Convergence to the Tracy-Widom distribution for longest paths in a directed random graph

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Abstract. We consider a directed graph on the 2-dimensional integer lattice, placing a directed edge from vertex (i_1, i_2) to (j_1, j_2), whenever $i_1 \le j_1$, $i_2 \le j_2$, with probability p, independently for each such pair of vertices. Let $L_{n,m}$ denote the maximum length of all paths contained in an $n \times m$ rectangle. We show that there is a positive exponent a, such that, if $m/n^a \to 1$, as $n \to \infty$, then a properly centered/rescaled version of $L_{n,m}$ converges weakly to the Tracy-Widom distribution. A generalization to graphs with non-constant probabilities is also discussed.

1. Introduction

Random directed graphs form a class of stochastic models with applications in computer science (Isopi and Newman, 1994), biology (Cohen and Newman, 1991; Newman, 1992; Newman and Cohen, 1986) and physics (Itoh and Krapivsky, 2012). Perhaps the simplest of all such graphs is a directed version of the standard Erdős-Rényi random graph (Barak and Erdős, 1984) on n vertices, defined as follows: For each pair $\{i, j\}$ of distinct positive integers less than n, toss a coin with probability of head equal to p, $0 < p < 1$, independently from pair to pair; if head shows up then introduce an edge directed from $\min(i, j)$ to $\max(i, j)$. There is a natural extension of this graph to the whole of $\mathbb{Z}$ studied in detail in Foss and Konstantopoulos (2003). In particular, if we define the asymptotic growth rate $C = C(p)$, as the a.s. limit of the maximum length of all paths between 1 and n divided by n, Foss and Konstantopoulos (2003) provide sharp bounds on $C(p)$ for all values of $p \in (0, 1)$.

A natural generalization arises when we replace the total order of the vertex set by a partial order, usually implied by the structure of the vertex set. In such a model, coins are tossed only for pairs of vertices which are comparable in this partial order. The canonical case is to consider, as a vertex set, the 2-dimensional integer lattice $\mathbb{Z} \times \mathbb{Z}$, equipped with the standard component-wise partial order: $(i_1, i_2) \prec (j_1, j_2)$ if the two pairs are distinct and $i_1 \le i_2$, $j_1 \le j_2$. Such a graph was considered in Denisov et al. (2012). In that paper, it was shown that if $L_{n,m}$ denotes

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the maximum length of all paths of the graph, restricted to $\{0,\dots,n\}\times\{1,\dots,m\}$, then there is a positive κ (depending on p and the fixed integer m), such that

$$\left(\frac{L_{[nt],m}-Cnt}{\kappa\sqrt{n}},\ t\ge 0\right)\xrightarrow[n\to\infty]{\text{(d)}}(Z_{t,m},\ t\ge 0), \tag{1.1}$$

where $Z_{\bullet,m}$ is the stochastic process defined in terms of m independent standard Brownian motions, $B^{(1)},\dots,B^{(m)}$, via the formula

$$Z_{t,m}:=\sup_{0=t_0<t_1\cdots<t_{m-1}<t_m=t}\sum_{j=1}^{m}[B^{(j)}_{t_j}-B^{(j)}_{t_{j-1}}],\quad t\ge 0.$$

One can speak of Z as a *Brownian directed percolation model*, the terminology stemming from the picture of a "weighted graph" on $\mathbb{R}\times\{1,\dots,m\}$ where the weight of a segment $[s,t]\times\{j\}$ equals the change $B^{(j)}_t-B^{(j)}_s$ of a Brownian motion. If a path from $(0,0)$ to (t,m) is defined as a union $\bigcup_{j=1}^m[t_{j-1},t_j]\times\{j\}$ of such segments, then Z represents the maximum weight of all such paths.

Baryshnikov (2001), answering an open question by Glynn and Whitt (1991), showed that

$$Z_{1,m}\overset{\text{(d)}}{=}\lambda_m,$$

where λ_m is the largest eigenvalue of a GUE matrix of dimension m. Since $Z_{\bullet,m}$ is 1/2-self-similar, we see that

$$Z_{t,m}\overset{\text{(d)}}{=}\sqrt{t}\lambda_m.$$

Now, fluctuations of λ_m around the centering sequence $2\sqrt{m}$ have been quantified by Tracy and Widom (1994) who showed the existence of a limiting law, denoted by F_{TW}:

$$m^{1/6}(\lambda_m-2\sqrt{m})\xrightarrow[m\to\infty]{\text{(d)}}F_{\text{TW}}.$$

A natural question then, raised in Denisov et al. (2012), is whether one can obtain F_{TW} as a weak limit of $L_{n,m}$ when n and m tend to infinity simultaneously. Our paper is concerned with resolving this question. To see what scaling we can expect, rewrite the last display, for arbitrary $t>0$, as

$$m^{1/6}(\frac{Z_{t,m}}{\sqrt{t}}-2\sqrt{m})\xrightarrow[m\to\infty]{\text{(d)}}F_{\text{TW}}.$$

A statement of the form $X(t,m)\xrightarrow[m\to\infty]{\text{(d)}}X$, where the distribution of $X(t,m)$ does not depend on the choice of $t>0$, implies the statement $X(t,m(t))\xrightarrow[t\to\infty]{\text{(d)}}X$, for any function $m(t)$ such that $m(t)\xrightarrow[t\to\infty]{}\infty$. Hence, upon setting $m=[t^a]$, we have

$$t^{a/6}\left(\frac{Z_{t,[t^a]}}{\sqrt{t}}-2\sqrt{t^a}\right)\xrightarrow[t\to\infty]{\text{(d)}}F_{\text{TW}}. \tag{1.2}$$

Therefore, it is reasonable to guess that, when a is small enough, an analogous limit theorem holds for a centered scaled version of the largest length $L_{n,[n^a]}$, namely that

$$n^{a/6}\left(\frac{L_{n,[n^a]}-c_1n}{c_2\sqrt{n}}-2\sqrt{n^a}\right)\xrightarrow[n\to\infty]{\text{(d)}}F_{\text{TW}}, \tag{1.3}$$

where c_1,c_2 are appropriate constants.

A stochastic model, bearing some resemblance to ours, is the so-called *directed last passage percolation model* on $\mathbb{Z}^d$ (the case $d=2$ being of interest here). We are given a collection of i.i.d. random variables indexed by elements of $\mathbb{Z}_+^d$. A path from the origin to the point $n \in \mathbb{Z}_+^d$ is a sequence of elements of $\mathbb{Z}_+^d$, starting from the origin and ending at n, such that the difference of successive members of the sequence is equal to the unit vector in the ith direction, for some $1 \leq i \leq d$. The weight of a path is the sum of the random variables associated with its members. Specializing to $d = 2$, let $L_{n,m}$ be the largest weight of all paths from $(0, 0)$ to (n, m). Assuming that the random variables have a finite moment of order larger than 2, Bodineau and Martin (2005) showed that (1.3) holds for all sufficiently small positive a (the threshold depending on the order of the finite moment). Independently, Baik and Suidan (2005) obtained the same result for random variables with a finite 4th moment and for $a < 3/14$. In both papers, partial sums of i.i.d. were approximated with Brownian motions, in the first case using the Komlós-Major-Tusnády (KMT) construction, while in the second using Skorokhod embedding.

To show that (1.3) holds for our model, we adopt the technique introduced in Denisov et al. (2012), which involves the existence of *skeleton points* on each line $\mathbb{Z} \times \{j\}$. Skeleton points are, by definition, random points which are connected with all the other points on the same line. In Denisov et al. (2012) Denisov, Foss and Konstantopoulos used this fact, together with the fact that, for finite m, one can pick skeleton points common to all m lines, in order to prove (1.1). However, when m tends to infinity simultaneously with n, it is not possible to pick skeleton points common to all lines. Modifying the definition of skeleton points enables us to give a new proof of (1.1), as well as to prove (1.3). To achieve the latter, we borrow the idea of KMT coupling from Bodineau and Martin (2005). However, we need to do some work in order to express the random variable $L_{n,m}$ in a way that resembles a maximum of partial sums.

Although we focus on the case where the edge probability p is constant, it is possible to consider a more general case, where the probability that a vertex $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$ connects to a vertex (j_1, j_2) depends on the distances $|j_1 - i_1|$ and $|j_2 - i_2|$ of the two vertices. This generalization is discussed in the last section of the article.

2. The one-dimensional directed random graph

We summarize below some properties of the directed Erdős-Rényi graph on $\mathbb{Z}$ with connectivity probability p taken from Foss and Konstantopoulos (2003). For $i < j$, let $L[i, j]$ be the maximum length of all paths with start and end points in the interval $[i, j]$. Then, for $i < j < k$, we have $L[i, k] \leq L[i, j] + L[j, k] + 1$. Since the distribution of the random graph is invariant under translations, and is also ergodic (the natural invariant σ-field is trivial), it follows, from Kingman's subadditive ergodic theorem, that there is a deterministic constant $C = C(p)$ such that

$$\lim_{n\to\infty} L[1, n]/n = C, \text{ a.s.} \tag{2.1}$$

In fact, $C = \inf_{n\geq 1} EL[1, n]/n$. The function $C(p)$ is not known explicitly; only bounds are known (Foss and Konstantopoulos, 2003, Thm. 10.1). For example, $0.5679 \leq C(1/2) \leq 0.5961$. We also know that there exists, almost surely, a random integer sequence $\{\Gamma_r, r \in \mathbb{Z}\}$ with the property that for all r, all $i < \Gamma_r$, and all $j > \Gamma_r$, there is a path from i to Γ_r and a path from Γ_r to j. The existence of such points, referred to as *skeleton points*, is not hard to establish (Denisov et al., 2012). Since the directed Erdős-Rényi graph is invariant under translations, so is the sequence of skeleton points, i.e., $\{\Gamma_r, r \in \mathbb{Z}\}$ has the same law as $\{n+\Gamma_r, r \in \mathbb{Z}\}$, for all $n \in \mathbb{Z}$. Moreover, it turns out that the sequence forms a stationary renewal process. If we enumerate the skeleton points according to $\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \cdots$, we have that $\{\Gamma_{r+1} - \Gamma_r, r \in \mathbb{Z}\}$ are independent random variables, whereas $\{\Gamma_{r+1} - \Gamma_r, r \neq 0\}$ are i.i.d. Stationarity implies that the law of the omitted difference $\Gamma_1 - \Gamma_0$ has a density which is proportional to the tail of the distribution of $\Gamma_2 - \Gamma_1$. In Denisov et al. (2012) it is shown that the distance $\Gamma_2 - \Gamma_1$ between two successive skeleton points has a finite 2nd moment. One can follow the same steps of the proof, to show that in our case, with constant probability p, this random variable has moments of all orders. Moreover, one can show that for some $\alpha > 0$ (the maximal such α depends on p) it holds that $Ee^{\alpha(\Gamma_2 - \Gamma_1)} < \infty$.

The rate λ_0 of the sequence of skeleton points can be expressed as an infinite product:

$$\lambda_0 := \frac{1}{E(\Gamma_2 - \Gamma_1)} = \prod_{k=1}^{\infty} (1 - (1-p)^k)^2. \tag{2.2}$$

For example, for $p = 1/2$, $\lambda_0 \approx 1/12$.

A central limit theorem for $L[1, n]$ is also available (Denisov et al., 2012, Thm. 2). If we let

$$\sigma_0^2 := \mathrm{var}(L[\Gamma_1, \Gamma_2] - C(\Gamma_2 - \Gamma_1)), \tag{2.3}$$

then

$$\frac{L[1, n] - Cn}{\sqrt{\lambda_0 \sigma_0^2 n}} \xrightarrow[n\to\infty]{\text{(d)}} N(0, 1), \tag{2.4}$$

where $N(0, 1)$ is a standard normal random variable. Note that $\sigma_0^2 \neq \mathrm{var}(L[\Gamma_1, \Gamma_2])$. Unfortunately, we have no estimates for σ_0^2, but, interestingly, there is a technique for estimating it, based on perfect simulation. This was briefly explained in Foss and Konstantopoulos (2003) in connection with an infinite-dimensional Markov chain which carries most of the information about the law of the directed Erdős-Rényi random graph.

In addition, it is shown in Foss and Konstantopoulos (2003) that C can also be expressed as

$$C = \frac{EL[\Gamma_1, \Gamma_2]}{E(\Gamma_2 - \Gamma_1)}. \tag{2.5}$$

In fact, if $\{\nu_r, r \in \mathbb{Z}\}$ is a random sequence of integers, defined on the same probability space as the one supporting the random graph, such that $\{\Gamma_{\nu_r}, r \in \mathbb{Z}\}$ is a stationary point process then

$$C = \frac{EL[\Gamma_{\nu_r}, \Gamma_{\nu_{r+1}}]}{E(\Gamma_{\nu_{r+1}} - \Gamma_{\nu_r})}.$$

The most important property of the skeleton points is that if γ is a skeleton point, and if $i \leq \gamma \leq j$, then a path with length $L[i, j]$ (a maximum length path) must necessarily contain γ. This crucial property will be used several times below, especially since, for every $i < j$, the following equality holds

$$L[\Gamma_i, \Gamma_j] = L[\Gamma_i, \Gamma_{i+1}] + L[\Gamma_{i+1}, \Gamma_{i+2}] + \cdots + L[\Gamma_{j-1}, \Gamma_j].$$

Furthermore, the restriction of the graph on the interval between two successive skeleton points is independent of the restriction on the complement of the interval;

hence the summands in the right-hand side of the last display are independent random variables.

3. Statement of the main result

It is clear from (2.4) that the constants c_1, c_2 in (1.3) should be as follows: $c_1 = C$, $c_2 = \sqrt{\lambda\sigma^2}$. Now we can formulate the main result.

Theorem 3.1. *Let C, λ_0, σ_0^2 be the quantities associated with the directed random graph on $\mathbb{Z}$ with connectivity probability p, defined by (2.1) (equivalently, (2.5)), (2.2), (2.3), respectively. Consider the directed random graph on $\mathbb{Z}\times\mathbb{Z}$ and let $L_{n,m}$ be the maximum length of all paths between two vertices in $[0, n] \times [1, m]$. Then, for all $0 < a < 3/14$,*

$$n^{a/6}\left(\frac{L_{n,[n^a]} - Cn}{\sqrt{\lambda_0\sigma_0^2}\sqrt{n}} - 2\sqrt{n^a}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} F_{TW}, \tag{3.1}$$

where F_{TW} is the Tracy-Widom distribution.

To prove this theorem, we will first define the notion of skeleton points for the graph on $\mathbb{Z}\times\mathbb{Z}$ and then prove pathwise upper and lower bounds for $L_{n,m}$ which depend on paths going through these skeleton points. This will be done in Section 4. In Section 5.1 we show that the difference between these bounds is of the order $o(n^b)$, where $b = (1/2) - (a/6)$ is the net exponent in the denominator of (3.1). We will then (Section 5.2) introduce a quantity $S_{n,m}$ which resembles a last passage percolation problem and show that it differs from $L_{n,m}$ by a quantity which is of the order $o(n^b)$, when $m = [n^a]$. The problem will then be translated to a last passage percolation problem (with the exception of random indices). This will finally, in Section 5.3 be compared to the Brownian directed percolation problem by means of strong coupling.

4. Skeleton points and pathwise bounds

Our model is a directed random graph G with vertices $\mathbb{Z}\times\mathbb{Z}$. For each pair of vertices $\boldsymbol{i}$, $\boldsymbol{j}$, such that $\boldsymbol{i} \prec \boldsymbol{j}$, toss an independent coin with probability of heads equal to p; if a head shows up introduce an edge directed from $\boldsymbol{i}$ to $\boldsymbol{j}$.

A path of length ℓ in the graph is a sequence $(\boldsymbol{i}_0, \boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell)$ of vertices $\boldsymbol{i}_0 \prec \boldsymbol{i}_1 \prec \ldots \prec \boldsymbol{i}_\ell$ such that there is an edge between any consecutive vertices.

We denote by $G_{n,m}$ the restriction of G on the set of vertices $\{0, 1, \ldots, n\} \times \{1, \ldots, m\}$. The random variable of interest is

$$L_{n,m} := \text{ the maximum length of all paths in } G_{n,m}.$$

We refer to the set $\mathbb{Z}\times\{j\}$ as "line j" or "jth line", and note that the restriction of G onto $\mathbb{Z}\times\{j\}$ is a directed Erdős-Rényi random graph. We denote this restriction by $G^{(j)}$. Typically, a superscript (j) will refer to a quantity associated with this restriction. For example, for $a \le b$,

$$\begin{aligned} L^{(j)}[a,b] := {} & \text{the maximum length of all paths in } G^{(j)} \\ & \text{with vertices between } (a,j) \text{ and } (b,j) \end{aligned}$$

and we agree that $L^{(j)}[a,b] = 0$ if $a \ge b$.

Clearly, the $\{G^{(j)}, j \in \mathbb{Z}\}$ are i.i.d. random graphs, identical in distribution to the directed Erdős-Rényi random graph. Therefore, for each $j \in \mathbb{Z}$,

$$\lim_{n\to\infty} L^{(j)}[1,n]/n = C, \text{ a.s.}$$

To establish upper and lower bounds for $L_{n,m}$, we need to slightly change the definition of a skeleton point in G.

Definition 4.1 (Skeleton points in G)**.** A vertex (i,j) of the directed random graph G is called skeleton point if it is a skeleton point for $G^{(j)}$ (for any $i' < i < i''$, there is a path from (i',j) to (i,j) and a path from (i,j) to (i'',j)) and if there is an edge from (i,j) to $(i,j+1)$.

Therefore, the skeleton points on line j are obtained from the skeleton point sequence of the directed Erdős-Rényi random graph $G^{(j)}$ by independent thinning with probability p. When we refer to skeleton points on line j, we shall be speaking of this thinned sequence. The elements of this sequence are denoted by

$$\cdots < \Gamma_{-1}^{(j)} < \Gamma_0^{(j)} \le 0 < \Gamma_1^{(j)} < \Gamma_2^{(j)} < \cdots$$

and have rate

$$\lambda = \frac{1}{E(\Gamma_2^{(j)} - \Gamma_1^{(j)})} = p\lambda_0 = p\prod_{k=1}^{\infty}(1-(1-p)^k)^2.$$

The associated counting process of skeleton points on line j is defined by

$$\Phi^{(j)}(t) - \Phi^{(j)}(s) = \sum_{r\in\mathbb{Z}} \mathbf{1}(s < \Gamma_r^{(j)} \le t), \quad s,t \in \mathbb{R}, \quad s \le t,$$

together with the agreement that

$$\Phi^{(j)}(0) = 0.$$

Note that we insist on having the parameter t in $\Phi^{(j)}(t)$ as an element of $\mathbb{R}$ (and not just $\mathbb{Z}$). We also let

$$X^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)},$$
$$Y^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)+1},$$

be the skeleton points on line j straddling t:

$$X^{(j)}(t) \le t < Y^{(j)}(t). \tag{4.1}$$

Next we prove upper and lower bounds for $L_{n,m}$. The set of dissections of the interval $[0,n] \subset \mathbb{R}$ in m non-overlapping, possibly empty intervals is denoted by

$$\mathcal{T}_{n,m} := \{\boldsymbol{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1} : \ 0 = t_0 \le t_1 \le \cdots \le t_{m-1} \le t_m = n\}.$$

Lemma 4.2. (Upper bound) *Define*

$$\overline{L}_{n,m} := \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}),\ Y^{(j)}(t_j)] + m. \tag{4.2}$$

Then $L_{n,m} \le \overline{L}_{n,m}$.

Proof: Let π be a path in $G_{n,m}$. Consider the lines visited by π, denoting their indices by $1 \le \nu_1 < \nu_2 < \cdots < \nu_J \le m$. Let (a_j, ν_j) and (b_j, ν_j) be the first and the last vertex of line ν_j in the path π. Then the length of π satisfies

$$|\pi| \le \sum_{j=1}^{J} L^{(\nu_j)}[a_j, b_j] + J - 1.$$

Since successive vertices in the path should be increasing in the order $\prec$, we have $b_{j-1} \le a_j$, $2 \le j \le J$. Hence, with $b_0 := 0$,

$$|\pi| \le \sum_{j=1}^{J} L^{(\nu_j)}[b_{j-1}, b_j] + J - 1 \le \sum_{j=1}^{J} L^{(\nu_j)}[X^{(\nu_j)}(b_{j-1}),\ Y^{(\nu_j)}(b_j)] + J - 1,$$

where we used (4.1). Since $J \le m$, we can extend $0 = b_0 \le b_1 \le \cdots \le b_J \le n$ to a dissection of $[0, n]$ into m non-overlapping intervals, showing that the right-hand side of the last display is bounded above by $\overline{L}_{n,m}$. Taking the maximum over all π in $G_{n,m}$, we obtain $L_{n,m} \le \overline{L}_{n,m}$, as required. □

Note that the existence and properties of skeleton points were not used in the proof of the upper bound, other than to ensure that the upper bound is a.s. finite.

Lemma 4.3. (Lower bound) *Define*

$$\Delta_n^{(j)} := \max_{0 \le i \le \Phi^{(j)}(n)} (\Gamma_{i+1}^{(j)} - \Gamma_i^{(j)}),$$

and

$$\underline{L}_{n,m} := \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}),\ X^{(j)}(t_j)] - \sum_{j=1}^{m} \Delta_n^{(j)}.$$

Then $L_{n,m} \ge \underline{L}_{n,m}$.

Proof: We will show that, for all $\boldsymbol{t} = (t_0, \ldots, t_n) \in \mathcal{T}_{n,m}$, there is a path π in $G_{n,m}$ with length $|\pi|$ satisfying

$$\sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}),\ X^{(j)}(t_j)] \le |\pi| + \sum_{j=1}^{m} \Delta_n^{(j)}. \tag{4.3}$$

Fix $\boldsymbol{t} \in \mathcal{T}_{n,m}$ and use the notation

$$I_j = [Y^{(j)}(t_{j-1}),\ X^{(j)}(t_j)] = [a_j, b_j], \quad j = 1, \ldots, m.$$

Note that $a_j \ge b_j$ if there is one or no skeleton points on the segment $(t_{j-1}, t_j] \times \{j\}$ and then $L^{(j)}(I_j) = 0$.

Given two skeleton points (x, i), (y, j) we say that there is a *staircase path* from (x, i) to (y, j) if there is a sequence of skeleton points

$$(x, i) = (x_0, i),\ (x_1, i+1), \ldots,\ (x_{j-i}, j) = (y, j),$$

such that $x = x_0 \le x_1 \le \cdots \le x_{j-i} = y$. See Figure 4.1. Clearly then, there is a path from (x, i) to (y, j) which jumps upwards by one step each time it meets a new skeleton point from the sequence. We denote this by

$$(x, i) \overset{s}{\rightsquigarrow} (y, j).$$

Among all the staircase paths from (x, i) to (y, j), we will consider the *best one*, defined by two properties:

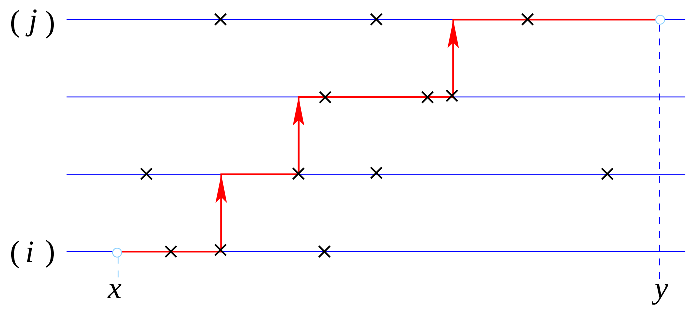

FIGURE 4.1. A staircase path from (x, i) to (y, j) jumps upwards at skeleton points (denoted by $\times$) but may skip several of them before deciding to make a jump

- Property 1: A best path from (x, i) to (y, j) jumps from line k to line $k+1$, $k = i, i+1, \dots, j-1$, at the first next skeleton point on line k, i.e. at the points x_0 and $x_{k-i+1} = Y^{(k+1)}(x_{k-i}),\ k = i, \dots, j-2$.
- Property 2: Every horizontal segment of a best path is a path of maximal length.

If all the intervals $I_1, \dots, I_m$ are empty, the left-hand side of (4.3) is zero and the inequality is trivially satisfied for any path π.

Otherwise, for a fixed $\boldsymbol{t} \in G_{n,m}$ we will construct a path π in $G_{n,m}$ for which (4.3) holds. Define a subsequence $\nu_1 < \nu_2 < \cdots$ of $1, \dots, m$, inductively, as follows:

$$\nu_1 := \inf\{1 \le j \le m : \ I_j \ne \varnothing\}, \tag{4.4}$$

$$\nu_r := \inf\{j > \nu_{r-1} : \ (b_{\nu_{r-1}}, \nu_{r-1}) \overset{s}{\rightsquigarrow} (b_j, j)\}, \quad r \ge 2. \tag{4.5}$$

See Figure 4.2 for an illustration. The procedure stops if one of the elements of the subsequence exceeds m or if the condition inside the infimum is not satisfied by a path in $G_{n,m}$.

Let J be the last index in the above defined sequence. Let π_1 be a path of maximum length from (a_{ν_1}, ν_1) to (b_{ν_1}, ν_1) and define, for $r = 2, 3, \dots, J$, a path π_r as a best staircase path from $(b_{\nu_{r-1}}, \nu_{r-1})$ to (b_{ν_r}, ν_r). Note that, for each $r = 2, 3, \dots, J,\ J \ge 2$, the end vertex of π_{r-1} is the start vertex of π_r. Therefore we can concatenate the paths $\pi_1, \dots, \pi_J$ to obtain a path π. This path starts from (a_{ν_1}, ν_1) and ends at (b_{ν_J}, ν_J).

Let

$$\pi^{(j)} := \text{ the restriction of path } \pi \text{ on line } j$$

and $|\pi^{(j)}|$ its length on line j. Also, for $j \ge \nu_1$ denote

$$(c_j, j) := \text{the first vertex on line } j \text{ of path } \pi.$$

Split the sum in the left-hand side of (4.3) along the elements of the subsequence $\{\nu_1, \dots, \nu_J\}$:

$$\sum_{j=1}^{m} L^{(j)}(I_j) = \sum_{r=1}^{J+1} \sum_{j=\nu_{r-1}+1}^{\nu_r} L^{(j)}(I_j) =: \sum_{r=1}^{J+1} G_r,$$

where we have conveniently set

$$\nu_0 := 0,\ \nu_{J+1} := m,$$

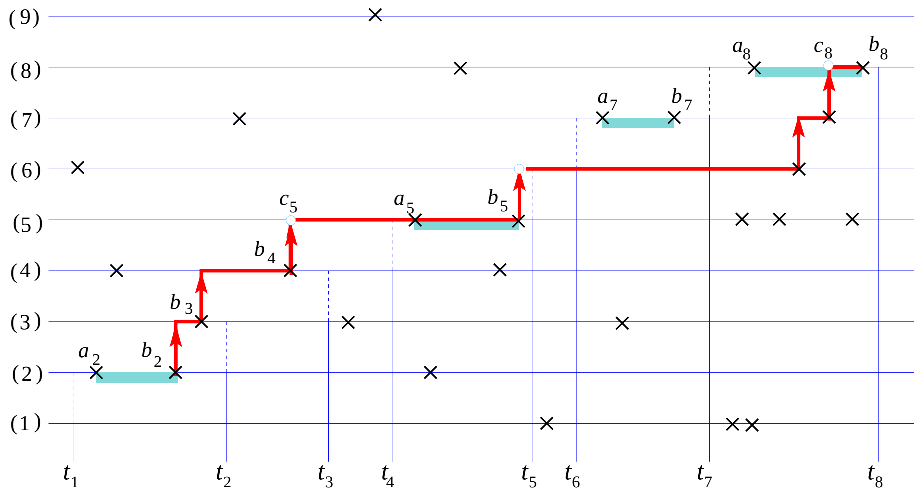

FIGURE 4.2. Illustration of the procedure defined by (4.4)-(4.5). There are four best staircase paths: the path from $(b_2, 2)$ to $(b_3, 3)$, the path from $(b_3, 3)$ to $(b_4, 4)$, the path from $(b_4, 4)$ to $(b_5, 5)$, and the path from $(b_5, 5)$ to $(b_8, 8)$. Observe that $I_j = (a_j, b_j)$, in the figure, are nonempty only for $j = 2, 5, 7$ and 8 (these are the highlighted intervals), but I_7 is not visited by the constructed path. Moreover, I_8 is only partly visited and the path enters I_8 at a point c_8 between b_8.

in order to take care of the first and last terms. By the defintion of ν_1, the intervals $I_1, I_2, \dots, I_{\nu_1 - 1}$ are empty and

$$G_1 = L^{(\nu_1)}(I_{\nu_1}) = |\pi^{(\nu_1)}|.$$

Assume now that $2 \le r \le J$, and write

$$G_r = \sum_{j=\nu_{r-1}+1}^{\nu_r} L^{(j)}(I_j) = \sum_{j=\nu_{r-1}+1}^{\nu_r - 1} L^{(j)}(I_j) + L^{(\nu_r)}(I_{\nu_r}).$$

Since π_{ν_r} is the path of maximal length from (c_{ν_r}, ν_r) to its end-vertex (b_{ν_r}, ν_r) (Property 2), if $c_{\nu_r} < a_{\nu_r}$ then $L^{(\nu_r)}(I_{\nu_r}) \le |\pi^{(\nu_r)}|$. Define in this case $I'_{\nu_r} = \emptyset$. Otherwise, we can write

$$L^{(\nu_r)}(I_{\nu_r}) \le L^{(\nu_r)}[a_{\nu_r}, Y^{(\nu_r)}(c_{\nu_r})] + L^{(\nu_r)}[Y^{(\nu_r)}(c_{\nu_r}), b_{\nu_r}].$$

Then, again because of Property 2, $L^{(\nu_r)}[Y^{(\nu_r)}(c_{\nu_r}), b_{\nu_r}] < |\pi^{(\nu_r)}|$ and it is left to find a bound on the interval $I'_{\nu_r} = [a_{\nu_r}, Y^{(\nu_r)}(c_{\nu_r})]$. Recall that by Property 1, depending whether j is a member of the sequence $\{\nu_r, r = 1, \dots, J\}$ or not, $c_{j+1} = b_j$ or $c_{j+1} = Y^{(j)}(c_j)$, respectively. Also, because of $I_j \subseteq [t_{j-1}, t_j]$, we know that $L^{(j)}(I_j) \le t_j - t_{j-1}$. Hence, if $\nu_r - \nu_{r-1} > 1$ it holds

$$\begin{aligned}\sum_{j=\nu_{r-1}+1}^{\nu_r-1} L^{(j)}(I_j) + L^{(j)}(I'_{\nu_r}) &\le \sum_{j=\nu_{r-1}+1}^{\nu_r-1} (t_j - t_{j-1}) + (Y^{(\nu_r)}(c_{\nu_r}) - t_{\nu_r-1}) \\ &\le Y^{(\nu_r)}(c_{\nu_r}) - b_{\nu_{r-1}} = \sum_{j=\nu_{r-1}+1}^{\nu_r} (Y^{(j)}(c_j) - c_j) \le \sum_{j=\nu_{r-1}+1}^{\nu_r} \Delta_n^{(j)}.\end{aligned}$$

Combining the above, we obtain

$$G_r \le \sum_{j=\nu_r+1}^{\nu_r-1} \Delta_n^{(j)} + |\pi^{(\nu_r)}|.$$

If $\nu_J = m$, then $G_{J+1} = 0$. Otherwise, we can extend the sequence $\{c_j, j = \nu_1, \nu_1+1, \dots, \nu_J\}$ defining iteratively $c_{\nu_J+1} := b_{\nu_J}$ and $c_{j+1} := Y^{(j)}(c_j)$ until $c_j > n$ for some j. Let K be the last index such that $c_K \le n$. As there was not possible to construct the best staircase path after the line ν_J, K is at most m. Similarly as above, for G_{J+1} it holds

$$\begin{aligned}G_{J+1} &= \sum_{j=\nu_J+1}^{m} L^{(j)}(I_j) \le \sum_{j=\nu_J+1}^{m} (t_j - t_{j-1}) \le n - b_{\nu_J} \\ &= \sum_{j=\nu_J+1}^{K} (Y^{(j)}(c_j) - c_j) \le \sum_{j=\nu_J+1}^{K} \Delta_n^{(j)}.\end{aligned}$$

Finally, we obtain

$$\sum_{j=1}^{m} L^{(j)}(I_j) \le \sum_{j=1}^{m} \Delta_n^{(j)} + \sum_{r=1}^{J} |\pi^{(\nu_r)}| \le \sum_{j=1}^{m} \Delta_n^{(j)} + |\pi|,$$

as required. □

5. Further estimates in probability and Brownian directed percolation

In the present section we prove Theorem 3.1 as a sequence of lemmas.

5.1. *Asymptotic coincidence of the two bounds.* Looking at (1.3), we can see that the correct scaling requires exponent

$$b := \frac{1}{2} - \frac{a}{6}$$

in the denominator and condition $a < 3/7$, which is equivalent to $a < b$.

In the following two lemmas we will not specifically use the definiton of b and condition on a. Both lemmas hold for more general $a, b > 0$, $0 < b - a < 1$.

Lemma 5.1. *With $b = (1/2) - (a/6)$ and $a < 3/7$,*

$$\frac{\overline{L}_{n,[n^a]} - \underline{L}_{n,[n^a]}}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0.$$

Proof: Let $\boldsymbol{t}$ be such that the maximum in the right-hand side of (4.2) is achieved. Then

$$\begin{aligned}
\overline{L}_{n,m} - \underline{L}_{n,m} &\le m + \sum_{j=1}^{m} \Delta_n^{(j)} \\
&\quad + \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_j)] - \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_j)] \\
&\le m + \sum_{j=1}^{m} \Delta_n^{(j)} \\
&\quad + \sum_{j=1}^{m} \left\{ L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_{j-1})] + L^{(j)}[X^{(j)}(t_j), Y^{(j)}(t_j)] \right\} \\
&\le m + \sum_{j=1}^{m} \Delta_n^{(j)} + 2 \sum_{j=1}^{m} \max_{0 \le i \le \Phi^{(j)}(n)} L^{(j)}[\Gamma_i^{(j)}, \Gamma_{i+1}^{(j)}] \\
&\le m + 3 \sum_{j=1}^{m} \Delta_n^{(j)}.
\end{aligned}$$

Hence

$$\frac{1}{n^b} E[\overline{L}_{n,[n^a]} - \underline{L}_{n,[n^a]}] \le \frac{n^a}{n^b} + 3 \frac{n^a}{n^b} E[\Delta_n^{(1)}].$$

Since $b > a$ and the random variables $\{\Gamma_{i+1}^{(1)} - \Gamma_i^{(1)}, i \ge 1\}$ have a finite $1/(b-a)$-th moment, the converegence to 0 for the second term above follows by Lemma A.1. □

5.2. *Centering.* We introduce the quantity

$$S_{n,m} := \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^{m} \left\{ L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - C[X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right\}.$$

This should be "comparable" to $L_{m,n} - Cn$ when $m = [n^a]$. Indeed, we have:

Lemma 5.2. *With $b = (1/2) - (a/6)$, and $a < 3/7$,*

$$\frac{S_{n,[n^a]} - (L_{n,[n^a]} - Cn)}{n^b} \xrightarrow[n \to \infty]{(\mathrm{p})} 0.$$

Proof: We begin by rewriting the numerator above as

$$\begin{aligned}
&S_{n,m} - (L_{n,m} - Cn) \\
&= \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \left\{ \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] + Cn - C \sum_{j=1}^{m} [X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right\} - L_{n,m}.
\end{aligned}$$

Upon writing $n = \sum_{j=1}^{m} (t_j - t_{j-1})$, for any $\boldsymbol{t} \in \mathcal{T}_{n,m}$, we have

$$\left| n - \sum_{j=1}^{m} [X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right| = \left| \sum_{j=1}^{m} [t_j - X^{(j)}(t_j)] - \sum_{j=1}^{m} [t_{j-1} - X^{(j)}(t_{j-1})] \right| \le 2 \sum_{j=1}^{m} \Delta_n^{(j)}.$$

Hence, on the one hand we have

$$\begin{aligned} S_{n,m} - (L_{n,m} - Cn) &\le \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^m L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} + 2C \sum_{j=1}^m \Delta_n^{(j)} \\ &\le \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^m L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_j)] - L_{n,m} + 2C \sum_{j=1}^m \Delta_n^{(j)} \\ &\le \overline{L}_{n,m} - \underline{L}_{n,m} + 2C \sum_{j=1}^m \Delta_n^{(j)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} S_{n,m} - (L_{n,m} - Cn) &\ge \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^m L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} - 2C \sum_{j=1}^m \Delta_n^{(j)} \\ &\ge \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^m L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} - 2C \sum_{j=1}^m \Delta_n^{(j)} \\ &\ge \underline{L}_{n,m} - \overline{L}_{n,m} - 2C \sum_{j=1}^m \Delta_n^{(j)} = -(\overline{L}_{n,m} - \underline{L}_{n,m}) - 2C \sum_{j=1}^m \Delta_n^{(j)}. \end{aligned}$$

Therefore,

$$|S_{n,m} - (L_{n,m} - Cn)| \le \overline{L}_{n,m} - \underline{L}_{n,m} + 2C \sum_{j=1}^m \Delta_n^{(j)}.$$

Thus, for $m = [n^a]$, the result follows by applying Lemma A.1 and Lemma 5.1. $\square$

Define now variance σ^2 as

$$\sigma^2 := \mathrm{var}(L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)}))$$

and observe that $\sigma^2 = \sigma_0^2/p$. We work with the quantity $\frac{1}{\sigma} S_{n,m}$, which can be rewritten as

$$\frac{1}{\sigma} S_{n,m} = \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^m \sum_{k=\Phi^{(j)}(t_{j-1})+1}^{\Phi^{(j)}(t_j)} \chi_k^{(j)},$$

where

$$\chi_k^{(j)} := \frac{1}{\sigma} \big\{ L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)}) \big\}.$$

Note that the random variables $\{\chi_k^{(j)}\}_{k \ge 1, j \ge 1}$, indexed by both k and j, are independent and that $\{\chi_k^{(j)}\}_{k \ge 2, j \ge 1}$ are identically distributed with zero mean and unit variance. The fact that the $\{\chi_1^{(j)}\}_{j \ge 1}$ do not have the same distribution will not affect the result, so we will not separately take care of it.

5.3. *Coupling with Brownian motion.* The term $\frac{1}{\sigma} S_{n,m}$ resembles a centered last passage percolation path weight, except that random indices are involved. Therefore, we start using the idea of strong coupling with Brownian motions, analogously to the proof in Bodineau and Martin (2005). Let $B^{(1)}, B^{(2)}, \dots$ be i.i.d. standard Brownian motions, and recall that

$$Z_{n,m} := \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^{m} [B^{(j)}_{t_j} - B^{(j)}_{t_{j-1}}]. \tag{5.1}$$

Define the random walks $R^{(1)}, R^{(2)}, \dots$ by

$$R^{(j)}_i = \sum_{k=1}^{i} \chi^{(j)}_k, \quad i = 0, 1, 2, \dots, \tag{5.2}$$

with $R^{(j)}_0 = 0$. With this notation, we have

$$\frac{1}{\sigma} S_{n,m} = \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^{m} [R^{(j)}_{\Phi^{(j)}(t_j)} - R^{(j)}_{\Phi^{(j)}(t_{j-1})}]. \tag{5.3}$$

Taking into account (1.2) and Lemma 5.2, it is evident that to prove Theorem 3.1 it remains to show that

$$\frac{\sigma^{-1} S_{n,[n^a]} - \sqrt{\lambda} Z_{n,[n^a]}}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0,$$

or, using the scaling property of Brownian motion, that:

Lemma 5.3. *For all $a < 3/14$,*

$$\frac{\sigma^{-1} S_{n,[n^a]} - Z_{\lambda n,[n^a]}}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0.$$

To show that the random walks are close enough to the Brownian motion we use the following version of the Komlós-Major-Tusnády strong approximation result (Komlós et al., 1976, Thm. 4):

Theorem 5.4. *For any $0 < r < 1$, $n \in \mathbb{Z}_+$ and $x \in \left[c_1(\log n)^{1/r}, c_2(n \log n)^{1/2}\right]$, starting with a probability space supporting independent Brownian motions $B^{(j)}$, $j = 1, 2, \dots$, we can jointly construct i.i.d. sequences $\chi^{(j)} = (\chi^{(j)}_1, \chi^{(j)}_2, \dots)$, $j = 1, 2, \dots$, with the correct distributions and, moreover, such that, with $R^{(j)}_i$ as in (5.2) above,*

$$P(\max_{1\le i\le n} |B^{(j)}_i - R^{(j)}_i| > x) \le Cn \exp\{-\alpha x^r\} \text{ for all } j = 1, 2, \dots,$$

where the constants C, c_1, c_2 are depending on α, r and distributions of $\chi^{(1)}_1$ and $\chi^{(1)}_2$.

In addition to this, in order to take care of the random indices appearing in (5.3), we need a convergence rate result for the counting processes $\{\Phi^{(j)}, j \ge 1\}$ which is proven in the appendix.

Proof of Lemma 5.3: From (5.3) and (5.1) we have

$$
\begin{aligned}
|\sigma^{-1}S_{n,m} - Z_{\lambda n,m}| &\le \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} \Big\{ \big|(R^{(j)}_{\Phi^{(j)}(t_j)} - R^{(j)}_{\Phi^{(j)}(t_{j-1})}) - (B^{(j)}_{\lambda t_j} - B^{(j)}_{\lambda t_{j-1}})\big| \Big\} \\
&= \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} \Big\{ \big|R^{(j)}_{\Phi^{(j)}(t_j)} - B^{(j)}_{\Phi^{(j)}(t_j)}\big| + \big|R^{(j)}_{\Phi^{(j)}(t_{j-1})} - B^{(j)}_{\Phi^{(j)}(t_{j-1})}\big| \\
&\qquad + \big|B^{(j)}_{\Phi^{(j)}(t_j)} - B^{(j)}_{\lambda t_j}\big| + \big|B^{(j)}_{\Phi^{(j)}(t_{j-1})} - B^{(j)}_{\lambda t_{j-1}}\big| \Big\} \\
&\le 2\sum_{j=1}^{m} \Big\{ \max_{0\le i\le n} \big|R^{(j)}_i - B^{(j)}_i\big| + \sup_{0\le s\le n} \big|B^{(j)}_{\Phi^{(j)}(s)} - B^{(j)}_{\lambda s}\big| \Big\} \\
&=: 2\sum_{j=1}^{m} U^{(j)}_n + 2\sum_{j=1}^{m} V^{(j)}_n,
\end{aligned}
$$

where

$$
U^{(j)}_n := \max_{0\le i\le n} \big|R^{(j)}_i - B^{(j)}_i\big|, \qquad V^{(j)}_n := \sup_{0\le s\le n} \big|B^{(j)}_{\Phi^{(j)}(s)} - B^{(j)}_{\lambda s}\big|.
$$

Therefore, it is enough to show that,

$$
\frac{1}{n^b}\sum_{j=1}^{[n^a]} U^{(j)}_n \xrightarrow[n\to\infty]{(\mathrm{p})} 0 \quad \text{and} \quad \frac{1}{n^b}\sum_{j=1}^{[n^a]} V^{(j)}_n \xrightarrow[n\to\infty]{(\mathrm{p})} 0.
$$

For the first convergence we will take into account the coupling estimate as in Theorem 5.4. That is, we will throughout assume that the random walks and Brownian motions have been constructed *jointly*. The second convergence will be established without this estimate, i.e., we will show that it is true, *regardless* of the joint construction of the Brownian motions and the random walks. This is because *the coupling we use is not detailed enough to give us information about the joint distribution of $B^{(j)}$ and the counting process* $\Phi^{(j)}$ (the latter is not a function of the random walks used in the coupling).

Proof of the first convergence. Let $\delta > 0$. We need to show that

$$
P\Big(\sum_{j=1}^{[n^a]} U^{(j)}_n > \delta n^b\Big) \to 0 \quad \text{as } n\to\infty.
$$

Let $\varepsilon < b - a$. Then

$$
P\Big(\sum_{j=1}^{[n^a]} U^{(j)}_n > \delta n^b\Big) \le P\Big(\max_{1\le j\le [n^a]} U^{(j)}_n \le n^\varepsilon, \sum_{j=1}^{[n^a]} U^{(j)}_n > \delta n^b\Big) + P\Big(\max_{1\le j\le [n^a]} U^{(j)}_n > n^\varepsilon\Big). \tag{5.4}
$$

The first term from right-hand side is zero for large n. We are allowed, for n large enough, to estimate the second term using Theorem 5.4 for an arbitrary $r \in (0,1)$ and $x = n^\varepsilon$ as

$$
P\Big(\max_{1\le j\le [n^a]} U^{(j)}_n > n^\varepsilon\Big) \le n^a P\Big(\max_{1\le i\le n} |B^{(1)}_i - R^{(1)}_i| > n^\varepsilon\Big) \le n^a C n \exp\{-\alpha n^{\varepsilon r}\} \to 0,
$$

as $n\to\infty$.

Proof of the second convergence. Let $1/4 < \varepsilon < b - a$. Replacing $U^{(j)}_n$ by $V^{(j)}_n$ in (5.4), we see that the first term is again zero for large n and it remains to show that second term converge to 0, i.e., it is enough to show the convergence for its upper-bound

$$n^a P\big(V_n^{(1)} > n^\varepsilon\big) \to 0,$$

as $n \to \infty$. Let $\gamma > 1$ and $1/2 < q < 2\varepsilon$. Then we can write

$$\begin{aligned} n^a P\big(V_n^{(1)} > n^\varepsilon\big) \le n^a P\big(&\sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| > \gamma n^q\big) \\ &+ n^a P\big(\sup_{0\le s\le n} |B^{(1)}_{\Phi^{(1)}(s)} - B^{(1)}_{\lambda s}| > n^\varepsilon,\ \sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| \le \gamma n^q\big). \end{aligned}$$

Corollary A.1 implies the convergence of the first term above

$$n^a P\big(\sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| > \gamma n^q\big) \le n^a \exp\{-\alpha(2n)^{qr}\} \to 0,$$

as $n \to \infty$, where $0 < r < 2 - 1/q$. Set $\varphi = \gamma n^q/\lambda$. For the second term using the fact that, under our condition, $\lambda s - \gamma n^q \le \Phi^{(1)}(s) \le \lambda s + \gamma n^q$ for $s \in [0, 2n]$, we have

$$\begin{aligned} & n^a P\big(\sup_{0\le s\le n} |B^{(1)}_{\Phi^{(1)}(s)} - B^{(1)}_{\lambda s}| > n^\varepsilon,\ \sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| \le \gamma n^q\big) \\ &\le n^a P\big(\max_{0\le k\le \lfloor n/\varphi\rfloor} \sup_{0\le s\le\varphi} |B^{(1)}_{\Phi^{(1)}(k\varphi+s)} - B^{(1)}_{\lambda(k\varphi+s)}| > n^\varepsilon, \\ &\qquad\qquad\qquad\qquad \sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| \le \gamma n^q\big) \\ &\le n^a P\big(\max_{0\le k\le \lfloor n/\varphi\rfloor} \sup_{\substack{0\le s\le\varphi \\ -\gamma n^q \le t\le \gamma n^q}} |B^{(1)}_{\lambda(k\varphi+s)+t} - B^{(1)}_{\lambda(k\varphi+s)}| > n^\varepsilon\big) \\ &\le n^a (n/\varphi + 1) P\big(\sup_{0\le s\le t\le 3\gamma n^q} |B_t - B_s| > n^\varepsilon\big) \\ &\le n^a (n/\varphi + 1) P\big(\sup_{0\le s\le 3\gamma n^q} |B_s| > n^\varepsilon/2\big) \\ &\le 4n^a (n/\varphi + 1) P(B_{3\gamma n^q} > n^\varepsilon/2) \qquad\qquad \text{(a)} \\ &\le 4n^a (n/\varphi + 1) \exp\{-n^{2\varepsilon}/(24\gamma n^q)\} \to 0, \qquad\qquad \text{(b)} \end{aligned}$$

as $n \to \infty$. The inequality (a) is the consequence of $P(\sup_{0<s<t} B_s > x) = 2P(B_t > x)$ and the inequality (b) is an estimate for the tail of the normal distribution. □

Remark 5.5. The condition $a < 3/14$ is equivalent to $b - a > 1/4$ and, thus, it ensures existence of an $\varepsilon > 1/4$ and later existance of $1/2 < q < 2\varepsilon$. Therefore, it is necessary for application of Lemma A.1, as well as for the convergence of the last estimate above.

6. Graph with non-constant edge probabilities

In Denisov et al. (2012), the authors consider a one-dimensional model with connectivity probabilities depending on the distance between points, i.e. there is an edge between the vertices i and j with probability $p_{|i-j|}$. To prove a central limit theorem for the maximal path length in that graph, two conditions are introduced:

- $0 < p_1 < 1$;
- $\sum_{k=1}^{\infty} k(1-p_1)\cdots(1-p_k) < \infty$.

The conditions guarantee the existence of skeleton points and finite variance, defined as in (2.3).

We can extend the idea of the graph on $\mathbb{Z}\times\mathbb{Z}$ keeping, whenever possible, the notation from the $\mathbb{Z}\times\mathbb{Z}$ graph with constant probability p.

Let $\{p_{i,j}, i, j \leq 0\}$ be a sequence of probabilities that satisfies the following:

- $0 < p_{1,0} < 1$;
- $0 < p_{0,1} < 1$;
- $\sum_{k=1}^{\infty} k^{r-1}(1-p_{1,0})(1-p_{2,0})\cdots(1-p_{k,0}) < \infty$ for some $r > 2$.

Then the vertices (i_1, i_2) and (j_1, j_2), $(i_1, i_2) \prec (j_1, j_2)$, are connected with probability $p_{j_1-i_1,j_2-i_2}$.

The conditions above are needed to ensure the existence of skeleton points, as in Definition 4.1, and a finite rth moment of the random variables $\chi_2, \chi_3, \ldots$ for some $r > 2$.

From now on, let r denote the order of the highest finite moment, i.e.

$$r = \sup\{q > 0 : \sum_{k=1}^{\infty} k^{q-1}(1-p_{1,0})(1-p_{2,0})\cdots(1-p_{k,0}) < \infty\}.$$

Now we can state the analogue of Theorem 3.1 for this more general setting:

Theorem 6.1. *For all* $a < \min(3/14, (r-2)/(3r/7+1))$,

$$n^{a/6}\left(\frac{L_{n,[n^a]} - Cn}{\sqrt{\lambda\sigma^2}\sqrt{n}} - 2\sqrt{n^a}\right) \xrightarrow[n\to\infty]{\text{(d)}} F_{TW}.$$

Proof: We follow the lines of the proof of Theorem 3.1, emphasizing the points one should be careful about or which need to be modified.

The construction of the upper and lower bounds for $L_{n,[n^a]}$ is the same as in Section 4. In Lemma 5.1 and Lemma 5.2, the term $n^a E[\Delta_1]/n^b$ converges to 0 if $a + 1/r < b$, which leads to the constraint $a < 6/7(1/2 - 1/r)$.

Next, we rewrite, as in Lemma 5.3,

$$|\sigma^{-1} S_{n,m} - Z_{\lambda n,m}| \leq 2\sum_{j=1}^{m} U_n^{(j)} + 2\sum_{j=1}^{m} V_n^{(j)}$$

and prove convergence of each term separately.

Proof of the first convergence. Instead of Theorem 5.4, we use the combination of results from Komlós et al. (1976) and Major (1976), as stated in Proposition 2 of Bodineau and Martin (2005), which allow us to couple Brownian motions $B^{(j)}$, $j = 1, 2, \ldots$ and random walks $R^{(j)}$, $j = 1, 2, \ldots$ so that

$$P\big(\max_{1\leq i\leq n} |B_i^{(j)} - R_i^{(j)}| > x\big) \leq Cnx^{-r}, \text{for all } n \in \mathbb{Z}_+, \text{ and all } x \in [n^{1/r}, n^{1/2}]. \tag{6.1}$$

Let $\varepsilon = 1/2$. We want to establish convergence of the same terms as in (5.4). Applying, on the first term, the Markov inequality and properties of the coupling with Brownian motion (6.1) yield:

$$P\big(\max_{1\le j\le [n^a]} U_n^{(j)} \le n^{1/2}, \sum_{j=1}^{[n^a]} U_n^{(j)} > \delta n^b\big) \le \frac{n^a}{\delta n^b} E\big[U_n^{(1)} \mathbf{1}(U_n^{(1)} \le n^{1/2})\big]$$

$$\le \frac{n^a}{\delta n^b}\left(n^{1/r} + \int_{n^{1/r}}^{n^{1/2}} P(U_n^{(1)} > x)dx\right) \le \frac{n^a}{\delta n^b}\left(n^{1/r} + \int_{n^{1/r}}^{n^{1/2}} Cnx^{-r}dx\right)$$

$$= \frac{n^a}{\delta n^b}\left(n^{1/r} - \frac{C}{r-1}n^{3/2-r/2} + \frac{C}{r-1}n^{1/r}\right) \le \left(1+\frac{C}{r-1}\right)\frac{n^a n^{1/r}}{\delta n^b} \to 0$$

as $n \to \infty$. For the second term we use again the coupling properties (6.1):

$$P\big(\max_{1\le j\le [n^a]} U_n^{(j)} > n^{1/2}\big) \le n^a P\big(\max_{1\le i\le n} |B_i^{(1)} - R_i^{(1)}| > n^{\varepsilon}\big) \le n^a Cnn^{-r/2} \to 0,$$

as $n \to \infty$ because $a + 1 - r/2 < 0$ due to our new constraint on a.

Proof of the second convergence. Here, the only change is the replacement of Lemma A.2 by part of Theorem 6.12.1 in Gut (2013), in our notation:

If $1/2 \le q < 1$, then for all $\varepsilon > 0$ it holds that $n^{qr-2}P(\max_{0\le k\le n}|\Gamma_k - \lambda k| > \varepsilon n^q) \to 0$ as $n \to \infty$.

One can, in the same fashion as before, prove the following analogue of Corollary A.1,

$$n^{qr-2}P\big(\sup_{0\le t\le n} |\Phi(t) - \lambda t| > \varepsilon n^q\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Likewise in Remark 5.5, condition $a < 3/14$ is necessary. Another constraint that occurs is $a \le qr - 2$ and it can be shown that this is satisfied if $a \le (r-2)/(3r/7+1)$. □

Appendix A.

Lemma A.1. Let $r \ge 1$ and $\{X_i, i \ge 1\}$ be a sequence of non-negative i.i.d. random variables such that $E|X_1|^r < \infty$. Then,

$$\frac{1}{n^{\frac{1}{r}}} E\left[\max_{1\le i\le n} X_i\right] \to 0 \text{ as } n \to \infty.$$

Proof: It suffices to prove the lemma for $r = 1$, since then the result for $r > 1$ follows easily by Jensen's inequality. Let $M_n = \max_{1\le i\le n} X_i$ and $S_n = X_1 + X_2 + \cdots + X_n$. Borel-Cantelli lemma, together with the assumption $EX < \infty$, give $X_n/n \to 0$, almost surely, as $n \to \infty$. An easy computation shows that then also $M_n/n \to 0$, almost surely, as $n \to \infty$.

On the other hand, we know that $M_n \le S_n$ for all n and that $\{S_n/n, n \ge 1\}$ is uniformly integrable. Therefore, M_n/n is also unformly integrable and $EM_n/n \to 0$ as $n \to \infty$ (Gut, 2013, Thm. 5.4.5, Thm. 5.5.2). □

The following lemma is an analogue of a result by Lanzinger (1998, Prop. 2), with slight improvement in the probability bound. Moreover, we specialize the lemma to our case.

Lemma A.2. Let $1/2 < q \le 1$ and $r < 2 - 1/q$. Suppose that $X, X_1, X_2, \dots$ are i.i.d. random variables such that $EX = \mu$ and $Ee^{\alpha|X|} < \infty$ for some $\alpha > 0$, and set $S_n = \sum_{i=1}^n X_k$. Then, for all $\varepsilon > 1$,

$$\exp\{\alpha n^{qr}\}P\big(\max_{1\le k\le n} |S_k - k\mu| > \varepsilon n^q\big) \to 0$$

as $n \to \infty$.

Proof: Without loss of generality we can assume that $\mu = 0$. We first show that, for all $\varepsilon > 1$, $\exp\{\alpha n^{qr}\}P(S_n > \varepsilon n^q) \to 0$ as $n \to \infty$.

For fixed $n \in \mathbb{Z}_+$, define $X'_k = X_k \mathbf{1}(X_k \le \varepsilon n^q)$ and set $S'_n = \sum_{k=1}^n X'_k$. Note that $EX' \le 0$ and that for α', where $\alpha' < \alpha$ and $\alpha' \varepsilon^r > \alpha$, it holds that $EX^2 e^{\alpha'|X|^r} < \infty$. We can write

$$P(S_n > \varepsilon n^q) \le P(S'_n > \varepsilon n^q) + nP(X > \varepsilon n^q). \tag{A.1}$$

We first find a bound for the moment generating function for X' in $\alpha'(\varepsilon n^q)^{r-1}$:

$$\begin{aligned} Ee^{\alpha'(\varepsilon n^q)^{r-1}X'} &\le 1 + \frac{1}{2}\alpha'^2(\varepsilon n^q)^{2(r-1)}(EX'^2 + EX'^2 e^{\alpha'(\varepsilon n^q)^{r-1}X'}\mathbf{1}\{X' > 0\}) \\ &\le 1 + \frac{1}{2}\alpha'^2(\varepsilon n^q)^{2(r-1)}(EX^2 + EX^2 e^{\alpha'|X|^r}) \\ &\le 1 + Cn^{2q(r-1)} \le \exp\{Cn^{-2q(1-r)}\}, \end{aligned}$$

where for the first inequality we used $e^y \le 1 + \max\{1, e^y\}y^2/2$ and for the last one $1 + y \le e^y$. Now, using Markov's inequality and the bound above for the first term in (A.1) yields

$$\begin{aligned} P(S'_n > \varepsilon n^q) = P(e^{\alpha'(\varepsilon n^q)^{r-1}S'_n} > e^{\alpha'(\varepsilon n^q)^r}) &\le e^{-\alpha'(\varepsilon n^q)^r}\left(Ee^{\alpha'(\varepsilon n^q)^{r-1}X'}\right)^n \\ &\le \exp\{-\alpha'(\varepsilon n^q)^r + Cnn^{-2q(1-r)}\}. \end{aligned}$$

For the second term in (A.1), again using Markov's inequality, we have

$$nP(X > \varepsilon n^q) = nP(e^{\alpha X^r} > e^{\alpha(\varepsilon n^q)^r}) \le ne^{-\alpha(\varepsilon n^q)^r}E(e^{\alpha|X|^r}).$$

Combining the two estimates finally establishes that

$$\begin{aligned} \exp\{\alpha n^{qr}\}P(S_n > \varepsilon n^q) &\le \exp\{-n^{qr}(\alpha'\varepsilon^r - \alpha - Cn^{1-2q+qr})\} \\ &\quad + n\exp\{-n^{qr}\alpha(\varepsilon^r - 1)\}Ee^{\alpha|X|^r} \to 0 \end{aligned}$$

as $n \to \infty$ because of the choice of r, α' and ε.

By symmetry, the same convergence rate holds also for $P(S_n < -\varepsilon n^q)$.

Thus, the statement of the theorem follows using Lévy inequality (Gut, 2013, Thm. 3.7.2) and the fact that for n large enough we can find ε' such that $1 < \varepsilon' < \varepsilon - \sqrt{2\sigma^2}n^{1/2-q}$:

$$\begin{aligned} P(\max_{1\le k\le n}|S_k| > \varepsilon n^q) \le 2P(|S_n| > \varepsilon n^q - \sqrt{2n\sigma^2}) &= 2P(|S_n| > n^q(\varepsilon - \sqrt{2\sigma^2}n^{1/2-q})) \\ &\le 2P(|S_n| > \varepsilon' n^q). \end{aligned}$$

□

Corollary A.1. Let $X, X_1, X_2, \ldots$ be positive, integer-valued i.i.d. random variables and suppose that the assumptions of Lemma A.2 are satisfied. Then, for the counting process Φ, where $\Phi(t) = \max\{n : S_n \le t\}$, it holds that

$$\exp\{\alpha n^{qr}\}P(\sup_{0\le t\le n}|\mu\Phi(t) - t| > \varepsilon n^q) \to 0 \text{ for all } \varepsilon > 1.$$

Proof: For the counting process we have $t - X_{\Phi(t)+1} \le S_{\Phi(t)} \le t$. Therefore, we can write

$$
\begin{aligned}
&\{\sup_{0\le t\le n} |\mu\Phi(t) - t| > \varepsilon n^q\} = \\
&= \{\sup_{0\le t\le n} (\mu\Phi(t) - t) > \varepsilon n^q\} \cup \{\inf_{0\le t\le n} (\mu\Phi(t) - t) < -\varepsilon n^q\} \\
&\subset \{\sup_{0\le t\le n} (\mu\Phi(t) - S_{\Phi(t)}) > \varepsilon n^q\} \cup \{\inf_{0\le t\le n} (\mu\Phi(t) - S_{\Phi(t)} - X_{\Phi(t)+1}) < -\varepsilon n^q\} \\
&\subset \{\sup_{0\le t\le n} (\mu\Phi(t) - S_{\Phi(t)}) > \varepsilon n^q\} \cup \{\inf_{0\le t\le n} (\mu\Phi(t) - S_{\Phi(t)}) < -\varepsilon' n^q\} \\
&\quad \cup \{\sup_{0\le t\le n} X_{\Phi(t)+1} > (\varepsilon - \varepsilon')n^q\} \\
&\subset \{\max_{1\le k\le n} |S_k - k\mu| > \varepsilon n^q\} \cup \{\max_{1\le k\le n} |S_k - k\mu| > \varepsilon' n^q\} \\
&\quad \cup \{\max_{1\le k\le n+1} X_k > (\varepsilon - \varepsilon')n^q\},
\end{aligned}
$$

where $1 < \varepsilon' < \varepsilon$. Thus,

$$
\begin{aligned}
&\exp\{\alpha n^{qr}\} P(\sup_{0\le t\le n} |\mu\Phi(t) - t| > \varepsilon n^q) \le \exp\{\alpha n^{qr}\} P(\max_{1\le k\le n} |S_k - k\mu| > \varepsilon n^q) \\
&+ \exp\{\alpha n^{qr}\} P(\max_{1\le k\le n} |S_k - k\mu| > \varepsilon' n^q) + (n+1)\exp\{\alpha n^{qr}\} P(X > (\varepsilon - \varepsilon')n^q).
\end{aligned}
$$

The first and the second term above converge to 0 as $n \to 0$ by Lemma A.2. We prove convergence to 0 for the third term using Markov's inequality,

$$(n+1)e^{\alpha n^{qr}} P(X > (\varepsilon - \varepsilon')n^q) \le (n+1)e^{\alpha(n^{qr} - (\varepsilon-\varepsilon')n^q)} E e^{\alpha X} \to 0 \text{ as } n \to \infty. \qquad \square$$

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