right)^{\frac{p_5-\varepsilon}{2}}\right]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-1}\right|^{(p_5-\varepsilon)\frac{p_4}{p_5-\varepsilon}}\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \\
\cdot\ & \left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_5-\varepsilon}{2}\frac{p_4}{p_4-p_5+\varepsilon}}\right)^{\frac{p_4-p_5+\varepsilon}{p_4}} \\
\leq\ & \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \\
\cdot\ & \left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{p_4(p_5-\varepsilon)}{2(p_4-p_5+\varepsilon)}\frac{p_4-p_5+\varepsilon}{p_4(p_5-\varepsilon)}}\right)^{\frac{p_4-p_5+\varepsilon}{p_4}\frac{p_4(p_5-\varepsilon)}{p_4-p_5+\varepsilon}} \\
=\ & \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{1}{2}}\right)^{p_5-\varepsilon} \\
\leq\ & \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}} \\
\cdot\ & \left\{\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{1}{2}}\right)+\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Y}(t)\theta(t)\right]^2 dt\right]^{\frac{1}{2}}\right)\right\}^{p_5-\varepsilon} \\
\leq\ & \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\left\{\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{p_3}{2}}\right)^{\frac{1}{p_3}}\right. \\
+\ & \left.\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{Y}(t)\right|\left(\int_0^T[\theta(t)]^2\,dt\right)^{\frac{1}{2}}\right]\right)\right\}^{p_5-\varepsilon} \\
\leq\ & \left(\mathbb{E}\left[\sup_{t\in 0,T}\left|M(t)^{-p_4}\right|\right]\right)^{\frac{p_5-\varepsilon}{p_4}}\left\{\left(\mathbb{E}\left[\int_0^T\left[\widetilde{Z}(t)\right]^2 dt\right]^{\frac{p_3}{2}}\right)^{\frac{1}{p_3}}\right. \\
+\ & \left.\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{Y}(t)\right|^{p_3}\right]\right)^{\frac{1}{p_3}}\left(\mathbb{E}\left[\int_0^T[\theta(t)]^2\,dt\right]^{\frac{p_3}{p_3-1}}\right)^{\frac{p_3-1}{p_3}}\right\}^{p_5-\varepsilon} \\
<\ & \infty
\end{aligned}
$$

$\square$

Let us now compare the above result with Theorem 4.1 of Yong (2006). It can be noted that all our assumptions are implied by those of Yong, i.e. we have solvability under weaker assumptions on the coefficients. The price we pay for this is that the space of terminal values is more *restricted* than in Yong (2006). Indeed, if in (4.6) we place α instead of α_0, $\alpha_0\beta/(2\alpha_0+\beta)$ instead of α, and $\sqrt{\beta}$ instead of β and β_0 (in order for our assumptions to be implied by those of Yong (2006)), then

$$p = \frac{\alpha_0\alpha\sqrt{\beta}}{\sqrt{\beta}(\alpha\alpha_0 - \alpha - \alpha_0) - 4\alpha\alpha_0 - \alpha\alpha_0/\sqrt{\beta}}$$

Thus, it can be seen that our p is *bigger* than that in Theorem 4.1 of Yong (2006), i.e. the space of our terminal values is *smaller*, which is to be expected since we are working under weaker assumptions. An important example of a BSDE considered in the next subsection (see equation (4.21)), requires $\xi = 1$ which does satisfy our requirements, and thus our result can be applied.

An immediate application of Theorem 8 is to market completeness, and as a consequence we obtain the completeness of the market under weaker assumptions on the coefficients as compared to Yong (2006). Thus, consider a market with one bond and n stocks, the prices of which are, respectively,

$$\begin{cases} dP_0(t) = & r(t)P_0(t)dt, \\ dP_i(t) = & P_i(t)[b_i(t)dt + \sigma_i'(t)dW(t)], \quad i = 1, ..., n, \\ P_i(0) > & 0, \quad i = 0, 1, ..., n. \end{cases} \tag{4.16}$$

The process $r(\cdot)$ is the *interest rate*, the processes $b_i(\cdot)$, $i = 1, ..., n$, are the *appreciation rates*, and the processes $\sigma_i'(t) = [\sigma_{i1}(t), ..., \sigma_{1d}(t)]$, $i = 1, ..., n$, are the *volatilities* of the stocks. If $\pi_i(t)$ denotes the value of the holdings in asset i at time t, then it can be shown (see, for example, Bingham & Kiesel (2004), Yong & Zhou (1999)) that the value $Y(t)$ of a *self-financing portfolio* is

$$\begin{cases} dY(t) = & [r(t)Y(t) + \pi'(t)(b(t) - r(t)\mathbb{1})]dt + \pi'(t)\sigma(t)dW(t), \quad t \in [0, T], \\ Y(0) = & Y_0 > 0. \end{cases} \tag{4.17}$$

Here $\pi(t) := [\pi_1(t), ..., \pi_n]'$, $b(t) := [b_1(t), ..., b_n(t)]'$, $\sigma(t) := [\sigma_1(t), ..., \sigma_n(t)]'$, and Y_0 is the investor's initial wealth. We assume that *rank* $\sigma(t) = d$, *a.e.*

$t \in [0,T]$ *a.s.*, which ensures that $n \geq d$ and $[\sigma'(t)\sigma(t)]^{-1}$ exists. If we define the processes $\theta(\cdot)$ and $Z(\cdot)$ as

$$\begin{aligned} \theta(t) &:= [\sigma'(t)\sigma(t)]^{-1}\sigma'(t)[b(t) - r(t)\mathbb{1}], \\ Z(t) &:= \sigma'(t)\pi(t), \end{aligned} \tag{4.18}$$

we can rewrite (4.17) as

$$\begin{cases} dY(t) = & [r(t)Y(t) + \theta'(t)Z(t)]dt + Z'(t)dW(t), \quad t \in [0,T], \\ Y(0) = & Y_0 > 0, \end{cases} \tag{4.19}$$

which is the *forward* version of (4.1). The following definition of market completeness is adapted from Yong (2006), whereas Corollary 1 follows from Theorem 8. Let

$$\begin{aligned} \Pi^p[0,T] \;:=\; & \big\{\pi(\cdot) \in \mathcal{L}^0_{\mathcal{F}}(0,T;\mathbb{R}^n) : \pi'(\cdot)(b(\cdot) - r(\cdot)\mathbb{1}) \in \mathcal{L}^p_{\mathcal{F}}(\Omega;\mathcal{L}^1(0,T;\mathbb{R})) \\ & \textit{and} \quad \sigma'(\cdot)\pi(\cdot) \in \mathcal{L}^p_{\mathcal{F}}(\Omega;\mathcal{L}^1(0,T;\mathbb{R}^d)\big\}. \end{aligned}$$

Definition 1. *Let $\mathcal{H} \subset L^0_{\mathcal{F}_T}(\Omega,\mathbb{R})$ and $\Pi \subset \Pi^0[0,T]$. The market (4.16) is $(\mathcal{H},\Pi)$-complete if for any $\xi \in \mathcal{H}$ there exists a solution pair $(Y(\cdot), Z(\cdot))$ to (4.1), with the process $Z(\cdot)$ being such that $\pi(\cdot)$ as given by (4.18) belongs to the set Π.*

Corollary 1. *If the coefficients $r(\cdot)$ and $\theta(\cdot)$ satisfy the conditions of Theorem 8, then the market (4.16) is $(L^{p^+}_{\mathcal{F}_T}(\Omega,\mathbb{R}), \Pi^{1^+})$-complete, with p given by (4.6).*

The above corollary thus gives sufficient conditions for market completeness when the coefficients are unbounded and satisfy weaker integrability conditions as compared to Yong (2006). Note that by establishing the solvability of the BSDE (4.1) we essentially also solve the pricing and hedging problem, since $Y(0)$ represents the price of the contingent claim ξ at time zero, whereas $Z(\cdot)$ represents the hedging strategy.

4.2. Riccati BSDE and asset management

We now consider the optimal investment problem with *power utility* for the market considered in the previous subsection. An explicit solution is found by first showing the solvability of a certain *Riccati* BSDE with unbounded coefficients, and then combining ideas from Lim & Zhou (2002) and Ferland & Waiter (2008). Thus, consider an investor with an initial capital Y_0, and the power utility

$$J(Z(\cdot)) := -\mathbb{E}[Y^{\lambda}(T)], \quad \lambda \in (0,1).$$

The optimal investment problem is the following *optimal control* problem:

$$\begin{cases} \min\limits_{Z(\cdot)\in\mathcal{A}} J(Z(\cdot)), \\ \\ s.t. \quad (4.19), \end{cases} \tag{4.20}$$

where $\mathcal{A}$ is a suitable admissible set of *controls* to be defined precisely after the following two results.

Lemma 3. *Let* $\eta := 2\lambda(1-\lambda)^{-1}+1$ *and the processes* $\widehat{r}(\cdot)$ *and* $\widehat{\theta}(\cdot)$ *be defined as*

$$\begin{aligned} \widehat{r}(t) &:= -\lambda\eta r(t) - \frac{\lambda\eta\theta'(t)\theta(t)}{1-\lambda}, \\ \widehat{\theta}(t) &:= -\frac{2\lambda}{1-\lambda}\theta(t). \end{aligned}$$

Let the processes $\widehat{r}(\cdot)$ *and* $\widehat{\theta}(\cdot)$ *satisfy the conditions of Theorem 8. Then the equation*

$$\begin{cases} dR(t) = R_1(t)dt + R_2'(t)dW(t), \quad t\in[0,T], \\ \\ R_1(t) := \widehat{r}(t)R(t) + \widehat{\theta}'(t)R_2(t), \\ \\ R(T) = 1, \end{cases} \tag{4.21}$$

has a unique solution pair $R(\cdot)\in\mathcal{L}^{1+}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$, $R_2(\cdot)\in\mathcal{L}^{1+}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d))$, *and* $R(t)>0$, $\forall t\in[0,T]$ *a.s.. If* $\widehat{r}(t)<0$ *a.e.* $t\in[0,t]$ *a.s., then* $R(t)\geq 1$ $\forall t\in[0,T]$ *a.s..*

Proof. Theorem 8 ensures the existence of a unique solution pair $R(\cdot)\in\mathcal{L}^{1+}_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$, $R_2(\cdot)\in\mathcal{L}^{1+}_{\mathcal{F}}(\Omega;\mathcal{L}^2([0,T];\mathbb{R}^d))$. Let $\widehat{M}(\cdot)$ denote the solution to the equation

$$\begin{cases} d\widehat{M}(t) = & -\widehat{r}(t)\widehat{M}(t)dt - \widehat{\theta}'(t)\widehat{M}(t)dW(t), \quad t \in [0,T], \\ \\ \widehat{M}(0) = & 1, \end{cases} \tag{4.22}$$

and $(\widehat{R}(\cdot), \widehat{R}_2(\cdot))$ be the unique solution pair of the equation

$$\begin{cases} d\widehat{R}(t) = & \widehat{R}_2'(t)dW(t), \quad t \in [0,T], \\ \\ \widehat{R}(T) = & \widehat{M}(T). \end{cases}$$

Due to $\widehat{R}(t) = \mathbb{E}\left[\widehat{M}(T)|\mathscr{F}_t\right]$ and (4.4), we have

$$\begin{aligned} R(t) &= \widehat{M}(t)^{-1}\widehat{R}(t) = \widehat{M}(t)^{-1}\mathbb{E}\left[\widehat{M}(T)|\mathscr{F}_t\right] \\ &= \mathbb{E}\left[e^{-\int_t^T \left[\widehat{r}(s)+\frac{1}{2}\widehat{\theta}'(s)\widehat{\theta}(s)\right]ds - \int_t^T \widehat{\theta}'(s)dW(s)} \middle| \mathscr{F}_t \right]. \end{aligned} \tag{4.23}$$

Since the process $\widehat{\theta}(\cdot)$ is assumed to satisfy (2.2), the following is a probability measure:

$$\widehat{\mathbb{P}}(A) := \int_A N(T)d\mathbb{P}(\omega), \quad \forall A \in \mathscr{F},$$

where

$$N(t) := e^{\frac{-1}{2}\int_0^t \widehat{\theta}'(s)\widehat{\theta}(s)ds - \int_0^t \widehat{\theta}(s)'dW(s)}.$$

We can now write (4.23) as

$$R(t) = \widehat{\mathbb{E}}\left[e^{-\int_t^T [\widehat{r}(s)]ds} \middle| \mathscr{F}_t \right] > 0, \tag{4.24}$$

where $\widehat{\mathbb{E}}[\,\cdot\,]$ is the expectation under the new probability measure $\widehat{\mathbb{P}}$. It is clear that $R(t) > 0$, $\forall t \in [0,T]$ *a.s.*, and if $\widehat{r}(t) < 0$, *a.e.* $t \in [0,t]$ *a.s.*, then $R(t) \geq 1$, $\forall t \in [0,T]$ *a.s.*. □

Remark 1. *The condition $\widehat{r}(t) < 0$ a.e. $t \in [0,T]$ a.s., is reasonable from the applications point of view (e.g. the interest rate $r(t) \geq 0$ a.e. $t \in [0,T]$ a.s.), and thus we assume it for the remainder of this section to ensure that $R(t) \geq 1$, $\forall t \in [0,T]$ a.s..*

Lemma 4. *Let the conditions of Lemma 3 hold. The processes* $Q(t) := R^{1/\eta}(t)$ *and* $Q_2(t) := \eta^{-1}Q^{1-\eta}(t)R_2(t)$ *are a solution pair to the Riccati BSDE:*

$$\begin{cases} dQ(t) = Q_1(t)dt + Q_2'(t)dW(t), \quad t \in [0,T], \\ Q_1(t) := -\lambda r(t)Q(t) - \dfrac{\lambda(Q_2(t)+\theta(t)Q(t))'(Q_2(t)+\theta(t)Q(t))}{2(1-\lambda)Q(t)}, \\ Q(T) = 1, \\ Q(t) > 0, \quad \forall t \in [0,T] \quad a.s.. \end{cases} \tag{4.25}$$

Proof. The differential of $R(\cdot)$ is

$$\begin{aligned} dR(t) =& dQ(t)^\eta \\ =& \Bigg\{ \eta Q(t)^{\eta-1}\left[-\lambda r(t)Q(t) - \frac{\lambda(Q_2(t)+\theta(t)Q(t))'(Q_2(t)+\theta(t)Q(t))}{2(1-\lambda)Q(t)}\right] \\ &+ \frac{\eta(\eta-1)}{2}Q(t)^{\eta-2}Q_2'(t)Q_2(t)\Bigg\}dt + \eta Q(t)^{\eta-1}Q_2'(t)dW(t) \\ =& \Bigg\{ -\lambda\eta r(t)Q(t)^\eta - \frac{\lambda\eta}{1-\lambda}Q(t)^{\eta-2}\big[Q_2'(t)Q_2(t) + 2Q_2'(t)Q(t)\theta(t) \\ &+ \theta'(t)\theta(t)Q(t)^2\big] + \frac{\eta(\eta-1)}{2}Q(t)^{\eta-2}Q_2'(t)Q_2(t)\Bigg\}dt + \eta Q(t)^{\eta-1}Q_2'(t)dW(t) \\ =& \left[-\left(\lambda\eta r(t) + \frac{\lambda\eta\theta'(t)\theta(t)}{1-\lambda}\right)R(t) - \frac{2\lambda}{1-\lambda}\theta'(t)R_2(t)\right]dt + R_2'(t)dW(t), \\ =& R_1(t)dt + R_2'(t)dW(t), \end{aligned}$$

which has a solution for $R(T) = 1$, as shown in the previous lemma. ∎

The admissible set of controls $\mathcal{A}$ is defined as:

$$\mathcal{A} := \{Z(\cdot) \in \mathcal{L}_{\mathcal{F}}^{0+}(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d))| \quad Y(t) > 0 \quad \forall t \in [0,T] \quad a.s., \quad and$$

$$[\lambda Q(\cdot)Z(\cdot)Y^{-1}(\cdot) + Q_2(\cdot)]Y(\cdot)^\lambda \in \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^d)\}.$$

The requirement of $Y(\cdot)$ being positive prevents *bankruptcy*, whereas the second requirement on the admissible set is a technical one implied by the method we use to solve the optimal investment problem.

Lemma 5. *Let the processes $\overline{r}(\cdot)$ and $\overline{\theta}(\cdot)$ be defined as*

$$\overline{r}(t) := \frac{1}{1-\lambda} r(t) + \frac{2+3\lambda}{2(1-\lambda)^2}\theta'(t)\theta(t),$$

$$\overline{\theta}(t) := \frac{1}{1-\lambda}\theta(t),$$

and be such that $\overline{r}(\cdot) \in \mathcal{L}_{\mathcal{F}}^1(\Omega; \mathcal{L}^1([0,T];\mathbb{R}))$, $\overline{\theta}(\cdot) \in \mathcal{L}_{\mathcal{F}}^2(\Omega; \mathcal{L}^2([0,T];\mathbb{R}^d))$. Let $\overline{r}(\cdot)$ satisfy (2.8) and $\overline{\theta}(\cdot)$ satisfy (2.2) for some constants $\overline{\alpha} > 0$ and $\overline{\beta} > 1$, respectively, such that $\overline{\alpha}\overline{\beta}^2 = 4(\overline{\beta}^2 + 2\overline{\alpha}\overline{\beta} - \overline{\alpha})$. If the assumptions of Lemma 3 hold, then $[(Q_2(\cdot) + \theta(\cdot)Q(\cdot))](1-\lambda)^{-1}Q^{-1}(\cdot)Y(\cdot) \in \mathcal{A}$.

Proof. If we choose $Z(t) = [(Q_2(t) + \theta(t)Q(t))](1-\lambda)^{-1}Q^{-1}(t)Y(t)$, then (4.19) becomes

$$\begin{cases} dY(t) = \left[r(t) + \theta'(t)\frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}\right] Y(t)dt + \frac{(Q_2'(t)+\theta'(t)Q(t))}{(1-\lambda)Q(t)} Y(t)dW(t), \ t \in [0,T], \\ Y(0) = Y_0. \end{cases} \tag{4.26}$$

We first show that this linear equation, the coefficients of which depend on the solution pair $(Q(\cdot), Q_2(\cdot))$, has a solution $Y(t) > 0$, $\forall t \in [0,T]$ *a.s.*. Let $\mu \in (0,1)$, and consider the equation

$$\begin{cases} dX(t) = \overline{r}(t)X(t)dt + \overline{\theta}'(t)X(t)dW(t), \quad t \in [0,T] \\ X(0) = Y(0)Q(0)^{-\mu}. \end{cases} \tag{4.27}$$

The process

$$X(t) := e^{\int_0^t [\overline{r}(s) - \frac{1}{2}\overline{\theta}'(s)\overline{\theta}(s)]ds + \int_0^t \overline{\theta}'(s)dW(s)}, \quad t \in [0,T]. \tag{4.28}$$

is a solution to (4.27) if it has enough integrability. It is sufficient for this purpose to show that $\overline{\theta}'(\cdot)X(\cdot)$ is a square integrable process. Thus, from Theorem 3, it follows that

$$\begin{aligned}\mathbb{E}\left[\int_0^T X(t)^2\overline{\theta}'(t)\overline{\theta}(t)dt\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^2\int_0^T \overline{\theta}'(t)\overline{\theta}(t)dt\right] \\ &\leq \frac{1}{2}\left(\mathbb{E}\left[\sup_{t\in[0,T]} X(t)^4\right]+\mathbb{E}\left[\int_0^T \overline{\theta}'(t)\overline{\theta}(t)dt\right]^2\right) < \infty.\end{aligned}$$

In order to show that (4.28) is the unique solution to (4.27), let us assume that $\overline{X}(\cdot)$ is another solution to (4.27). By Itô's formula we obtain

$$\begin{aligned}dX(t)^{-1}\overline{X}(t) &= [-r(t)+\theta'(t)\theta(t)]M(t)^{-1}\overline{M}(t)dt - \theta'(t)M(t)^{-1}\overline{M}(t)dW(t) \\ &\quad +M(t)^{-1}[r(t)\overline{M}(t)dt+\theta'(t)\overline{M}(t)dW(t)] \\ &\quad -\theta'(t)\theta(t)M(t)^{-1}\overline{M}(t)dt = 0. \qquad (4.29)\end{aligned}$$

Hence we have

$$\begin{cases} dM(t)^{-1}\overline{M}(t) = 0, \quad t\in[0,T], \\ M(0)^{-1}\overline{M}(0) = 1, \end{cases}$$

which gives $M(t)^{-1}\overline{M}(t) = 1$, i.e. $M(t) = \overline{M}(t)$, for all $t \in [0,T]$, a.s.. By applying Itô's formula to $Y(t) := X(t)Q(t)^{\mu}$, it can be shown that it satisfies (4.26). Moreover, since both $X(\cdot)$ and $Q(\cdot)$ are positive, so is $Y(\cdot)$. The proof of the uniqueness of $Y(\cdot)$ can be shown in a similar way to the uniqueness of $X(\cdot)$.

We now show that the process $[(Q_2(\cdot)+\theta(\cdot)Q(\cdot))](1-\lambda)^{-1}Q^{-1}(\cdot)Y(\cdot)$ is square integrable:

$$\mathbb{E}\left[\int_0^T [\lambda Q(t)U(t)+Q_2(t)]'[\lambda Q(t)U(t)+Q_2(t)]Y(t)^{2\lambda}dt\right]$$

$$\leq \mathbb{E}\left[\int_0^T \sup_{t\in[0,T]} Y(t)^{2\lambda}[\lambda Q(t)U(t)+Q_2(t)]'[\lambda Q(t)U(t)+Q_2(t)]dt\right]$$

$$= \mathbb{E}\left[\sup_{t\in[0,T]} Y(t)^{2\lambda} \int_0^T [\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]dt\right]$$

$$\leq \frac{1}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]} Y(t)^{4\lambda}\right] + \frac{1}{2}\,\mathbb{E}\left\{\int_0^T [\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]dt\right\}^2.$$

For the first term on the right hand side, let $Y(t) = X(t)Q(t)^{\mu}$ for any $\mu = \frac{1}{1-\lambda} \in (0,1)$, we obtain:

$$\mathbb{E}\left[\sup_{t\in[0,T]} Y(t)^{4\lambda}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{4\lambda} Q(t)^{4\mu\lambda}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{4\lambda} R(t)^{\frac{4\mu\lambda}{\eta}}\right]$$

$$\leq \mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{4\lambda} \cdot \sup_{t\in[0,T]} R(t)^{\frac{4\lambda}{1+\lambda}}\right] \leq \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]} X(t)^{8} + \sup_{t\in[0,T]} R(t)^{8}\right] < \infty.$$

For the second term on the right hand side we have:

$$\mathbb{E}\left\{\int_0^T [\lambda Q(t)U(t) + Q_2(t)]'[\lambda Q(t)U(t) + Q_2(t)]dt\right\}^2$$

$$= \mathbb{E}\Bigg\{\int_0^T \left[1 + \frac{2\lambda}{1-\lambda} + \frac{\lambda^2}{(1-\lambda)^2}\right] Q_2'(t)Q_2(t)$$

$$+ \left[\frac{2\lambda}{1-\lambda} + \frac{2\lambda^2}{(1-\lambda)^2}\right] Q_2'(t)\theta(t)Q(t) + \frac{\lambda^2}{(1-\lambda)^2}\theta'(t)\theta(t)Q(t)^2 dt\Bigg\}^2$$

$$\leq 3k_1^2\,\mathbb{E}\left[\int_0^T Q_2'(t)Q_2(t)dt\right]^2 + 3k_2^2\,\mathbb{E}\left[\int_0^T Q_2'(t)Q(t)\theta(t)dt\right]^2$$

$$+ 3k_3^2\,\mathbb{E}\left[\int_0^T \theta'(t)\theta(t)Q(t)^2 dt\right]^2$$

$$\leq \frac{3k_1^2}{\eta^4}\,\mathbb{E}\left[\int_0^T R_2'(t)R_2(t)dt\right]^2 + \frac{3k_3^2}{4}\,\mathbb{E}\left[\int_0^T \left[(\theta'(t)\theta(t))^2 + R(t)^{\frac{4}{\eta}}\right] dt\right]^2$$

$$+\frac{3k_2^2}{4\eta^2}\,\mathbb{E}\left[\int_0^T\left[R_2'(t)R(t)+\frac{1}{2}(\theta'(t)\theta(t))^2+\frac{1}{2}R(t)^{\frac{4}{\eta}}\right]dt\right]^2$$

$$\leq c_1\,\mathbb{E}\left[\int_0^T R_2'(t)R_2(t)dt\right]^2+c_2\,\mathbb{E}\left[\int_0^T\frac{1}{2}(\theta'(t)\theta(t))^2dt\right]^2+c_3\,\mathbb{E}\left[\int_0^T R(t)^{\frac{4}{\eta}}dt\right]^2$$

$$\leq c_1\,\mathbb{E}\left[\int_0^T R_2'(t)R_2(t)dt\right]^2+c_2\,\mathbb{E}\left[\int_0^T\frac{1}{2}(\theta'(t)\theta(t))^2dt\right]^2+c_3T^2\,\mathbb{E}\left[\sup_{t\in[0,T]}R(t)^8dt\right]<\infty,$$

where $k_1=1+\frac{2\lambda}{1-\lambda}+\frac{\lambda^2}{(1-\lambda)^2}$, $k_2=\frac{2\lambda}{1-\lambda}+\frac{2\lambda^2}{(1-\lambda)^2}$ $k_3=\frac{\lambda^2}{(1-\lambda)^2}$ and $c_1=\frac{3k_1^2}{\eta^4}+\frac{9k_2^2}{4\eta^2}$, $c_2=\frac{9k_2^2}{4\eta^2}+6k_3^2$, $c_3=\frac{3k_3^2}{2}+\frac{9k_2^2}{16\eta^2}$. □

Our next result gives the solution to the optimal investment problem (4.20) in an explicit closed-form. The optimal trading strategy turns out to be a linear function of wealth (see equation (4.30)), which is in accordance with the case of the market with bounded coefficients of Ferland and Waiter (2008), which it generalises, and immediately ensures the positivity of the wealth process $Y(\cdot)$.

Theorem 9. *Let the conditions of Lemma 5 hold. The optimal investment problem (4.20) has a unique solution given by*

$$Z^*(t)=\frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}Y(t).\tag{4.30}$$

The corresponding optimal cost is $J(Z^*(\cdot))=-Q(0)Y^\lambda(0)$.

Proof. From the previous lemma, we know that $Z^*(\cdot)\in\mathcal{A}$. The differential of $Q(\cdot)Y^\lambda(\cdot)$ is:

$$\begin{aligned}dQ(t)Y(t)^\lambda=&\,Q_1(t)Y(t)^\lambda dt+Q_2'(t)Y(t)^\lambda dW(t)+\lambda Y(t)^{\lambda-1}Q_2'(t)Z(t)dt\\&+Q(t)[\lambda r(t)Y(t)^\lambda+\lambda Y(t)^{\lambda-1}\theta'(t)Z(t)\\&+\frac{1}{2}\lambda(\lambda-1)Y(t)^{\lambda-2}Z'(t)Z(t)]dt+\lambda Q(t)Y(t)^{\lambda-1}Z'(t)dW(t)\end{aligned}$$

$$=\Bigg\{Q_1(t)Y(t)^\lambda + \lambda r(t)Q(t)Y(t)^\lambda + Y(t)^\lambda\Bigg[\lambda Q_2'(t)U(t)$$

$$+\lambda Q(t)\theta'(t)U(t) + \frac{\lambda(\lambda-1)}{2}Q(t)U'(t)U(t)\Bigg]\Bigg\}dt$$

$$+\left[Q_2'(t)Y(t)^\lambda + \lambda Q(t)Y(t)^\lambda U'(t)\right]dW(t),$$

where $U(t) := Z(t)/Y(t)$. After integration and taking the expectation, this becomes:

$$\begin{aligned}-\mathbb{E}\left[Y(T)^\lambda\right] &= -Q(0)Y^\lambda(0) - \mathbb{E}\int_0^T [Q_1(t)Y(t)^\lambda + \lambda r(t)Q(t)Y(t)^\lambda]dt \\ &- \mathbb{E}\int_0^T Y(t)^\lambda\Bigg[(\lambda Q_2'(t) + \lambda\theta'(t)Q(t))U(t) + \frac{\lambda(\lambda-1)}{2}Q(t)U'(t)U(t)\Bigg]dt.\end{aligned}$$

By the *completion of squares* method, we obtain:

$$-\mathbb{E}\left[Y(T)^\lambda\right] = -Q(0)Y^\lambda(0)$$

$$-\mathbb{E}\int_0^T Y(t)^\lambda\left[Q_1(t) + \lambda r(t)Q(t) + \frac{\lambda(Q_2(t)+\theta(t)Q(t))'(Q_2(t)+\theta(t)Q(t))}{2(1-\lambda)Q(t)}\right]dt$$

$$+\frac{\lambda(1-\lambda)}{2}\mathbb{E}\int_0^T Y(t)^\lambda Q(t)\left[U(t) - \frac{Q(t)^{-1}(Q_2(t)+\theta(t)Q(t))}{1-\lambda}\right]'$$
$$\cdot\left[U(t) - \frac{Q(t)^{-1}(Q_2(t)+\theta(t)Q(t))}{1-\lambda}\right]dt$$

$$= -Q(0)Y^\lambda(0) + \frac{\lambda(1-\lambda)}{2}\mathbb{E}\int_0^T Y(t)^\lambda Q(t)\left[U(t) - \frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}\right]'$$

$$\cdot\left[U(t) - \frac{(Q_2(t)+\theta(t)Q(t))}{(1-\lambda)Q(t)}\right]dt \geq -Q(0)Y^\lambda(0),$$

with equality if and only if $U(t) = \frac{Q_2(t)+\theta(t)Q(t)}{(1-\lambda)Q(t)}$, $a.e.\ t \in [0,T]\ a.s.$. $\square$

5. Conclusions

We have given several integrability results for the exponential process with unbounded coefficients under the Kazamaki type conditions. While using this weaker condition as compared to a Novikov type condition of Yong, we obtained the same integrability in most cases. These results were then applied to the solvability of linear and Riccati BSDEs, which in turn were used to solve the problems of market completeness (and also of pricing and hedging) and asset management with power utility. We expect that our results will be useful in solving related problems, such as the problem of optimal consumption, the mean-variance asset management, the linear-quadratic control problem, the risk-sensitive control problem, the bond pricing problem, all in the setting of unbounded coefficients, through our results on linear and Riccati BSDEs.

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