



Attacks to some verifiable multi-secret sharing schemes and two improved schemes



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ABSTRACT

Secret sharing plays an important role in protecting confidential information from being lost, destroyed, or falling into wrong hands. Verifiable multi-secret sharing enables a dealer to share multiple secrets among a group of participants such that the deceptive behaviors of the dealer and the participants can be detected. In this paper, we analyze the security of several recently proposed verifiable multi-secret sharing schemes. We show that these schemes cannot withstand some deceptive behaviors of the dealer, and hence fails to satisfy the basic requirement of secure verifiable secret sharing schemes. After that, we present two improved verifiable multi-secret sharing schemes. Our new schemes can not only resist cheating by the dealer or participants, but also remove the use of private channels.

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1. Introduction

Secret sharing plays an important role in protecting important information from getting lost, destroyed, or falling into wrong hands. It has many practical applications, such as safeguarding very confidential information, opening a bank vault, launching a missile, etc. In 1979, the first (t, n) threshold secret sharing schemes were proposed by Shamir [29] and Blakley [2] independently. In a (t, n) threshold secret sharing scheme, a secret can be shared among n participants such that t or more participants can reconstruct the secret, but $t - 1$ or fewer participants can not. In real applications, it is known that traditional secret sharing schemes like Shamir's and Blakley's cannot solve the following problems:

- (1) Only one secret can be shared during one secret sharing process, they cannot be used to share multiple secrets simultaneously.
- (2) The shadows of participants are not reusable. Once the secret has been reconstructed, all shadows will no longer be private.
- (3) Deceptive behaviors of a dishonest dealer cannot be detected. A dishonest dealer may distribute a fake shadow to a certain participant, and then that participant would subsequently never obtain the true secret.
- (4) Deceptive behaviors of a malicious participant cannot be prevented in the process of reconstruction. A malicious participant may provided a fake shadow to cheat the other participants to prevent them from reconstructing the true secret.
- (5) Private channels are required for the communications between the dealer and participants.
- (6) The dealer knows all shadows of participants. The shadows of participants are not reusable for different dealers.

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To solve some of the above problems, the concepts of multi-secret sharing (MSS) [15,16] and verifiable secret sharing (VSS) [6] have been introduced. A number of multi-secret sharing schemes and verifiable secret sharing schemes [5,7–9,23,33,35] have been presented. Using multi-secret sharing schemes [4,13,22], the drawback in (1) can be removed. To deal with the drawback in (2), Jackson et al. [20] have further introduced multi-use secret sharing schemes. The difference is that the shadow kept by each participant in a multi-use scheme is reusable after secret reconstruction.

To overcome the problems both in (1) and (2), He and Dawson [15] proposed a MSS scheme. One year later, they [16] put forward a dynamic MSS scheme based on two-variable one-way function. Two other practical MSS schemes were presented by Chien et al. [5] in 2000 and Yang et al. [33] in 2004 respectively. Pang and Wang [23] pointed out although the reconstruction in Yang's scheme is easier than in Chien's scheme, more public values are required when $p < t$. They also demonstrated an improved scheme based on interpolation method. However, Li et al. [21] pointed out Pang et al.'s scheme needs to generate a polynomial of degree $(n+p-1)$ in both the secret distribution and the secret reconstruction. And hence the efficiency is unfavorable when p is very large. They presented a new (t, n) threshold multi-secret sharing scheme using interpolation method. Some vulnerabilities of MSS schemes using polynomial interpolation were shown by Sahasranand et al. in [28]. They pointed out that a scheme based on interpolation method cannot be used to implement a (k, t, n) scheme when the k secrets to be distributed are inherently generated from a polynomial of degree less than $k - 1$, and the scheme does not work if all of the secrets to be shared are the same, etc. Although the drawbacks in (1) and (2) may be eliminated in these MSS schemes, the problems in (3)–(6) still exist.

To do away with the drawback in (3), Chor et al. [6] have initiated the study of verifiable secret sharing (VSS). In a VSS scheme, participants are able to verify that their shadows are consistent, and cheating by a dishonest dealer can be detected. VSS is now a fundamental tool in cryptographic research [14]. In order to resolve the problem in (4), cheating immune secret sharing [24–26,34] and publicly verifiable secret sharing (PVSS) [31] were investigated. The PVSS scheme presented by Stadler can detect not only cheating by the dealer but also the cheating by any participants.

Taking the problems in (1)–(4) into consideration, Harn [13] has introduced a (t, n) threshold verifiable multi-secret sharing (VMSS) scheme which can detect both malicious dealer and dishonest participants. In Harn's scheme, every participant keeps only one reusable shadow for sharing any set of secrets. However, Lin and Wu [22] pointed out that Harn's scheme suffered from the problems of large amount $(n! / ((n-t)! \cdot t!))$ of modular exponentiations and running interactive verification protocol to verify the validity of shadows. Chen et al. [4] presented an alternative (t, n) VSS scheme to avoid the disadvantages in Harn's scheme. However, Lin and Wu [22] showed that Chen et al.'s scheme is inefficient because the dealer has to record all participants' shadows and take $2n$ modular exponentiations to compute an n -dimensional verification vector for each shared secret. Lin and Wu put forward a (t, n) threshold VMSS scheme (LW scheme) based on the intractability of factorization and the hardness of the discrete logarithm problem modulo a composite [22]. In [17], He and Wu have indicated that LW scheme can't resist cheating by participants, because a malicious participant can provide a fake subshadow to cheat other honest participants. An improvement of the LW scheme was given by Chang et al. [3]. The improved scheme not only successfully overcomes the drawbacks of LW scheme, but also is computationally more efficient than the other VMSS schemes. Unfortunately, Huang et al. [18] identified that Chang's VMSS scheme could not withstand conspiracy attack. They showed that any $t + 1$ participants can conspire to compute the system's secret \mathbf{R} or $\phi(N)$ with high probability. Subsequently, these malicious participants could reconstruct the shared secret independently.

In 2004, Yang et al. [33] proposed a relatively efficient multi-secret sharing scheme (YCH scheme). But Shao and Cao pointed out that this scheme does not enjoy the property of verifiability, and presented a modified scheme (SC scheme) [30] by adding the property of verifiability based on Feldman's [12] VSS scheme. Note that, in the SC scheme, all shadows are computed by the dealer and private channels are required for the dealer to distribute shadows to participants. So the problems in (5) and (6) remain unsolved. In 2006, Zhao et al. [35] introduced a practical verifiable multi-secret sharing scheme (ZZZ scheme) based on YCH scheme and Hwang–Chang's scheme (HC scheme) [19]. The verification phase of the ZZZ scheme is the same as that of the HC scheme. By making use of the techniques of public key cryptography [32], e.g. RSA cryptosystem [27] and Diffie–Hellman key agreement [10] method, the ZZZ scheme and HC scheme realized secret sharing without private channels. This property is particularly significant in applications where private channels are hard to set up. In these schemes, each participant chooses his secret shadow by himself. Hence the problems in (5) and (6) could be solved simultaneously. Similar to ZZZ scheme, the VMS schemes presented in [8,9] were also dealt with the problems in (5) and (6). For simplicity, we call the scheme in [8] MS scheme, the type 1 scheme in [9] the MS1 scheme, the type 2 scheme in [9] the MS2 scheme.

Although it was claimed that these schemes (ZZZ scheme, MS scheme, MS1 scheme, MS2 scheme) could identify cheating by both the dealer and the participants, unfortunately, we find that their claims are wrong.

In this paper, we analyze the security drawbacks of these verifiable multi-secret sharing schemes, including the ZZZ scheme, MS scheme, MS1 scheme, MS2 scheme. We demonstrate how a dishonest dealer can cheat a participant without being detected in all these schemes. So these schemes cannot withstand cheating by dishonest dealer.

In addition, taking into account all the problems (1) to (6), we also propose two new verifiable multi-secret sharing schemes. Our new schemes have the following features:

- (1) The dealer can arbitrarily give any set of secrets for sharing, and only one shadow, which is reusable, should be kept by each participant. This solves the problems in (1) and (2).
- (2) Every participant can detect any cheating by the dealer. This solves the problem in (3).
- (3) Every participant can detect the cheating by any other participants by using a non-interactive protocol. This solves the problem in (4).

- (4) The dealer and the participants communicate through public channel. This solves the problem in (5).
 (5) The dealer does not know each participant's shadow. The shadows of participants can be reusable for the different round of sharing. This solves the problem in (6).

With these features, our new schemes can be applied in many practical situations, such as authenticating an electronic funds transfer.

The rest of the paper is organized as follows. In [Section 2](#) we revisit the concepts of RSA encryption scheme and homogeneous linear recursion (HLR) which will be building blocks in constructing our new VMSS schemes. Then we describe the security model for VMSS schemes. The security analysis and attacks on several verifiable multi-secret sharing schemes are shown in [Section 3](#). Our new verifiable multi-secrets sharing scheme without a private channel is depicted in [Section 4](#) followed by security analysis. In [Section 5](#) we give performance analysis of our new VMSS schemes. Finally, [Section 6](#) concludes our paper.

2. Preliminaries

2.1. RSA encryption scheme

RSA is one of the best known public-key encryption schemes named after its inventors Rivest, Shamir and Adleman [27]. It is the first practical realization of public-key encryption scheme based on the notion of one-way trapdoor function introduced by Diffie and Hellman [10,11]. The security of RSA is based on the hardness of large integer factorization.

The RSA encryption scheme is specified as follows:

Key setup: A user Alice performs the following steps to generate her public and private keys.

1. choose two random prime numbers p and q such that $|p| \approx |q|$;
2. compute $N = pq$;
3. compute $\phi(N) = (p - 1)(q - 1)$;
4. choose a random integer $e < \phi(N)$ such that $\gcd(e, \phi(N)) = 1$, and compute the integer d such that

$$ed \equiv 1 \pmod{\phi(N)};$$
5. publish (N, e) as her public key, safely destroy p, q and $\phi(N)$, and keep d as her private key.

Encryption: To send a confidential message $m \in Z_N$ to Alice, the sender Bob creates the ciphertext c as follows

$$c \leftarrow m^e \pmod{N}.$$

Decryption: To decrypt a ciphertext c , Alice computes

$$m \leftarrow c^d \pmod{N}.$$

2.2. Homogeneous linear recursion (HLR)

In this section we briefly introduce homogeneous linear recursion which forms the mathematical background of our second scheme. A detailed description of homogeneous linear recursion can be found in [1].

Definition 1. Let t be a positive integer and $c_1, c_2, \dots, c_t, a_1, a_2, \dots, a_t$ be real numbers. A homogeneous linear recursion of degree t is defined by the equations

$$[HLR] \begin{cases} u_0 = c_1, u_1 = c_2, \dots, u_{t-1} = c_t, \\ u_{i+t} + a_1 u_{i+t-1} + \dots + a_t u_i = 0 \quad (i \geq 0) \end{cases}$$

where c_1, c_2, \dots, c_t and a_1, a_2, \dots, a_t are constants.

Definition 2. We define the auxiliary equation for [HLR] to be

$$x^t + a_1 x^{t-1} + \dots + a_t = 0.$$

We shall assume that the auxiliary equation has t roots, which will certainly be the case if we work in the field \mathbb{C} of complex numbers. However, the t roots do not need to be distinct, and we shall suppose that the distinct values are $\alpha_1, \alpha_2, \dots, \alpha_l$, occurring with multiplicities m_1, m_2, \dots, m_l , respectively. In other words, the auxiliary equation can be rewritten as

$$(x - \alpha_1)^{m_1} (x - \alpha_2)^{m_2} \dots (x - \alpha_l)^{m_l} = 0,$$

where $m_1 + m_2 + \dots + m_l = t$.

Lemma 1. Suppose sequence $\{u_i\}$ is defined by [HLR], and the auxiliary equation has roots $\alpha_1, \alpha_2, \dots, \alpha_l$ with multiplicities m_1, m_2, \dots, m_l . Then

$$u_i = p_1(i) \alpha_1^i + p_2(i) \alpha_2^i + \dots + p_l(i) \alpha_l^i,$$

where, for $j = 1, 2, \dots, l$, $p_j(i)$ is an expression of the form $A_0 + A_1 i + \dots + A_{(m_j-1)} i^{(m_j-1)}$. In other words, $p_j(i)$ is a polynomial function of i with degree at most $m_j - 1$.

2.3. Security model of VMSS schemes

A (t, n) VMSS scheme without private channels involves a dealer D and a set $M = \{M_1, M_2, \dots, M_n\}$ of n participants. It is composed of the following 4 phases.

1. *Initialization*: The dealer sets up the system parameters which may include its private information and some public data authentically available to all participants. All participants choose their secret shadows and compute the corresponding public information respectively. Then each participant authenticates its identity and public information to the dealer. After that the dealer publishes the public parameters, the identities and public information of all participants.
2. *Construction*: For a set of secrets to be shared, the dealer D computes subshadows and corresponding verification information for all participants. D authentically publishes all verification information.
3. *Verification*: Each participant gets its subshadow from verification information. Then it checks the validity of its subshadow using the public information.
4. *Recovery*: t or more participants cooperate to recover the shared secrets. Each of them supplies its subshadow to the others. They verify the validity of each subshadow. When at least t valid subshadows are collected, they can compute all shared secrets using a predetermined algorithm.

With regard to the security, a (t, n) VMSS scheme without private channels must satisfy the following requirements.

1. *Correctness*: If the dealer and the participants act honestly, any t or more participants can reconstruct the secret correctly during the execution of the reconstruction algorithm.
2. *Verifiability*:
 - Any deceptive behavior of the dealer can be identified in the verification phase.
 - In the recovery phase, a dishonest participant who supplies a fake subshadow can be identified by the others.
3. *Privacy*: Any collusion of less than t participants cannot obtain any of the shared secrets.

According to these security requirements, in the security model of VMSS schemes, we characterize three types of adversaries. One is a dishonest dealer who aims to cheat some participants by distributing to them invalid subshadows. We say a dishonest dealer succeeds if it distributes an invalid subshadow to a participant without being detected with a non-negligible probability. The second is a cheating participant who aims to submit a fake subshadow without being detected in the recovery phase. Such an adversary succeeds if it submits a fake subshadow without being detected with a non-negligible probability. Another is an adversary who is not the dealer but corrupts up to $t - 1$ participants. An adversary of this kind gets complete control of up to $t - 1$ corrupted participants and aims to extract some information of the shared secrets. We say such an adversary succeeds if it can get some information of the shared secrets other than those can be induced from public information and the information owned by the corrupted participants with a non-negligible probability.

Definition 3. A (t, n) VMSS scheme without private channels is said secure if it satisfies *Correctness* and no adversary can succeed with a non-negligible probability.

3. Security analysis and attacks to several verifiable multi-secret sharing schemes

3.1. Security analysis and attack on ZZZ scheme [35]

3.1.1. Brief review of ZZZ scheme

• Initialization phase

Let P_1, P_2, \dots, P_k denote k secrets to be shared. Firstly, the dealer D chooses two large strong primes, p and q , computes $N = pq$. D randomly chooses an integer g from the interval $[N^{1/2}, N]$ such that g is relatively prime to p and q . D publishes $\{g, N\}$.

Let $M = \{M_1, M_2, \dots, M_n\}$ be the set of participants. Each participant M_i in M randomly chooses an integer s_i from the interval $[2, N]$ as her/his own secret shadow and computes $R_i = g^{s_i} \text{ mod } N$, then M_i provides R_i and her/his identity information ID_i , to the dealer D . D must ensure that $R_i \neq R_j$ for all $i \neq j$. Once $R_i = R_j$, D should demand these participants to choose different secret shadows until R_i 's are different for $i = 1, 2, \dots, n$. D publishes $\{(ID_i, R_i)\}$.

• Construction phase

The dealer D performs the following steps

- (1) Randomly choose an integer s_0 from the interval $[2, N]$ such that s_0 is relatively prime to $(p - 1)$ and $(q - 1)$. Then D computes f such that $s_0 \times f = 1 \text{ mod } \phi(N)$, where $\phi(N)$ is the Euler phi-function;
- (2) Compute $R_0 = g^{s_0} \text{ mod } N$ and $I_i = R_i^{s_0} \text{ mod } N$, ($i = 1, 2, \dots, n$);
- (3) Publish $\{R_0, f\}$.

In case $k \leq t$,

- Choose a prime Q and construct $(t - 1)$ th degree polynomial $h(x) \text{ mod } Q$,

$$h(x) = P_1 + P_2x + \dots + P_kx^{k-1} + a_1x^k + a_2x^{k+1} + \dots + a_{t-k}x^{t-1} \text{ mod } Q,$$

where $0 < P_1, P_2, \dots, P_k, a_1, a_2, \dots, a_{t-k} < Q$;

- Compute $y_i = h(I_i) \text{ mod } Q$ for $i = 1, 2, \dots, n$;

– Publish (y_1, y_2, \dots, y_n) .

In case $k > t$,

– Choose a prime Q and construct $(k - 1)$ th degree polynomial $h(x) \bmod Q$,

$$h(x) = P_1 + P_2x + \dots + P_kx^{k-1} \bmod Q,$$

where $0 < N, P_1, P_2, \dots, P_k < Q$;

– Compute $y_i = h(I_i) \bmod Q$ for $i = 1, 2, \dots, n$;

– Compute $h(i) \bmod Q$ for $i = 1, 2, \dots, k - t$;

– Publish $(y_1, y_2, \dots, y_n, h(1), h(2), \dots, h(k - t))$.

• Recovery and verification phase

Without loss of generality, suppose t or more members M_1, M_2, \dots, M_t of M collaboratively recover the secrets P_1, P_2, \dots, P_k , they execute the following steps:

(1) M_i supplies $I'_i = R_0^{s_i} \bmod N$, where s_i is the shadow of M_i ;

(2) Anybody can verify the validity of I'_i provided by M_i : if $I_i'^f = R_i \bmod N$, then I'_i is true; otherwise I'_i is false and M_i may be a cheater;

(3) Recover the secrets: The polynomial $h(x) \bmod Q$ can be uniquely determined as follows:

$k \leq t$

$$\begin{aligned} h(x) &= \sum_{i=1}^t y_i \prod_{j=1, j \neq i}^t \frac{x - I'_j}{I'_i - I'_j} \bmod Q \\ &= P_1 + P_2x + \dots + P_kx^{k-1} + a_1x^k + a_2x^{k+1} + \dots + a_{t-k}x^{t-1} \bmod Q. \end{aligned}$$

$k > t$

$$\begin{aligned} h(x) &= \sum_{i=1}^t y_i \prod_{j=1, j \neq i}^t \frac{x - I'_j}{I'_i - I'_j} \prod_{l=1}^{k-t} \frac{x - I_l}{I'_i - I_l} + \sum_{i=1}^{k-t} h(i) \prod_{j=1, j \neq i}^{k-t} \frac{x - j}{i - j} \prod_{l=1}^t \frac{x - I_l}{i - I_l} \bmod Q \\ &= P_1 + P_2x + \dots + P_kx^{k-1} \bmod Q. \end{aligned}$$

3.1.2. Analysis and attack to ZZZ scheme

We notice that, when collaboratively reconstruct the shared secrets, only the validity of I_i provided by each M_i is checked using equation $I_i^f = R_i \bmod N$. But the consistence of I_i with y_i is not verified. Based on this observation, a dishonest dealer D can cheat M_i by using an invalid y_i which is inconsistent with $I_i = R_0^{s_i} \bmod N$. The attack comes as follows:

In the **Construction phase**, the dishonest dealer D performs:

In case $k \leq t$

• D chooses a prime Q and construct $(t - 1)$ th degree polynomial $h(x) \bmod Q$,

$$h(x) = P_1 + P_2x + \dots + P_kx^{k-1} + a_1x^k + a_2x^{k+1} + \dots + a_{t-k}x^{t-1} \bmod Q$$

where $0 < P_1, P_2, \dots, P_k, a_1, a_2, \dots, a_{t-k} < Q$;

• Assume D wants to cheat participant M_i , he/she first randomly chooses $J_i \neq I_i$ and computes $y'_i = h(J_i) \bmod Q$, correctly computes the other y_j for $j = 1, 2, \dots, n, j \neq i$;

• D publishes $(y_1, y_2, \dots, y_{i-1}, y'_i, y_{i+1}, \dots, y_n)$.

In case $k > t$

• D chooses a prime Q and construct $(t - 1)$ th degree polynomial $h(x) \bmod Q$,

$$h(x) = P_1 + P_2x + \dots + P_kx^{k-1} + a_1x^k + a_2x^{k+1} + \dots + a_{t-k}x^{t-1} \bmod Q,$$

where $0 < P_1, P_2, \dots, P_k, a_1, a_2, \dots, a_{t-k} < Q$;

• Assume D wants to cheat participant M_i , he/she first randomly chooses $J_i (J_i \neq I_i)$ and computes $y'_i = h(J_i) \bmod Q$, correctly computes the other y_j for $j = 1, 2, \dots, n, j \neq i$;

• D publishes $(y_1, y_2, \dots, y'_i, \dots, y_n, h(1), h(2), \dots, h(k - t))$.

In the **Recovery and verification phase**, the honest M_i will supply $I'_i = R_0^{s_i} \bmod N$ which is really valid, since it does hold that $I_i'^f = R_i \bmod N$. As any participant can by no means find out y'_i is inconsistent with $I'_i = R_0^{s_i} \bmod N$, i.e. $y'_i \neq h(I_i)$, in the reconstruction phase, M_i and his collaborators will use the fake y'_i . As a result they will not recover the true secrets. In addition, when this happens, it is hard for the collaborators to identify which y_i is fake.

[Note]: Obviously, such attacks can also be extended to MS scheme [8].

3.2. Security analysis and attacks to MS1 and MS2 schemes [9]

3.2.1. Analysis and attack to MS1 scheme

For space limitation, we omit the description of the original MS1 scheme. Please refer to [9] for details. The security drawbacks of the MS1 scheme is similar to that of the ZZZ scheme. We notice that, when collaboratively reconstruct the shared secrets, only the validity of $l_i = f(r, s_i)$ provided by each M_i is checked using equation $g^l_i = G_i \pmod p$. While the consistence of l_i with the [*MS1] homogeneous linear recursive formula is not verified. Based on this observation, in the generation of sequence $\{u_i\}$, a dishonest dealer D can cheat M_i by replacing the valid $l_i = f(r, s_i)$ with an invalid $l'_i = f(r, s'_i)$, where $s'_i \neq s_i$. And any participants are not able to detect. The attack comes as follows:

D randomly chooses $J_i (J_i \neq s_i)$ and computes:

$$\begin{cases} l'_i = f(r, J_i); \\ l_i = f(r, s_i); \\ G_i = g^{l_i} \pmod p. \end{cases}$$

If $1 \leq i \leq t$

- D replaces the l_i with the l'_i to calculate the following [HLR] equations:

$$[*] \begin{cases} u_0 = I_1, u_1 = I_2, \dots, u'_{i-1} = l'_i, \dots, u_{t-1} = I_t, \\ u_{j+t} + a_1 u_{j+t-1} + \dots + a_t u_j = 0 \pmod q \quad (j \geq 0) \end{cases}$$

- D computes u_j for $t \leq j \leq n+k$;
- D computes $y_j = I_j - u_{j-1}$ for $t < j \leq n$ and $r_j = P_j - u_{j+n}$ for $1 \leq j \leq k$;
- D publishes $(r, G_1, G_2, \dots, G_i, \dots, G_n, r_1, r_2, \dots, r_k, y_{t+1}, y_{t+2}, \dots, y_n)$.

If $t < i \leq n$

- D considers [HLR](Homogeneous linear recursion) which is defined by the equations

$$[*] \begin{cases} u_0 = I_1, u_1 = I_2, \dots, u_{t-1} = I_t, \\ u_{j+t} + a_1 u_{j+t-1} + \dots + a_t u_j = 0 \pmod q \quad (j \geq 0) \end{cases}$$

- D computes $u_j, t \leq j \leq n+k$;
- D replaces the l_i with the l'_i to compute $y'_i = l'_i - u_{i-1}$, correctly computes the other y_j for $t < j \leq n, j \neq i$ and $r_j = P_j - u_{j+n}$ for $1 \leq j \leq k$.
- D publishes $(r, G_1, G_2, \dots, G_i, \dots, G_n, r_1, r_2, \dots, r_k, y_{t+1}, y_{t+2}, \dots, y'_i, \dots, y_n)$.

Since M_i cannot find out $l_i = f(r, s_i)$ is replaced with an invalid $l'_i = f(r, s'_i)$ by D , in the reconstruction phase, he and his collaborators will use the subshadow $l_i = f(r, s_i)$ to recover the secrets. As a result they will fail to recover the true secrets. (Note that any t or more honest participants without M_i can recover the true secrets.) In addition, when this happens, it is hard for the collaborators to identify which $l_i = f(r, s_i)$ is replaced by D . This fact indicates that the MS1 scheme cannot withstand cheating by the dealer.

3.2.2. Analysis and attack to MS2 scheme

For the complete description of the MS2 scheme, please refer to [9]. We notice that, when collaboratively reconstruct the shared secrets, only the validity of $l_i = R_i^{s_i} \pmod N$ provided by each M_i is checked using equation $(l_i)^f = R_i \pmod N$, while the consistence of l_i with the sequence $\{u_i\}$ generated from Homogeneous linear recursive formula [*MS2] is not verified. Based on this observation, a dishonest dealer D can cheat M_i by replacing the valid $l_i = R_i^{s_i}$ with an invalid $l'_i = (J_i)^{s_i}$ ($J_i \neq R_i$) in generating the sequence $\{u_i\}$ or $\{y_i\}$. The attack comes as follows:

D randomly chooses $J_i (J_i \neq R_i)$ and computes:

$$\begin{cases} l'_i = (J_i)^{s_i} \pmod N; \\ l_i = (R_i)^{s_i} \pmod N. \end{cases}$$

If $1 \leq i \leq t$

- D replaces the l_i with the l'_i to compute the following [HLR] equations:

$$[*] \begin{cases} u_0 = I_1, u_1 = I_2, \dots, u'_{i-1} = l'_i, \dots, u_{t-1} = I_t, \\ u_{i+t} + a_1 u_{i+t-1} + \dots + a_t u_i = 0 \pmod q \quad (i \geq 0) \end{cases}$$

- D computes u_i for $t \leq i \leq n+k$;
- D computes $y_i = l_i - u_{i-1}$ for $t < i \leq n$ and $r_i = P_i - u_{i+n}$ for $1 \leq i \leq k$;
- D publishes $(R_0, f, r_1, r_2, \dots, r_k, y_{t+1}, y_{t+2}, \dots, y_n)$.

If $t < i \leq n$

- D considers the sequence $\{u_i\}$ which is defined by the formulas:

$$[*] \begin{cases} u_0 = I_1, u_1 = I_2, \dots, u_{t-1} = I_t, \\ u_{i+t} + a_1 u_{i+t-1} + \dots + a_t u_i = 0 \pmod q \quad (i \geq 0) \end{cases}$$

- D computes u_i for $t \leq i \leq n+k$;
- D replaces the I_i with the I'_i to compute $y'_i = I'_i - u_{i-1}$, correctly computes the other y_j for $t < j \leq n, j \neq i$ and $r_i = P_i - u_{i+n}$ for $1 \leq i \leq k$;
- D publishes $(R_0, f, r_1, r_2, \dots, r_k, \mathcal{Y}_{t+1}, \mathcal{Y}_{t+2}, \dots, \mathcal{Y}'_i, \dots, \mathcal{Y}_n)$.

Since M_i cannot find out $I_i = R_i^{s_0}$ is replaced by D in the generation of sequence $\{u_i\}$ or $\{y_i\}$, in recovery phase, he and his collaborators will use the true I_i that is inconsistent with sequence $\{u_i\}$ or $\{y_i\}$. As a result they will not recover the true secrets. (But any t or more honest participants without M_i can recover the true secrets.) In addition, when this kind of cheating occurs, it is hard for the collaborators to identify which $I_i = R_i^{s_0}$ is replaced by D . This fact indicates that the MS2 scheme also cannot withstand cheating by the dealer.

4. New VMSS schemes

To overcome the security drawbacks of the above analyzed VMSS schemes, we propose two new VMSS schemes in this section. By adding some consistence checking measures, our new schemes effectively get rid of the security flaws in ZZZ scheme, MS scheme, MS1 scheme, and MS2 scheme. Our first scheme is based on the ZZZ scheme introduced in [35] and Feldman's VSS scheme [12], and the second is based on the MS2 scheme [9]. The two new schemes can not only resist cheating by the dealer or participants, but also remove the use of private channels.

4.1. Description of our scheme 1

Let D be the dealer, $M = \{M_1, M_2, \dots, M_n\}$ be the set of participants, $t (< n)$ be the threshold.

• Initialization phase

In this phase, the dealer (denoted as D) first creates a public notice board (NB) which is used for storing necessary public information. The participants can access the information on NB. But the contents on the board can only be modified or updated by D . Let λ be the security parameter. For our context, it should be chosen as the security parameter for a secure RSA cryptosystem.

Initialization of the dealer D , D performs the following:

- (1) Choose two large strong primes p_0 and q_0 ($p_0 > q_0$) with bit-length $\lambda/2$ satisfying the requirement of a secure RSA public key cryptosystem [32], and compute $N = p_0 q_0$ of bit-length λ .
- (2) Compute $\phi(N) = (p_0 - 1)(q_0 - 1)$ which is Euler's function, then safely destroy p_0, q_0 .
- (3) Choose primes q, Q such that $Q|(q - 1)$, and the bit-length of Q is at least $\lambda/2$. Then randomly choose an element g of Z_q^* with order Q .
- (4) D publishes (λ, N, Q, q, g) on NB.

Initialization of participants:

- (1) Each M_i with identity information ID_i chooses two strong primes p_i and q_i ($p_i > q_i$) of bit-length $\lambda/2$, and computes $N_i = p_i q_i$ which satisfy $(N_i > N)$.
- (2) M_i computes $\phi(N_i) = (p_i - 1)(q_i - 1)$.
- (3) M_i randomly chooses an integer e_i which is coprime to $\phi(N_i)$ and computes the integer d_i such that $e_i d_i = 1 \pmod{\phi(N_i)}$.
- (3) M_i provides authentically (ID_i, e_i, N_i) to D through a public channel, and keeps his shadow d_i secret.

D puts $(ID_i, e_i, N_i), i = 1, 2, \dots, n$, on NB.

[Note]: After the initialization phase, the information on NB can be reusable. The dealer D does not know any participant's shadow, so the shadow can be reusable for multiple rounds of sharing even with different dealers.

• Construction phase

Let P_1, P_2, \dots, P_k be the k secrets to be shared, $0 < P_i < Q, i = 1, 2, \dots, k$. The dealer D performs the following steps:

- (1) In case $k \leq t$
 - (1) D constructs a polynomial $f(x) \pmod Q$ of degree $(t - 1)$:

$$f(x) = P_1 + P_2 x + \dots + P_k x^{k-1} + a_1 x^k + a_2 x^{k+1} + \dots + a_{t-k} x^{t-1} \pmod Q.$$

- (2) D generates subshadow Y_i for participant M_i : D randomly chooses n different integers C_1, C_2, \dots, C_n such that $0 < C_i < Q, i = 1, 2, \dots, n$ and computes $Y_i = f(C_i) \pmod Q$ for $i = 1, 2, \dots, n$.
- (3) D computes H_i for $i = 1, 2, \dots, n$,

$$H_i = Y_i^{e_i} \pmod{N_i}.$$

(4) D computes A_i for $i = 1, 2, \dots, t$,

$$A_i = g^{\beta_i} \bmod q \quad (1 \leq i \leq k),$$

$$A_i = g^{\alpha_i - k} \bmod q \quad (k < i \leq t).$$

(5) D publishes $(C_1, C_2, \dots, C_n, H_1, H_2, \dots, H_n, A_1, A_2, \dots, A_t)$ on NB.

(2) In case $k > t$

(1) D constructs a polynomial $f(x) \bmod Q$ of degree $(k - 1)$:

$$f(x) = P_1 + P_2x + \dots + P_kx^{k-1} \bmod Q.$$

(2) D generates subshadow Y_i for participant M_i , D randomly chooses distinct integers C_1, C_2, \dots, C_n such that $0 < C_i < Q, i = 1, 2, \dots, n$, and computes $Y_i = f(C_i) \bmod Q$ for $i = 1, 2, \dots, n$.

(3) D picks $k - t$ distinct minimum integers $\eta_1, \eta_2, \dots, \eta_{k-t}$ from $\mathbb{Z}_Q^* - \{C_i \mid i = 1, 2, \dots, n\}$, computes $f(\eta_i) \bmod Q$ for $i = 1, 2, \dots, k - t$.

(4) D computes H_i for $i = 1, 2, \dots, n$:

$$H_i = Y_i^{e_i} \bmod N_i.$$

(5) D computes $A_i = g^{\beta_i} \bmod q, i = 1, 2, \dots, k$.

(6) D puts $(C_1, C_2, \dots, C_n, H_1, H_2, \dots, H_n, \eta_1, \eta_2, \dots, \eta_{k-t}, f(\eta_1), f(\eta_2), \dots, f(\eta_{k-t}), A_1, A_2, \dots, A_k)$ on NB.

• **Verification phase**

Each participant M_i gets his subshadow by computing $Y_i = (H_i)^{d_i} \bmod N_i$ for $i = 1, 2, \dots, n$. The validity of subshadows and their consistency with the information published by D on NB can be verified by each participant M_i as follows.

(1) If $k \leq t, M_i$ checks

$$g^{Y_i} \stackrel{?}{=} \prod_{l=0}^{t-1} (A_{l+1})^{(C_i)^l} \bmod q$$

(2) If $k > t, M_i$ checks

$$g^{Y_i} \stackrel{?}{=} \prod_{l=0}^{k-1} (A_{l+1})^{(C_i)^l} \bmod q$$

$$g^{f(\eta_j)} \stackrel{?}{=} \prod_{l=0}^{k-1} (A_{l+1})^{(\eta_j)^l} \bmod q, \quad j = 1, 2, \dots, k - t$$

If M_i 's verification is successful, M_i believes the subshadow Y_i he has got is valid and is consistent with the public information on NB. If no participant fails in the verification, D is thought honest.

Note that we add the consistency detection of all subshadows in the verification phase. This is necessary for preventing the dealer from distributing a fake subshadow to a participant. The weakness of the schemes attacked in the previous section is mainly due to lack of such a detection.

• **Recovery phase**

Suppose t arbitrary participants $\{M_i\}_{i \in I} (I \subseteq \{1, 2, \dots, n\})$ pool their subshadows Y_i to reconstruct the shared secrets. Each participant $M_i (i \in I)$ can check whether others' secret subshadows are valid and consistent by the following equations:

$$g^{Y_i} \stackrel{?}{=} \prod_{l=0}^{k-1} (A_{l+1})^{(C_i)^l} \bmod q, \quad j \in I.$$

Once t valid subshadows are collected, the polynomial $f(x) \bmod Q$ can be uniquely determined as follows:

(1) If $k \leq t$

$$\begin{aligned} f(x) &= \left(\sum_{i=1}^t Y_i \prod_{j=1, j \neq i}^t \frac{x - C_j}{C_i - C_j} \right) \bmod Q \\ &= P_1 + P_2x + \dots + P_kx^{k-1} + a_1x^k + a_2x^{k+1} + \dots + a_{t-k}x^{t-1} \bmod Q \end{aligned}$$

(2) If $k > t$

$$\begin{aligned} f(x) &= \left(\sum_{i=1}^t Y_i \prod_{j=1, j \neq i}^t \frac{x - C_j}{C_i - C_j} \prod_{l=1}^{k-t} \frac{x - \eta_l}{C_i - \eta_l} + \sum_{i=1}^{k-t} f(\eta_i) \prod_{j=1, j \neq i}^{k-t} \frac{x - \eta_j}{\eta_i - \eta_j} \prod_{l=1}^t \frac{x - C_l}{\eta_i - C_l} \right) \bmod Q \\ &= P_1 + P_2x + \dots + P_kx^{k-1} \bmod Q \end{aligned}$$

The structure of our scheme 1 is similar to that of the ZZZ scheme. The main difference lies in we use the RSA encryption system while the ZZZ scheme uses Diffie–Hellman key exchange. A comparison of the two scheme is shown in Fig. 1. In the ZZZ

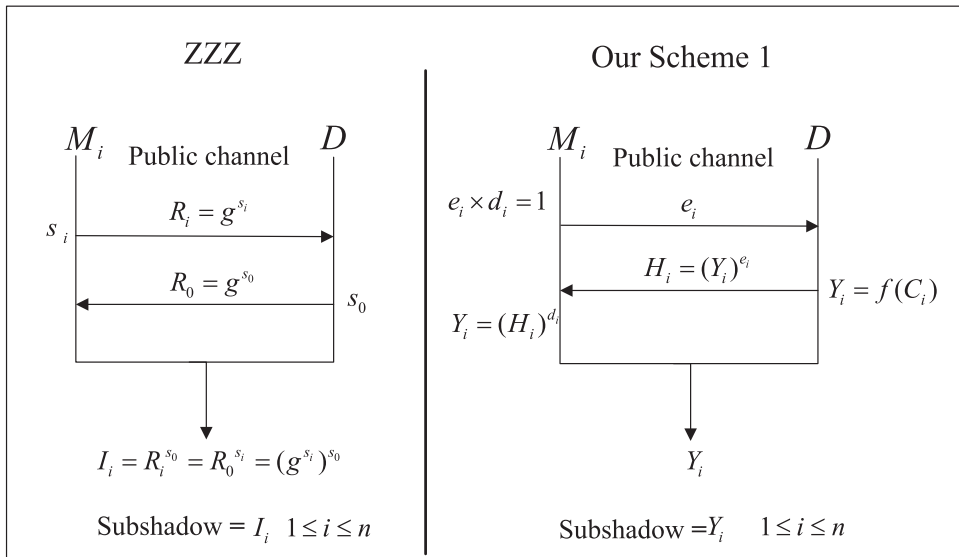


Fig. 1. The difference between our scheme 1 and ZZZ scheme.

scheme, participant M_i chooses its secret shadow s_i and computes its subshadow as $I_i = R_0^{s_i} \bmod N$ using the public information $R_0 = g^{s_0} \bmod N$ published by the dealer D . M_i also sends $R_i = g^{s_i} \bmod N$ to D as its public information corresponding to its secret shadow s_i . D computes the subshadow of M_i as $I_i = R_i^{s_0} \bmod N$ which is used in the computation of public information $y_i = h(I_i)$. While in our new scheme, participant M_i generates a RSA public and private key pair $((e_i, N_i), d_i)$. It sets the private key d_i as its secret shadow and computes its subshadow as $Y_i = H_i^{d_i} \bmod N_i$ using the public information $H_i = Y_i^{e_i} \bmod N_i$ published by the dealer D . Where D computes the subshadow Y_i of M_i as $Y_i = f(C_i) \bmod Q$ from a publicly known $C_i \in Z_Q$. D also publishes commitments to the coefficients of the polynomial $f(x)$. To compare the two scheme, please note in the ZZZ scheme, three modular exponentiation $I_i = R_0^{s_i} \bmod N$, $R_i = g^{s_i} \bmod N$, and $I_i = R_i^{s_0} \bmod N$ are needed for M_i and D to compute the same subshadow I_i for participant M_i . Whereas in our new scheme, only two modular exponentiation $Y_i = H_i^{d_i} \bmod N_i$ and $H_i = Y_i^{e_i} \bmod N_i$ are needed for this purpose. More importantly, in the ZZZ scheme the dealer D does not commit to the public information y_i . This weakness makes it possible for the dealer D to cheat a participant M_i by publishing a fake y_i as pointed in our attack. While in our new scheme, this weakness is eliminated by requiring the dealer publishing commitment to the polynomial $f(x)$.

4.2. Security analysis

The security of our scheme 1 is based on the discrete logarithm problem and the large integer factorization problem which are assumed to be hard. We analyze the security of our scheme 1 from three aspects as formulated in the security model.

1. **Correctness:** If the dealer and the participants are honest, any t or more participants can correctly reconstruct the set of secrets using the recovery algorithm. This can be proved by the following equations.

• If $k \leq t$

$$f(x) = \left(\sum_{i=1}^t Y_i \prod_{j=1, j \neq i}^t \frac{x - C_j}{C_i - C_j} \right) \bmod Q$$

$$= P_1 + P_2x + \dots + P_kx^{k-1} + a_1x^k + a_2x^{k+1} + \dots + a_{t-k}x^{t-1} \bmod Q$$

• If $k > t$

$$f(x) = \left(\sum_{i=1}^t Y_i \prod_{j=1, j \neq i}^t \frac{x - C_j}{C_i - C_j} \prod_{l=1}^{k-t} \frac{x - \eta_l}{C_i - \eta_l} + \sum_{i=1}^{k-t} f(\eta_i) \prod_{j=1, j \neq i}^{k-t} \frac{x - \eta_j}{\eta_i - \eta_j} \prod_{l=1}^t \frac{x - C_l}{\eta_i - C_l} \right) \bmod Q$$

$$= P_1 + P_2x + \dots + P_kx^{k-1} \bmod Q$$

2. **Verifiability:**

The following theorem shows that the dealer can not pass through verification if he distributes inconsistent subshadows.

Theorem 1. Suppose the discrete logarithm in Z_q^* with base g is intractable. Then the probability for the dealer successfully distributes a fake subshadow to any participant is negligible, and the success probability for a participant M_i in submitting a fake subshadow in the recovery phase without being detected is also negligible.

Proof. For any participant M_i , its valid subshadow is an element $Y_i \in Z_Q^*$ such that $Y_i = f(C_i) \bmod Q$, $g^{Y_i} = \prod_{l=0}^{t-1} (A_{l+1})^{(C_i)^l} \bmod q$, if $k \leq t$; or $Y_i = f(C_i) \bmod Q$, $g^{Y_i} = \prod_{l=0}^{k-1} (A_{l+1})^{(C_i)^l} \bmod q$, $g^{f(\eta_j)} = \prod_{l=0}^{k-1} (A_{l+1})^{(\eta_j)^l} \bmod q$, $j = 1, 2, \dots, k - t$, if $k > t$.

Without loss of generality, just consider the case $k \leq t$. Assume that the dealer distributes an invalid subshadow Y'_i to M_i that passes the verification. Then D has to find a $Y'_i \neq f(C_i)$ such that $g^{Y'_i} = \prod_{l=0}^{t-1} (A_{l+1})^{(C_i)^l} \bmod q$. As $g^{Y'_i} = \prod_{l=0}^{t-1} (A_{l+1})^{(C_i)^l} \bmod q$ implies the equation $Y'_i = f(C_i) \bmod Q$ holds with probability 1, we conclude that the probability for the dealer D successfully cheats any participant is negligible. \square

Now consider the success probability for a participant M_i in submitting a fake subshadow in the recovery phase without being detected. Let Y_i be M_i 's true subshadow obtained from the dealer. To successfully cheat in the recovery phase, M_i has to find a $Y'_i \neq Y_i, Y'_i \in Z_Q^*$ such that $g^{Y'_i} = \prod_{l=0}^{t-1} (A_{l+1})^{(C_i)^l} \bmod q$. Since $g^{Y'_i} = \prod_{l=0}^{t-1} (A_{l+1})^{(C_i)^l} \bmod q$ implies $Y'_i = f(C_i) = Y_i \bmod Q$ with probability 1, we know that M_i 's success probability in submitting a fake subshadow in the recovery phase without being detected is negligible.

3. Privacy:

To demonstrate that no useful information about the set of shared secrets are revealed to an adversary corrupting at most $t - 1$ participants, we give the following two theorems with brief proofs. The first one shows that the open commitments do not reveal any useful information about the set of secrets and the subshadows, and the second one implies the confidentiality of the set of shared secrets against an adversary who corrupts up to $t - 1$ participants.

Theorem 2. *The adversary E can not get any useful information about $\{P_1, P_2, \dots, P_k\}$ and the subshadows possessed by any participants from the public information under the assumptions that the RSA cryptosystem used in the system is secure, and the discrete logarithm problem in Z_q^* with respect to the base g is intractable. i.e. the commitments $A_i, i = 1, 2, \dots, t$ (or k), and $H_i, i = 1, 2, \dots, n$ do not reveal any useful information about the set of secrets and the subshadows.*

Proof. In case $k \leq t$, the public commitments are $A_i = g^{P_i} \bmod q$ for $i = 1, \dots, k$, $A_i = g^{a_i-k} \bmod q$ for $i = k + 1, \dots, t$, $H_i = Y_i^{e_i} \bmod N_i$ for $i = 1, 2, \dots, n$. While in case $k > t$, the public commitments are $A_i = g^{P_i} \bmod q$ for $i = 1, \dots, k$, $H_i = Y_i^{e_i} \bmod N_i$ for $i = 1, 2, \dots, n$. As computing the discrete logarithm to the base g is difficult, the adversary can not derive any useful information about the secrets and the polynomial $f(x)$ from the open commitments. Secondly, to derive a subshadows Y_i from H_i without knowing M_i 's private key, one needs to break the RSA encryption scheme. So, the public information leaks no useful information about the set of shared secrets and the subshadows. \square

Theorem 3. *An adversary corrupting up to $t - 1$ participants cannot derive any subshadow kept by an honest participant and consequently cannot get useful information about the set of shared secrets.*

Proof. We learn that the adversary cannot get any useful information about the secret polynomial $f(x)$ from **Theorem 2**. Nevertheless according to the algorithm of construction, to acquire the subshadows of those honest participants, the adversary has no choice but compute $f(x)$ merely using the subshadows of the corrupted ones. Without loss of generality we suppose that the corrupted participants are $\{M_1, \dots, M_{t-1}\}$. The adversary has to compute all coefficients of $f(x)$ from the following system of linear equations in Z_Q :

- If $k \leq t$

$$\begin{cases} P_1 + P_2C_1 + \dots + P_kC_1^{k-1} + a_1C_1^k + \dots + a_{t-k}C_1^{t-1} = Y_1 \\ P_1 + P_2C_2 + \dots + P_kC_2^{k-1} + a_1C_2^k + \dots + a_{t-k}C_2^{t-1} = Y_2 \\ \vdots \\ P_1 + P_2C_{t-1} + \dots + P_kC_{t-1}^{k-1} + a_1C_{t-1}^k + \dots + a_{t-k}C_{t-1}^{t-1} = Y_{t-1} \end{cases}$$

i.e.

$$\begin{bmatrix} 1 & C_1 & \dots & C_1^{k-1} & C_1^k & \dots & C_1^{t-1} \\ 1 & C_2 & \dots & C_2^{k-1} & C_2^k & \dots & C_2^{t-1} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 1 & C_k & \dots & C_k^{k-1} & C_k^k & \dots & C_k^{t-1} \\ 1 & C_{k+1} & \dots & C_{k+1}^{k-1} & C_{k+1}^k & \dots & C_{k+1}^{t-1} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 1 & C_{t-1} & \dots & C_{t-1}^{k-1} & C_{t-1}^k & \dots & C_{t-1}^{t-1} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \\ a_1 \\ \vdots \\ a_{t-k} \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \\ Y_{k+1} \\ \vdots \\ Y_{t-1} \end{bmatrix}$$

• If $k > t$

$$\begin{cases} P_1 + P_2C_1 + \dots + P_kC_1^{k-1} = Y_1 \\ P_1 + P_2C_2 + \dots + P_kC_2^{k-1} = Y_2 \\ \vdots \\ P_1 + P_2C_{t-1} + \dots + P_kC_{t-1}^{k-1} = Y_{t-1} \\ P_1 + P_2\eta_1 + \dots + P_k\eta_1^{k-1} = f(\eta_1) \\ \vdots \\ P_1 + P_2\eta_{k-t} + \dots + P_k\eta_{k-t}^{k-1} = f(\eta_{k-t}) \end{cases}$$

i.e.

$$\begin{bmatrix} 1 & C_1 & C_1^2 & \dots & C_1^{k-1} \\ 1 & C_2 & C_2^2 & \dots & C_2^{k-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & C_{t-1} & C_{t-1}^2 & \dots & C_{t-1}^{k-1} \\ 1 & \eta_1 & \eta_1^2 & \dots & \eta_1^{k-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \eta_{k-t} & \eta_{k-t}^2 & \dots & \eta_{k-t}^{k-1} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_t \\ P_{t+1} \\ \vdots \\ P_k \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{t-1} \\ f(\eta_1) \\ \vdots \\ f(\eta_{k-t}) \end{bmatrix}$$

These are systems of linear equations where the rank of coefficient matrices is less than the number of variables. That means it has at least $Q > 2^{\lambda/2}$ answers and the probability for the adversary to pick out the genuine $\{P_1, P_2, \dots, P_k, a_1, \dots, a_{t-k}\}$ are not more than $1/Q \leq \lambda/2$. Accordingly the probability to calculate the subshadow of any uncorrupted participant is not more than $1/2^{\lambda/2}$, which is a negligible function of the security parameter λ . \square

From the above analysis, we can draw the conclusion that our scheme 1 is a secure VMSS scheme.

4.3. Scheme 2

In this section we present another new VMSS scheme based on the homogeneous linear recursion, the RSA cryptosystem and the discrete logarithm problem. Let D be the dealer, $M = \{M_1, M_2, \dots, M_n\}$ be the set of participants, $t(t < n)$ be the threshold.

• Initialization phase

In this phase, the dealer D first creates a public notice board (NB) which is used for storing necessary public information. The participants can access the information on NB. But the contents on the board can only be modified or updated by D .

Initialization of D :

- (1) On input a security parameter λ , D chooses two $\lambda/2$ bit strong primes p_0 and q_0 ($p_0 > q_0$), satisfying the requirement of a secure RSA public key cryptosystem [32], and computes $N = p_0q_0$.
- (2) D computes $\phi(N) = (p_0 - 1)(q_0 - 1)$, then safely destroy p_0, q_0 .
- (3) D randomly chooses two primes Q, q , such that $Q|(q - 1)$, and the bit-length of Q is at least $\lambda/2$. D also selects an element g of Z_q^* with order Q such that the discrete logarithm problem with base g in Z_q^* is infeasible.
- (4) D randomly chooses another integer $\alpha \neq 0$ and establishes the auxiliary equation:

$$(x - \alpha)^t = x^t + a_1x^{t-1} + \dots + a_t = 0.$$

- (5) D publishes $(\lambda, N, Q, q, g, \alpha)$ on NB.

Initialization of participants:

- (1) Each M_i with identity information ID_i chooses two strong primes p_i and q_i ($p_i > q_i$), satisfying the requirement of a secure RSA public key cryptosystem [32], and computes $N_i = p_iq_i$ which satisfy $(N_i > N)$.
- (2) M_i computes $\phi(N_i) = (p_i - 1)(q_i - 1)$.
- (3) M_i randomly chooses an integer e_i which is coprime to $\phi(N_i)$ and computes the integer d_i such that $e_id_i = 1 \pmod{\phi(N_i)}$.
- (4) M_i authentically provides (ID_i, e_i, N_i) to D through a public channel, and keeps his shadow d_i secret.

D puts $(ID_i, e_i, N_i), i = 1, 2, \dots, n$, on NB.

we note that, similar to scheme 1, the information on NB can be reusable after the initialization phase. The dealer D does not know any participant's shadow, so the shadow can be reusable for multiple rounds of sharing even with different dealers.

• Construction phase

Let $P_1, P_2, \dots, P_k \in Z_Q$ denote k secrets to be shared. D chooses at random an integer a_i such that $Q > a_i$ for $i = 1, 2, \dots, t$. Then D performs the following steps to generate a subshadow u_i for each participant M_i :

- (1) Randomly choose $C_i \in Z_Q^*$ for $i = 1, 2, \dots, t$.

(2) Set up an [HLR] by the equations

$$[*] \begin{cases} u_1 = C_1, u_2 = C_2, \dots, u_t = C_t, \\ u_{i+t} + a_1 u_{i+t-1} + \dots + a_t u_i = 0 \pmod Q \quad (i \geq 1) \end{cases}$$

and compute $u_i, t < i \leq n + k$.

(3) Compute $Y_i = P_i - u_{n+i} \pmod Q$ for $i = 1, 2, \dots, k$.

(4) Compute $H_i = (u_i)^{e_i} \pmod{N_i}$ and $T_i = g^{u_i} \pmod q$ for $1 \leq i \leq n$.

(5) Publish $(H_1, H_2, \dots, H_n, T_1, T_2, \dots, T_n, Y_1, Y_2, \dots, Y_k)$ on NB.

• **Verification phase**

Each participant M_i can get its subshadow by computing $u_i = (H_i)^{d_i} \pmod{N_i}$ for $i = 1, 2, \dots, n$. The validity and consistency of M_i 's subshadow u_i with the information published by D on NB can be verified as follows:

$$T_{t+i} \prod_{j=1}^t (T_{t+i-j})^{a_j} \stackrel{?}{=} 1 \pmod q$$

$$T_i \stackrel{?}{=} g^{u_i} \pmod q$$

If M_i 's verification is successful, M_i believes the subshadow u_i he has got is valid and is consistent with the public information on NB. If no participant fails in the verification, D is thought honest.

Similar to that in scheme 1, we add the consistency detection of all subshadows in the verification phase. This is necessary for removing the weakness of the schemes attacked in the previous section.

• **Recovery phase**

Assume that t or more arbitrary participants $\{M_i\}_{i \in I} (I \subseteq \{1, 2, \dots, n\})$ pool together their subshadows u_i to reconstruct the shared secrets. Each participant M_i can check whether the subshadows provided by the others are valid by the following equations:

$$g^{u_j} \stackrel{?}{=} T_j \pmod q, \quad j \in I.$$

If there are at least t valid subshadows, the shared secrets can be correctly reconstructed. Suppose they use t valid subshadows $\{u_i | i \in J \subset I\}$, they can get the following simultaneous equations using Lemma 1 in Section 2:

$$z_0 + z_1 i + \dots + z_{t-1} i^{t-1} = u_i \alpha^{-i} \pmod Q, \quad i \in J.$$

Solving the equations (or equivalently using the technique of Lagrange interpolation), they get (in Z_Q) $z_0 = A_0, z_1 = A_1, \dots, z_{t-1} = A_{t-1}$. Now, they have

$$u_i = (A_0 + A_1 i + \dots + A_{t-1} i^{t-1}) \alpha^i \pmod Q, \quad \forall i \geq t.$$

Hence they can reconstruct the shared secrets:

$$P_i = Y_i + u_{n+i} \pmod Q, \quad i = 1, 2, \dots, k.$$

The difference between our scheme 2 and MS1 and MS2 schemes is shown in Fig. 2.

As seeing in Fig. 2, in scheme SM1, a participant M_i chooses its secret shadow s_i , and encrypts it using the public key of the dealer D and sends the cipher text to D . After that both D and M_i can compute the subshadow $l_i = f(r, s_i)$ for M_i . No information is provided for detecting whether the l_i used in the [HLR] sequence is the same as the true subshadow of M_i . In scheme SM2, a participant M_i chooses its secret shadow s_i , computes its subshadow as $l_i = R_0^{s_i} \pmod N$ using the public information $R_0 = g^{s_0} \pmod N$ published by the dealer D and sends $R_i = g^{s_i} \pmod N$ to D as its commitment to its secret shadow s_i . D computes the subshadow of M_i as $l_i = R_i^{s_0} \pmod N$ which is used in the generation of the [HLR] sequence. Three modular exponentiations are needed for the computation. No information for verifying whether correct l_i is used in the computation of the [HLR] sequence. While in our scheme 2, M_i selects its secret shadow d_i and keeps it from the dealer. The subshadow of M_i is encrypted using M_i 's public key corresponding to d_i and is provided in the public information. Only two modular exponentiations are involved for the transmission of a subshadow. We require the dealer publish some information for verifying the consistence of participant's subshadow and the [HLR] sequence so that cheating behavior of the dealer can be detected.

4.4. Security analysis

The security of our scheme 2 is based on the discrete logarithm problem and the large integer factorization problem which are assumed to be hard.

1. **Correctness:** If the dealer and the participants are honest, any t or more participants can reconstruct the set of the shared secrets in the recovery phase. This fact can be shown as follows.

Suppose $\{M_i, i \in I\}$ be a set of at least t honest participants. Let $\{u_i, i \in I\}$ be their corresponding subshadows obtained from the honest dealer D . Using their subshadows, they can get the following equations

$$z_0 + z_1 i + \dots + z_{t-1} i^{t-1} = u_i \alpha^{-i} \pmod Q, \quad i \in I.$$

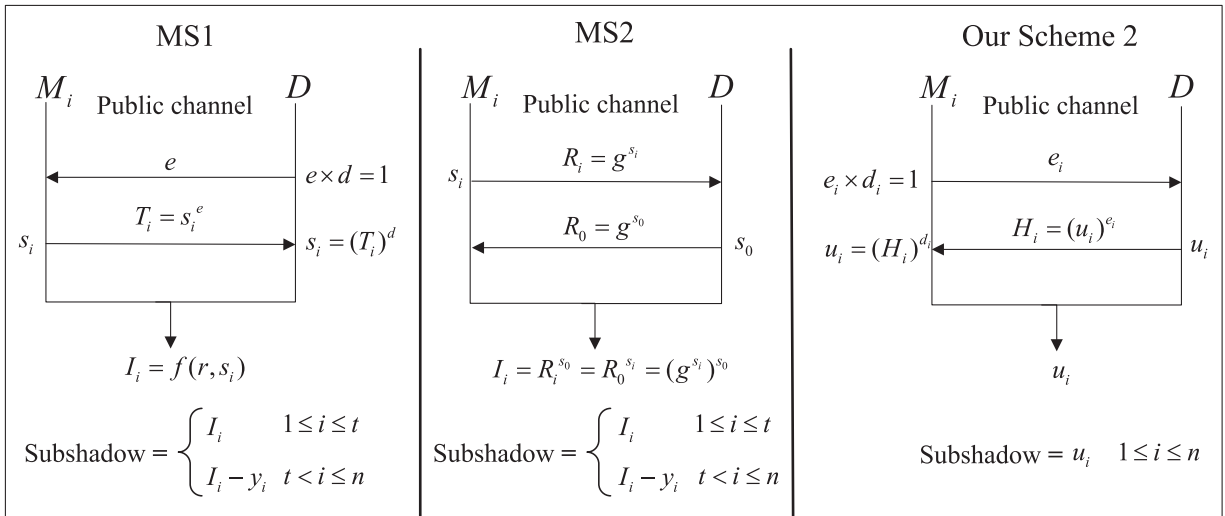


Fig. 2. The difference of our scheme 2 with MS1 and MS2 schemes.

Solving this system of equations (or using the technique of Lagrange interpolation), they get

$$u_i = (A_0 + A_1i + \dots + A_{t-1}i^{t-1})\alpha^i \text{ mod } Q, \quad \forall i \geq t.$$

Hence they can reconstruct the shared secrets:

$$P_i = Y_i + u_{n+i} \text{ mod } Q, \quad i = 1, 2, \dots, k.$$

2. Verifiability:

The following theorem shows that the dealer cannot pass through verification if he distributes inconsistent subshadows. And a dishonest participant is unable to submit a fake subshadow without being detected in the recovery phase.

Theorem 4. *The probability for the dealer successfully distributing a fake subshadow to any participant without being detected is negligible. At the same time, the probability for a dishonest participant submitting a fake subshadow without being detected in the recovery phase is negligible.*

Proof. Assume M_i 's valid subshadow is u_i , and the dealer successfully distributes a fake subshadow u'_i to M_i . Then $T_i = g^{u'_i} = g^{u'_i} \text{ mod } q$, and $T_{t+i} \prod_{j=1}^t (T_{t+i-j})^{a_j} = 1 \text{ mod } q$. Since $u_i, u'_i \in Z_Q$, this implies that the probability for $u'_i \neq u_i$ is negligible. Similarly, if a dishonest participant M_i submits a fake subshadow u'_i without being detected in the recovery phase, then we have $T_i = g^{u'_i} = g^{u_i} \text{ mod } q$. Hence, $u'_i = u_i \text{ mod } Q$ holds with probability 1. This means the probability for a participant M_i successfully submitting a fake subshadow without being detected in the recovery phase is negligible. \square

3. Privacy:

Theorem 5. *Assume that computing discrete logarithm in Z_q^* is difficult and the RSA encryption scheme is secure. Then the public information $T_i, H_i, i = 1, \dots, n$, do not reveal any useful information about the set of shared secrets and the subshadows of participants.*

Proof. We know that $T_i = g^{u_i} \text{ mod } q, H_i = (u_i)^{e_i} \text{ mod } N_i$ for $i = 1, \dots, n$. As computing the discrete logarithm to the base g is difficult, the adversary cannot derive any useful information about subshadows from the open commitments T_1, T_2, \dots, T_n . Second, Since H_i is the RSA encryption of M_i 's subshadow u_i , to derive u_i from H_i the adversary E has to break the RSA encryption scheme. Under the assumption that the RSA encryption scheme used in our construction is secure the adversary gets no useful information about u_1, u_2, \dots, u_n from H_1, H_2, \dots, H_n . Without the knowledge of at least t subshadows of the participants, the adversary cannot compute any of $u_{n+1}, u_{n+2}, \dots, T_{n+k}$ due to the definition and properties of linear recursion sequence u_j . Hence the adversary gets no useful information about the set of shared secrets $u_{n+1} + Y_1, u_{n+2} + Y_2, \dots, u_{n+k} + Y_k$ from public information. \square

Theorem 6. *An adversary corrupts up to $t - 1$ participants cannot derive any subshadow kept by any honest participant and consequently cannot get any of the shared secrets.*

Proof. We learn that the adversary cannot get any useful information about the [HLR] from Theorem 5. Nevertheless according to the algorithm in the construction phase, to acquire the subshadows of those honest participants, the adversary has no choice but compute the [HLR] merely using the subshadows of the corrupted ones. Without loss of generality we suppose

Table 1
Analysis of computation cost.

Scheme	Initialization		Construction Dealer D	Verification Each M_i	Recovery Each M_i
	Dealer D	Each M_i			
ZZZ	0	1	$n + 1$	–	t
MS	$2n$	1	0	–	$t - 1$
MS1	n	1	n	–	$t - 1$
MS2	0	1	$n + 1$	–	t
Scheme 1	0	0	$n + t, k \leq t$	$t + 1, k \leq t$	$(t - 1)(k + 1)$
Scheme 2	0	0	$2n$	2	$t - 1$

that the corrupted participants are $\{M_1, \dots, M_{t-1}\}$. The adversary has to compute the sequence u_i of [HLR] from the following system of equations:

$$\begin{cases} u_1 = C_1, u_2 = C_2, \dots, u_{t-1} = C_{t-1} \\ u_{i+t} + a_1 u_{i+t-1} + \dots + a_t u_i = 0 \pmod Q, i = 1, 2, \dots, n - t. \end{cases}$$

Namely, to compute any other subshadow u_j , the adversary should solve the following system of linear equations using the $t - 1$ subshadow $u_1 = C_1, u_2 = C_2, \dots, u_{t-1} = C_{t-1}$:

$$u_{i+t} + a_1 u_{i+t-1} + \dots + a_t u_i = 0 \pmod Q, i = 1, 2, \dots, n - t$$

Note that this system of linear equations consists of $n - t$ equations, and there are $n - t + 1$ variables u_t, u_{t+1}, \dots, u_n . So the rank of the coefficient matrix is less than the number of variables. That means it has not less than Q answers and the probability for the adversary to pick out the correct sequence u_{t-1+i} ($i \geq 1$) used in share distribution is not more than $1/Q$. Accordingly the probability to calculate the subshadow of any uncorrupted participant is not more than $1/Q$. As $Q > 2^{\lambda/2}$, this probability is not more than $1/2^{\lambda/2}$, which is a negligible function of the security parameter λ . \square

From the above analysis, we know that our scheme 2 is a secure VMSS without private channels.

5. Performance analysis

5.1. Computation cost

In comparing the computational cost of our schemes with some other schemes of the same type, we list the amount of the most time consuming operations in each phase of these schemes. The most time consuming operations we consider here is modular exponentiation. Table 1 shows the main computational cost of our new schemes and the four schemes analyzed in Section 3.

As shown in Table 1, in the initialization and construction phases, our scheme 2 is the most efficient. While our scheme1 is slightly more efficient than MS and MS1 schemes (assume $n > k$), and less efficient than ZZZ and MS2 schemes. All the six schemes have almost the same computational cost in the recovery phase. Since we add consistence test to prevent cheating by the dealer, the verification phases of over new schemes require more modular exponentiations than in the other schemes. We note that such added computational cost is necessary for preventing cheating by the dealer. Without this distinctive feature of preventing cheating by the dealer, a so called verifiable secret sharing scheme will lose its real meaning for "verifiability". As demonstrated in Section 3, it is the lack of such tests that makes the other four schemes vulnerable to cheating by the dealers. As a whole, our new schemes are efficient and have a comparable computational cost with respect to similar existing verifiable multi-secret sharing schemes.

5.2. Communication cost

In Table 2, we list the communication cost in the initialization and construction phases of the six schemes, ZZZ scheme, MS scheme, MS1 scheme, MS2 scheme, and our new schemes. The communication cost of the six schemes in the recovery phase is almost the same. Table 2 indicates that our two new schemes are nearly as communication efficient as the other four schemes in the initialization phase, but slightly less efficient in the construction phase. This inefficiency is due to we require the dealer publish some redundant information for testing the consistence of the public information with the shadows and subshadows of participants. Our analysis in Section 3 reveals that the insecurity of the other four schemes is exactly resulted from lacking of such redundant information.

5.3. Main performance features

We also compare the main performance features of our new schemes with the other four schemes cited above. We consider six main functionalities of a VMSS scheme.

Table 2
Analysis of communication cost.

Scheme	Initialization phase		Construction phase
	D Broadcast	M_i to D	D Broadcast
ZZZ	(g, N) $(ID_i, R_i), i = 1, 2, \dots, n$	(ID_i, R_i) $i = 1, 2, \dots, n$	$(R_0, f), (y_1, y_2, \dots, y_n), k \leq t$ $(R_0, f), (y_1, \dots, y_n, h(1), \dots, h(k-t)), k > t$
MS	$(e, N, g, p), (r, G_i), i = 1, 2, \dots, n$ $(ID_i, T_i), i = 1, 2, \dots, n$	(ID_i, T_i) $i = 1, 2, \dots, n$	$(y_1, y_2, \dots, y_n), k \leq t$ $(y_1, y_2, \dots, y_n, h(1), h(2), \dots, h(k-t)), k > t$
MS1	(e, N, g, q, α) $(ID_i, T_i), i = 1, 2, \dots, n$	(ID_i, T_i) $i = 1, 2, \dots, n$	$(r, G_1, G_2, \dots, G_n, r_1, r_2, \dots, r_k, y_{t+1}, y_{t+2}, \dots, y_n)$
MS2	(N, g, q, α) $(i, T_i), i = 1, 2, \dots, n$	(i, T_i) $i = 1, 2, \dots, n$	$(R_0, f, r_1, r_2, \dots, r_k, y_{t+1}, y_{t+2}, \dots, y_n)$
Scheme 1	(λ, N, Q, q, g) $(ID_i, e_i, N_i), i = 1, 2, \dots, n$	(ID_i, e_i, N_i) $i = 1, 2, \dots, n$	$(C_1, \dots, C_n, H_1, \dots, H_n, A_1, \dots, A_t), k \leq t$ $(C_1, \dots, C_n, H_1, \dots, H_n, \eta_1, \dots, \eta_{k-t}, A_1, \dots, A_k, f(\eta_1), \dots, f(\eta_{k-t})), k > t$
Scheme 2	$(\lambda, N, Q, q, g, \alpha)$ $(ID_i, e_i, N_i), i = 1, 2, \dots, n$	(ID_i, e_i, N_i) $i = 1, 2, \dots, n$	$(H_1, H_2, \dots, H_n, T_1, T_2, \dots, T_n, Y_1, Y_2, \dots, Y_k)$

Table 3
Performance features.

Functionality	ZZZ scheme	MS scheme	MS1 scheme	MS2 scheme	Our scheme 1	Our scheme 2
1	NO	NO	NO	NO	YES	YES
2	YES	YES	YES	YES	YES	YES
3	YES	YES	YES	YES	YES	YES
4	YES	YES	YES	YES	YES	YES
5	YES	YES	YES	YES	YES	YES
6	YES	NO	NO	YES	YES	YES

- Functionality 1: Resist cheating by the dealer D
- Functionality 2: Resist cheating by dishonest participants M_i
- Functionality 3: Without secret channel
- Functionality 4: Reconstruct multi-secrets parallelly
- Functionality 5: Reuse of the secret shadows
- Functionality 6: Reuse of the secret shadows for multiple rounds of sharing even with different dealers.

Table 3 shows that both of our new schemes possess all the six main performance features. While the other four schemes do not have functionality 1, i.e. the cannot resist cheating by a dishonest dealer. The MS scheme and MS1 scheme do not possess functionality 6. This means that the secret shadows of participants will useless after the recovery phase. So, the initialization of participants has to be executed in every round of sharing even if the group of participants is not changed. Our two new schemes effectively overcome this inconvenience. They allow participants reuse their secret shadows in different rounds of multi-secret sharing even with different dealers. In this way, a participant could run the initialization of participant only once, and could use the information generated in this execution of initialization in many rounds of multi-secret sharing no matter the dealers in these round of sharing are different. This feature enables the participants to greatly reduce the cost of initialization in multiple rounds of multi-secret sharing.

6. Conclusion

Verifiable multi-secret sharing schemes provide practical techniques for sharing multiple secrets in a group of participants so that cheating behavior of a dealer or a participant can be detected. They are important tools in keeping multiple secrets such as cryptographic keys, and in designing secure multi-party protocols. In this paper, we begin with re-analyze the security of four recently proposed VMSS schemes. Our analysis reveals that all these schemes are subject to cheating by dishonest dealers. So these schemes do not satisfy the basic security requirement of verifiable secret sharing schemes. We notice that the security drawback of these schemes is induced by lacking of the consistence test of the information published by a dealer with the subshadows of participants. Based on the analysis, we further put forward two improved VMSS schemes. In our new schemes, we require the dealer to publish some redundant information for the necessary consistence checking. The security analysis and performance analysis of our new schemes demonstrate that they are secure and efficient verifiable multi-secret sharing schemes withstanding cheating by the dealer or a participant, requiring no secret channels, allowing parallel reconstruct of multiple secrets and reuse of shadows in different rounds of sharing even with different dealers.

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