SYMMETRIES OF TILINGS OF LORENTZ SPACES

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ABSTRACT. We study tilings of the 3-dimensional simply connected Lorentz manifold of constant curvature. This manifold is modelled on the Lie group

$$G = \widetilde{\mathrm{SU}(1,1)} \cong \widetilde{\mathrm{SL}(2,\mathbb{R})},$$

equipped with the Killing form. The tilings are produced by the fundamental domain construction introduced by the second author. The construction gives Lorentz polyhedra as fundamental domains for the action by left multiplication of a discrete co-compact subgroup of finite level. We determine the symmetry groups of these tilings and discuss the connection with the Seifert fibration of the quotient space. We then give an explicit description of the symmetry group of the tiling in the case when the discrete subgroup is a lift of a triangle group.

1. INTRODUCTION

Tilings of the plane, the sphere and the hyperbolic plane are a beautiful and well-studied subject, see [6] for an overview. We are interested in tilings of

$$G = \widetilde{\mathrm{SU}(1,1)} \cong \widetilde{\mathrm{SL}(2,\mathbb{R})},$$

the model for one of Thurston's eight 3-dimensional geometries [21, 19, 11]. The Killing form on the simply connected Lie group G induces a bi-invariant Lorentz metric. Thurston's geometry of G can be interpreted as a sub-geometry of its Lorentz geometry. We will describe a construction of tilings of G by Lorentz polyhedra and study symmetries of these tilings.

In this paper we will study tilings of G that stem from the fundamental domain construction introduced by the second author in [15, 16]. Let Γ be a discrete subgroup acting on G by left multiplication. The level of Γ is the index of $\Gamma \cap Z(G)$ as a subgroup of the centre $Z(G)$ of G. Note that $Z(G)$ is an infinite cyclic group. [15, 16] introduced a construction of polyhedral fundamental domains for the action of a discrete subgroup of finite index on G by left multiplication, generalizing the results of Fisher [8] and Balke et al. [2]. We will discuss properties of the tilings of G by these fundamental domains. Multiplication by elements of Γ on the left induces an isometric action on the tiling that is simply transitive. However there

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may be other isometries of G that preserve the tiling. We give a description of the full symmetry group of the tiling, i.e. the group of all Lorentz isometries of G that preserve the tiling. We then discuss the Seifert fibration of the quotient $\Gamma\backslash G$ and explain a connection between certain symmetries of the tiling and the exceptional fibres of the Seifert fibration. Finally we give a more detailed description of the symmetry group of the tiling for the lifts of triangle groups.

The fundamental domain construction leads to a description of the quotients $\Gamma\backslash G$ as polyhedra with faces identified according to certain gluing rules on the boundary. There is a connection with singularity theory which was the starting point for the investigation of the quotients $\Gamma\backslash G$, namely that these quotients can be identified with the links of quasi-homogeneous Gorenstein surface singularities as shown by Dolgachev in [7]. For instance the links of the singularities in the series E, Z and Q as described by Arnold, Gusein-Zade and Varchenko in [1] correspond to quotients of the form $\Gamma\backslash G$, where $\Gamma = \Gamma(p,q,r)^k$ is a lift of a triangle group, see section 6 for the definition and [5] for a detailed discussion of this connection. Links of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities on the other hand lead to bi-quotients of the form $\Gamma_1\backslash G/\Gamma_2$ for discrete subgroups Γ_1 and Γ_2 with cyclic Γ_2 as discussed in [17]. The fundamental domain construction can also be generalized to the action of Γ_1 on the left and Γ_2 on the right as shown in [18, 3, 4] and it would be of interest to study the symmetries of the tilings by fundamental domains in this case.

Another interesting interpretation of Thurston's geometry of $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ is the projective interpretation introduced by Molnár in [11], where the geometry is modelled on the projective space by a special group of collineations. Molnár's approach gives a unified framework for all eight Thurston's geometries and emphasises their connections in the sense of Felix Klein. Using this interpretation, Molnár and Prok [12] and Molnár, Prok and Szirmai [13] studied tilings of G by simplices. Molnár, Szirmai and Vesnin in [14] used particular tilings of G by prisms to obtain geodesic ball packings and translation ball packings of very high density.

The paper is organised as follows: In section 2 we discuss the Lorentz geometry of $\mathrm{SU}(1,1)$ and its covering Lie groups and describe their isometry groups. In section 3 we discuss the fibration of G over the hyperbolic plane $\mathbb{D}$ and those isometries of G that preserve the fibration. The construction of the tiling by fundamental domains is outlined in section 4. In section 5 we describe the symmetries of the tiling. The description of the group of all symmetries of the tiling by fundamental domains is given in Theorem 16. Finally in sections 6 and 7 we focus on the special case where the group Γ is a lift of a triangle group. In Theorem 20 in section 6 we show that the group of all symmetries of the tiling for a lift Γ of a triangle group is a product of the group of left translations by elements of Γ and a particular dihedral group. In section 7 we discuss which fibres of the fibration $\Gamma\backslash G \to \mathbb{D}$ are fixed point sets of symmetries of the tiling and determine their orders.

2. Geometry of $G = \widetilde{\mathrm{SU}(1,1)}$

The group $\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm 1\}$ can be identified with the group $\mathrm{Isom}^+(\mathbb{D})$ of orientation-preserving isometries of the disc model $\mathbb{D}$ for the hyperbolic plane.

Consider the universal covering map $G \to \mathrm{PSU}(1,1)$. The kernel of this homomorphism is the centre $Z(G) \cong \mathbb{Z}$ of the group G. Therefore, for each natural number k there is a connected k-fold covering of $\mathrm{PSU}(1,1)$ which is unique up to Lie group isomorphism and is given by

$$G_k = G/(k \cdot Z(G)).$$

Given a base-point $x \in \mathbb{D}$ and a real number t, let $\rho_x(t) \in \mathrm{PSU}(1,1)$ denote the rotation through the angle t about the point x. This leads to a homomorphism $\rho_x : \mathbb{R} \to \mathrm{PSU}(1,1)$, which lifts to the unique homomorphism $R_x : \mathbb{R} \to G$ into the universal covering group. The element $R_x(2\pi)$ does not depend on x and is one of the two generators of the centre $Z(G)$.

The group $\mathrm{SU}(1,1)$ can be identified with

$$\bar{G} = \left\{(z,w) \in \mathbb{C}^2 \;\middle|\; |z|^2 - |w|^2 = -1\right\} \quad \text{via} \quad \begin{pmatrix} \bar{w} & z \\ \bar{z} & w \end{pmatrix} \mapsto (z,w).$$

We will now give a description of the group $\mathrm{Isom}(G)$ of Lorentz isometries of G following [10]. The product $G \times G$ acts on G via

$$(g,h) \cdot x = gxh^{-1}$$

by Lorentz isometries since the metric is bi-invariant. Therefore left translation $L_g(h) = gh$, right translation $R_g(h) = hg$ and conjugation $K_g(h) = ghg^{-1}$ are isometries of G. Let $\bar{\varepsilon} : \bar{G} \to \bar{G}$ be given by $\bar{\varepsilon}(z,w) = (-z, \bar{w})$. Let $\varepsilon : G \to G$ be the lift of $\bar{\varepsilon}$ with $\varepsilon(e) = e$. The map ε is the inversion $\varepsilon(g) = g^{-1}$. Let $\bar{\eta} : \bar{G} \to \bar{G}$ be given by the conjugation with the reflection in the real axis, i.e. $\bar{\eta}(z,w) = (\bar{z}, \bar{w})$. Let $\eta : G \to G$ be the lift of $\bar{\eta}$ with $\eta(e) = e$.

Proposition 1. *The connected component* $\mathrm{Isom}_0(G)$ *of the identity in the Lorentzian isometry group* $\mathrm{Isom}(G)$ *of* G *is*

$$\mathrm{Isom}_0(G) = \{L_a \circ R_b \;|\; a,b \in G\} \cong (G \times G)/\{(g,g) : g \in Z(G)\}.$$

The full Lorentzian isometry group of G *has four components corresponding to time- and/or space-reversals. The group of orientation-preserving isometries is*

$$\mathrm{Isom}^+(G) = \mathrm{Isom}_0(G) \times \langle \eta \rangle.$$

The full isometry group of G *is*

$$\mathrm{Isom}(G) = \mathrm{Isom}^+(G) \times \langle \varepsilon \rangle.$$

The stabilizers of the identity in the isometry groups are

$$\begin{aligned} \mathrm{Isom}_0(G)_e &= \{K_g \;|\; g \in G\} \cong G, \\ \mathrm{Isom}^+(G)_e &= \mathrm{Isom}_0(G)_e \times \langle \eta \rangle, \\ \mathrm{Isom}(G)_e &= \mathrm{Isom}^+(G)_e \times \langle \varepsilon \rangle. \end{aligned}$$

The elements of $\mathrm{Isom}^+(G)_e$ *act on* G *through automorphisms, while the elements of* $\mathrm{Isom}^-(G)_e = \mathrm{Isom}(G)_e \setminus \mathrm{Isom}^+(G)_e$ *act on* G *through anti-automorphisms. A similar description can be obtained for the isometry groups* $\mathrm{Isom}(G_k)$ *of finite covering groups* G_k.

Let us describe the action of the isometries of G on the hyperbolic plane $\mathbb{D}$. The group G acts on $\mathbb{D}$ since $\mathrm{Isom}^+(\mathbb{D}) \cong \mathrm{PSU}(1,1)$. The group $\mathrm{Isom}_0(G)_e$ acts on $\mathbb{D}$ by $K_g(x) = g(x)$. The isometry η acts on $\mathbb{D}$ as a reflection in the real axis. Combining these actions we obtain an action of $\mathrm{Isom}^+(G)_e = \mathrm{Isom}_0(G)_e \times \langle\eta\rangle$ on $\mathbb{D}$.

Remark. By projecting the isometries of G to $\bar{G} \cong G_2 \cong \mathrm{SU}(1,1)$ we obtain the description of the isometry group of $\bar{G}$ and its subgroups. In this case, which was considered in [2], subgroups of the isometry group can be identified with some well known matrix groups

$$\mathrm{Isom}(\bar{G}) \cong \mathrm{O}(2,2), \quad \mathrm{Isom}^+(\bar{G}) \cong \mathrm{SO}(2,2),$$
$$\mathrm{Isom}(\bar{G})_e \cong \mathrm{O}(1,2), \quad \mathrm{Isom}^+(\bar{G})_e \cong \mathrm{SO}(1,2), \quad \mathrm{Isom}_0(\bar{G})_e \cong \mathrm{SU}(1,1) \cong \bar{G}.$$

3. Fibration of G

The simply connected Lie group $G = \widetilde{\mathrm{SU}(1,1)} \cong \widetilde{\mathrm{SL}(2,\mathbb{R})}$ with a left-invariant Riemannian metric is one of the eight three-dimensional Thurston geometries, see [21, 19]. If we fix an arbitrary point u in $\mathbb{D}$ then the group

$$H = G \times (\mathrm{Isom}^+(G)_e)_u$$

acts transitively on G with compact point stabilizers. So we have identified the geometry of $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ according to Thurston with a subgeometry of the Lorentzian geometry of $G = \widetilde{\mathrm{SU}(1,1)}$. Moreover, the Lorentzian isometry group $\mathrm{Isom}(G)$ acts transitively on the set of all such geometric structures on G. Since we have restricted the group of Lorentzian isometries we should have further objects invariant with respect to H, in addition to geodesics and totally geodesic submanifolds. The choice of H was equivalent to the choice of a point u in $\mathbb{D}$. But any choice of a point $u \in \mathbb{D}$ induces a fibration $\xi_u : G \to \mathbb{D}$ given by $g \mapsto g(u)$. The fibres of $\xi_u : G \to \mathbb{D}$ are the left cosets of the stabilizer G_u. For additional geometric insight into this fibration see [11], p. 272. In this section our aim is to describe the group $\mathrm{Isom}_{fib}(G)$ of all isometries of G that are compatible with the fibration $\xi_u : G \to \mathbb{D}$, i.e. map fibres to fibres.

Lemma 2. *For any $\varphi \in \mathrm{Isom}^+(G)_e$ and $g \in G$ we have*

$$K_{\varphi(g)} = \varphi \circ K_g \circ \varphi^{-1}.$$

Proof. Any isometry $\varphi \in \mathrm{Isom}^+(G)_e$ is an automorphism of G, hence for $h \in G$ we have

$$\begin{aligned} K_{\varphi(g)}(h) &= \varphi(g) \cdot h \cdot \varphi(g^{-1}) = \varphi(g \cdot \varphi^{-1}(h) \cdot g^{-1}) \\ &= \varphi(K_g(\varphi^{-1}(h))) = (\varphi \circ K_g \circ \varphi^{-1})(h). \end{aligned}$$ □

Lemma 3. *For any $\varphi \in \mathrm{Isom}^+(G)_e$ and $g \in G_u$ the point $\varphi(u)$ is a fixed point of the action of $\varphi(g)$ on $\mathbb{D}$.*

Proof. The formula $K_{\varphi(g)} = \varphi \circ K_g \circ \varphi^{-1}$ from Lemma 2 and the fact that $g(u) = u$ imply

$$\varphi(g)(\varphi(u)) = (\varphi \circ g \circ \varphi^{-1})(\varphi(u)) = \varphi(g(u)) = \varphi(u). \quad \square$$

Proposition 4. *The group of isometries of G that preserve the fibration ξ_u, orientation and identity is $\mathrm{Isom}^+_{fib}(G)_e = (\mathrm{Isom}^+(G)_e)_u$.*

Proof. Let $\varphi \in \mathrm{Isom}^+(G)_e$. Our aim is to show that φ preserves the fibration ξ_u if and only if $\varphi(u) = u$.

Suppose that φ preserves the fibration ξ_u, then the image $\varphi(G_u)$ of the fibre $\xi_u^{-1}(u) = G_u$ under φ must also be a fibre of ξ_u. Both fibres $\varphi(G_u)$ and G_u contain $\varphi(e) = e$, hence $\varphi(G_u) = G_u$. Using Lemma 3 we obtain that $\varphi(u)$ is a fixed point for every $g \in G_u$, i.e. every element in G_u fixes both points u and $\varphi(u)$ in $\mathbb{D}$, which is only possible if $\varphi(u) = u$.

On the other hand suppose that $\varphi(u) = u$. Lemma 3 says that $\varphi(u) = u$ is a fixed point of $\varphi(g)$ for every $g \in G_u$, hence $\varphi(G_u) \subset G_u$. We also have $\varphi^{-1}(u) = u$, hence $\varphi^{-1}(G_u) \subset G_u$ and $G_u \subset \varphi(G_u)$. We conclude that $\varphi(G_u) = G_u$, so that φ preserves the fibre $G_u = \xi_u^{-1}(u)$. As φ is an automorphism, it must then map fibres to fibres: $\varphi(a \cdot G_u) = \varphi(a) \cdot G_u$. Thus φ preserves the fibration ξ_u. $\square$

Lemma 5. *For a subgroup Γ of G and a point $u \in \mathbb{D}$ we have $N(\Gamma_u) \subset G_u$, hence $\Gamma_u \lhd \Gamma$ if and only if $\Gamma = \Gamma_u$. In the case $\Gamma = G$ this means that G_u is not a normal subgroup of G.*

Proof. Let $g \in \Gamma_u$ and $a \in N(\Gamma_u)$, then the element aga^{-1} is in Γ_u and hence $(aga^{-1})(u) = u$. On the other hand Lemma 3 implies that $K_a(u) = a(u)$ is a fixed point of $K_a(g) = aga^{-1}$. Therefore aga^{-1} has two fixed points u and $a(u)$ in $\mathbb{D}$, which is impossible for an isometry of $\mathbb{D}$ unless $u = a(u)$. Thus every element of $N(\Gamma_u)$ fixes u and therefore belongs to G_u. As a consequence, if $\Gamma_u \lhd \Gamma$ then $\Gamma \subset N(\Gamma_u) \subset G_u$ and therefore

$$\Gamma = \Gamma \cap G_u = \Gamma_u.$$

The other direction, if $\Gamma = \Gamma_u$ then $\Gamma_u \lhd \Gamma$, is obvious. Now if we take $\Gamma = G$, the group G contains elements that do not fix u, hence $G \neq G_u$ and therefore G_u is not normal in G. $\square$

Proposition 6. *The group of isometries of G that preserve the fibration ξ_u and identity is* $\mathrm{Isom}_{fib}(G)_e = (\mathrm{Isom}^+(G)_e)_u$.

Proof. Our aim is to show that elements of $\mathrm{Isom}^-(G)_e$ do not preserve the fibration ξ_u. If an isometry φ preserves the fibres of ξ_u, then $\varphi(a \cdot G_u)$ must be a fibre and must contain $\varphi(a)$, hence $\varphi(a \cdot G_u) = \varphi(a) \cdot G_u$ for all $a \in G$, in particular $\varphi(G_u) = G_u$. On the other hand any isometry $\varphi \in \mathrm{Isom}^-(G)_e$ acts on G as an anti-automorphism, hence $\varphi(a \cdot G_u) = \varphi(G_u) \cdot \varphi(a)$ for all $a \in G$. Combining these results we get $\varphi(a) \cdot G_u = G_u \cdot \varphi(a)$ for all $a \in G$. This is equivalent to G_u being normal in G in contradiction to Lemma 5. Thus elements of $\mathrm{Isom}^-(G)_e$ do not preserve the fibration ξ_u. $\square$

Theorem 7. *The group of fibration preserving isometries of G is*

$$\mathrm{Isom}_{fib}(G) = \{L_a \mid a \in G\} \times (\mathrm{Isom}^+(G)_e)_u.$$

In particular, all elements of this group are orientation-preserving.

Proof. It is clear that $\{L_a \mid a \in G\} \subset \mathrm{Isom}_{fib}(G)$. For $\varphi \in \mathrm{Isom}_{fib}(G)$ and $a = \varphi(e)$ we have $(L_{a^{-1}} \circ \varphi)(e) = a^{-1} \cdot \varphi(e) = e$, hence $L_{a^{-1}} \circ \varphi \in \mathrm{Isom}_{fib}(G)_e = (\mathrm{Isom}^+(G)_e)_u$. Therefore any element of $\mathrm{Isom}_{fib}(G)$ can be written as a product of a left translation L_a and an element $L_{a^{-1}} \circ \varphi$ in $(\mathrm{Isom}^+(G)_e)_u$. $\square$

The centre $Z(G) = \langle R_u(2\pi k)\rangle$ is contained in the fibre $\xi_u^{-1}(u) = G_u$ of the fibration $\xi_u : G \to \mathbb{D}$, hence we obtain induced fibrations $\xi_u : G_k \to \mathbb{D}$ on the k-fold coverings $G_k = G/(k \cdot Z(G))$. The fibres of $\xi_u : G_k \to \mathbb{D}$ are circles, the left cosets of the stabilizer $(G_k)_u$, and this gives G_k the structure of a Seifert fibration. Similar arguments as in the case of the universal covering G show that the group of fibration preserving isometries of the finite covering G_k is

$$\mathrm{Isom}_{fib}(G_k) = \{L_a \mid a \in G_k\} \times (\mathrm{Isom}^+(G_k)_e)_u.$$

4. The Construction of Fundamental Domains

We will now briefly describe the elements of the fundamental domain construction [15, 16] that we will need for the symmetry investigation. To construct the fundamental domains, the group G is considered as a hypersurface embedded in the $\mathbb{R}_+$-bundle $L = \mathbb{R}_+ \times G$. We consider the 4-dimensional real vector space $\mathbb{R}^4 \cong \mathbb{C}^2$ with the symmetric bilinear form

$$\langle (z_1, w_1), (z_2, w_2)\rangle = \mathrm{Re}(z_1\bar{z}_2 - w_1\bar{w}_2)$$

of signature $(2, 2)$. The group $\mathrm{SU}(1, 1)$ consists of matrices of the form

$$\begin{pmatrix} \bar{w} & z \\ \bar{z} & w \end{pmatrix}$$

with determinant $|w|^2 - |z|^2 = 1$ which act on $\mathbb{D}$ via Möbius transformations

$$x \mapsto \frac{\bar{w}x + z}{\bar{z}x + w}.$$

Recall that the quadric

$$\bar{G} = \left\{a \in \mathbb{C}^2 \mid \langle a, a\rangle = -1\right\} = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 - |w|^2 = -1\right\}$$

can be identified with the group $\mathrm{SU}(1, 1)$ via

$$(z, w) \mapsto \begin{pmatrix} \bar{w} & z \\ \bar{z} & w \end{pmatrix}.$$

The cone over $\bar{G}$ can be described as

$$\begin{aligned}\bar{L} &= \{\lambda \cdot a \mid \lambda > 0,\ a \in \bar{G}\} \\ &= \left\{a \in \mathbb{C}^2 \mid \langle a, a\rangle < 0\right\} = \left\{(z, w) \in \mathbb{C}^2 \mid |z| < |w|\right\}.\end{aligned}$$

The bilinear form on $\mathbb{C}^2$ induces a pseudo-Riemannian metric of signature $(2, 2)$ on $\bar{L}$ and of signature $(2, 1)$ on $\bar{G}$. The form on $\bar{G}$ is proportionate to the Killing form on $\mathrm{SU}(1, 1)$. The universal covering $G \to \bar{G}$ extends to the universal covering $L \to \bar{L}$. The covering space L inherits canonically a pseudo-Riemannian metric from $\bar{L}$. There exist canonical trivializations $\bar{L} \cong \bar{G} \times \mathbb{R}_+$ and $L \cong G \times \mathbb{R}_+$. The isometries of G lift to isometries of L. The universal covering $\pi : G \to \bar{G}$ can be described as

$$G = \{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid |z|^2 = r^2 - 1\}, \qquad \pi(z, \alpha, r) = (z, re^{i\alpha}).$$

For $(z, \alpha, r) \in G$, the positive real number r can be computed from z, hence we can also identify G with $\mathbb{C} \times \mathbb{R}$ via $(z, \alpha, r) \mapsto (z, \alpha)$.

Let Γ be a discrete subgroups of finite level in G. For the fundamental domain construction, the hypersurface G is replaced with its piece-wise totally geodesic polyhedral model, specially adopted to the action of Γ. The fundamental domains for the action of Γ on the model polyhedron are its faces, which are then projected onto G to obtain fundamental domains for the action of Γ on G. To explain how this polyhedral model is constructed, we need to describe the totally geodesic subspaces in L. For $g \in G$, consider the intersection with $\bar{L}$ of the affine tangent space of $\bar{G}$ at the point $\pi(g)$

$$\bar{E}_{\pi(g)} = \{a \in \bar{L} \mid \langle \pi(g), a\rangle = -1\}$$

and the intersection with $\bar{L}$ of the half-space of $\mathbb{C}^2$ bounded by $\bar{E}_{\pi(g)}$ and not containing the origin

$$\bar{I}_{\pi(g)} = \{a \in \bar{L} \mid \langle \pi(g), a\rangle \leqslant -1\}.$$

The sets $\bar{E}_{\pi(g)}$ and $\bar{I}_{\pi(g)}$ are simply connected and even contractible, hence their pre-images under the covering map π consist of infinitely many connected components, one of them containing g. Let E_g and I_g be those connected components of $\pi^{-1}(\bar{E}_{\pi(g)})$ and $\pi^{-1}(\bar{I}_{\pi(g)})$ respectively that contain g. The three-dimensional submanifold E_g divides L into two connected components, the closure of one of which is I_g. Let H_g be the closure of the other component of the complement of E_g in L. The boundaries of I_g and of H_g are equal to E_g.

We can now describe the polyhedral model of G. We assume the existence of a fixed point u of the action of Γ on $\mathbb{D}$. For every point x in the orbit $\Gamma(u)$ of the point u under the action of Γ on $\mathbb{D}$, let Q_x be the prism-like polyhedron

$$Q_x = \bigcap_{\substack{g\in\Gamma \\ g(u)=x}} H_g.$$

Let P be the union of these prisms

$$P = \bigcup_{x\in\Gamma(u)} Q_x.$$

The boundary ∂P of P is the polyhedral model of G. We obtain fundamental domains F_g for the action of Γ on ∂P as faces of ∂P:

$$F_g = \operatorname{Cl}\operatorname{Int}(E_g \cap \partial P), \quad g \in \Gamma.$$

These faces F_g are then projected under the bundle map $\pi : L \to G$ to Lorentz polyhedra $\mathcal{F}_g$ which are fundamental domains for the action of Γ on G:

$$\mathcal{F}_g = \pi(F_g), \quad g \in \Gamma.$$

The interior of the fundamental domain F_e can be described as

$$F_e^\circ = (E_e \cap \partial Q_u)^\circ - \bigcup_{x\in\Gamma(u)\setminus\{u\}} Q_x.$$

5. Symmetries of the tiling

Let Γ be a co-compact non-cyclic discrete subgroup of G of finite level k that is invariant under the isometry η. The construction described in section 4 yields a tiling of ∂P by fundamental domains F_g, $g \in \Gamma$. We will assume that the fixed point used in the construction of fundamental domains is $u = 0$ in $\mathbb{D}$. This tiling is then projected under the bundle map $\pi : L \to G$ to a tiling of G by Lorentz polyhedra $\mathcal{F}_g$, $g \in \Gamma$. We can assume without loss of generality that the fixed point $u \in \mathbb{D}$ used in the construction is at the origin. The group Γ acts on G by left multiplication, and this action extends to an action on L by isometries. This induces an action on P and on ∂P and finally an action of Γ on the tiling of G by $\mathcal{F}_g$ which is simply transitive. However, there may be other isometries of L that preserve the tiling of G by $\mathcal{F}_g$, i.e. map fundamental domains to fundamental domains. It suffices to determine the isometries that preserve the linear model ∂P of G and the tiling of ∂P by F_g.

Definition. An isometry φ of L is a *symmetry* of the tiling of ∂P by F_g if it maps every tile F_g to another tile F_h, i.e. for every $g \in \Gamma$ there exists $h \in \Gamma$ such that $\varphi(F_g) = F_h$. Let Symm be the group of all symmetries of the tiling of ∂P by F_g.

Lemma 8. *For any isometry φ of L and any $g \in G$ we have*

$$\varphi(E_g) = E_{\varphi(g)}, \quad \varphi(I_g) = I_{\varphi(g)}, \quad \varphi(H_g) = H_{\varphi(g)}.$$

Proof. By definition E_g is the 3-dimensional totally geodesic submanifold of L which is tangent to G at the point g. The isometry φ will map E_g to the 3-dimensional totally geodesic submanifold of L which is tangent to G at the point $\varphi(g)$, hence $\varphi(E_g) = E_{\varphi(g)}$. Similarly $\varphi(H_g) = H_{\varphi(g)}$ and $\varphi(I_g) = I_{\varphi(g)}$. $\square$

Lemma 9. *If $\varphi \in \mathrm{Symm}$ then $\varphi(\Gamma) = \Gamma$ and*

$$\varphi(F_g) = F_{\varphi(g)} \quad \textit{for every } g \in \Gamma.$$

Proof. If $\varphi \in \mathrm{Symm}$ then for any $g \in \Gamma$ there exists $h \in \Gamma$ such that

$$\varphi(F_g) = F_h \subset E_h.$$

On the other hand $F_g \subset E_g$ and therefore $\varphi(F_g) \subset \varphi(E_g)$. According to Lemma 8, we have $\varphi(E_g) = E_{\varphi(g)}$. We conclude that the 3-dimensional polyhedron F_g is mapped under the isometry φ to the intersection of two 3-dimensional subspaces E_h and $E_{\varphi(g)}$. This is only possible if $\varphi(g) = h$, hence $\varphi(g) \in \Gamma$ and $\varphi(F_g) = F_{\varphi(g)}$ for every $g \in \Gamma$. We proved that $\varphi(\Gamma) \subset \Gamma$ for every $\varphi \in \mathrm{Symm}$. Applying this result to $\varphi^{-1} \in \mathrm{Symm}$ we also have $\varphi^{-1}(\Gamma) \subset \Gamma$ and therefore $\Gamma \subset \varphi(\Gamma)$. Combining this results we see that $\varphi(\Gamma) = \Gamma$. $\square$

First let us observe that we can focus on those symmetries that preserve the identity:

Proposition 10. *The group of all symmetries of the tiling is*

$$\mathrm{Symm} = \{L_a \mid a \in \Gamma\} \times \mathrm{Symm}_e\,.$$

Proof. For any $a \in \Gamma$ the left multiplication L_a with a is a symmetry of the tiling

$$L_a(F_g) = a \cdot F_g = F_{a \cdot g}.$$

For an element $\varphi \in \mathrm{Symm}$ the isometry $L_{(\varphi(a))^{-1}} \circ \varphi$ is in Symm_e, hence

$$\mathrm{Symm} = \{L_a \mid a \in \Gamma\} \times \mathrm{Symm}_e\,. \quad \square$$

For a symmetry φ in Symm_e we have $\varphi(e) = e$, and therefore Lemma 9 implies $\varphi(F_e) = F_{\varphi(e)} = F_e$. We will now look at the way different types of isometries act on F_e. From the description of the fundamental domain F_e in section 4 we see that it is contained in the set $E_e \cap \partial Q_u$. The image of this set under the projection $L \to \bar{L}$ is the layer between two planes

$$\left\{(z, w) \in \bar{L} \;\middle|\; \mathrm{Re}(w) = 1,\ |\mathrm{Im}(w)| \leqslant \tan\vartheta\right\},$$

where $\vartheta = \frac{\pi k}{2p}$ and p is the order of the point u as a fixed point of the image of Γ in $\mathrm{PSU}(1,1)$. The points $R_u(\pm\vartheta)$ project to the points $(0, 1 \pm i\tan\vartheta)$ in $\bar{L}$ which are situated on the top and bottom planes of this layer respectively.

Lemma 11. *The points $R_u(\pm\vartheta)$ are contained in F_e.*

Proof. The points $R_u(\pm\vartheta)$ project to $(0, 1 \pm i\tan\vartheta)$ under π. Lemma 1(i) in [16] states that if $(z, w) \in \pi(Q_x)$ then

$$|w| - |z| \leqslant \sec\vartheta \cdot \sqrt{1 - |x|^2}.$$

For $(z, w) = (0, 1 \pm i\tan\vartheta)$ we have

$$|w| - |z| = \sec\vartheta > \sec\vartheta \cdot \sqrt{1 - |x|^2}.$$

Hence these points are not contained in any $\pi(Q_x)$ and therefore they are contained in $\pi(F_e)$ and their pre-images $R_u(\pm\vartheta)$ are contained in F_e. □

Lemma 12. *The way isometries act on $R_u(t)$ for $t \in \mathbb{R}$ is:*

(1) If $g \in G$ is an elliptic element with fixed point u then the conjugation K_g fixes the image of R_u point-wise, otherwise K_g maps each of the points $R_u(\tan\vartheta)$ and $R_u(-\tan\vartheta)$ to a point outside of the set $E_e \cap \partial Q_u$.

(2) If $x \in \mathbb{D} \cap \mathbb{R}$ then $\eta(R_x(t)) = R_x(-t)$.

(3) The isometries η and ε interchange the points $R_u(t)$ and $R_u(-t)$.

Proof. We have:

(1) If $g \in G$ projects to the element $(a, b) \in \bar{G}$ then the conjugation of $(0, 1+i\omega)$ by (a, b) gives

$$\begin{pmatrix} \bar{b} & a \\ \bar{a} & b \end{pmatrix} \cdot \begin{pmatrix} 1 - i\omega & 0 \\ 0 & 1 + i\omega \end{pmatrix} \cdot \begin{pmatrix} b & -a \\ -\bar{a} & \bar{b} \end{pmatrix}$$
$$= \begin{pmatrix} 1 - i(1 + 2|a|^2)\omega & 2ia\bar{b}\omega \\ -2i\bar{a}b\omega & 1 + i(1 + 2|a|^2)\omega \end{pmatrix}.$$

Thus the action of K_g on $(0, 1 \pm i\tan\vartheta)$ is

$$(0, 1 \pm i\tan\vartheta) \mapsto (\pm 2ia\bar{b}\tan\vartheta, 1 \pm i(1 + 2|a|^2)\tan\vartheta).$$

For the images of $(0, 1 \pm i\tan\vartheta)$ under K_g to stay between the planes $\mathrm{Im}(w) = \pm\tan\vartheta$ we require

$$(1 + 2|a|^2) \cdot \tan\vartheta \leqslant \tan\vartheta.$$

This is only possible if $a = 0$. For $(a, b) \in \bar{G}$ with $a = 0$ we must have $|b| = 1$. Hence (a, b) is of the form $(0, e^{it})$ for some $t \in \mathbb{R}$, so that g must be an elliptic element that fixes $u = 0$. If on the other hand $g \in G$ is an elliptic element that fixes u then $g = R_u(s)$ for some $s \in \mathbb{R}$ and $K_g(R_u(t)) = R_u(s)R_u(t)R_u(-s) = R_u(t)$ for all t.

(2) The isometry $\bar{\eta}$ is the conjugation with the reflection in the real axis, hence $\bar{\eta}(\rho_x(t))$ is a rotation with the fixed point $\bar{x}$. If x is on the real axis then $\bar{x} = x$. It is clear geometrically that $\bar{\eta}(\rho_x(t))$ rotates through the same angle as $\rho_x(t)$ but in the opposite direction, hence $\bar{\eta}(\rho_x(t)) = \rho_x(-t)$. Therefore both $\eta(R_x(t))$ and $R_x(-t)$ have the same projection under the covering map $G \to \mathrm{PSU}(1,1)$ and coincide at $t = 0$, hence they coincide everywhere.

(3) The fixed point $u = 0$ is on the real axis in $\mathbb{D}$, hence $\eta(R_u(t)) = R_u(-t)$. For the isometry ε we have $\varepsilon(R_u(t)) = (R_u(t))^{-1} = R_u(-t)$.

□

Proposition 13. *It holds*

$$\mathrm{Symm}_e^+ \subset \mathcal{K} \times \langle\eta\rangle \text{ and } \mathrm{Symm}_e \subset \mathcal{K} \times \langle\eta\rangle \times \langle\varepsilon\rangle, \text{ where } \mathcal{K} = \{K_g \mid g \in N(\Gamma)_u\}.$$

Proof. Every isometry $\varphi \in \mathrm{Symm}_e$ can be written as $\varphi = K_g \circ \psi$ for some $g \in G$ and $\psi \in \langle\eta, \varepsilon\rangle$. Lemma 11 says that $R_u(\pm\tan\vartheta) \in F_e$. Every symmetry in Symm_e preserves F_e, hence φ must leave $R_u(\pm\tan\vartheta)$ inside F_e. According to Lemma 12 the isometries η, ε and K_g with $g \in G_u$ leave the points $R_u(\pm\tan\vartheta)$ inside F_e, while the conjugations K_g with $g \notin G_u$ move $R_u(\pm\tan\vartheta)$ outside F_e. Hence we must have $g \in G_u$. Now Lemma 9 says that $\varphi(\Gamma) = \Gamma$. We assumed that η preserves Γ, while ε always preserves Γ since $\varepsilon(g) = g^{-1}$, therefore $\psi(\Gamma) = \Gamma$. Thus we have $\Gamma = \varphi(\Gamma) = K_g(\psi(\Gamma)) = K_g(\Gamma) = g \cdot \Gamma \cdot g^{-1}$, i.e. $g \in N(\Gamma)$. In summary, every element of Symm_e is of the form $\varphi = K_g \circ \psi$ with $g \in G_u \cap N(\Gamma) = N(\Gamma)_u$ and $\psi \in \langle\eta, \varepsilon\rangle$. For $\varphi \in \mathrm{Symm}_e^+$ we will in addition have $\psi \in \langle\eta\rangle$. □

Proposition 14. *It holds*

$$\mathrm{Symm}_e^+ = \mathcal{K} \times \langle\eta\rangle, \text{ where } \mathcal{K} = \{K_g \mid g \in N(\Gamma)_u\}.$$

An isometry $\varphi \in \mathrm{Isom}^+(G)_e$ is in Symm_e^+ if and only if $\varphi(\Gamma) = \Gamma$ and $\varphi(G_u) = G_u$.

Proof. We know from Proposition 13 that $\mathrm{Symm}_e^+ \subset \mathcal{K} \times \langle\eta\rangle$. Note that all isometries in $\mathcal{K} \times \langle\eta\rangle$ preserve Γ and G_u: conjugations K_g with $g \in N(\Gamma)_u$ preserve Γ since $g \in N(\Gamma)$ and preserve G_u since $g \in G_u$, the isometry η preserves Γ by assumption and preserves G_u since $\eta(R_u(t)) = R_u(-t)$. It remains to show that if $\varphi \in \mathrm{Isom}^+(G)_e$ preserves Γ and G_u then $\varphi \in \mathrm{Symm}_e^+$. Lemma 8 says that $\varphi(H_g) = H_{\varphi(g)}$ for all $g \in G$. For a point $x \in \Gamma(u)$ and $g \in \Gamma$ such that $g(u) = x$ we have

$$Q_x = \bigcap_{\substack{h\in\Gamma \\ h(u)=x}} H_h = \bigcap_{h \in g\cdot\Gamma_u} H_h,$$

$$\varphi(Q_x) = \bigcap_{h \in g\cdot\Gamma_u} \varphi(H_h) = \bigcap_{h \in g\cdot\Gamma_u} H_{\varphi(h)} = \bigcap_{k \in \varphi(g\cdot\Gamma_u)} H_k.$$

Since $\varphi \in \mathrm{Isom}^+(G)_e$ is an automorphism and φ preserves $\Gamma_u = \Gamma \cap G_u$, we have

$$\varphi(g \cdot \Gamma_u) = \varphi(g) \cdot \varphi(\Gamma_u) = \varphi(g) \cdot \Gamma_u$$

and therefore

$$\varphi(Q_x) = \bigcap_{k \in \varphi(g\cdot\Gamma_u)} H_k = \bigcap_{k \in \varphi(g)\cdot\Gamma_u} H_k = Q_{\varphi(g)(u)} = Q_{\varphi(x)}.$$

Finally $\varphi(\Gamma) = \Gamma$ implies $\varphi(\Gamma(u)) = \Gamma(u)$. It follows that

$$\begin{aligned} \varphi(P) &= \varphi\left(\bigcup_{x\in\Gamma(u)} Q_x\right) = \bigcup_{x\in\Gamma(u)} \varphi(Q_x) = \bigcup_{x\in\Gamma(u)} Q_{\varphi(x)} \\ &= \bigcup_{y\in\varphi(\Gamma(u))} Q_y = \bigcup_{y\in\Gamma(u)} Q_y = P \end{aligned}$$

and therefore

$$\varphi(\partial P) = \partial P.$$

For any $g \in \Gamma$ we have that φ maps F_g to another fundamental domain

$$\begin{aligned}\varphi(F_g) &= \varphi\big(\operatorname{Cl}\operatorname{Int}(E_g \cap \partial P)\big) = \operatorname{Cl}\operatorname{Int}(\varphi(E_g) \cap \varphi(\partial P))\\ &= \operatorname{Cl}\operatorname{Int}(E_{\varphi(g)} \cap \partial P) = F_{\varphi(g)}\end{aligned}$$

with $\varphi(g) \in \Gamma$, hence $\varphi \in \mathrm{Symm}_e^+$. □

Proposition 15. *It holds*

$$\mathrm{Symm}_e = \mathrm{Symm}_e^+ = \mathcal{K} \times \langle\eta\rangle, \text{ where } \mathcal{K} = \{K_g \mid g \in N(\Gamma)_u\}.$$

An isometry $\varphi \in \mathrm{Isom}(G)_e$ is in Symm_e if and only if $\varphi \in \mathrm{Isom}^+(G)_e$, $\varphi(\Gamma) = \Gamma$ and $\varphi(G_u) = G_u$.

Proof. According to Proposition 13, any isometry φ in Symm_e^- is of the form $\varphi = \psi \circ \varepsilon$ for some $\psi \in \mathcal{K} \times \langle\eta\rangle$. Proposition 14 then implies $\psi \in \mathrm{Symm}_e^+$. Now we have $\varphi \in \mathrm{Symm}_e^-$ and $\psi \in \mathrm{Symm}_e^+$, hence $\varepsilon = \psi^{-1} \circ \varphi \in \mathrm{Symm}_e^-$. Lemma 9 implies that for every $g \in \Gamma$

$$\varepsilon(F_g) = F_{\varepsilon(g)} = F_{g^{-1}} = g^{-1} \cdot F_e.$$

On the other hand ε is an anti-automorphism, hence

$$\varepsilon(F_g) = \varepsilon(g \cdot F_e) = \varepsilon(F_e) \cdot \varepsilon(g) = F_e \cdot g^{-1}.$$

Thus we have $F_e \cdot g^{-1} = g^{-1} \cdot F_e$ for every $g \in \Gamma$ and therefore

$$K_g(F_e) = g \cdot F_e \cdot g^{-1} = g \cdot g^{-1} \cdot F_e = F_e.$$

However using Lemma 12 as in the proof of Proposition 13 we see that $K_g(F_e) \subset F_e$ implies $g \in G_u$. Thus we must have $\Gamma \subset G_u$ and therefore $\Gamma = \Gamma_u$ in contradiction to the assumption that Γ is not cyclic. □

Combining Propositions 10 and 15 we obtain the following theorem:

Theorem 16. *The group of all symmetries of the tiling of G by $\mathcal{F}_g$ is*

$$\mathrm{Symm} = \{L_a \mid a \in \Gamma\} \times \{K_g \mid g \in N(\Gamma)_u\} \times \langle\eta\rangle.$$

In particular, $\mathrm{Symm} \subset \mathrm{Isom}^+(G)$.

Recall that according to Theorem 7 the group of all fibration preserving isometries of G is

$$\mathrm{Isom}_{fib}(G) = \{L_a \mid a \in G\} \times (\mathrm{Isom}^+(G)_e)_u.$$

From Theorem 16 we can see that the group Symm of symmetries of the tiling is a subgroup of $\mathrm{Isom}_{fib}(G)$, i.e. symmetries of the tiling preserve the fibration ξ_u.

The group G can be identified with a subgroup of $\mathrm{Isom}_0(G)$ via its action on itself by conjugation: we identify $g \in G$ with $K_g \in \mathrm{Isom}_0(G)$. The group Symm_e can then be interpreted as follows

Proposition 17. *It holds*

$$\mathrm{Symm}_e = N_{\mathrm{Isom}^+(G)}(\{K_g \mid g \in \Gamma\})_u.$$

Proof. For $\varphi \in \mathrm{Isom}^+(G)_e$ and $g \in G$, Lemma 2 says that $\varphi \circ K_g \circ \varphi^{-1} = K_{\varphi(g)}$, hence φ normalizes $\{K_g \mid g \in \Gamma\}$ in $\mathrm{Isom}^+(G)$ if and only if $\varphi(\Gamma) = \Gamma$. Recall that every $\varphi \in \mathrm{Isom}^+(G)_e$ is of the form $\varphi = K_g \circ \psi$ for some $g \in G$ and $\psi \in \langle \eta \rangle$. Note that η preserves Γ and fixes u, while K_g preserves Γ if $g \in N(\Gamma)$ and fixes u if $g \in G_u$, hence $\varphi = K_g \circ \psi$ normalizes $\{K_g \mid g \in \Gamma\}$ if and only if $g \in N(\Gamma)_u$. According to Proposition 15 this is equivalent to $\varphi \in \mathrm{Symm}_e$. □

How do elements of Symm_e act on F_e? The isometries K_g with $g \in N(\Gamma)_u$ act as rotations about the axis $\pi(G_u)$. The other elements of Symm_e act on $\mathbb{D}$ as reflections and on F_e as half-turns about axes perpendicular to $\pi(G_u)$. The fibration $\xi_u : G \to \mathbb{D}$ maps such an axis onto the fixed point set of the corresponding reflection. In general, the group Symm contains, besides its fixed point free elements including the elements of Γ, rotations about fibres of ξ_u and half-turns about axes perpendicular to those. The fibration ξ_u maps those axes to elliptic fixed points and reflection lines respectively. The corresponding motions of $\mathbb{D}$ leave the orbit $\Gamma(u)$ invariant.

6. Symmetries of the tiling for triangle groups

In this section we consider discrete subgroups of G whose projections to $\mathrm{PSU}(1,1)$ are triangle groups. A triangle group $\Gamma(p,q,r)$ is the subgroup of orientation-preserving isometries in the group generated by three reflexions in the sides of a hyperbolic triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$. This group can be also described as the group generated by three rotations as follows: Consider a hyperbolic triangle in $\mathbb{D}$ with vertices $u = 0$, v and w and angles

$$\alpha_u = \frac{\pi}{p}, \quad \alpha_v = \frac{\pi}{q} \quad \text{and} \quad \alpha_w = \frac{\pi}{r}.$$

The triangle group $\Gamma(p,q,r)$ is the group generated by the rotations

$$\rho_u(2\pi/p) \quad \text{and} \quad \rho_v(2\pi/q).$$

If we use three rotations

$$a = \rho_u(2\pi/p), \quad b = \rho_v(2\pi/q), \quad c = \rho_w(2\pi/r)$$

as generators, we get the presentation

$$\Gamma(p,q,r) \cong \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle.$$

The following existence result for discrete subgroups of finite level in G can be found in [15] (Section 2.8, Theorem 38).

Proposition 18. *There exists a subgroup of G of level k whose projection to* $\mathrm{PSU}(1,1)$ *is $\Gamma(p,q,r)$ if and only if*

$$\gcd(k,p) = \gcd(k,q) = \gcd(k,r) = 1 \quad pqr - pq - qr - rp \equiv 0 \mod k.$$

If this condition is satisfied then there exists exactly one such subgroup which we will denote by $\Gamma(p,q,r)^k$. The group $\Gamma(p,q,r)^k$ is generated by the elements

$$R_u(\tfrac{2\pi}{p}) \cdot U^a, \quad R_v(\tfrac{2\pi}{q}) \cdot U^b \quad \text{and} \quad U^k,$$

where $U = R_u(2\pi)$ is a generator of the centre of G and a and b are integers such that $pa + 1 \equiv qb + 1 \equiv 0 \mod k$.

We are interested in the symmetry group Symm of the tiling of G by fundamental domains $\mathcal{F}_g$ in the case $\Gamma = \Gamma(p,q,r)^k$. For computations in G we need the following notion of the argument of an element in G:

Definition. In section 2 we parametrized G as

$$G = \{(z,\alpha,r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \bigm| |z|^2 = r^2 - 1\}.$$

The argument map is $\arg : G \to \mathbb{R}$ given by $\arg(z,\alpha,r) = \alpha$.

We will summarize the basic properties of the argument map as derived in [15] in the following proposition:

Proposition 19. *Let $x \in \mathbb{D}$ and $t \in \mathbb{R}$.*

(1) $\arg(R_0(2t)) = t$.
(2) $\arg(R_x(2\pi k)) = \pi k$ *for any* $k \in \mathbb{Z}$.
(3) $\arg(R_x(2t)) = \pi k$ *for some* $k \in \mathbb{Z}$ *if and only if* $t = \pi k$.
(4) $\arg(R_x(2t)) \in (0,\pi)$ *for* $t \in (0,\pi)$.
(5) $\arg(g \cdot R_0(2t)) = \arg(R_0(2t) \cdot g) = \arg(g) + \arg(R_0(2t)) = \arg(g) + t$.

Proof. For convenience of the reader we will sketch the proofs of these properties.

(1) Both expressions $\arg(R_0(2t))$ and t have the same projection e^{it} under the covering map $\mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and coincide at $t = 0$, hence they coincide everywhere.
(2) For any $x \in \mathbb{D}$ we have $R_x(2\pi k) = R_0(2\pi k)$ and $\arg(R_0(2\pi k)) = \pi k$.
(3) If $\arg(R_x(2t)) = \pi k$ for some $k \in \mathbb{Z}$, then the image (z,w) of $R_x(2t)$ in $\bar{G}$ satisfies $w \in \mathbb{R}$. We have $0 = (\bar{w}x+z)-(\bar{z}x+w)x$ as $R_x(2t)$ fixes the point x. For real w this becomes $0 = z - \bar{z}x^2$. If $z \neq 0$ this implies $|z| = |\bar{z}x^2| < |z|$ as $|x| < 1$. Thus we must have $z = 0$ and therefore $w = \pm 1$. Hence the projection $\rho_x(2t)$ of $R_x(2t)$ to $\mathrm{PSU}(1,1)$ is the identity map and therefore $2t = 2\pi k$ for some $k \in \mathbb{Z}$.
(4) $\arg(R_x(2t))$ is a continuous function of t that changes from 0 at $t = 0$ to π at $t = 2\pi$ and can only be equal to a multiple of π if t is a multiple of 2π, hence $\arg(R_x(2t))$ is in $(0,\pi)$ for $t \in (0,2\pi)$.
(5) All four expressions have the same projection under $\mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and coincide at $t = 0$, hence they coincide everywhere. □

Theorem 20. *Let $\Gamma = \Gamma(p,q,r)^k$ be a hyperbolic triangle group. We can assume without loss of generality that the triangle is positioned so that $u = 0$, that v is on the positive real axis and w is in the upper half-plane. Consider the symmetries of the tiling of G by fundamental domains $\mathcal{F}_g$. Here we assume that the fixed point used in the construction of fundamental domains $\mathcal{F}_g$ is $u = 0$. Let δ be the conjugation by the element $R_u(\alpha)$, where $\alpha = \frac{2\pi}{p}$ if $q \neq r$ and $\alpha = \frac{\pi}{p}$ if $q = r$. Then the group of symmetries of the tiling that preserve the identity is the dihedral group*

$$\mathrm{Symm}_e = \langle \delta \rangle \times \langle \eta \rangle.$$

The order of Symm_e *is $2p$ if $q \neq r$ and $4p$ if $q = r$. The group of all symmetries of the tiling is*

$$\mathrm{Symm} = \Gamma \times \mathrm{Symm}_e,$$

where Γ corresponds to the left multiplication by elements of Γ.

Proof. The triangle group $\Gamma(p,q,r)^k$ is generated by the elements $R_u(\frac{2\pi}{p})\cdot U^a$, $R_v(\frac{2\pi}{q})\cdot U^b$ and U^k, where U, a and b are as in Proposition 18. First let us check that this group is invariant under η so that the results of section 5 can be applied. Proposition 12(2) says that for a point x on the real axis in $\mathbb{D}$ the isometry η maps $R_x(t)$ to its inverse $R_x(-t)=(R_x(t))^{-1}$. The points u and v are on the real axis, hence η maps the generators of Γ and of Γ_u to their inverses and therefore preserves Γ and Γ_u.

According to Proposition 15, $\mathrm{Symm}_e=\{K_g \bigm| g\in N(\Gamma)_u\}\times\langle\eta\rangle$, hence our aim is to describe $N(\Gamma)_u$. Elements of G_u are of the form $R_u(\theta)$ for $\theta\in\mathbb{R}$. Suppose that $R_u(\theta)\in N(\Gamma)_u$, i.e. the conjugation by $R_u(\theta)$ preserves Γ, then, after projection to $\mathrm{PSU}(1,1)$, we have that $\rho_u(\theta)\rho_v(\frac{2\pi}{q})\rho_u^{-1}(\theta)$ is in $\Gamma(p,q,r)$ and has $(\rho_u(\theta))(v)$ as a fixed point. In the case $q\neq r$ the only fixed points of elements of $\Gamma(p,q,r)$ at the same distance to u as v are the images of v under the generator $\rho_u(\frac{2\pi}{p})$, hence θ must be a multiple of $\frac{2\pi}{p}$. In the case $q=r$ the images of $w=(\rho_u(\frac{\pi}{p}))(v)$ under the generator $\rho_u(\frac{2\pi}{p})$ are at the same distance to u as the images of v, hence θ must be a multiple of $\frac{\pi}{p}$. Thus we proved that $N(\Gamma)_u\subset\langle R_u(\alpha)\rangle$.

On the other hand $\langle R_u(\frac{2\pi}{p})\rangle\subset N(\Gamma)_u$ since the conjugation with $R_u(\frac{2\pi}{p})$ fixes $R_u(\frac{2\pi}{p})\cdot U^a$ and U^k and

$$\begin{aligned}&R_u(\tfrac{2\pi}{p})\cdot\big(R_v(\tfrac{2\pi}{q})\cdot U^b\big)\cdot R_u^{-1}(\tfrac{2\pi}{p})\\&=\big(R_u(\tfrac{2\pi}{p})\cdot U^a\big)\cdot\big(R_v(\tfrac{2\pi}{q})\cdot U^b\big)\cdot\big(R_u(\tfrac{2\pi}{p})\cdot U^a\big)^{-1}\in\Gamma(p,q,r)^k.\end{aligned}$$

Now suppose that $q=r$ and that δ is the conjugation with $R_u(\alpha)=R_u(\frac{\pi}{p})$. It is easy to see geometrically that the isometry $\rho_u(\frac{\pi}{p})\rho_v(\frac{2\pi}{q})\rho_u(\frac{\pi}{p})$ fixes v and rotates $\bar{w}$ to w, hence

$$\rho_u(\tfrac{\pi}{p})\rho_v(\tfrac{2\pi}{q})\rho_u(\tfrac{\pi}{p})=\rho_v^{-1}(\tfrac{2\pi}{q})=\rho_v^{q-1}(\tfrac{2\pi}{q}).$$

Thus the elements $R_u(\frac{\pi}{p})R_v(\frac{2\pi}{q})R_u(\frac{\pi}{p})$ and $R_v^{q-1}(\frac{2\pi}{q})$ have the same projection to $\mathrm{PSU}(1,1)$. Direct computation using hyperbolic trigonometry shows that the element $R_v(\frac{2\pi}{q})$ corresponds to the element (z,w) in $\mathrm{SU}(1,1)$ with

$$w=\cos\tfrac{\pi}{q}+i\cosh(\mathrm{dist}(0,v))\sin\tfrac{\pi}{q},$$

where dist is the hyperbolic metric on $\mathbb{D}$. In the case $q=r$ we obtain

$$w=\frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{2p}}\cdot\exp\left(i\cdot\left(\tfrac{\pi}{2}-\tfrac{\pi}{2p}\right)\right).$$

Hence $\arg\left(R_v(\frac{2\pi}{q})\right)$ and $\frac{\pi}{2}-\frac{\pi}{2p}$ coincide up to an integer multiple of 2π. On the other hand Proposition 19 implies that $\arg\left(R_v(\frac{2\pi}{q})\right)\in(0,\pi)$, hence

$$\arg\left(R_v(\tfrac{2\pi}{q})\right)=\tfrac{\pi}{2}-\tfrac{\pi}{2p}.$$

Similarly

$$\arg\left(R_v^{q-1}(\tfrac{2\pi}{q})\right)=\tfrac{\pi}{2}+\tfrac{\pi}{2p}.$$

Proposition 19 implies

$$\arg\left(R_u(\tfrac{\pi}{p})R_v(\tfrac{2\pi}{q})R_u(\tfrac{\pi}{p})\right)=\arg\left(R_v(\tfrac{2\pi}{q})\right)+\tfrac{\pi}{p}=\tfrac{\pi}{2}+\tfrac{\pi}{2p}=\arg\left(R_v^{q-1}(\tfrac{2\pi}{q})\right).$$

Thus $R_u(\frac{\pi}{p})R_v(\frac{2\pi}{q})R_u(\frac{\pi}{p})$ and $R_v^{q-1}(\frac{2\pi}{q})$ have the same projection to PSU(1,1) and the same argument, therefore

$$R_u(\tfrac{\pi}{p})R_v(\tfrac{2\pi}{q})R_u(\tfrac{\pi}{p}) = R_v^{q-1}(\tfrac{2\pi}{q}).$$

It follows that

$$\delta(R_v(\tfrac{2\pi}{q})) = R_u(\tfrac{\pi}{p})R_v(\tfrac{2\pi}{q})R_u^{-1}(\tfrac{\pi}{p}) = R_v^{q-1}(\tfrac{2\pi}{q})R_u^{-1}(\tfrac{2\pi}{p}).$$

Furthermore conjugation with elements of the cyclic group G_u fixes G_u point-wise:

$$\delta\left(R_u(\tfrac{2\pi}{p})\right) = R_u(\tfrac{2\pi}{p}), \quad \delta(U) = U.$$

This shows that δ maps the set of generators

$$R_u(\tfrac{2\pi}{p})\cdot U^a, \quad R_v(\tfrac{2\pi}{q})\cdot U^b \quad \text{and} \quad U^k$$

to

$$R_u(\tfrac{2\pi}{p})\cdot U^a, \quad R_v^{q-1}(\tfrac{2\pi}{q})R_u^{-1}(\tfrac{2\pi}{p})\cdot U^b \quad \text{and} \quad U^k,$$

which is also a set of generators of Γ since

$$R_v^{q-1}(\tfrac{2\pi}{q})R_u^{-1}(\tfrac{2\pi}{p})\cdot U^b = \left(R_v(\tfrac{2\pi}{q})\cdot U^b\right)^{q-1}\left(R_u(\tfrac{2\pi}{p})\cdot U^a\right)^{-1}\left(U^k\right)^{\frac{a+2b-bq}{k}},$$

where $\frac{a+2b-bq}{k}$ is an integer since $pa+1 \equiv qb+1 \equiv 0 \mod k$ and $pqr-pq-qr-rp \equiv 0 \mod k$ imply $a+2b-bq \equiv 0 \mod k$. Therefore $\langle R_u(\frac{\pi}{p})\rangle \subset N(\Gamma)$.

Thus we proved that $N(\Gamma)_u = \langle R_u(\alpha)\rangle$ and therefore $\mathrm{Symm}_e = \langle\delta\rangle \times \langle\eta\rangle$. Finally Theorem 16 states that $\mathrm{Symm} = \Gamma \times \mathrm{Symm}_e$. □

Remark. The computation of $N(\Gamma)_u$ for a triangle group $\Gamma = \Gamma(p,q,r)^k$ in Theorem 20 can be considered in the context of the classification of maximal triangle groups. Under the projection $G \to \mathrm{PSU}(1,1)$ the group $\Gamma = \Gamma(p,q,r)^k$ will map to $\bar{\Gamma} = \Gamma(p,q,r)$ and the group $N(\Gamma)$ will map to a subgroup of $N(\bar{\Gamma})$. The normalizer of a non-abelian Fuchsian group is a Fuchsian group (see for example Theorem 2.3.8 in [9]). On the other hand, according to Singerman [20], any Fuchsian group that contains a triangle group is itself a triangle group, hence $\Gamma(p,q,r) \subset N(\Gamma(p,q,r))$ is an inclusion of triangle groups. and such inclusions have been classified in [20].

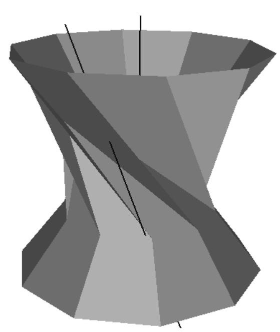

Figure 1: Fundamental domain for $\Gamma(5,3,3)^2$ with two exceptional fibres.

Figures 1 and 2 show examples of fundamental domains F_e for the triangle groups $\Gamma(5,3,3)^2$ and $\Gamma(7,3,3)^2$. We can describe the action of the symmetry group of the tiling on F_e. The conjugation δ by the element $R_u(\alpha)$ acts on F_e as a rotations

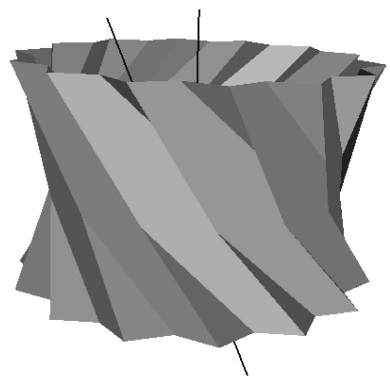

Figure 2: Fundamental domain for $\Gamma(7,3,3)^2$ with two exceptional fibres.

about the vertical line. The symmetry η acts on $\mathbb{D}$ as the reflection in the real axis and on F_e as a half-turn about an axis which is perpendicular to the vertical line and projects to the real axis in $\mathbb{D}$. In each figure for $\Gamma(p,q,r)^k$ we also include two lines that correspond to the fibres $\xi_u^{-1}(u)$ and $\xi_u^{-1}(v)$, where u and v are the vertices of the triangle with the angles $\frac{\pi}{p}$ and $\frac{\pi}{q}$ respectively. The fibres $\xi_u^{-1}(u)$ and $\xi_u^{-1}(v)$ are the vertical line and the non-vertical line respectively. The fibre $\xi_u^{-1}(u)$ coincides with the rotational symmetry axis of F_e. All other fibres that meet F_e and are fixed point-wise by some symmetry are images of these two under the symmetries of F_e.

7. Isotropy groups of Seifert fibres

Let $\Gamma = \Gamma(p,q,r)^k$, $q \neq r$, be a triangle group of level k that corresponds to a triangle with vertices $u = 0$, v and w with angles π/p, π/q and π/r as in section 6. In this section we study the connection between the symmetries of the tiling and the Seifert fibration of $\Gamma\backslash G$. We will work in the k-fold covering of $\mathrm{PSU}(1,1)$ given by $G_k = G/(k \cdot Z(G))$. The fibration $\xi_u : G_k \to \mathbb{D}$ is compatible with the action of Γ on G_k by left multiplication, hence it induces a Seifert fibration structure on the quotient $\Gamma\backslash G \cong \Gamma\backslash G_k$. We are interested in those fibres that are fixed point sets of symmetries of the tiling of G by $\mathcal{F}_g$ projected to G_k. Figures 1 and 2 show examples of such fibres that meet $\mathcal{F}_e$. We observe that if $a \in G$ is a fixed point of $\varphi \in \mathrm{Symm}$ then $x = \xi_u(a) \in \mathbb{D}$ is a fixed point of the action of φ on $\mathbb{D}$. We will now investigate when the fibre $\xi_u^{-1}(x)$ over an elliptic fixed point x of an element of Γ is a fixed point set of a symmetry of the tiling and if it is, what is the largest order of a symmetry that fixes this fibre point-wise.

Definition. For $x \in \mathbb{D}$ let

$$\mathrm{Symm}(x) = \{\varphi \in \mathrm{Symm} \mid \varphi(a) = a \quad \text{for all } a \in \xi_u^{-1}(x)\}$$

be the group of all symmetries of the tiling that fix the fibre $\xi_u^{-1}(x)$ point-wise.

Lemma 21. *Let $x \in \mathbb{D}$. An orientation-preserving isometry of G that fixes the fibre $\xi_u^{-1}(x)$, $x \in \mathbb{D}$, point-wise is of the form $g \mapsto R_x(\beta) \cdot g \cdot R_u(-\beta)$ for some $\beta \in \mathbb{R}$.*

Proof. First we consider the case $x = u$. If an orientation-preserving isometry fixes $\xi_u^{-1}(u) = \{R_u(t) \mid t \in \mathbb{R}\}$ point-wise then it fixes $e = R_u(0) \in \xi_u^{-1}(u)$ and hence belongs to $\mathrm{Isom}^+(G)_e = \{K_g \mid g \in G\} \times \langle\eta\rangle$. Note that $K_g(R_u(t)) = R_{g(u)}(t)$ and $\eta(R_u(t)) = R_u(-t)$, hence $K_g \circ \eta$ does not preserve the fibre $\xi_u^{-1}(u)$, while K_g preserves $\xi_u^{-1}(u)$ point-wise if and only if $g(u) = u$, i.e. $g = R_u(\beta)$ for some $\beta \in \mathbb{R}$. Thus any orientation-preserving isometry that fixes $\xi_u^{-1}(u)$ point-wise is of the form $K_{R_u(\beta)} = L_{R_u(\beta)} \circ R_{R_u(-\beta)}$ for some $\beta \in \mathbb{R}$.

For a general $x \in \mathbb{D}$ and an element $T \in \xi_u^{-1}(x)$, we have $\xi_u^{-1}(x) = T \cdot \xi_u^{-1}(u)$. If an orientation-preserving isometry φ preserves the fibre $\xi_u^{-1}(x)$ point-wise then $L_{T^{-1}} \circ \varphi \circ L_T$ preserves the fibre $\xi_u^{-1}(u)$ point-wise, hence φ is given by

$$\varphi = L_T \circ K_{R_u(\beta)} \circ L_{T^{-1}} = L_{T \cdot R_u(\beta) \cdot T^{-1}} \circ R_{R_u(-\beta)} = L_{R_x(\beta)} \circ R_{R_u(-\beta)}. \qquad \square$$

Proposition 22. *The group* $\mathrm{Symm}(u)$ *is a cyclic group of order* p *which is generated by the conjugation* $\delta = K_{R_u(\alpha)}$*, where* $\alpha = \frac{2\pi}{p}$*.*

Proof. Let $\varphi \in \mathrm{Symm}(u)$. Theorem 16 implies that $\varphi \in \mathrm{Isom}^+(G)$ and hence Lemma 21 implies that $\varphi = K_{R_u(\beta)}$ for some $\beta \in \mathbb{R}$. On the other hand the fibre $\xi_u^{-1}(u)$ contains e, hence $\varphi \in \mathrm{Symm}_e$. Theorem 20 implies that $\mathrm{Symm}_e = \langle\delta\rangle \times \langle\eta\rangle$. Thus $\mathrm{Symm}(u) = \langle\delta\rangle$. The order of δ is equal to p. $\square$

Proposition 23. *For the vertices* v *and* w *of the triangle, the groups* $\mathrm{Symm}(v)$ *and* $\mathrm{Symm}(w)$ *are cyclic groups*

$$\mathrm{Symm}(v) = \left\langle g \mapsto R_v\left(\tfrac{2\pi}{\gcd(p,q)}\right) \cdot g \cdot R_u\left(-\tfrac{2\pi}{\gcd(p,q)}\right)\right\rangle,$$
$$\mathrm{Symm}(w) = \left\langle g \mapsto R_w\left(\tfrac{2\pi}{\gcd(p,r)}\right) \cdot g \cdot R_u\left(-\tfrac{2\pi}{\gcd(p,r)}\right)\right\rangle$$

of orders $\gcd(p,q)$ *and* $\gcd(p,r)$ *respectively.*

Proof. Let $\varphi \in \mathrm{Symm}(v)$. Theorem 16 implies that $\varphi \in \mathrm{Isom}^+(G)$ and hence Lemma 21 implies that $\varphi = L_{R_v(\beta)} \circ R_{R_u(-\beta)}$ for some $\beta \in \mathbb{R}$. On the other hand Theorem 20 implies that $\mathrm{Symm} = \{L_a \mid a \in \Gamma\} \times \langle\delta\rangle \times \langle\eta\rangle$. Thus

$$\varphi = L_{R_v(\beta)} \circ R_{R_u(-\beta)} = L_a \circ \delta^m = L_{a \cdot R_u(m\alpha)} \circ R_{R_u(-m\alpha)}$$

for some $a \in \Gamma$ and $m \in \mathbb{Z}$, i.e.

$$R_u(\beta) = R_u(m\alpha) \quad \text{and} \quad R_v(\beta) \in \Gamma \cdot R_u(m\alpha).$$

Then $\beta = m\alpha = \frac{2\pi m}{p} \mod 2\pi k$. Recall that $\Gamma(p,q,r)^k$ is generated by the elements $R_u(\frac{2\pi}{p}) \cdot U^a$, $R_v(\frac{2\pi}{q}) \cdot U^b$ and U^k, where $U = R_u(2\pi) = R_v(2\pi)$, $ap+1 = bq+1 = 0 \mod k$ as in Proposition 18. Hence

$$R_u(m\alpha) = R_u\left(\tfrac{2\pi m}{p}\right) = \left(R_u\left(\tfrac{2\pi}{p}\right) \cdot U^a\right)^m \cdot U^{-am} \in \Gamma \cdot U^{-am}$$

and therefore

$$R_v(\beta) \in \Gamma \cdot R_u(m\alpha) = \Gamma \cdot U^{-ma} = \Gamma \cdot R_v^{-qam}\left(\tfrac{2\pi}{q}\right).$$

Thus $R_v(\beta) = R_v(m\alpha) = R_v\left(\frac{2\pi m}{p}\right)$ must be of the form $R_v^n\left(\frac{2\pi}{q}\right)$ for some $n \in \mathbb{Z}$, i.e. $\frac{m}{p} - \frac{n}{q}$ is an integer. If p and q are co-prime, this implies $m = 0 \mod p$ and hence $\beta = 2\pi m/p$ is an integer multiple of 2π, $R_v(\beta) = R_u(\beta) \in \langle U\rangle$ and $\varphi = \mathrm{Id}$, i.e. $\gcd(p,q) = 1$ implies $\mathrm{Symm}(v) = \{\mathrm{Id}\}$. More generally $\frac{m}{p} - \frac{n}{q} \in \mathbb{Z}$ implies that

m is a multiple of p/d, where $d = \gcd(p,q)$. Therefore $\beta = 2\pi m/p$ is an integer multiple of $2\pi/d$. On the other hand let $\beta = 2\pi/d$, $m = p/d$ and $n = q/d$ then

$$L_{R_v(\beta)} \circ R_{R_u(-\beta)} = L_{R_v(\beta)\cdot R_u(-\beta)} \circ K_{R_u(\beta)} = L_{R_v(\beta)\cdot R_u(-\beta)} \circ \delta^m.$$

The element

$$\begin{aligned} &R_v(\beta) \cdot R_u(-\beta) \\ &= R_v^n\left(\tfrac{2\pi}{q}\right) \cdot R_u^{-m}\left(\tfrac{2\pi}{p}\right) = \left(R_v\left(\tfrac{2\pi}{q}\right)\cdot U^b\right)^n \cdot \left(R_u\left(\tfrac{2\pi}{p}\right)\cdot U^a\right)^{-m} \cdot U^{ma-nb} \end{aligned}$$

is in Γ if $ma - nb = 0 \mod k$. We have

$$ma - nb = \frac{pa - qb}{d}.$$

We know that $pa = qb = -1 \mod k$ and $\gcd(d,k) = 1$, hence $ma - nb = 0 \mod k$. Therefore $L_{R_v(\beta)} \circ R_{R_u(-\beta)}$ is of the form $L_a \circ \delta^m$ with $a \in \Gamma$ and hence belongs to Symm. Thus Symm(v) is a cyclic group generated by $L_{R_v(\beta)} \circ R_{R_u(-\beta)}$, where $\beta = 2\pi/d$. The order of this group is $d = \gcd(p,q)$. The same reasoning applies to Symm(w). □

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