International Journal of Foundations of Computer Science

Reachability Problems for One-Dimensional Piecewise Affine Maps *

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Received (Day Month Year)
Accepted (Day Month Year)
Communicated by (xxxxxxxxxx)

Piecewise affine maps (PAMs) are frequently used as a reference model to discuss the frontier between known and open questions about the decidability for reachability questions. In particular, the reachability problem for one-dimensional PAM is still an open problem, even if restricted to only two intervals. As the main contribution of this paper we introduce new techniques for solving reachability problems based on p-adic norms and weights as well as showing decidability for two classes of maps. Then we show the connections between topological properties for PAM's orbits, reachability problems and representation of numbers in a rational base system. Finally we construct an example where the distribution properties of well studied sequences can be significantly disrupted by taking fractional parts after regular shifts. The study of such sequences could help with understanding similar sequences generated in PAMs or in well known Mahler's 3/2 problem.

Keywords: Reachability problems; piecewise affine maps (PAMs); β-expansion; p-adic analysis.

1. Introduction

There remain a number of quite basic models/fragments of classes of programs over the reals for which we have fundamental difficulties in the design of verification

*This research is supported by EPSRC grant "Reachability problems for words, matrices and maps" (EP/M00077X/1)

tools. One of them is the model of iterative map that appears in many different contexts, including discrete-event/discrete-time/hybrid systems, qualitative biological models, chaos-based cryptography, etc [1, 9, 18, 30].

The one-dimensional piecewise-affine iterative map is a very rich mathematical object and one of the simplest dynamical systems exhibiting complex dynamics on a set of rational numbers. A function $f\colon \mathbb{Q} \to \mathbb{Q}$ is a one-dimensional piecewise-affine map (PAM) if f is of the form $f(x) = a_i x + b_i$ for $x \in X_i$ where all coefficients a_i,b_i and the extremities of a finite number of bounded intervals X_i are rational numbers. Let us consider the sequence of iterations starting from a rational point x: x, $f(x)$, $f^2(x) = f(f(x))$, and so on. The reachability for PAM is the problem to decide for a given f and two rational points x and y whether y is reachable from x. In other words, is there an $n \in \mathbb{N}$ such that $f^n(x) = y$?

The decidability of the reachability problem for one dimensional PAM is a long standing open problem, which is related to other challenging questions in the theory of computation, number theory and linear algebra [8, 16, 17]. The model of PAMs plays a crucial role in the recent research on verification and analysis of hybrid systems [2, 3], timed automata [2], control systems [10, 11], representation of numbers in a rational base (β-expansions) [23, 28], discounted sum automata [12], etc. The computations in PAMs have a natural geometrical interpretation as pseudo-billiard system [21] and Hierarchical Piecewise Constant Derivative (HPCD) system [3]. The reachability problem for one-dimensional PAMs is still open even if restricted to a map with only two intervals [2, 3, 6, 13].

There are several examples of one-dimensional iterative maps with undecidable reachability problems. In particular the reachability problem is undecidable for one dimensional piecewise rational maps [21] or maps which are based on the following basis of elementary functions $\{x^2, x^3, x^{1/2}, x^{1/3}, 2x, x+1, x-1\}$ [21]. On the other hand the reachability problem for PAMs in dimension one is open if f is replaced by linear rational functions (i.e. of the form $f(x) = \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}$), but it is known to be undecidable for maps with rational functions of degree two, i.e of the form $f(x) = \frac{a_i \cdot x^2 + b_i \cdot x + e_i}{c_i \cdot x + d_i}$ [21], where $a_i, b_i, c_i, d_i, e_i \in \mathbb{Q}$. The original undecidability result for piecewise rational functions of degree two has been shown in [21] where two of the finite number of intervals were unbounded such as $(l, +\infty)$ and $(-\infty, r)$. Later the result was improved by constructing undecidability proof for a map with a finite number of bounded intervals and only one unbounded interval $(l, +\infty)$ in [5]. On the other hand some simplifications or additional constrains on affine maps can make the system to be more predictable with decidable reachability problems, see [2, 3, 6, 13].

The primary goal of this paper is to demonstrate new approaches for solving reachability problems in PAMs, connecting reachability questions with topological properties of maps and widening connections with other important theoretical computer science problems. First, we show new techniques for decidability of the reachability problem in PAMs based on p-adic norms and weights. We illustrate these techniques showing decidability of two classes of PAMs. The algorithm in Theorem 8 solves the point-to-point reachability problem for two-interval injective PAM under the assumption that a PAM has bounded invariant densities. Our numerical experiments show that the sequence of densities converge to invariant bounded functions. It is not yet clear whether it holds for all PAMs or if not whether this property can be algorithmically checked in general. The approximation of the invariant measures and densities is a very active field of research in dynamical systems. For example the first implementable method for guaranteeing the accuracy of invariant measures for a subclass of iterative maps was proposed in [19]. The author of [19] has developed a computer-assisted method for computing the rigorous upper and lower bounds of invariant densities of uniformly expanding maps on the interval [0,1].

Following the proposed approach based on p-adic weights in Theorem 9 we define another fragment of PAMs for which the reachability problem is decidable. In particular we remove the condition on bounded invariant densities and injectivity of PAM and consider a PAM f with a constraint on linear coefficients in affine maps. This class of PAMs is also related to encoding of rational numbers in the rational base (β-expansions). The decidability of the point-to-point problem for this class is shown in Theorem 9 and the decidability of the point-to-set problem for the same class would give an answer to an open problem related to β-expansions. Finally, in Theorem 12 we prove by means of p-adic norms the decidability of the point-to-point reachability problem in iterative systems based on nonlinear polynomial functions.

Then we establish several connections between topological properties for PAM's orbits, the reachability problems in PAMs and representation of numbers in a rational base system. We show that the reachability problems for above objects are tightly connected to questions about distribution of the fractional parts in the generated sequences and moreover about distribution of the fractional parts after regular shifts.

In order to study the densities of orbits in a PAM we also suggest an interpretation of the reachability problems in a more general framework, where a sequence of numbers generated by a map is hitting some dynamical intervals. This concept helps to represent the mechanism of the transformations in PAMs and to illustrate several topological aspects which are in the core of reachability questions. Then we show that if we consider the problem about the distribution of numbers in the sequence from $\mathbb{R}/\mathbb{Z}$, which is similar to the problem about the distribution from $\mathbb{Q}/\mathbb{Z}$ sequence in PAMs or 3/2-Mahler's sequence, we have a strange anomaly. Namely we show that the uniform distribution of orbits in maps on $\mathbb{R}/\mathbb{Z}$ may not always remain uniform or even dense when taking a fractional part after regular shifts. The same question about the numbers from $\mathbb{Q}/\mathbb{Z}$, generated by a PAM, has not been extensively studied yet, but is relevant in the context of solving original reachability problems in PAMs which are presented in this paper.

2. Preliminaries and notation

In what follows we use traditional denotations $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ for sets of natural, integers, positive integers, primes, rational and real numbers, respectively. Let us denote by $S^1 = \mathbb{Q}/\mathbb{Z}$ the unit circle which consists only of rational numbers. By $\{x\}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ we denote the fractional part[a] of a number, floor and ceiling functions.

Let Y be a set of numbers and x a single number. We define their addition and multiplication as follows: $Y + x = x + Y = \{x + y | y \in Y\}$ and $xY = Yx = \{xy | y \in Y\}$. The application of a function $f : X \to Y$ to a set $X' \subseteq X$ is defined as $f(X') = \{f(x) | x \in X'\}$. If $f : X \to 2^Y$ and $X' \subseteq X$, we define $f(X') = \bigcup_{x \in X'} f(x)$.

***p*-adic norms and weights:** Let us consider an arbitrary finite subset of prime numbers $\mathbb{F} = \{p_1, p_2, \ldots, p_k\} \subset \mathbb{P}$ in ascending order and

$$m = p_1 p_2 \ldots p_k.$$

For a given prime p any nonzero rational number x can be represented by $x = p^{\alpha_p}(r/s)$, where r and s are integers not divisible by p, and α_p is a unique integer. The p-adic norm of x is then defined by $|x|_p = p^{-\alpha_p}$. The p-adic weight of x is defined as $\|x\|_p = \log_p(|x|_p) = -\alpha_p$. The following properties of p-adic weights directly follow from the properties of p-adic norm:

$$\|x\|_p = \|y\|_p \Rightarrow \|x + y\|_p \leq \|x\|_p. \tag{1}$$

$$\|x\|_p < \|y\|_p \Rightarrow \|x + y\|_p = \|y\|_p, \tag{2}$$

$$\|x \cdot y\|_p = \|x\|_p + \|y\|_p, \tag{3}$$

$$\|x^r\|_p = r\|x\|_p, \tag{4}$$

If there is a prime $p \notin \mathbb{F}$ such that $\|x\|_p > 0$, then we define $\|x\|_m = +\infty$, otherwise $\|x\|_m = \max\limits_{p \in F} \|x\|_p$. $\|x\|_m$ will be called the *m-weight* of x. The *m-vector-weight* of x is the vector $(\|x\|)_m = (\|x\|_{p_1}, \|x\|_{p_2}, \ldots, \|x\|_{p_k})^T$. If $\|x\|_m > 0$ then the m-weight of a number x is the number of digits after the radix point in the representation of x in base m, i.e. x can be written as $x = y \cdot m^{-\|x\|_m}$, where y is an integer having non-zero last digits in its m-ary representation. If $\|x\|_m \leq 0$, then x is an integer number. Without loss of generality we consider from now on only rationals x for which $\|x\|_m < +\infty$.

Example 1. *For example let us consider a fraction* $\frac{12}{49}$*, then* $\left|\frac{12}{49}\right|_2 = \left|\frac{2^2 3}{7^2}\right|_2 = \frac{1}{2^2}$*,* $\left\|\frac{12}{49}\right\|_2 = -2$*,* $\left\|\frac{12}{49}\right\|_{42} = \max\left\{\left\|\frac{12}{49}\right\|_2, \left\|\frac{12}{49}\right\|_3, \left\|\frac{12}{49}\right\|_7\right\} = \max\{-2, -1, 2\} = 2$*, and the m-weight of a* $\frac{12}{49}$ *is equal to* $\left(\left\|\frac{12}{49}\right\|\right)_{42} = (-2, -1, 2)^T$.

[a] It will be clear from the context if brackets are used in other conventional ways, for example, to indicate a set of numbers.

Lemma 2. *Given a finite subset of prime numbers $\mathbb{F}$, then for any positive integer $a \in \mathbb{Z}^+$ there is an integer b such that if $\|x\|_m < a$ then $\|x\|_p \geq b$ for all $p \in \mathbb{P}$.*

Proof. We formally show now that for any rational number $x = \frac{\xi}{\mu}$ in $[0,1]$ if the powers of primes in the factorization of the denominator μ are bounded then the powers of primes in the factorization of the numerator ξ are also bounded.

Let us define two sums of weights:

$\alpha = -\sum_{p\in\mathbb{P},\|x\|_p\leq 0} \|x\|_p$ (the sum of all powers of primes in the numerator ξ)
$\beta = \sum_{p\in\mathbb{F},\|x\|_p\geq 1} \|x\|_p$ (the sum of all powers of primes in the denominator μ).

Taking into account that 2 is the smallest prime and p_k is the largest prime in $\mathbb{F}$, we have the following inequality:

$$\frac{2^\alpha}{p_k^{ka}} \leq \frac{2^\alpha}{p_k^\beta} \leq x \leq 1.$$

From it follows that $2^\alpha \leq p_k^{ka}$ and therefore $-\alpha \geq -ka\log_2 p_k$. If we set $b = -\alpha$ we have $\|x\|_p \geq b$ for all $p \in \mathbb{P}$. □

Corollary 3. *For any $a \in \mathbb{Z}$, there is only a finite number of rational numbers $x \in [0,1]$ for which $\|x\|_m < a$.*

3. Reachability problem for iterative maps

PAMs are frequently used as a reference model to show the openness of the reachability questions in other systems. The reachability problem for one-dimensional PAM is still open even if restricted to only two intervals. As the main contribution of this paper we introduce new techniques for solving reachability problems based on p-adic norms and weights as well as showing the connections between topological properties for PAM's orbits, reachability problems and representation of numbers in a rational base system.

Definition 4. *We say that $f : S^1 \to S^1$ is a one-dimensional PAM if function f is defined on a finite family of (rational) intervals $\{X_1, \ldots, X_l\}$ such that $S^1 = X_1 \cup X_2 \cup \ldots \cup X_l$ and where $f(x) = a_i x + b_i$, $a_i, b_i \in \mathbb{Q}$ for $x \in X_i$.*

If the intervals $\{X_1, \ldots, X_l\}$ are disjoint we say that the PAM is *deterministic*, otherwise the PAM is non-deterministic. By default the PAM is understood to be *deterministic*.

For a PAM f, let A_f denotes the matrix with values (a_{ji}), where $a_{ji} = \|a_i\|_{p_j}$, $1 \leq i \leq l$, $1 \leq j \leq k$. The rank of A_f is denoted by $\text{rank}(A_f)$. We define the derivative of a PAM f as $f'(x) = a_i$ for $x \in X_i$, $1 \leq i \leq l$.

Generally speaking, a PAM does not need to correspond to $f : X \to X$, where $X = S^1$. However if the set X is a union of any finite number of bounded intervals we always can scale it to S^1. If $f : X \to X$ is such that $X \not\subseteq [0,1)$, and $X \subseteq [a,b)$, then by applying conjugation $h(x) = \frac{x-a}{b-a}$ we can reduce the original reachability problem for f to the reachability problem for the mapping $g = h \circ f \circ h^{-1}$ from $[0,1)$ to $[0,1)$.

If f is a deterministic PAM, an orbit (trajectory) of a point x is denoted by $O_f(x)$ and will be understood either as a set $O_f(x) = \{f^i(x) | i \in \mathbb{Z}^+\}$ or as a sequence, i.e. $O_f(x) : \mathbb{Z}^+ \to S^1$, $O_f(x)(i) = f^i(x)$. We will say that a point y is reachable from a point x if $y \in O_f(x)$.

In general the reachability problem for PAM can be defined as follows. Given a PAM f, $x \in S^1$ and $Y \subseteq S^1$, decide whether the intersection $Y \cap O_f(x)$ is empty. If Y is a finite union of intervals, we name the reachability problem as *point-to-set (interval)* problem. If Y is a one element set, the reachability problem is known as *point-to-point* reachability. Also in a similar way it is possible to define set-to-point and set-to-set reachability problems.

In this paper we mainly focus on the analysis of one-dimensional PAMs and by the *reachability problem for PAM* we understand the point-to-point reachability. We will explicitly state the type of the problem when we need to refer to other type of reachability questions.

In this paper we also consider a special type of PAMs known as complete PAMs.

Definition 5. *A deterministic PAM $f : S^1 \to S^1$ in the definition 4 is complete if $f(X_i) = S^1$ for any $i = 1, \ldots, k$.*

We show in this paper that for a complete PAM on two intervals the reachability problem is decidable. Moreover the reachability problem for any number of intervals can be reduced to the vector reachability problem for 2×2 matrices over rational numbers. The vector reachability problem for a matrix semigroup S is defined as follows: given two vectors x and y and a finite collection of matrices $M_1, ..., M_n$ generating S, decide whether there exists a matrix $M \in S$ such that $M \cdot x = y$. At the moment this problem for 2×2 matrices is open in general and it has been proved to be decidable so far only for matrices from $SL(2, \mathbb{Z})$ [24]. However there is some hope that the problem is decidable for two dimensional matrices over $\mathbb{Z}$, $\mathbb{Q}$ and maybe even over $\mathbb{C}$, following new techniques developed in [25, 26]. In particular it is known that there is no injective semigroup morphism from pairs of words over any finite alphabet (with at least two elements) into complex 2×2 matrices [14], which means the exact encoding of the Turing Machine computation into 2×2 matrices cannot be used directly for proving undecidability in 2×2 matrix semigroups over $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$. This horizon of decidability has been recently extended in [20] by showing that there is no injective semigroup morphism from pairs of words to $SL(3, \mathbb{Z})$.

Proposition 6. *The reachability problem for a complete one dimensional PAM $f : S^1 \to S^1$ on k intervals can be reduced to the vector reachability problem for a matrix semigroup generated by k matrices from $\mathbb{Q}^{2 \times 2}$.*

Proof. Let us assume that a PAM f is defined on k intervals $\{X_1, \ldots, X_k\}$ such that

$S^1 = X_1 \cup X_2 \cup \ldots \cup X_k$ with affine functions $f_1, f_2, \ldots, f_k$. Then we compute inverse functions which are in the form $f_i^{-1}(x) = g_i(x) = a_i x + b_i$. Now each function $g_i(x)$ maps S^1 into X_i. Let us assume that the original reachability problem was to check that a point y is reachable from a point x_0, i.e. if $y \in O_f(x_0)$. As we consider the problem for complete PAM we now can reformulate the problem as backward reachability as follows. Is a point x_o reachable from a point y by any finite sequence of applications of affine functions from the set $\{g_i(x) \,|i \in \{1, ..., k\}\}$, i.e. in any order. If the original reachability problem has a positive answer after l iterations then there exists a sequence of visited intervals $X_{i_1}, X_{i_2}, ..., X_{i_l}$, i.e. with indexes $i_1, i_2, ..., i_l$. This implies that applying affine functions from a set $\{g_i(x) \,|i \in \{1, ..., k\}\}$ with the reverse order of indexes $i_l, ..., i_2, i_1$ should map y to x_0. If any finite sequence of applications of affine functions from a set $\{g_i(x) \,|i \in \{1, ..., k\}\}$ does not map y to x_0 then y is not reachable from x_0 in the original reachability problem. Finally the set of affine functions applied in non-deterministic way could be represented as the vector reachability problem from a vector $\overline{y} = \begin{pmatrix} y \\ 1 \end{pmatrix}$ to a vector $\overline{x_0} = \begin{pmatrix} x_0 \\ 1 \end{pmatrix}$ in the matrix semigroup generated by k upper triangular matrices $M_1, \ldots, M_k$, where $M_i = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$. Thus $y \in O_f(x_0)$ iff $\overline{x_0}$ is reachable from $\overline{y}$. □

Overall the Proposition 6 shows that if the vector reachability problem for 2×2 matrices semigroup over rational number or specific its subcase of matrices of the form $M_i = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$ is decidable then the reachability for a complete one dimensional PAM would be decidable. Moreover if hypothetically someone could prove undecidability for point-to-point reachability in complete one dimensional PAM then it would lead to the undecidability of the vector reachability in 2×2 matrices semigroup over rationals.

4. Decidability using p-adic norms

It is well known in dynamical systems research that, due to complexity of orbits in iterative maps, it is less useful, and perhaps misleading, to compute the orbit of a single point and it is more reasonable to approximate the statistics of the underlying dynamics [15, 27]. This information is encoded in the so-called *invariant measures*, which specify the probability to observe a typical trajectory within a certain region of state space and their corresponding *invariant densities*.

Let us consider a density as an ensemble of initial starting points (i.e. initial conditions). The action of the dynamical system on this ensemble is described by the Perron-Frobenius operator. Invariant densities are those ensembles which are fixed under the linear Perron-Frobenius operator and in other words they are eigenfunctions with eigenvalue one [15].

Formally, by an ensemble A we understand an enumerated set (sequence) of points in phase space. We can associate to an ensemble its distribution function and density function: Let I be a set of points. We denote by $F_I^A(n) = |\{i \in \mathbb{Z}^+ | i \leq n, A(i) \in I\}|$ the number of elements in the sequence A which belong to the set I and with indexes less or equal to n. The distribution function of the ensemble A is defined as $\Phi_A(x) = \lim_{n\to\infty} \frac{F^A_{(-\infty,x)}(n)}{n}$. The density function ϕ_A of the ensemble A is defined as $\phi_A(x) = \Phi'_A(x)$.

Suppose an ensemble A_0 with density ϕ_0 is given. If we apply PAM f to each point of the ensemble, we get a new ensemble A_1 with some density distribution ϕ_1. We say that the function ϕ_1 is obtained from ϕ_0 using the Perron-Frobenius or transfer operator, which we denote by L_f. It is known that

$$\phi_1(x) = L_f(\phi_0)(x) = \sum_{y \in f^{-1}(x)} \frac{\phi_0(y)}{|f'(y)|}.$$

If $\phi_1 = \phi_0$ we say that ϕ_0 is a f-invariant density function or an eigenfunction of the transfer operator L_f.

4.1. *One-dimensional affine maps*

We prove that if for an injective PAM f there exists a bounded invariant density function then the reachability problem for f is decidable. Here is an example of a simple injective PAM f with non-trivial behavior that is generating the points which are powers of 2 (obeying Benford's law) with some shifts:

$$f(x) = \begin{cases} 2x & \text{if } x \in [\frac{1}{10}, \frac{1}{2}]; \\ x/5 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

As we show below the injectivity of a PAM leads to a simpler expression for Perron-Frobenius operator and to an explicit connection between the bounds for an invariant density function and the bounds for a product of multiplicative coefficients of affine functions in two interval PAM.

Lemma 7. *Let f be an injective PAM, and ϕ be a f-invariant density function. Assume there are $K_{min} > 0$ and $K_{max} < +\infty$ such that for any x from the domain of f the following inequality holds: $K_{min} < \phi(x) < K_{max}$. Then for an arbitrary segment of the orbit $x_1, x_2, \ldots x_{n+1}$, where $x_{i+1} = f(x_i)$, we have $\frac{K_{min}}{K_{max}} \leq |c_1 \cdot c_2 \cdot \ldots \cdot c_n| \leq \frac{K_{max}}{K_{min}}$, where $c_i = a_j$ if $x_i \in X_j$.*

Proof. Let ϕ be an eigenfunction of the Perron-Frobenius operator for an injective PAM f. Then injectivity of f and the fact that $y = f(x)$ implies that $\phi(y) = \frac{\phi(x)}{|f'(x)|}$. We denote $f'(x_i)$ by c_i. Then $\phi(x_{n+1}) = \frac{\phi(x_1)}{|c_1 \cdot c_2 \cdot \ldots \cdot c_n|}$ and $|c_1 \cdot c_2 \cdot \ldots \cdot c_n| = \frac{\phi(x_1)}{\phi(x_{n+1})}$. Now we can bound $|c_1 \cdot c_2 \cdot \ldots \cdot c_n|$ by $\frac{K_{min}}{K_{max}}$ and $\frac{K_{max}}{K_{min}}$. □

Now we can explicit a first fragment of PAMs with decidable reachability problem, where decidability is based on the analysis of p-adic weights during application of injective PAMs.

Theorem 8. *Given an injective PAM f with two intervals and the existence of a f-invariant density function ϕ such that there are $K_{min} > 0$ and $K_{max} < +\infty$ and $K_{min} < \phi(x) < K_{max}$ for all x in the domain of f. Then the reachability problem for f is decidable.*

Proof. A PAM f with two intervals X_1 and X_2 is defined as follows: $f(x) = a_i x + b_i$, if $x \in X_i$. Note that we only consider PAMs over rational numbers where a starting point as well as all coefficients and borders of intervals are rational numbers.

Let us consider an arbitrary pair of points $x, y \in X$ such that $y \in O_f(x)$ and the sequence of points $x_1, x_2, \ldots, x_{n+1}$ from the orbit $O_f(x)$ such that $x_1 = x$, $x_{n+1} = y$, $x_{i+1} = f(x_i)$, $1 \leq i \leq n$. We will show that there exists a computable upper bound M on the length of the shortest reachability sequence from x to y. This bound can be computed from the following quantitites: K_{min}, K_{max} a_1, a_2, b_1, b_2, x, y.

Let us consider the p-adic weights $\|x_i\|_m$ for all $i \in \{1, \ldots, n+1\}$ in the path $x_1, x_2, \ldots, x_{n+1}$ where $x_1 = x$ and $x_{n+1} = y$. If the p-adic weights are bounded from above by some value M_1 then we can restrict the length of a reachability path and its bound M as follows: $n < M = m^{M_1}$ as $\|x_i\|_m$ is the number of digits after the radix point in the representation of x_i in base m. If an orbit's segment of numbers with p-adic weights is bounded by M_1 and the length of the orbit is greater than m^{M_1} then this orbit always contains at least two identical numbers and the orbit loops. On the other hand we will show that if on any orbit the p-adic weights are exceeding the bound M_1 then the weights cannot be reduced to the level of M_1 later.

So in order to prove a computable upper bound on $\|x_i\|_m$ it is sufficient to prove that for any $p \in \mathbb{F}$ p-adic weights (i.e. logarithmic norms) $\|x_i\|_p$ are bounded from above by a computable number M_2 for all $i \in \{1, \ldots, n+1\}$. Note that by the definition of the logarithmic m-norm, $M_2 = M_1$.

Let us define a number $h = \max\{\|b_1\|_m + 1, \|b_2\|_m + 1, \|x_1\|_m, \|x_{n+1}\|_m\}$. Let us take an arbitrary $p \in \mathbb{F}$ and select a subsequence x_j, x_{j+1}, $\ldots$, x_r in the sequence x_1, x_2, $\ldots$, x_{n+1} such that its elements $x \in \{x_{j+1}, x_{j+2}, \ldots, x_{r-1}\}$ have the following property $\|x\|_p > h$, but at the same time $\|x_j\|_p \leq h$ and $\|x_r\|_p \leq h$. For simplicity, but without loss of generality, we assume that $j = 1$, $r = n+1$ and $\|x_1\|_p = \|x_{n+1}\|_p = h$.

Later we will either find a computable upper bound on n or we will show a computable bound M_2 on p-adic weights based on $K_{min}, K_{max}, a_1, a_2$ and h. As stated above, this is enough for proving the theorem.

Let us define $x_{i+1} = f(x_i) = c_i x_i + d_i$, where $c_i \in \{a_1, a_2\}$, $d_i \in \{b_1, b_2\}$. From the properties (1) and (2) of the p-adic weights and the fact that $\|x_i\|_p > \|b_j\|_p$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2\}$ follows that $\|x_{i+1}\|_p = \|x_i\|_p + \|c_i\|_p$. This implies $\|x_{n+1}\|_p = \|x_1\|_p + \sum_{i=1}^{n} \|c_i\|_p = \|x_1\|_p + \sum_{i=1}^{2} \alpha_i \|a_i\|_p$ for some non-negative numbers α_1 and α_2, where $\alpha_1 + \alpha_2 = n$. Taking into account that $\|x_1\|_p = \|x_{n+1}\|_p$ we have $\sum_{i=1}^{2} \alpha_i \|a_i\|_p = 0$.

Let r be the greatest common divisor of α_1 and α_2. We define $\alpha = \alpha_1/r$ and $\beta = \alpha_2/r$, i.e. α and β are the smallest non-negative integers such that $\alpha\|a_1\|_p + \beta\|a_2\|_p = 0$. Thus, we obtain $\sum_{i=1}^{2} \alpha_i\|a_i\|_p = r(\alpha\|a_1\|_p + \beta\|a_2\|_p)$. By Lemma 7 we have: $\frac{K_{min}}{K_{max}} \leq |c_1 \cdot c_2 \cdot \ldots \cdot c_n| \leq \frac{K_{max}}{K_{min}}$, i.e. $\frac{K_{min}}{K_{max}} \leq |a_1^\alpha a_2^\beta|^r \leq \frac{K_{max}}{K_{min}}$,

Now we consider two cases $|a_1^\alpha a_2^\beta| \neq 1$ and $|a_1^\alpha a_2^\beta| = 1$. The first case corresponds to linearly independent columns of the matrix A_f, rank$(A) = 2$, and the second case to linear dependence, rank$(A_f) < 2$.

Let $|a_1^\alpha a_2^\beta| \neq 1$, then either $|a_1^\alpha a_2^\beta| > 1$ or $|a_1^\alpha a_2^\beta| < 1$. If $|a_1^\alpha a_2^\beta| > 1$ then from $|a_1^\alpha a_2^\beta|^r \leq \frac{K_{max}}{K_{min}}$ it follows that $r \leq \frac{\ln \frac{K_{max}}{K_{min}}}{\ln |a_1^\alpha a_2^\beta|}$. On the other hand if $|a_1^\alpha a_2^\beta| < 1$, then from $|a_1^\alpha a_2^\beta|^r \geq \frac{K_{min}}{K_{max}}$ it follows $r \leq \frac{\ln \frac{K_{min}}{K_{max}}}{\ln |a_1^\alpha a_2^\beta|}$.

Now suppose that $|a_1^\alpha a_2^\beta| = 1$. The only non-trivial case is when $a_1 \neq 1$ and $a_2 \neq 1$ as the cases where $|a_1| = 1$, $|a_2| = 1$ or both correspond to the trivial case of piecewise-affine mapping (i.e. with no more than one linear factor).

Without loss of generality, we assume that $a_1 > 1$ and $a_2 < 1$. Note also that $\|a_2\|_p = -\frac{\alpha}{\beta}\|a_1\|_p$. Let us now consider an arbitrary subsequence of consecutive points $x_1x_2 \ldots x_{j+1}$, $j \leq n$ of the original reachability path. Assuming that α_1, α_2 are such that $|c_1c_2 \ldots c_j| = |a_1|^{\alpha_1}|a_2|^{\alpha_2}$ we have that

$$\|x_{j+1}\|_p = \|x_1\|_p + \alpha_1\|a_1\|_p + \alpha_2\|a_2\|_p = (\alpha_1 - \alpha_2\frac{\alpha}{\beta})\|a_1\|_p.$$

On the other hand from Lemma 7 we know that: $\frac{K_{min}}{K_{max}} \leq |a_1|^{\alpha_1}|a_2|^{\alpha_2} \leq \frac{K_{max}}{K_{min}}$. Since $|a_1^\alpha a_2^\beta| = 1$,then $|a_2| = |a_1|^{-\frac{\alpha}{\beta}}$ and

$$\frac{K_{min}}{K_{max}} \leq |a_1|^{\alpha_1 - \alpha_2\frac{\alpha}{\beta}} \leq \frac{K_{max}}{K_{min}}.$$

Taking into account assumption that $|a_1| > 1$, we get $\alpha_1 - \alpha_2\frac{\alpha}{\beta} \leq \frac{\ln \frac{K_{max}}{K_{min}}}{\ln |a_1|}$.

Now it follows that:

$$\|x_{j+1}\|_p = \|x_1\|_p + (\alpha_1 - \alpha_2\frac{\alpha}{\beta})\|a_1\|_p \leq h + \left(\frac{\ln \frac{K_{max}}{K_{min}}}{\ln |a_1|}\right)\|a_1\|_m.$$

Thus, we have shown a computable upper bound for p-adic weights of the orbital elements that can reach a point y. Finally, in view of provided reasoning, we have shown that the reachability problem for this type of PAMs is decidable. □

The theorem could be applied for a larger class of PAMs if more information were known about the convergence of density functions under the action of the Perron-Frobenius operator. Let us call an ensemble A to be statistically fixed with respect to f, if $\phi_A = L_f(\phi_A)$. E.g. if someone could show that in injective PAM all statistically fixed ensembles have identical distribution functions then Theorem 8 could be applied to show decidability of injective PAMs.

Following the proposed approach based on p-adic weights we define another fragment of PAMs for which the reachability problem is decidable. In particular, we remove the condition on eigenfunction of the transfer operator and injectivity of PAM and consider a PAM f with only constraints on linear coefficients in affine maps. More specifically we require that the powers of prime numbers from prime factorizations of linear coefficients have the same signs (i.e. two sets of prime numbers used in nominator and denominator are disjoint).

Theorem 9. *The reachability problem for a PAM f is decidable if every row of a matrix A_f contains values of the same sign, (i.e. $a_{ji} \cdot a_{j'i} \geq 0$, for all i, j such that $1 \leq i \leq l$, $1 \leq j, j' \leq k$).*

Proof. Let us consider a PAM f of the form $f(x) = a_i x + b_i$ for $x \in X_i$ where all coefficients a_i,b_i and the extremities of a finite number of bounded intervals X_i are rational numbers. Let us define $h = \max\{\|b_1\|_m, \|b_2\|_m, \ldots, \|b_l\|_m\}$. The condition of the theorem means that for any prime $p \in \mathbb{F}$ all linear coefficients of the map f have non-zero p-adic weights of the same sign.

In this case, if p-adic weights of linear coefficients of f are non-negative, then for any $x \in X$ from $\|x\|_p > h$ it follows that $\|f(x)\|_p \geq \|x\|_p$ and therefore $\|f(x)\|_m \geq \|x\|_m$ (i.e. m-adic weight does not decrease). If p-adic weight of linear coefficients of the mapping are negative, then for any $x \in X$ we have $\|f(x)\|_p \leq \max\{\|x\|_p, h\}$.

Thus, in the sequence of reachable points for an orbit of a map f either all points of the orbit have m-adic weights bounded from above by h, then we have a cyclic orbit, or from some moment when m-adic weight of a reachable point exceeds h it does not decrease and again, either orbit loops or m-adic weight increases indefinitely.

Thus, in order to decide whether y is reachable, i.e. $y \in O_f(x)$, it is sufficient to start generating a sequence of reachable points in the orbit $O_f(x)$ and wait for one of the events, where either 1) a point in the orbit is equal to y *(y is reachable)*, or 2) the orbit will loop and $y \notin O_f(x)$ *(y is not reachable)*, or 3) a point x' is reachable, such that $\|x'\|_m > \max\{h, \|y\|_m\}$, and then $y \notin O_f(x)$ *(y is not reachable).* □

Corollary 10. *The reachability problem for complete PAMs with two intervals is decidable.*

Proof. The condition of a PAM with two intervals $f : S^1 \to S^1$ to be complete means that $S^1 = X_1 \cup X_2$ and $f(X_1) = f(X_2) = S^1$. Thus, if $X_1 = \left[0, \frac{m}{n}\right]$ and $X_2 = \left[\frac{m}{n}, 1\right)$, then $f(x) = a_1 x + b_1$, where $a_1 = \pm\frac{n}{m}$, when $x \in X_1$, and $f(x) = a_2 x + b_2$, where $a_2 = \pm\frac{n}{n-m}$ at $x \in X_2$, $m, n \in \mathbb{N}$, $\gcd(m, n) = 1$. It is clear that n, m, $n - m$ are relatively prime. So the conditions of Theorem 9 are satisfied. □

Note that the coefficients of complete PAMs with three intervals would not satisfy the conditions of Theorem 9, so some new techniques should be developed to solve more general case. However as we formulated early in Proposition 6 the reachability problem for n-interval one dimensional complete PAM can be reduced to the special case of vector reachability in 2×2 matrix semigroups generated by n matrices over rational numbers, which is still open in general [24].

4.2. *Nonlinear polynomial maps*

We call polynomial functions $f(x) = \sum_{i=0}^{n} a_i x^i$ with rational coefficients, where $a_n \neq 0$, $n > 1$, as *nonlinear polynomial functions*. Let us analyze the reachability in iterative nonlinear polynomial maps in terms of p-adic norms $|x|_p$, $p \in \mathbb{P}$, and the absolute value (the Archimedean norm) $|x|_\infty = |x|$.

Lemma 11. *If f is a nonlinear polynomial function then there is a constant $c > 0$ such that for any $p \in \mathbb{P} \cup \{\infty\}$ and for any x, $|x|_p > c$, the inequality $|f(x)|_p > |x|_p$ holds.*

Proof. First let us consider the case with the Archimedean norm $|x|$ and let c_∞ be the maximal absolute value x such that $|f(x)| = |x|$. If $|f(x)| \neq |x|$ for all x then $c_\infty = 0$. It is clear that $|f(x)| > |x|$ if $|x| > c_\infty$.

Let us consider now p-adic norms. From the definition of the p-adic norm it follows that $|xy|_p = |x|_p |y|_p$ and if $|x|_p > |y|_p$ then $|x + y|_p = |x|_p$.

Let us assume that $f(x) = \sum_{i=0}^{n} a_i x^i$, $a_n \neq 0$, $n > 1$, and $|x|_p > \max\left\{ \frac{|a_i|_p}{|a_j|_p} : 0 \leq i < j \leq n, a_j \neq 0 \right\}$, then $\left|a_j x^j\right|_p > \left|a_i x^i\right|_p$ for $i < j$. From this it follows that $|f(x)|_p = \left|\sum_{i=0}^{n} a_i x^i\right|_p = |a_n x^n|_p$. Since $|x|_p > \frac{1}{\sqrt[n-1]{|a_n|_p}}$ if and only if $|a_n x^n|_p > |x|_p$ we conclude that if $c_p = \max\left\{ \frac{1}{\sqrt[n-1]{|a_n|_p}}, \frac{|a_i|_p}{|a_n|_p} : 0 \leq i \leq n-1 \right\}$ and $|x|_p > c_p$ then $|f(x)|_p > |x|_p$.

Hence, if $c = \max\{c_p : p \in \mathbb{P} \cup \{\infty\}\}$ and $|x|_p > c$ then $|f(x)|_p > |x|_p$ for all $p \in \mathbb{P} \cup \{\infty\}$. □

Theorem 12. *The point-to-point reachability problem for piecewise deterministic or non-deterministic maps over nonlinear polynomial functions is decidable.*

Proof. Let Φ be a finite set of *nonlinear polynomial functions*. We show now that the reachability problem from the initial rational point x' to the target rational point x'' is decidable.

Following Lemma 11 there exists a constant c such that for every polynomial $f \in \Phi$ and all points x such that $|x|_p > c$ we have that $|f(x)|_p > |x|_p$ for all $p \in \mathbb{P} \cup \{\infty\}$. We also assume that $c > |x'|_p$ and $c > |x''|_p$, $p \in \mathbb{P} \cup \{\infty\}$.

In order to solve the reachability problem we need to explore all possible (deterministic or non-deterministic) orbits starting from the initial point x'. Since the number of rational points with limited Archimedean and p-adic norms is finite we eventually should observe for every orbit only one of the following events: 1) the orbit is visiting the same points again and the target point x'' has not been met; 2) the Archimedean or one of p-adic norms is exceeding the constant c (from now on this norm will never decrease), i.e. the target point x'' has not been met; 3) the target point x'' has been met.

$\square$

5. PAM representation of β-expansions

The target discounted-sum 0-1 problem [29, 12] is defined as follows: *Given a rational non-integer $\beta > 1$ and a number $x \in [0, 1]$ decide whether there is a sequence $w : \mathbb{N} \to \{0, 1\}$ of zeros and ones such that $x = \sum_{i=1}^{\infty} w(i)\frac{1}{\beta^i}$.*

For any $x \in S^1$, there exists a β-expansion $w : \mathbb{N} \to \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ such that $x = \sum_{i=1}^{\infty} w(i)\frac{1}{\beta^i}$. If $w(i) \in \{0, 1\}$ for all i, we call it its $(0, 1) - \beta$**-expansion**. As a consequence, when $\beta \leq 2$ the answer to the target discounted-sum problem is always positive. Therefore, the only interesting case is when $\beta > 2$. We denote $D = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$. Then the minimal and maximal numbers, which are representable in the basis β with digits from the alphabet D, are $min = \sum_{i=1}^{\infty} 0\frac{1}{\beta^i} = 0$ and $max = \sum_{i=1}^{\infty} (\lceil\beta\rceil - 1)\frac{1}{\beta^i} = \frac{\lceil\beta\rceil - 1}{\beta - 1}$. When $\beta > 2$ then $max < 2$. Let us denote by X_d the interval $[\frac{min+d}{\beta}, \frac{max+d}{\beta})$ for each $d \in D$. If β is a non-integer number then intervals X_d and X_{d+1} intersect. Also taking into account that $max < 2$ the intervals X_d and X_{d+2} have no common points. Finally from the above construction we get the lemma:

Lemma 13. *If β is rational non-integer number and $\beta > 2$ then it holds:*

(1) $X_d \cap X_{d+1} \neq \emptyset$ for $d < \lceil\beta\rceil - 1$;
(2) $X_d \cap X_{d+2} = \emptyset$ for $d < \lceil\beta\rceil - 2$;
(3) $[min, max) = \cup_{d \in D} X_d$.

Proposition 14. *For any β-expansion there is a non-deterministic PAM where a symbolic dynamic of visited intervals (i.e. a sequence of symbols associated with intervals) from an initial point x_0 corresponds to its representation in base β.*

Proof. Let us define the piecewise affine mapping $f \subseteq [min, max) \times [min, max)$ as follows $f = \{(x, \beta x - d) | x \in X_d, d \in D\}$. It directly follows from this definition that $f(X_d) = [min, max)$. Let us consider an orbit $x(i) = f^i(x)$, $i \in \mathbb{Z}^+$. Let $d_i \in D$ be such that $x(i) \in X_{d_i}$. Then for any $n \in \mathbb{N}$ $min < \left|\beta^n x - d_1\beta^{n-1} - d_2\beta^{n-2} - \ldots - d_n\beta^0\right| < max$, and in other form $\frac{min}{\beta^n} < \left|x - \sum_{i=1}^{n}(d_i\frac{1}{\beta^i})\right| < \frac{max}{\beta^n}$. So $\left|x - \sum_{i=1}^{n}(d_i\frac{1}{\beta^i})\right| \to 0$, $n \to \infty$, and therefore $x = \sum_{i=1}^{\infty}(d_i\frac{1}{\beta^i})$. Let us consider it in other direction. Let $x = \sum_{i=1}^{\infty}(d_i\frac{1}{\beta^i})$, then the sequence $\mathbf{x}(i)$, where $\mathbf{x}(0) = x$, $\mathbf{x}(i+1) = \beta\mathbf{x}(i) - d_i$, is the orbit of x in PAM f. Let us name the constructed map as the β**-expansion PAM**. $\square$

The nondeterministic β-expansion can be translated into deterministic maps corresponding to *greedy* and *lazy* expansions as follows:

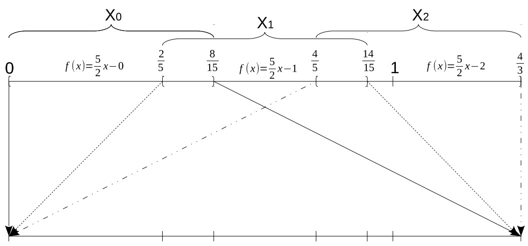

Fig. 1. A non-deterministic PAM for $\frac{5}{2}$-expansion

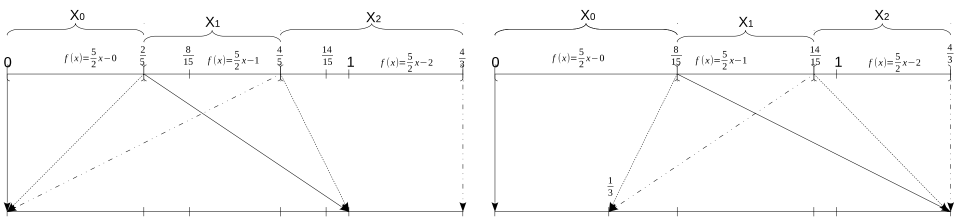

Fig. 2. Deterministic greedy (on the left) and lazy (on the right) $\frac{5}{2}$-expansion PAM

Definition 15. *A function $f : [min, max) \to [min, max)$ is the greedy β-expansion PAM if the domain $[min, max)$ is divided into intervals X'_d, $d \in \{0, 1, \dots, \lceil\beta\rceil - 1\}$ such that $X'_{\lceil\beta\rceil-1} = X_{\lceil\beta\rceil-1}$, $X'_{d-1} = X_{d-1} - X_d$, $d \in \{1, 2, \dots, \lceil\beta\rceil - 1\}$ and $f(x) = \beta x - d$ iff $x \in X'_d$.*

Since $X_d = [\frac{min+d}{\beta}, \frac{max+d}{\beta})$ then $X'_d = [\frac{min+d}{\beta}, \frac{min+d+1}{\beta}) = [\frac{d}{\beta}, \frac{d+1}{\beta})$ and the length of the interval X'_d is equal to $\frac{1}{\beta}$, $d < \lceil\beta\rceil - 1$.

Definition 16. *A function $f : [min, max) \to [min, max)$ is the lazy β-expansion PAM, if the domain $[min, max)$ is divided into intervals X''_d, $d \in \{0, 1, \dots, \lceil\beta\rceil - 1\}$, such that $X''_0 = X_0$, $X''_d = X_d - X_{d-1}$, $d \in \{1, 2, \dots, \lceil\beta\rceil - 1\}$, and $f(x) = \beta x - d$ iff $x \in X''_d$.*

Proposition 17. *Let f and g be greedy and lazy β-expansion PAM respectively. Then f and g are (topologically) conjugate by the homeomorphism $h : h(x) = h^{-1}(x) = max - x$, i.e. $f = h \circ g \circ h$.*

Proof. The statement holds since $X'_d = max - X''_{\lceil\beta\rceil-1-d}$, $d \in \{0, 1, \dots, \lceil\beta\rceil - 1\}$ □

We would like to highlight that the questions about reachability as well as representation of numbers in rational bases are tightly connected with questions about the density of orbits in PAMs. Moreover if the density of orbits are the same for all non-periodic points then it may be possible to have a wider application of p-adic techniques provided in the beginning of the paper. Let us formulate a hypothesis that goes along with our experimental simulations in PAMs:

Hypothesis 1. *The orbit of any rational point in any expanding*[b] *deterministic PAM is either finite or dense on the whole domain.*

Lemma 18. *Any* $(0,1)-\beta$*-expansion is greedy.*

Proof. Let f be a β-expansion PAM. Assume that there is a point x and the orbit $\mathbf{x}(i)$, where $\mathbf{x}(0)=x$, $\mathbf{x}(i+1)=\beta\mathbf{x}(i)-d_i$ in the map f such that $d_i\in\{0,1\}$ for all $i\in\mathbb{N}$ and the orbit does not correspond to the β greedy expansion of x.

The intersection of intervals X_0 and X_1 is an interval $X_{01}=[\frac{\min+1}{\beta},\frac{\max}{\beta})$. Applying a map $y=\beta x$ to X_{01} we see that X_{01} is scaled into $[\min+1,\max)=[1,\frac{\lceil\beta\rceil-1}{\beta-1})$. The interval $[1,\frac{\lceil\beta\rceil-1}{\beta-1})$ does not have any common points with X_0 as the point 1 lies on the right side of the left border of the interval $X_2=[\frac{\min+2}{\beta},\frac{\max+2}{\beta})$ and by Lemma 13 $X_i\cap X_{i+2}=\emptyset$.

Note that when $x>\frac{1}{\beta-1}$ we have $\beta x-1>x$. Let us assume that for some i $\mathbf{x}(i)\in X_{01}$ and $\mathbf{x}(i+1)=\beta\mathbf{x}(i)-0$, i.e. we did not followed a greedy expansion and therefore $\mathbf{x}(i+1)\in[1,\frac{\lceil\beta\rceil-1}{\beta-1})$. Then $\mathbf{x}(i+2)=\beta\mathbf{x}(i+1)-1>\mathbf{x}(i+1)$ and $\mathbf{x}(i+2)\notin X_0$, etc. In this case starting from $\mathbf{x}(i+1)$ there is monotonically increasing sequence of orbital points in the interval X_1. So points in such orbit should eventually leave the interval X_1 and reach X_d, where $d>1$. This gives us a contradiction with the original assumption. □

Corollary 19. $(0,1)-\beta$*-expansion is unique and greedy.*

Proof. The statement is obvious since the greedy expansion can be expressed by a deterministic map. □

Theorem 20. *If Hypothesis 1 holds then a non-periodic* $(0,1)-\beta$*-expansion does not exist.*

Proof. Any $(0,1)-\beta$-expansion can be constructed by expanding deterministic greedy β-expansion PAM. If the orbit of a rational point in greedy β-expansion PAM is non-periodic, then by Hypothesis 1 it should be dense and therefore should intersect all intervals and cannot provide $(0,1)-\beta$-expansion. □

Theorem 21. *If Hypothesis 1 holds then for any rational number its deterministic* β*-expansion is either eventually periodic or it contains all possible patterns (finite subsequences of digits) from* $\{0,1,\ldots,\lceil\beta\rceil-1\}$.

Proof. The statement is obvious as Hypothesis 1 implies that the orbit is either periodic or it is dense and the dense orbit is visiting all intervals. □

[b]I.e. with linear coefficients that are greater than 1

It seems that the point-to-interval problem is harder than the point-to-point reachability problem for the expanding PAMs, as for example Theorem 9 gives an algorithm for the point-to-point reachability problem in the β-expansion PAMs, but not for the point-to-interval reachability that is equivalent to the β-expansion problem.

Note that in the β-expansion PAMs all linear coefficients are the same, so the density of the orbit corresponds to the density of the following sequence $\mathbf{x}(n) = f^n(x_0)$, where $f(x) = \{\beta x\}$. For example when $\beta = \frac{5}{2}$ and $x_0 = 1$ we get the sequence:

$$\{\tfrac{5}{2}\}, \{\tfrac{5}{2}\{\tfrac{5}{2}\}\}, \{\tfrac{5}{2}\{\tfrac{5}{2}\{\tfrac{5}{2}\}\}\}, \ldots$$

The question about the distribution of a similar sequence $\{\frac{3}{2}\}$, $\{\frac{3^2}{2^2}\}$, $\{\frac{3^3}{2^3}\}$, ... , where the integer part is removed once after taking a power of a fraction (for example 3/2) is known as Mahler's 3/2 problem, that is a long standing open problem in analytic number theory.

6. Density of orbits and its geometric interpretation

We pointed out in the previous section that knowing more properties about the distribution and density of points on the orbit could be useful for studying β-expansion and closely related reachability problems in PAMs. In this section we provide a geometrical interpretation of classical Equidistribution theorem, Benford's Law phenomenon and its generalization introducing the concept of dynamical intervals, which are used to express the behavior of iterative maps taking some fractional part of generated numbers.

The generalization is based on considering a point on the orbit in some base p, where p is a product of all prime numbers used in denominators of coefficients (in such affine or linear systems), interpreting a mapping as an update of a sequence of symbols (in base p) by integer multiplication/addition/subtraction and replacing the division operation by some simple shifts of updated sequence of symbols.

Although we are not answering original questions about the distribution and density of original problems with shifts we show an example where the distribution properties of well studied sequences can be significantly disrupted by taking fractional parts after regular shifts which is similar to iteration in PAMs or in well known Mahler's 3/2 problem.

It is well known as Equidistribution theorem that $\mathbf{x}(n) = \{\alpha n\}$, where α is an irrational number, has an uniform distribution. Let us give some geometric interpretation of the orbit density. Consider the Cartesian plane with the y-axis x and and the x-axis y (just swapping their places). Now let us divide the set of lines $x = n$, $n \in \mathbb{N}$, by integer points on the segments of the unit length. The set of points (y, x), where $m \leq y < m + 1$, $x = n$, i.e. the interval $[m, m + 1) \times n$ on the line $x = n$, will be denoted by $S_{m,n}$, $m \in \mathbb{Z}^+$, $n \in \mathbb{N}$. In other words, $S_{m,n} = (m + [0, 1)) \times n$. Let $I \subseteq [0, 1)$ be an interval then we denote by $I_{m,n}$ the set $(m + I) \times n$, see Figure 3

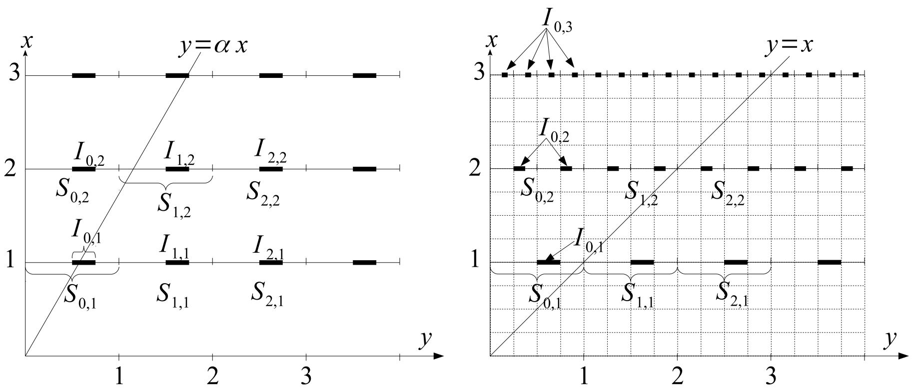

Fig. 3. Left: An example for two sets $S_{m,n}$ and $I_{m,n}$; Right: A dynamic interval $I(n)$.

for illustration.

Two points of the plane are defined to be equivalent if they belong to the same line passing through the origin. We call α as homogeneous coordinate of a point (y, x) if $y = \alpha x$. By $H(I)$ we denote the set of homogeneous coordinates of all points from $\bigcup\limits_{m\in\mathbb{Z}^+, n\in\mathbb{N}} I_{m,n}$. The sequence $\mathbf{x}(n) = \{\alpha n\}$ is dense in $[0, 1)$ if and only if for any interval $I \subseteq [0, 1)$ there are m and n, such that the line $y = \alpha x$ intersects the set $I_{m,n}$. It is known that $[0, 1) - H(I) \subseteq \mathbb{Q}$ for any interval $I \subseteq [0, 1)$, i.e. for any irrational $\alpha > 0$ the line $y = \alpha x$ intersects the set $\bigcup\limits_{m\in\mathbb{Z}^+, n\in\mathbb{N}} I_{m,n}$. Moreover in the case of irrational factors it is known that the frequency of occurrence of $\mathbf{x}(n) = \{\alpha n\}$ in the interval I is equal to its length.

In some sense the interval I, in the above example, can be named as *static* because it does not change in time n. However in order to study and describe previously mentioned problems such as the target discounted-sum problem, PAMs reachability problems, the Mahler's 3/2 problem we require the notion of "dynamic intervals".

Let $\mathbf{x}$ be a sequence of numbers from $[0, 1)$ and $\mathbf{x}(n)$ is the n-th element in the sequence. What is the distribution of a sequence $\mathbf{x}'(n) = \left\{p^{\mathbf{k}(n)}\mathbf{x}(n)\right\}$ for $n = 1, ..., \infty$, where $\mathbf{k} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a non-decreasing sequence? For example, if $\mathbf{k}(n) = n - 1$ and the number $\mathbf{x}(n)$ has in the base p the following form $\mathbf{x}(n) = 0.a_{n,1}a_{n,2} \ldots a_{n,n}a_{n,n+1} \ldots$, then $\mathbf{x}'(n) = 0.a_{n,n}a_{n,n+1} \ldots$.

Let us assume that $I \subseteq S^1$ and $\mathbf{k} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a non-decreasing sequence, $p \in \mathbb{N}$. Now we define "dynamical intervals" as an evolving infinite sequence $\mathbf{I}(1)$, $\mathbf{I}(2)$, $\mathbf{I}(3)$, … :

$$\mathbf{I}(1) = I, \mathbf{I}(n) = \bigcup_{j=0}^{p^{\mathbf{k}(n)}-1} \tfrac{I+j}{p^{\mathbf{k}(n)}}.$$

By $F^{\mathbf{x}}_{\mathbf{I}}(n) = |\{i \in \mathbb{Z}^+ | i \leq n, \mathbf{x}(i) \in \mathbf{I}(i)\}|$ we denote a function representing a frequency of hitting dynamical interval $\mathbf{I}$ by the sequence $\mathbf{x}$. In contrast to $F_I^{\mathbf{x}}(n)$ which only counts the number of hittings to a fixed interval I, our new function $F_{\mathbf{I}}^{\mathbf{x}}$ counts the number of hittings when both points and intervals are changing in time.

Proposition 22. *The following equation holds:* $F_I^{\mathbf{x}'}(n) = F_{\mathbf{I}}^{\mathbf{x}}(n)$.

Let us consider the Benford's Law phenomenon, i.e. the phenomenon that significant digit distribution in real data are not occurring randomly. For example the sequence p^1, p^2, p^3, ... satisfies Benford's Law, under the condition that $\log_{10} p$ is an irrational number, which is a consequence of the *Equidistribution theorem*. It gives us the fact that each significant digit of numbers in (p^n) sequence will correspond to the interval $\mathbb{R}/\mathbb{Z}$ and the length of the interval related to the frequency for each appearing digit.

However the question about the distribution of the sequence $\{(3/2)^n\}$ is different in the way that it is not about the distribution of the first digits of 3^n in base 2, i.e. not about the distribution of the sequence $\frac{3^n}{2^{\lceil n \log_2 3 \rceil}}$, but related to the sequence of digits after some shift of the number $\frac{3^n}{2^{\lceil n \log_2 3 \rceil}}$ corresponding to the multiplication by a power of 2.

So in the above notations the distribution of numbers in the sequence $\mathbf{x}'(n) = \{(3/2)^n\}$ corresponds to the $F_I^{\mathbf{x}'}(n)$ for the logarithmic (Benford's law) distributed sequence $\mathbf{x}(n) = \frac{3^n}{2^{\lceil n \log_2 3 \rceil}}$, $p = 2$ and $\mathbf{k}(n) = \lceil n \log_2 3 \rceil - n$.

Now we will show that even if the sequence $\{\alpha\}$, $\{2\alpha\}$, $\{3\alpha\}$, $\ldots$ is uniformly distributed on the circle $\mathbb{R}/\mathbb{Z}$, the irrationality of α is not enough to guarantee uniform distribution or even density of the sequence $\mathbf{x}'(n)$ on the circle corresponding to the linear shifts $\mathbf{k}(n) = n$.

Theorem 23. *Let us define* $\alpha = \sum_{i=1}^{\infty} \frac{1}{2^{\Delta_i}}$ *where* $\Delta_1 = 1$, $\Delta_{i+1} = 2^{\Delta_i} + \Delta_i$, $i \geq 1$ *(http://oeis.org/A034797). Then for all* $n \in \mathbb{N} \cup \{0\}$ *the sequence* $\{2^n n\alpha\}$ *is not dense in the interval* $[0, 1]$ *and* $\{2^n n\alpha\} < \frac{1}{2}$.

Proof. Let us consider a sequence Δ, with initial elements $\Delta_1 = 1$, $\Delta_2 = 3$, $\Delta_3 = 11$, $\Delta_4 = 2059$ etc.

Let us prove that when $0 \leq n \leq \Delta_i$ the inequality $\{2^n n\alpha\} < \frac{1}{2}$ if and only if $\{2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}}\} < \frac{1}{2}$. The implication from left to right follows from $\{2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}}\} < \{2^n n\alpha\}$. Let us show that it also holds in other direction. Assume that $\{2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}}\} < \frac{1}{2}$ then $\{2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}} + x\} < \frac{1}{2}$ for any $0 \leq x \leq \frac{1}{2^{\Delta_i + 1 - n}}$. In this case it is enough to show that $2^n n\alpha - 2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}} < \frac{1}{2^{\Delta_i + 1 - n}}$ holds when $n \leq \Delta_i$. In fact we have that

$$2^n n\alpha - 2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}} = 2^n n \sum_{j=i+1}^{\infty} \frac{1}{2^{\Delta_j}} < 2^n n \frac{1}{2^{\Delta_{i+1} - 1}} =$$

$$= \frac{1}{2^{\Delta_{i+1} - n - \log_2(n) - 1}} \leq \frac{1}{2^{\Delta_{i+1} - \Delta_i - \log_2(\Delta_i) - 1}} == \frac{1}{2^{2^{\Delta_i} - \log_2(\Delta_i) - 1}}.$$

Finally we have $\frac{1}{2^{2^{\Delta_i - \log_2(\Delta_i)-1}}} < \frac{1}{2^{\Delta_i+1-n}}$ when $n \leq \Delta_i$ and $i > 1$.

Now for proving the theorem it is enough to show that for $0 \leq n \leq \Delta_i$ the inequality $\{2^n n \sum_{j=1}^{i} \frac{1}{2^{\Delta_j}}\} < \frac{1}{2}$ holds. In the case $i = 1$ the statement is trivial. Let us assume that it holds for $i = k-1$. We show now that it holds for $i = k$, i.e. we show that for $\Delta_{k-1} \leq n \leq \Delta_k$ inequality $\{2^n n \sum_{j=1}^{k} \frac{1}{2^{\Delta_j}}\} < \frac{1}{2}$ holds, or taking into account $\Delta_{k-1} \leq n$ we also have inequality $\{2^n n \frac{1}{2^{\Delta_k}}\} < \frac{1}{2}$.

For $\Delta_{k-1} \leq n \leq \Delta_k - \Delta_{k-1} - 1$ we get $2^n n \frac{1}{2^{\Delta_k}} \leq 2^{\Delta_k - \Delta_{k-1}-1}(\Delta_k - \Delta_{k-1} - 1)\frac{1}{2^{\Delta_k}}$. Since $\Delta_k = 2^{\Delta_{k-1}} + \Delta_{k-1}$, then $2^{\Delta_k - \Delta_{k-1}-1}(\Delta_k - \Delta_{k-1} - 1)\frac{1}{2^{\Delta_k}} = (2^{\Delta_{k-1}} - 1)\frac{1}{2^{\Delta_{k-1}+1}} = \frac{1}{2} - \frac{1}{2^{\Delta_{k-1}+1}} < \frac{1}{2}$.

$$
\begin{array}{lllcccccccccccl}
 & & & \Delta_1 & & \Delta_2 & & & & & & & & \Delta_3 & \\
\mathbf{x}(0) & = & 0, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\mathbf{x}(1) & = & 0, & \boxed{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
\mathbf{x}(2) & = & 0, & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
\mathbf{x}(3) & = & 0, & 0 & 1 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots \\
\mathbf{x}(4) & = & 0, & 1 & 0 & 0 & \boxed{0} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
\mathbf{x}(5) & = & 0, & 1 & 0 & 1 & 0 & \boxed{0} & 0 & 0 & 0 & 1 & 0 & 1 & \ldots \\
\mathbf{x}(6) & = & 0, & 1 & 1 & 0 & 0 & 0 & \boxed{0} & 0 & 0 & 1 & 1 & 0 & \ldots \\
\mathbf{x}(7) & = & 0, & 1 & 1 & 1 & 0 & 0 & 0 & \boxed{0} & 0 & 1 & 1 & 1 & \ldots \\
\mathbf{x}(8) & = & 0, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \ldots \\
\mathbf{x}(9) & = & 0, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \boxed{0} & 0 & 1 & \ldots \\
\mathbf{x}(10) & = & 0, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \boxed{1} & 0 & \ldots \\
\mathbf{x}(11) & = & 0, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \boxed{1} & \ldots
\end{array}
\qquad
\begin{array}{ll}
\mathbf{x}'(0) & = 0,\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
\mathbf{x}'(1) & = 0,\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ \ldots \\
\mathbf{x}'(2) & = 0,\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ \ldots \\
\mathbf{x}'(3) & = 0,\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ \ldots \\
\mathbf{x}'(4) & = 0,\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ \ldots \\
\mathbf{x}'(5) & = 0,\ 0\ 0\ 0\ 1\ 0\ 1\ \ldots \\
\mathbf{x}'(6) & = 0,\ 0\ 0\ 1\ 1\ 0\ \ldots \\
\mathbf{x}'(7) & = 0,\ 0\ 1\ 1\ 1\ \ldots \\
\mathbf{x}'(8) & = 0,\ 0\ 0\ 0\ \ldots \\
\mathbf{x}'(9) & = 0,\ 0\ 1\ \ldots \\
\mathbf{x}'(10) & = 0,\ 0\ \ldots \\
\mathbf{x}'(11) & = 0,\ 0\ldots
\end{array}
$$

Consider the case when $\Delta_k - \Delta_{k-1} \leq n \leq \Delta_k$, i.e. $2^{\Delta_{k-1}} \leq n \leq 2^{\Delta_{k-1}} + \Delta_{k-1}$ and define $m = n - 2^{\Delta_{k-1}}$. We have that $0 \leq m \leq \Delta_{k-1}$ and $2^n n \frac{1}{2^{\Delta_k}} = 2^{m+2^{\Delta_{k-1}}}(m + 2^{\Delta_{k-1}})\frac{1}{2^{2^{\Delta_{k-1}}+\Delta_{k-1}}} = 2^m(m + 2^{\Delta_{k-1}})\frac{1}{2^{\Delta_{k-1}}} = 2^m m \frac{1}{2^{\Delta_{k-1}}} + 2^m$. Therefore $\{2^n n \frac{1}{2^{\Delta_k}}\} = \{2^m m \frac{1}{2^{\Delta_{k-1}}}\}$, where $0 \leq m \leq \Delta_{k-1}$. Now by the induction we have that $\{2^m m \frac{1}{2^{\Delta_{k-1}}}\} < \frac{1}{2}$. $\square$

7. Conclusion

In this paper we investigate natural long-standing open reachability problem for one-dimensional piecewise-affine maps (PAM) which plays a pivotal role for hybrid systems reachability and is related to some difficult number-theoretic problems. Let us summarize our contributions. First we have shown a simple reduction of reachability problem for a subclass of PAMs to an (open) reachability question for 2×2 matrix semigroups. Then we provided decidability for two subclasses of PAMs, connected a deep and natural question on existence/boundedness of invariant densities with decidability of the reachability problems in PAMs and open problems on beta-expansions. Although the question about the distributions for PAM orbits remains open we have unexpectedly shown that in some systems, operating with irrational numbers, the uniform distribution of original orbits in maps may not remain uniform or even dense when taking the fractional part after regular shifts. This makes the questions about PAMs even more "mysterious" as it is not clear whether such property may hold for a sequence of points generated by PAMs, β-expansion and Mahler's problem.

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