CATASTROPHE BONDS AS INNOVATIONS IN AN AGENT-BASED CAPM

Thesis submitted in accordance with the requirements
of the University of Liverpool for the degree of
Doctor of Philosophy

by

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April 2018

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Declaration

All sentences or passages quoted in this project dissertation from other people's work have been specifically acknowledged by clear cross referencing to author, work and page(s). I understand that failure to do this amounts to plagiarism and will be considered grounds for failure in this module and the degree examination as a whole. In addition, Chapter 2 is a joint working paper with Professor Jan Wenzelburger.

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Abstract

This thesis studies the effect of catastrophe (CAT) bonds as innovations in a financial market. It provides a new and alternative approach to the description of CAT bonds, by utilizing an agent-based CAPM to analyse the behaviour of CAT bonds in a financial-market setting that comprises key features of an insurance industry. First, a model of CAT bonds as new assets in the single-period CAPM is developed. In this model, a CAT bond price is determined by a market clearing condition. The effects of CAT bonds on the financial market are discussed. Conditions under which investors include new assets in their portfolios to increase utility are established. The premium and coupon of a CAT bond are determined, which provides an equilibrium theoretical foundation of the premium calculation principle from actuarial science. The notion of transferable insurance risk is developed and conditions for the transferability of catastrophe risk from the insurance industry to the financial market by means of CAT bonds are established. In a second step, CAT bonds are introduced in an agent-based multi-period CAPM with investors holding heterogeneous beliefs regarding future payoffs of assets. The minimum premium and coupon are determined. The notion of transferable risk is adapted to an agent-based framework. The concept of perceived Pareto superiority is introduced. Stochastic difference equations are derived to describe the co-evolution of asset and CAT bond prices. The changes in the market price of risk are analysed. The concept of a perfect forecasting rule is generalized to CAT bonds. Finally, a compound Poisson process that describes the evolution of insurance losses related to catastrophe risk is introduced. The loss that CAT bonds cover at each time period is simulated and discussed according to two different scenarios. By applying the loss process, the effects of CAT bonds on the financial market are simulated in terms of investors' behaviour and market risk.

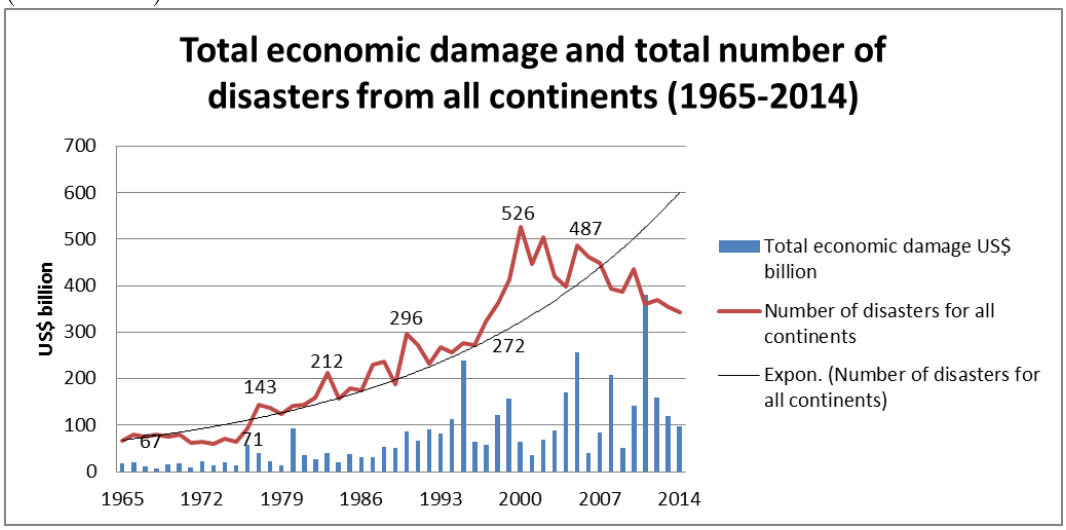
Chapter 1

Introduction

Catastrophic events such as floods, earthquakes and droughts threaten people's lives, properties, and economies. They cause many deaths and have severe negative financial impacts on the global economy every year. The frequency of disasters and corresponding damage has increased significantly, since 1994. As shown in Figure 1.0.1, the total number of reported disasters has been increasing since 1974. In terms of financial costs, catastrophic losses have increased from \$58.78 billion in 1976 to \$380.09 billion in 2011[1], over a six-fold increase. Relevant insured losses have climbed significantly as a result. Given these circumstances, catastrophe bonds (CAT bonds) were designed in the mid-1990s to improve reinsurance capacity and reduce the negative impacts on the (re-)reinsurance industry, such as the availability and price of insurance policies [137]. In the following Section 1.1, CAT bonds and their role in the market are discussed.

[1] The annual value of economic damage was calculated in 2014 by considering the inflation rate

Figure 1.0.1: Total economic damage and total number of disasters on all continents (1965-2014)



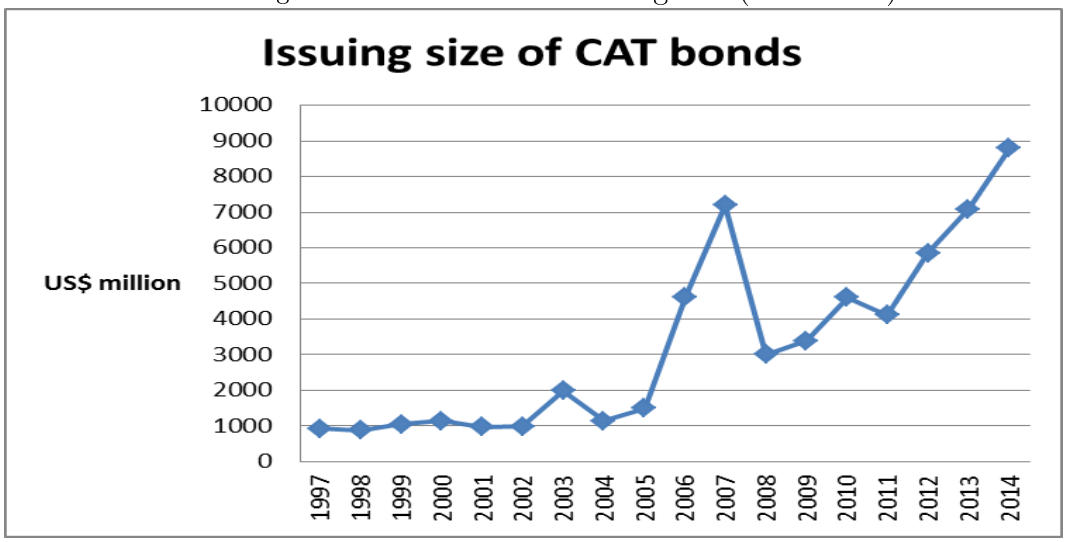
('All continents' includes Africa, the Americas, Asia, Europe, and Oceania.) Data source: Emergency Events Database (EMN-DAT), Centre for Research on the Epidemiology of Disasters (CRED), [Online], Available at: http://www.emdat.be/disaster_trends/index.html

1.1 CAT bonds market

CAT bonds are usually issued by (re-)insurance companies[2] and used to transfer potential catastrophe risk to the financial market. Unlike corporate bonds issued by (re-)insurance companies, CAT bonds are isolated from issuers' financial positions such as changing profits or obligations [137]. Unlike traditional financial instruments, CAT bonds' payoffs are not connected with financial factors that affect market volatility such as interest rates or economic policies. CAT bonds' payoffs, as insurance-linked securities, are linked to specific insured losses from catastrophic events. Hence, a CAT bond contract has a default term: once a pre-defined catastrophic event occurs, the bond holder may lose the coupon partially or as a whole

[2]CAT bonds are issued by governments to prevent economic loss resulting from possible natural disasters.

Figure 1.1.1: CAT bonds issuing size (1997-2014)

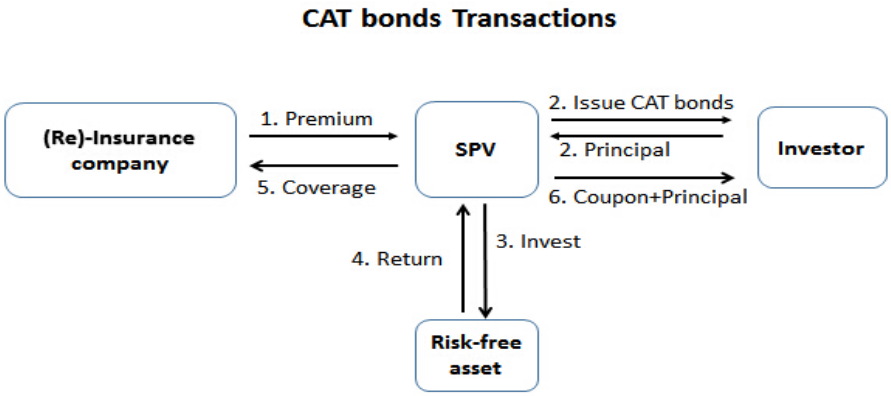


Data resource: Aon Benfield [7, 8, 9, 10, 11]

and even the principal. However, because CAT bonds cover a final threshold of insured losses, only a significant catastrophic event will trigger the default terms such as Hurricane Katrina in 2005 [122]. Because the default of CAT bonds can only be triggered by severe catastrophic events, CAT bonds are regarded to have zero correlation with the financial market and therefore attract investors' attention [52]. CAT bonds are regarded as ideal financial investments to diversify portfolio risk and have become increasingly popular since 2005 [122]. Figure 1.1.1 displays the issuing size of CAT bonds from 1997 to 2014. In the first quarter of 2016, the issuance of CAT bonds reached $2 billion, the highest amount on record, and a 35% increase compared to the previous record set in the first quarter of 2015 [54].

CAT bonds are prominent in the US market and have gradually become more popular in Japan and Europe. Currently, more countries are offering CAT bonds to mitigate the costs of severe natural catastrophes and raise reconstruction funds

Figure 1.1.2: CAT bonds related transactions



(Transactions happen in order: 1, 2, ..., 6.) Source: Swiss Re Capital Markets [138]

to decrease potential economic losses [122]. For example, since 2006, Mexico has sold a series of three-year CAT bonds in the market to raise hundreds of millions of dollars to cover catastrophe risks [122]. China issued its first CAT bonds in 2015 and raised $50 million for protection [14]. Chile's government is considering CAT bonds to be used as protection from catastrophe risks. Chile suffered a devastating earthquake that killed more than 500 people and caused around $30 million of losses in 2010 [122]. An overview of CAT bond-related transactions is given in Figure 1.1.2. When a (re-)insurance company intends to use CAT bonds to transfer catastrophe risk, they first set up a special purpose vehicle (SPV) and the (re-)insurance company is named as a sponsor. The SPV is a legal entity independent of the sponsor, which ensures CAT bonds to be isolated from any obligations of the sponsor company, thus protecting the rights of CAT bond holders. The role of the SPV is to issue CAT bonds. The relationship between the SPV and sponsor can be described by an insurance contract. The sponsor cedes the final threshold of its insured loss from a

pre-defined catastrophe to the SPV and passes a certain premium to it (as depicted in Procedure 1 in Figure 1.1.2). The SPV promises to cover this loss once the pre-defined catastrophic event occurs. The SPV then issues CAT bonds to investors (often institutional investors) in the financial market to raise money for the coverage of possible losses (see Procedure 2 in Figure 1.1.2). The duration of CAT bonds is usually defined as 12 months with an average coupon rate of 7% - 9% (in 2016), much higher than the one-year US Treasury rate of 0.8% (in 2016) or corporate bonds [87]. If the default term is triggered, investors will lose part or all of their coupon and possibly their principal. If the default term is not triggered, investors receive coupons as risk compensation and get their principal back at maturity. After selling CAT bonds, the SPV then invests the funds received from investors and the sponsor into a risk-free asset, such as money market funds or Treasury bills (Procedures 3 and 4 in Figure 1.1.2). If a pre-agreed event happens, the SPV will use the gross return of this fund to cover the insurance loss and use the remaining money to pay back investors; if no such event occurs, this money is used to pay for the coupons and investors' principal [138].

1.2 Previous research on CAT bonds

Existing research on CAT bonds focuses primarily on pricing. A small number of papers have also examined the economic benefits of CAT bonds (e.g., Barrieu and Louberge [15], Kish [87]). Pricing-related research on CAT bonds has spanned different fields, namely financial economics, financial mathematics, and actuarial science. An introduction into research in these fields, followed by a discussion is

provided later.

As noted above, CAT bonds can be considered as reinsurance contracts that protect sponsors (re-insurance companies) from the risks of potential natural disasters by transferring this risk to the financial market. In actuarial science, professionals focus on how to set premiums for CAT bonds. The premium, in the insurance industry, is generally regarded as the price of the insurance policy and is calculated by the premium calculation principle

$$\text{premium}=\text{expected loss}+\text{risk load}, \tag{1.2.1}$$

where risk load is a load for risk margin and expenses. As introduced in Section 1.1, the SPV receives a certain amount of the premium and issues CAT bonds to raise money for coverage of the sponsor. The premium rate of the bond is risk compensation for investors; thus, the coupon rate of a CAT bond is the sum of the premium rate and risk-free rate [70]. Actuarial research considers the premium to be the main determining factor of a CAT bond's value. Almost all CAT bond-related research from actuarial science has focused on how to stipulate the premium based on the premium calculation principle (1.2.1). For example, Lane uses an empirical approach to determine how the 'pricing mechanism' of insured risk has been set by markets. He uses the Cobb-Douglas production function as a market utility function to measure insured risk, Furthermore, he employs the conditional expected loss (CEL) and probability of first dollar loss (PFL) as variables to describe the respective severity and frequency of insured loss. By applying the Cobb-Douglas production function to CEL and PFL as variables, the premium formula then becomes

$$\text{Premium} = \gamma(PFL)^{\alpha} \times (CEL)^{\beta}.$$

Lane [94] then ascertains γ, α and β, according to insurance-linked securities (ILS) market data from 1999. Similarly, Major and Kreps [105] choose a log-linear function to demonstrate the relationship between the premium and expected loss by conducting empirical analysis. Bodoff and Gan [23] use a linear function of expected loss to describe the market price of CAT bonds. In addition, they use this function and past data to study the effects of various perils and zones on CAT bond prices. (For similar research, see Berge [17] as cited in Galeotti et al. [70]; Lane and Mahul [95]; and Dieckmann [60]. Galeotti et al. [70] compare and test the accuracy of different premium calculation models for CAT bonds and examine the empirical validity of the models. Unlike the above methods, which fit empirical data to determine the relationship between the premium, the expected loss, and the risk load, Wang [146] applies probability transforms of Wang's [145] probability transform to extend the Sharpe ratio to evaluate risk with a skewed distribution. This allows for an evaluation of CAT bonds returns that are skewed distributed.

The study of CAT bonds from the perspective of actuarial science focuses mainly on premium valuation. This is so because in actuarial science, the premium for a reinsurance contract determines the value of a contract. As CAT bonds are regarded as reinsurance contracts, researchers in actuarial science then take the premium that the (re-)insurance company passes to issue CAT bonds as the value of CAT bonds. However, the premium itself cannot determine the final value of a CAT bond. This is so because CAT bonds are traded in the financial market, so that the valuation of

CAT bonds is also affected by the financial market as Wang [146] point out.

As financial instruments, CAT bonds have attracted interest in the financial literature with regard to pricing. The no-arbitrage pricing method is commonly used to assign fair prices to financial derivatives, such as options and futures. By that method, the price of an option equals the price of the portfolio, which replicates the payoff of the option at maturity. Cox and Pederson [48] show that with exposure to catastrophe risk, the market is incomplete, which contradicts the requirement for the arbitrage pricing model that the market should be complete. In addition, Aase [1] states that arbitrage pricing with the assumption of unpredictable jumps at random time points results in many equivalent martingale measures and hence no unique price. Due to market incompleteness, models that are based on no-arbitrage pricing, such as the Black-Scholes (BS) model, are not appropriate for valuing CAT bonds.

In the literature, two main approaches have been applied to solve these problems. One is Merton [111] who assumes that the risk caused by a jump process can be diversified away. The BS model assumes that the underlying stock returns are represented by a stochastic process with a continuous sample path. He states that changes in stock prices can be modelled via a jump process reflecting the impact of information that causes stock price changes of "extraordinary magnitude". Hence he changes the "continuous stochastic process" in the BS model to a continuous jump process and shows that the BS solution is not valid, in this instance. Then he shows proves that no portfolio can eliminate the risk caused by the jump process. Hence, the BS cannot provide fair option prices. In other words, one cannot price options via a jump process because an arbitrage opportunity will be created.

Merton [111] solves this problem by deriving a pricing formula under the assump-

tions of the CAPM. In the CAPM, the beta coefficient measures the sensitivity of the expected excess returns of assets to the expected excess return of the market. If the jump process is caused by information that only affects specific companies or industries, rather than the whole market, then the beta of the asset return caused by the jump process is 0. Therefore, the expected excess return rate caused by the jump process is simply a risk-free rate. Under the assumption that jump risk is non-systematic, Merton [111] is able to generate a solution related to option prices. Yet, as he points out, this solution holds only as long as the CAPM holds; if the CAPM does not hold, the solution is invalid. He also derives another solution from Ross [123]. Both solutions are based on the assumption that the jump component of a risk process can be diversified away.

Vaugirard [143] follows Merton [111] and adapts a framework with underlying state variables to describe the market price of natural risk. He assumes that investors are neutral towards jump risk in line with Merton's [111] assertion that risk caused by the jump process can be diversified away. Vaugirard [143] shows the existence of well-defined arbitrage-free prices for CAT bonds by computing a first-passage time distribution. He also applies a numerical analysis to demonstrate the sensitivity of insurance-linked bond prices to risk exposure. Following Vaugirard's [143] model, Nowak and Romaniuk [115] extended it to develop a general valuation formula to price more types of CAT bonds; Lai, Parcollet and Lamond [92] add the effect of currency exchange risk to this pricing formula.

In addition to Merton [111], another method that is used to cope with the problems caused by jump processes when the no-arbitrage pricing is applied is Cox and Ross [47]. Cox and Ross [47] find explicit option valuation formulas by assuming

that the jump amplitude (i.e. the coefficient to describe a Poisson process) is a non-random function at a jump. Following their method, Lee and Yu [96] study the effect of interest rates to the CAT bond price and discuss hazard, basis and default risks. They use a Monte Carlo simulation to estimate CAT bond prices and conclude that both moral hazard and basis risk may drive down CAT bonds prices substantially.

As an alternative approach to this literature, Baryshnikov, Mayo, and Taylor [16] price CAT bonds as threshold bonds that trigger default in the event of catastrophe. They point out that it is incorrect to apply traditional securities pricing methods to CAT bonds because they assume that the underlying contingent processes are log-normal distributed. They price CAT bonds as defaultable bonds so that the CAT bond price is the expected value of the discounted payoff of CAT bonds following the catastrophic event. They use a compound Poisson process to describe catastrophic events' happening and define a continuous payoff process. To the best of my knowledge, their contribution is the first paper to connect CAT bond pricing to catastrophic event processes. It has significantly influenced subsequent studies. For instance, Burnecki, Kukula, and Taylor [36] following on previous results, use a numerical analysis to demonstrate that there is a relationship between CAT bond prices and the threshold level and time to expiry. They also demonstrate how CAT bond prices change due to different claim distributions. However, their findings indicate that the method proposed by Baryshnikov, Mayo, and Taylor [16] depends largely on the precise determination of the claim (disaster) arrival process and the claim distribution, which is quite practical. Similarly, Ma and Ma [101], Schmidt [129], Jiang and Dassios [53] also use a (compound) Poisson process to describe the flow of claims-related catastrophe risk in their research.

Contrary to researchers who add additional assumptions to deal with the incompleteness of markets created by catastrophe risk in order to use arbitrage-free pricing, researchers such as Aase [1] and Cox and Pederson [48] apply equilibrium theory to solve the pricing problems that the no-arbitrage pricing method cannot cope with. Aase [1] proposes a scenario in which CAT bonds are issued by insurers who are seeking protection as reinsurance contracts; on the other side, investors require compensation for bearing this catastrophe risk by holding CAT bonds. He then applies the aggregation theorem of Rubinstein [125], which claims that under certain conditions, even though the market may be incomplete, equilibrium prices will be determined as if they were determined in a complete (Arrow-Debreu) market. So equilibrium prices of CAT bonds, which optimise insurers risk sharing intention and provide the relevant compensation for investors, are obtained. Aase [1] provides an idea on how to value and analyse CAT bonds by considering the insurance industry and financial market. Cox and Pederson [48] apply the representative agent technique [3] used to price uncertain cash-flows in an incomplete market setting to derive an equilibrium price formula of bonds linked to a catastrophe occurrence. They then recast the equilibrium valuation formula as a standard form of "risk-neutral expectation" by removing the form of utility function and aggregate endowment process applied in the representative-agent technique. It shows that pricing CAT bonds in the equilibrium environment not only solves the incomplete market problems that the no-arbitrage pricing method faces but can also provide a pricing formula that corresponds to the no-arbitrage pricing framework. The research from Aase [1] and Cox and Pederson [48] provide a foundation for this thesis' main theme.

[3] See Huang and Litzenberger [82], Karatzas [86], Magill and Quinzii [104], or Panjer et al [117] for details on the theory of the representative agent.

1.3 Research Motivation

The discussion above showed that much of the research on CAT bonds thus far is mainly concerned with pricing and comes either from actuarial science or finance. There are still a number of problems that are unsolved in the literature. With regard to the pricing of CAT bonds, actuarial science claims that the premium for a CAT bond contract determines its value. Researchers in the field of finance, however, price CAT bonds as financial instruments traded in the financial market. Given the characteristics of a CAT bond, as proposed by Wang [146], any valuation needs to consider the insurance and the financial market to ensure a comprehensive analysis of them and hence a proper valuation. Research by Aase [1] is a good starting point on this issue. He describes how insurers are willing to pay a risk premium for protection while, on the other side, investors require compensation for bearing this catastrophe risk by holding CAT bonds. Thus, this scenario can be applied in a model of CAT bonds to allow for an analysis of the insurance industry and the financial market as a whole. In addition, as Baryshnikov, Mayo and Taylor [16] point out, most research in the finance literature prices CAT bonds by assuming that they are already traded in the market. It raises questions on conditions under when issuers are willing to issue CAT bonds and under what conditions investors in a financial market are willing to purchase them. The advantage of CAT bonds, especially to diversify investors' portfolios' risk, is mentioned frequently. However, the question of how investors behave in a financial market with CAT bonds has not been addressed.

This thesis applies an agent-based CAPM to study the behaviour of investors in a financial market in which ordinary financial securities such as stocks, are traded

alongside CAT bonds. The model allows for a comprehensive analysis of both the financial and the insurance markets. Decisions on the premium that (re-)insurance companies pass to the issuer, the promised payoff of a CAT bond, and the appropriate issuing size of CAT bonds are discussed. The market prices of CAT bonds are determined in temporary market equilibria in which supply and demand for CAT bonds are equilibrated. The reasons why investors are willing to purchase CAT bonds and include them in their portfolios are discussed. To this end, a mechanism that explains how and when catastrophe risk can be successfully transferred to a financial market is developed. By constructing an agent-based CAPM with CAT bonds in a setting that comprises an insurance market, this thesis fills a research gap on CAT bonds. The model allows for an analysis of the design of CAT bonds as well as the behaviour of CAT bonds' investors.

1.4 Thesis organization

This thesis consists of 6 chapters. Chapter 1 gives an introduction to the state of current research on CAT bonds. It includes a brief introduction to CAT bonds and its market, previous research on CAT bonds, and research motivation.

In Chapter 2, CAT bonds are considered as innovations in a traditional CAPM. In this static two-period model, the effect of introducing a CAT bond to the financial market is discussed. Under certain conditions, the market price of risk will increase after the introduction of the bond as a new asset. It is shown that in such a case investors will include this new asset into their portfolios to increase their utility. The

premium that is passed from a (re-)insurance company and the coupon of a CAT bond are determined by taking into account the issuer's constraints and investors' requirements. The premium formula verifies the premium calculation principal from actuarial science with a constraint on the admissible risk load for the financial market. A new concept, called transferable insurance risk, is introduced to allow this thesis to analyse the conditions under which catastrophe risk is transferable from the insurance industry to the financial market by means of CAT bonds.

Chapter 3 reviews an agent-based CAPM with heterogeneous beliefs which among others was developed by Böhm and Chirella [24], Brock and Hommes [30], Chiarella and He [41, 42] and Wenzelburger [147]. It discusses arbitrage opportunities in this model as well as the valuation of redundant assets. In Chapter 4, CAT bonds are introduced in the agent-based CAPM with investors holding heterogeneous beliefs. The design of a CAT bond is developed and a premium that is passed from (re-)insurance companies and a promised payoff to investors is determined. An example illustrating how an issuer determines the coupon and premium before issuing CAT bonds is provided. In each time period, investors have heterogeneous beliefs regarding future payoffs of assets. The concept of perceived Pareto superiority is introduced to explain why investors will include CAT bonds in their portfolios. The notion of transferable risk is generalised to the agent-based CAPM. Stochastic difference equations that describe the co-evolution of stock and CAT bond prices are developed and discussed. These prices are, in essence, driven by the way investors form forecasts about the future evolution of the markets. Following on from Wenzelburger [147], a perfect forecasting rule for future asset prices is derived which allows for rational expectations for one of the investors as a special case of a generally non-linear

stochastic dynamic system.

In Chapter 5, the compound Poisson process typically used to describe the evolution of insurance losses related to catastrophic events is introduced. Two different ways of how CAT bonds cover loss are simulated and discussed. By applying the loss process, the effects of CAT bonds on the financial market are simulated in terms of investors' behaviour and market risk. Finally, a number of conclusions are given in Chapter 6.

In summary, this thesis provides a new and alternative approach for the description of CAT bonds, using an agent-based CAPM to analyse investors' behaviour when CAT bonds are introduced as innovations in a financial-market setting that comprises key features of an insurance market.

Chapter 2

CAT bonds in the CAPM

2.1 Introduction

In recent decades, financial markets have seen a significant increase in the types of financial innovations available. Allen and Gale [4], Tufano [141], and Piazza [119] have discussed the advantages of innovations to firms and governments. There is also increasing academic research on modeling financial innovations. Allen and Gale [4], Rahi [121], and Demange [57] have studied the design of innovations from the issuer side. (For more information on research of financial innovations please read the survey article of Duffie and Rahi [63]). Allen and Gale [4] and Rahi [121] study securitization in terms of maximising the issuer's profit. Demange [57] takes private information into account. Some researchers have studied the effect of financial innovations on markets. Biais et al. [19] show that loss-making issuers bring more risk to investors and hence increase instability in financial markets. Wenzelburger [151] shows that the introduction of put options decreases market risk. Simsek

[136] defines speculative variance as a portfolio's risk. When traders disagree on beliefs, financial innovations increase the speculative variance. In addition to risk changing, the literature also studies how new assets affect market equilibrium and investors' welfare. Diamond [59]claims that adding new assets to a one-commodity model cannot lead to a new equilibrium if everyone is worse off than in the previous equilibrium. Hart [76] uses an example to show that adding a new asset will make all agents worse off in an equilibrium. However, Elul [67] uses a general equilibrium model with incomplete markets to show that it is possible to introduce a new asset to make every agent better-off under certain constraints. Cass and Citanna [38] have similar results.

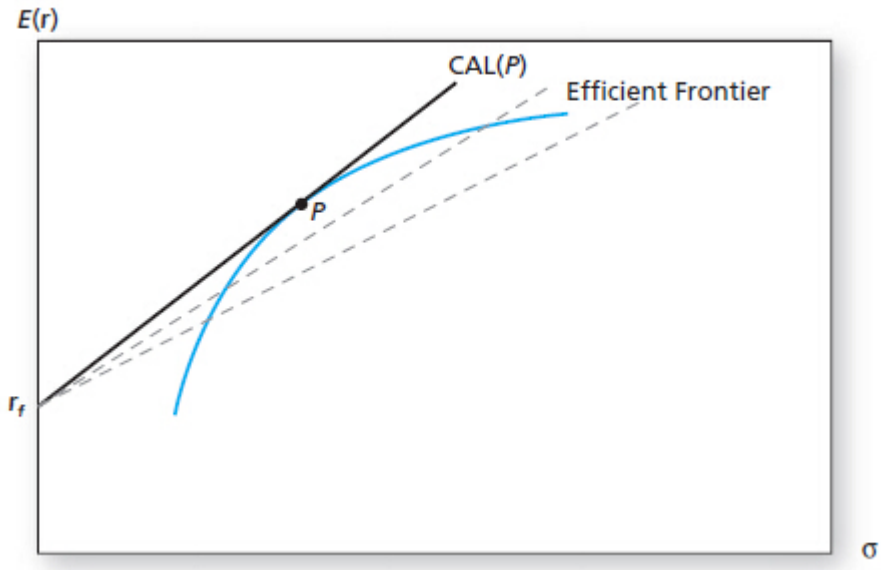
This chapter will provide an alternative and novel way to design CAT bonds and explore how CAT bonds affect market equilibria and investors' welfare. CAT bonds can typically not be replicated with traded financial assets in the market [48]. In the literature, Mayers [109], Breeden [28], and Grossman and Shiller [73] show that equilibria exists in incomplete markets with agents having non-traded endowments. Oh [116] shows that in incomplete markets with non-traded endowments, the relationship between the risk of the asset and the expected returns are positive, so investors have higher expectations on returns if the asset is riskier. Unlike Oh [116], Wenzelburger and Koch-Medina [88] establish the existence of CAPM equilibria with non-traded assets in a more general setting. The chapter will also follow Wenzelburger and Koch-Medina [88] in applying the capital asset pricing model (CAPM) to study how an issuer, namely a (re-)insurance company, can design CAT bonds, how insurance risk is transferred to the financial market, and when investors include CAT bonds into their portfolio. Section 2.2 introduces the formulation of the CAPM

and core theorems developed by Böhm and Chiarella [24], and Wenzelburger [150]. Section 2.3 explores the trading conditions of innovations in the model and derives a CAPM equilibrium with innovations and determines the equilibrium price of innovations. Based on the framework developed in Section 2.3, CAT bond contracts are designed in Section 2.4, by considering the issuer (seller) and the investors (buyers) following the idea of Aase [1]. Unlike innovations in Allen and Gale [4] are designed to increase profit for issuers, CAT bonds are designed to transfer issuers' potential insurance risk from catastrophic events to the financial markets. The promised payoff and premiun are decided in Subsection 2.4.1. Subsection 2.4.2 then studies the issuance conditions for CAT bonds. The risk load that is constrained by conditions in the financial markets is discussed in Subsection 2.4.3. The last Subsection 2.4.4 illustrates a special case with linear mean-variance preference.

2.2 Prerequisites

Markowitz's portfolio selection model [108] demonstrates how mean-variance optimisers select optimal portfolios. The model firstly determines a set of efficient portfolios which provide the maximum expected return with a given risk or the minimum risk by giving an expected return. This set is called the 'efficient frontier of risky assets'. Portfolios that lie on the efficient frontier are called efficient portfolios. They provide the best risk-return combinations. Markowitz [108] defines a line in a mean-variance panel, that starts from the risk free rate $(0, r_f)$ and is tangent with the efficient frontier is the capital allocation line (CAL) (see Figure 2.2.1), Portfolios

Figure 2.2.1: Capital allocation line (CAL)



Source: Bodie el at. [22]

along this CAL provides the highest reward-to-volatility ratio as this CAL has the highest slop among capital allocation lines.The optimal portfolio is determined by the tangent point at which the CAL is tangent to the efficient frontier. Different individuals with different risk aversions will choose their own mixture between the optimal portfolio of risky assets and risk-free assets. Therefore the separation principle is developed; that is the portfolio choice problem can be separated into two independent steps. The first step is to determine the optimal mix of risky assets, i.e., the optimal portfolio of risky assets, which is independent of individual preferences. The second step is to determine the optimal amount of funds invested into the risk-free asset, which depends on individual preferences.

The capital asset pricing model (CAPM) developed by Sharpe [131], Lintner [99]

and Mossion [112] and based on the Markowitz portfolio theory assumes that individuals are alike as much as possible, including the fact that they all agree on the same estimation of mean and co-variances of asset returns. The optimal portfolio will turn out to be the market portfolio – a summation of all assets based on an equilibrium argument. Individuals with different risk aversion will choose different mixtures of market portfolio and risk-free assets for their portfolios. In the CAPM, the beta of an asset is defined as the ratio of the covariance between the risky asset and the market portfolio to the variance of the market portfolio. The beta coefficient thus measures the volatility, or systematic risk, of a risky asset in comparison to the market as a whole. The security market line (SML) in the panel of betas and asset returns show a linear relation between beta coefficients and asset returns. Therefore, the expected excess rate of return of a risky asset is proportional to the market risk premium and the price of this risky asset can be calculated. This is how the CAPM value risky assets.

Böhm and Chrila [24] and Böhm and Wenzelburger [26] explain the CAPM from another point of view and unveil a number of its hidden features. They endow each investor with a utility function and discuss the demand behaviour of investors. Following this model framework, Chrilla et al [43, 44] study investors' behaviour with heterogeneous beliefs.

The next section will adopt the CAPM framework introduced in Böhm and Chiarella [24] and Wenzelburger [150] and some well-known results on the CAPM.

2.2.1 Model setting

Consider a two-period model in which an investor need to transfer initial wealth into the future. Investment opportunities consist of K risky assets and one risk-free bond. All prices and payoffs are denominated in a non-storable consumption good that serves as the numeraire. The K risky assets are characterised by stochastic payoffs $q = (q_1, \ldots, q_K)$, where $q_k \geq 0$ denotes the payoff per unit of the k-th asset. The risk free bond pays a constant return $R_f = 1 + r_f > 0$ per unit. A portfolio is given by a vector $(x, y) \in \mathbb{R}^K \times \mathbb{R}$. The vector $x = (x_1, \ldots, x_K) \in \mathbb{R}^K$ represents the portfolio of risky assets, where x_k $(k = 1, \ldots, K)$ denotes the number of shares of the k-th risky asset and the scalar y describes the number of risk-free bonds. The total stock of risky assets is $x_m \in \mathbb{R}^K_+$ and is referred to as the market portfolio of the economy.

The initial wealth of a typical investor consists of $w_0 > 0$ units of the consumption good and it is assumed that consumption takes place in the second period only. Assume for simplicity that there are no short-sale constraints. If $p = (p_1, \ldots, p_K) \in \mathbb{R}^K$ denotes the price vector of the risky assets, the investor's budget constraint is

$$w_0 = y + \langle p, x \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^K$. Second-period wealth is $w_1 = R_f y + \langle q, x \rangle$. Substituting for y, the investor's second period wealth associated with the portfolio of risky assets $x \in \mathbb{R}^K$ takes the form

$$w_1 = R_f w_0 + \langle q - R_f p,\, x \rangle.$$

Define the vector of expected payoffs and the covariance matrix of the payoffs by $\bar{q} = \mathbb{E}[q]$ and $\Sigma = (\Sigma_{lk}) \in \mathcal{M}_K$, respectively, where $\Sigma_{lk} := Cov[q_l, q_k]$ is the covariance between the l-th and the k-th asset. Σ is a symmetric $K \times K$ matrix. To ensure that none of the assets are redundant, we assume in addition that Σ is positive definite, meaning that all eigenvalues are strictly positive.

Let $\pi := E[q] - R_f p$ be the vector of expected excess returns. The expected second-period wealth associated with a portfolio of risky assets $x \in \mathbb{R}^K$ and its standard deviation are given by

$$E[w_1] = \mu_w(w_0, \pi, x) := R_f w_0 + \langle \pi,\, x \rangle$$

and

$$Var[w_1] = \sigma_w(x) := \sqrt{\langle x, \Sigma\, x \rangle}$$

respectively. The investment behaviour of a typical investor is described by risk preferences that depend on expected future wealth and its standard deviation only.

Assumption 1. *An investor is characterised by a utility function $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, which is a function of the mean μ and the standard deviation of future wealth σ, and a probability distribution for future payoffs. The utility function U is continuously differentiable, strictly increasing in μ, strictly decreasing in σ, and strictly quasi-concave. The probability distribution is characterised by a vector of expected payoffs $\bar{q} \in \mathbb{R}^K_+$ and a positive definite covariance matrix Σ.*

Relative to the CAPM literature, the importance of Assumption 1 lies in the fact

that the probability distribution for future payoffs may be arbitrary, so long as its first two moments are finite. With Assumption 1, the portfolio selection problem of an investor takes the form

$$\max_{x \in \mathbb{R}^K} U(\mu_w(w_0, \pi, x),\, \sigma_w(x)). \tag{2.2.1}$$

The solution to (2.2.1) is an optimal portfolio of risky assets x_*. To establish the existence and uniqueness of x_*, the concept of a limiting slope is needed. In the $\mu - \sigma$ plane, the slope of any indifference curve of U is given by the marginal rate of substitution between risk and return

$$\mathrm{MRS}(\mu, \sigma) = -\frac{\frac{\partial U}{\partial \sigma}(\mu, \sigma)}{\frac{\partial U}{\partial \mu}(\mu, \sigma)},$$

which measures the risk aversion of the investor characterised by U. The limiting slope of the indifference curve starting in $(R_f w_0,\, 0)$ is defined as

$$\rho_U = \sup\big\{\mathrm{MRS}(\mu, \sigma) :\, (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ \; s.t.\; U(\mu, \sigma) = U(R_f w_0, 0)\big\}.$$

Intuitively, the limiting slope is the steepest slope of an indifference curve.

2.2.2 Two-fund separation theorem

The two-fund separation theorem dates back to Tobin [140] and Lintner [99]. It states that the solution to the optimisation problem (2.2.1) will be a mean-variance efficient portfolio. Recall to this end the classical mean-variance optimisation problem of maximising expected future wealth, given a prescribed level of risk σ. Formally, it is

given by

$$\max_{x \in \mathbb{R}^K} \mu_w(w_0, \pi, x) \, s.t. \, \sigma_w(x) \leq \sigma. \tag{2.2.2}$$

Following Markowitz [108], Böhm and Chiarella [24] formulate a unique portfolio that solves (2.2.2)

$$x_{\text{eff}}(\sigma, \pi, \Sigma) = \frac{\sigma}{\sqrt{\langle \pi, \Sigma^{-1}\pi \rangle}} \Sigma^{-1}\pi. \tag{2.2.3}$$

The portfolio (2.2.3) has the highest expected second-period wealth, given the upper bound σ for the standard deviation of second-period wealth. This is the reason why the portfolio $x_{\text{eff}}(\sigma, \pi, \Sigma)$ is referred to as (mean-variance) efficient. The efficient frontier in terms of wealth describes the loci of all efficient risk-return combinations. It is formally defined by a straight line

$$\sigma \mapsto \mu_w(w_0, \pi, x_{\text{eff}}(\sigma, \pi, \Sigma)) = R_f w_0 + \rho\sigma, \tag{2.2.4}$$

where the slope $\rho = \sqrt{\langle \pi, \Sigma^{-1}\pi \rangle}$ is referred to as the price of risk. The efficient frontier is used to formulate the two-fund separation theorem as follows.

Theorem 1. *Under the hypotheses of Assumption 1, let w_0 and $0 \neq \pi \in \mathbb{R}^K$ with $\sqrt{\langle \pi, \Sigma^{-1}\pi \rangle} < \rho_U$ be arbitrary. Then there exists a unique solution $x_* \in \mathbb{R}^K$ to the problem (2.2.1). The solution is the efficient portfolio*

$$x_* = x_{\text{eff}}(\sigma_*, \pi, \Sigma), \tag{2.2.5}$$

$$\sigma_* = \arg\max_{\sigma \geq 0} U(R_f w_0 + \sqrt{\langle \pi, \Sigma^{-1}\pi \rangle}\, \sigma, \sigma)$$

that determines the optimal level of risk the investor is willing to assume and

$$\mu_* = R_f w_0 + \sqrt{\langle \pi, \Sigma^{-1}\pi \rangle}\, \sigma_*$$

her expected level of future wealth.

Note that the optimal level of risk σ_* is equal to the standard deviation of the optimal portfolio (2.2.5), i.e., $\sigma_* = \sigma_w(x_*)$ and that the optimal risk-return characteristics (σ_*, μ_*) of the investor lies on the efficient frontier $\mu = R_f w_0 + \rho\sigma$. The situation of Theorem 1 is illustrated in Figure 2.2.2.

It will turn out to be useful subsequently to define the investor's willingness to assume risk by

$$Var(w_0, \rho) := \arg\max_{\sigma \geq 0} U(R_f w_0 + \rho\, \sigma, \sigma), \quad \rho \in [0, \rho_U), \tag{2.2.6}$$

so that optimal risk is $\sigma_* = Var(w_0, \rho)$ with the price of risk being $\rho = \sqrt{\langle \pi, \Sigma^{-1}\pi \rangle}$ as before.

2.2.3 CAPM equilibrium

Consider now an asset market with $i = 1, \ldots, I$ investors whose preferences over risk and return are all characterised by Assumption 1. Suppose that investors know the true probability distribution of the future payoffs, which are characterised by vector of mean payoff $\bar{q}$ and the covariance matrix Σ. Let $w_0^{(i)}$ be the initial wealth of investor i and $Var^{(i)}$ denote her willingness to assume risk, as defined in (2.2.6),

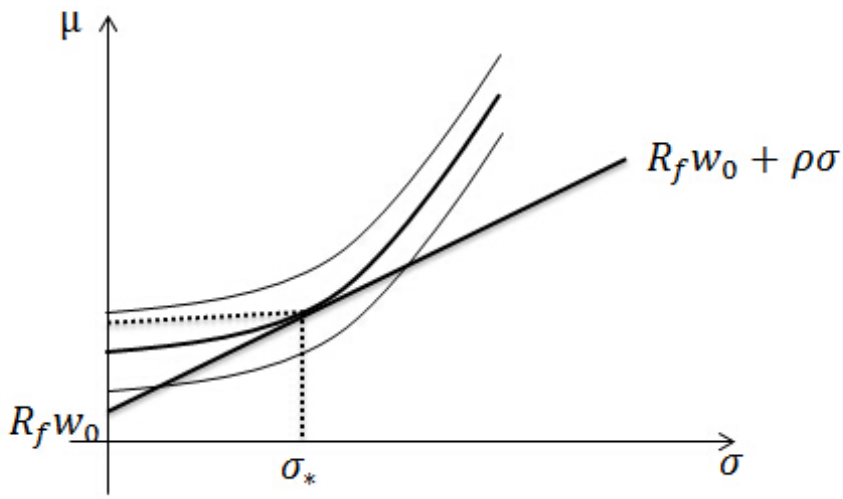


Figure 2.2.2: Offer curves.

derived from a utility function $U^{(i)}$ which obeys Assumption 1. A CAPM equilibrium may now be defined as follows.

Definition 1. A CAPM equilibrium consists of a price vector $p_* \in \mathbb{R}^K$ and a portfolio allocation $x_*^{(1)}, \ldots, x_*^{(I)} \in \mathbb{R}^K$ of risky assets such that the following holds:

(i) Each $x_*^{(i)}$, $i = 1, \ldots, I$ is individually optimal, i.e.it solves the utility maximisation problem

$$\max_{x \in \mathbb{R}^K} U^{(i)}(\mu_w(w_0^{(i)}, \pi_*, x),\, \sigma_w(x)),$$

where $\pi_* = \bar{q} - R_f p_*$ is the expected equilibrium excess return;

(ii) The allocation is feasible, i.e. it satisfies the market clearing condition for risky assets

$$\sum_{i=1}^{I} x_*^{(i)} = x_m.$$

Following Wenzelburger [150] we may now define aggregate willingness to assume the risk of all investors by the sum

$$\phi(\rho) = \sum_{t=1}^{I} \varphi^i(w_0^{(i)},\, \rho), \quad \rho \in [0, \bar{\rho}), \tag{2.2.7}$$

where $\bar{\rho} = \min\{\rho_{U^i} : i = 1, \ldots, I\}$ is the minimum of all limiting slopes $\rho_U^{(i)}$, with $\rho_U^{(i)}$ denoting the limiting slope of $U^{(i)}$. Then

$$\sigma_{\max} = \sup\{\phi(\rho) : \rho \in [0, \bar{\rho})\}$$

is the upper bound of risk that investors are collectively willing to accept. The upper bound $\sigma_{\max}$ may be finite or infinite. Setting $\sigma_m = \sqrt{\langle x_m, \Sigma\, x_m\rangle}$, this yields the following existence result, which is developed by Wenzelburger [150].

Theorem 2. *[Existence and Uniqueness of Equilibria] Assume that aggregate willingness to assume risk $\phi : [0, \bar{\rho}) \to \mathbb{R}_+$ is continuous with respect to ρ. Given $(\bar{q}, \Sigma)$ and endowments $w_0^1, \ldots, w_0^I$, the following holds:*

(i) For every $x_m \in \mathbb{R}_+^K$ with $0 < \sigma_m < \sigma_{\max}$ there exists a CAPM equilibrium with

$$p_* = \tfrac{1}{R_f}\left(E[q] - \tfrac{\rho_*}{\sigma_m}\Sigma\, x_m\right) \tag{2.2.8}$$

and

$$x_*^i = \frac{Var^i(w_0^i, \rho_*)}{\sigma_m} x_m, \quad i = 1, \ldots, I,$$

where $\rho_* \in [0, \bar{\rho})$ *is a solution to* $\phi(\rho) = \sigma_m$*;*

(ii) If ϕ *is strictly monotonically increasing, then the equilibrium is uniquely determined.*

Proof. See Appendix A. □

While Theorem 2 will be the workhorse of this chapter, it is convenient to have the following more amenable pricing formula for one particular risky asset k ($k = 1, \ldots, K$).

Corollary 1. *The equilibrium price of the* k*-th asset* ($k = 1, \ldots, K$) *takes the form*

$$p_{k*} = \frac{1}{R_f} \left(E[q_k] - \frac{\rho_*}{\sigma_m} Cov[q_k,\, e_m] \right), \tag{2.2.9}$$

where $e_m = \langle q, x_m \rangle$ *is the market payoff.*

Proof. In coordinate form, formula (2.2.8) reads

$$p_{k*} = \frac{1}{R_f} \left(E[q_k] - \frac{\rho_*}{\sigma_m} \sum_{l=1}^{K} \Sigma_{kl} x_{lm} \right) = \frac{1}{R_f} \left(E[q_k] - \frac{\rho_*}{\sigma_m} Cov[q_k,\, e_m] \right),$$

noting that $\Sigma_{kl} = Cov[q_k, q_l]$ and $e_m = \sum_{l=1}^{K} q_l x_{lm}$.

□

Corollary 1 may be used to evaluate any payoff e that can be replicated with a portfolio of one risk-free and K risky assets so that $e = \langle q, x_e \rangle + R_f y_e$. The

evaluation $\mathcal{V}(e)$ with respect to this financial structure is simply defined to be the price of the replicating portfolio (x_e, y_e). Using the fact the price of the risk-free bond was normalised to 1, Corollary 1 yields

$$\begin{aligned} \mathcal{V}(e) &= \langle p_*, x_e \rangle + y_e \\ &= \tfrac{1}{R_f}\left(E[e] - \tfrac{\rho_*}{\sigma_{\mathfrak{m}}} Cov[e, e_{\mathfrak{m}}]\right). \end{aligned}$$

It is well-known that the existence of CAPM equilibria requires that aggregate bond holdings in equilibrium may be negative. This implies that some investors may have to issue bonds. In our setting, the budget constraint for investor i is $w_0^i = y_*^i + \langle p_*,\, x_*^i \rangle$ so that aggregate bond holdings becomes

$$\sum_{i=1}^{I} y_*^i = \sum_{i=1}^{I} w_0^i - \langle p_*,\, x_{\mathfrak{m}} \rangle = \sum_{i=1}^{I} w_0^i - \mathcal{V}(e_{\mathfrak{m}}). \tag{2.2.10}$$

Thus, aggregate bond holdings amounts to the difference between aggregate initial wealth and the market value of the market payoff.

Example 1. Consider investors with linear mean-variance preferences. The utility function of investor i is $U^i(\mu, \sigma) = \mu - \frac{1}{2a^i}\sigma^2$, where $a^i > 0$ describes her risk tolerance. Her willingness to assume risk is $\varphi^i(w_0^i, \rho) = a^i \rho$. Using (2.2.7), aggregate willingness to assume risk becomes the linear function

$$\phi(\rho) = (a^1 + \cdots + a^I)\rho.$$

It follows from Theorem 2 that $\rho_* = \frac{\sigma_{\mathfrak{m}}}{a^1 + \cdots + a^I}$ so that the CAPM equilibrium takes

the form

$$p_{k*} = \frac{1}{R_f}\left(E[q_k] - \frac{1}{a^1+\cdots+a^I} Cov[q_k, e_{\mathfrak{m}}]\right), \quad k = 1, \ldots, K$$

with

$$x_*^i = \left(\frac{a^i}{a^1+\cdots+a^I}\right) x_{\mathfrak{m}}, \quad i = 1, \ldots, I.$$

The valuation of the market payoff becomes

$$\mathcal{V}(e_{\mathfrak{m}}) = \frac{1}{R_f}\left(E[e_{\mathfrak{m}}] - \frac{1}{a^1+\cdots+a^I} Var[e_{\mathfrak{m}}]\right)$$

and hence is independent of initial endowments w_0^i. Aggregate bond holdings (2.2.10) are thus negative, whenever aggregate wealth is less than the market value of the market payoff.

2.3 Innovations in the CAPM

Before studying CAT bonds in the CAPM, a generalised model with innovations will be discussed in this section. Subsection 2.3.1 gives the set up of the new market when one type of innovations is introduced. In this thesis, an INNOVATION is understood as a non-redundant asset. Then, in Subsection **??**, the conditions that the innovations can be issued into the market are discussed. This subsection gives the setting of a market with innovations following Wenzelburger and Koch-Medina [88].

2.3.1 Model setting

This subsection introduces the model setting when a type of innovations is introduced into a financial market based on the CAPM formulation in Section 2.2. The setup builds on Wenzelburger and Koch-Medina [88] but is simplified in assuming that investors' endowments are fully tradable.

In this context, an INNOVATION is introduced to an existing market in which one risk-free and K risky assets are traded. If q_0 denotes the stochastic payoff of such an innovation, the financial structure of the market is the augmented payoff vector $q^+ = (q_0, q_1, \ldots, q_K)$. We assume that the innovation alters the payoffs of neither the incumbent assets q_k, $k = 1, \ldots, K$, nor the risk-free return R_f. The vector of expected payoffs then becomes $\overline{q}^+ = (\bar{q}_0, \overline{q}_1, \ldots, \overline{q}_K)$ and the new covariance structure is represented by a symmetric $(K+1) \times (K+1)$ matrix

$$\Sigma^+ = \begin{pmatrix} \Sigma & b^\top \\ b & Var(q_0) \end{pmatrix}, \tag{2.3.1}$$

where $b = (b_1, \ldots, b_K)$ is the vector of the covariance of stocks and innovation with

$$b_k = Cov[q_0, q_k], \quad k = 1, \ldots, K,$$

represents the covariance of stock q_k and the innovation q_0.

Since by assumption, the payoff of the innovation cannot be replicated with a portfolio of the K incumbent assets, the new covariance matrix Σ^+ is positive definite, so that its inverse is well defined. Hence, with the innovation added to the market, the financial structure changes to $(\overline{q}^+, \Sigma^+)$.

The innovation is assumed to be in positive net-supply, $x_0 > 0$, so that its total payoff becomes $Q_0 = x_0 \cdot q_0$. The new market portfolio is $x_m^+ = (x_0, x_m) \in \mathbb{R}^{K+1}$, the new market payoff becomes $e_m^+ = e_m + Q_0$, and the market risk is $\sigma_m^+ = \sqrt{Var[e_m^+]}$.

Under this environment, the conditions on if innovations can be issued into the financial market are discussed in the following subsection.

2.3.2 The trading condition

This subsection discusses the conditions that enable innovations to be traded in the market, i.e. investors are willing to purchase the innovations. Investors will trade innovations only if they are not worse off by including these innovations in their portfolios. So the trading condition of innovations requires that at least one investor is better off by including innovations into their portfolios. Before discussing if the innovations are preferred by investors, a new concept - Pareto-superiority - is defined in the CAPM environment to describe the situation that one portfolio is preferred by the other one in terms of risk-return allocations.

Definition 2. Pareto-superiority

An allocation of risk and return $\left\{(\mu_1^{(1)}, \sigma_1^{(1)}), \cdots, (\mu_1^{(I)}, \sigma_1^{(I)})\right\}$ is Pareto-superior to an allocation of risk and return $\left\{(\mu_2^{(1)}, \sigma_2^{(1)}), \cdots, (\mu_2^{(I)}, \sigma_2^{(I)})\right\}$ if

$$U^{(i)}(\mu_1^{(i)}, \sigma_1^{(i)}) \geq U^{(i)}(\mu_2^{(i)}, \sigma_2^{(i)}) \quad \text{for all} \quad i = 1, \ldots, I$$

with at least one inequality being strict.

Definition 2 is the standard definition of Pareto-superiority in microeconomics. If the risk allocation with innovations is Pareto-superior to the allocation without

innovations, then at least one investor will purchase the innovations. It implies innovations are preferred and hence will be traded in the financial market. By the model setting, investors are holding homogenous beliefs, so they have the same expected mean and variance on innovation' payoffs. In this case, if one investor is willing to purchase innovations to gain a higher utility, other investors must be willing to purchase them too. This can be explained by the Two-fund separation theorem 1, at equilibrium, all investors will hold a portion of the market portfolio, which contains all risky assets in the market. Therefore a new efficient frontier will be formed. Mathematically, it can be expressed as,

$$\mu_w^+ = R_f w_0 + \rho^+ \sigma^+,$$

with new slope (market price of risk) defined as

$$\rho^+ = \sqrt{\langle \pi^+, (\Sigma^+)^{-1} \pi^+ \rangle}.$$

Therein,$\pi^+ = (\pi, \tau) := \overline{q}^+ - R_f p^+$ is the vector of excess returns of the market augmented by the innovation, $p^+ = (p_0, p_1^+, \ldots, p_K^+)$ is the price vector of risky assets at equilibrium, and p_0 denotes the price of innovations. Figures 2.3.1 and 2.3.2 show that if the slope of the new efficient frontier in the market with innovations is higher than in the original market without innovations, investors will end with higher utility as Proposition 1 states.

However, whether or not an investor will use the innovation to decrease the risk of her future wealth will depend on her preferences. This observation is illustrated in

Figure 2.3.1: Aggregate willingness to consume risk is increasing with market price of risk

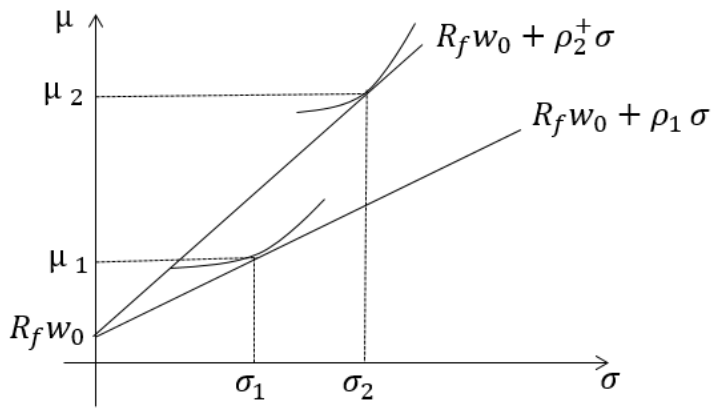


Figure 2.3.2: Aggregate willingness to consume risk is decreasing with market price of risk

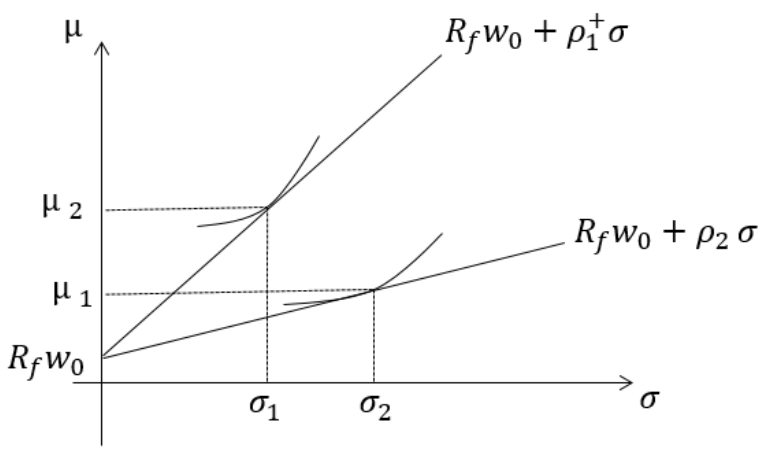


Figure 2.3.1. An investor whose willingness to assume risk is increasing in the price of risk, will use an innovation to invest into a riskier portfolio, thereby increasing her exposure to risk. An investor whose willingness to assume risk is decreasing in the price of risk will use the innovation to reduce the risk of her asset holdings. The standard case with linear mean-variance preferences $U(\mu, \sigma) = \mu - \frac{1}{2a}\sigma^2$, where a denotes the risk tolerance, is illustrated in Figure 2.3.1. The willingness to assume risk is proportional to the price of risk, $\varphi(w_0, \rho) = a\rho$, and therefore increasing in the price of risk. The case with a decreasing risk exposure is illustrated in Figure 2.3.2 and obtained for $U(\mu, \sigma) = -e^{-a\mu} - \sigma$. For more discussion see Wenzelburger [150].

Proposition 1. *If the price of risk of the market with the innovation, ρ^+, is higher than in the market without the innovation, ρ, the optimal portfolio with innovations is preferred to the portfolio without innovations. Mathematically, it is*

$$U(\mu_*^+, \sigma_*^+) > U(\mu_*, \sigma_*),$$

where the inequality is strict whenever $\rho^+ > \rho$. And vice versa.

Proof. If $\rho^+ \geq \rho$, the slope of the efficient frontier for the market with the innovation is steeper than that the slope of the efficient frontier without the innovation. The two-fund separation theorem now implies that the utility level $U(\mu_\star^+, \sigma_\star^+)$ the investor will attain when including the innovation into his portfolio will ceteris paribus be as least as high as the level $U(\mu_\star, \sigma_\star)$ of the optimal portfolio without the innovation.

□

Proposition 1 also implies that if the market price of risk ρ^+ in the market with innovations is equal to the original ρ, that is $\rho^+ = \rho$, then there is no difference for the investor to hold innovations in his portfolio or not. So there is a lack of motivation for this investor to trade innovations in the market. Therefore, if all investors are in the same situation, then the demand of innovations will be equivalent to zero.

Based on CAPM assumptions, Definition 2 and Proposition 1, a relation between the market price of risk and Pareto superiority can be generated in the following Proposition.

Corollary 2. *In the standard CAPM, if $\rho^+ > \rho$, the risk-return allocation in the new market with innovations is Pareto-superior to the risk-return allocation in the original market and vise versa.*

Proof. Based on the CAPM assumptions, all investors hold the same beliefs on the future payoffs. By the two fund-separation theorem 1, at equilibrium, every investor will hold a portion of market portfolio that contains all of risky assets in the market. In other words, every investor will hold a mixture portfolio along the capital market line (CML). If the slop of CML in the new market $\rho+$ is higher than the original one ρ , then all investors have a higher utility therefore, according to the Definition 2, the risk-return allocation in the new market with innovations is Pareto-superior to the risk-return allocation in the original market.

□

If Proposition 2 is satisfied, the innovations are preferred by the investors and hence traded in the market.

The following Proposition shows the mathematical condition to ensure $\rho^+ > \rho$.

Proposition 2. *Let* $\pi^+ = (\pi, \tau)$ *and* $\Sigma^+ = \begin{pmatrix} \Sigma & b^\top \\ b & Var(q_0) \end{pmatrix}$ *be given, where* Σ^+ *is assumed to be positive definite. Then the price of risk of the market with the innovation,* ρ^+*, is higher than in the market without the innovation,* ρ*. In particular,*

$$(\rho^+)^2 = \rho^2 + \frac{(\tau - \langle b, \Sigma^{-1}\pi \rangle)^2}{r_c - \langle b, \Sigma^{-1} b \rangle} \geq \rho^2,$$

where the inequality is strict whenever $\tau \neq \langle b, \Sigma^{-1}\pi \rangle$*.*

Proof. See Appendix A. □

Proposition 1 or Corollary 2 shows the condition to identify if the innovations are preferred by investors. If innovations are preferred by investors, innovations will be traded in the financial market.

2.3.3 CAPM Equilibrium

The above subsection has discussed situations when investors are willing to purchase innovations. When the innovations are traded in the market, this subsection will investigate the existence and of unique market equilibrium with innovations. In addition, the equilibrium price of innovations is given and the original risky asset prices changes is discussed.

By the Theorem 2, the existence and uniqueness of CAPM equilibria with innovations is now straightforward.

Proposition 3. *Let all hypothesis of Theorem 2 be satisfied and assume, in addition, that* $\sigma_m < \sigma_m^+ < \sigma_{\max}$ *and that aggregate willingness to assume risk* $\phi(\rho)$ *is strictly increasing in* ρ*. Then there exists a uniquely determined CAPM equilibrium with*

$$p_*^+ = \frac{1}{R_f}\left(\bar{q}^+ - \frac{\rho_*^+}{\sigma_m^+}\Sigma^+ x_m^+\right)$$

and

$$x_*^{(i)+} = \frac{Var^{(i)}(w_0^{(i)}, \rho_*^+)}{\sigma_m^+} x_m^+, \quad i = 1, \ldots, I,$$

where $\rho_*^+ \in [0, \bar{\rho})$ *solves* $\phi(\rho_*^+) = \sigma_m^+$.

The proof of Theorem 3 is immediate from Theorem 2. Following this, the proposition below shows the relation of the risk of the portfolio and the market price of risk.

Lemma 1. *Assume that in the standard CAPM, investors' aggregate willingness to assume risk* $\phi(\rho)$ *is strictly increasing in* ρ*, if the risk of the portfolio with innovations* σ_*^+*, is larger than the risk of the portfolio without innovations,* σ_**, the market price of risk with innovations,* ρ^+*, is higher than the market price of risk without innovations,* ρ*, and vise versa.*

Proof. Under the standard CAPM setting, the assumption that investors' aggregate willingness to assume risk $\phi(\rho)$ is strictly increasing in ρ ensures that the increasing of aggregate consumed risk $\sigma = \phi(\rho)$ leads to the increasing of ρ. So $\sigma_*^+ > \sigma_*$ leads to $\rho^+ > \rho$.

□

By the Lemma 1 and Corollary 2, the following corollary can be generated on the relation of portfolio risk changes and Pareto-superiority.

Lemma 2. *Assume that in the standard CAPM, investors' aggregate willingness to assume risk $\phi(\rho)$ is strictly increasing in ρ, if the risk of the portfolio with innovations, σ_*^+, is larger than the risk of the portfolio without innovations, σ_*, then the risk-return allocation in the new market with innovations is Pareto-superior to the risk-return allocation in the original market without innovations.*

Proof. This can be achieved directly by Corollary 2 and Lemma 1.

□

This Lemma will be used in the next subsection to generate innovations issuing conditions.

Using the market payoff of the market with the innovation, $e_{\mathfrak{m}}^+ = e_{\mathfrak{m}} + Q_0$, Proposition 3, yields the equilibrium price p_{0*}. This result is stated in the following corollary.

Corollary 3. *(i) The equilibrium price of the innovation is p_{0*}*

$$p_{0*} = \frac{1}{R_f}\left(E[q_0] - \frac{\rho_*^+}{\sigma_m^+} Cov[q_0, e_m^+]\right);$$

(ii) The equilibrium price of the k-th incumbent asset $(k = 1, \ldots, K)$ is

$$p_k^+ = \frac{1}{R_f}\left(E[q_k] - \frac{\rho_*^+}{\sigma_m^+} Cov[q_k, e_m^+]\right).$$

Corollary 3 reveals that the CAPM equilibrium price of the innovation obeys the usual principle. It is obtained by subtracting a risk premium from the discounted expected payoff, where the premium depends on the payoff's correlation with the market payoff, $e_{\mathfrak{m}}^{+}$, and the market price of risk ρ_{*}^{+}.

Corollary 3 also shows that the introduction of an innovation has two effects on the CAPM equilibrium prices of the incumbent assets. The first and most obvious effect is a price change which is due to a change in the correlation structure of the financial market from $Cov[q_k, e_{\mathfrak{m}}]$ to $Cov[q_k, e_{\mathfrak{m}}^{+}]$. The second effect is caused by the risk-taking behaviour of the investors which changes the ratio $\rho_{*}/\sigma_{\mathfrak{m}}$ to $\rho_{*}^{+}/\sigma_{\mathfrak{m}}^{+}$. The interplay between these two effects determines whether the price of an incumbent security rises or falls after the introduction of the innovation.

Lemma 3. *(i)The price change of the k-th risky asset $(k = 1, \ldots, K)$ is*

$$p_{k*}^{+} - p_{k*} = \frac{1}{R_f}\left(\left(\frac{\rho_*}{\sigma_m} - \frac{\rho_*^{+}}{\sigma_m^{+}}\right) Cov[q_k, e_m] - \frac{\rho_*^{+}}{\sigma_m^{+}} Cov[q_k, Q_0]\right),$$

where $e_m = \langle q, x_m \rangle$ is the market payoff of the incumbent assets;

(ii)Aggregate bond holdings of investors take the form

$$\sum_{i=1}^{I} y_*^{(i)+} = \sum_{i=1}^{I} y_*^{(i)} + \mathcal{V}(e_m) - \mathcal{V}^{+}(e_m^{+}).$$

Proof. Lemma (i) follows from Corollaries 1 and 3, noting that $e_m^{+} = e_m + Q_0$. Lemma (ii) follows from formula (2.2.10).

□

The equilibrium allocation of risky assets changes because of the price change and the fact that investors will allocate funds to the innovation. Lemma 3 (ii) shows that the change in aggregate bond holdings amounts to the change in the valuation of the market payoff. That is, aggregate bond holdings are reduced whenever the value, $\mathcal{V}^+(e_m^+)$, of the new market payoff, e_m^+, at new prices is higher than the value, $\mathcal{V}(e_m)$, of the old market payoff, e_m, at the old prices. In such a situation the innovations are financed by a reduction in bond holdings and risky asset holdings if the all risky asset price changes in Lemma 3 (i) are not zero. If the price changes are zero, the innovations are financed by the reduction in bond holdings. The following example explains the above discussions in a special case.

Example 2. For linear mean-variance preferences as in Example 1, one has

$$\begin{aligned}\mathcal{V}^+(e_m^+) - \mathcal{V}(e_m) &= \tfrac{1}{R_f}\left(E[Q_0] - \tfrac{1}{a^1+\cdots+a^I} Cov[Q_0, e_m^+]\right) \\ &= p_0 x_{0m}.\end{aligned}$$

The investment into the innovation is thus completely financed by a reduction of the investment in risk-free bonds.

The total amount of funds, $p_0 x_{0m}$, invested in the innovation is determined by the price of the innovation given in formula (i) of Corollary 3. The resulting valuation formula follows from the linearity of the pricing formula and is stated in the following corollary.

Corollary 4. *The total amount of funds $p_0 x_{0m}$ invested into the innovation is*

$$\mathcal{V}^+(Q_0) = \frac{1}{R_f}\left(E[Q_0] - \frac{\rho_*^+}{\sigma_m^+} Cov[Q_0, e_m^+]\right).$$

Observe from Corollary 4 that positive equilibrium prices with innovations obtain, whenever the covariance of the innovation with the market, $Cov[Q_0, e_{\mathbf{m}}^+]$, is either negative or, if positive, sufficiently small. This Corollary will be used in the next subsection to derive positive asset prices.

2.3.4 Innovation issuance

After the discussion of the trading conditions of new assets in the CAPM and the situation of the CAPM equilibrium with innovations, this subsection discusses the conditions under which the innovations can be issued into the financial market. The Subsection 2.3.2 has discussed the condition that make innovations tradable in the financial market. However, that condition can not guarantee innovations can be traded successfully in the market in the CAPM. It is well known that the properties of CAPM equilibrium allow for arbitrage opportunities. It implies the situation that risky asset (including innovations) prices are not positive may exist.

The following example shows that arbitrage opportunities exist in the CAPM following the method of Philip and Jonathan [66].

Example 3. [Arbitrage opportunity] This example takes the valuation formula of innovations in Corollary 4 as an example to derive arbitrage opportunities. The valuation formula can be rearranged as

$$
\begin{aligned}
\mathcal{V}(Q_0) &= \tfrac{1}{R_f}\left[E[(Q_0)] - \tfrac{\rho_*^+}{\sigma_m^+} Cov[Q_0],\, e_m^+\right] \\
&= \tfrac{1}{R_f}\left(E[Q_0] - \tfrac{\rho_*^+}{\sigma_m^+}\left(E[Q_0 \cdot e_m^+] - E[Q_0]E[e_m^+]\right)\right) \\
&= \tfrac{1}{R_f}\left(E[Q_0] - \tfrac{\rho_*^+}{\sigma_m^+}\left(E[Q_0 \cdot e_m^+] - E[Q_0]E[e_m^+]\right)\right) \\
&= \tfrac{1}{R_f} E\left(Q_0 - \tfrac{\rho_*^+}{\sigma_m^+}\left(Q_0 \cdot e_m^+ - Q_0 \cdot E[e_m^+]\right)\right) \\
&= E\left[Q_0\left(1 - \tfrac{\rho_*^+}{\sigma_m^+}\left(e_m - E[e_m^+]\right)\right)\right]/R_f, \qquad (2.3.2)
\end{aligned}
$$

which is the same form as formula (5)[1] in Philip and Jonathan [66]. According to Philip and Jonathan [66], an arbitrage opportunity can be found based on formula (2.3.2). For example, if $e_m^+ > E[e_m^+] + \frac{\sigma_m^+}{\rho_*^+}$ or $prob(e_m^+ > E[e_m^+] + \frac{\sigma_m^+}{\rho_*^+}) > 0$ and let the payoff $Q_0 = \left[\frac{\rho_*^+}{\sigma_m^+}(e_m^+ - E[e_m^+]) - 1\right]^{-1} > 0$, then the valuation of Q_0 that is $\mathcal{V}(Q_0)$ or in other words $p_0 x_{0m}$ will be $-1 < 0$, which creates an arbitrage opportunity. Following Theorem 1 in Philip and Jonathan [66], if the market portfolio generates sufficiently large returns in some states, that is, $prob(e_m^+ > E[e_m^+] + \frac{\sigma_m^+}{\rho_*^+}) > 0$, there exists an arbitrage opportunity.

This example shows that arbitrage opportunities exist. It also shows the price of innovations can be negative. In this case, this innovation cannot rise positive money. It also has no meaning economically. Therefore, to ensure innovations have positive prices is essential. According this example, any risky asset price can be negative. So another condition that ensures innovations can be issued and traded in the market successfully is to set all risky assets prices are non-negative.

[1] See Philip and Jonathan [66] p. 237.

Using Corollary 3, however, it is not difficult to see that asset prices are positive whenever the equilibrium price of risk ρ_*^+ satisfies

$$E[e] > \frac{\rho_*^+}{\sigma_{\mathfrak{m}}^+} Cov[e, e_{\mathfrak{m}}^+] \tag{2.3.3}$$

for all payoffs $e = q_0, q_1, \ldots, q_K$. Condition (2.3.3) places an upper bound on the equilibrium price of risk. Following Levy [97], define the essential risk of of a payoff e by

$$\frac{Cov[e, e_{\mathfrak{m}}^+]}{E[e]}$$

and the maximum price of risk by

$$\rho_{\max} := \frac{\sigma_{\mathfrak{m}}^+}{\max\left\{\frac{Cov[q_1, e_{\mathfrak{m}}^+]}{E[q_1]}, \ldots, \frac{Cov[q_K, e_{\mathfrak{m}}^+]}{E[q_K]}, \frac{Cov[q_0, e_{\mathfrak{m}}^+]}{E[q_0]}\right\}}. \tag{2.3.4}$$

It then follows from (2.3.3) that all asset prices are positive, whenever $\rho_*^+ < \rho_{\max}$. The upper bound $\rho_{\max}$ is determined by the asset with the highest essential risk.[2] These findings yield the following condition for the positivity of equilibrium asset prices, see also Wenzelburger and Koch-Medina [88].

Lemma 4. *Suppose that aggregate willingness to assume risk ϕ is increasing in ρ. Then equilibrium asset prices of the CAPM with the innovation are positive, if and only if*

$$\phi(\rho_{\max}) > \sigma_{\mathfrak{m}}^+,$$

[2] A reasonable approximation for the upper bound is the asset with the highest coefficient of variation $\frac{\sqrt{Var[q]}}{E[q]}$ of those assets that correlate positively with the market.

where $\rho_{\max} := \frac{\sigma_{\mathfrak{m}}^{+}}{\max\left\{\frac{Cov[q_1, e_{\mathfrak{m}}^{+}]}{E[q_1]}, \ldots, \frac{Cov[q_K, e_{\mathfrak{m}}^{+}]}{E[q_K]}, \frac{Cov[q_0, e_{\mathfrak{m}}^{+}]}{E[q_0]}\right\}}$.

Proof. The price of the k-th asset is

$$p_{k*}^{+} = \frac{1}{R_f}\left(E[q_k] - \frac{\rho_*^{+}}{\sigma_{\mathfrak{m}}^{+}} Cov[q_k,\, e_{\mathfrak{m}}^{+}]\right)$$

Hence $p_{k*}^{+} > 0$ if and only if

$$E[q_k] > \frac{\rho_*^{+}}{\sigma_{\mathfrak{m}}^{+}} Cov[q_k,\, e_{\mathfrak{m}}^{+}] = \rho_*^{+} Cor[q_k, e_{\mathfrak{m}}^{+}]\sqrt{Var[q_k]}. \tag{2.3.5}$$

Equation (2.3.5) holds automatically if $Cor[q_k, e_{\mathfrak{m}}^{+}] \leq 0$. If $Cor[q_k, e_{\mathfrak{m}}^{+}] > 0$, (2.3.5) is satisfied if $\frac{E[q_k]}{\sqrt{Var[q_k]}Cor[q_k, e_{\mathfrak{m}}^{+}]} > \rho_*^{+}$. An analogous argument applies for the innovation. Since in equilibrium $\phi(\rho_*^{+}) = \sigma_{\mathfrak{m}}^{+}$, the assertion now follows from the monotonicity of ϕ.

□

Lemma 4 states that asset prices remain positive, whenever investors are willing to assume the new market risk $\sigma_{\mathfrak{m}}^{+}$ for a sufficiently low price of risk. Whether or not an innovation can be introduced into the market without creating an obvious arbitrage opportunity will thus depend on investors' preferences over risk and return.

Therefore according to the Lemma 2 and Lemma 4, the following theorem is generated to define the conditions of innovations issuance.

Theorem 3. *[Issuance conditions]Under the hypotheses of Proposition 3, let*

$$\overline{q}^{+} = (\overline{q_0}, \overline{q}) \quad \textit{and} \quad \Sigma^{+} = \begin{pmatrix} \Sigma & b^{\top} \\ b & Var(q_0) \end{pmatrix}$$

with

$$b = \big(Cov[q_0, q_1], \cdots, Cov[q_0, q_K]\big)$$

be given. Assume, in addition, that the following holds:

(i) $Var[Q_0] > -2Cov[e_{\mathfrak{m}}, Q_0]$;

(ii) Aggregate willingness to assume risk $\phi(\rho)$ is increasing in ρ.

(iii) $\phi(\rho_{\max}) > \sigma_{\mathfrak{m}}^{+}$ with $\rho_{\max} := \frac{\sigma_{\mathfrak{m}}^{+}}{\max\left\{\frac{Cov[q_1, e_{\mathfrak{m}}^{+}]}{E[q_1]}, \ldots, \frac{Cov[q_K, e_{\mathfrak{m}}^{+}]}{E[q_K]}, \frac{Cov[q_0, e_{\mathfrak{m}}^{+}]}{E[q_0]}\right\}}$.

Proof. Condition (i) Market risk with innovations is

$$(\sigma_{\mathfrak{m}}^{+})^2 = Cov[e_{\mathfrak{m}} + Q_0, e_{\mathfrak{m}} + Q_0] = \sigma_{\mathfrak{m}}^2 + 2Cov[e_{\mathfrak{m}}, Q_0] + Var[Q_0].$$

So if, and only if, $Var[Q_0] > -2Cov[e_{\mathfrak{m}}, Q_0]$, the market risk with innovations is larger than the market risk without innovations.

□

Condition (i) ensures market risk in the market with the innovation is larger than market risk in the market without the innovation, i.e., $\sigma_{\mathfrak{m}}^{+} > \sigma_{\mathfrak{m}}$. Condition (ii) assumes aggregate willingness $\phi(\rho)$ is increasing in the price of risk, so based on condition (i), the equilibrium price of risk ρ_{*}^{+} in the market with the innovation is higher

than the equilibrium price of risk ρ_* in the market without it.[3] Based on Lemma 2, the CAPM equilibrium with the innovation, pertaining to $(\overline{q}^+, \Sigma^+)$ is Pareto superior to the CAPM equilibrium without the innovation, pertaining to $(\overline{q}, \Sigma)$. All investors are willing to trade and include the innovations in their portfolios. Condition (iii) ensures this model is economically meaningful by assuming all asset prices are positive based on Lemma 4.

2.4 CAT bonds in the CAPM

This section is going to use the framework of the CAPM with innovations that is developed from Section 2.3 to design CAT bonds. In this thesis, a CAT bond is an insurance-linked asset designed in the spirit of Canabarro et al. [37].

As Figure 2.4.1 illustrates, this bond is issued by a new agent in period 0, also referred to as a sponsor. This sponsor may be thought of as an insurance or reinsurance company that wants to raise funds from the financial markets in order to cover the costs of certain insurance claims. These claims are triggered by well-specified natural catastrophes which may occur within the time span of period 0 and period 1. The accumulated claims are determined by a random variable L and capped at the maximum coverage L_{max}. The CAT bond thus specifies an insurance contract between the sponsor and those financial investors who buy the bond. In order to administrate the bonds and isolate its financial obligations from all other

[3] It is intuitively clear that $\phi(\rho)$ is increasing in ρ whenever sufficiently many investors increase risk with ρ. Indeed, it is well known that $\varphi^i(w_0^i, \rho)$ is increasing in ρ, if the marginal rate of substitution between risk and return $\mathrm{MRS}^i(\mu, \sigma)$ satisfies

$$\frac{\partial \mathrm{MRS}^i}{\partial \mu}(\mu, \sigma)\sigma < 1 \quad \text{for all } \sigma > 0,$$

e.g., see Hens et al. [77] or Wenzelburger [150].

obligations of the company, the sponsor sets up a so-called Special Purpose Vehicle (SPV) to carry the work to issue CAT bonds.

In order to introduce the general design of a CAT bond, the following assumption is adopted.

Assumption 2. *The accumulated insurance claims arising from catastrophic events between period 0 and 1 are described by a random variable L with the following properties:*

(i)The probability density function of L has compact support, given by $[0, L_{max}]$;

(ii)The claim L does not lie in the marketed space $\mathcal{M}$;

(iii)The covariance of L with the market payoff $e_{\mathfrak{m}}$ satisfies

$$Cov[L, e_{\mathfrak{m}}] < Var[L].$$

The first assumption is adopted for notational convenience. As upper bounds for claims are a priori unknown, the upper limit L_{max} is typically set by the designer of the catastrophe bond. The second one implies that the insurance claims cannot be replicated with financial assets of the financial market. It is plausible to assume that the overall effect of claims that derive from natural catastrophes on financial markets is negative. The third assumption is slightly more general in assuming that the covariance of the insurance claims with the financial market is not larger that its variance.

Assume x_{cm} ($x_{cm} > 0$) CAT bonds are issued into the financial market, with price p_c and payoff q_c. The total payoff of CAT bonds is $Q_c = q_c x_{cm}$ in period 1. This

Figure 2.4.1: Cash flows of the SPV.

	Sponsor	SPV	Investors
t=0	$\theta \longrightarrow$	$\theta + \mathcal{V}^+(Q_c)$	$\longleftarrow \mathcal{V}^+(Q_c)$
t=1	$L \longleftarrow$	$(\theta + \mathcal{V}^+(Q_c))R_f$	$\longrightarrow Q_c$

payoff is determined as follows. If no specified catastrophic event occurs that triggers insurance claims, the total promised payoff C to financial investors consists of the principal $\mathcal{V}^+(Q_c)$ obtained from financial investors with a coupon r_c so that

$$C = \mathcal{V}^+(Q_c)(1 + r_c). \tag{2.4.1}$$

If catastrophic events occur as specified, the funds of financial investors will be used to cover the loss of the sponsor up to the maximum coverage L_{max}. Accumulated claims L will be subtracted from the promise (2.4.1) so that the total payoff to financial investors in period 1 becomes

$$Q_c = C - L. \tag{2.4.2}$$

A CAT bond is thus a defaultable bond with payoff (2.4.2), where the default is defined by catastrophic events embodied in the random variable L.

Bondholders will not be asked to pledge additional capital in the second period and hence will at most lose their principal. Since $0 \leq L \leq L_{max}$, this is guaranteed whenever the promised repayment C is larger than the maximum loss L_{max}, that is $C \geq L_{max}$, which is assumed for the remainder of this chapter.

With the introduction of Q_c, the market payoff of the financial market changes to

$$e_m^+ = e_m + Q_c$$

so that market risk becomes $\sigma_m^+ = \sqrt{Var[e_m^+]}$.

The covariance structure of the financial market which is augmented by the catastrophe bond as the innovation is then represented by a symmetric $(K+1)\times(K+1)$ matrix as

$$\Sigma^+ = \begin{pmatrix} \Sigma & b^\top \\ b & Var(q_c) \end{pmatrix}, \tag{2.4.3}$$

where $b = (b_1, \ldots, b_K)$ with

$$b_k = Cov[q_k, q_c], \quad k = 1, \ldots, K.$$

Since by Assumption 2 (ii) the insurance claim L cannot be replicated with financial assets, the new covariance matrix Σ^+ is positive definite and hence invertible.

2.4.1 Promised payoff and premium

This subsection discusses how the SPV decides the promised payoff C of CAT bonds to investors and the premium θ that the re-insurance company passes to the SPV in order to have coverage at time 1.

As introduced above, the SPV issues CAT bonds to raise funds and cover losses for sponsors up to L_{max} at time 1. Applying Corollary 4, the amount of funds received from the financial market for Q_c is computed as

$$\mathcal{V}^+(Q_c) = \tfrac{1}{R_f}\left(E[Q_c] - \tfrac{\rho_*^+}{\sigma_m^+} Cov[Q_c, e_m^+]\right), \tag{2.4.4}$$

where ρ_*^+ is the equilibrium market price of risk, determined in the market for risk

by equating demand and supply for risk, that is, by $\phi(\rho_*^+) = \sigma_m^+$. Inserting (2.4.2), $\mathcal{V}^+(Q_c)$ is

$$\begin{aligned} \mathcal{V}^+(Q_c) &= \tfrac{1}{R_f}\left(C - E[L] + \tfrac{\rho_*^+}{\sigma_m^+} Cov[L, e_m^+]\right) \\ &= \tfrac{C}{R_f} - \mathcal{V}^+(L), \end{aligned} \tag{2.4.5}$$

with

$$\mathcal{V}^+(L) = \tfrac{1}{R_f}\left(E[L] - \tfrac{\rho_*^+}{\sigma_m^+} Cov[L, e_m^+]\right). \tag{2.4.6}$$

Eq. (2.4.5) shows that the funds of the SPV obtained from financial investors will be equal to the discounted promise C/R_f minus the market valuation (2.4.6) of the aggregate loss L. It shows that how much funds the SPV can raise is mainly determined by the promised payoff C and investors' valuation of loss L. To avoid any none meaningful situations, assume the market valuation of the loss $\mathcal{V}^+(L)$ is non-negative. It is obvious from (2.4.5) that the total amount of funds raised in the financial market is positive if, and only if, the promised payoff is $C > R_f \mathcal{V}^+(Q_c)$.

If the SPV wants to have L_{max} at time 0 by issuing CAT bonds. That is

$$\begin{aligned} \mathcal{V}^+(Q_c) &\geq L_{max} \\ \tfrac{C}{R_f} - \mathcal{V}^+(L) &\geq L_{max} \\ C &\geq R_f(L_{max} + \mathcal{V}^+(L)). \end{aligned}$$

It shows the promised payoff should be at least larger than the sum of L_{max} and

$R_f\mathcal{V}^+(L)$. The minimum promised payoff is $C = R_f(L_{max} + \mathcal{V}^+(L))$ if the SPV wants to have fund of L_{amx} at time 1. After receiving the promised payoff, the CAT bond's price can be defined as below.

Lemma 5. *If under the hypotheses of Theorem 4, $C = R_f\big(L_{max} + \mathcal{V}^+(L)\big)$, then the price per unit of the CAT bond is*

$$p_0 = \tfrac{L_{max}}{x_{cm}},$$

where x_{cm} is the total amount of bonds issued.

Proof. Observe from (2.4.6) that $\mathcal{V}^+(L)$ is independent of the promised repayment C. This is so because market risk is $\sigma_{\mathfrak{m}}^+ = \sqrt{Var[e_{\mathfrak{m}} - L]}$ and hence independent of C so that the market price of risk ρ_*^+ is independent of C. Moreover, the covariance of the loss with the financial market augmented by the catastrophe bond $Cov[e_{\mathfrak{m}}^+, L]$ is independent of P as well. Setting $P = R_f\big(h + \mathcal{V}^+(L)\big)$, the catastrophe-bond price per unit take the form

$$p_{0*} = \tfrac{\mathcal{V}^+(Q_c)}{x_{cm}} = \tfrac{L_{max}}{x_{cm}}.$$

□

As illustrated in Table 2.4.1, in period 0, the SPV receives the insurance premium θ from the sponsor and funds $\mathcal{V}^+(Q_c)$ from the financial market as given in (2.4.4). The SPV then invests all of its funds into the risk-free asset. Its proceeds in period 1 are then $R_f(\mathcal{V}^+(Q_c) + \theta)$. These funds must be used to repay financial investors and to cover all claims from the sponsor up to the amount of L_{max}. The profit made

by the SPV computes as

$$\begin{aligned}\Pi_{\text{SPV}} &= R_f(\mathcal{V}^+(Q_c)+\theta)-(Q_c+L)\\ &= R_f\left(\tfrac{C}{R_f}-\mathcal{V}^+(L)+\theta\right)-C\\ &= R_f(\theta-\mathcal{V}^+(L)). \qquad (2.4.7)\end{aligned}$$

As seen from (2.4.7), this profit is deterministic and determined by the difference between the insurance premium θ and the market valuation of the hazard risk $\mathcal{V}^+(L)$. To ensure that the SPV has non-negative profit, $\Pi_{\text{SPV}} \geq 0$, the insurance premium θ must be higher than the market valuation $\mathcal{V}^+(L)$.

Observe from (2.4.6) that $\mathcal{V}^+(L)$ is independent of the promised repayment C. This is so because market risk is $\sigma_m^+ = \sqrt{Var[e_m - L]}$ and hence independent of C so that the market price of risk ρ_*^+ is independent of C. Moreover, the covariance of the loss with the financial market augmented by the catastrophe bond $Cov[e_m^+, L]$ isalso independent of C . As a consequence, the premium θ and the profit of the SPV (2.4.7) are independent of the promised repayment C so that the SPV will be able to cover the insurance claim L whenever $\theta \geq \mathcal{V}^+(L)$. A sponsor who sets up the SPV to finance its own hazard risk will set $\theta = \mathcal{V}^+(L)$ because higher premia would not increase the overall profit of the sponsor.[4]

2.4.2 CAT bond issuance

This subsection discusses how the insurance risk can be securitised in financial

[4]The sponsor could use the SPV to finance hazard risks of other insurance companies. Charging $\theta > \mathcal{V}^+(L)$ would then be profitable. This case, however, is more complex and left for future research.

products and then is transferred from a (re-)insurance company to the financial market based on the ideas of Aase [1].

For the insurance risk to be transferred, the first requirement is that one (re-)insurance company has the incentive to transfer it to the financial market. To address this condition, a standard assumption that is adopted in insurance economics to describe the risk-preferences of a (re-)insurer is shown below.

Assumption 3. *The sponsor's uncertain future wealth is given by the random variable w_S. Preferences of the sponsor are assumed to be given by a von-Neumann Morgenstern utility function $U_S(w_S)$, where U_S is assumed to be strictly increasing and strictly concave.*

Suppose the (re-)insurance company considers transferring its potential insurance risk L ($0 \leq L \leq L_{max}$) by an reinsurance method. This reinsurance contract needs a premium θ to cover L. Instead of paying θ, the (re-)insurance company could invest the same amount into financial assets to increase its capital. For simplicity, assume that the (re-)insurance company is only allowed to invest into risk-free assets. A (re-)insurance company characterised by Assumption 3 will then insure L for the insurance premium θ if it is better off doing so, i.e. if

$$E[U_S(w_S - L)] < E[U_S(w_S - R_f\theta)]. \tag{2.4.8}$$

In insurance economics, a participation constraint of the form (2.4.8) is standard except that we allow for gains from a risk-free intertemporal investment.

Note that if the sponsor is risk-neutral and has utility function $U_S(w_s) = aw_S + b$

for some positive constants a and b. In this case the participation constraint (2.4.8) amounts to

$$R_f \mathcal{V}^+(L) < E[L]. \tag{2.4.9}$$

Inserting (2.4.6), we see that (2.4.9) is equivalent to

$$\frac{\rho_*^+}{\sigma_m^+} Cov[L, e_m^+] > 0.$$

This violates Condition (ii) of Theorem 4.

So, if a (re-)insurance company's expectation of its utility with paying an insurance contract is higher than its expected utility without re-insuring this risk, this (re-)insurance company is willing to purchase the reinsurance contract. As this thesis is mainly studying the risk transfer mechanism of CAT bonds, this subsection discusses the conditions on how to transfer an insurance risk by securitising this risk as financial products and make it tradable in the financial market. After the (re-)insurance company is willing to transfer this risk by passing premium θ, there are still two conditions to be satisfied, as Subsection 2.3.4 shows. One is that these insurance-linked securities are accepted by the financial market, i.e. whether they can be traded in the financial market. The other condition is that the introduction of this product will not create negative prices of other financial assets in the market. Overall, the following definitions conclude the conditions on how this insurance risk can be securitised as financial products and hence be transferred into the financial market

Definition 3. [Transferable insurance risk]

An insured risk, represented by a random variable L, can be securitised in an insurance-linked securities and traded in the financial market if the following conditions hold:

(i) The sponsor is risk-averse and better off transferring the hazard risk to the financial market by paying premium θ

$$E[U_S(w_S - L)] < E\big[U_S\big(w_S - \theta\big)\big].$$

(ii) The equilibrium allocation of risk and return $\big\{(\sigma_*^{1+}, \mu_*^{1+}), \cdots, (\sigma_*^{I+}, \mu_*^{I+})\big\}$ in the market with catastrophe bond is Pareto superior to the equilibrium allocation of risk and return in the market without catastrophe bond $\big\{(\sigma_*^{(1)}, \mu_*^{(1)}), \cdots, (\sigma_*^{(I)}, \mu_*^{(I)})\big\}$, that is,

$$U^{(i)}(\mu_*^{(i)+}, \sigma_*^{(i)+}) \geq U^{(i)}(\mu_*^{(i)}, \sigma_*^{(i)}) \quad \text{for all } i = 1, \ldots, I,$$

where at least one inequality is strict;

(iii) All asset prices (including this insurance-linked security prices) in an equilibrium with the insurance-linked securities are positive.

Definition 3 determines three essential conditions on ensuring the insurance risk is transformed into the financial market. The following content will explore how the insurance risk can be transferred to the financial market by means of CAT bonds. The premium and promised payoff that is achieved from Subsection 2.4.1 are applied. That is the promised payoff that the insurance company passed to the SPV and the promised payoff of CAT bonds to the investors are set as

$$\begin{cases} \theta = \mathcal{V}^+(L) \\ C = R_f(L_{max} + \mathcal{V}^+(L)). \end{cases}$$

So that for the condition (i) at Definition 3, if the (re-)insurance company's utility satisfies

$$E[U_S(w_S - E(L))] < E\big[U_S\big(w_S - R_f\mathcal{V}^+(L)\big)\big].$$

It means this (re-)insurance company is willing to pass $R_f\mathcal{V}^+(L)$ to transfer its potential risk $E(L)$.

A prerequisite for the transferability of hazard risk is that the opportunity cost of the insurance contract $R_f\mathcal{V}^+(L)$ is less than the maximum claim L_{max}. Otherwise, the insurer would be better off keeping this potential debt L on its balance sheet. The next Lemma shows that this requirement is met, if the essential risk of the payoff $L_{max} - L$ in the sense of Levy [97], given by

$$\frac{Cov[L_{max}-L, e_m^+]}{E[L_{max}-L]}$$

is sufficiently small in relation to market risk. Since $Cov[L, e_{\mathfrak{m}}^+]$ was assumed to be negative, the essential risk of $L_{max} - L$ is positive.

Lemma 6. *Under the hypotheses of Proposition 3, the opportunity cost $R_f\mathcal{V}^+(L)$ is less than the maximum claim L_{max}, if and only if the essential risk of the loss L is sufficiently small in relation to market risk σ_m^+, that is:*

$$\phi\left(\frac{E[L_{max}-L]\sigma_m^+}{Cov[L_{max}-L, e_m^+]}\right) > \sigma_m^+.$$

Proof. Inserting (2.4.6), $R_f\mathcal{V}^+(L) < L_{max}$ if and only if

$$L_{max} - E[L] > -\frac{\rho_*^+}{\sigma_*^+}Cov[L, e_m^+] = \frac{\rho_*^+}{\sigma_*^+}Cov[L_{max} - L, e_m^+]$$

or, equivalently,

$$\frac{E[L_{max}-L]\sigma_m^+}{Cov[L_{max}-L,e_m^+]} > \rho_*^+.$$

Using the fact that ϕ is strictly increasing, the assertion follows.

□

It remains to establish conditions under which CAT bonds provide a Pareto improvement of the risk allocation of the economy as stipulated in Definition 3. Subsection 2.3.4 discussed the issuing conditions of innovations. That is in a CAPM setting the market price of risk must increase so that financial investors are willing to accept the extra risk of adding the innovation to their portfolios. The SPV who manages the bonds has to ensure that financial investors are repaid and the insurance claim is covered up to the maximum loss under all possible circumstances. The following theorem now provides conditions under which this is the case so that the insurance claim L is transferable to the financial market by means of issuing CAT bonds.

Theorem 4. *The insurance claim L satisfying Assumption 2 is transferable to the financial market in the sense of Definition 3, if the following conditions hold:*

(i) The participation constraint of the sponsor

$$E[U_S(w_S - L)] < E\big[U_S\big(w_S - R_f\mathcal{V}^+(L)\big)\big];$$

(ii) The correlation of L with the market payoff of the financial assets e_m satisfies $Var[L] > 2Cov[L, e_m]$;

(iii) Aggregate willingness to assume risk $\phi(\rho)$ is increasing in ρ and satisfies

$$\phi\left(\rho_{\max}\right) > \sigma_m^+,$$

where

$$\rho_{\max} := \frac{\sigma_m^+}{\max\left\{\frac{Cor[q_0, e_m^+]}{E[q_0]}, \ldots, \frac{Cor[q_K, e_m^+]}{E[q_K]}\right\}};$$

(iv) The essential risk of the loss L is sufficiently small in relation to market risk σ_m^+, that is,

$$\phi\left(\frac{E[L_{max} - L]\sigma_m^+}{Cov[L_{max} - L, e_m^+]}\right) > \sigma_m^+.$$

Proof. Condition (i) can be achieved directly from the Definition 3 and the premium setting achieved in the Subsection 2.4.1, that is $\theta = R_f \mathcal{V}^+(L)$.

Condition (ii) follows the result from the Section 2.3, the market risk of a financial market in which catastrophe bonds are traded is the variance of $e_{\mathfrak{m}}^+$ so that

$$\sigma_{\mathfrak{m}}^+ = \sqrt{Var[e_{\mathfrak{m}}^+]} = \sqrt{\sigma_{\mathfrak{m}}^2 - 2Cov[e_{\mathfrak{m}}, L] + Var[L]}. \tag{2.4.10}$$

Hence market risk will increase, that is $\sigma_{\mathfrak{m}}^+ > \sigma_{\mathfrak{m}}$ if and only if Condition (ii) holds. By the assumption of Condition (iii) that the aggregate willingness to assume risk $\phi(\rho)$ is increasing in ρ and according to Lemma 2, the risk allocation in the new market with CAT bonds is Pareto superior to the market without CAT bonds. It implies that investors are willing to hold CAT bonds into their portfolios and hence

CAT bonds can be traded in the market.

Conditions (iii) and (iv) are achieved from Lemma 4 and Lemma 6 respectively.

□

The three conditions of the theorem may be interpreted as follows. The first one ensures that one re-insurance company has the intention to transfer its risk by giving premium. Condition (ii) ensures that market risk is increasing, and with Condition (iii) together ensures that the market portfolio with CAT bonds is Pareto-superior to the market portfolio without CAT bonds. So that CAT bonds are traded in the market. Condition (iii) guarantees that all risky assets have positive prices.

2.4.3 Risk load

For the remainder of the paper, assume that all conditions of Theorem 4 are satisfied so that catastrophe risk L is transferable in the sense of Definition 3. This subsection applies the setting $C = R_f\big(L_{max} + \mathcal{V}^+(L)\big)$ to discuss the constraint condition of risk load from the financial market, which gives a complement condition to the literature in actuarial science.

Consider a zero-profit SPV that sets out to raise enough funds from the financial market to cover all possible losses between date 0 and the maturity date 1 up to the upper bound L_{max}. With the promise C set to $R_f\big(L_{max} + \mathcal{V}^+(L)\big)$, the payoff to financial investors will always be positive, as they will at most lose the amount L_{max}. Indeed, since

$$Q_c = R_f(L_{max} + \mathcal{V}^+(L)) - L \geq r_f L_{max} + R_f \mathcal{V}^+(L) > 0,$$

they will at least receive the interest on their principal and their risk evaluation with interest. The coupon rate r_c is

$$r_c = \frac{R_f(L_{max}+\mathcal{V}^+(L))}{L_{max}} - 1 = r_f + \frac{R_f\mathcal{V}^+(L)}{L_{max}}, \tag{2.4.11}$$

in which the ratio $\frac{R_f\mathcal{V}^+(L)}{L_{max}}$ may be interpreted as an intertemporal *insurance-premium rate* for holding CAT bonds instead of safe bonds. Since the insurance premium paid by the sponsor is set to the financial market valuation of the catastrophe risk, $\theta = \mathcal{V}^+(L)$, it follows from (2.4.6) that this insurance-premium rate takes the form[5]

$$\frac{R_f\mathcal{V}^+(L)}{L_{max}} = \frac{E[L]}{L_{max}} - \frac{\rho_*^+}{\sigma_{\mathfrak{m}}^+}\frac{Cov[L,e_{\mathfrak{m}}^+]}{L_{max}}. \tag{2.4.12}$$

In insurance economics, an insurance-premium rate is, in general, defined by the ratio $\tau = \frac{\vartheta}{L_{max}}$, where ϑ denotes the insurance premium and L_{max} the cover. The premium rate is typically assumed to be of the form

$$\tau = E[L]/L_{max} + \Lambda, \tag{2.4.13}$$

where Λ is interpreted as a load for the risk margin and expenses of the insurance industry. Galeotti et al. [70] consider a variety of premium calculation formulas which calculate the risk load Λ using principles from insurance economics. Our analysis now shows that the risk load Λ is determined by the willingness of financial investors to assume the risk imposed by the claim L. A comparison of (2.4.12) with (2.4.13) shows that CAPM investors will only assume the additional catastrophe

[5] Observe that our insurance-premium rate accounts for the fact that the insurance premium is invested into risk-free bonds.

risk, if the risk load satisfies

$$\Lambda \geq \frac{\rho_*^+}{\sigma_{\mathfrak{m}}^+}\big(Var[L] - Cov[L, e_{\mathfrak{m}}]\big)\frac{1}{L_{max}}. \tag{2.4.14}$$

Otherwise, the profit of the SPV would be negative. Hence, any premium calculation formula must respect the financial market evaluation of the catastrophe risk, which is embodied by the price of risk ρ_*^+ appearing in (2.4.14). This subsection closes with a remark on an alternative CAT bond design.

Remark 1. Assume that $\mathcal{V}^+(L)$ is less than the maximum loss L_{max}. An alternative choice for a zero-profit SPV then is to set $C = R_f L_{max}$, because funds from the financial market are positive, $\mathcal{V}^+(Q_c) = L_{max} - \mathcal{V}^+(L) > 0$. By Lemma 6, this choice is possible if the essential risk of $L_{max} - L$ is sufficiently low. The coupon rate is

$$c = \frac{P}{\mathcal{V}^+(Q_c)} - 1 = r_f + \frac{R_f \mathcal{V}^+(L)}{L_{max} - \mathcal{V}^+(L)}.$$

This coupon rate is higher than (2.4.11) and the minimum total payoff to investors is $r_f L_{max}$ in the event of a maximum loss $L = L_{max}$. However, total funds received in period 0 are less than L_{max} so that the SPV could cover the maximum loss at maturity only.

2.4.4 A special example

Consider now investors with linear mean-variance preferences and a risk-neutral sponsor. Assume, furthermore, that the insurance loss is uncorrelated with the finan-

cial market, i.e., $Cov[e_m, L] = 0$. The introduction of catastrophe bonds increases market risk by the hazard risk $Var[L]$, so that

$$\sigma_m^+ = \sqrt{\sigma_m^2 + Var[L]}.$$

The utility function of investor i is $U^{(i)}(\mu, \sigma) = \mu - \frac{1}{2a^i}\sigma^2$, where $a^{(i)} > 0$ is i's risk tolerance. Since $\sum_{i=1}^{I} a^{(i)}\rho_*^+ = \sigma_m^+$, the financial market valuation of the insurance claim L is

$$\mathcal{V}^+(L) = \frac{1}{R_f}\left(E[L] + \frac{1}{a}Var[L]\right), \tag{2.4.15}$$

where $\sum_{i=1}^{I} a^{(i)} = a$ is aggregate risk tolerance. Formula (2.4.15) shows that $R_f\mathcal{V}^+(L) < L_{max}$ if, and only if,

$$\frac{Var[L]}{L_{max} - E[L]} < a.$$

Hence, a sponsor whose insurance premium is $\theta = \mathcal{V}^+(L)$ has an incentive to transfer her insurance risk to the financial market only, if aggregate risk tolerance of the financial investors is sufficiently large. The equilibrium price of a CAT bond is $p_0 = \frac{L_{max}}{x_{cm}}$.

Since $\frac{\rho_*}{\sigma_m} = \frac{1}{a} = \frac{\rho_*^+}{\sigma_m^+}$ and $Cov[L, e_m] = 0$, the introduction of the CAT bond has no effect on asset prices and asset holdings of the incumbent assets, because investor i's asset holdings are

$$x_*^{(i)+} = \frac{a^{(i)}}{a}x_m^+ = \begin{pmatrix} x_*^{(i)} \\ \frac{a^{(i)}}{a}x_{cm} \end{pmatrix}.$$

The orthogonality of e_m and Q_c implies that

$$\mathcal{V}^+(e_m^+) = \mathcal{V}(e_m) + \mathcal{V}(Q_c) = \mathcal{V}(e_m) + L_{max}.$$

Analogously to (2.2.10), aggregate bond holdings of financial investors become

$$\sum_{i=1}^{I} y_*^{(i)+} = \sum_{i=1}^{I} w_0^{(i)} - \mathcal{V}^+(e_m^+) = \sum_{i=1}^{I} y_*^{(i)} - L_{max}.$$

Hence, aggregate bond holdings of financial investors are reduced by exactly the amount L_{max} needed to cover the maximal loss of the sponsor. Financial investors will thus cover the insurance risk by reducing their investment into the risk-free bond while keeping their original positions of risky assets.

Summarising these findings, hazard risk in this example will be financed by reducing investment into the risk-free bond. In particular, financial investors will issue risk-free bonds whenever their initial wealth is not large enough to bear the costs of natural hazards.

2.5 Conclusion

This chapter has provided a novel way of modelling financial innovations as new assets and developed a new model that allows for an analysis of CAT bonds by including the insurance industry into the financial market. The effect of the innovation on the financial market was analysed and discussed. Under a certain condition, the market price will increase after the introduction of the new asset and hence investors

will include the new asset into their portfolios. Under this situation, their utilities increase. Considering the utility of issuers and the conditions in the financial market, premium and coupon rates are determined. A new concept, called transferable insurance risk, was developed and the conditions that allow for a transfer of catastrophic risk from the insurance industry to financial markets by issuing CAT bonds were discussed.

Chapter 3

An Agent-based CAPM

3.1 Introduction

The traditional CAPM developed by Sharpe [131], Lintner [99], Mossion [112], and Merton [110] assumes all investors are rational mean-variance optimisers, meaning that all investors analyse securities in the same way and share the same economic view of the world. In other words, investors' estimations of the probability distribution of future cash flows from investing in the available securities are identical. That is, everyone holds homogeneous beliefs regarding the means and variances of each asset. However, many researchers argue that this assumption contradicts the fact that individuals have different capacities, such as the ability to access and process information. Some individuals are able to access information and, hence, make a fairly correct prediction about future prices, while others may have limited capability and hence have 'wrong' estimations of future payoffs. Kurz [90] has defined the term 'rational beliefs', which is compatible with data and satisfying a certain technical condition. The traditional view such as Muth [114] states that people who are rational, or have rational behaviour, have rational expectations. The notion of 'rational beliefs' distinguishes between 'rational expectations' and 'rational

behaviour', implying that agents who are following their rational behaviour (for example, maximising their utility) can have different predictions (beliefs) for future payoffs of traded assets.

Vissing-Jorgensen [144], provides evidence from survey data that heterogeneous beliefs in future payoffs exist. By using UBS data, she shows that different percentages of individual investors have different thoughts about the current stock market, such as the market was over-valuated, undervalued, unsure or fairly valued. Investors' different opinions will therefore drive the fluctuations in the market. Brock, Hommes and Wagener [31] estimate a two-type agent (fundamentalists and chartists) asset-pricing model, with yearly S&P 500 data to explain how a world-wide stock market bubble was triggered in the late 1990s.

Agent-based models based on heterogeneous beliefs have attracted the interest of many researchers. Some researchers have shown the effects of heterogeneity in financial markets. For example, Anderson, Ghysels and Juergens [5] study the effect of heterogeneous beliefs on asset returns and consider whether these elements are essential to a pricing model. In their setting, agents have the same preferences but interpret information differently or exercise different skills to access information. They show that the ability to obtain or interpret information affects investors' expected return and, hence, influences return volatility. Kraus and Smith [89] also consider two types of investors who have different private information and construct different current portfolios by holding different beliefs on future stocks prices. In this situation, they show that the introduction of options increases the uncertainty of stock prices in the next period and enhances the volatility of stock prices. In both examples, investors receive different information. Research along these lines

includes Varian [142], Harris and Raviv [75]. In addition to heterogeneous beliefs, Jarrow Robert [83] differentiates between heterogeneous and homogeneous beliefs when short sale constraints are in place in a mean-variance environment.

Zeeman [152] provides an early example of a heterogeneous-agent model. More examples are provided by Haltiwanger and Waldmann [74], DeLong et al. [56], and Dacorogna et al. [51]. In terms of asset-pricing, Brock and Hommes [28] investigate a discounted present value asset pricing model with heterogeneous beliefs. They define that agents can revise their beliefs in each period according to past realised profits. They show that under a certain condition, heterogeneous agents lead to chaotic asset price fluctuations. Chiarella et al. [44] believe that asset prices are not only driven by rational investors but also by 'boundedly rational behaviour'. They present a boundedly rational heterogeneous agent (BRHA) model, which is based on the traditional one-period CAPM. Investors have heterogeneous beliefs about future payoffs from risky assets and different risk preferences. Their expectations are driven by their observations of past asset returns. Thus asset prices are driven by investors' different expectations. This model can accommodate market features such as volatility clustering, which standard models with homogeneous beliefs cannot describe. Similar research (see Li [98], Jouini and Napp [85] and Wenzelburger [151]) applies a similar model to study the effect of put options on the financial market. Chiarella et al. [45] later use an evolutionary CAPM (ECAPM) with heterogeneous beliefs to study the spillover effects of the rational behaviour of agents.

The next section will introduce the agent-based CAPM with heterogeneous beliefs developed by Brock and Hommes [30], Böhm and Chiarella [24] and Wenzelburger [151]. This model will be applied in Chapter 3 to design CAT bonds and analyse the

interaction of agents with heterogeneous beliefs. The model framework is introduced in Sections 3.2 to 3.4. In Section 3.5, the concept of a reference portfolio in Wenzelburger [147] is introduced, helping to generate an insightful form of the pricing formula. Section 3.6 provides a framework to price redundant assets based on the model.

3.2 Model setting

Consider a multi-period model with time period $t = 1, .., T$. Assume that there are K risky assets (indexed by $k = 1, ..., K$) and one risk-less bond traded in the financial market, where $k = 1, .., K$ represents the kth stock. Define $R_f = (1 + r_f)$ (r_f is risk-less rate). The risk-less bond pays a return rate R_f at each period t. The risky assets are traded at prices $p_t = (p_{1t}, \ldots, p_{Kt}) \in \mathbb{R}_+^K$ of period t. The gross return of risky assets at period t are $q_t = (q_{1t}, ..., q_{Kt})$.

Assume that there are I investors in financial market indexed by $i = 1, 2, ..., I$. They cannot store their consumption goods directly and need to transfer their wealth across periods. They are characterised by

- a utility function $U_t^{(i)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, which is continuously differential, strictly increasing in $\mu^{(i)}$, strictly decreasing in $\sigma^{(i)}$, and strictly quasi-concave.
- a subjective probability distribution $v_t^{(i)} \in Prob(\mathbb{R}^K)$ for future cum-dividend asset prices (gross returns) parameterised by a pair $(q_t^{(i)}, V_t^{(i)}) \in \mathbb{R}^K \times \mathcal{M}_K$ of conditional mean value of gross return $q_t^{(i)} = \mathbb{E}_t^{(i)}[q_{t+1}] \in \mathbb{R}^K$ and asso-

ciated covariance matrix $\Sigma_t^{(i)} = \Pi_{kl,t}^{(i)} = Cov_t[q_{k,t+1}^{(i)}, q_{l,t+1}^{(i)}] \in \mathcal{M}_K$ *with* $k = 1, ..., K$ *and* $l = 1, ..., K$.

- at each end of period, investors are allowed to adjust their portfolio, based on their subjective beliefs $(q_t^{(i)}, \Sigma_t^{(i)})$ regarding future gross returns of the assets.

Assume that the gross return is defined by $q_t^{(i)} = p_t^{(i)} + d_{t+1}^{(i)}$. Let $((x_{1t}^{(i)}, ..., x_{Kt}^{(i)}), y_t^{(i)}) = (x_t^{(i)}, y_t^{(i)}) \in \mathbb{R}^K \times \mathbb{R}$ be the portfolio held by investor i at period t, where $x_{kt}^{(i)}$ is the number of risky asset k $(k = 1, ..., K)$ and $y_t^{(i)}$ is the number of risk-less bond in portfolio.

Individual i's initial wealth at time $t-1$, $t = 1, .., T$ is defined as

$$w_{t-1}^{(i)} = y_{t-1}^{(i)} R_f + \langle q_{t-1}^{(i)}, x_t^{(i)} \rangle.$$

Given p_{t-1}, the future wealth is

$$\begin{aligned} w_t^{(i)} &= R_f w_{t-1}^{(i)} + \langle q_t^{(i)} - R_f p_{t-1}, x_{t-1}^{(i)} \rangle \\ &= R_f w_{t-1}^{(i)} + \langle \pi_t^{(i)}, x_{t-1}^{(i)} \rangle. \end{aligned}$$

Let $\pi_t^{(i)} = q_t^{(i)} - R_f p_{t-1}$ be expected vector of excess returns by investor i.

The subjectively expected value of investor i's future wealth at time t is

$$\mu^{(i)}(w_{t-1}^{(i)}, \pi_t^{(i)}, x_{t-1}^{(i)}) = R_f w_{t-1}^{(i)} + \langle \pi_t^{(i)}, x_{t-1}^{(i)} \rangle$$

with associated subjective variance

$$\sigma(x_{t-1}^{(i)}) = \langle x_{t-1}^{(i)}, V_t^{(i)} x_{t-1}^{(i)} \rangle^{\frac{1}{2}}.$$

In each period t, a group of noise traders who demand the random quantity $\xi_t \in \mathbb{R}^K$ of shares is active in the asset market. The probabilistic prerequisites on the exogenous noise and the exogenous dividend process are stipulated in the following assumption.

Assumption 4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ an increasing family of sub-σ-algebras of $\mathcal{F}$.*

1. The dividend payments are described by a $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ adapted stochastic process $\{d_t\}_{t\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{D} \in \mathbb{R}_+^K$.

2. The noise traders' transactions are governed by a $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ adapted stochastic process $\{\xi_t\}_{t\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^K$, which is uncorrelated with the dividend process $\{d_t\}_{t\in\mathbb{N}}$ defined in 1.

3.3 Two-fund separation theorem

Consider the utility maximisation problem at period t

$$\max_{x_{t-1}^{(i)} \in \mathbb{R}^{K+1}} U^{(i)}(\mu_t^{(i)}(w_{t-1}^{(i)}, \pi_t^{(i)}, x_{t-1}^{(i)}),\ \sigma_t^{(i)}(x_{t-1}^{(i)})),\ t = 1, ..., T \tag{3.3.1}$$

with condition that

$$\max_{x_{t-1}^{(i)} \in \mathbb{R}^{K+1}} \mu_t^{(i)}(w_{t-1}^{(i)}, \pi_t^{(i)}, x_{t-1}^{(i)}), \ s.t. \, \sigma_t^{(i)}(x_{t-1}^{(i)}) \leq \sigma_t^{(i)}, \ with \ t = 1, ..., T \tag{3.3.2}$$

where $\sigma_t^{(i)} \geq 0$.

A unique portfolio that solves this problem is efficient at period t:

$$x_{\text{eff},t}^{(i)}(\sigma_t^{(i)}, \pi_t^{(i)}, V_t^{(i)}) = \frac{\sigma_t^{(i)}}{\langle \pi_t^{(i)}, \, (V_t^{(i)})^{-1} \pi_t^{(i)} \rangle^{\frac{1}{2}}} (V_t^{(i)})^{-1} \pi_t^{(i)}, \ (t = 1, ..., T). \tag{3.3.3}$$

which produces the highest expected wealth $\mu_{w_t}^{(i)}$ given an upper bound $\sigma_t^{(i)}$ for the standard deviation of wealth $\sigma_{w_t}^{(i)}$, according to Markowitz [108].

The efficient frontier in terms of wealth at period t is defined by

$$\mu^{(i)}(w_{t-1}^{(i)}, \pi_t^{(i)}, x_{\text{eff},t-1}^{(i)}) = R_f w_{t-1}^{(i)} + \rho_t^{(i)} \sigma_t^{(i)}, \tag{3.3.4}$$

where $\rho_t^{(i)} = \langle \pi_t^{(i)}, (V_t^{(i)})^{-1} \pi_t^{(i)} \rangle^{\frac{1}{2}}$ is the price of risk holding by investor i in period t. It is used to formulate a two-fund separation theorem which dates back to Tobin [140] and Lintner [99].

Theorem 5. *Two-fund separation theorem*

Investor's utility function $U^{(i)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, *is continuously differentiable, strictly increasing in* μ_t, *decreasing in* σ_t, *and strictly quasi-concave. There exists a unique*

efficient portfolio $x_{t*}^{(i)} \in \mathbb{R}^K$ *to the problem (3.3.1). It is*

$$x_{t*}^{(i)} = x_{\text{eff}}^{(i)}(\sigma_{t*}^{(i)}, \pi_t^{(i)}, V_t^{(i)}), \tag{3.3.5}$$

$$= \frac{\sigma_{t*}^{(i)}}{\langle \pi_t^{(i)},\, (V_t^{(i)})^{-1}\pi_t^{(i)} \rangle^{\frac{1}{2}}} (V_t^{(i)})^{-1}\pi_t^{(i)} \tag{3.3.6}$$

where

$$\sigma_{t*}^{(i)} = \arg\max_{\sigma_t^i \geq 0} U^{(i)}(R_f w_{t-1}^{(i)} + \rho_t^{(i)}\sigma_t^{(i)}, \sigma_t^{(i)})$$

determines optimal risk σ_{t*}.

The following example illustrates a simple case to apply two-fund separation theorem.

Example 4. Linear utility function is

$$U^{(i)} = \mu_t^{(i)} - \frac{1}{2a^{(i)}}(\sigma_t^{(i)})^2$$

where $a > 0$ denotes risk tolerance. Then the risk $\sigma_{t*}^{(i)} = a^{(i)}\rho_t^{(i)}$ is a linear and hence subjective function with limiting slope $\rho_U = \infty$.

3.4 Temporary equilibrium map

Considering a simple case: the utility function for every investor is linear that is

$$U^{(i)}(\mu_t^{(i)},\, \sigma_t^{(i)}) = \mu_t^{(i)} - \frac{1}{2a^{(i)}}(\sigma_t^{(i)})^2.$$

with a^i is risk aversion of investor i. Accordingly, by understanding the Example 4 the gross-demand function for risky assets at period t is

$$x_t^{(i)} = a^{(i)}(V_t^{(i)})^{-1}\pi_t^i.$$

The total amount of risky assets at period t is defined as $(x_{1m}, ..., x_{Km}) = x_m \in \mathbb{R}^K$ and referred to as the market portfolio of the economy. Notation x_k $(k = 0, .., K)$ refers to the total number of kth assets.

The market clearing condition is

$$\sum_{i=1}^{I} x_t^{(i)} + \xi_t = x_m. \tag{3.4.1}$$

Solving the market-clearing condition (3.4.1) for p_t yields a temporary equilibrium map, which is related to arbitrary beliefs $(q_t^{(i)}, V_t^{(i)})$ rather than the past price process.

Theorem 6. *For solving the market-clearing condition for p_t, there exists a uniquely temporary equilibrium determined by*

$$p_t = G(\xi_t, a^{(i)}, q_t^{(i)}, V_t^{(i)}) = \sum_{i=1}^{I} A_t^{(i)} q_t^{(i)} - A_t(x_m - \xi_t), \tag{3.4.2}$$

where $A_t := \frac{1}{R_f}\left(\sum_{i=1}^{I} a^{(i)}(V_t^{(i)})^{-1}\right)^{-1}$ *and* $A_t^{(i)} := a^{(i)}A_t(V_t^{(i)})^{-1}$ *are all well-defined and invertible.*

Proof. By market clearing condition

$$
\begin{aligned}
\sum_{i=1}^{I} x_t^{(i)} + \xi_t &= x_m \\
\sum_{i=1}^{I} a^{(i)} (V_t^{(i)})^{-1} (q_t^{(i)} - R_f p_t) &= x_m - \xi_t
\end{aligned}
$$

By solving for p_t gives us

$$
\begin{aligned}
p_t &= \frac{1}{R_f} \left(\sum_{i=1}^{I} a^{(i)} (V_t^{(i)})^{-1} \right)^{-1} \cdot \left[\sum_{i=1}^{I} A_t^{(i)} (V_t^{(i)})^{-1} q_t^{(i)} - (x_m - \xi_t) \right] \\
&= \sum_{i=1}^{I} a^{(i)} q_t^{(i)} - A_t (x_m - \xi_t).
\end{aligned}
$$

Define $A_t = \frac{1}{R_f} \left(\sum_{i=1}^{I} a^{(i)} (V_t^{(i)})^{-1} \right)^{-1}$, $A_t^{(i)} = a^{(i)} A_t (V_t^{(i)})^{-1}$, $i = 1, 2$ and A_t, $A_t^{(i)}$ are all well-defined and invertible.

□

3.5 Reference portfolio

When investors hold heterogeneous beliefs, different investors will form their own efficient frontiers for efficient risky assets and then form their own capital market lines. Therefore, investors will form different optimal risky portfolios and the aggregate of those optimal risky portfolios cannot exactly be market portfolio. So the term 'market portfolio' shown in the standard CAPM cannot be used when investors

hold heterogeneous beliefs. In the literature, Jouini and Napp [85] define consensus belief (namely average belief) to represent the effect of the aggregated heterogeneous beliefs in order to study equilibrium prices in terms of the homogeneous belief model. Wenzelburger [149] use the preference portfolio to solve the same problem. This section is going to introduce the concept of the reference portfolio, introduced by Wenzelburger [147], which will be useful to derive the price formula under the heterogeneous situation. A reference portfolio is defined as

$$x_{ref,t} = \frac{1}{\rho_t}(V_t)^{-1}\pi_t, \tag{3.5.1}$$

where $\rho_t = \langle \pi_t, V_t^{-1}\pi_t \rangle^{\frac{1}{2}}$ is market price of risk and the risk of the reference portfolio is $\sigma_t = \langle x_{ref,t}, V_t x_{ref,t} \rangle$ and set as 1. The payoff of reference portfolio at time period $t+1$ is defined as $e_{t+1}^{ref} = \langle q_{t+1}, x_{ref,t} \rangle$.

This $x_{ref,t}$ is an efficient portfolio, which brings the highest Sharp ratio for the holder in the real market[1]. The Two Fund Separation Theorem (Theorem 5) shows that the higher the Sharpe ratio (market price of risk) of a portfolio, the higher the expected utility of an investment. Hence, $x_{ref,t}$ achieves the highest utility in the market. It indirectly reflects the interaction of investors who hold heterogeneous beliefs and hence ingeniously avoids the difficulties associated with those beliefs. If investors hold homogeneous beliefs, this portfolio is collinear with market portfolio, which verifies its efficient property. Note that this portfolio is a 'fictitious' portfolio and may not have any investors who hold it.

Lemma 7. *By using the reference portfolio, the temporary equilibrium prices p_t sat-*

[1] For more detailed information regarding to the proof, see Wenzelburger [147]

isfies the equation

$$p_t = \frac{1}{R_f}\left[\mathbb{E}_t[q_{t+1}] - \rho_t V_t x_{ref,t}\right],\ t = 0, 1, 2.... \tag{3.5.2}$$

Proof. This formula is generated using the definition of reference portfolio (see formula (3.5.1) to obtain the equation of expected excess return

$$\pi_t = \rho_t V_t x_{ref,t}.$$

Then, with the definition of $\pi_t = E_t[q_{t+1}] - R_f p_t$, formula (3.5.2) is achieved.

□

This Lemma 7 helps to derive an expression of the temporary equilibrium price of the *kth* asset which is not directly linked with investors' heterogeneous beliefs. It is shown in the Corollary 5 below.

Corollary 5. *A temporary equilibrium price of the kth asset in period t satisfies*

$$p_{kt} = \frac{1}{R_f}\left[E_t[q_{k,t+1}] - \rho_t Cov_t[q_{k,t+1}, e^{ref}_{t+1}]\right],$$

where $e^{ref}_{t+1} = \langle q_{t+1}, x_{ref,t}\rangle$ *is defined as a reference payoff. (In the following context, we use market payoff directly for simplification.)*

Proof. By Lemma 7, the kth asset price could be written as

$$\begin{aligned}
p_{kt} &= \frac{1}{R_f}\left[E_t(q_{k,t+1}) - \rho_t\left(V_t \cdot x_{ref,t}\right)_k\right], \\
&= \frac{1}{R_f}\left[E_t(q_{k,t+1}) - \rho_t \sum_{l=1}^{K} Cov_t(q_{k,t+1}, q_{l,t+1}) \cdot x_{ref,lt}\right] \\
&= \frac{1}{R_f}\left[E_t(q_{k,t+1}) - \rho_t Cov_t(q_{k,t+1}, \sum_{l=1}^{K} q_{l,t+1} \cdot x_{ref,lt})\right] \\
&= \frac{1}{R_f}\left[E_t(q_{k,t+1}) - \rho_t Cov_t(q_{k,t+1}, e_{ref,t+1})\right] \\
&= \frac{1}{R_f}\left[E_t(q_{k,t+1}) - \rho_t Cov_t(q_{k,t+1}, e_{ref,t+1})\right].
\end{aligned}$$

□

3.6 Pricing redundant assets

No arbitrage pricing method here to achieve the price formula of redundant asset in CAPM.

Define a redundant asset Γ (one-period maturity) with a payoff $q_{\Gamma,t+1}$ at time $t+1$, which can be replicated by a portfolio $(x_{\Gamma,t+1}, y_{\Gamma,t+1})$ constructed by stocks and risk-free bonds. So $q_{\Gamma,t+1} = \langle x_{\Gamma,t+1}, q_{t+1}\rangle + R_f y_{\Gamma,t+1}$. At equilibrium, the unique evaluation of Γ at time t is $p_{\Gamma,t} = \langle p_t, x_{\Gamma,t+1}\rangle + y_{\Gamma,t+1}$. By applying Corollary 5 and the property that the covariance between risk-free bonds and e^{ref}_{t+1} is 0, this evaluation of the redundant asset Γ is

$$
\begin{aligned}
p_{\Gamma,t} &= \langle p_t, x_{\Gamma,t+1}\rangle + y_{\Gamma,t+1} \\
&= \frac{1}{R_f} E_t\left[\langle x_{\Gamma,t+1}, q_{t+1}\rangle\right] - \rho_t Cov_t\left[\langle x_{\Gamma,t+1}, q_{t+1}\rangle, e_{t+1}^{ref}\right] \\
&\quad + \frac{1}{R_f} E_t\left[R_f y_{\Gamma,t+1}\right] - \rho_t Cov_t\left[R_f y_{\Gamma,t+1}, e_{t+1}^{ref}\right] \\
&= \frac{1}{R_f} E_t\left[\langle x_{\Gamma,t+1}, q_{t+1}\rangle + R_f y_{\Gamma,t+1}\right] - \rho_t Cov_t\left[\langle x_{\Gamma,t+1}, q_{t+1}\rangle + R_f y_{\Gamma,t+1}, e_{t+1}^{ref}\right] \\
&= \frac{1}{R_f}\left[E_t(q_{\Gamma,t+1}) - \rho_t Cov_t(q_{\Gamma,t+1}, e_{t+1}^{ref})\right] \qquad (3.6.1)
\end{aligned}
$$

Hence, we applied the replicating method in CAPM to get the pricing formula of any arbitrary asset. It extends the Corollary 5 further to arbitrary payoff in the market. That is Corollary 6 below.

Corollary 6. *Assume a redundant asset Γ (one-period maturity) with an arbitrary payoff $q_{\Gamma,t+1}$, which can be replicated by stocks and risk-free bonds. Under the assumption of no arbitrage opportunity, the equilibrium price of this redundant asset Γ satisfies*

$$
p_{\Gamma,t} = \mathcal{V}_t(q_{\Gamma,t+1}) = \frac{1}{R_f}\left(E_t(q_{\Gamma,t+1}) - \rho_t Cov_t(q_{\Gamma,t+1}, e_{ref,t+1})\right).
$$

The following content shows that the arbitrage opportunity exists in multi-period CAPM. Amending the method from Philip and Jonathan [66], the valuation formula (3.6.1) can be rearranged as

$$
\begin{aligned}
p_{\Gamma,t} &= \frac{1}{R_f}\left(E_t[q_{\Gamma,t+1}] - \rho_t Cov_t[q_{\Gamma,t+1}, e_{ref,t+1}]\right) \\
&= \frac{1}{R_f}\left[E_t[q_{\Gamma,t+1}] - \rho_t\left(E_t[q_{\Gamma,t+1}, e_{ref,t+1}] - E_t[q_{\Gamma,t+1}]E_t[e_{ref,t+1}]\right)\right] \\
&= \frac{1}{R_f}E_t\left[q_{\Gamma,t+1} - \rho_t\left(q_{\Gamma,t+1}e_{ref,t+1} - q_{\Gamma,t+1}E_t[e_{ref,t+1}]\right)\right] \\
&= \frac{1}{R_f}E_t\left[q_{\Gamma,t+1}\left(1 - \rho_t(e_{ref,t+1} - E_t[e_{ref,t+1}])\right)\right]. \qquad (3.6.2)
\end{aligned}
$$

This formula (3.6.2) can be applied to show that the arbitrage opportunity exists. For example: if $e_{ref,t+1} > E_t[e_{ref,t+1}] + \frac{1}{\rho_t}$, (in other words, $prob(e_{ref,t+1} > E_t[e_{ref,t+1}] + \frac{1}{\rho_t}) > 0$) and let random payoff be specific such as

$$q_{\Gamma,t+1} = [\rho_t(e_{ref,t+1} - E_t[e_{ref,t+1}]) - 1]^{-1} > 0,$$

then the price $p_{\Gamma,t} = -\frac{1}{R_f} < 0$, which creates the arbitrage opportunity.

This shows the equilibrium prices in CAPM are typically not arbitrage-free.

3.7 Conclusion

This chapter introduced the agent-based CAPM with heterogeneous beliefs, which was developed by several authors, including Chiarella et al. [41, 42, 43] and Wenzelburger [147, 149]. In particular, Section 3.6 discussed a CAPM-based pricing formula for redundant assets along with its limitations and arbitrage opportunities. This model will be extended in the next chapter to include CAT bonds, analyse investors' behaviour, and investigate the co-evolution of asset and CAT bond prices.

Chapter 4

CAT bonds in an Agent-based CAPM

Chapter 3 has introduced an agent-based asset pricing model with heterogeneous beliefs. This chapter will apply this model to design CAT bonds and to analyse the interaction between investors who hold heterogeneous beliefs on future prices. Section 4.1 gives the model setting of a market with CAT bonds following the model setting introduced in Chapter 3. Section 4.2 discusses the design of CAT bonds. The price of CAT bonds is derived in subsection 4.2.1 then the setting of premium and promised payoff is discussed in Subsection 4.2.2. An example is provided in Subsection 4.2.2 showing the way insurance companies set up the SPV to determine the premium and promised payoff. Section 4.3 focusses on the content of CAT bonds issuance. In this section, a definition of perceived Pareto superiority is proposed and conditions for the transferable risk are developed. The issuing condition for CAT bonds is concluded in Theorem 7. Wenzelburger's [149] concept of perfect forecasting rule is introduced in Section 4.4. Stochastic difference equations are generated to describe investor behaviour. Using the results of CAT bonds payoffs and promised payoffs in Sections 4.1 and 4.2, the catastrophic loss is embedded into the stochastic difference equations.

4.1 Model setting

Based on the model introduced in Chapter 3, when the CAT bonds are issued at time period t, the vectors of notations in the model shown in Chapter 3 are extended. Let subscript 'c' denote CAT bonds, 's' denote stocks. To distinguish the market with CAT bonds from the original market without CAT bonds, superscript '+' is added to the notations used in the original market.

Let p_{ct} denote the equilibrium price of CAT bonds, q_{ct} is its payoff, ξ_{ct} is the noise trader of CAT bonds and $p_{st}^{+} = (p_{1t}^{+}, ..., p_{Kt}^{+})$ denote the equilibrium price of stocks. The vector $q_{st}^{+} = (q_{1t}^{+}, ..., q_{Kt}^{+})$ is the payoff of stocks and ξ_{st} is the noise trader of stocks. The price vector at each time is $p_t^{+} = (p_{ct}, p_{st}^{+})$, payoff is $q_t^{+} = (q_{ct}, q_{st}^{+})$ and noise trader is $\xi_t^{+} = (\xi_{ct}, \xi_{st})$.

Investor i's beliefs on the risky asset payoffs are

$$q_t^{(i)+} = \mathbb{E}_t^{(i)}[q_{t+1}^{+}]$$

and

$$\Sigma_t^{(i)+} = \begin{pmatrix} V_{11,t}^{(i)} & \Sigma_{12,t}^{(i)} \\ \Sigma_{21,t}^{(i)} & \Sigma_{22,t}^{(i)} \end{pmatrix} \tag{4.1.1}$$

respectively, where $\Sigma_t^{(i)+}$ is a positive definite symmetric $(K+1) \times (K+1)$ matrix and well defined with

$$V_{11,t}^{(i)} = Var_t^{(i)}[q_{c,t+1}],$$

$$\Sigma_{12,t}^{(i)} = [b_{1t}^{(i)} \ \dots \ b_{Kt}^{(i)}], \ b_{kt}^{(i)} = Cov_t^{(i)}[q_{k,t+1}, q_{c,t+1}],$$

$$\Sigma_{22,t}^{(i)} = Var_t^{(i)}[q_{s,t+1}] = \left(Cov_t^{(i)}[q_{k,t+1}, q_{l,t+1}]\right)_{kl}, \ k = 1, \dots, K, \ l = 1, ..., K.$$

So $\Sigma_{12,t}^{(i)} = (\Sigma_{21,t}^{(i)})^\top$are the covariance vectors of stocks and CAT bonds, and $\Sigma_{22,t}^{(i)}$ is the covariance matrix of stock payoffs.

The subjective expected excess return of risky assets is $\pi_t^{(i)+} = E_t^{(i)}[q_{t+1}^+] - R_f p_t^{(i)+}$. The subjective market price of risk is $\rho_t^{(i)+} = \langle \pi_t^{(i)+}, (\Sigma_t^{(i)+})^{-1} \pi_t^{(i)+} \rangle^{\frac{1}{2}}$.

The objective covariance Σ_t^+ is given by

$$\Sigma_t^+ = \begin{pmatrix} V_{11,t} & \Sigma_{12,t} \\ \Sigma_{21,t} & \Sigma_{22,t} \end{pmatrix}, \tag{4.1.2}$$

where $V_{11,t} = Var_t[q_{c,t+1}]$, $\Sigma_{12,t} = (\Sigma_{21,t})^\top = [b_{1t} \ \dots \ b_{Kt}]$, $b_{kt} = Cov_t[q_{k,t+1}, q_{c,t+1}]$, $k = 1, \dots, K.$, $V_{22,t} = \Sigma_t[q_{s,t+1}] = (Cov_t[q_{k,t+1}, q_{l,t+1}])_{kl}$.

The objective market price of risk is $\rho_t^+ = \langle \pi_t^+, (V_t^+)^{-1} \pi_t^+ \rangle^{\frac{1}{2}}$ and the expected excess return of risky assets is $\pi_t^+ = E_t[q_{t+1}^+] - R_f p_t^+$.

The reference portfolio is

$$x_{ref,t}^+ = \frac{1}{\rho_t^+} (V_t^+)^{-1} \pi_t^+,$$

Its payoff is defined as

$$e_{t+1}^{ref+} = x_{ref,t}^+ \cdot q_{t+1}^+.$$

4.2 Design of CAT bonds

As Chapter 2 has discussed CAT bonds transactions, CAT bonds related transactions in multi-periods are illustrated in Diagram 4.2.1. In each time period, the sponsor (the re-insurance company) passes premium θ_t to the SPV – a non-profit intermediate, to issue CAT bonds. The SPV then raise funds by selling bonds to investors in the financial market. A CAT bond contract promises investors a payoff of c_t subject to no pre-agreed events having occurred, and at maturity investors will have their principle returned. However, if a specific event occurs before maturity, investors will lose their promised payoff and even part of their principle. This is because the fund will be used to cover the loss of the sponsor at time $t+1$. The total number of CAT bonds issuing at each time is $x_{cm} > 0$. The total fund used by the SPV to cover the loss of the sponsor is L_{t+1} ($0 \leq L_{t+1} \leq L_{max}$, L_{max} is the maximum loss that the SPV covers) at time $t+1$. So each CAT bond contract will cover l_{t+1} and $l_{t+1} = L_{t+1}/x_m$ ($0 \leq l_{t+1} \leq l_{max}$, l_{max} is the maximum loss covered in one CAT bond contract). The payoff $q_{c,t+1}$ of one CAT bonds contract is defined as

$$q_{c,t+1} = c_t - l_{t+1} \geq 0, \tag{4.2.1}$$

which requires $c_t \geq l_{max}$. The total payoff to investors is $Q_{c,t+1} = q_{c,t+1} \cdot x_{cm}$.

According to this payoff formula (4.2.1), the conditional expected value and its associated variance[1] of a CAT bond payoff are

$$E_t[q_{c,t+1}] = c_t - E_t[l_{t+1}] \tag{4.2.2}$$

[1] The $E_t[q_{c,t+1}]$ is for the conditional expectation $E[q_{c,t+1}|\mathcal{F}_t]$ and $Var_t[q_{c,t+1}]$ for conditional second moment $Var[q_{c,t+1}|\mathcal{F}_t]$.

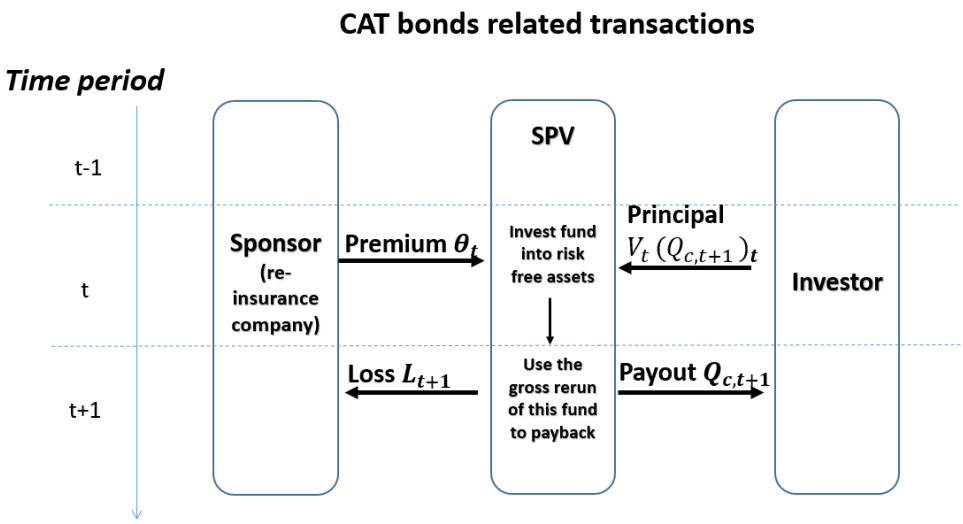


Figure 4.2.1: CAT bonds related transactions in a multi-period model

and

$$Var_t[q_{c,t+1}] = Var_t[l_{t+1}], \tag{4.2.3}$$

respectively. This shows that the mean and variance of the CAT bond payoff is determined by the mean and variance of the linked catastrophic loss.

4.2.1 CAT bond price

By the Corollary 5 of temporary equilibrium price of the kth asset, the equilibrium price of CAT bonds in period t must satisfy

$$p_{ct} = \frac{1}{R_f}\left[E_t[q_{c,t+1}] - \rho_t^+ Cov_t[q_{c,t+1}, e_{t+1}^{ref+}]\right], \tag{4.2.4}$$

where $e_{t+1}^{ref+} = \langle x_{ref,t}^+, q_{t+1}^+ \rangle$ is the payoff of the reference portfolio.

Substituting $q_{c,t+1}$ (4.2.1) and $E_t[q_{c,t+1}]$ (4.2.2) into the above equilibrium formula (4.2.4), gives

$$\begin{aligned} p_{ct} &= \frac{1}{R_f}\left[c_t - E_t[l_{t+1}] - \rho_t^+ Cov_t[(c_t - l_{t+1}), e_{t+1}^{ref+}]\right] \\ &= \frac{1}{R_f}c_t - \frac{1}{R_f}\left(E_t[l_{t+1}] - \rho_t^+ Cov_t[l_{t+1}, e_{t+1}^{ref+}]\right) \end{aligned}$$

So the price of CAT bonds is

$$p_{ct} = \frac{1}{R_f}c_t - \mathcal{V}_t(l_{t+1}), \tag{4.2.5}$$

where

$$\mathcal{V}_t(l_{t+1}) = \frac{1}{R_f}\left[E_t[l_{t+1}] - \rho_t Cov_t[l_{t+1}, e_{t+1}^{ref+}]\right] \tag{4.2.6}$$

is the valuation of the loss l_{t+1} at time period t from the financial market[2].

This price formula (4.2.5) shows the relation bewteen the market price of one CAT bond contract with its promised payoff c_t and the market valuation of the loss l_{t+1} that one CAT bond contract may cover. This indicates that the market value

[2] This is derived from the Corollary 5.

of each single period CAT bond contract equals the difference between the present value of the promised payoff c_t and the market valuation of the insured loss covered by a CAT bond at maturity. This formula will be used in the next subsection to determine the premium that the sponsor needs to pass to the SPV and the CAT bond's promised payoff to investors.

4.2.2 Promised payoff and premium

According to the transactions related to CAT bonds show in the illustration 4.2.1, the sponsor provides an amount of premium θ_t to ensure a coverage up to L_{max}. The SPV then issues CAT bonds into the financial market with a pre-determined coupon c_t of each CAT bond contract. This implies that to issue CAT bonds, determination of the minimum premium and coupon for each CAT bond is essential. This subsection discusses how to decide with the minimum premium and coupon.

Define $C_t = c_t \cdot x_{cm}$ as the total promised payoff to investors at time period t; $L_{max} = l_{max} \cdot x_{cm}$ is the total maximum loss that CAT bonds will cover; the loss that all issued CAT bonds will cover at $t+1$ is $L_{t+1} = l_{t+1} \cdot x_{cm}$ $(0 \leq L_{t+1} \leq L_{max})$, the total payoff of CAT bonds at maturity is $Q_{c,t+1} = q_{c,t+1} \cdot x_{cm} = C_t - L_{t+1}$[3].

The principle, i.e., the total money received from issuing CAT bonds, is denoted by $\mathcal{V}_t(Q_{c,t+1})$ as this the market valuation of CAT bond payoffs $Q_{c,t+1}$ at time t. By applying the price formula of CAT bonds (4.2.5), the principle is

[3] By applying the definition (4.2.1) of $q_{c,t+1}$: $q_{c,t+1} = c_t - l_{t+1}$.

$$
\begin{aligned}
\mathcal{V}_t(Q_{c,t+1}) &= p_{ct} \cdot x_{cm} \\
&= \frac{1}{R_f}\left(C_t - R_f \mathcal{V}_t(L_{t+1})\right) \\
&= \frac{1}{R_f} C_t - \mathcal{V}_t(L_{t+1}), \qquad (4.2.7)
\end{aligned}
$$

which equals the discounted total promised payoff minus the market valuation of the total loss at time $t+1$. This formula shows that only when the promised payoff C_t is larger than $R_f \mathcal{V}_t(L_{t+1})$, are the investors willing to purchase CAT bonds. Otherwise, investors will not buy and the SPV will be unable to raise money by issuing CAT bonds. So the minimum requirement to raise money is when the promised payoff $C_t > R_f \mathcal{V}_t(L_{t+1})$. To avoid a non-meaningful situation, one basic assumption is necessary, as stated in Assumption 5.

Assumption 5. *The market valuation of the total insured loss that CAT bonds will cover is always positive, that is*

$$\mathcal{V}_t(L_{t+1}) > 0.$$

This assumption excludes the situation in which the promised payoff is non-positive. It also implies investors' expected insurance loss covered by one CAT bond should be larger than the value of $\rho_t^+ Cov_t[l_{t+1}, e_{t+1}^{ref+}]$. That is

$$E_t[l_{t+1}] \geq \rho_t^+ Cov_t[l_{t+1}, e_{t+1}^{ref+}],$$

by applying the formula of $\mathcal{V}_t(l_{t+1})$ (4.2.6).

As Figure 4.2.1 shows, the SPV invests the fund from sale of CAT bonds into risk-free assets. At maturity, the gross value of this fund should cover any loss up to L_{max}. For a simple analysis, assume all the money required to cover the loss is from issuing CAT bonds. So, mathematically, the requirement to generate a positive fund by selling CAT bonds is

$$\mathcal{V}_t(Q_{c,t+1}) \geq \frac{1}{R_f} L_{max}. \tag{4.2.8}$$

Substituting (4.2.7) into the inequality formula (4.2.8) gives the minimum requirement of the promised payoff C_t to all CAT bonds investors

$$\begin{aligned} C_t - R_f \mathcal{V}_t(L_{t+1}) &\geq L_{max} \\ C_t &\geq L_{max} + R_f \mathcal{V}_t(L_{t+1}). \end{aligned} \tag{4.2.9}$$

This formula satisfies the minimum requirement of C_t to ensure investors are willing to purchase CAT bonds, $C_t > R_f \mathcal{V}_t(L_{t+1})$. Formula (4.2.9) shows that if the SPV requires L_{max} at maturity, the promised payoff should be larger than the sum of the total money the SPV wants to raise and the minimum requirement of the promised payoff. So the minimum promised payoff that the SPV needs to set is

$$C_{t,min} = L_{max} + R_f \mathcal{V}_t(L_{t+1})$$

if the SPV wants to have L_{max} at $t+1$ from issuing cat bonds.

Therefore, for each CAT bond contract, the promised payoff to the investor is

$c_t \geq l_{max} + R_f \mathcal{V}_t(l_{t+1})$. The minimum promised payoff of each contract should satisfy $c_{t,min} = l_{max} + R_f \mathcal{V}_t(l_{t+1})$.

As the illustration shows, after raising enough money, the SPV needs to use this fund to cover the loss (if a specific event happens) and use the rest of the money to pay it back to investors at maturity. Note that, at period t, the SPV also receive the premium from the sponsor. The SPV invests all the funds received into the risk-free asset, and at maturity, the realised fund in period $t+1$ is $R_f(\theta_t + \mathcal{V}_t(Q_{c,t+1}))$[4].

At time $t+1$, the SPV allocates money to the sponsor to cover any loss and payoffs to investors, that is $(L_{t+1} + Q_{c,t+1})$ in total. So at time $t+1$, the profit of SPV is denoted as

$$\Pi_{t+1} = R_f \left[(\theta_t + \mathcal{V}_t(Q_{c,t+1}))\right] - (L_{t+1} + Q_{c,t+1}). \tag{4.2.10}$$

By substituting the valuation $\mathcal{V}_t(Q_{c,t+1})$ formula (4.2.7) and $Q_{c,t+1} = C_t - L_{t+1}$ into the above formula, the promised payoff can be canceled and formula 4.2.10 rewritten as

$$\begin{aligned} \Pi_{t+1} &= R_f\theta_t + C_t - R_f\mathcal{V}_t(L_{t+1}) - C_t \\ &= R_f\left(\theta_t - \mathcal{V}_t(L_{t+1})\right). \end{aligned}$$

This gives the relation between SPV profit and the premium and market valuation of loss. It shows that the profit of the SPV has a zero correlation with the promised

[4]From the materials, the SPV dismisses after its function at each fund raising. The remaining profit will be passed back to the sponsor. The sponsors can be a non-profit charity. To consider a simple case, this thesis assumes that the sponsor is (re-)insurance company that set up a SPV for all of fund raising.

payoff, but correlates instead with the premium from the sponsor and the financial market valuation of the insurance loss. To ensure the SPV receive a non-negative profit i.e. avoid bankruptcy, the sponsor must at least pass a minimum premium $\theta_t = \mathcal{V}_t(L_{t+1})$ to the SPV. That is because

$$\begin{aligned} \Pi_{t+1} &\geq 0 \\ \theta_t &\geq \mathcal{V}_t(L_{t+1}). \end{aligned}$$

This inequality defines the minimum premium.

In conclusion, the requirements of promised payoff and premium are achieved below

$$\begin{cases} C_t \geq & L_{max} + R_f \mathcal{V}_t(L_{t+1}) \\ \theta_t \geq & \mathcal{V}_t(L_{t+1}) \end{cases} \tag{4.2.11}$$

For each CAT bond contract, the promised payoff must satisfy $c_t \geq l_{max} + R_f \mathcal{V}_t(l_{t+1})$.

The following illustrates how the SPV designs CAT bonds by applying the above results.

Example 5. Assume a (re-)insurance company intends to issue CAT bonds to diversify its catastrophic risk and set up a SPV. This example shows how the SPV decides the premium θ_t and coupon c_t of each CAT bond contract.

By applying the results derived previously as formula (4.2.11) shows, the SPV

needs to estimate the market value of loss $\mathcal{V}_t^e(L_{t+1})$[5]. The requirements of premium θ_t and total promised payoff C_t are

$$\begin{cases} C_t \geq & L_{max} + R_f \mathcal{V}_t^e(L_{t+1}) \\ \theta_t \geq & \mathcal{V}_t^e(L_{t+1}). \end{cases}$$

The SPV can also set minimum values, that is

$$\begin{cases} C_t & = L_{max} + R_f \mathcal{V}_t^e(L_{t+1}) \\ \theta_t & = \mathcal{V}_t^e(L_{t+1}) \end{cases}. \tag{4.2.12}$$

For each CAT bond, if no pre-agreed event occurs, the coupon is $c_t = l_{max} + R_f \mathcal{V}_t^e(l_{t+1})$.

However, the SPV faces a situation in which its estimation may not be exactly equal to the real value in the market and may lead to an unsuccessful bond issuance. The following shows why this approximation value matters.

According to the formula of $\mathcal{V}_t(Q_{c,t+1})$ (4.2.7), the total money that the SPV can receive by issuing CAT bonds is

$$\mathcal{V}_t(Q_{c,t+1}) = \frac{1}{R_f} C_t - \mathcal{V}_t(L_{t+1}).$$

Substituting for the C_t that the SPV sets, the received money is

[5]Methods of the estimation will not be discussed in this thesis.

$$
\begin{aligned}
\mathcal{V}_t(Q_{c,t+1}) &= \frac{1}{R_f}\left(L_{max} + R_f\mathcal{V}_t^e(L_{t+1}) - R_f\mathcal{V}_t(L_{t+1})\right) \\
&= \frac{1}{R_f}L_{max} + \left(\mathcal{V}_t^e(L_{t+1}) - \mathcal{V}_t(L_{t+1})\right). \qquad (4.2.13)
\end{aligned}
$$

Substituting for C_t and θ_t set by the SPV, the realised profit Π_{t+1} of the SPV (4.2.10) is

$$
\begin{aligned}
\Pi_{t+1} &= R_f\left(\theta_t + \mathcal{V}_t(Q_{c,t+1})\right) - (L_{t+1} + Q_{c,t+1}) \\
&= R_f\left(\mathcal{V}_t^e(L_{t+1}) + \frac{1}{R_f}L_{max} + (\mathcal{V}_t^e(L_{t+1}) - \mathcal{V}_t(L_{t+1}))\right) - C_t \\
&= R_f\left(\mathcal{V}_t^e(L_{t+1}) + \frac{1}{R_f}L_{max} + (\mathcal{V}_t^e(L_{t+1}) - \mathcal{V}_t(L_{t+1}))\right) - (L_{max} + R_f\mathcal{V}_t^e(L_{t+1})) \\
&= R_f\left(\mathcal{V}_t^e(L_{t+1}) - \mathcal{V}_t(L_{t+1})\right). \qquad (4.2.14)
\end{aligned}
$$

From formulas (4.2.13) and (4.2.14), if the estimated value exceeds larger than the real value, that is $\mathcal{V}_t^e(L_{t+1}) \geq \mathcal{V}_t(L_{t+1})$, the SPV raises $\mathcal{V}_t(Q_{c,t+1})$ or more and will be able to cover L_{max} and obtain a non-negative profit. If $\mathcal{V}_t^e(L_{t+1}) < \mathcal{V}_t(L_{t+1})$, the SPV cannot raise money from the financial market then contracts between the sponsor, the SPV and investors cannot be implemented. This problem can be solved with further research on the determination of the market valuation and the amount of additional funding or reserve required in advance.

4.3 CAT bond issuance

Section 4.2.2 has discussed how the SPV sets the premium charged to the sponsor, and the promised payoff to investors in order to raise the required money. However, such conditions are not enough to ensure a successful CAT bond issuance. A proper decision about the premium and the promised payoff is crucial during the preparation stage for issuing CAT bonds. This section, together with the result from Subsection 4.2.2, will discuss the conditions that decide a successful CAT bond issuance.

Chapter 2 has discussed the conditions under which CAT bonds can be issued in a single period CAPM. In that environment, investors hold homogeneous beliefs about future payoffs of financial products. A series of concepts to study CAT bonds issuing conditions such as transferable risk, Pareto superiority have been introduced in Chapter 2. This section will follow a framework similar to that in Chapter 2 and investigate the issuing conditions for the agent-based CAPM with heterogeneous beliefs.

4.3.1 Transferable insurance risk

This subsection demonstrates how an insurance risk can be transferred from (re-)insurance companies to the financial market by means of a financial innovation.

Firstly, the (re-)insurance company must have the intention to purchase this financial innovation to transfer its potential risk, assuming an re-insurance company has utility $U_s(w_{s,t})$ at time t as the Assumption 3 defines in Chapter 2, where $w_{s,t}$ is the value of the company's asset. This (re-)insurance company has its potential

insurance risk $L_{t+1}(0 \leq L_{t+1} \leq L_{max})$ at time $t+1$. Assuming that there is one financial innovation that can transfer this risk into the financial market by charging premium θ_t, then the (re-)insurance company is willing to purchase this financial innovation to transfer its potential risk as long as the expected utility of purchasing it is higher than covering this risk itself.[6] Mathematically, it is:

$$E_t[U_s(w_{s,t+1} - R_f\theta_t)] > E_t[U_s(w_{s,t+1} - L_{t+1})].$$

Therefore, if the (re-)insurance company uses this financial innovation to transfer its risk, it will lose the opportunity cost of θ_t; that is, the gross value of investing θ_t into risk-free assets. If the (re-)insurance company decides to cover loss by itself, it will face the uncertain loss L_{t+1} at time $t+1$. Consequently, if the above inequality holds, then the (re-)insurance company is willing to use this innovation to transfer its risk.

However, how can this innovation be traded in the financial market? The following content will discuss the trading condition for this innovation.

Chapter 2 has defined Pareto superiority to describe the state of innovations preferred by investors and which are, hence, traded in the market. However, under the environment of the agent-based CAPM heterogeneous beliefs, this concept is no longer applicable. This is because investors' beliefs are no longer the same as the market objective means and variances of asset payoffs, so the notion of perceived Pareto superiority is defined below to describe the situation in which innovations are preferred by at least one investor.

[6] Let us assume that there is only one way for a (re-)insurance company to reinsurance its potential risk for simplicity.

Definition 4. [Perceived Pareto superiority]

Consider the situation in which a new financial innovation is introduced into the financial market. The original market is set as market 1 and the new market with innovations is market 2. An allocation of risk and return

$$\{(\mu_{1t}^{(i)}, \sigma_{1t}^{(i)}), \cdots, (\mu_{1t}^{(I)}, \sigma_{1t}^{(I)})\}$$

in market 1 is *Perceived Pareto Superior* to an allocation of risk and return

$$\{(\mu_{2t}^{(i)}, \sigma_{2t}^{(i)}), \cdots, (\mu_{2t}^{(I)}, \sigma_{2t}^{(I)})\}$$

in market 2 if the utility of market 1 is larger than the utility of the market 2

$$U^{(i)}(\mu_{1t}^{(i)}, \sigma_{1t}^{(i)}) \geq U(\mu_{2t}^{(i)}, \sigma_{2t}^{(i)}) \text{ for all } i = 1, ..., I$$

with at least one inequality being strict.

If one investor perceives a higher utility by having this innovation in his portfolio, the innovation will be traded in the market. As investors have different beliefs on future payoffs, they will form their own efficient frontiers; therefore, they form their own capital market lines (CML) of the market. The following proposition describes the relation of the perceived slope of CML and utility, which can be used to discover whether innovations are preferred in the market.

Proposition 4. *The optimal portfolio in the market with innovations (denoted by market 1) is preferred to the portfolio in the market without innovations (denoted by market 2) at time period t for individual i,*

$$U^{(i)}(\mu_{1t}^{(i)}, \sigma_{1t}^{(i)}) \geq U(\mu_{2t}^{(i)}, \sigma_{2t}^{(i)}),$$

if the market price of risk in market 1 is higher than in market 2

$$\rho_{1t}^{(i)} > \rho_{2t}^{(i)}.$$

This proposition shows that if the perceived market price of risk (slope of CML) in the market with innovations is higher than in the market without innovations, then investor i believes that he will have a higher utility by having this innovation; therefore, he will buy it at time t.

Similarly to Chapter 2, Chapter 3 also shows that arbitrage opportunities exist in the agent-based CAPM. In that case, asset prices may be negative after introducing this innovation. Therefore, to make this model economically meaningful, ensuring all asset prices (including this innovation) are positive is essential.

By considering the above discussion, the conditions for the insurance risk that can be transferred by means of a financial innovation are defined below.

Definition 5. [*Transferable insurance risk*] The potential insurance risk L_{t+1} ($0 \leq L_{t+1} \leq L_{max}$) can be transferred by a financial innovation that charges insurance company premium θ_t to the financial market if the following conditions hold:

(i) The (re-)insurance company is willing to transfer its risk L_{t+1} if

$$E_t[U_s(w_{s,t+1} - R_f\theta_t)] > E_t[U_s(w_{s,t+1} - E_t(L_{t+1}))].$$

(ii) The risk-return allocation at equilibrium $\left\{(\sigma_{t*}^{(1)+}, \mu_{t*}^{(1)+})...(\sigma_{t*}^{(I)+}, \mu_{t*}^{(I)+})\right\}$ in the market with this innovation is perceived Pareto superior to it in the market without

this innovation $\left\{(\sigma_{t*}^{(1)}, \mu_{t*}^{(1)})...(\sigma_{t*}^{(I)}, \mu_{t*}^{(I)})\right\}$.

(iii) The introduction of this innovation will not lead to negative risky asset prices

In Definition 5, condition (i) shows that the (re-)insurance company is willing to transfer its potential insurance risk L_{t+1} by paying θ_t. Conditions (ii) and (iii) ensure this innovation can be introduced and traded in the financial market. Condition (ii) can be regarded as the trading condition. This means at least one investor would like to include this innovation in his portfolio and, hence, this innovation is traded in the market. Condition (iii) ensures the model is meaningful by introducing this innovation.

The definition of how the insurance risk can be transferred into the financial market by an innovation will help to decide the issuing conditions of CAT bonds in the next subsection.

4.3.2 Issuance conditions

This subsection will discuss conditions in which CAT bonds can be issued successfully to the financial market based on framework of transferable insurance risk, as defined in subsection 4.3.1, and the result of the minimum premium and promised payoff achieved from Subsection 4.2.2.

Subsection 4.2.2 has discussed the minimum premium θ_t and promised coupon c_t of each CAT bond contract, which ensure the SPV is able to play its role and raise the required money. In that subsection, Example 5 shows how the SPV sets these values, based on discussion of the minimum premium and promised payoff, that is

$$\begin{cases} \theta_t & = \mathcal{V}_t^e(L_{t+1}) \\ C_t & = L_{max} + R_f \mathcal{V}_t^e(L_{t+1}) \end{cases}$$

or $c_t = l_{max} + R_f \mathcal{V}_t^e(l_{t+1})$ for each CAT bond contract. This subsection will apply these results to discuss the issuing conditions for CAT bonds.

Subsection 4.3.1 defines the issuing condition that firstly, the re-insurance company is willing to use CAT bonds to transfer its risk. If its expected utility of purchasing CAT bonds is larger than its expected utility without purchasing them, the re-insurance company then would like to use. Thus, applying the result of minimum premium above, if the following inequality meets

$$E_t \left[U_s \left(w_{s,t+1} - R_f \mathcal{V}_t^e(L_{t+1}) \right) \right] > E_t \left[U_s(w_{s,t+1} - L_{t+1}) \right].$$

Then the (re-)insurance company will to use CAT bonds to transfer its risk. The (re-)insurance company then need to set up a SPV to issue CAT bonds and pass it a minimum premium at time t in order to receive a coverage of L_{t+1} up to L_{max} at time $t+1$.[7]

The following content discusses the conditions required to ensure that CAT bonds are issued into the financial market.

Introducing CAT bonds into the financial market may affect stock prices. Chapter 2 assumes that the price is not affected. This subsection examines the case where the stock prices may change. The literature identifies two ways in which stocks prices change. However, there is as yet, nothing on how stock prices are affected by CAT

[7] In this thesis, the operational cost such as administration fees for setting up a SPV are not considered.

bonds. Further work is needed. This thesis is a start and assumes that stock prices in the new market change as [8]

$$p_{st}^{+} = p_{st} + \varsigma_t,$$

where ς_t is a $1 \times K$ vector that $\varsigma_t = \{\varsigma_{1t}, \varsigma_{2t}, ..., \varsigma_{Kt}\}$ and ς_{kt} $(k = 1, 2, ..., K\ )$ denotes the kth stock price change. So the payoff of stocks in the market is

$$q_{st}^{+} = q_{st} + \zeta_t,$$

where ζ_t is a $1 \times K$ vector that $\zeta_t = \{\zeta_{1t}, \zeta_{2t}, ..., \zeta_{Kt}\}$ and ζ_{kt} $(k = 1, 2, ..., K\ )$ denotes the kth stock payoff change.

In this setting, investors' beliefs about the stock payoffs stocks at $t+1$ are giben by

$$E_t(q_{s,t+1}^{(i)+}) = E_t(q_{s,t+1}^{(i)}) + \zeta_t^{(i)},$$

$$\Sigma_{22,t}^{(i)+} = \Sigma_{st}^{(i)},$$

where $\zeta_t^{(i)}$ is a $1 \times K$ vector that $\zeta_t^{(i)} = \{\zeta_{1t}^{(i)}, \zeta_{2t}^{(i)}, ..., \zeta_{Kt}^{(i)}\}$ and $\zeta_{kt}^{(i)}$ $(k = 1, 2, ..., K\ )$ denotes the investors' expectation in the kth stock payoff change.

As discussed in Subsection 4.3.1, if one investor perceives a higher utility by including CAT bonds in his portfolio at equilibrium, then he will buy CAT bonds in the market. CAT bonds, therefore, are traded in the market. In this case, Proposition 4 implies this investor perceives a higher market price of risk (i.e. the slope of the

[8] In this way, the variance of payoff is the same as in the market without CAT bonds, which makes calculations easier.

CML). By applying the price of stocks, the following proposition provides conditions for which the perceived market price of risk with CAT bonds will be higher than in a market without CAT bonds.

Proposition 5. *[Perceived market price of risk] The perceived market price of risk for the investor i in the market with CAT bonds is higher than in the original market without CAT bonds, hence*

$$\rho_t^{(i)+} > \rho_t^{(i)},\ i = 1, 2, ...I$$

provided

$$2\langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)} \pi_{st}^{(i)+} \rangle + 2\langle \pi_t^{(i)}, V_t^{(i)}(\zeta_t^{(i)} - R_f \varsigma_t) \rangle$$

$$+\langle (\zeta_t^{(i)} - R_f \varsigma_t), V_t^{(i)}(\zeta_t^{(i)} - R_f \varsigma_t) \rangle + \langle \pi_{ct}^{(i)}, \Sigma_{11,t}^{(i)} \pi_{ct}^{(i)} \rangle \geq 0.$$

Proof. See Appendix B.

□

The above proposition ensures CAT bonds are traded in the market. However, as defined in Definition 5, to make this model economically meaningful, the introduction of CAT bonds should not make any asset prices non-positive. The following proposition provides conditions for positive asset prices in the market with CAT bonds.

Proposition 6. *[Positive asset prices] In the market with CAT bonds, if the conditional expected value of the kth asset payoff is larger than value of* $\rho_t^+ Cov_t[q_{k,t+1}^+, e_{t+1}^{ref+}]$*, that is*

$$E_t[q_{k,t+1}^+] \quad > \quad \rho_t^+ Cov_t[q_{k,t+1}^+, e_{t+1}^{ref+}],$$

then price of this kth asset is positive.

Proof. By Corollary 5, the kth risky price in the market with CAT bonds, is

$$p_{kt}^+ = \frac{1}{R_f}\left[E_t[q_{k,t+1}^+] - \rho_t^+ Cov_t[q_{k,t+1}^+, e_{t+1}^{ref+}]\right].$$

To ensure a positive risky asset price,

$$\begin{aligned} p_{kt}^+ &> 0 \\ E_t[q_{k,t+1}^+] &> \rho_t^+ Cov_t[q_{k,t+1}^+, e_{t+1}^{ref+}] \end{aligned}$$

□

Considering a (re-)insurance company with utility $U_s(w_{s,t})$ at time t where $w_{s,t}$ is its asset value and it has uncertain insurance loss L_{t+1} $(0 \leq L_{t+1} \leq L_{max})$, the following theorem demonstrates how this insurance risk can be transferred to the financial market by means of CAT bonds by applying the results of premium and promised payoff.[9] This theorem will apply the result that the minimum premium θ_t

[9] As noted above, this thesis does not consider any costs such as administration costs or management fees that

that the company passes to the SPV is

$$\theta_t = \mathcal{V}_t^e(L_{t+1}),$$

and the minimum promised payoff for each CAT bond contract set by the SPV is

$$C_t = L_{max} + R_f \mathcal{V}_t^e(L_{t+1}),$$

where L_{t+1} $(0 \leq L_{t+1} \leq L_{\max})$ is insured loss at time $t+1$ and $\mathcal{V}_t^e(L_{t+1})$ is the estimation of the market valuation of possible loss at time t+1 defined as

$$\mathcal{V}_t^e(L_{t+1}) = \frac{1}{R_f}\left[E_t[L_{t+1}] - (\rho_t^e)^+ Cov_t^e[L_{t+1}, e_{t+1}^{ref+}]\right].$$

Theorem 7. *The insurance risk L_{t+1} $(0 \leq L_{t+1} \leq L_{\max})$ can be transferred from the (re-)insurance company through issuing CAT bonds if the following conditions hold:*

(I)The utility of the (re-)insurance company at time t satisfies

$$E_t\left[U_s\left(w_{s,t+1} - R_f \mathcal{V}_t^e(L_{t+1})\right)\right] > E_t\left[U_s\left(w_{s,t+1} - L_{t+1}\right)\right]$$

(II) Individual i (i=1,...,I) has the belief that the conditional covariance of financial assets payoff $\Sigma_t^{(i)+}$ is positive definite.

(III)The conditional expected value of every kth asset payoff satisfies

$$E_t[q_{k,t+1}^+] \quad > \quad \rho_t^+ Cov_t[q_{k,t+1}^+, e_{t+1}^{ref+}],$$

(IV)The new market is perceived Pareto superior to the original market if

are generated by setting up and managing the SPV.

$$2\langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)} \pi_{st}^{(i)+} \rangle + 2\langle \pi_t^{(i)}, V_t^{(i)}(\zeta_t^{(i)} - R_f \varsigma_t) \rangle + \langle (\zeta_t^{(i)} - R_f \varsigma_t), V_t^{(i)}(\zeta_t^{(i)} - R_f \varsigma_t) \rangle \geq 0,$$

where $\varsigma_t = p_{st}^{+} - p_{st}$, $\zeta_t^{(i)} = E[q_{st}^{(i)+}] - E[q_{st}^{(i)}]$.

Proof. Condition (I) is achieved directly by following Definition 5 and the setting of premium θ_t of CAT bonds. Condition (II) is to ensure that the CAT bonds are financial innovations that cannot be replicated in the market. Condition (III) is from Proposition 6 to ensure positive asset prices. Condition (IV) is derived from Definition 4 Propositions 4 and 5. Definition 4 and Proposition 4 imply that when investor's perceived market price of risk in the market with CAT bonds is higher than in the market without CAT bonds, the risk-return allocation in the market with CAT bonds is Pareto-superior to the market without CAT bonds. Then Condition (IV) can be achieved directly according to Proposition 5.

□

Theorem 7 demonstrates that conditions in which a (re-)insurance company can transfer an insurance risk successfully by issuing CAT bonds into the financial market. Condition (I) ensures that the insurance company is willing to transfer its insurance risk after comparing the conditional utility of purchasing CAT bonds with the utility without purchasing CAT bonds. Condition (II) ensures that the CAT bonds cannot be replicated by any assets in the market. In other words, the insurance risk cannot be replicated by any financial asset in the financial market. Therefore, CAT bonds can be designed as innovations and are issued to the financial market. Condition (III) ensures that the introduction of CAT bonds will not lead negative asset prices to make this model meaningful. Condition (IV) ensures at least one investor perceives that the CAT bond will bring higher return rates therefore purchase it.

4.4 Stochastic difference equations

This section will study investor behaviour in the market in relation to CAT bonds and how their interactions affect asset prices. In the literature, Grandmont [72] proposes a forecasting rule to describe how investors form their expectations and, hence, a complete description of the evolution of asset prices can be achieved. Some researchers apply a genetic algorithm as a learning and expectation device to update investors' beliefs, see Marimon [106], Arifovic [12, 13], Bullard and Dufy [33]. This section will apply the concept of a perfect forecasting rule that is introduced by Wenzelburger [147] to depict rational expectations for investors who can correctly predict the first and second moments of future payoffs. In the literature on agent-based models with heterogeneous beliefs, researchers usually assume that there are only two investors in the market, see Haltiwanger and Waldmann [74], Brock and Hommes [30] and Wenzelburger [148]. This section will therefore assume that there are only two investors in the financial market. Let us assume investor $i = 1$ is a trend follower, while investor $i = 2$ is an expert who can make perfect forecasts regarding the future payoffs of stocks. In addition, for easier calculation, we assume that there is only one type of stock (that is $K = 1$) and CAT bonds that are traded in the financial market.

Subsection 4.4.1 gives the price formulas of CAT bonds and stocks in the financial market under these two assumptions. The price formulas reveal an inderdependence concerning the heterogeneity of investors. Subsection 4.4.2 will apply the price formulas to show the existence of perfect forecasting rules for first and second moments, and then develop dynamic difference equations to describe investors' behaviour. The

difference equations will be linked to the catastrophic loss that is transferred to the market by CAT bonds.

4.4.1 Price formulas

Under the assumption that there is only one type of stock in the market, the covariance matrix for future payoffs can be reduced to

$$\Sigma_t^{(i)} = \begin{pmatrix} V_{11,t}^{(i)} & \Sigma_{12,t}^{(i)} \\ \Sigma_{12,t}^{(i)} & V_{22,t}^{(i)} \end{pmatrix}.$$

That is, the covariance matrix of stocks becomes the variance of this one type of stock.

Based on the assumption that there are only two investors in the market, by Theorem 6, the equilibrium price of the risky asset p_t can be written as

$$p_t = A_t^{(1)} q_t^{(1)} + A_t^{(2)} q_t^{(2)} - A_t(x_m - \xi_t) \tag{4.4.1}$$

with

$$A_t = \frac{1}{R_f}\left(a^{(1)}(\Sigma_t^{(1)})^{-1} + a^{(2)}(\Sigma_t^{(2)})^{-1}\right)^{-1}, \tag{4.4.2}$$

$$A_t^{(1)} = a^{(1)} A_t (\Sigma_t^{(1)})^{-1} = \left(I + \frac{a^{(2)}}{a^{(1)}}\Sigma_t^{(1)}(\Sigma_t^{(2)})^{-1}\right)^{-1}, \tag{4.4.3}$$

$$A_t^{(2)} \;=\; a^2 A_t (\Sigma_t^2)^{-1} = \left(I + \frac{a^{(1)}}{a^{(2)}}\Sigma_t^{(2)}(\Sigma_t^{(1)})^{-1}\right)^{-1}. \tag{4.4.4}$$

Observe that

$$A_t^{(1)} + A_t^{(2)} = \frac{1}{R_f} I,$$

and that A_t , $A_t^{(1)}$, $A_t^{(2)}$ are all well defined 2×2 matrices, which are invertible. Specifically,

$$A_t = \begin{pmatrix} A_{11,t} & A_{12,t} \\ A_{21,t} & A_{22,t} \end{pmatrix}, \tag{4.4.5}$$

with

$$A_{11,t} = \frac{V_{11,t}^{(2)} \det \Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) V_{11,t}^{(1)} \det \Sigma_t^{(2)}}{a^{(2)} R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) \Sigma_t^{(2)})}, \tag{4.4.6}$$

$$A_{12,t} = A_{21,t} = \frac{\Sigma_{12,t}^{(2)} \det \Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) \Sigma_{12,t}^{(1)} \det \Sigma_t^{(2)}}{a^{(2)} R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) \Sigma_t^{(2)})}, \tag{4.4.7}$$

$$A_{22,t} = \frac{V_{22,t}^{(2)} \det \Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) V_{22,t}^{(1)} \det \Sigma_t^{(2)}}{a^{(2)} R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) \Sigma_t^{(2)})}; \tag{4.4.8}$$

and

$$A_t^{(i)} = \begin{pmatrix} A_{11,t}^{(i)} & A_{12,t}^{(i)} \\ A_{21,t}^{(i)} & A_{22,t}^{(i)} \end{pmatrix}, \ i = 1, 2$$

with

$$A_{11,t}^{(1)} = \frac{(a^{(1)}/a^{(2)}) \left(V_{22,t}^{(1)} V_{11,t}^{(2)} - \Sigma_{12,t}^{(1)} \Sigma_{12,t}^{(2)} \right) + (a^{(1)}/a^{(2)})^2 \det \Sigma_t^{(2)}}{R_f \det(\Sigma_t^1 + (a^{(1)}/a^2) \Sigma_t^2)},$$

$$A_{12,t}^{(1)} = \frac{(a^{(1)}/a^{(2)})(V_{11,t}^{(1)}\Sigma_{12,t}^{(2)} - \Sigma_{12,t}^{(1)}V_{11,t}^{(2)})}{R_f \det(\Sigma_t^1 + (a^{(1)}/a^{(2)})\Sigma_t^{(2)})}, \tag{4.4.9}$$

$$A_{21,t}^{(1)} = \frac{(a^{(1)}/a^{(2)})(V_{22,t}^{(1)}\Sigma_{12,t}^{(2)} - \Sigma_{12,t}^{(1)}V_{22,t}^{(2)})}{R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)})\Sigma_t^{(2)})}, \tag{4.4.10}$$

$$A_{22,t}^{(1)} = \frac{(a^{(1)}/a^{(2)})(V_{11,t}^{(1)}V_{22,t}^{(2)} - \Sigma_{12,t}^{(1)}\Sigma_{12,t}^{(2)}) + (a^{(1)}/a^{(2)})^2 \det \Sigma_t^{(2)}}{R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)})\Sigma_t^{(2)})}. \tag{4.4.11}$$

In addition, $A_{11,t}^{(2)} = \frac{1}{R_f} - A_{11,t}^{(1)},\ A_{12,t}^{(2)} = -A_{12,t}^{(1)},\ A_{21,t}^{(2)} = -A_{21,t}^{(1)},\ A_{22,t}^{(2)} = \frac{1}{R_f} - A_{22,t}^{(1)}$.

By substituting the definitions of $A_t^{(1)}$ (formula (4.4.4)), $A_t^{(2)}$ (formula (4.4.3)) and A_t (formula 4.4.5) into the price formula (4.4.1), the equilibrium price formula of CAT bonds and stocks can be derived as

$$p_{ct} = A_{11,t}^{(1)}(q_{ct}^{(1)} - q_{ct}^{(2)}) + \frac{1}{R_f}q_{ct}^{(2)} + A_{12,t}^{(1)}q_{st}^{(1)} + A_{12,t}^{(2)}q_{st}^{(2)} - A_{11,t}(x_{cm} - \xi_{ct}) - A_{12,t}\left(x_{sm} - \xi_{st}\right), \tag{4.4.12}$$

$$p_{st} = A_{21,t}^{(1)}(q_{ct}^{(1)} - q_{ct}^{(2)}) + A_{22,t}^{(1)}q_{st}^{(1)} + A_{22,t}^{(2)}q_{st}^{(2)} - A_{12,t}\left(x_{cm} - \xi_{ct}\right) - A_{22,t}\left(x_{sm} - \xi_{st}\right) \tag{4.4.13}$$

respectively.

The price laws – formulas (4.4.12) and (4.4.13) – show that, at equilibrium, the price of CAT bonds and stocks are linked to each other by investors' subjective beliefs regarding the future payoffs of stocks and CAT bonds. Both the prices of CAT bonds

and stocks are affected by investors' beliefs about CAT bonds and stocks and the noise traders from both markets.

As both investors receive the same information of catastrophic events released from a particular institute, it is reasonable to assume that both investors have the same expectations of CAT bond payoffs, given information available at time t. Under this situation, investors' beliefs about CAT bonds are the same; that is

$$q_{ct}^{(1)} = q_{ct}^{(2)} = c_t - E_t(l_{t+1}), \tag{4.4.14}$$

and

$$V_{11,t}^{(1)} = V_{11,t}^{(2)} = Var_t(l_{t+1}). \tag{4.4.15}$$

Applying the result from Subsection 4.3 on issuance conditions, when the SPV wants to rise L_{max} by issuing CAT bonds, the promised payoff c_t to each CAT bond contract is

$$c_t = l_{max} + R_f \mathcal{V}_t^e(l_{t+1}),$$

where $\mathcal{V}_t^e(l_{t+1})$ is the estimated valued and defined by the formula (4.2.6)

$$\mathcal{V}_t^e(l_{t+1}) = \frac{1}{R_f}\left[E_t[l_{t+1}] - \rho_t^e Cov_t^e[l_{t+1}, e_{t+1}^{ref+}]\right].$$

So, investors' beliefs about CAT bonds for the first moment can be written as

$$
\begin{aligned}
q_{ct}^{(2)} &= l_{max} + R_f \mathcal{V}_t^e(l_{t+1}) - E_t(l_{t+1}) \\
&= l_{max} - \rho_t^e Cov_t^e[l_{t+1}, e_{t+1}^{ref+}].
\end{aligned}
$$

This shows that the expected value of a CAT bond' payoff is reduced to a CAT bond's maximum covered loss minus a term. That term is the estimated values of market price of risk and the correlation of insurance loss and market payoffs by the SPV.

In this case, the CAT bond price formula (4.4.12) and the stock price formula (4.4.13) can be written as

$$
p_{ct} = \frac{1}{R_f} q_{ct}^{(2)} + A_{12,t}^{(1)} q_{st}^{(1)} + A_{12,t}^{(2)} q_{st}^{(2)} - A_{11,t}(x_{cm} - \xi_{ct}) - A_{12,t}\left(x_{sm} - \xi_{st}\right), \quad (4.4.16)
$$

$$
p_{st} = A_{22,t}^{(1)} q_{st}^{(1)} + A_{22,t}^{(2)} q_{st}^{(2)} - A_{12,t}\left(x_{cm} - \xi_{ct}\right) - A_{22,t}\left(x_{sm} - \xi_{st}\right) \quad (4.4.17)
$$

respectively, with

$$
q_{ct}^{(2)} = l_{max} - \rho_t^e Cov_t^e[l_{t+1}, e_{t+1}^{ref+}].
$$

This demonstrates that when investors' beliefs about CAT bonds are the same as the objective mean and variance of CAT bonds, prices of CAT bonds are affected by the SPV's estimated market values. Both prices are affected by investors' beliefs

about stocks, the covariance between the insured loss with stocks, the variance of insured loss and the noise traders (ξ_{ct}, ξ_{st}) from both markets.

Together with forecasting rules for first and second moments, equations (4.4.16) and (4.4.17) form a system of stochastic difference equations that describe the co-evolutions of stocks and CAT bonds prices.

4.4.2 Perfect forecasting rules

This subsection will apply price formulas achieved in the 4.4.1 to look into the existence of perfect forecasting rules for first and second moments.

For an easier calculation and clear analysis, the probabilistic prerequisites for the exogenous noise and the exogenous dividend process are stipulated in the assumptions from Wenzelburger [149]. In addition, assume that there is no correlation of the noise traders with the loss process.

Assumption 6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ an increasing family of sub-σ-algebras of $\mathcal{F}$.*

(1) Assume that the process of noise traders $\{\xi_t\}_{t\in N}$ are governed by an exogenous i.i.d. process; with $E_t(\xi_{c,t+1}) = E_t(\xi_{s,t+1}) = 0$, $Var_t(\xi_{c,t+1}) = Var_t(\xi_{s,t+1}) = 1$ and $Cov_t(\xi_{ct,}\xi_{st}) = 0$.

(2) Assume that the dividend payments $\{d_t\}$ is a $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ adapted stochastic process, its conditional mean and variance are $E_t(d_{t+1}) = d_{t+1}$ and $V_t(d_{t+1}) = 0$.

(3) Assume the loss process $\{L_t\}$ has no correlation with the noise traders from the stock market $Cov_t(\xi_{s,t+1}, L_{t+1}) = 0$.

Assumption 6 (1) assumes that the mean and variance of the noise traders in both CAT bonds and stocks market have mean 0 and variance 1. Assumption (2) defines the dividend is known at time t (3) defines that there is no correlation of the insured loss with the noise traders from the stock market.

So, by formula (4.4.17), the conditional expected value and variance of the stock price is

$$E_{t-1}(p_{st}) = A^{(1)}_{22,t}q^{(1)}_{st} + A^{(2)}_{22,t}q^{(2)}_{st} - A_{12,t}x_{cm} - A_{22,t}x_{sm}, \tag{4.4.18}$$

$$\begin{aligned} Var_{t-1}(p_{st}) &= (A_{12,t})^2 Var_{t-1}(\xi_{ct}) + (A_{22,t})^2 Var_{t-1}(\xi_{st}) \\ &= (A_{12,t})^2 + (A_{22,t})^2, \end{aligned} \tag{4.4.19}$$

respectively.

The conditional covariance between the stock and the CAT bond is

$$\begin{aligned} Cov_{t-1}[q_{st}, q_{ct}] &= Cov_{t-1}[(p_{st} + d_t), (c_{t-1} - l_t)] \\ &= -Cov_{t-1}[l_t, p_{st}]. \end{aligned} \tag{4.4.20}$$

as $q_{st} = p_{st} + d_t$ and $q_{ct} = c_{t-1} - l_t$.

By substituting the stock price formula (4.4.17) into (4.4.20), one receives

$$
\begin{aligned}
Cov_{t-1}[q_{st}, q_{ct}] &= -Cov_{t-1}[l_t, (A_{12,t} \cdot \xi_{ct} + A_{22,t} \cdot \xi_{st})] \\
&= -A_{12,t} Cov_{t-1}[l_t, \xi_{ct}] - A_{22,t} Cov_{t-1}[l_t, \xi_{st}] \\
&= -A_{12,t} Cov_{t-1}[l_t, \xi_{ct}]. \qquad (4.4.21)
\end{aligned}
$$

As the investor $i = 2$ can make a perfect forecast, its expected mean and variance for the stock payoff should equal the objective ones. In addition to that, its expected covariance between the stock and the CAT bond should equal the objective conditional covariance. Mathematically, they are

$$
\begin{cases}
\text{(I)} \;\; q_{st}^{(2)} & = E_t(q_{s,t+1}), \\
\text{(II)} \;\; V_{22,t}^{(2)} & = Var_t(q_{s,t+1}), \\
\text{(III)} \;\; V_{12,t}^{(2)} & = Cov_t(q_{st}, q_{ct}).
\end{cases} \qquad (4.4.22)
$$

As $q_{st}^{(2)} = p_{st}^{(2)} + d_{t+1}$, the condition (I) can be rewritten as

$$
p_{st}^{(2)} = E_t(p_{s,t+1}). \qquad (4.4.23)
$$

Hence, according to the conditional expected value of the stock price (formula (4.4.18)), formula (4.4.23) equals to

$$
p_{s,t-1}^{(2)} = A_{22,t}^{(1)} q_{st}^{(1)} + A_{22,t}^{(2)} q_{st}^{(2)} - A_{12,t} x_{cm} - A_{22,t} x_{sm}.
$$

Solving for $q_{st}^{(2)}$, the perfect forecasting rule of stock future payoff for the first moment is

$$q_{st}^{(2)} = \frac{1}{A_{22,t}^{(2)}} \left(p_{s,t-1}^{(2)} - A_{22,t}^{1} q_{st}^{1} + A_{12,t} x_{cm} + A_{22,t} x_{sm} \right). \tag{4.4.24}$$

As $q_{st}^{(2)} = p_{st}^{(2)} + d_{t+1}$,

$$p_{st}^{(2)} = \frac{1}{A_{22,t}^{(2)}} \left(p_{s,t-1}^{(2)} - A_{22,t}^{(1)} q_{st}^{(1)} + A_{12,t} x_{cm} + A_{22,t} x_{sm} \right) - d_{t+1},$$

which is the investor $i = 2$'s perfect forecasting rule for the first moment of the stock future price. This result gives the proposition below.

Proposition 7. *The perfect forecasting rule for the first moment of the stock payoffs by investor $i = 2$ is*

$$q_{st}^{(2)} = \frac{1}{A_{22,t}^{(2)}} \left(p_{s,t-1}^{(2)} - A_{22,t}^{1} q_{st}^{1} + A_{12,t} x_{cm} + A_{22,t} x_{sm} \right).$$

By substituting the forecasting rule (4.4.24) into the stock price formula (4.4.17), the stock price is constituted by the perfect forecasting rule of stock prices, noise traders and matrices $A_{12,t}$, $A_{22,t}$, that is

$$p_{st} \;=\; p_{s,t-1}^{(2)} + (A_{12,t}\xi_{ct} + A_{22,t}\xi_{st}). \tag{4.4.25}$$

This formula shows how investors' beliefs affect the stock price. It is decided by

investor $i = 2$'s forecasting of the stock price and noise traders from both CAT bonds and stocks markets. In addition, the extent of noise traders' effects are decided by both investors' beliefs about the risks of stocks, and the conditional covariance of CAT bonds and stocks.

The following seeks to identify a perfect forecasting rule for the conditional covariance of stocks and CAT bonds and the conditional variance of stocks. By substituting the conditional variance formula of stocks (4.4.19) and conditional covariance formula (4.4.21) into conditions (II) and (III) respectively, conditions (II) and (III) become

$$\text{(II) } V^{(2)}_{22,t-1} = (A_{12,t})^2 + (A_{22,t})^2, \tag{4.4.26}$$

$$\text{(III) } V^{(2)}_{12,t-1} = -A_{12,t} Cov_{t-1}[l_t, \xi_{ct}], \tag{4.4.27}$$

respectively.

Consider a general case in which investors' beliefs regarding to the conditional covariance of CAT bonds and stocks are not zero; that is $V^{(1)}_{12,t} \neq V^{(2)}_{12,t} \neq 0$. So, from condition (III), the expression of $A_{12,t}$ can be written as

$$A_{12,t} = -\frac{\Sigma^{(2)}_{12,t-1}}{Cov_{t-1}[l_t, \xi_{ct}]}. \tag{4.4.28}$$

Using this expression of $A_{12,t}$, the condition (II) can be rewritten as

$$V^{(2)}_{22,t-1} - \left(\frac{\Sigma^{(2)}_{12,t-1}}{Cov_{t-1}[l_t, \xi_{ct}]} \right)^2 = (A_{22,t})^2 \tag{4.4.29}$$

By substituting the definition of $A_{12,t}$ (4.4.7) and $A_{22,t}$ (4.4.8) into formulas (4.4.28)

and (4.4.29),simultaneous equations with two arguments $\Sigma^{(2)}_{12,t}$ and $V^{(2)}_{22,t}$ are achieved:

$$\begin{cases} -\frac{\Sigma^{(2)}_{12,t-1}}{Cov_{t-1}[l_t,\xi_{ct}]} & = \frac{\Sigma^{(2)}_{12,t}\det\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(1)}_{12,t}\det\Sigma^{(2)}_t}{a^{(2)}R_f\det(\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(2)}_t)} \\ \pm\sqrt{V^{(2)}_{22,t-1}-\left(\frac{\Sigma^{(2)}_{12,t-1}}{Cov_{t-1}[l_t,\xi_{ct}]}\right)^2} & = \frac{V^{(2)}_{22,t}\det\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})V^{(1)}_{22,t}\det\Sigma^{(2)}_t}{a^{(2)}R_f\det(\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(2)}_t)}. \end{cases}$$

Solving these two equations by replacing $\det\Sigma^{(2)}_t$ and $\det(\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(2)}_t)$

$$\det\Sigma^{(2)}_t = V^{(2)}_{11,t}V^{(2)}_{22,t}-\Sigma^{(2)}_{12,t}\Sigma^{(2)}_{21,t}$$

and

$$\det(\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(2)}_t) =$$

$$\left(V^{(1)}_{11,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{11,t}\right)\left(V^{(1)}_{22,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)-\left(\Sigma^{(1)}_{12,t}+\frac{a^{(1)}}{a^{(2)}}\Sigma^{(2)}_{12,t}\right)\left(\Sigma^{(1)}_{21,t}+\frac{a^{(1)}}{a^{(2)}}\Sigma^{(2)}_{21,t}\right),$$

respectively. Several results are achieved.

Proposition 8. *The perfect forecasting rule for the second moment and the covariance are calculated under three situations. Define* $(a^{(1)}/a^{(2)}) = D$, $V^{(1)}_{11,t} = V^{(2)}_{11,t} = n \neq 0$, $\Sigma^{(1)}_{12,t} = \Sigma^{(1)}_{21,t} = m \neq 0$, $V^{(1)}_{22,t} = w \neq 0$. *Note that* $\Sigma^{(2)}_{12,t} = \Sigma^{(2)}_{21,t}$ *so this Proposition only shows the value of* $\Sigma^{(2)}_{12,t}$.

$$\text{Situation1, } a^{(1)}HR_fn(1+D)-Dmn=0,\ P_1na^{(1)}(1+D)-\det\Sigma^1_t-Dwn\neq 0$$

$$\begin{cases} \Sigma_{12,t}^{(2)} & = \frac{(2Ha^{(1)}R_f m + \det\Sigma_t^1) \pm \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\ V_{22,t}^{(2)} & = \frac{2a^{(1)}mP_1X_1 - (Dw - P_1a^{(1)}D)X_1^2 - (a^2\det\Sigma_t^1 + a^{(1)}nw)P_1}{(P_1na^{(1)} + P_1Da^{(1)}n - \det\Sigma_t^1 - Dwn)} \end{cases}$$

$$\begin{cases} \Sigma_{12,t}^{(2)} & = \frac{(2Ha^{(1)}R_f m + \det\Sigma_t^1) \pm \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\ V_{22,t}^{(2)} & = \frac{2a^{(1)}mP_2X_1 - (Dw - P_2a^{(1)}D)X_1^2 - (a^2\det\Sigma_t^1 + a^{(1)}nw)P_2}{(P_2na^{(1)} + P_2Da^{(1)}n - \det\Sigma_t^1 - Dwn)} \end{cases}$$

if $\Delta_1 = (2Ha^{(1)}R_f m + \det\Sigma_t^1)^2 + 4a^{(1)}H^2R_f^2(a^{(2)}\det\Sigma_t^1 + a^{(1)}nw) \geq 0$.

Situation 2, $a^{(1)}HR_f n(1+D) - Dmn \neq 0$, $P_1na^{(1)}(1+D) - \det\Sigma_t^1 - Dwn = 0$.

$$\begin{cases} \Sigma_{12,t}^{(2)} & = \frac{2P_1a^{(1)}m \pm \sqrt{\Delta_{2(P1)}}}{2D(w - P_1a^{(1)})} \\ V_{22,t}^{(2)} & = \frac{(Ha^{(1)}R_fD - Dm)X_{1P1}^2 + (2Ha^{(1)}R_f m + \det\Sigma_t^1)X_{1P1} - (a^{(2)}\det\Sigma_t^1 + a^{(1)}nw)HR_f}{(a^{(1)}HR_f(1+D) - Dm)n} \end{cases}$$

if $\Delta_{2(P1)} = 4P_1^2(a^{(1)})^2m^2 - 4DP_1(w - P_1a^{(1)})(a^{(2)}\det\Sigma_t^1 + a^{(1)}nw) \geq 0$.

$$\begin{cases} \Sigma_{12,t}^{(2)} & = \frac{2P_1a^{(1)}m \pm \sqrt{\Delta_{2(P2)}}}{2D(w - P_1a^{(1)})} \\ V_{22,t}^{(2)} & = \frac{(Ha^{(1)}R_fD - Dm)X_{1P2}^2 + (2Ha^{(1)}R_f m + \det\Sigma_t^1)X_{1P2} - (a^{(2)}\det\Sigma_t^1 + a^{(1)}nw)HR_f}{(a^{(1)}HR_f(1+D) - Dm)n} \end{cases}$$

if $\Delta_{2(P2)} = 4P_2^2(a^{(1)})^2m^2 - 4DP_2(w - P_2a^{(1)})(a^{(2)}\det\Sigma_t^1 + a^{(1)}nw) \geq 0$.

Situation 3 $a^{(1)}HR_f n(1+D) - Dmn \neq 0$, $P_1na^{(1)}(1+D) - \det\Sigma_t^1 - Dwn \neq 0$

$$\begin{cases} \Sigma_{12,t}^{(2)} & = \frac{-\left(2Ha^{(1)}R_f m + \det\Sigma_t^1 - 2P_2a^{(1)}mW_{P2}\right) + \sqrt{\Delta_{3(P2)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_2a^{(1)})W_{P2}\right)} \\ V_{22,t}^{(2)} & = \frac{2a^{(1)}mP_2X_{1P2} - D(w - P_2a^{(1)})X_{1P2}^2 - (a^{(2)}\det\Sigma_t^1 + a^{(1)}nw)P_2}{(P_2na^{(1)}(1+D) - \det\Sigma_t^1 - Dwn)}. \end{cases}$$

if $\Delta_{3(P2)} = \left(2Ha^{(1)}R_f m + \det\Sigma_t^1 - 2P_2a^{(1)}mW_{P2}\right)^2 \geq 0$.

Proof. See Appendix B.

□

Therefore, dynamic difference equations are

$$\begin{cases} p_{ct} & = \frac{1}{R_f} q_{ct}^{(2)} + p_{s,t-1}^{(2)} + (A_{12,t}^{(1)} - A_{22,t}^{(1)}) q_{st}^{(1)} + (A_{12,t} - A_{11,t}) x_{cm} + \\ & (A_{22,t} - A_{12,t}) x_{sm} + A_{11,t}\xi_{ct} + A_{12,t}\xi_{st} \\ p_{st} & = p_{s,t-1}^{(2)} + A_{12,t}\xi_{ct} + A_{22,t}\xi_{st} \\ p_{st}^{(2)} & = \frac{1}{A_{22,t}^{(2)}} \left(p_{s,t-1}^{(2)} - A_{22,t}^{(1)} q_{st}^{(1)} + A_{12,t} x_{cm} + A_{22,t} x_{sm} \right) - d_{t+1} \end{cases} \tag{4.4.30}$$

with $\Sigma_{12,t}^{(2)}$ and $V_{22,t}^{(2)}$ are defined in Proposition 8.

4.4.3 Special case with zero correlation

This subsection will consider a special case in which both investors ($i = 1, 2$) believe that there is no conditional correlation of CAT bonds and stocks, and will explore how such beliefs affect the market.

If both investors think that there is no correlation between CAT bonds and stocks, that is $\Sigma_{12,t}^{(1)} = \Sigma_{12,t}^{(2)} = 0$, by the definition of $A_{12,t}$ (4.4.7), $A_{12,t}^{(1)}$ (4.4.9), the values of $A_{12,t}$, $A_{12,t}^{(1)}$ and $A_{12,t}^{(2)}$ ($A_{12,t}^{(2)} = -A_{12,t}^{(1)}$) are all equal to zero, that is

$$A_{12,t} = A_{12,t}^{(1)} = A_{12,t}^{(2)} = 0.$$

The covariance matrices of$\Sigma_t^{(1)}$ and $\Sigma_t^{(2)}$ become

$$\Sigma_t^{(1)} = \begin{pmatrix} V_{11,t}^{(1)} & 0 \\ 0 & V_{22,t}^{(1)} \end{pmatrix}$$

$$\Sigma_t^{(2)} = \begin{pmatrix} V_{11,t}^{(2)} & 0 \\ 0 & V_{22,t}^{(2)} \end{pmatrix}.$$

So the determinant of $\Sigma_t^{(1)}$ and $\Sigma_t^{(2)}$ are simply

$$\det \Sigma_t^{(1)} = V_{11,t}^{(1)} V_{22,t}^{(1)},$$

$$\det \Sigma_t^{(2)} = V_{11,t}^{(2)} V_{22,t}^{(2)}.$$

The determinant of $\det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)})\Sigma_t^{(2)})$ is

$$\det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)})\Sigma_t^{(2)}) = V_{11,t}^{(1)} \left(1 + \frac{a^{(1)}}{a^{(2)}}\right) \left(V_{22,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)}\right),$$

with the assumption (see Subsection 4.4.1) that investors hold the same beliefs on CAT bonds: $V_{11,t}^{(1)} = V_{11,t}^{(2)}$. Therefore, the formula of $A_{22,t}$ (4.4.8) can be written as

$$\begin{aligned} A_{22,t} &= \frac{V_{22,t}^{(2)} \det \Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) V_{22,t}^{(1)} \det \Sigma_t^{(2)}}{a^{(2)} R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)})\Sigma_t^{(2)})} \\ &= \frac{V_{22,t}^{(2)} V_{11,t}^{(1)} V_{22,t}^{(1)} (1 + \frac{a^{(1)}}{a^{(2)}})}{a^{(2)} R_f V_{11,t}^{(1)} \left(1 + \frac{a^{(1)}}{a^{(2)}}\right) \left(V_{22,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)}\right)} \end{aligned}$$

If $a^{(1)} \neq -a^{(2)}$, $A_{22,t}$ can be written as

$$A_{22,t} = \frac{1}{R_f\left(\frac{a^{(2)}}{V^{(2)}_{22,t}} + \frac{a^{(1)}}{V^{(1)}_{22,t}}\right)}. \tag{4.4.31}$$

Formula $A^{(1)}_{22,t}$ (4.4.11) can be written as

$$\begin{aligned}
A^{(1)}_{22,t} &= \frac{\frac{a^{(1)}}{a^{(2)}}V^{(1)}_{11,t}V^{(2)}_{22,t} + (\frac{a^{(1)}}{a^{(2)}})^2 V^{(2)}_{11,t}V^{(2)}_{22,t}}{R_f V^{(1)}_{11,t}\left(1+\frac{a^{(1)}}{a^{(2)}}\right)\left(V^{(1)}_{22,t} + \frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)} \\
&= \frac{\frac{a^{(1)}}{a^{(2)}}V^{(1)}_{11,t}V^{(2)}_{22,t}\left(1+\frac{a^{(1)}}{a^{(2)}}\right)}{R_f V^{(1)}_{11,t}\left(1+\frac{a^{(1)}}{a^{(2)}}\right)\left(V^{(1)}_{22,t} + \frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)} \\
&= \frac{\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}}{R_f\left(V^{(1)}_{22,t} + \frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)}.
\end{aligned}$$

If $a^{(1)} \neq -a^{(2)}$, $A^{(1)}_{22,t}$ becomes

$$A^{(1)}_{22,t} = \frac{1}{a^{(2)}R_f\left(\frac{V^{(1)}_{22,t}}{V^{(2)}_{22,t}} + \frac{a^{(1)}}{a^{(2)}}\right)}. \tag{4.4.32}$$

As the relation of $A^{(1)}_{22,t}$ and $A^{(2)}_{22,t}$ is $A^{(2)}_{22,t} = \frac{1}{R_f} - A^{(1)}_{22,t}$, so $A^{(2)}_{22,t}$ can be derived based on formula (4.4.32) as

$$
\begin{aligned}
A_{22,t}^{(2)} &= \frac{1}{R_f} - \frac{\frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)}}{R_f \left(V_{22,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)} \right)} \\
&= \frac{V_{22,t}^{(1)}}{R_f \left(V_{22,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)} \right)} \\
&= \frac{1}{R_f \left(1 + \frac{a^{(1)}}{a^{(2)}} \frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}} \right)}. \qquad (4.4.33)
\end{aligned}
$$

The formula of $A_{11,t}$ (4.4.6) can be written as

$$
\begin{aligned}
A_{11,t} &= \frac{V_{11,t}^{(2)} \det \Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) V_{11,t}^{(1)} \det \Sigma_t^{(2)}}{a^{(2)} R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) \Sigma_t^{(2)})} \\
&= \frac{V_{11,t}^{(2)} V_{11,t}^{(1)} \left(V_{22,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)} \right)}{a^{(2)} R_f V_{11,t}^{(1)} \left(1 + \frac{a^{(1)}}{a^{(2)}} \right) \left(V_{22,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{22,t}^{(2)} \right)} \\
&= \frac{V_{11,t}^{(2)}}{R_f \left(a^{(1)} + a^{(2)} \right)} \qquad (4.4.34)
\end{aligned}
$$

Based on the simplified results of $A_{22,t}^{(2)}$ (4.4.33), $A_{22,t}^{(1)}$ (4.4.32), $A_{12,t} = 0$ and $A_{22,t}$ (4.4.31), the perfect forecast for the stock price $p_{st}^{(2)}$ (4.4.30) is

$$
\begin{aligned}
p_{st}^{(2)} &= \frac{1}{A_{22,t}^{(2)}}\left(p_{s,t-1}^{(2)} - A_{22,t}^{(1)}q_{st}^{(1)} + A_{12,t}x_{cm} + A_{22,t}x_{sm}\right) - d_{t+1}, \\
&= \frac{1}{A_{22,t}^{(2)}}\left(p_{s,t-1}^{(2)} - A_{22,t}^{(1)}q_{st}^{(1)} + A_{22,t}x_{sm}\right) - d_{t+1}, \\
&= R_f\left(1 + \frac{a^{(1)}}{a^{(2)}}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}}\right)\left(p_{s,t-1}^{(2)} - \frac{q_{st}^{(1)}}{a^{(2)}R_f\left(\frac{V_{22,t}^{(1)}}{V_{22,t}^{(2)}} + \frac{a^{(1)}}{a^{(2)}}\right)} + \frac{V_{22,t}^{(1)}x_{sm}}{a^{(2)}R_f\left(\frac{V_{22,t}^{(1)}}{V_{22,t}^{(2)}} + \frac{a^{(1)}}{a^{(2)}}\right)}\right) - d_{t+1}, \\
&= R_f\left(1 + \frac{a^{(1)}}{a^{(2)}}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}}\right)p_{s,t-1}^{(2)} - \frac{V_{22,t}^{(2)}}{a^{(2)}}.x_{sm} + \frac{q_{st}^{(1)}V_{22,t}^{(2)}}{a^{(2)}V_{22,t}^{(1)}} - d_{t+1}, \\
&= \frac{1}{a^{(2)}}\left[R_f p_{s,t-1}^{(2)}\left(a^{(1)}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}} + a^{(2)}\right) - V_{22,t}^{(2)}x_{sm} + q_{st}^{(1)}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}}\right] - d_{t+1},
\end{aligned}
$$

and $q_{st}^{(2)}$ is

$$
\begin{aligned}
q_{st}^{(2)} &= p_{st}^{(2)} + d_{t+1} \\
&= \frac{1}{a^{(2)}}\left[R_f p_{s,t-1}^{(2)}\left(a^{(1)}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}} + a^{(2)}\right) - V_{22,t}^{(2)}x_{sm} + q_{st}^{(1)}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}}\right].
\end{aligned}
$$

So, the perfect forecast rule for the first moment is simplified and only related to the variance of stocks from both investors' beliefs. Under the assumption that investors believe that there is no correlation of CAT bonds and stocks, it shows investor $i = 2$'s perfect forecast for the first moment is no longer related to the variance of CAT bonds.

After having the perfect forecast rule for the first moment, the investor $i = 2$'s perfect beliefs for the conditional covariance and variance for the asset payoff will be

calculated. Under this assumption, the requirement (iii) (4.4.27) will always hold, and condition (ii) (4.4.26) becomes

$$V_{22,t-1}^{(2)} = (A_{22,t})^2.$$

By the definition of $A_{22,t}$ (4.4.8), the above formula can be written as

$$\sqrt{V_{22,t-1}^{(2)}} = \frac{V_{22,t}^{(2)} \det \Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) V_{22,t}^{(1)} \det \Sigma_t^{(2)}}{a^{(2)} R_f \det(\Sigma_t^{(1)} + (a^{(1)}/a^{(2)}) \Sigma_t^{(2)})}.$$

Solving for $V_{22,t}^{(2)}$,

$$\begin{aligned} V_{22,t}^{(2)} &= \frac{V_{22,t}^{(1)} \sqrt{V_{22,t-1}^{(2)}}}{\frac{V_{11,t}^{(1)} V_{22,t}^{(1)} + (\frac{a^{(1)}}{a^{(2)}}) V_{22,t}^{(1)} V_{11,t}^{(2)}}{R_f a^{(2)} \left(V_{11,t}^{(1)} + \frac{a^{(1)}}{a^{(2)}} V_{11,t}^{(2)} \right)} - \frac{a^{(1)}}{a^{(2)}} \sqrt{V_{22,t-1}^{(2)}}}, \\ &= \frac{a^{(2)}}{\frac{1}{R_f \sqrt{V_{22,t-1}^{(2)}}} - \frac{a^{(1)}}{V_{22,t}^{(1)}}}. \end{aligned}$$

Compared with Proposition 8, this perfect forecast rule for the second moment is simplified: only the investor $i = 1$'s belief about the variance of stocks matters.

Proposition 9. *Assume that investors hold the belief that there is no correlation of the CAT bonds and stocks, mathematically the following equation holds*

$$A_{12,t} = A_{12,t}^{(1)} = A_{12,t}^{(2)} = 0.$$

The perfect rules for the first moment and second moment are

$$q_{st}^{(2)} = \frac{1}{a^{(2)}}\left[R_f p_{s,t-1}^{(2)}\left(a^{(1)}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}} + a^{(2)}\right) - V_{22,t}^{(2)} x_{sm} + q_{st}^{(1)}\frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}}\right]$$

and

$$V_{22,t}^{(2)} = \frac{a^{(2)}}{\frac{1}{R_f\sqrt{V_{22,t-1}^{(2)}}} - \frac{a^{(1)}}{V_{22,t}^{(1)}}}$$

respectively.

Proof. See Apendix B.

□

By the assumption that $A_{12,t} = A_{12,t}^{(1)} = A_{12,t}^{(2)} = 0$, the prices of CAT bonds (4.4.12) and stocks (4.4.13) become

$$\begin{cases} p_{ct} &= \frac{1}{R_f} q_{ct}^{(2)} - A_{11,t}(x_{cm} - \xi_{ct}) \\ p_{st} &= p_{s,t-1}^{(2)} + A_{22,t}\xi_{st}. \end{cases}$$

By substituting $A_{11,t}$ (4.4.34) and $A_{22,t}$ (4.4.31) into the above joint formulas,

$$\begin{cases} p_{ct} &= \frac{1}{R_f} q_{ct}^{(2)} - \frac{V_{11,t}^{(2)}}{R_f\left(a^{(1)}+a^{(2)}\right)}(x_{cm} - \xi_{ct}) \\ p_{st} &= p_{s,t-1}^{(2)} + \frac{1}{R_f\left(\frac{a^{(2)}}{V_{22,t}^{(2)}} + \frac{a^{(1)}}{V_{22,t}^{(1)}}\right)}\xi_{st}. \end{cases}$$

So, under the assumption that $V_{12,t}^{(1)} = V_{12,t}^{(2)} = 0$, the stochastic difference equations in this system are shown below,

$$
\begin{cases}
p_{ct} & = \frac{1}{R_f} q_{ct}^{(2)} - \frac{V_{11,t}^{(2)}}{R_f\left(a^{(1)}+a^{(2)}\right)}(x_{cm} - \xi_{ct}) \\
p_{st} & = p_{s,t-1}^{(2)} + \frac{1}{R_f\left(\frac{a^{(2)}}{V_{22,t}^{(2)}} + \frac{a^{(1)}}{V_{22,t}^{(1)}}\right)} \xi_{st}. \\
p_{st}^{(2)} & = \frac{1}{a^{(2)}} \left[R_f p_{s,t-1}^{(2)} \left(a^{(1)} \frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}} + a^{(2)} \right) - V_{22,t}^{(2)} x_{sm} + q_{st}^{(1)} \frac{V_{22,t}^{(2)}}{V_{22,t}^{(1)}} \right] - d_{t+1} \\
V_{22,t}^{(2)} & = \frac{a^{(2)}}{a^{(1)}} \frac{V_{22,t}^{(1)} \sqrt{V_{22,t-1}^{(2)}}}{(R_f a^{(1)})^{-1} V_{11,t}^{(1)} - \sqrt{V_{22,t-1}^{(2)}}}.
\end{cases}
$$

It shows that when investors hold belief that there is no correlation between stocks and CAT bonds then the prices of stock and CAT bonds are no longer related to each other. It shows how investors' beliefs drive price movements in the financial market. Each asset price is only related to investors' beliefs about this specific asset. It also can be confirmed by the objective conditional covariance that there is no correlation between CAT bonds and stocks.

The objective conditional covariance between the payoff of stocks and CAT bonds (4.4.21) is

$$
Cov_{t-1}(q_{ct}, q_{st}) = -A_{12,t} Cov_{t-1}[l_t, \xi_{ct}] = 0,
$$

which shows that investors' beliefs regarding the covariance between stocks and CAT bonds drives the real covariance in the maker in the same direction.

Proposition 10. *If investors hold the same belief that there is no correlation between stocks and CAT bonds, the covariance of stocks and CAT bonds in the real market will be no longer correlated.*

This shows that investors' subjective beliefs change the prices of CAT bonds and stocks and drive the market in the direction of investors' beliefs. This result implies that the objective covariance matrix in the market is

$$\Sigma_t = \left(\begin{array}{c|c} V_{11,t} & 0 \\ \hline 0 & V_{22,t} \end{array} \right),$$

which is the same as investors' believe.

4.5 Conclusion

This chapter investigated CAT bonds in the agent-based CAPM with individuals holding heterogeneous beliefs. The premium and coupon were determined. An example showed how the SPV can apply the results of this thesis to determine the coupon and premium before issuing the CAT bond. The notion of transferable insurance risk in an agent-based model was defined. In each time period, investors have different beliefs about future payoffs. The concept of perceived Pareto superiority was defined. The interaction between two different investors were investigated by means of stochastic difference equations. Following Wenzelburger [148], the perfect forecasting rule was derived.

Chapter 5

Loss process with CAT bonds

This chapter aims to introduce the catastrophic loss process in Section 5.1, which is commonly used in actuarial science and CAT bonds research such as by Aase [1], Burneck et al. [36] and Ma and Ma [101]. Then Section 5.2 embeds this loss process to CAT bonds to simulate the effect of CAT bonds to the financial market. Subsection 5.2.1 simulates and discusses two ways how CAT bonds cover losses from catastrophic events. Following the discussion in Subsection 5.2.1, Subsection 5.2.2 explores how investors' behaviour changes when CAT bonds are introduced into the financial market. Finally, the changes in market risk are illustrated in the last subsection.

5.1 Prerequisites

This section introduces a typical process that is used to simulate insurance loss.[1] It consists of two main components. One is characterising incidence frequency, which is

[1] The main content of this section is from Cizek, P. el at. [46] and Shreve[133]

usually modelled by the claim arrival process, and will be introduced in Subsection 5.1.1.[2] As claims occur when incidents happen, the claim arriving process describes the flow of the catastrophic event. The second is used to describe the severity of loss from the occurrence of an event. It is usually called loss distribution, or claim size distribution, and will be introduced in Subsection 5.1.2. After providing a general understanding of the claim arrival process and loss distribution, Subsection 5.1.3 introduces the aggregate claim loss process.[3]

5.1.1 Claim arrival processes

The claim arrival process depicts the number of claims occurring from certain catastrophic events over a time period. In other words, it depicts the number of the occurring catastrophic events occurring over in a time period as claims happen when a catastrophic event happens. Researchers Ma and Ma [101], Schmidt [129], Jiang and Dassios [53] use a doubly stochastic Poisson process (or Cox Process) to describe the flow of claims related to catastrophic risk. This subsection introduces the Cox process. The introduction of exponential random variable and homogeneous Poisson process are used to provide an understanding of the Cox Process.

Firstly, we introduce the concept of the exponential distribution, which is used to describe the time between two events for the homogeneous Poisson process.

Definition 6. [133][exponential random variable/exponential distribution] Define τ as an exponential random variable (or τ has the exponential distribution) if its density

[2] Term 'claims' refer to insurance claims that are requested from the insurance policy holders to the insurance company.

[3] Note that notations used in this section are independent from the rest of the thesis.

function is

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and the cumulative distribution function

$$F(t) = \mathbb{P}\{\tau \leq t\} = 1 - e^{-\lambda t},\ t \geq 0$$

where λ is a positive constant.

The mean of random variable is $E[\tau] = \frac{1}{\lambda}$[4]. One property of the exponential distribution is memorylessness. This can be explained by the following example.

Example 6. [133][memorylessness property] Consider a restaurant that opens at time 0. The time waiting for the customers to arrive is exponentially distributed with mean $\frac{1}{\lambda}$ (λ is a positive constant). Suppose, no customer arrives during the time interval $[0, s]$, and after time t , the first customer arrives. The probability that the restaurant at time s will have to wait an additional t time units for the first customer to arrive is $P\{\tau > t + s | \tau > s\} = \mathbb{P}\{\tau > t\} = e^{-\lambda t}$. It shows that the probability of waiting to have the first customer at time s is the same as the probability of waiting at time 0.

The mean $\frac{1}{\lambda}$ means the average waiting time for customers to arrive. In other

[4]For a detailed calculation, see Shreve [133] p. 462.

words, the customer arrives at an average rate λ per unit time. After understanding the exponential distribution and its property, the following definition explains the homogeneous Poisson process, which applies the property of exponential distribution and is commonly used to model random events.

Definition 7. [133][Homogeneous Poisson Process] Considering a sequence $\{\tau_1, \tau_2, ...\}$ of independent exponential random variables to denote the inter-arrivals times between each event happenings. Define arrival times (or jump times) as

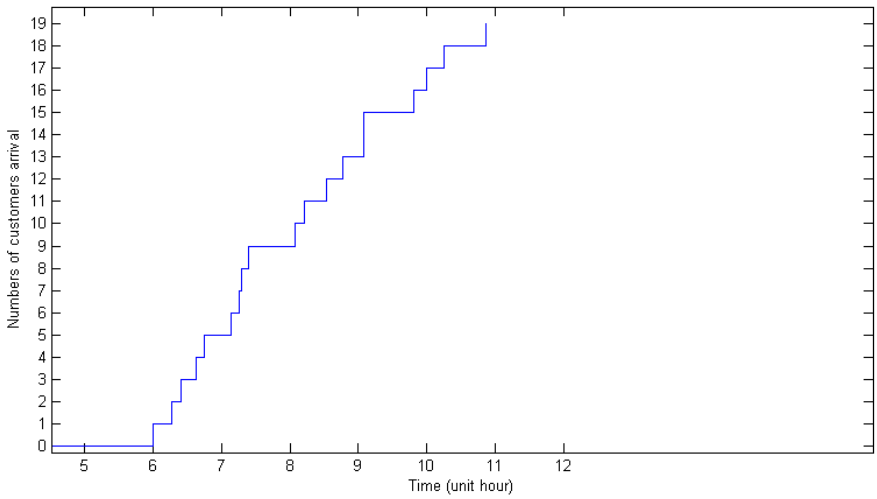
$$T_n = \sum_{k=1}^{n} \tau_k. \tag{5.1.1}$$

The Poisson Process $N(t)$ counts the number of events by jumping one unit at each arrival time T_n $(n = 1, 2, ...)$. More precisely,

$$N(t) = \begin{cases} 0 & if\ 0 \leq t < T_1 \\ 1 & if\ T_1 \leq t < T_2 \\ \vdots & \\ n & if\ T_n \leq t < T_{n+1} \\ \vdots & \end{cases}$$

By the property of exponential random variables, the expected time between each jump is $\frac{1}{\lambda}$, and the jumps arrive at an average rate of λ per unit time. The positive constant λ is called the intensity of the homogeneous Poisson process $N(t)$.

Figure 5.1.1: Customers arriving from 6 p.m. to 11 p.m.



The following example explains how the homogeneous Poisson process is used to model the arrival of customers in a restaurant.

Example 7. Continuing with Example 6, the waiters in the restaurant count the number of customers coming from time 6 p.m. to 11 p.m. Assuming the customers comes into the restaurant with a constant average rate 3 per hour, then a process that models the customer arriving from 6 p.m. to 11 p.m. can be illustrated in Figure 5.1.1.

Understanding the homogeneous Poisson process will help to understand the Cox process, which is a non-homogeneous Poisson process. Its intensity is a stochastic process rather than a constant λ, so it is reasonable to apply it to model Poisson process to describe catastrophic events, because the event does not come in a constant

rate per time unit. The definition of Cox process is given below.

Definition 8. [53][Cox process] Define a stochastic process $m(t)$

$$m(t) = \int_0^t m(u)du < \infty \ almost\ surely.$$

The Poisson process is generated by using the process $\{m(t)\}$ as its intensity process is called the Cox process, where $m(t)$ is called an intensity function.[5]

The following example is a Cox process that is used to model a catastrophic event.

Example 8. This example applies the intensity function generated by Burneck et al. [36]. They apply the loss dataset from catastrophic events provided by Property Claim Services[6] and achieve an intensity function $m(t) = 35.32 + 2.32 \cdot 2\pi \cdot sin(2\pi(t - 0.2))$. This example show the Cox process (Figure 5.1.2) that is used to describe catastrophic events with intensity function $m(t)$. It shows the flow of catastrophic events over 10 years.

5.1.2 Claim size distribution

After the introduction of the process that is used to model the flow of catastrophic events, this subsection introduces the severity of loss/claims for each incident. Claim size distribution is used to describe the probability distribution of the claim size

[5] For a srick mathematic definition of the Cox Process, see Dassios and Jang [53]

[6] Property Claim Services (PCS) is the internationally recognized authority on insured property losses from catastrophes in the United States, Puerto Rico, and the U.S. Virgin Islands.

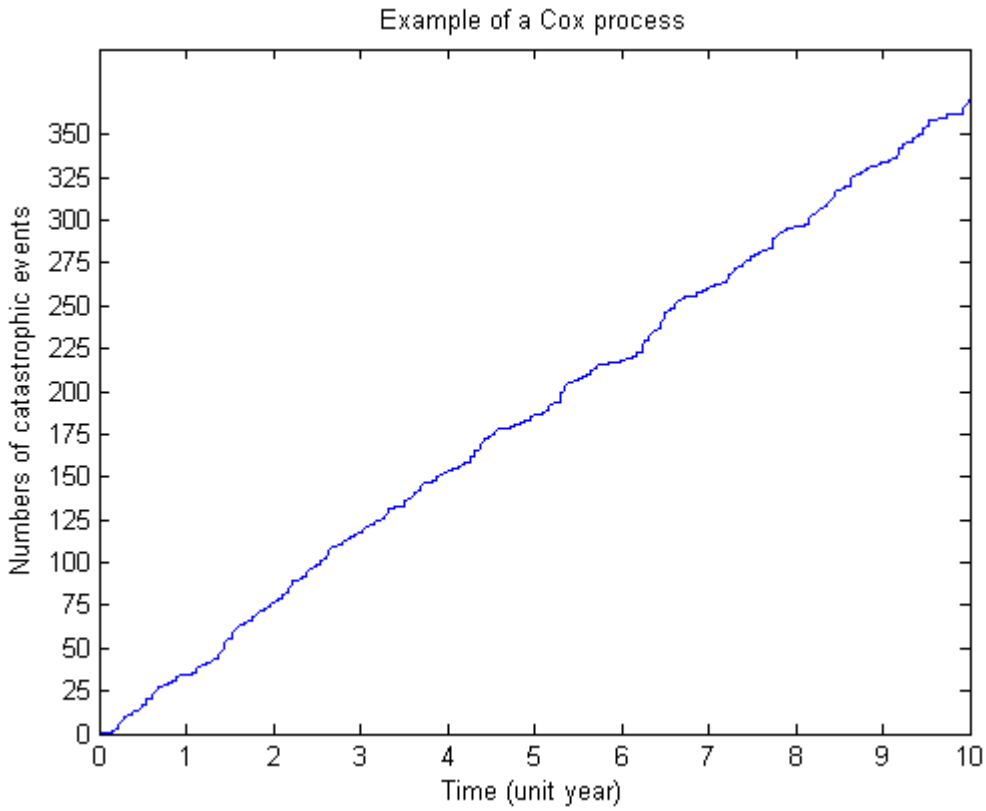


Figure 5.1.2: Example of a Cox process

occurring in catastrophic events. The distribution of claims from catastrophes is strongly skewed. Typical distributions are log-normal, Pareto, Burr, Weibull, and Gamma distributions that have a heavy-tailed distribution.

A heavy-tailed distribution has a 'heavier tail' than an exponential distribution. To explain heavy-tailed distribution, a function called complementary cumulative distribution function (ccdf) is needed. The definition below defines ccdf.[7]

Definition 9. [46][Complementary cumulative distribution function] Defining a random variable x with cumulative distribution function (cdf)

$$F(x) = \mathbb{P}\{x \leq X\},$$

its complementary cumulative distribution function (ccdf) is defined as

$$\bar{F}(x) = \mathbb{P}\{x > X\} = 1 - F(x).$$

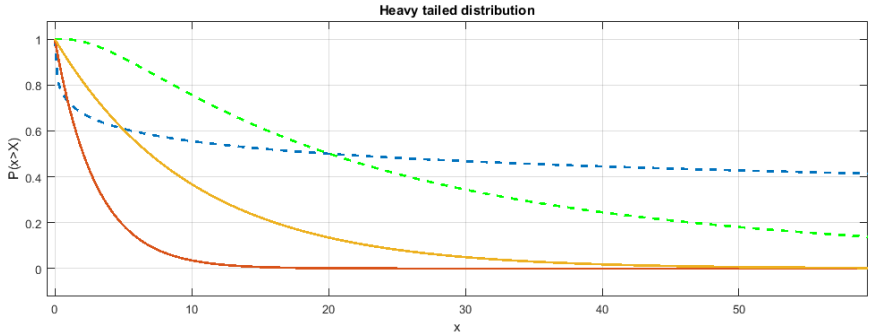
So the ccdf is the probability that the value of a random variable x is larger than a value X.

A heavy-tailed distribution has a 'heavier tail' than an exponential distribution. It implies that the ccdf of a heavy-tailed distribution decays slower than the ccdf of exponential distribution.

This implies that it has a higher probability of having a very large value. The following example gives the comparison of a log-normal distribution and exponential

[7] For a strict mathematic definition, see Shreve [133].

Figure 5.1.3: Example of a heavy- tailed distribution



The green dotted line denotes ccdf of log-normal distribution with $\mu = 3$, $\sigma = 1$; the blue dotted line denotes ccdf of log-normal distribution with $\mu = 3$, $\sigma = 1$.
The red line is the ccdf of exponential distribution with the mean 3; The brown line is the ccdf of exponential distribution with the mean 10.

distribution to show the log-normal distribution has the 'heavy tail' property.

Example 9. The log-normal distribution cdf of is given by

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right), y > 0.$$

The ccdf of exponential distribution is

$$\bar{F}(x) = e^{-\lambda x},$$

Figure 5.1.3 shows that log-normal distribution is a heavy-tailed distribution because it has a heavier tail than the exponential distribution.

5.1.3 Aggregate claim process

This subsection explains the aggregate claim loss, which is modelled by a compound Poisson process. Subsection 5.1.1 shows that a Poisson process jumps at one unit each time. The compound Poisson process allows for the jump size to be random. It is used to model the aggregate claim loss from a series insured incidents. Researchers such as Burneck et al. [36] and Ma and Ma [101] apply the compound doubly stochastic Poisson process to describe the aggregate claim process related to catastrophic loss

$$L_{At} = \begin{cases} \sum_{i=1}^{N_t} Z_i & N_t > 0 \\ 0 & N_t = 0 \end{cases},$$

where N_t is the claim arriving process with density $m(t)$ and $\{Z_i\}$ is the sequence of positive independent and identically distributed random variables with distribution function $F(z) = P(Z_i < z)$. $\{Z_i\}$ denotes the amount of claims incurred by each event in the time flow $\{\tau_i\},\ i = 1, 2, ...N_t$.

Example 10. Applying the claim arrival process in Example (8) and assuming that the claim size distribution is lognormal distributed with mean $\mu = 18.3806$ and variance $\sigma = 1.1052$, then an aggregate loss process can be plotted as shown in Figure 5.1.4.

5.2 The effects of CAT bonds

The price formulas of CAT bonds and stocks derived in Subsection 4.4.1 show how

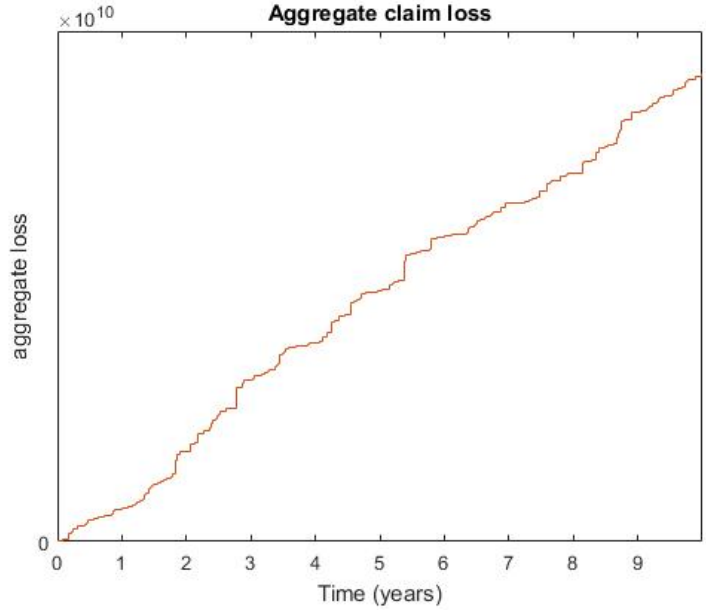


Figure 5.1.4: An aggregate claim process

the catastrophic losses are transferred into the financial market by means of CAT bonds and affect risky asset prices. That is

$$\begin{cases} p_{ct} & = \frac{1}{R_f} q_{ct}^{(2)} + A_{12,t}^{(1)} q_{st}^{(1)} + A_{12,t}^{(2)} q_{st}^{(2)} - A_{11,t}(x_{cm} - \xi_{ct}) - A_{12,t}\left(x_{sm} - \xi_{st}\right), \\ p_{st} & = A_{22,t}^{(1)} q_{st}^{(1)} + A_{22,t}^{(2)} q_{st}^{(2)} - A_{12,t}\left(x_{cm} - \xi_{ct}\right) - A_{22,t}\left(x_{sm} - \xi_{st}\right) \\ q_{ct}^{(1)} & = q_{ct}^{(2)} = c_t - E_t(l_{t+1}), \\ V_{11,t}^{(1)} & = V_{11,t}^{(2)} = V_t(l_{t+1}), \\ c_t & = l_{max} + R_f \mathcal{V}_t^e(l_{t+1}), \end{cases}$$

where $\mathcal{V}_t^e(l_{t+1}) = \frac{1}{R_f}\left[E_t[l_{t+1}] - \rho_t^e Cov_t^e[l_{t+1}, e_{t+1}^{ref+}]\right]$. This section will apply the loss process introduced above to investigate the effect of CAT bonds on the financial

market by a number of simulations. Subsection 5.2.1 illustrates how the catastrophic losses are embedded in the CAT bonds in discrete multi-time periods. In addition, two different cases are discussed about what kind of loss a CAT bond contract can cover. Subsection 5.2.2 shows how investors' demands for a risky asset changed after CAT bonds were introduced. The final subsection shows how the market risk changes when CAT bonds are introduced.

5.2.1 Loss process of CAT bonds

This subsection simulates the loss process that CAT bonds bring into the financial market in a discrete multi-time period. Most research applies the Cox process, introduced above to plot the loss of CAT bonds. However, from the reports on CAT bonds published by financial consultant companies [122], CAT bonds only cover the loss from a severe catastrophic event. That is why the return rate on CAT bonds is high and therefore attracts investors. This subsection will simulate CAT bond losses in two situations. One is that CAT bonds cover the aggregate loss at each time period. The second situation is that CAT bonds only cover the loss caused by a certain event when its loss exceeds a certain level.

The first case is illustrated in Figure 5.2.1. At each time period, once the aggregate loss exceeds a certain value, CAT bonds will use the fund raised from investors to cover the exceeded loss.

Figure 5.2.2 shows the situation in which CAT bonds only cover a specific loss that exceeds a certain level at each time period. Unlike the first scenario, Figure 5.2.2 simulates the losses occurred by each event rather than an aggregate loss of a number of events at each time period. So as long as the loss occurred from a specific

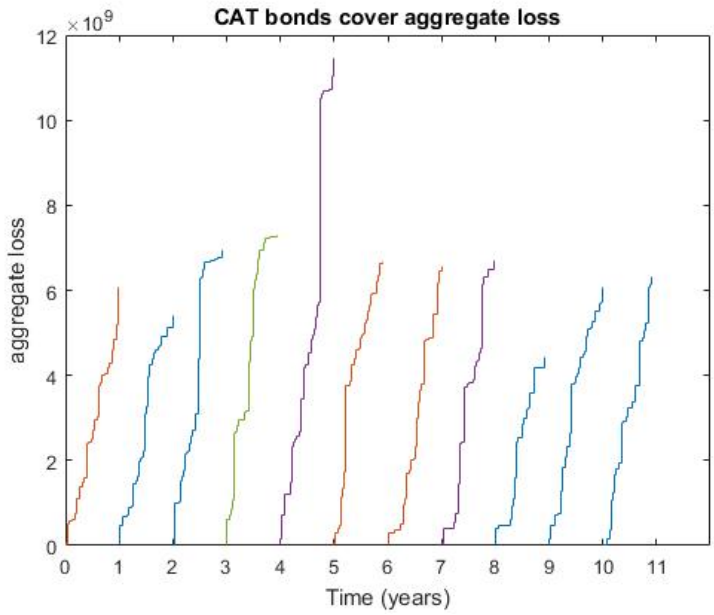


Figure 5.2.1: CAT bonds cover aggregate loss

event exceeds a certain level, the funds received by selling CAT bonds will be used to cover this certain loss and then the CAT bond contract stops before maturity.

The following example shows that when a certain loss exceeds the threshold D, the CAT bond contract stops.[8]

Example 11. Considering the second situation discussed above and setting the threshold is set as $D = 5 \times 10^8$, the CAT bonds cover the loss above D. The maximum loss that CAT bonds are set to cover is $L_{max} = 100 \times 10^6$. The time period for the CAT bond contract is one year. Once a pre-agreed event happens, the loss reaches thresh-

[8] Bond holders always get their payoff at maturity. So even if any pre-agreed event happens before the maturity, investors have to wait untill maturity to receive their reduced payoff or principal back.

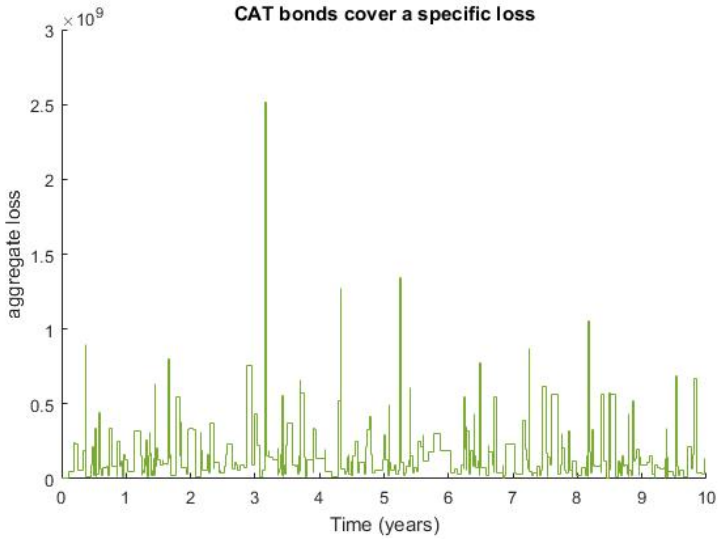


Figure 5.2.2: CAT bonds cover a single loss

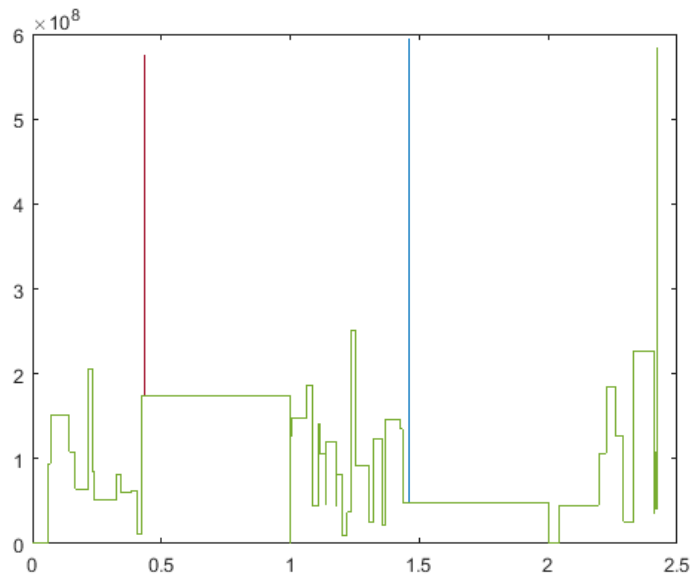
old D, and then the CAT bonds contract stops. (This means the CAT bond contract only provides coverage to the (re-)insurance company for the first pre-defined event.) Investors should wait until maturity to receive payoffs. Figure 5.2.3 shows once a loss occurred from a single event reaches D, the loss process stops. This means these three years all trigger the default term. The loss of CAT bonds at each maturity is $L_1 = 5.761 \times 10^8 - D$, $L_2 = 5.959 \times 10^8 - D$, $L_3 = 5.845 \times 10^8 - D$.

The following section will use the second scenario to calculate the effect of the insurance loss transferred by CAT bonds to the financial market.

5.2.2 Investment behaviour

After identifying the loss that CAT bonds bring to the market at each time period, this subsection shows the effect of CAT bonds on investor i's behaviour. After

Figure 5.2.3: CAT bonds cover loss exceeds 5×10^8



theoretical study of the perfect forecast rule for investors, as defined by Theorem 1, the amount of risky asset demanded by investor i at time t is

$$\begin{aligned} x_*^{(i)} &= x_{eff}^{(i)}(\sigma_*^{(i)}, \pi^{(i)}, \Sigma^{(i)}) \\ &= \frac{(\Sigma^{(i)})^{-1}\pi^{(i)}}{\langle \pi^{(i)}, (\Sigma V^{(i)})^{-1}\pi^{(i)}\rangle^{\frac{1}{2}}}\sigma_*^{(i)}, \\ &= \kappa^{(i)}\sigma_*^{(i)} \end{aligned}$$

where $\pi^{(i)} = q^{(i)} - R_f p$ is the expected vector of excess returns by investor i, $\rho^{(i)} = \langle \pi^{(i)}, (V^{(i)})^{-1}\pi^{(i)}\rangle^{\frac{1}{2}}$ is the price of risk holding by investor i in period t and the parameter $\kappa^{(i)}$ is defined as

$$\kappa^{(i)} = \frac{(\Sigma^{(i)})^{-1}\pi^{(i)}}{\langle \pi^{(i)}, (\Sigma^{(i)})^{-1}\pi^{(i)} \rangle^{\frac{1}{2}}}.$$

Figure 5.2.4 shows the investor's demand for stocks with the changes in their willingness to consume risk when there are only two stocks in the market. Figure 5.2.5 shows the investor's demand for the same stocks and CAT bonds when CAT bonds are introduced into the market. Comparing Figures 5.2.4 and 5.2.5, when CAT bonds are introduced the demand for one particular stock decreases. Both figures also illustrate how the demand changes according to the investor's willingness to consume risk.

5.2.3 Market risk

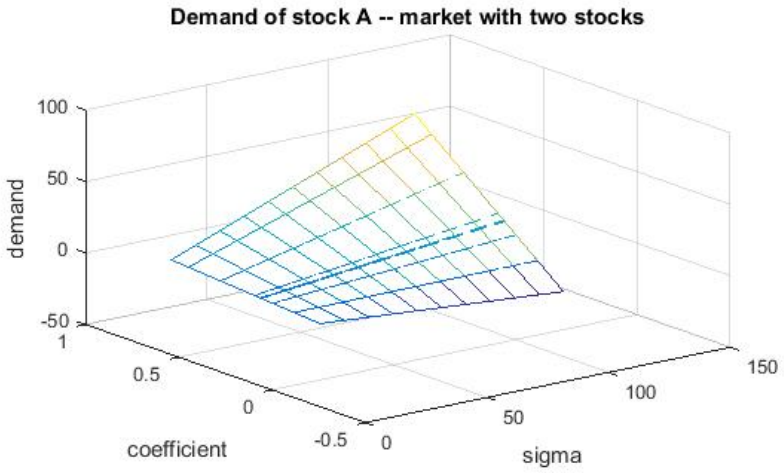
This subsection plots the changes in market risk when CAT bonds are introduced into the market.

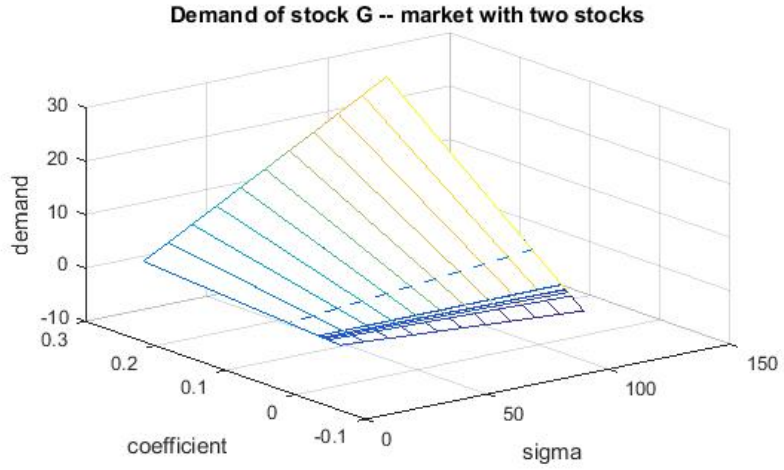
The market risk is defined as $\sigma_{mt} = \sqrt{< x_m, \Sigma_t x_m >}$. Figure 5.2.6 shows the market risk without CAT bonds in a multi-period. When CAT bonds are introduced, the market risk and its volatility increase dramatically as Figure 5.2.7 shows.

5.3 Conclusion

This chapter introduced aggregate loss process that is commonly used to depict the aggregate loss from catastrophic events. A discussion on how CAT bonds cover catastrophic loss at each time period was given, showing an appropriate way to embed this loss process in CAT bonds. The effect of CAT bonds on investors' demand for a risky asset was simulated. When CAT bonds are introduced into the market, the

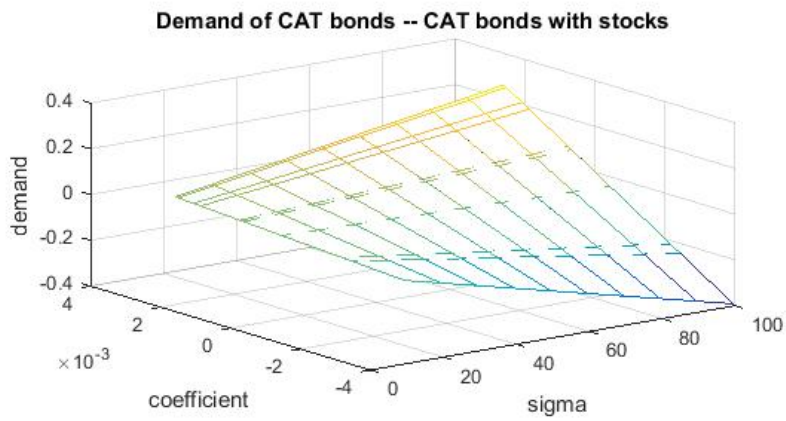
Figure 5.2.4: Demand of stocks in the market with two stocks

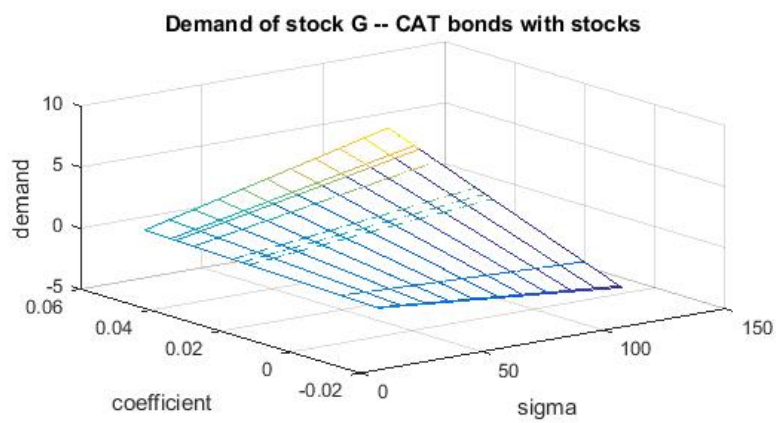




Term 'coefficient' refers to $\kappa^{(i)}$ and 'sigma' is the risk that investor is willingness to consume.

Figure 5.2.5: Demand of CAT bonds and stocks in the market with CAT bonds





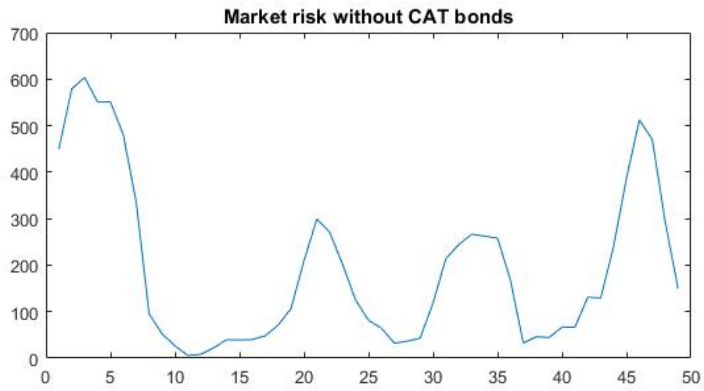


Figure 5.2.6: Market risk without CAT bonds

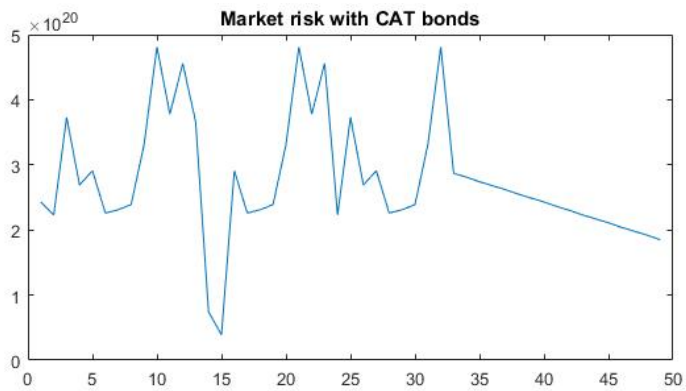


Figure 5.2.7: Market risk with CAT bonds

demand for other risky assets decreases dramatically. In addition, the changes of market risk were simulated when CAT bonds are introduced into the market. The introduction of CAT bonds not only increased the market risk but also increased its volatility.

Chapter 6

Conclusions

This thesis provides a new and alternative approach for modeling CAT bonds by using an agent-based CAPM. Investors' behaviour is analysed when CAT bonds are introduced as innovations in a financial market setting that comprises key features of an insurance market. This framework allows for an analysis of the design of CAT bonds and an exploration of the mechanism that transfers catastrophe risk from an insurance market to a financial market.

By incorporating elements of an insurance market into a financial market, this thesis fills a research gap identified by Wang [146], who points out that any valuation of a CAT bond needs to consider both the insurance and the financial market to ensure a comprehensive analysis and thus a proper valuation of CAT bonds. This thesis started from a traditional static CAPM, studied the introduction of a CAT bond as a new asset added to existing investment opportunities in Chapter 2. The existence and uniqueness of a CAPM equilibrium was established and hence the equilibrium price of a CAT bond as an innovation was determined. The trading conditions under which investors in the financial market are willing to purchase CAT bonds were discussed. Introducing the notion of Pareto superiority in terms

of allocations of risk and return, it was shown that the optimal portfolio with CAT bonds is preferred by an investor, if the corresponding risk-return allocation in the new market with CAT bonds is Pareto superior to the allocation in the market without CAT bonds. If one investor prefers to add CAT bonds to his portfolio, by the two fund separation theorem, all investors will have CAT bonds in their portfolios due to the assumption that investors hold homogeneous beliefs. More generally, this thesis showed that if the slope of the capital market line (i.e. the market price of risk) in the new market is steeper than it in the original market without innovations, then the innovations are preferred and hence traded by investors in the financial market. Formal conditions that determine whether or not the market price of risk with innovations is higher are developed. These findings are in accordance with Cass and Citanna [38] and Elul [67], who show when it is possible to introduce a new asset to make every agent better off. Based on the CAPM with innovations, the design of CAT bonds was studied with regard to the premium that a (re-)insurance company needs to pay, the amount of funds the issuer can raise, and the promised payoff of a CAT bond that financial investors demand. In this setting, the price of a CAT bond was determined as the equilibrium price of an innovation. Considering (re-)insurance companies to be risk averse, the conditions under which a reinsurance company is willing to reinsure catastrophe risk by using CAT bonds was developed. Together with the conditions ensuring that investors are willing to purchase CAT bonds, these conditions stipulate the issuing conditions of CAT bonds, i.e., the conditions under which catastrophe risk is transferable from the insurance industry to the financial market via CAT bonds.

As the standard CAPM neither explains situations in which investors hold het-

erogeneous beliefs nor how asset prices evolve over time, the thesis introduced CAT bonds into an agent-based multi-period CAPM in Chapter 4. The design of CAT bonds developed for the static CAPM was generalised for an agent-based CAPM with investors holding heterogeneous beliefs. To do so, the notion of perceived Pareto superiority was introduced to describe the situation in which an investor's perceived risk-return allocation in a market with CAT bonds has a higher utility than his perceived risk-return allocation in the market without CAT bonds. If the risk-return allocation in the new market is perceived Pareto superior to the market without CAT bonds, CAT bonds are traded in the financial market.

The issuing conditions for CAT bonds were generalised to the dynamic setting. Stochastic difference equations that describe the co-evolution of stock and CAT bond prices were developed and discussed. These equations are generally non-linear and allow for a range of complex dynamic behaviours. As is common in agent-based finance, it turns out that these prices are, in essence, driven by the way in which investors form forecasts about the future evolution of the markets. As a special case, a system which allows for one investor to have rational expectations was formulated, generalising the notion of a perfect forecasting rule introduced in Wenzelburger [147]. The nature of the stochastic difference equations reveals that catastrophe risk will correlate with the return of financial assets as soon as investors hold heterogeneous beliefs. In Chapter 5, a compound Poisson process that describes the aggregate catastrophe claim loss is introduced. By embedding this loss process with CAT bonds at each time period, the effects of CAT bonds to investors' behaviour and the market risk are simulated and discussed.

This thesis leaves some interesting topics to be explored in future research. The

CAPM and the agent-based CAPM is based on the assumption that investors are characterised by mean-variance preferences over future payoffs. However, the mean and the variance may not describe the return process of CAT bonds accurately enough. This is so, because the returns on CAT bonds are defined as a promised payoff to investors minus the losses that are linked to natural disasters. These losses are generally not normally distributed, so that the mean and variance describe this loss process insufficiently. This fact suggests two directions to pursue in future research. One direction could focus on how to transform the loss process so that it can be evaluated by a mean and variance pattern. Wang [145], taking an actuarial science viewpoint, finds that the traditional Sharpe ratio does not reflect the asset returns that have skewed distributions. He provides a probability transformation, which can extend the Sharpe ratio to evaluate risk with a skewed distribution. As the Sharpe ratio is calculated as the expected excess return to standard deviation, it means that Wang's transformation could improve the way in which the mean-variance pattern evaluates catastrophe risk that has a skewed distribution. This might improve the accuracy with which the return on a CAT bond could be evaluated by a mean-variance pattern. Wang's method is important for possible empirical applications and combining his transformation with the CAPM would be an interesting research topic. The second research direction is to extend the agent-based CAPM to a model that takes skewness as a third moment into account. In this way, the properties of loss processes could be transferred more accurately into the financial market, leading to a more precise valuation of catastrophe risks.

Appendix A

Proofs of Chapter 2

Proof. (Theorem 2) We briefly review the proof in Wenzelburger [150]. By Theorem 1, the equilibrium condition for the asset market takes the form

$$\phi(\sqrt{\langle \pi_*, \Sigma^{-1}\pi_* \rangle})\, x_{\text{eff}}(1) = x_{\mathfrak{m}}. \tag{A.0.1}$$

Taking standard deviations (A.0.1) implies

$$\phi(\rho_*) = \sqrt{\langle x_{\mathfrak{m}}, \Sigma\, x_{\mathfrak{m}} \rangle} = \sigma_{\mathfrak{m}} \tag{A.0.2}$$

with $\rho_* := \sqrt{\langle \pi_*, \Sigma^{-1}\pi_* \rangle}$. This means that in equilibrium, aggregate willingness to assume risk $\phi(\rho_*)$ must be equal to the aggregate risk of the market. Vice versa, if ρ_* solves (A.0.2), then $\pi_* = \frac{\rho_*}{\sigma_{\mathfrak{m}}} \Sigma\, x_{\mathfrak{m}}$ solves (A.0.1).

□

Proof. (Proposition 2) Define $x_e^+ = (\Sigma^+)^{-1}\pi^+$ and write $x_e^+ = \binom{x_e}{z_e}$ with $x_e \in \mathbb{R}^K$ and $z_e \in \mathbb{R}$. Using (2.3.1), we see that $\Sigma^+ x_e^+ = \pi^+$ implies

$$\begin{aligned} \pi &= \Sigma\, x_e + z_e b, \\ \tau &= \langle b, x_e \rangle + c z_e. \end{aligned}$$

Solving for $\binom{x_e}{z_e}$, we get

$$x_e = \Sigma^{-1}\pi - z_e\, \Sigma^{-1} b \in \mathbb{R}^K \tag{A.0.3}$$

$$z_e = \frac{\tau - \langle b, \Sigma^{-1}\pi \rangle}{c - \langle b, \Sigma^{-1} b \rangle} \in \mathbb{R}. \tag{A.0.4}$$

Note that the denominator in (A.0.4) is positive because Σ^+ is symmetric and positive definite implying that

$$0 < \left\langle \begin{pmatrix} -\Sigma^{-1} b \\ 1 \end{pmatrix}, \Sigma^+ \begin{pmatrix} -\Sigma^{-1} b \\ 1 \end{pmatrix} \right\rangle = c - \langle b, \Sigma^{-1} b \rangle. \tag{A.0.5}$$

Using (A.0.3) and (A.0.4), the market price of risk takes the form

$$\begin{aligned} (\rho^+)^2 &= \langle x_e^+, \Sigma^+ x_e^+ \rangle = \langle x_e^+, \pi^+ \rangle = \rho^2 + z_e(\tau - \langle b, \sigma^{-1}\pi \rangle) \\ &= \rho^2 + \tfrac{(\tau - \langle b, \Sigma^{-1}\pi \rangle)^2}{c - \langle b, \Sigma^{-1} b \rangle}. \end{aligned} \tag{A.0.6}$$

It follows from (A.0.5), that the second term in (A.0.6) strictly positive whenever $\tau \neq \langle b, \Sigma^{-1}\pi \rangle$. This completes the proof.

□

Appendix B

Proofs of Chapter 4

Proof. (Proposition 5) At equilibrium, investor i forms his own capital allocation line (CAL) and has his perceived market price of risk $\rho_t^{(i)+}$(the slope of CAL). The market price of risk in the market with CAT bonds is defined as

$$\rho_t^{(i)+} = \langle \pi_t^{(i)+}, \Sigma_t^{(i)+} \pi_t^{(i)+} \rangle^{\frac{1}{2}}.$$

In that formula, $\pi_t^{(i)+}$is the expected excess return of investor i and defined as

$$\pi_t^{(i)+} = E_t^{(i)}(q_{t+1}^+) - R_f p_t^+ = \begin{pmatrix} \pi_{ct}^{(i)} \\ \pi_{st}^{(i)+} \end{pmatrix}.$$

with $\pi_{ct}^{(i)}$ denoting the investor's expected excess return from CAT bonds and $\pi_{st}^{(i)+}$ the vector of the investor's expected excess return from stocks. For easier calculation, firstly, let us calculate $(\rho_t^{(i)+})^2$.

$$
\begin{aligned}
(\rho_t^{(i)+})^2 &= \langle \pi_t^{(i)+}, \Sigma_t^{(i)+}\pi_t^{(i)+}\rangle \\
&= \left\langle \begin{pmatrix} \pi_{ct}^{(i)} \\ \pi_{st}^{(i)+} \end{pmatrix}, \begin{pmatrix} V_{11,t}^{(i)} & \Sigma_{12,t}^{(i)} \\ \Sigma_{21,t}^{(i)} & \Sigma_{22,t}^{(i)} \end{pmatrix} \begin{pmatrix} \pi_{ct}^{(i)} \\ \pi_{st}^{(i)+} \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \pi_{ct}^{(i)} \\ \pi_{st}^{(i)+} \end{pmatrix}, \begin{pmatrix} V_{11,t}^{(i)}\pi_{ct}^{(i)} + \Sigma_{12,t}^{(i)}\pi_{st}^{(i)+} \\ \Sigma_{21,t}^{(i)}\pi_{ct}^{(i)} + \Sigma_{22,t}^{(i)}\pi_{st}^{(i)+} \end{pmatrix} \right\rangle \\
&= \left\langle \pi_{ct}^{(i)}, \left(V_{11,t}^{(i)}\pi_{ct}^{(i)} + \Sigma_{12,t}^{(i)}\pi_{st}^{(i)+}\right)\right\rangle + \left\langle \pi_{st}^{(i)+}, \left(\Sigma_{21,t}^{(i)}\pi_{ct}^{(i)} + \Sigma_{22,t}^{(i)}\pi_{st}^{(i)+}\right)\right\rangle \\
&= \langle \pi_{ct}^{(i)}, V_{11,t}^{(i)}\pi_{ct}^{(i)}\rangle + \langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)}\pi_{st}^{(i)+}\rangle + \langle \pi_{st}^{(i)+}, \Sigma_{21,t}\pi_{ct}^{(i)}\rangle + \langle \pi_{st}^{(i)+}, \Sigma_{22,t}^{(i)}\pi_{st}^{(i)+}\rangle
\end{aligned}
$$

where $\langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)}\pi_{st}^{(i)+}\rangle = \langle \pi_{st}^{(i)+}, \Sigma_{21,t}^{(i)}\pi_{ct}^{(i)}\rangle$. This is because $\pi_{ct}^{(i)}$ is a scalar, so the above formula can be rewritten as

$$
(\rho_t^{(i)+})^2 = \langle \pi_{ct}^{(i)}, V_{11,t}^{(i)}\pi_{ct}^{(i)}\rangle + 2\langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)}\pi_{st}^{(i)+}\rangle + \langle \pi_{st}^{(i)+}, \Sigma_{22,t}^{(i)}\pi_{st}^{(i)+}\rangle.
$$

In order to have $(\rho_t)^2$ ($(\rho_t)^2 = \langle \pi_t^{(i)}, \Sigma_t^{(i)}\pi_t^{(i)}\rangle$), substituting formulas $E_t(q_{s,t+1}^{(i)+}) = E_t(q_{s,t+1}^{(i)}) + \zeta_t^{(i)}$, and $\Sigma_{22,t}^{(i)} = \Sigma_t^{(i)}$ into the above formula. The last part equals

$$
\begin{aligned}
\langle \pi_{st}^{i+}, V_{22,t}^{i}\pi_{st}^{(i)+}\rangle &= \left\langle \pi_t^{(i)} + (\zeta_t^{(i)} - R_f\varsigma_t), \Sigma_t^{(i)}\left(\pi_t^{(i)} + (\zeta_t^{(i)} - R_f\varsigma_t)\right)\right\rangle \\
&= \left\langle \pi_t^{(i)} + (\zeta_t^{(i)} - R_f\varsigma_t), \Sigma_t^{(i)}\pi_t^{(i)} + \Sigma_t^{(i)}(\zeta_t^{(i)} - R_f\varsigma_t)\right\rangle
\end{aligned}
$$

$$
= \langle \pi_t^{(i)}, \Sigma_t^{(i)}\pi_t^{(i)}\rangle + \langle \pi_t^{(i)}, \Sigma_t^{(i)}(\zeta_t^{(i)} - R_f\varsigma_t)\rangle + \langle (\zeta_t^{(i)} - R_f\varsigma_t), \Sigma_t^{(i)}\pi_t^{(i)}\rangle + \langle (\zeta_t^{(i)} - R_f\varsigma_t), \Sigma_t^{(i)}(\zeta_t^{(i)} - R_f\varsigma_t)\rangle.
$$

As $\langle \pi_t^{(i)}, \Sigma_t^{(i)}(\zeta_t^{(i)} - R_f\varsigma_t)\rangle = \langle (\zeta_t^{(i)} - R_f\varsigma_t), \Sigma_t^{(i)}\pi_t^{(i)}\rangle$, this formula can be written as

$$\langle \pi_{st}^{(i)+}, \Sigma_{22,t}^{(i)} \pi_{st}^{(i)+} \rangle = \langle \pi_t^{(i)}, \Sigma_t^{(i)} \pi_t^{(i)} \rangle + 2\langle \pi_t^{(i)}, \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle + \langle (\zeta_t^{(i)} - R_f \varsigma_t), \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle.$$

As $\langle \pi_t^{(i)}, \Sigma_t^{(i)} \pi_t^{(i)} \rangle = (\rho_t^{(i)})^2$, replacing the term $\langle \pi_t^{(i)}, \Sigma_t^{(i)} \pi_t^{(i)} \rangle$, it gives

$$\langle \pi_{st}^{(i)+}, \Sigma_{22,t}^{(i)} \pi_{st}^{(i)+} \rangle = (\rho_t^{(i)})^2 + 2\langle \pi_t^{(i)}, \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle + \langle (\zeta_t^{(i)} - R_f \varsigma_t), \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle.$$

So we derives the formula including $(\rho_t^{(i)+})^2$ and $(\rho_t^{(i)})^2$, that is

$$\begin{aligned} (\rho_t^{(i)+})^2 &= (\rho_t^{(i)})^2 + \langle \pi_{ct}^{(i)}, V_{11,t}^{(i)} \pi_{ct}^{(i)} \rangle + 2\langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)} \pi_{st}^{(i)+} \rangle + 2\langle \pi_t^{(i)}, \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle \\ &\quad + \langle (\zeta_t^{(i)} - R_f \varsigma_t), \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle \end{aligned}$$

As $\langle \pi_{ct}^{(i)}, V_{11,t}^{(i)} \pi_{ct}^{(i)} \rangle \geq 0$, so as long as the remaining part is larger than zero, that is

$$2\langle \pi_{ct}^{(i)}, \Sigma_{12,t}^{(i)} \pi_{st}^{(i)+} \rangle + 2\langle \pi_t^{(i)}, \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle + \langle (\zeta_t^{(i)} - R_f \varsigma_t), \Sigma_t^{(i)} (\zeta_t^{(i)} - R_f \varsigma_t) \rangle \geq 0.$$

then the perceived market price of risk in the market with CAT bonds is larger than in the market without CAT bonds, that is

$$\rho_t^{(i)+} > \rho_t^{(i)}.$$

□

Proof. (Proposition 8). To discover the existence of $V_{22,t}^{(2)}$ from the following formulas:

$$\begin{cases} -\frac{\Sigma^{(2)}_{12,t-1}}{Cov_{t-1}[l_t,\xi_{ct}]} & = \frac{\Sigma^{(2)}_{12,t}\det\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(1)}_{12,t}\det\Sigma^{(2)}_t}{a^2R_f\det(\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(2)}_t)} \ (i) \\ \pm\sqrt{V^{(2)}_{22,t-1}-\left(\frac{\Sigma^{(2)}_{12,t-1}}{Cov_{t-1}[l_t,\xi_{ct}]}\right)^2} & = \frac{V^{(2)}_{22,t}\det\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(1)}_{22,t}\det\Sigma^{(2)}_t}{a^{(2)}R_f\det(\Sigma^{(1)}_t+(a^{(1)}/a^{(2)})\Sigma^{(2)}_t)} \ (ii) \end{cases} \tag{B.0.1}$$

To solve this function and get result of $\Sigma^{(2)}_{12,t}$ and $V^{(2)}_{22,t}$, the determinant of $\det(\Sigma^{(1)}_t + (a^{(1)}/a^{(2)})\Sigma^{(2)}_t)$ and $\det\Sigma^{(2)}_t$ are needed. By the definition of determinant, $\det(\Sigma^{(1)}_t + (a^{(1)}/a^{(2)})\Sigma^{(2)}_t)$ equals to

□

$$\det(\Sigma^{(1)}_t + (a^{(1)}/a^{(2)})\Sigma^{(2)}_t) =$$

$$\left(V^{(1)}_{11,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{11,t}\right)\left(\Sigma^{(1)}_{22,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)-\left(\Sigma^{(1)}_{12,t}+\frac{a^{(1)}}{a^{(2)}}\Sigma^{(2)}_{12,t}\right)\left(\Sigma^{(1)}_{21,t}+\frac{a^{(1)}}{a^{(2)}}\Sigma^{(2)}_{21,t}\right)$$

and $\det\Sigma^{(2)}_t$ is

$$\det\Sigma^{(2)}_t = V^{(2)}_{11,t}V^{(2)}_{22,t} - \Sigma^{(2)}_{12,t}\Sigma^{(2)}_{21,t}.$$

To make this calculation easier to read, set $(a^{(1)}/a^{(2)}) = D$, $V^{(2)}_{22,t} = Y > 0$, $\Sigma^{(2)}_{12,t} = \Sigma^{(2)}_{21,t} = X > 0$, $V^{(1)}_{11,t} = V^{(2)}_{11,t} = n \neq 0$, $\Sigma^{(1)}_{12,t} = \Sigma^{(1)}_{21,t} = m \neq 0$, $\Sigma^{(1)}_{22,t} = w \neq 0$. So

$$\begin{aligned} \det(\Sigma^{(1)}_t + (a^{(1)}/a^{(2)})\Sigma^{(2)}_t) &= (n+Dn)(w+DY)-(m+DX)(m+DX) \\ &= nw + nDY + Dnw + D^2nY - m^2 - D^2X^2 - 2mDX \\ &= \det\Sigma^{(1)}_t + D(nY + nw + DnY - DX^2 - 2mX) \end{aligned}$$

and

$$\det \Sigma_t^{(2)} \quad = \quad nY - X^2.$$

Set $-\frac{\Sigma_{12,t-1}^{(2)}}{Cov_{t-1}[l_t,\xi_{ct}]} = A_{12,t} = H$, $\sqrt{V_{22,t-1}^{(2)} + H^2} = P_1$ and $-\sqrt{V_{22,t-1}^{(2)} + H^2} = P_2$, so the function (B.0.1) can be expressed as

$$\begin{cases} H & = \frac{X \det \Sigma_t^{(1)} + Dm(nY - X^2)}{a^2 R_f(\det \Sigma_t^{(1)} + nDY + Dnw + D^2nY - D^2X^2 - 2mDX)} \ (i) \\ P1 & = \frac{Y \det \Sigma_t^{(1)} + Dw(nY - X^2)}{a^2(\det \Sigma_t^{(1)} + nDY + Dnw + D^2nY - D^2X^2 - 2mDX)} \ (ii) \end{cases} \tag{B.0.2}$$

and

$$\begin{cases} H & = \frac{X \det \Sigma_t^{(1)} + Dm(nY - X^2)}{a^2 R_f(\det \Sigma_t^{(1)} + nDY + Dnw + D^2nY - D^2X^2 - 2mDX)} \ (i) \\ P2 & = \frac{Y \det \Sigma_t^{(1)} + Dw(nY - X^2)}{a^2(\det \Sigma_t^{(1)} + nDY + Dnw + D^2nY - D^2X^2 - 2mDX)} \ (ii). \end{cases} \tag{B.0.3}$$

Now solve the simultaneous equations (B.0.2). For the first equation (i) from (B.0.2), multiply $a^2R_f(\det \Sigma_t^{(1)} + nDY + Dnw + D^2nY - D^2X^2 - 2mDX)$ at both sides, then

$$Ha^2R_f(\det \Sigma_t^{(1)} + nDY + Dnw + D^2nY - D^2X^2 - 2mDX) \quad = \quad X \det \Sigma_t^{(1)} + Dm(nY - X^2)$$

$$Ha^2R_f \det \Sigma_t^{(1)} + Ha^2R_f nDY + Ha^2R_f Dnw + Ha^2R_f D^2nY - Ha^2R_f D^2X^2 - 2Ha^2R_f mDX =$$

$$X \det \Sigma_t^{(1)} + DmnY - DmX^2$$

As $a^{(2)}D = a^{(2)}\frac{a^{(1)}}{a^{(2)}} = a^{(1)}$, this formula can be written as

$$(Ha^{(2)}R_f \det \Sigma_t^{(1)} + a^{(1)}HR_f nw) + a^{(1)}HR_f nY + a^{(1)}HR_f DnY - Ha^{(1)}R_f DX^2$$

$$-2Ha^{(1)}R_f mX - DmnY - X \det \Sigma_t^{(1)} + DmX^2 = 0$$

$$HR_f(a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw) + Y\left(a^{(1)}HR_f n(1+D) - Dmn\right) = (Ha^{(1)}R_f D - Dm)X^2$$

$$+(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)})X$$

$$Y(a^{(1)}HR_f n(1+D) - Dmn) = (Ha^{(1)}R_f D - Dm)X^2 + (2Ha^{(1)}R_f m + \det \Sigma_t^{(1)})X$$

$$-(Ha^{(2)}R_f \det \Sigma_t^{(1)} + a^{(1)}HR_f nw).$$

The second formula from (B.0.2) can be rewritten as

$$(Pna^{(1)}+PDa^{(1)}n-\det\Sigma_t^{(1)}-Dwn)Y = 2Pa^{(1)}mX-(Dw-Pa^{(1)}D)X^2-(Pa^{(2)}\det\Sigma_t^{(1)}+a^{(1)}Pnw).$$

So the problem of solving simultaneous equations (B.0.2) is transferred into solving the following simultaneous equations

$$\begin{cases} (a^{(1)}HR_fn(1+D)-Dmn)Y & = (Ha^{(1)}R_f-m)DX^2+(2Ha^{(1)}R_fm+\det\Sigma_t^{(1)})X \\ & -HR_f(a^{(2)}\det\Sigma_t^{(1)}+a^{(1)}nw)\ (i) \\ (P_1na^{(1)}(1+D)-\det\Sigma_t^{(1)}-Dwn)Y & = 2P_1a^{(1)}mX-(w-Pa^{(1)})DX^2 \\ & -P_1(a^{(2)}\det\Sigma_t^{(1)}+a^{(1)}nw)\ (ii) \end{cases} \tag{B.0.4}$$

There are four possible situations to solve these simultaneous equations (B.0.4):

Situation 1, $a^{(1)}HR_fn(1+D)-Dmn=0$ and $P_1na^{(1)}(1+D)-\det\Sigma_t^{(1)}-Dwn=0$. In this case, Y can be an arbitrary number, which is not suitable to this case so this situation is discarded.

Situation 2, $a^{(1)}HR_fn(1+D)-Dmn=0$, $P_1na^{(1)}(1+D)-\det\Sigma_t^{(1)}-Dwn\neq 0$.

Consider the first condition. If $a^{(1)}HR_fn(1+D)-Dmn=0$, that is

$$a^{(1)}HR_fn(1+D)-Dmn=0.$$

Cancel n (as $n\neq 0$) from both sides,

$$
\begin{aligned}
a^{(1)}HR_f(1+D) - Dm &= 0 \\
a^{(1)}HR_fD - Dm &= -a^{(1)}HR_f.
\end{aligned}
$$

Then the first formula from (B.0.4) can be written as

$$
\begin{aligned}
(Ha^{(1)}R_fD - Dm)X^2 + (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})X - (Ha^{(2)}R_f \det \Sigma_t^{(1)} + a^{(1)}HR_fnw) &= 0 \\
a^{(1)}HR_fX^2 - (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})X + HRf(a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw) &= 0.
\end{aligned}
$$

So it becomes a quadratic equation in one unknown variable$X$. If

$$
\Delta_1 = (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})^2 + 4a^{(1)}H^2R_f^2(a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw) \geq 0,
$$

then there are two solutions of X[1],

$$
\begin{cases}
X_1 &= \frac{(2Ha^{(1)}R_fm + \det \Sigma_t^{(1)}) + \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\
X_2 &= \frac{(2Ha^{(1)}R_fm + \det \Sigma_t^{(1)}) - \sqrt{\Delta_1}}{2a^{(1)}HR_f}
\end{cases}. \tag{B.0.5}
$$

Now consider the second condition, to see if there are any constraints on the solution of X.

If $P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn \neq 0$, then the second formula from (B.0.4) is

[1] If $\Delta_1 = 0$, $X_1 = X_2$.

$$Y_{P1} = \frac{2a^{(1)}mP_1X - (Dw - P_1a^{(1)}D)X^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P}{(P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)}.$$

If $\Delta_1 \geq 0$, by substituting the results of $X_1$and X_2 respectively, there are two results of Y

$$Y_{1P1} = \frac{2a^{(1)}mP_1X_1 - (Dw - P_1a^{(1)}D)X_1^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)}$$

$$Y_{2P1} = \frac{2a^{(1)}mP_1X_2 - (Dw - P_1a^{(1)}D)X_2^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)}.$$

Similarly, equation (B.0.3) will obtain similar results. To conclude, all of the results are

$$\begin{cases} X_1 & = \frac{(2Ha^{(1)}R_fm + \det \Sigma_t^{(1)}) + \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\ Y_{1P1} & = \frac{2a^{(1)}mP_1X_1 - (Dw - P_1a^{(1)}D)X_1^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

$$\begin{cases} X_2 & = \frac{(2Ha^{(1)}R_fm + \det \Sigma_t^{(1)}) - \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\ Y_{2P1} & = \frac{2a^{(1)}mP_1X_2 - (Dw - P_1a^{(1)}D)X_2^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

$$\begin{cases} X_1 & = \frac{(2Ha^{(1)}R_fm + \det \Sigma_t^{(1)}) + \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\ Y_{1P2} & = \frac{2a^{(1)}mP_2X_1 - (Dw - P_2a^{(1)}D)X_1^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_2}{(P_2na^{(1)} + P_2Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

$$\begin{cases} X_2 & = \frac{(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)}) - \sqrt{\Delta_1}}{2a^{(1)}HR_f} \\ Y_{2P2} & = \frac{2a^{(1)}mP_2X_2 - (Dw - P_2a^{(1)}D)X_2^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_2}{(P_2na^{(1)} + P_2Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}.$$

Situation 3, $a^{(1)}HR_fn(1+D) - Dmn \neq 0$, $P_1na^{(1)}(1+D) - \det \Sigma_t^{(1)} - Dwn = 0$.

Consider the first condition, $a^{(1)}HR_fn(1+D) - Dmn \neq 0$, then a formula of Y expressed by X can be derived as

$$Y = \frac{(Ha^{(1)}R_fD - Dm)X^2 + (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})X - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_fD - Dm)n}. \tag{B.0.6}$$

Consider the second condition, $P_1na^{(1)}(1+D) - \det \Sigma_t^{(1)} - Dwn = 0$, then the second formula from (B.0.4) becomes a quadratic equation in one unknown variable X:

$$(Dw - P_1a^{(1)}D)X^2 - 2P_1a^{(1)}mX + (P_1a^2 \det \Sigma_t^{(1)} + a^{(1)}Pnw) = 0$$

If

$$\Delta_{2(P1)} = 4P_1^2(a^{(1)})^2m^2 - 4(Dw - P_1a^{(1)}D)(P_1a^2 \det \Sigma_t^{(1)} + a^{(1)}Pnw) \geq 0$$

then

$$X_{1P1} = \frac{2P_1a^{(1)}m + \sqrt{\Delta_{2(P1)}}}{2D(w - P_1a^{(1)})}$$

$$X_{2P1} = \frac{2P_1 a^{(1)} m - \sqrt{\Delta_{2(P1)}}}{2D(w - P_1 a^{(1)})}.$$

So according to the formula of Y expressed by X (B.0.6), the results of X and Y are

$$\begin{cases} X_{1P1} & = \frac{2P_1 a^{(1)} m + \sqrt{\Delta_{2(P1)}}}{2D(w - P_1 a^{(1)})} \\ Y_{X1P1} & = \frac{(Ha^{(1)} R_f D - Dm) X_{1P1}^2 + (2Ha^{(1)} R_f m + \det \Sigma_t^{(1)}) X_{1P1} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)} nw) HR_f}{(a^{(1)} HR_f + a^{(1)} HR_f D - Dm) n} \end{cases}$$

and

$$\begin{cases} X_{2P1} & = \frac{2P_1 a^{(1)} m - \sqrt{\Delta_{2(P1)}}}{2D(w - P_1 a^{(1)})} \\ Y_{X2P1} & = \frac{(Ha^{(1)} R_f D - Dm) X_{2P1}^2 + (2Ha^{(1)} R_f m + \det \Sigma_t^{(1)}) X_{2P1} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)} nw) HR_f}{(a^{(1)} HR_f + a^{(1)} HR_f D - Dm) n} \end{cases}$$

Similarly, from formula (B.0.3), the results are

$$\begin{cases} X_{1P2} & = \frac{2P_1 a^{(1)} m + \sqrt{\Delta_{2(P2)}}}{2D(w - P_1 a^{(1)})} \\ Y_{X1P2} & = \frac{(Ha^{(1)} R_f D - Dm) X_{1P2}^2 + (2Ha^{(1)} R_f m + \det \Sigma_t^{(1)}) X_{1P2} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)} nw) HR_f}{(a^{(1)} HR_f + a^{(1)} HR_f D - Dm) n} \end{cases}$$

and

$$\begin{cases} X_{2P2} & = \frac{2P_2 a^{(1)} m - \sqrt{\Delta_{2(P2)}}}{2D(w - P_2 a^{(1)})} \\ Y_{X2P2} & = \frac{(Ha^{(1)} R_f D - Dm) X_{2P2}^2 + (2Ha^{(1)} R_f m + \det \Sigma_t^{(1)}) X_{2P2} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)} nw) HR_f}{(a^{(1)} HR_f + a^{(1)} HR_f D - Dm) n}. \end{cases}$$

Situation 4, $a^{(1)}HR_fn(1+D) - Dmn \neq 0$, $P_1na^{(1)}(1+D) - \det \Sigma_t^{(1)} - Dwn \neq 0$.

Under this situation, a formula of Y expressed by X can be derived from the first formulas from (B.0.4), that is

$$Y = \frac{(Ha^{(1)}R_fD - Dm)X^2 + (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})X - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_fD - Dm)n}. \tag{B.0.7}$$

Similarly, the second formula from (B.0.4) can derive a formula of Y expressed by X

$$Y = \frac{2a^{(1)}mP_1X - (Dw - P_1a^{(1)}D)X^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1na^{(1)} + P_1Da^{(1)}n - \det \Sigma_t^{(1)} - Dwn)}. \tag{B.0.8}$$

Then by (B.0.6)=(B.0.8) to solve X:

$$\frac{(Ha^{(1)}R_fD - Dm)X^2 + (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})X - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_fD - Dm)n} =$$

$$\frac{2a^{(1)}mP_1X - (w - P_1a^{(1)})DX^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1na^{(1)}(1+D) - \det \Sigma_t^{(1)} - Dwn)}$$

$$\Big((Ha^{(1)}R_fD - Dm)X^2 + (2Ha^{(1)}R_fm + \det \Sigma_t^{(1)})X - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f\Big) =$$

$$\frac{(a^{(1)}HR_f(1+D)-Dm)n}{(P_1na^{(1)}(1+D)-\det\Sigma_t^{(1)}-Dwn)}\Big(2P_1a^{(1)}mX-(w-P_1a^{(1)})DX^2-(a^2\det\Sigma_t^{(1)}+a^{(1)}nw)P_1\Big)$$

Set $\frac{(a^{(1)}HR_f(1+D)-Dm)n}{(P_1na^{(1)}(1+D)-\det\Sigma_t^{(1)}-Dwn)}=W_{P1}$, then this formula becomes

$$\Big((Ha^{(1)}R_fD-Dm)X^2+(2Ha^{(1)}R_fm+\det\Sigma_t^{(1)})X-(a^{(2)}\det\Sigma_t^{(1)}+a^{(1)}nw)HR_f\Big)=$$

$$2P_1a^{(1)}mW_{P1}X-(w-P_1a^{(1)})DW_{P1}X^2-(a^2\det\Sigma_t^{(1)}+a^{(1)}nw)P_1W_{P1}$$

$$\Big((Ha^{(1)}R_f-m)DX^2+(2Ha^{(1)}R_fm+\det\Sigma_t^{(1)})X-(a^{(2)}\det\Sigma_t^{(1)}+a^{(1)}nw)HR_f\Big)$$

$$-2P_1a^{(1)}mW_{P1}X+(w-P_1a^{(1)})DW_{P1}X^2+(a^2\det\Sigma_t^{(1)}+a^{(1)}nw)P_1W_{P1}=0$$

$$\big((Ha^{(1)}R_f-m)+(w-Pa^{(1)})W_{P1}\big)\,DX^2+\Big(2Ha^{(1)}R_fm+\det\Sigma_t^{(1)}-2P_1a^{(1)}mW_{P1}\Big)\,X$$

$$-\Big((a^{(2)}\det\Sigma_t^{(1)}+a^{(1)}nw)HR_f+(a^2\det\Sigma_t^{(1)}+a^{(1)}nw)P_1W_{P1}\Big)=0$$

If

$$\Delta_{3(P1)} = \left(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)} - 2P_1 a^{(1)} m W_{P1}\right)^2$$

$$+4D\left(Ha^{(1)}R_f - m + (w - P_1 a^{(1)})W_{P1}\right)$$

$$\left((a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f + (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1 W_{P1}\right) \geq 0,$$

then the solutions of X are

$$X_{1P1} = \frac{-\left(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)} - 2P_1 a^{(1)} m W_{P1}\right) + \sqrt{\Delta_{3(P1)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_1 a^{(1)})W_{P1}\right)}$$

$$X_{2P1} = \frac{-\left((2Ha^{(1)}R_f m + \det \Sigma_t^{(1)}) - 2P_1 a^{(1)} m W_{P1}\right) - \sqrt{\Delta_{3(P1)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_1 a^{(1)})W_{P1}\right)}.$$

As there are two expressions of Y , the possible results are

$$\begin{cases} X_{1P1} & = \frac{-\left(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)} - 2P_1 a^{(1)} m W_{P1}\right) + \sqrt{\Delta_{3(P1)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_1 a^{(1)})W_{P1}\right)} \\ Y_{X1P1} & = \frac{2a^{(1)} m P_1 X_{1P1} - (Dw - P_1 a^{(1)} D)X_{1P1}^2 - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1 n a^{(1)} + P_1 D a^{(1)} n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

or $Y'_{X1P1} = \frac{(Ha^{(1)}R_f D - Dm)X_{1P1}^2 + (2Ha^{(1)}R_f m + \det \Sigma_t^{(1)})X_{1P1} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_f D - Dm)n}$

$$\begin{cases} X_{2P1} & = \frac{-\left((2Ha^{(1)}R_f m + \det \Sigma_t^{(1)}) - 2P_1 a^{(1)} m W_{P1}\right) - \sqrt{\Delta_{3(P1)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_1 a^{(1)})W_{P1}\right)} \\ Y_{X2P1} & = \frac{2a^{(1)} m P_1 X_{2P1} - (Dw - P_1 a D)X_{2P1}^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_1}{(P_1 n a^{(1)} + P_1 D a^{(1)} n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

or $Y'_{X2P1} = \frac{(Ha^{(1)}R_f D - Dm)X_{2P1}^2 + (2Ha^{(1)}R_f m + \det \Sigma_t^{(1)})X_{2P1} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_f D - Dm)n}$.

Similarly, to solve simultaneous equations (B.0.3), set $\frac{(a^{(1)}HR_f(1+D) - Dm)n}{(P_2 n a^{(1)}(1+D) - \det \Sigma_t^{(1)} - Dwn)} =$

W_{P2} and if

$$\Delta_{3(P2)} = \left(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)} - 2P_2 a^{(1)} m W_{P2}\right)^2$$

$$+4D\left(Ha^{(1)}R_f - m + (w - P_2 a^{(1)})W_{P2}\right)$$

$$\left((a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f + (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_2 W_{P2}\right) \geq 0$$

then

$$\begin{cases} X_{1P2} & = \frac{-\left(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)} - 2P_2 a^{(1)} m W_{P2}\right) + \sqrt{\Delta_{3(P2)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_2 a^{(1)})W_{P2}\right)} \\ Y_{X1P2} & = \frac{2a^{(1)} m P_2 X_{1P2} - (Dw - P_2 a^{(1)} D)X_{1P2}^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_2}{(P_2 n a^{(1)} + P_2 D a^{(1)} n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

or $Y'_{X1P2} = \frac{(Ha^{(1)}R_f D - Dm)(X_{1P2})^2 + (2Ha^{(1)}R_f m + \det \Sigma_t^{(1)})X_{1P2} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_f D - Dm)n}$

$$\begin{cases} X_{2P2} & = \frac{-\left((2Ha^{(1)}R_f m + \det \Sigma_t^{(1)}) - 2P_2 a^{(1)} m W_{P2}\right) - \sqrt{\Delta_{3(P2)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_2 a^{(1)})W_{P2}\right)} \\ Y_{X2P2} & = \frac{2a^{(1)} m P_2 X_{2P2} - (Dw - P_2 a D)X_{2P2}^2 - (a^2 \det \Sigma_t^{(1)} + a^{(1)}nw)P_2}{(P_2 n a^{(1)} + P_2 D a^{(1)} n - \det \Sigma_t^{(1)} - Dwn)} \end{cases}$$

or $Y'_{X2P2} = \frac{(Ha^{(1)}R_f D - Dm)(X_{2P2})^2 + (2Ha^{(1)}R_f m + \det \Sigma_t^{(1)})X_{2P2} - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)HR_f}{(a^{(1)}HR_f + a^{(1)}HR_f D - Dm)n}$.

By checking, only when $X = X_{1P2}$, result $Y_{X1P2} = Y'_{X1P2}$, so under situation 4 the result is

$$\begin{cases} X_{1P2} & = \frac{-\left(2Ha^{(1)}R_f m + \det \Sigma_t^{(1)} - 2P_2 a^{(1)} m W_{P2}\right) + \sqrt{\Delta_{3(P2)}}}{2D\left((Ha^{(1)}R_f - m) + (w - P_2 a^{(1)})W_{P2}\right)} \\ Y_{X1P2} & = \frac{2a^{(1)} m P_2 X_{1P2} - (Dw - P_2 a^{(1)} D)X_{1P2}^2 - (a^{(2)} \det \Sigma_t^{(1)} + a^{(1)}nw)P_2}{(P_2 n a^{(1)}(1+D) - \det \Sigma_t^{(1)} - Dwn)}. \end{cases}$$

Proof. (Proposition 9) After having the perfect forecast rule for the first moment, the investor $i = 2$'s perfect beliefs for the conditional covariance and variance for the asset payoff are going to be calculated. Under this assumption, requirement (iii) (4.4.27) will always hold and condition (ii) (4.4.26) becomes

$$V^{(2)}_{22,t-1} = (A_{22,t})^2.$$

By the definition of $A_{22,t}$ (4.4.8), the above formula can be written as

$$\sqrt{V^{(2)}_{22,t-1}} = \frac{V^{(2)}_{22,t} \det \Sigma^{(1)}_t + (a^{(1)}/a^{(2)}) V^{(1)}_{22,t} \det \Sigma^{(2)}_t}{a^{(2)} R_f \det(\Sigma^{(1)}_t + (a^{(1)}/a^{(2)})\Sigma^{(2)}_t)}.$$

Substituting

$$\det(\Sigma^{(1)}_t + (a^{(1)}/a^{(2)})\Sigma^{(2)}_t) = \left(V^{(1)}_{11,t} + \frac{a^{(1)}}{a^{(2)}} V^{(2)}_{11,t}\right) \det \left(V^{(1)}_{22,t} + \frac{a^{(1)}}{a^{(2)}} V^{(2)}_{22,t}\right),$$

$$\det \Sigma^{(2)}_t = V^{(2)}_{11,t} \det(V^{(2)}_{22,t})$$

into the above formula, it gives

$$\sqrt{V^{(2)}_{22,t-1}} \left(V^{(1)}_{11,t} + \frac{a^{(1)}}{a^{(2)}} V^{(2)}_{11,t}\right) \det \left(V^{(1)}_{22,t} + \frac{a^{(1)}}{a^{(2)}} V^{(2)}_{22,t}\right) a^{(2)} R_f$$

$$= V^{(2)}_{22,t} \det \Sigma^{(1)}_t + \frac{a^{(1)}}{a^{(2)}} V^{(1)}_{22,t} V^{(2)}_{11,t} \det V^{(2)}_{22,t}.$$

Solving for $V^{(2)}_{22,t}$,

$$\sqrt{V^{(2)}_{22,t-1}}\left(V^{(1)}_{11,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{11,t}\right)\det\left(V^{(1)}_{22,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)a^{(2)}R_f-(\frac{a^{(1)}}{a^{(2)}})V^{(1)}_{22,t}V^{(2)}_{11,t}\det V^{(2)}_{22,t}=V^{(2)}_{22,t}\det V^{1}_{t},$$

as $\det\Sigma^{(1)}_t = V^{(1)}_{11,t}V^{(1)}_{22,t} > 0$, so it becomes

$$\left(\sqrt{V^{(2)}_{22,t-1}}\left(V^{(1)}_{11,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{11,t}\right)\det\left(V^{(1)}_{22,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)a^{(2)}R_f-(\frac{a^{(1)}}{a^{(2)}})V^{(1)}_{22,t}V^{(2)}_{11,t}\det(V^{(2)}_{22,t})\right)$$

$$(\det\Sigma^{(1)}_t)^{-1}=V^{(2)}_{22,t},$$

$$\sqrt{V^{(2)}_{22,t-1}}\left(V^{(1)}_{11,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{11,t}\right)\det\left(V^{(1)}_{22,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{22,t}\right)a^{(2)}R_f(\det\Sigma^{(1)}_t)^{-1}$$

$$-(\frac{a^{(1)}}{a^{(2)}})V^{(1)}_{22,t}V^{(2)}_{11,t}\det(V^{(2)}_{22,t})(\det\Sigma^{(1)}_t)^{-1}=V^{(2)}_{22,t},$$

Solving for $V^{(2)}_{22,t}$,

$$V^{(2)}_{22,t} = \frac{V^{(1)}_{22,t}\sqrt{V^{2}_{22,t-1}}}{\frac{\det\Sigma^{(1)}_t+(\frac{a^{(1)}}{a^{(2)}})V^{(1)}_{22,t}V^{(2)}_{11,t}}{R_f a^{(2)}\left(V^{(1)}_{11,t}+\frac{a^{(1)}}{a^{(2)}}V^{(2)}_{11,t}\right)}-\frac{a^{(1)}}{a^{(2)}}\sqrt{V^{(2)}_{22,t-1}}}.$$

By substituting $\det\Sigma^{(1)}_t$ into the above formula, it gives the perfect forecasting rule for the second moments:

$$
\begin{aligned}
V_{22,t}^{(2)} &= \frac{V_{22,t}^{(1)}\sqrt{V_{22,t-1}^{(2)}}}{\frac{V_{11,t}^{(1)}V_{22,t}^{(1)}+(\frac{a^{(1)}}{a^{(2)}})V_{22,t}^{(1)}V_{11,t}^{(2)}}{R_f a^{(2)}\left(V_{11,t}^{(1)}+\frac{a^{(1)}}{a^{(2)}}V_{11,t}^{(2)}\right)} - \frac{a^{(1)}}{a^{(2)}}\sqrt{V_{22,t-1}^{(2)}}}, \\
&= \frac{a^{(2)}}{\frac{1}{R_f\sqrt{V_{22,t-1}^{(2)}}} - \frac{a^{(1)}}{V_{22,t}^{(1)}}}.
\end{aligned}
$$

□

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