EXPANDING THURSTON MAPS AS QUOTIENTS

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Abstract. A *Thurston map* is a branched covering map $f\colon S^2 \to S^2$ that is *postcritically finite*. *Mating of polynomials*, introduced by Douady and Hubbard, is a method to *geometrically* combine the Julia sets of two polynomials (and their dynamics) to form a rational map. We show that for every *expanding* Thurston map f every sufficiently high iterate $F = f^n$ is obtained as the mating of two polynomials. One obtains a concise description of F via *critical portraits*. The proof is based on the construction of the invariant Peano curve from [Meyb]. As another consequence we obtain a large number of fractal tilings of the plane and the hyperbolic plane.

Contents

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1. Introduction

Douady and Hubbard observed in the early 80's (see [Dou83]) that the Julia set of certain rational maps "contains" the Julia sets of some polynomials. This prompted them to introduce the notion of *mating of polynomials*. This is a method to geometrically combine the Julia sets of two polynomials and the dynamics defined on them. Somewhat surprisingly this operation "often" (though certainly not always) results in a rational map. The dynamics of such a rational map may then be described in terms of the dynamics of polynomials (which is much better understood). The main result of the present paper is that for any postcritically finite rational map f whose Julia set is the whole Riemann sphere, every sufficiently high iterate $F = f^n$ is obtained as a mating.

An excellent introduction to matings can be found in [Mil04]. There are several different ways to define matings (not all of them equivalent). An overview is given in [MP]. We focus here on the *topological mating*, the definition of which is given below. This is possibly the most commonly used notion of mating.

1.1. The Carathéodory semi-conjugacy of a polynomial Julia set.

Let P be a monic polynomial (i.e., the coefficient of the leading term is 1) of degree $d \geq 2$

with connected and locally connected filled Julia set $\mathcal{K}$. *Böttcher's theorem* (see for example [Mil06, §9] or [CG93, II.4]) asserts that P is conformally conjugate to z^n in a neighborhood of ∞, i.e., there is a conformal map $\phi\colon U \to V$, where U, V are neighborhoods of ∞, such that $\phi(\infty) = \infty$ and

$$\phi(z^d) = P(\phi(z)), \tag{1.1}$$

for all $z \in U$. We may choose $\phi'(\infty) := \lim_{z\to\infty} z/\phi(z) > 0$ (in fact then $\phi'(\infty) = 1$). This makes ϕ unique. The conjugacy ϕ may be extended conformally to the whole domain of attraction of ∞ (i.e., to the complement of the filled Julia set $\mathcal{K}$ of P). The extended map $\overline{\phi}\colon \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \mathcal{K}$ then is the Riemann map of the simply connected domain $\widehat{\mathbb{C}} \setminus \mathcal{K}$.

Since the Julia set $\mathcal{J} = \partial\mathcal{K}$ of P is assumed to be locally connected it follows from Carathéodory's theorem (see for example [Mil99, Theorem 17.14]) that $\overline{\phi}$ extends continuously to

$$\sigma\colon S^1 = \partial\overline{\mathbb{D}} \to \partial\mathcal{K} = \mathcal{J}, \tag{1.2}$$

where $\mathcal{J}$ is the *Julia set* of P. Since σ is the extension of the conjugacy $\overline{\phi}$ it follows that $\sigma(z^d) = P(\sigma(z))$ for all $z \in S^1$, i.e., the following diagram commutes

$$\begin{CD} S^1 @>z^d>> S^1 \\ @V\sigma VV @VV\sigma V \\ \mathcal{J} @>P>> \mathcal{J}. \end{CD} \tag{1.3}$$

Note however that σ will not be injective in general. We call the map σ the *Carathéodory semi-conjugacy* of $\mathcal{J}$.

We remind the reader that every postcritically finite polynomial has connected and locally connected filled Julia set (see for example [Mil99, Theorem 19.7]). Thus the description above holds in this case.

1.2. **Mating of polynomials.** Consider two monic polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ (called the *white* and the *black* polynomial) of the same degree $d \geq 2$ with connected and locally connected Julia sets. Let $\sigma_{\mathtt{w}}, \sigma_{\mathtt{b}}$ be the Carathéodory semi-conjugacies of their Julia sets $\mathcal{J}_{\mathtt{w}}, \mathcal{J}_{\mathtt{b}}$.

Glue the filled Julia sets $\mathcal{K}_{\mathtt{w}}, \mathcal{K}_{\mathtt{b}}$ (of $P_{\mathtt{w}}, P_{\mathtt{b}}$) together by identifying $\sigma_{\mathtt{w}}(z) \in \partial\mathcal{K}_{\mathtt{w}}$ with $\sigma_{\mathtt{b}}(\bar{z}) \in \partial\mathcal{K}_{\mathtt{b}}$. More precisely, we consider the disjoint union of $\mathcal{K}_{\mathtt{w}}, \mathcal{K}_{\mathtt{b}}$, and let $\mathcal{K}_{\mathtt{w}} \perp\!\!\!\perp \mathcal{K}_{\mathtt{b}}$ be the quotient obtained from the equivalence relation generated by $\sigma_{\mathtt{w}}(z) \sim \sigma_{\mathtt{b}}(\bar{z})$ for all $z \in S^1 = \partial\mathbb{D}$. The complex conjugation $\bar{z}$ is customary here, though not essential: identifying $\sigma_{\mathtt{w}}(z)$ with $\sigma_{\mathtt{b}}(z)$ amounts to the mating of $P_{\mathtt{w}}$ with $\overline{P_{\mathtt{b}}(\bar{z})}$. The *topological mating of* $P_{\mathtt{w}}, P_{\mathtt{b}}$ is the map

$$P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}\colon \mathcal{K}_{\mathtt{w}} \perp\!\!\!\perp \mathcal{K}_{\mathtt{b}} \to \mathcal{K}_{\mathtt{w}} \perp\!\!\!\perp \mathcal{K}_{\mathtt{b}},$$

given by

$$P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}|_{\mathcal{K}_i} = P_i,$$

for $i = \mathtt{w}, \mathtt{b}$. It follows from (1.3) that $x_{\mathtt{w}} \sim x_{\mathtt{b}} \Rightarrow P_{\mathtt{w}}(x_{\mathtt{w}}) \sim P_{\mathtt{b}}(x_{\mathtt{b}})$ (for all $x_{\mathtt{w}} \in \mathcal{K}_{\mathtt{w}}, x_{\mathtt{b}} \in \mathcal{K}_{\mathtt{b}}$). This shows that the map $P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}$ is well defined. If a map is topologically conjugate to a map $P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}$ we say it is obtained as a (topological) mating.

Most results obtained so far have focused on the question of when the mating of two polynomials results in a map that is (topologically conjugate to) a rational map. This has been completely answered in the quadratic, postcritically finite case by Rees, Tan, and Shishikura (see [Ree92], [Tan92], [Shi00]). See also the result on matings of Siegel disk polynomials by Yampolsky-Zakeri [YZ01]. Here we consider the question whether a given rational map is obtained as a mating.

1.3. **Main results.** The first main result of this paper is the following.

Theorem 1.1. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritcally finite rational map such that its Julia set is the whole sphere, i.e., $\mathcal{J}(f) = \widehat{\mathbb{C}}$. Then every sufficiently high iterate $F = f^n$ arises as the topological mating of two polynomials.*

Note that mating of polynomials may result in maps that are branched covering maps $f\colon S^2 \to S^2$, but are not (topologically conjugate to) rational maps. A branched covering map f of the sphere S^2 is called a *Thurston map* if it is *postcritically finite* (i.e., each critical point has finite orbit). We furthermore assume that f is *expanding* in a suitable sense (see Section 2 for definitions and more background). Recall that a *periodic critical point* (of a Thurston map f) is a critical point c, such that $f^k(c) = c$ for some $k \geq 1$. A postcritically finite rational map is expanding if and only if it has no periodic critical points if and only if its Julia set is the whole sphere. In general however, an expanding Thurston maps may have periodic critical points. Theorem 1.1 is a special case of the following more general theorem.

Theorem 1.2. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map f without periodic critical points. Then every sufficiently high iterate $F = f^n$ is obtained as a topological mating of two polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$.*

The polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are postcritically finite, where each critical point is *strictly* preperiodic (i.e., $P_{\mathtt{w}}, P_{\mathtt{b}}$ have no periodic critical points).

We also prove a version of this theorem in the case when f is allowed to have periodic critical points. It is relatively easy to show that in this case no iterate $F = f^n$ can be (i.e., is topologically conjugate to) the mating of two polynomials.

It is however possible to slightly alter the mating construction, so that a result similar to Theorem 1.2 holds. Namely we collapse the closure of each bounded Fatou component of $P_{\mathtt{w}}$ and $P_{\mathtt{b}}$ (i.e., each Fatou component distinct from the basins of attraction of ∞). In addition we need to take the *closure* of the equivalence relation. An equivalence relation $\sim$ on a compact metric space S is called *closed* if it is closed as a subset of the product space $S \times S$. If $\sim$ is not closed the quotient $S/\!\sim$ fails to be Hausdorff.

Formally we consider the equivalence relation on the disjoint union of $\mathcal{K}_{\mathtt{w}}, \mathcal{K}_{\mathtt{b}}$ generated by the following,

$$\sigma_{\mathtt{w}}(z) \sim \sigma_{\mathtt{b}}(\bar{z}), \quad \text{for all } z \in S^1$$

and for any bounded Fatou component A of $P_{\mathtt{w}}$ or $P_{\mathtt{b}}$,

$$x \sim y, \quad \text{for all } x, y \in \operatorname{clos} A.$$

Since $\sim$ may not be closed in general, we consider the closure $\widehat{\sim}$ of $\sim$ (see Lemma 4.5). Let $\mathcal{K}_{\mathtt{w}} \widehat{\amalg} \mathcal{K}_{\mathtt{b}}$ be the quotient of (the disjoint union of) $\mathcal{K}_{\mathtt{w}}, \mathcal{K}_{\mathtt{b}}$ with $\widehat{\sim}$. We will show that the maps $P_{\mathtt{w}}, P_{\mathtt{b}}$ descend to this quotient, meaning that the following is well defined.

$$P_{\mathtt{w}} \widehat{\amalg} P_{\mathtt{b}} \colon \mathcal{K}_{\mathtt{w}} \widehat{\amalg} \mathcal{K}_{\mathtt{b}} \to \mathcal{K}_{\mathtt{w}} \widehat{\amalg} \mathcal{K}_{\mathtt{b}},$$
$$P_{\mathtt{w}} \widehat{\amalg} P_{\mathtt{b}}([x]) := \begin{cases} [P_{\mathtt{w}}(x)], & x \in \mathcal{K}_{\mathtt{w}}; \\ [P_{\mathtt{b}}(x)], & y \in K_{\mathtt{b}}. \end{cases} \tag{1.4}$$

Theorem 1.3. *Let f be an expanding Thurston map. Then every sufficiently high iterate $F = f^n$ is topologically conjugate to a map $P_{\mathtt{w}} \widehat{\amalg} P_{\mathtt{b}}$ as above.*

The polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are postcritically finite, furthermore their Fatou sets are *separated*. This means that two distinct bounded Fatou components of $P_{\mathtt{w}}$ (of $P_{\mathtt{b}}$) have disjoint closures.

Note that if each critical point of $P_{\mathtt{w}}, P_{\mathtt{b}}$ is strictly preperiodic there are no bounded Fatou components, i.e., $P_{\mathtt{w}} \widehat{\amalg} P_{\mathtt{b}} = P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ in this case. Theorem 1.2 may thus be viewed as a special case of Theorem 1.3.

It is easy to find examples of Thurston maps $f \colon S^2 \to S^2$ (which may be rational) such that no iterate $F = f^n$ is obtained as a mating of polynomials. For example this is the case when f is a hyperbolic, postcritically finite rational map which is not a polynomial and has exactly three postcritical points; see [Meya].

In the theorems above we do not only show existence of the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ into which F "unmates", but the polynomials may be explicitly computed via a simple algorithm. More precisely their *critical portraits* are computed explicitly. Roughly—and somewhat incorrectly—speaking this is the set of external angles at the critical points. Such a critical portrait determines a monic centered polynomial uniquely. This was introduced in [BFH92] and generalized in [Poi09].

It has long been known that a rational map F may arise as the mating of polynomials in several distinct ways. This phenomenon of *shared mating* was first observed by Wittner in his thesis [Wit88]. From our construction it follows that shared matings are in fact abundant in our setting. The algorithm to find the critical portraits together with several examples of shared matings are presented in [Meya].

1.4. **Invariant Peano curves.** This paper is a direct continuation of the paper [Meyb], where the following theorem was proved.

Theorem 1.4. *Let $f \colon S^2 \to S^2$ be an expanding Thurston map. Then for every sufficiently high iterate $F = f^n$ there is a* Peano curve *$\gamma \colon S^1 \to S^2$ (onto) such that $F(\gamma(z)) = \gamma(z^d)$. Here $d = \deg F$. This means that the following diagram commutes.*

$$\begin{tikzcd} S^1 \arrow[r, "z^d"] \arrow[d, "\gamma"'] & S^1 \arrow[d, "\gamma"] \\ S^2 \arrow[r, "F"'] & S^2 \end{tikzcd}$$

Thus F admits a description akin to the description of polynomials obtained from the Carathéodory semi-conjugacy (1.3).

It should be pointed out that this theorem follows from the respective theorems in Section 1.3. Namely the Carathéodory semi-conjugacy $\sigma_{\mathtt{w}} \colon S^1 \to \mathcal{J}_{\mathtt{w}} \subset \mathcal{K}_{\mathtt{w}}$ (similarly $\sigma_{\mathtt{b}} \colon S^1 \to \mathcal{J}_{\mathtt{b}}$) descends to (the quotient) $\mathcal{K}_{\mathtt{w}} \amalg \mathcal{K}_{\mathtt{b}}$. If we are in the situation of Theorem 1.2 there is a homeomorphism $h\colon \mathcal{K}_{\mathtt{w}} \amalg \mathcal{K}_{\mathtt{b}} \to S^2$. Then the Peano curve $\gamma\colon S^1 \to S^2$ from Theorem 1.4 is given by the composition

$$S^1 \xrightarrow{\sigma_{\mathtt{w}}} \mathcal{J}_{\mathtt{w}} \hookrightarrow \mathcal{K}_{\mathtt{w}} \amalg \mathcal{K}_{\mathtt{b}} \xrightarrow{h} S^2.$$

Similarly if we are in the case of Theorem 1.3.

The theorems from Section 1.3 however are proved using Theorem 1.4. The iterate $F = f^n$ needed in Theorem 1.2 and Theorem 1.3 is the same as the one from Theorem 1.4. Several sufficient conditions for the existence of an invariant Peano curve γ for an expanding Thurston map f are given in [Meyb]. These conditions thus are sufficient to show that f (not some iterate) arises as a mating. An overview of these conditions is given in [Meya].

1.5. **The map F as a quotient.** Using the invariant Peano curve $\gamma\colon S^1 \to S^2$ from Theorem 1.4, an *equivalence relation* on S^1 is defined by

$$z \sim w \Leftrightarrow \gamma(z) = \gamma(w), \tag{1.5}$$

for all $z, w \in S^1$. Elementary topology yields that $S^1/\sim$ is homeomorphic to S^2 and that $z^d/\sim\colon S^1/\sim \to S^1/\sim$ is topologically conjugate to the map F.

Theorem 1.5. *Let $\gamma\colon S^1 \to S^2$ be an invariant Peano curve for the expanding Thurston map $F = f^n$ as in Theorem 1.4 and $\sim$ be the equivalence relation on S^1 induced by γ as above. Then the following diagram commutes,*

$$\begin{tikzcd} S^1/\sim \arrow[r, "z^d/\sim"] \arrow[d, "h"'] & S^1/\sim \arrow[d, "h"] \\ S^2 \arrow[r, "F"'] & S^2. \end{tikzcd}$$

Here the homeomorphism $h\colon S^1/\sim \to S^2$ is given by $h\colon [s] \mapsto \gamma(s)$, for all $s \in S^1$.

Thus the equivalence relation $\sim$ contains all the information to recover F up to topological conjugacy. It turns out that the equivalence relation $\sim$ may be constructed from *finite data*, more precisely from two finite families of finite sets of rational numbers. Thus these families encode the map F up to topological conjugacy by the previous theorem.

The proper setting is as follows. The Peano curve γ from Theorem 1.4 was constructed as the limit of *approximations* γ^n. The approximations have finitely many points where they touch themselves, but they never cross themselves. Thus $S^2 \setminus \gamma^n$ has two components which are colored *white* and *black*. We define equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ on S^1 by $s \overset{n,\mathtt{w}}{\sim} t$ ($s \overset{n,\mathtt{b}}{\sim} t$) whenever $\gamma^n(s) = \gamma^n(t)$ and γ^n touches itself at s, t in the white component (black component). It turns out that

- $\sim$ can be recovered from the equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ as a limit (defined in a suitable sense, see Theorem 4.7).
- The equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ can be *inductively obtained* from the "initial ones" $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ (Theorem 5.10).
- The equivalence classes of $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ form a *critical portrait* in the sense of [Poi09], see Definition 5.12. Such a critical portrait is a (finite) family of finite sets of rational angles.

Thus we obtain the following additional main result of this paper. The map $F = f^n$ is the (same) iterate of the expanding Thurston map f from Theorem 1.4.

Theorem 1.6. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then every sufficiently high iterate $F = f^n$ admits a description via two critical portraits.*

This provides an effective way to describe F. The description of F as in Theorem 1.5 may be viewed as a two-sided version of the viewpoint introduced by Douady-Hubbard and Thurston ([DH84], [DH85], [Thu09], see also [Ree92] and [Kel00]), namely the combinatorial description of Julia sets in terms of *external rays*.

An outline of the proof Theorem 1.2 is as follows. Consider the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ given by the critical portraits (i.e., by $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$) as outlined above. The Carathéodory semi-conjugacy $\sigma_{\mathtt{w}}\colon S^1 \to \mathcal{J}_{\mathtt{w}}$ induces an equivalence relation $\overset{\mathtt{w}}{\approx}$ on S^1 as above by

$$z \overset{\mathtt{w}}{\approx} w \;:\Leftrightarrow\; \sigma_{\mathtt{w}}(z) = \sigma_{\mathtt{w}}(w), \tag{1.6}$$

for all $z, w \in S^1$. The equivalence relation $\overset{\mathtt{b}}{\approx}$ on S^1 is defined similarly via $\sigma_{\mathtt{b}}$. The equivalence relations $\overset{\mathtt{w}}{\approx}$ and $\overset{\mathtt{b}}{\approx}$ generate the equivalence relation $\approx$ on S^1. It is relatively easy to show that $z^d/\!\approx\colon S^1/\!\approx \to S^1/\!\approx$ is topologically conjugate to the mating $P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}$. The proof of Theorem 1.2 will be obtained by showing that $\approx$ equals the equivalence relation $\sim$ induced by the invariant Peano curve γ as in (1.5).

1.6. **Further results.** We also prove the following theorem here.

Theorem 1.7. *The invariant Peano curve $\gamma\colon S^1 \to S^2$ from Theorem 1.4 maps normalized Lebesgue measure on S^1 to the measure of maximal entropy on S^2 with respect to F.*

By "normalized" it is meant that the total mass is 1, i.e., that the measure is a probability measure. The *measure of maximal entropy* (also called the *Lyubich* or *Brolin* measure) is the unique invariant probability measure that maximizes the (measure theoretic) entropy. It can be defined as the weak limit of $1/d^n \sum_{y \in F^{-n}(x_0)} \delta_y$, for any point $x_0 \in S^2$. Note that the measure of maximal entropy of $F = f^n$ equals the measure of maximal entropy of f.

Finally we note that each invariant Peano curve $\gamma\colon S^1 \to S^2$ induces a *fractal tiling*. Namely we divide the circle S^1 into $d = \deg F$ arcs and consider their images by γ. If F is a rational map these fractal tilings may be lifted to the *orbifold covering*, which is either the plane $\mathbb{C}$ or the hyperbolic plane $\mathbb{H}$.

1.7. **Outline of this paper.** In Section 2 we recall the setup from [BM]. Namely one picks a Jordan curve $\mathcal{C}$ containing all postcritical points. Then $F^{-n}(\mathcal{C})$ decomposes the sphere S^2 into *n-tiles*. This in turn allows for a combinatorial description of F.

In Section 3 the *construction* of the invariant Peano curve γ from Theorem 1.4 as given in [Meyb] is *reviewed*. In particular γ was constructed as a limit of *approximations* γ^n, whose properties we list.

Some elementary facts about equivalence relations are provided in Section 4.

In Section 5 we introduce *equivalence relations* $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$. They describe the self-intersections of the approximations γ^n. It is shown that these equivalence relations are obtained inductively, i.e., from $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$. These "initial equivalence relations" $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ form a *critical portrait* in the sense of [Poi09]. Furthermore the map F is completely determined from them, up to topological conjugacy.

In Section 6 we investigate the *sizes* of the *equivalence classes* induced by γ. More precisely we show that if F does not have periodic critical points, the size of such equivalence classes is bounded by some number $N < \infty$. If F has periodic critical points, we show that at least one equivalence class is finite.

In Section 7 we show that F is *obtained as a mating*, for the case when F has no periodic critical points; i.e., Theorem 1.2 is proved.

The case when F has periodic critical points, i.e., Theorem 1.3, is proved in Section 8.

In Section 9 we show that the invariant Peano curve γ maps *Lebesgue measure* on the circle S^1 to the *measure of maximal entropy* of F (on S^2), i.e., prove Theorem 1.7.

In Section 10 we illustrate the fractal tilings obtained from the construction. This also shows explicitly the invariant Peano curve for some examples.

We conclude the paper in Section 11 with some open questions.

1.8. **Notation.** The circle is denoted by S^1, the 2-sphere by S^2. By $\operatorname{int} U$ we denote the interior, by $\operatorname{clos} U$ the closure of a set U. The cardinality of a (finite) set S is denoted by $\#S$.

It will often be convenient to identify S^1 with $\mathbb{R}/\mathbb{Z}$; the map $z^d\colon S^1 \to S^1$ then is denoted by

$$\boxed{\mu = \mu_d\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad \mu(t) = dt (\operatorname{mod} 1),}$$

the n-th iterate is $\mu^n(t) = d^n t (\operatorname{mod} 1)$.

For two non-negative expressions A, B we write $A \lesssim B$ if there is a constant $C > 0$ such that $A \leq CB$. We refer to C as $C(\lesssim)$. Similarly we write $A \asymp B$ if $A/C \leq B \leq CA$ for a constant $C \geq 1$, we refer to C as $C(\asymp)$.

- The *n-iterate* of a map f is denoted by f^n.
- $F = f^n$ is the iterate of the expanding Thurston map f from Theorem 1.4.
- By $\operatorname{crit} = \operatorname{crit}(f)$, $\operatorname{post} = \operatorname{post}(f)$ we denote the *set of critical* and *postcritical points* (see Section 2.1).
- The *degree* of F is denoted by d, the *number of postcritical points* by k.
- The *local degree* of F at $v \in S^2$ is denoted by $\deg_F(v)$, see Definition 2.1.
- *Upper indices* indicate the *order* of an object. For example some preimage of some object U^0 by F^n will be denoted by U^n. Also maps and other objects associated with such objects U^n will have an upper index "n".
- $\mathcal{C}$ is a Jordan curve containing all postcritical points.
- $X^0_{\mathtt{w}}, X^0_{\mathtt{b}}$ are the white/black 0-tiles, i.e., the closures of the two components of $S^2 \setminus \mathcal{C}$.
- *Lower indices* "w" or "b" indicate whether an object is colored "white" or "black", i.e., if it is mapped eventually to $X^0_{\mathtt{w}}$ or $X^0_{\mathtt{b}}$; or closely related to such objects.
- A visual metric is denoted by ϱ, see Section 2.3.
- The *sets of all n-tiles, -edges, -vertices* are denoted by $\mathbf{X}^n, \mathbf{E}^n, \mathbf{V}^n$ (Section 2.2).
- γ^n is the *n-th approximation of the Peano curve* γ (Section 3.1).
- A point $\alpha^n \in S^1$ such that $\gamma^n(\alpha^n) \in \mathbf{V}^n$ is called an *n-angle*. The *set of all n-angles* is denoted by $\mathbf{A}^n$ (Section 3.1).

- An n-arc a^n is a closed interval in $\mathbb{R}/\mathbb{Z} = S^1$ that is mapped by γ^n (homeomorphically) to an n-edge.
- H^n are the *pseudo-isotopies* from which the approximations γ^n were constructed, see Section 3.2.
- $\pi_{\mathtt{w}} \cup \pi_{\mathtt{b}}$ is a cnc-partition (of a set $\{0, 1, \dots, 2n-1\}$). It describes the connection at a vertex, i.e., which white/black tiles are "connected" at v (see Section 3.3).
- By $\sim$ we denote the equivalence relation on S^1 induced by the invariant Peano curve ($s \sim t \Leftrightarrow \gamma(s) = \gamma(t)$), by $\overset{n}{\sim}$ the equivalence relation induced by γ^n (Section 4.3).
- The *join* of two equivalence relations $\overset{a}{\sim}, \overset{b}{\sim}$ is denoted by $\overset{a}{\sim} \vee \overset{b}{\sim}$, their *meet* by $\overset{a}{\sim} \wedge \overset{b}{\sim}$ (Section 4.1).
- The *equivalence relations* $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ describe where γ^n "touches itself on the white/black side" (see Definition 5.1). Their equivalence classes are denoted by $[\alpha]_{n,\mathtt{w}}$, $[\alpha]_{n,\mathtt{b}}$.
- $S^2_{\mathtt{w}}, S^2_{\mathtt{b}}$ are the *white/black hemispheres*, i.e., the components of $S^2 \setminus S^1$. They are equipped with the hyperbolic metric.
- $\mathcal{L}^n_{\mathtt{w}}, \mathcal{L}^n_{\mathtt{w}}$ are the *laminations* associated to $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$. Namely for each equivalence class $[\alpha]_{n,\mathtt{w}}$ there is a *leaf* $L \in \mathcal{L}^n_{\mathtt{w}}$, given as the hyperbolically convex hull of $[\alpha]_{n,\mathtt{w}}$. A white/black *n-gap* is the closure of one component of $S^2_{\mathtt{w}} \setminus \bigcup \mathcal{L}^n_{\mathtt{w}}$, or of $S^2_{\mathtt{w}} \setminus \bigcup \mathcal{L}^n_{\mathtt{w}}$ respectively. The *set of white/black n-gaps* is denoted by $\mathbf{G}^n_{\mathtt{w}}, \mathbf{G}^n_{\mathtt{b}}$ (Section 5.2).
- The *Carathéodory semi-conjugacy* is denoted by σ (see Section 1.1). The equivalence relation induced by σ is denoted by $\approx$ (1.6). The semi-conjugacies, equivalence relations of the white, black polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are denoted by $\sigma_{\mathtt{w}}, \sigma_{\mathtt{b}}$ and $\overset{\mathtt{w}}{\approx}, \overset{\mathtt{b}}{\approx}$. Identifying additionally closures of bounded Fatou components yields the equivalence relations $\overset{\mathcal{F},\mathtt{w}}{\approx}, \overset{\mathcal{F},\mathtt{b}}{\approx}$ (see (7.2), (7.3)).

2. Expanding Thurston maps

Here some material from [BM] is reviewed.

2.1. Definition of expanding Thurston maps.

Definition 2.1. A *Thurston map* is an orientation-preserving, postcritically finite, branched cover of the sphere $f\colon S^2 \to S^2$. This means that locally f can be written as $z \mapsto z^q$, $q \geq 1$ (after suitable local, orientation preserving, homeomorphic changes of coordinates in domain and range). More precisely for each point $v \in S^2$ there exists a $q \in \mathbb{N}$, (open) neighborhoods V, W of $v, w = f(v)$ and orientation preserving homeomorphisms $\varphi\colon V \to \mathbb{D}$, $\psi\colon W \to \mathbb{D}$ with $\varphi(v) = 0$, $\psi(w) = 0$ satisfying

$$\psi \circ f \circ \varphi^{-1}(z) = z^q,$$

for all $z \in \mathbb{D}$. The integer $q = \deg_f(v) \geq 1$ is called the *local degree* of the map at v. A point $c \in S^2$ at which the local degree $\deg_f(c) \geq 2$ is called a *critical point*. The set of all critical points is denoted by $\mathrm{crit} = \mathrm{crit}(f)$. *Postcritically finiteness* means that the *set of postcritical points*

$$\mathrm{post} = \mathrm{post}(f) := \bigcup_{j \geq 1} f^j(\mathrm{crit})$$

is finite.

Fix a Jordan curve $\mathcal{C} \supset \mathrm{post}$. The Thurston map is called *expanding* if

$$\operatorname{mesh} f^{-n}(\mathcal{C}) \to 0 \text{ as } n \to \infty.$$

Here mesh $f^{-n}(\mathcal{C})$ is the maximal diameter of a component of $S^2 \setminus f^{-n}(\mathcal{C})$. This is independent of the chosen curve $\mathcal{C}$ (see [BM, Lemma 6.1]). It is equivalent to the notion of expansion by Haïssinsky-Pilgrim in [HP09] (see [BM, Proposition 6.2]).

If f is a rational map it is expanding if and only if its Julia set is the whole sphere if and only if f has no periodic critical points, see[BM, Proposition 2.3].

We will however not assume that f is a rational map; in fact it should be emphasized that we make no assumption on the smoothness of f.

In particular we allow expanding Thurston maps that have *periodic critical points*. An example of such a map is obtained as follows. Consider a postcritically finite rational map g whose Julia set is a Sierpiński carpet (the closure of each Fatou component is a Jordan domain, closures of distinct Fatou components are disjoint). Identify the closure of each Fatou component, the map descends to this quotient. The quotient map is an expanding Thurston map with periodic critical points.

The map

$$F = f^n$$

will always denote the iterate from Theorem 1.4. Note that $\operatorname{post}(F) = \operatorname{post}(f)$. Throughout this paper we denote by

$$\boxed{d := \deg F = (\deg f)^n, \quad k := \#\operatorname{post}.}$$

2.2. **Tiles and edges.** Fix a Jordan curve $\mathcal{C} \supset \operatorname{post}$ (this was the first step to construct the Peano curve γ) and give it an orientation. In [BM, Chapter 13] it was shown that we can choose $\mathcal{C}$ to be F-invariant ($F(\mathcal{C}) \subset \mathcal{C}$), but we do not assume this here. We *color* the components of $S^2 \setminus \mathcal{C}$ *white* and *black*, such that $\mathcal{C}$ is positively oriented as the boundary of the white component. The closures of the white/black components of $S^2 \setminus \mathcal{C}$ are called the white/black 0-tiles, denoted by $X^0_{\mathtt{w}}, X^0_{\mathtt{b}}$. Similarly the closure of each component of $S^2 \setminus F^{-n}(\mathcal{C})$ is called an *n-tile*. It is colored white if it is mapped to $X^0_{\mathtt{w}}$ by F^n, black if mapped to $X^0_{\mathtt{b}}$ by F^n. The set of all n-tiles is denoted by $\mathbf{X}^n$. The restricted map

$$F^n \colon X^n \to X_j \text{ is a homeomorphism} \tag{2.1}$$

for each $X^n \in \mathbf{X}^n$, here $j \in \{\mathtt{w}, \mathtt{b}\}$ (see [BM, Prop 5.17]). In particular each n-tile is a closed Jordan domain.

Any point $v \in F^{-n}(\operatorname{post})$ is called an *n-vertex*, the set of all n-vertices is $\mathbf{V}^n := F^{-n}(\operatorname{post})$. Each n-tile contains k n-vertices in its boundary. Sometimes a postcritical point is also called a 0-*vertex* for convenience. The set of all n-vertices is denoted by $\mathbf{V}^n$. Note that

$$\operatorname{post} = \mathbf{V}^0 \subset \mathbf{V}^1 \subset \dots .$$

Thus an n-vertex $v \in S^2$ is a m-vertex for all $m \geq n$. Expansion implies that the union of the sets $\mathbf{V}^n$ is dense.

The postcritical points divide the curve $\mathcal{C}$ into k (closed) arcs, called the 0-*edges*. Similarly the closure of a component of $F^{-n}(\mathcal{C}) \setminus \mathbf{V}^n$ is called an *n-edge*. Each n-edge is mapped by F^n homeomorphically to some 0-edge. The boundary of each n-tile consists of k n-edges. The set of all n-edges is denoted by $\mathbf{E}^n$.

Recall that n-tiles are colored white and black. The n-tiles tile the sphere S^2 in a *checkerboard fashion*. This means that two n-tiles which share an n-edge are colored differently. Put differently, at each n-vertex v an even number of n-tiles intersect, their colors alternate around v.

2.3. **The visual metric.** In [BM, Chapter 7] *visual metrics* on S^2 for an expanding Thurston map F were considered. These are metrics defined in combinatorial terms, i.e., in terms of the tiles defined above.

Assume the Jordan curve $\mathcal{C} \supset \operatorname{post}$ has been fixed, and tiles are defined as above. Let

$$m(x,y) := \min\{n \in \mathbb{N} \mid \text{ there exists } \textit{disjoint } n\text{-tiles } X^n \ni x, Y^n \ni y\}, \tag{2.2}$$

for all distinct $x, y \in S^2$; we set $m(x,x) = \infty$ (for all $x \in S^2$). A metric ϱ on S^2 is a *visual metric* for F if there is a constant $\Lambda > 1$ such that

$$\varrho(x,y) \asymp \Lambda^{-m(x,y)}, \tag{2.3}$$

for all $x, y \in S^2$. Here we set $\Lambda^{-\infty} = 0$. The constant $C(\asymp) = C(\mathcal{C})$ is independent of x, y. The number Λ is called the *expansion factor* of ϱ. Visual metrics are not unique, but distinct visual metrics $\varrho, \widetilde{\varrho}$ are *snowflake equivalent* (i.e., $\varrho^\alpha \asymp \widetilde{\varrho}$ for some $\alpha > 0$).

We note the following. Let $A \subset S^2$ be an n-tile or an n-edge. Then

$$\operatorname{diam} A \asymp \Lambda^{-n}, \tag{2.4}$$

where $C = C(\asymp)$ is a constant independent of A, the diameter is measured with respect to a visual metric ϱ (with expansion factor $\Lambda > 1$).

*** check: is this needed? *** it was shown that S^2 can be equipped with a *visual metric*, denoted by $|x-y|_{\mathcal{S}}$, with respect to which F is a *local similarity.*

Theorem 2.2 ([BM])**.** *There is a constant $\Lambda > 1$ such that the following holds. For every $x \in S^2$ there is a neighborhood U_x of x such that*

$$\frac{|F(x) - F(y)|_{\mathcal{S}}}{|x-y|_{\mathcal{S}}} = \Lambda \quad \textit{for all } y \in U_x.$$

Then (see [BM, Section 8])

3. The invariant Peano curve and its approximations

The paper present is a direct continuation of [Meyb], as mentioned in the introduction. In particular we do not only use the main result (i.e., Theorem 1.4), but in fact the whole construction of the invariant Peano curve γ. We outline the construction here. Results of [Meyb] will be used freely.

A Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post} \subset \mathcal{C}$ is fixed. A visual metric ϱ for F with expansion factor $\Lambda > 1$ is defined via $\mathcal{C}$ and fixed. Metrical terms below are defined in terms of ϱ.

It will often be convenient to identify S^1 with $\mathbb{R}/\mathbb{Z}$. We write $\mu = \mu_d \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ for the map $t \mapsto dt(\text{mod}1)$ (which is conjugate to $z^d \colon \partial\mathbb{D} \to \partial\mathbb{D}$). We still write $\gamma \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ for the invariant Peano curve, slightly abusing notation. The same abuse of notation applies to the approximations γ^n defined below.

3.1. **The approximations γ^n.** The curve γ is constructed as the limit of curves $\gamma^n\colon S^1 \to S^2$, called the *n-th approximation* (of the Peano curve). The approximations satisfy the following.

- As a set $\gamma^0 = \mathcal{C}$, more precisely
$$\gamma^0\colon S^1 \to \mathcal{C}$$
is a homeomorphism.
- The γ^n cover all n-edges. This means when γ^n is viewed as a set it holds
$$\gamma^n = \bigcup \mathbf{E}^n.$$
- A point $\alpha^n \in S^1$ such that $\gamma^n(\alpha^n) \in \mathbf{V}^n$ is called an *n-angle*. Each n-angle is *rational* (here we identify S^1 with $\mathbb{R}/\mathbb{Z}$). The *set of all n-angles*
$$\mathbf{A}^n := (\gamma^n)^{-1}(\mathbf{V}^n)$$
is a finite set.
- For $m \geq n$ it holds that $\gamma^m = \gamma^n$ on $\mathbf{A}^n$,
$$\gamma^n|_{\mathbf{A}^n} = \gamma^m|_{\mathbf{A}^n} = \gamma|_{\mathbf{A}^n}.$$
Thus (recall that $\mathbf{V}^0 \subset \mathbf{V}^1 \subset \dots$)
$$\mathbf{A}^0 \subset \mathbf{A}^1 \subset \dots.$$
- The n-angles divide the circle S^1 into (closed) *n-arcs*. Each n-arc is mapped by γ^n homeomorphically to an n-edge. Conversely for each n-edge E^n there is a unique n-arc $a^n \subset S^1$, such that $\gamma^n(a^n) = E^n$.
- Each n-arc is mapped homeomorphically to an $(n-1)$-arc by $z^d\colon S^1 \to S^1$. More precisely, we have the following commutative diagram:
$$\begin{tikzcd} \mathbf{A}^{n+1} \subset \mathbb{R}/\mathbb{Z} \arrow[r, "\mu"] \arrow[d, "\gamma^{n+1}"'] & \mathbf{A}^n \subset \mathbb{R}/\mathbb{Z} \arrow[d, "\gamma^n"] \\ \mathbf{V}^{n+1} \subset S^2 \arrow[r, "F"'] & \mathbf{V}^n \subset S^2. \end{tikzcd} \tag{3.1}$$
See [Meyb, Lemma 4.5 (2)], as well as Remark 3.1. Thus there is a constant $C(\lesssim) > 0$ such that
$$\operatorname{diam} a^n \lesssim d^{-n}$$
for each n-arc a^n. Recall that the diameter is measured with respect to the visual metric ϱ.
- The curve γ^n touches itself, but does not intersect itself. This means for any $\epsilon > 0$ there is a Jordan curve $\gamma^n_\epsilon\colon S^1 \to S^2$ such that
$$\|\gamma^n - \gamma^n_\epsilon\|_\infty < \epsilon.$$
Here $\|\gamma^n - \gamma^n_\epsilon\|_\infty := \max\{\varrho(\gamma^n(t), \gamma^n_\epsilon(t)) \mid t \in S^1\}$ is the supremums norm with respect to some fixed visual metric ϱ.
- The curves γ^n converge uniformly to γ. More precisely
$$\|\gamma^n - \gamma\|_\infty \lesssim \Lambda^{-n}. \tag{3.2}$$

- Each n-edge E^n is contained in the boundary of a (unique) white n-tile $X^n_{\mathtt{w}}$. We orient $\partial X^n_{\mathtt{w}}$ mathematically positively, this induces an orientation on E^n. Equivalently we may define the orientation on E^n as follows. Each 0-edge $E^0 \subset \mathcal{C}$ inherits an orientation from the orientation of $\mathcal{C}$. Since F^n maps E^n to some 0-edge E^0 homeomorphically, we call pull back the orientation of 0-edges to each n-edge. Note that F maps positively oriented n-edges to positively oriented $(n-1)$-edges by definition.

  A distinct way to define the orientation of n-edges is as follows. Recall that for each n-edge E^n there is exactly one n-arc $a^n \subset S^1$ which is mapped homeomorphically to E^n by γ^n. Thus $E^n = \gamma^n(a^n)$ inherits the orientation of $a^n \subset S^1$. The following holds for the approximations γ^n:

  ($\star$) The two orientations on E^n described above agree.

Remark 3.1. Note that the approximations $\tilde{\gamma}^n \colon \mathbb{R}/\mathbb{Z} \to S^2$ parametrized as in [Meyb, Section 4.2] converge to $\tilde{\gamma} \colon \mathbb{R}/\mathbb{Z} \to S^2$, which semi-conjugates F to $\widetilde{\mu}(t) = dt + \theta_0 (\mathrm{mod} 1)$. The approximations

$$\gamma^n(t) := \tilde{\gamma}^n(t - \theta_0/(d-1))$$

converge to

$$\gamma(t) = \tilde{\gamma}(t - \theta_0(d-1)),$$

which is the desired Peano curve, i.e., semi-conjugates F to $\mu(t) = dt (\mathrm{mod} 1)$, see [Meyb, Lemma 4.1] and the subsequent remark. Let $\{\tilde{\alpha}^n_j\} \subset S^1$ be the points from [Meyb, Section 4.2] (that are mapped by $\tilde{\gamma}^n$ to n-vertices). Then the points

$$\alpha^n_j := \tilde{\alpha}^n_j + \theta_0/(d-1)$$

are the n-angles.

3.2. **Pseudo-isotopies.** The approximations γ^n described in the last section were constructed as follows. A *pseudo-isotopy* $H \colon S^2 \times [0,1] \to S^2$ is a homotopy, that is an isotopy an $[0,1)$ (i.e., it ceases to be a homeomorphism only at $t=1$); if H is constant on a set A it is an *isotopy rel. A*. In [Meyb] a pseudo-isotopy H^0 rel. post $= \mathbf{V}^0$ is constructed that deforms γ^0 to γ^1, where γ^0, γ^1 are as in the last section. It is possible to *lift* H^0 by F^n to H^n. This is a pseudo-isotopy rel. $\mathbf{V}^n$ and deforms γ^n to γ^{n+1}.

3.3. **Connections.** The first approximation γ^1 (more precisely the pseudo-isotopy H^0) was constructed as follows. At each 1-vertex v several white 1-tiles intersect. We declare which white 1-tiles are *connected* at v; see Figure 1 for an illustration. Connections (of white 1-tiles at v) are *non-crossing*. Furthermore the resulting *white tile graph* forms a *spanning tree*. "Following the outline" of this tree yields the first approximation γ^1. There is one more ingredient: the resulting curve γ^1 has to be "in the right homotopy class". This means there has to be a pseudo-isotopy H^0 rel. post as in Section 3.2 that deforms γ^0 to γ^1.

Formally let $X_0, \dots X_{2n-1}$ be the 1-tiles intersecting in a 1-vertex v, ordered mathematically positively around v. The white 1-tiles have even index, the black ones odd index. We consider a decomposition $\pi_{\mathtt{w}} = \pi_{\mathtt{w}}(v)$ of $\{0, 2, \dots, 2n-2\}$ (i.e., of indices corresponding to white 1-tiles around v); and a decomposition $\pi_{\mathtt{b}} = \pi_{\mathtt{b}}(v)$ of $\{1, 3, \dots, 2n-1\}$ (i.e., indices corresponding to black 1-tiles around v). They satisfy the following:

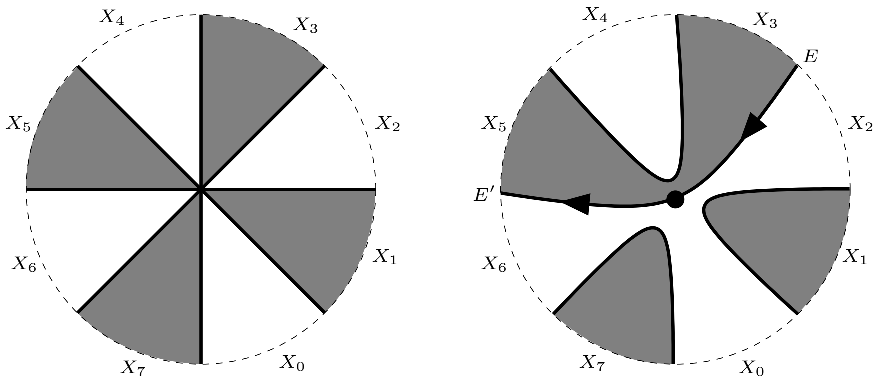


FIGURE 1. Connection at a vertex.

- They are *decompositions*. This means $\pi_{\mathtt{w}} = \{b_1, \dots, b_N\}$, where each *block* b_i is a subset of $\{0, 2, \dots, 2n-2\}$, $b_i \cap b_j = \emptyset$ $(i \neq j)$, and $\bigcup b_i = \{0, 2, \dots, 2n-2\}$. Similarly for $\pi_{\mathtt{b}}$.
- The decompositions $\pi_{\mathtt{w}}, \pi_{\mathtt{b}}$ are *non-crossing*. This means the following. Two distinct blocks $b_i, b_j \in \pi_{\mathtt{w}}$ are *crossing* if there are numbers $a, c \in b_i$, $b, d \in b_j$ and
$$a < b < c < d.$$
Each partition $\pi_{\mathtt{w}}, \pi_{\mathtt{b}}$ does not contain any (pair of) crossing blocks.
- The partitions $\pi_{\mathtt{w}}, \pi_{\mathtt{b}}$ are *complementary*. This means the following. Given $\pi_{\mathtt{w}}$, the partition $\pi_{\mathtt{b}}$ is the unique, biggest partition (of $\{1, 3, \dots, 2n-1\}$) such that $\pi_{\mathtt{w}} \cup \pi_{\mathtt{b}}$ is a non-crossing partition of $\{0, 1, \dots, 2n-1\}$.

A partition $\pi_{\mathtt{w}} \cup \pi_{\mathtt{b}}$ as above is called a *complementary non-crossing partition*, or cnc-partition. A *connection* (of 1-tiles) assigns to each 1-vertex a cnc-partition as above. Two 1-tiles $X_i, X_j \ni v$ are said to be *connected at v* if the indices i, j are contained in the *same* block of $\pi_{\mathtt{w}} \cup \pi_{\mathtt{b}}$. Note that tiles of different color are never connected. The 1-tile X_i is *incident* to the block $b \ni i$ of $\pi_{\mathtt{w}} \cup \pi_{\mathtt{b}}$. In the example illustrated in Figure 1 we have $\pi_{\mathtt{w}} = \{\{0, 2, 6\}, \{4\}\}$, $\pi_{\mathtt{b}} = \{\{1\}, \{3, 5\}, \{7\}\}$.

Remark 3.2. In the construction of the "initial pseudo-isotopy" H^0 we need additional data to the one described above. Namely in the case when v is a postcritical point, we need to "say where v is in the connection". This will be of no importance in the present paper.

3.4. **Succeeding edges.** The connection of 1-tiles from Section 3.3 can be used to define which 1-edges are *succeeding* at some 1-vertex v. Indeed this is the main purpose of the connection. Figure 1 serves again as an illustration.

Let the connection at a 1-vertex v be given by $\pi_{\mathtt{w}}(v) \cup \pi_{\mathtt{b}}(v)$. Two indices $i, j \in b \in \pi_{\mathtt{w}}(v) \cup \pi_{\mathtt{b}}(v)$ are called *succeeding* (in b), if b does not contain any index $i+1, i+2, \dots, j-1$. Indices are taken $\bmod 2n$ here, where $2n$ is the number of 1-tiles containing v.

Consider two positively oriented (as boundaries of white 1-tiles) 1-edges E, E'; where E has terminal point v, E' has initial point v. E, E' are *succeeding at v* (with respect to the given connection) if $E \subset X_i$, $E' \subset X_j$ and i, j are succeeding indices of a block $b \in \pi_{\mathtt{w}}(v)$ (thus X_i, X_j are *white* 1-tiles). The first approximation

γ^1 (viewed as an Eulerian circuit in $\bigcup \mathbf{E}^1$) may be given by the connection,

the 1-edges E, E' are *succeeding* in γ^1 if and only if
they are succeeding with respect to the connection.

3.5. **Connection of n-tiles.** The connection of 1-tiles can be *lifted* to a connection of n-tiles (see [Meyb, Section 8]). Thus at each n-vertex v a cnc-partition $\pi^n_{\mathtt{w}}(v) \cup \pi^n_{\mathtt{b}}(v)$ (as in Section 3.3) is defined. Succeeding n-edges are defined as in Section 3.4. As before

the n-edges E, E' are *succeeding* in γ^n if and only if
they are succeeding with respect to the connection of n-tiles.

3.6. **The connection graph.** The connection of n-tiles can be used to defined the *n-th white connection graph*. It is constructed as follows. For each *white* n-tile X there is a vertex $c(X)$ (thought of as the *center* of the n-tile X); for each n-vertex v and block $b \in \pi^n_{\mathtt{w}}(v)$ there is a vertex $c(v, b)$. The vertex $c(X)$ is connected to $c(v, b)$ if (and only if) X is incident to b at v. The connection of n-tiles satisfies the following.

The n-th white connection graph is a tree.

4. Equivalence relations

After these preparations we begin with the proof of the main theorems. As outlined in the introduction we consider the equivalence relation $\sim$ induced by the invariant Peano curve γ. First some elementary material is reviewed. The main result here is Theorem 4.7, which says that $\sim$ may be obtained from the equivalence relations $\overset{n}{\sim}$ induced by the approximations γ^n.

4.1. **The lattice of equivalence relations.** Equivalence relations on a set S can be *partially ordered* in a natural way, namely $\overset{b}{\sim}$ is *bigger* than $\overset{a}{\sim}$ ($\overset{b}{\sim} \geq \overset{a}{\sim}$) if

$$s \overset{a}{\sim} t \Rightarrow s \overset{b}{\sim} t, \tag{4.1}$$

for all $s, t \in S$. Equivalently, each equivalence class of $\overset{a}{\sim}$ is a subset of some equivalence class of $\overset{b}{\sim}$; equivalently if we view equivalence relations as subsets of $S \times S$ this means $\overset{b}{\sim} \supset \overset{a}{\sim}$.

The set of all equivalence relations on S forms a *lattice*, when equipped with this partial ordering. This means that *join* and *meet* are well defined. Recall that the join $\overset{a}{\sim} \vee \overset{b}{\sim}$ is the smallest equivalence relation bigger than $\overset{a}{\sim}$ and $\overset{b}{\sim}$. If $\overset{\vee}{\sim} := \overset{a}{\sim} \vee \overset{b}{\sim}$, then

$s \overset{\vee}{\sim} t$ if and only if
there are $s_1, \dots, s_N \in S$ such that
$s = s_1 \overset{a}{\sim} s_2 \overset{b}{\sim} s_3 \dots s_{N-2} \overset{a}{\sim} s_{N-1} \overset{b}{\sim} s_N = t,$

for all $s, t \in S$.

The meet $\overset{a}{\sim} \wedge \overset{b}{\sim}$ is the biggest equivalence relation smaller than $\overset{a}{\sim}$ and $\overset{b}{\sim}$. If $\overset{\wedge}{\sim} := \overset{a}{\sim} \wedge \overset{b}{\sim}$, then

$$s \overset{\wedge}{\sim} t \text{ if and only if } s \overset{a}{\sim} t \text{ and } s \overset{b}{\sim} t. \tag{4.2}$$

for all $s, t \in S$. When $\overset{a}{\sim}, \overset{b}{\sim}$ are viewed as subsets of $S \times S$ this means that $\overset{\wedge}{\sim} = \overset{a}{\sim} \cap \overset{b}{\sim}$.

4.2. **Closed equivalence relations.** We consider an equivalence relation $\sim$ on a topological space S.

Definition 4.1 (Closed equivalence relation)**.** An equivalence relation $\sim$ on a compact metric space S is called *closed* if one of the following equivalent conditions holds.

(CE 1) $\{(s,t) \mid s \sim t\} \subset S \times S$ is closed.

(CE 2) For any two convergent sequences in S; $s_n \to s_0, t_n \to t_0$ it holds

$$s_n \sim t_n \text{ for all } n \geq 1 \Rightarrow s_0 \sim t_0.$$

(CE 3) The projection map

$$\pi \colon S \to S/\!\sim \ \text{ given by } s \mapsto [s]$$

is closed.

The proof that the above conditions are equivalent is straightforward and left as an exercise.

Remark 4.2. If $\sim$ is closed as above it follows that each equivalence class is compact. Indeed by (CE 3) the set $[s] = \pi^{-1}(\pi(s))$ is closed, thus compact (for all $s \in S$).

Remark 4.3. The set of equivalence classes $\{[s] \mid s \in S\}$ forms a *decomposition* of S (i.e., a set of disjoint subsets of S whose union is S). Conversely, each decomposition can be viewed as an equivalence relation. An equivalence relation is closed if and only if the induced decomposition is *upper semicontinuous.* Property (CE 3), together with the requirement that each equivalence class is compact, is the general definition of upper semicontinuity in any topological space.

Remark 4.4. The closedness/upper semicontinuity of $\sim$ should be viewed as the minimal requirement that the quotient space (or *decomposition* space) $S/\!\sim$ has a "reasonable" topology. For example (S is a compact metric space) $\sim$ is closed if and only if

(CE 4) $S/\!\sim$ is Hausdorff.

The necessity follows immediately from (CE 2), see [Dav86, Proposition 2.1] for the sufficiency (this is the standard reference on decomposition spaces). Several other equivalent conditions for an equivalence relation to be closed may be found in [MP, Lemma 2.2].

Lemma 4.5 (Closure of equivalence relation)**.** *Let $\sim$ be an equivalence relation on a compact metric space S. Then there is a unique smallest closed equivalence relation $\widehat{\sim}$ bigger than $\sim$. We call $\widehat{\sim}$ the* closure *of $\sim$.*

Proof. Consider the family $\{\overset{\alpha}{\sim}\}_{\alpha \in I}$ of all closed equivalence relations bigger than $\sim$. This family is non-empty. Let $\widehat{\sim}$ be their meet

$$s \mathrel{\widehat{\sim}} t := \bigwedge \overset{\alpha}{\sim} \; = \; \bigcap \overset{\alpha}{\sim} \, .$$

In the last expression each $\overset{\alpha}{\sim}$ is viewed as a subset of $S \times S$. Clearly $\widehat{\sim}$ is the unique smallest closed equivalence relation bigger than $\sim$. $\square$

Note that $\{(s,t) \mid s \mathrel{\widehat{\sim}} t\}$ is generally *not* the closure of $\{(s,t) \mid s \sim t\}$, which may fail to be transitive.

4.3. **Equivalence relation induced by γ.** A surjection $h\colon S \to S'$ induces an equivalence relation in a natural way. Under mild assumptions the quotient $S/\sim$ is homeomorphic to S'.

Lemma 4.6 (Equivalence relation induced by a map)**.** *Let S, S' be compact Hausdorff spaces, and $h\colon S \to S'$ a continuous surjection. Define an equivalence relation on S by $s \sim t$ if and only if $h(s) = h(t)$. Then*

- *$S/\sim$ is homeomorphic to S'. The homeomorphism is given by $\varphi\colon [s] \mapsto h(s)$;*

Assume now furthermore that there are continuous maps $f\colon S \to S$, $g\colon S' \to S'$ such that $h \circ f = g \circ h$, i.e., the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "h"'] & S \arrow[d, "h"] \\ S' \arrow[r, "g"] & S'. \end{tikzcd}$$

Then

- *the map $f/\sim : S/\sim \to S/\sim$ given by $f/\sim : [s] \mapsto [f(s)]$ is well defined;*
- *it holds $\varphi \circ f/\sim = g \circ \varphi$, i.e., φ is a topological conjugacy between $f/\sim$ and g.*

Proof. The first statement is [HY88, Theorem 3-37], the second statement follows immediately from the commutative diagram. Finally let $[s] \in S/\sim$ be arbitrary, then

$$g \circ \varphi([s]) = g \circ h(s) = h \circ f(s) = \varphi([f(s)]) = \varphi \circ f/\sim ([s]),$$

i.e., the third statement holds. $\square$

From now on the equivalence relation $\sim$ on S^1 is the one induced by γ,

$$s \sim t :\Leftrightarrow \gamma(s) = \gamma(t),$$

for all $s, t \in S^1$. The previous lemma together with Theorem 1.4 now yields Theorem 1.5.

Consider now the equivalence relations $\overset{n}{\sim}$ induced by γ^n and their join $\overset{\infty}{\sim}$,

$$s \overset{n}{\sim} t \text{ if and only if } \gamma^n(s) = \gamma^n(t); \tag{4.3}$$

$$\overset{\infty}{\sim} := \bigvee \overset{n}{\sim}, \text{ meaning } s \overset{\infty}{\sim} t \text{ if and only if } s \overset{n}{\sim} t \text{ for some } n; \tag{4.4}$$

for all $s, t \in S^1$.

Theorem 4.7. *The equivalence relation $\sim$ on S^1 induced by γ (1.5) is the closure of $\overset{\infty}{\sim}$.*

To prove this theorem we will need some preparations first. Consider two points $s, t \in S^1$ such that $\gamma(s) = \gamma(t)$. We want to show that s, t are equivalent with respect to the closure of $\overset{\infty}{\sim}$. Recall that $\gamma^n(s), \gamma^n(t)$ are contained in some n-edges by construction.

Lemma 4.8. *There is a constant N (independent of s, t, and n) such that*

$$\gamma^n(s), \gamma^n(t) \textit{ can be joined by at most } N \textit{ } n\textit{-edges},$$

for all $s, t \in S^1$ with $\gamma(s) = \gamma(t)$.

Proof. Since ϱ is a visual metric for F (see Section 2.3) it holds $\varrho(x,y) \asymp \Lambda^{-m}$ for all $x, y \in S^2$. Here $\Lambda > 1$ is the expansion factor of ϱ and $m = m(x,y)$ is the smallest number for which there exist disjoint m-tiles $X^m \ni x, Y^m \ni y$.

Fix $s, t \in S^1$ with $\gamma(s) = \gamma(t)$. Since γ^n converges uniformly to γ as in (3.2) it follows that

$$\varrho(\gamma^n(s), \gamma^n(t)) \lesssim \Lambda^{-n},$$

with a constant $C(\lesssim)$ independent of s, t, and n. Thus, there is a constant n_0 such that the $(n - n_0)$-tiles

$$X^{n-n_0} \ni \gamma^n(s),\ Y^{n-n_0} \ni \gamma^n(t) \text{ are not disjoint.}$$

We now want to cover X^{n-n_0}, Y^{n-n_0} by n-tiles. The number required may be unbounded (since we do not assume that $\mathcal{C}$ is F-invariant). Given an n-vertex v, let $W^n(v)$ be the union of all n-tiles containing v (this is the closure of an n-*flower* as defined in [BM, Chapter 5.4]). Then

$$\text{every } (n - n_0)\text{-tile can be covered by } M \text{ sets } W^n(v),$$

where the number M is independent of n, n_0 (see [BM, Lemma 5.29]). Clearly in any set $W^n(v)$ we can connect any two n-edges $E_1, E_2 \subset W^n(v)$ by at most $2k$ n-edges in $W^n(v)$. Thus $\gamma^n(s), \gamma^n(t)$ can be connected by at most $4kM$ n-edges. □

Proof of Theorem 4.7. Fix $s, t \in S^1$ with $\gamma(s) = \gamma(t)$. According to the last lemma let

$$E_1^n, \dots, E_N^n, \text{ with } \gamma^n(s) \in E_1^n, \gamma^n(t) \in E_N^n,$$

be a chain of n-edges for each n. We can assume that E_1^n, E_N^n are the images of n-arcs containing s, t by γ^n. By taking a subsequence we can assume that N, the number of n-edges in this chain, is the same for all n.

Let $[u_j^n, v_j^n] \subset \mathbb{R}/\mathbb{Z}$ be the n-arc that is mapped to E_j^n, $\gamma^n([u_j^n, v_j^n]) = E_j^n$. Then

$$v_j^n \overset{n}{\sim} u_{j+1}^n \text{ for } j = 1, \dots N - 1.$$

Taking subsequences, we can assume that all the sequences $(v_j^n)_{n\in\mathbb{N}}$, $(u_j^n)_{n\in\mathbb{N}}$ converge. Thus it follows from (2.4) (for all $j = 1, \dots, N$) that

$$\lim_n u_j^n = \lim_n v_j^n =: v_j \text{ and}$$
$$\lim_n v_1^n = s, \quad \lim_n v_N^n = t.$$

Now let $\widehat{\sim}$ be the closure of $\overset{\infty}{\sim}$. Then

$$s = v_1 \mathbin{\widehat{\sim}} v_2 \mathbin{\widehat{\sim}} \dots \mathbin{\widehat{\sim}} v_N = t.$$

Hence $s \mathbin{\widehat{\sim}} t$, meaning that $\widehat{\sim}$ is bigger than $\sim$. □

5. The white and black equivalence relations

In view of Theorem 1.5 and Theorem 4.7, it is possible to recover the map F (up to topological conjugacy) from the equivalence relations $\overset{n}{\sim}$, i.e., the self intersections of the approximations γ^n. Since our ultimate goal is to "unmate" the map F into two polynomials, we will decompose each equivalence relation $\overset{n}{\sim}$ into two equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ (on S^1). The equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ can be obtained inductively from $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$. Thus it is possible to recover F up to topological conjugacy from $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$, i.e., from finite data.

Roughly speaking $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ describe where γ^n touches itself "in the white component", or "in the black component". We have

$$\overset{n}{\sim} = \overset{n,\mathtt{w}}{\sim} \vee \overset{n,\mathtt{b}}{\sim}.$$

These two sequences of equivalence relations are closely related to the two polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ into which F unmates. In fact $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ yield the "critical portraits" of polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$. From the sequence $\overset{n,\mathtt{w}}{\sim}$ we construct an equivalence relation $\overset{\mathtt{w}}{\sim}$. More precisely $\overset{\mathtt{w}}{\sim}$ will be the closure of the join of the $\overset{n,\mathtt{w}}{\sim}$ (recall that $\sim$ was obtained from the relations $\overset{n}{\sim}$ in exactly the same way). We will then show that $\overset{\mathtt{w}}{\sim}$ is the equivalence relation induced by the Carathéodory semi-conjugacy $\sigma_{\mathtt{w}} \colon S^1 \to \mathcal{J}_{\mathtt{w}}$ of $P_{\mathtt{w}}$. The corresponding statements are obtained in exactly the same way for the black polynomial $P_{\mathtt{b}}$.

The equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ will be *non-crossing*. This means the following. Given two distinct equivalence classes $[s]_{n,\mathtt{w}}, [t]_{n,\mathtt{w}} \subset \mathbb{R}/\mathbb{Z}$ of $\overset{n,\mathtt{w}}{\sim}$ there are no points $a, c \in [s]_{n,\mathtt{w}}$, $b, d \in [t]_{n,\mathtt{w}}$ such that

$$0 \leq a < b < c < d < 1. \tag{5.1}$$

Note that sometimes the term "unlinked" has been used for what we call non-crossing.

It will be convenient to represent the equivalence relations geometrically in two different ways. View $S^1 \subset S^2$ as the equator. The components (hemispheres) of $S^2 \setminus S^1$ are denoted by $S^2_{\mathtt{w}}, S^2_{\mathtt{b}}$. The circle S^1 is positively oriented as the boundary of the *white* hemisphere $S^2_{\mathtt{w}}$ (negatively oriented as boundary of the *black* hemisphere $S^2_{\mathtt{b}}$). We equip $S^2_{\mathtt{w}}, S^2_{\mathtt{w}}$ with the hyperbolic metric (solely to be able to talk about *hyperbolic geodesics*).

For each equivalence class $[s]_{n,\mathtt{w}} \subset S^1 \subset S^2$ of $\overset{n,\mathtt{w}}{\sim}$ there is a *leaf*, which is the hyperbolically convex hull of $[s]_{n,\mathtt{w}}$ in $S^2_{\mathtt{w}}$. The *set of all leaves* is the *lamination* $\mathcal{L}^n_{\mathtt{w}}$. Similarly the lamination $\mathcal{L}^n_{\mathtt{b}}$ is defined in terms of $\overset{n,\mathtt{b}}{\sim}$ (with leaves in $S^2_{\mathtt{b}}$). That $\overset{n,\mathtt{w}}{\sim}$ is non-crossing is equivalent with the fact that distinct leaves in $\mathcal{L}^n_{\mathtt{w}}$ are disjoint.

The second way to geometrically represent the equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ is via *gaps*. A *white n-gap* is the closure of one component of $S^2_{\mathtt{w}} \setminus \bigcup \mathcal{L}^n_{\mathtt{w}}$. Gaps will correspond to tiles, leaves to vertices, and arcs to edges.

5.1. **An example.** The construction will be illustrated first for the example g from [Meyb, Section 1.4]. It is a Lattès map and may be obtained as follows. Consider the equivalence relation $z \simeq \pm z + m + ni$, for all $m, n \in \mathbb{Z}$, on $\mathbb{C}$. The map $z \mapsto 2z$ descends to the quotient $\mathbb{C}/\simeq$. Then g is topologically conjugate to $2z/\simeq \, : \mathbb{C}/\simeq \to \mathbb{C}/\simeq$. The degree of the map is 4. This map is best represented as follows. Glue two unit squares together along their boundaries to form a *pillow*, which is a topological sphere. Divide each side of this pillow (i.e., each unit square) into four squares. Map each of these small squares to one of the sides as indicated in Figure 2 of the pillow yields the map g (up to topological conjugacy). Note that the vertices as the pillow are the postcritical points of the map. We choose the boundary of the pillow as the curve $\mathcal{C}$. The two sides of the pillow then are the two 0-tiles. The 8 small squares are the 1-tiles.

The 0-th approximation of the invariant Peano curve is $\gamma^0 = \mathcal{C}$. The first approximation γ^1 is shown in the left of Figure 3. It should be emphasized that we

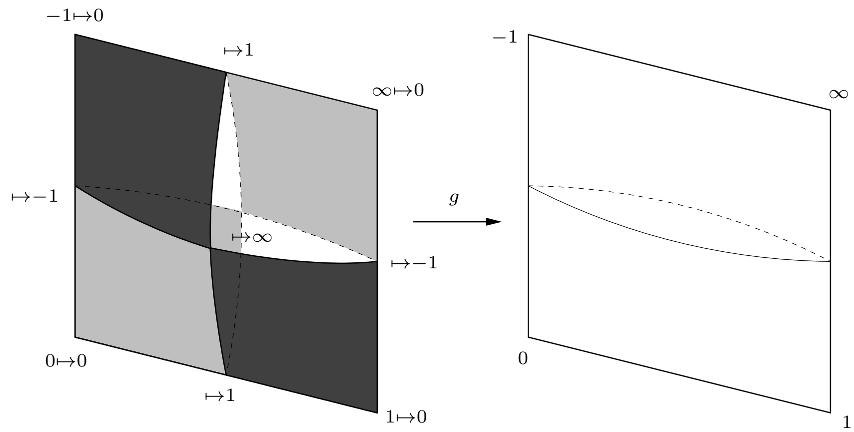


FIGURE 2. The Lattès map g.

could have used several other choices for γ^1. Each of these would have resulted in a different invariant Peano curve and different pairs of polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ into which g unmates.

For illustrative purposes it is best to lift objects to $\mathbb{C}$, technically speaking we work in the *orbifold covering*. The (lifts of the) approximations γ^1, γ^0 are illustrated in Figure 4. Practically speaking this means we cut the pillow from Figure 3 so that the the two halves of the "back of the pillow" are folded to the left and right, to obtain the picture in the lower left of Figure 4.

The parametrization of γ^1 and γ^0 is shown as well. Additionally the connections of the 1-tiles is shown (see Section 3.5). Namely the 4 white 1-tiles are connected at the 3 1-vertices in the middle. The 1-angles 2/16 and 10/16 are both mapped by γ^1 to the point in the middle. The set $\{2/16, 10/16\}$ forms one equivalence class of $\overset{1,\mathtt{w}}{\sim}$. The non-trivial equivalence classes (i.e., the ones which are not singletons) are

$$\text{equivalence classes of } \overset{1,\mathtt{w}}{\sim}\colon\ \left\{\frac{2}{16}, \frac{10}{16}\right\}, \left\{\frac{3}{16}, \frac{7}{16}\right\}, \left\{\frac{11}{16}, \frac{15}{16}\right\}. \tag{5.2}$$

The curve γ^1 also "touches itself in the black component". This however is more easily seen in Figure 3 than in the orbifold covering, i.e., in Figure 4. The corresponding non-trivial equivalence classes of $\overset{1,\mathtt{b}}{\sim}$ are

$$\text{equivalence classes of } \overset{1,\mathtt{b}}{\sim}\colon\ \left\{\frac{1}{16}, \frac{5}{16}\right\}, \left\{\frac{6}{16}, \frac{14}{16}\right\}, \left\{\frac{9}{16}, \frac{13}{16}\right\}. \tag{5.3}$$

We now view $S^1 = \mathbb{R}/\mathbb{Z}$ (i.e., the domain of γ^1, γ^0) as the boundary of a disk, which we identify with the hemisphere $S^2_{\mathtt{w}}$. The hyperbolic geodesic (in $S^2_{\mathtt{w}}$) connecting the two points of one equivalence class of $\overset{1,\mathtt{w}}{\sim} \subset S^1$ is a *leaf*. The lamination $\mathcal{L}^1_{\mathtt{w}}$ is the set of these three leaves.

A *white* 1-*gap* G is the closure of one component of $S^2_{\mathtt{w}} \setminus \bigcup \mathcal{L}^1_{\mathtt{w}}$. The set of all white 1-gaps is denoted by $\mathbf{G}^1_{\mathtt{w}}$.

Similarly we connect the points of one equivalence class of $\overset{1,\mathtt{b}}{\sim}$ by a hyperbolic geodesic in the outside of the circle (which we identify with the hemisphere $S^2_{\mathtt{b}}$). The lamination $\mathcal{L}^1_{\mathtt{b}}$ is the set of the three leaves thus obtained, the closure of each component of $S^2_{\mathtt{b}}$ is a black 1-gaps. This is illustrated in the top left of Figure 4.

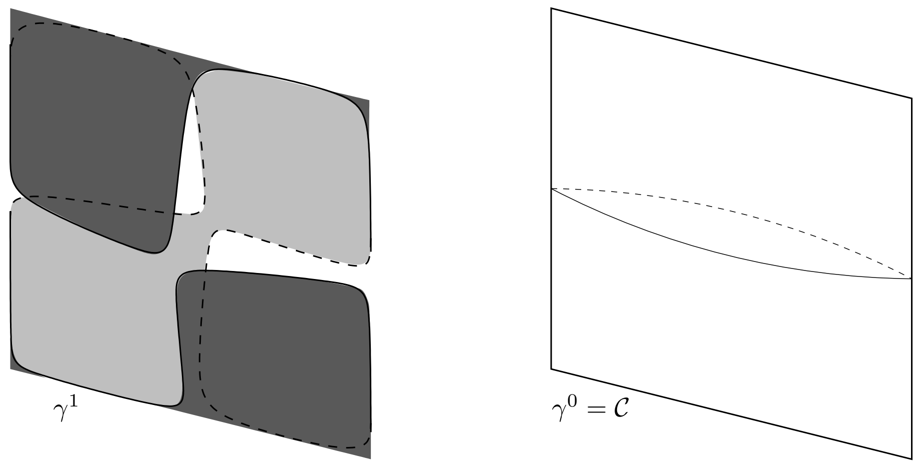


FIGURE 3. First approximation of invariant Peano curve.

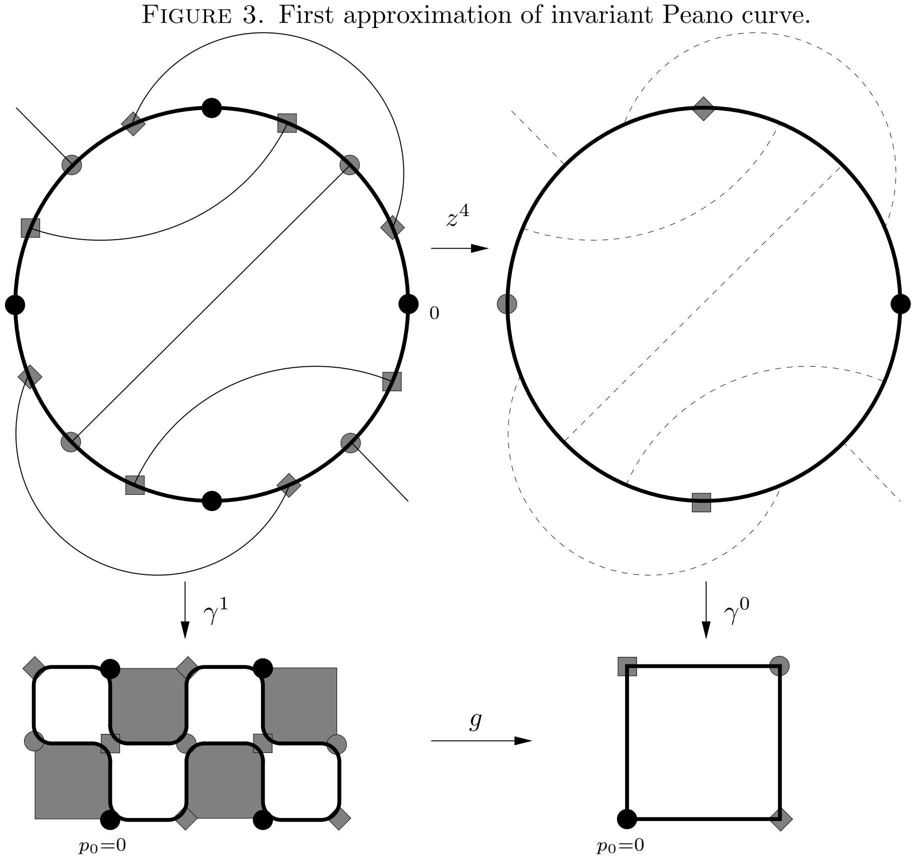


FIGURE 4. The laminations $\mathcal{L}^1_{w,g}, \mathcal{L}^1_{b,g}$.

Recall that each 1-arc was mapped by γ^1 to a 1-edge. Each white/black 1-gaps correspond to a white/black 1-tile. Note that each 1-gap G has 4 1-arcs in its boundary, one of each type. Thus $G \cap S^1$ is mapped by $\mu(t) = 4t$. Furthermore these four 1-arcs are mapped by γ^1 to 1-edges contained in the same 1-tile. Each 1-leave corresponds to a 1-vertex, i.e., each equivalence class (of $\overset{1,\mathtt{w}}{\sim}$ or of $\overset{1,\mathtt{b}}{\sim}$) is mapped by γ^1 to a 1-vertex. Note however that several distinct 1-leaves may be mapped to the same 1-vertex (this corresponds to distinct critical points being identified in the mating). Thus the combinatorial description via tiles, edges, and vertices is given via gaps, arcs, and leaves.

The equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ are constructed in the same fashion. They can however be obtained inductively from $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ as follows. Consider a *white* 1-*gap* G. The following holds for all $s,t \in S^1 = \mathbb{R}/\mathbb{Z}$:

$$
\begin{aligned}
&s \overset{2,\mathtt{w}}{\sim} t \text{ if and only if}\\
&\qquad \text{there is a white 1-gap } G \ni s,t \text{ and } \mu(s) \overset{1,\mathtt{w}}{\sim} \mu(t);
\end{aligned}
$$

thus the second white equivalence relation $\overset{2,\mathtt{w}}{\sim}$ is obtained from $\overset{1,\mathtt{w}}{\sim}$. In the same fashion all equivalence relations $\overset{n,\mathtt{w}}{\sim}$ can be inductively constructed from $\overset{1,\mathtt{w}}{\sim}$; similarly all $\overset{n,\mathtt{b}}{\sim}$ can be constructed from $\overset{1,\mathtt{b}}{\sim}$. Thus the lists (5.2), (5.3) contain all information to recover the map g up to topological conjugacy (by Theorem 4.7 and 1.5). They are called the white and black *critical portraits* of g.

5.2. **Definition of $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$.** We now define the equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ in general. They will be defined in terms of which white/black n-tiles are connected at some n-vertex, i.e., in terms of the *connection of n-tiles* (see Section 3.5).

Let v be an n-vertex. Consider one block $b \in \pi^n_{\mathtt{w}}(v) \cup \pi^n_{\mathtt{b}}(v)$ (from the cnc-partition defining the connection at v). Let $X_0, \dots, X_{2m-1}$ be the n-tiles containing v. We call an n-edge $E \ni v$ *incident to b at v* if

$$E \subset X_i, \qquad \text{where } i \in b.$$

Each n-edge is incident to exactly one block $b \in \pi^n_{\mathtt{w}}(v)$ and one block $c \in \pi^n_{\mathtt{b}}(v)$. Succeeding n-edges E, E' (at v) are incident to the same white/black block (see [Meyb, Lemma 6.13]).

Consider an n-angle $\alpha^n_j \in S^1$ such that $\gamma^n(\alpha^n_j) = v$. It is called *incident to the block $b \in \pi^n_{\mathtt{w}}(v) \cup \pi^n_{\mathtt{b}}(v)$ at v* if the n-arc $a^n_j = [\alpha^n_j, \alpha^n_{j+1}]$ (as well as $a^n_{j-1} = [\alpha^n_{j-1}, \alpha^n_j]$) is mapped by γ^n to an n-edge incident to b at v. In this case we also say that the n-edges $E = \gamma^n(a^n_{j-1})$ and $E' = \gamma^n(a^n_j)$ are incident to the n-angle α^n_j (at v) and vice versa. Recall that the set of n-angles $\mathbf{A}^n = \{\alpha^n_j\} \subset S^1$ is the set of all points that are mapped by γ^n to some n-vertex.

Definition 5.1 (Equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$)**.** Let the connection at any n-vertex v be given by the cnc-partition $\pi^n_{\mathtt{w}}(v) \cup \pi^n_{\mathtt{b}}(v)$.

Define the equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ on $\mathbf{A}^n$ by

$$\alpha \overset{n,\mathtt{w}}{\sim} \alpha' :\Leftrightarrow \alpha, \alpha' \text{ are incident to the same block } b \in \pi^n_{\mathtt{w}}(v), \text{ at some } v \in \mathbf{V}^n;$$

$$\alpha \overset{n,\mathtt{b}}{\sim} \alpha' :\Leftrightarrow \alpha, \alpha' \text{ are incident to the same block } b \in \pi^n_{\mathtt{b}}(v), \text{ at some } v \in \mathbf{V}^n;$$

for all $\alpha, \alpha' \in \mathbf{A}^n$. Thus there is a one-to-one correspondence between blocks $b \in \pi^n_{\mathtt{w}}(v)$ and equivalence classes with respect to $\overset{n,\mathtt{w}}{\sim}$; analogously between blocks $b \in \pi^n_{\mathtt{b}}(v)$ and equivalence classes with respect to $\overset{n,\mathtt{b}}{\sim}$. Note that n-angles α, α' such that $\gamma^n(\alpha) \neq \gamma^n(\alpha')$ are never equivalent with respect to $\overset{n,\mathtt{w}}{\sim}$ or $\overset{n,\mathtt{b}}{\sim}$.

It will be convenient to consider the equivalence relations $\overset{0,w}{\sim}, \overset{0,b}{\sim}$ on $\mathbf{A}^0$ as well. They are defined to be trivial, meaning that each equivalence class is a singleton.

We can view $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ as equivalence relations on S^1. Namely each $s \in S^1 \setminus \mathbf{A}^n$ is equivalent to itself and only to itself with respect to both $\overset{n,\mathtt{w}}{\sim}$ and $\overset{n,\mathtt{b}}{\sim}$. Let us record that $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$ *generate* the equivalence relation $\overset{n}{\sim}$, which is an immediate consequence of [Meyb, Lemma 6.2],

$$\overset{n}{\sim} = \overset{n,\mathtt{w}}{\sim} \vee \overset{n,\mathtt{b}}{\sim}. \tag{5.4}$$

The angles of each equivalence class $[\alpha^n]_{n,\mathtt{w}} \subset S^1$ inherit the cyclical ordering from S^1.

Definition 5.2 (Succeeding)**.** Two n-angles $\alpha^n, \tilde{\alpha}^n$ are called *succeeding* (with respect to $\overset{n,\mathtt{w}}{\sim}$) if

$$\alpha^n \overset{n,\mathtt{w}}{\sim} \tilde{\alpha}^n \quad \text{and}$$
$$\tilde{\alpha}^n \text{ is the successor to } \alpha^n \text{ in } [\alpha^n]_{n,\mathtt{w}}.$$

This means that the open arc $(\alpha^n, \tilde{\alpha}^n) \subset S^1$ does not contain any n-angle from $[\alpha^n]_{n,\mathtt{w}}$. The n-angle $\tilde{\alpha}^n$ is the *predecessor* to α^n.

A finite sequence $\alpha^n_1, \dots, \alpha^n_k$ is called *succeeding* (with respect to $\overset{n,\mathtt{w}}{\sim}$) if $\alpha^n_j, \alpha^n_{j+1}$ are succeeding with respect to $\overset{n,\mathtt{w}}{\sim}$ (for all $1 \leq j \leq k-1$). Clearly $\alpha^n \overset{n,\mathtt{w}}{\sim} \tilde{\alpha}^n$ if and only if there is a sequence of succeeding n-angles from α^n to $\tilde{\alpha}^n$ ($\mathbf{A}^n$ is a finite set).

Two succeeding angles correspond to a hyperbolic geodesic which forms a boundary arc of some leaf of the lamination.

The nect lemma shows that the cyclical ordering may The following lemma gives an alternative way to construct $\overset{n,\mathtt{w}}{\sim}$.

Lemma 5.3. *Consider two n-angles $\alpha^n_i, \alpha^n_j \in S^1$ that are mapped by γ^n to the same n-vertex v. Then*

α^n_i, α^n_j *are* succeeding

if and only if

The n-arcs $a^n_i = [\alpha^n_i, \alpha^n_{i+1}], a^n_{j-1} = [\alpha^n_{j-1}, \alpha^n_j]$ are mapped by γ^n to n-edges in the same white n-tile X.

In this case the n-edge $E' = \gamma^n(a^n_i)$ succeeds $E = \gamma^n(a^n_{j-1})$ in ∂X.

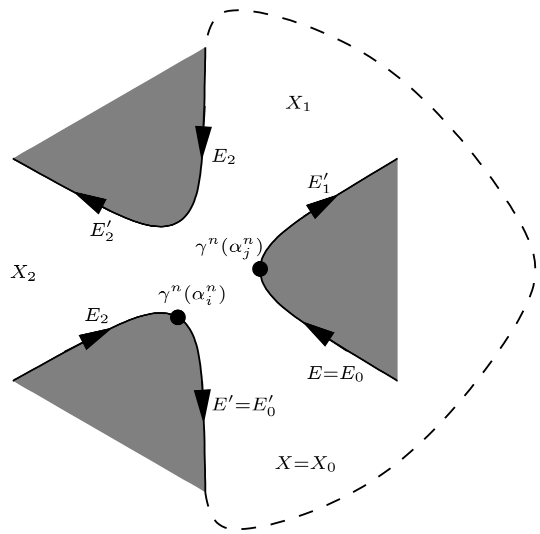


FIGURE 5. Illustration to Lemma 5.3

Here ∂X is oriented as positively as boundary of X as usual. The situation is illustrated in Figure 5.

Proof. Recall from Section 3.1 that the curve γ^n traverses each n-edge E positively as boundary of the white n-tile that E is contained in.

Note that both conditions in the statement of the lemma imply that α_i^n, α_j^n are incident to the same block $b \in \pi_{\mathtt{w}}^n(v)$.

Let $X_0, \dots, X_{m-1}$ be the white n-tiles incident to b (at v), cyclically ordered around v. Let $E_j, E_j' \subset X_j$ be the n-edges with terminal/initial point v. Then the cyclical order of the n-edges incident to b around v is $E_0', E_0, E_1', E_1, \dots, E_{m-1}', E_{m-1}$. Recall from Section 3.5 that n-edges are succeeding in γ^n (at v) if and only if they are succeeding with respect to $\pi_{\mathtt{b}}^n(v) \cup \pi_{\mathtt{w}}^n(v)$. Thus E_l is succeeded by E_{l+1}' in γ^n (index l is taken mod m), by definition (see Section 3.4). This means that E_l and E_{l+1}' are both incident to the same n-angle α incident to b. The proof thus will be finished if we can show that in γ^n each n-edge E_l' is followed by E_l, i.e., in γ^n between E_l' and E_l there is no other n-edge incident to b.

Recall that γ^n is a non-crossing Eulerian circuit. This means we can change γ^n slightly in a neighborhood of each n-vertex to obtain a Jordan curve γ_ϵ^n. If E_l' would be followed in γ^n by E_m (where $m \neq l$) (as shown in Figure 5) this would not be possible. Indeed let α be the n-angle incident to E_l' and β be the n-angle incident to E_m at v. We can connect $\gamma_\epsilon^n(\alpha)$ and $\gamma_\epsilon^n(\beta)$ by a an arc that does not cross γ_ϵ^n to form a Jordan curve Γ. Both components of Γ then contain parts of the curve γ_ϵ^n, which is a contradiction.

$\square$

Note that the previous lemma remains valid if $[\alpha_i^n]_{n,\mathtt{w}}$ consists of a single point, then of course $\alpha_i^n = \alpha_j^n$.

5.3. **Laminations and gaps.** We now define the laminations and gaps. These are different geometric representations of the equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$.

We view $S^1 \subset S^2$ as the equator. Recall that the two hemispheres $S^2_{\mathtt{w}}, S^2_{\mathtt{b}}$ (i.e., complementary components of $S^2 \setminus S^1$) are equipped with the hyperbolic metric.

Definition 5.4 (Laminations $\mathcal{L}^n_{\mathtt{w}}, \mathcal{L}^n_{\mathtt{b}}$)**.** Let $\alpha \in \mathbf{A}^n$ and $[\alpha]_{n,\mathtt{w}} \subset S^1$ be its equivalence class with respect to $\overset{n,\mathtt{w}}{\sim}$. A *leaf* $L = L([\alpha]_{n,\mathtt{w}})$ is the hyperbolically convex hull of $[\alpha]_{n,\mathtt{w}}$ in $S^2_{\mathtt{w}}$. If $[\alpha]_{n,\mathtt{w}}$ corresponds to the block $b \in \pi^n_{\mathtt{w}}(v)$ we sometimes write $L = L(v, b)$ to indicate "where the leaf comes from". The set of all such leaves is the *white lamination* $\mathcal{L}^n_{\mathtt{w}}$.

If $\#[\alpha]_{n,\mathtt{w}} = m$ the leaf $L = L([\alpha]_{n,\mathtt{w}})$ is an *ideal m-gon*. If $[\alpha]_{n,\mathtt{w}}$ consists of two points the corresponding leaf L is the *hyperbolic geodesic* (in $S^2_{\mathtt{w}}$) between these points. If $[\alpha]_{n,\mathtt{w}}$ consists of a single point the corresponding leaf is $L = \{\alpha\}$.

The *black lamination* $\mathcal{L}^n_{\mathtt{b}}$ (containing m-gons in $S^2_{\mathtt{b}}$) is constructed similarly from the equivalence classes of $\overset{n,\mathtt{b}}{\sim}$.

Remark 5.5. A *lamination* is usually defined to be a closed set of disjoint *geodesics*. One obtains a lamination in this standard sense by taking the boundaries of all our leaves. We use this slightly non-standard terminology, since we do not want to make a distinction between hyperbolic geodesics and ideal m-gons.

Remark 5.6. We will formulate the following lemmas only for the *white* equivalence relations $\overset{n,\mathtt{w}}{\sim}$, and the corresponding white laminations $\mathcal{L}^n_{\mathtt{w}}$. In each case there is an obvious analog for *black* equivalence relations $\overset{n,\mathtt{b}}{\sim}$, and the corresponding black laminations $\mathcal{L}^n_{\mathtt{b}}$. More precisely one has to replace "white" by "black", and each index "$\mathtt{w}$" by "$\mathtt{b}$". Furthermore in Lemma 5.9 "cyclically" has to be replaced by "anti-cyclically".

Recall that $\mu\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $\mu(t) = dt(\text{mod}1)$. The *image* of an ideal m-gon L (in $S^2_{\mathtt{w}}$ or $S^2_{\mathtt{b}}$) with vertices $\alpha^n_{j_1}, \dots, \alpha^n_{j_m}$ by μ is the ideal $\widetilde{m}$-gon $\tilde{L}$ with vertices $\mu(\alpha^n_{j_1}), \dots, \mu(\alpha^n_{j_m})$. The ideal $\widetilde{m}$-gon $\tilde{L}$ lies in the same hemisphere (in $S^2_{\mathtt{w}}$ or $S^2_{\mathtt{b}}$) as L. We write $\mu(L) = \tilde{L}$.

Lemma 5.7 (Properties of $\overset{n,\mathtt{w}}{\sim}$ and $\mathcal{L}^n_{\mathtt{w}}$)**.** *The equivalence relations $\overset{n,\mathtt{w}}{\sim}$ and the laminations $\mathcal{L}^n_{\mathtt{w}}$ satisfy the following.*

($\mathcal{L}^n$ 1) *The non-trivial equivalence classes are in $\mathbb{Q}$ and are mapped by the iterate μ^n to a single point,*

$$[\alpha^n]_{n,\mathtt{w}} \subset \mathbb{Q}, \quad \mu^n([\alpha^n]_{n,\mathtt{w}}) = \{\alpha^0\} \in \mathbf{A}^0,$$

for all $\alpha^n \in \mathbf{A}^n$.

($\mathcal{L}^n$ 2) *The equivalence relations $\overset{n,\mathtt{w}}{\sim}$ are* non-crossing *(see (5.1)). This means that the leaves in $\mathcal{L}^n_{\mathtt{w}}$ are* disjoint.

($\mathcal{L}^n$ 3) *Let $\{L_j\} = \mathcal{L}^n_{\mathtt{w}}$, and each L_j an ideal m_j-gon. Then*

$$\sum_j m_j = kd^n, \quad \sum_j (m_j - 1) = d^n - 1.$$

($\mathcal{L}^n$ 4) *The lamination $\mathcal{L}^{n+1}_{\mathtt{w}}$ is mapped by μ to $\mathcal{L}^n_{\mathtt{w}}$. This means*

$$L^{n+1} \in \mathcal{L}^{n+1}_{\mathtt{w}} \Rightarrow \mu(L^{n+1}) \in \mathcal{L}^n_{\mathtt{w}},$$

equivalently

$$\mu([\alpha^{n+1}]_{n+1,\mathtt{w}}) = [\mu(\alpha^{n+1})]_{n,\mathtt{w}} = [\alpha^n]_{n,\mathtt{w}},$$

for all $\alpha^{n+1} \in \mathbf{A}^{n+1}$*, where* $\alpha^n = \mu(\alpha^{n+1}) \in \mathbf{A}^n$*.*

($\mathcal{L}^n$ 5) *For each leaf* $L^n \in \mathcal{L}^n_{\mathtt{w}}$ *there is a leaf* $L^{n+1} \in \mathcal{L}^{n+1}_{\mathtt{w}}$*, such that* $\mu(L^{n+1}) = L^n$*. More precisely let* $L^{n+1}_1, \dots, L^{n+1}_N$ *be the leaves that are mapped by* μ *to* L^n*. Let* $L^n = L^n([\alpha^n]_{n,\mathtt{w}})$ *be an ideal* m*-gon (*$\#[\alpha^n]_{n,\mathtt{w}} = m$*) and each* $L^{n+1}_j = L^{n+1}([\alpha^{n+1}_j]_{n,\mathtt{w}})$ *an ideal* m_j*-gon (*$\#[\alpha^{n+1}_j] = m_j$*). Each* m_j *is a multiple of* m *and*

$$\sum_j m_j = d\, m.$$

($\mathcal{L}^n$ 6) *Restricted to* $\mathbf{A}^n$ *it holds*

$$\overset{n+1,\mathtt{w}}{\sim} = \overset{n,\mathtt{w}}{\sim}, \text{ equivalently}$$

$$[\alpha^n]_{n,\mathtt{w}} = [\tilde{\alpha}^n]_{n,\mathtt{w}} \Leftrightarrow [\alpha^n]_{n+1,\mathtt{w}} = [\tilde{\alpha}^n]_{n+1,\mathtt{w}},$$

for all $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$*,* $n \geq 0$*. In particular*

$$[\alpha^n]_{n,\mathtt{w}} \subset [\alpha^n]_{n+1,\mathtt{w}}, \quad \text{equivalently } L^n \subset L^{n+1},$$

where $L^n = L^n([\alpha^n]_{n,\mathtt{w}}) \in \mathcal{L}^n_{\mathtt{w}}$*,* $L^{n+1} = L^{n+1}([\alpha^n]_{n+1,\mathtt{w}}) \in \mathcal{L}^{n+1}_{\mathtt{w}}$*. This means that*

$$\overset{1,\mathtt{w}}{\sim} \leq \overset{2,\mathtt{w}}{\sim} \leq \dots .$$

Note also that this means that distinct 0*-angles* $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$ *are not equivalent with respect to any* $\overset{n,\mathtt{w}}{\sim}$*.*

We now shift our attention to the *complements* of the laminations.

Definition 5.8 (Gaps)**.** The closure of one component of $S^2_{\mathtt{w}} \setminus \bigcup \mathcal{L}^n_{\mathtt{w}}$ ($S^2_{\mathtt{w}} \setminus \bigcup \mathcal{L}^n_{\mathtt{w}}$) is called a white (black) *gap* or n-gap. The set of all white n-gaps is denoted by $\mathbf{G}^n_{\mathtt{w}}$.

Let $E_0, \dots, E_{k-1}$ be the 0-edges ordered cyclically on $\mathcal{C}$, meaning mathematically positively as boundary of the white 0-tile $X^0_{\mathtt{w}}$. Each n-edge E^n is said to be of *type* j if $F^n(E^n) = E_j$. Similarly each n-arc $a^n \subset S^1$ is of type j if $\gamma^n(\alpha^n)$ is, i.e., if $F^n(\gamma^n(a^n)) = E_j$.

In the same fashion let $p_0, \dots, p_{k-1}$ be the postcritical points labeled cyclically on $\mathcal{C}$. Each n-vertex v is of *type* j if $F^n(v) = p_j$. Note that v is also an $(n+m)$-vertex (for each $m \geq 0$), and might be of different type as such. A leaf $L = L(v, b) \in \mathcal{L}^n_{\mathtt{w}}$ is of *type* j if v is.

Lemma 5.9 (Properties of gaps)**.** *We have the following properties.*

($\mathbf{G}$ 1) *There is one white* n*-gap for each white* n*-tile,*

$$\#\mathbf{G}^n_{\mathtt{w}} = d^n.$$

($\mathbf{G}$ 2) *Each gap* $G \in \mathbf{G}^n_{\mathtt{w}}$ *has* k n-arcs $\subset S^1$ *in its boundary,* one of each type. *Their types are* cyclically ordered *as boundary of* G*. Equivalently* G *intersects* k leaves $L \in \mathcal{L}^n_{\mathtt{w}}$, one of each type, cyclically ordered *on* ∂G*.*

($\mathbf{G}$ 3) *Consider two* n*-arcs* $a^n, b^n \subset S^1$*. Then*

$$a^n, b^n \textit{ are in the boundary of the } \text{same } \textit{gap } G \in \mathbf{G}^n_{\mathtt{w}}$$

if and only if

$$\gamma^n(a^n), \gamma^n(b^n) \textit{ are contained in the } \text{same } \textit{white } n\textit{-tile.}$$

(**G** 4) *The $(n+1)$-gaps are mapped to n-gaps by μ. That means for each gap $G^{n+1} \in \mathbf{G}^{n+1}_{\mathtt{w}}$ there is a gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$ such that*

$$\mu(G^{n+1} \cap S^1) = G^n \cap S^1.$$

Furthermore μ is injective on the interior of $G^{n+1} \cap S^1$.

(**G** 5) *Every gap $G^{n+1} \in \mathbf{G}^{n+1}_{\mathtt{w}}$ is contained in a (unique) gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$,*

$$G^n \supset G^{n+1}.$$

(**G** 6) *There is a constant n_0 such that the following holds. Let $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ be* not equivalent *with respect to $\overset{n,\mathtt{w}}{\sim}$. Then for $m \geq n + n_0$ no gap $G^m \in \mathbf{G}^m_{\mathtt{w}}$ contains points from both sets $[\alpha^n]_{m,\mathtt{w}}, [\tilde{\alpha}^n]_{m,\mathtt{w}}$.*

Properties (**G** 1), (**G** 2), and (**G** 2) show that each gap $G \in \mathbf{G}^n_{\mathtt{w}}$ corresponds to a white n-tile.

Proof of Lemma 5.7 and Lemma 5.9. ($\mathcal{L}^n$ 2) Consider an equivalence class $[\alpha]_{n,\mathtt{w}}$, where $\alpha \in \mathbf{A}^n$. Let $b \in \pi^n_{\mathtt{w}}(v)$ be the block corresponding to $[\alpha]_{n,\mathtt{w}}$ ($\tilde{\alpha} \in [\alpha]_{n,\mathtt{w}} \Leftrightarrow \tilde{\alpha}$ incident to b at v).

Recall from Section 3.6 that the n-th white *connection graph* Γ^n is a *tree.*

Consider an n-arc $[\beta, \beta'] \subset \mathbb{R}/\mathbb{Z} = S^1$ (between two consecutive n-angles $\beta, \beta' \in \mathbf{A}^n$) in one component of $S^1 \setminus [\alpha]_{n,\mathtt{w}}$. Let $d \in \pi^n_{\mathtt{w}}(w), d' \in \pi^n_{\mathtt{w}}(w')$ be the blocks associated to $[\beta]_{n,\mathtt{w}}, [\beta']_{n,\mathtt{w}}$. Then $c(w,d), c(w',d')$ (the vertices in the connection graph Γ^n associated to d, d') are in the *same component* of $\Gamma^n \setminus c(v,b)$. Indeed $\gamma^n([\beta,\beta'])$ is contained in a white n-tile incident to both d, d'.

Assume there is an equivalence class $[\alpha']_{n,\mathtt{w}}$ ($\alpha' \in \mathbf{A}^n$) containing n-angles in *distinct* components of $S^1 \setminus [\alpha]_{n,\mathtt{w}}$ (i.e., $[\alpha]_{n,\mathtt{w}}, [\alpha']_{n,\mathtt{w}}$ are *crossing*). Let $b' \in \pi^n_{\mathtt{w}}(v')$ be the associated block. Then in Γ^n the vertex $c(v',b')$ connects distinct components of $\Gamma^n \setminus c(v,b)$. Thus Γ^n is not a tree, which is a contradiction.

(**G** 2) Fix a white n-tile X. Let $E_0, \dots, E_{k-1}$ be the n-edges in the boundary of X (ordered mathematically positively in ∂X). Consider the n-arc $a^n_{j-1} = [\alpha^n_{j-1}, \alpha^n_j] \subset S^1$ that is mapped by γ^n to E_0. Let $G \in \mathbf{G}^n_{\mathtt{w}}$ be the gap having a^n_{j-1} in its boundary. Consider the n-arc $a^n_i = [\alpha^n_i, \alpha^n_{i+1}]$ that is the cyclical successor to a^n_{j-1} in ∂G. Note that α^n_i is the n-angle *preceding* α^n_j with respect to $\overset{n,\mathtt{w}}{\sim}$. From Lemma 5.3 it follows that $\gamma^n(a^n_i) = E_1$. Continuing in this fashion yields the claim (note γ^n maps *exactly one* n-arc to each n-edge).

(**G** 3) This follows directly from the previous argument.

(**G** 1) From the above it is clear that there is a *bijection* between white n-tiles and white n-gaps. Thus there are exactly d^n such components.

($\mathcal{L}^n$ 3) The first equality follows from the fact that $\sum m_j = \#\mathbf{A}^n$, the number of n-angles, which in turn equals kd^n.

Recall from Section 3.6 the definition of the *connection graph.* Given a white n-tile X let $G = G(X) \in \mathbf{G}^n_{\mathtt{w}}$ be the corresponding gap according to (**G** 3), meaning that $\gamma^n(G \cap S^1) = \partial X$. From Definition 5.1 it follows that the vertices $c(X)$, $c(v,b)$ (of the n-th white connection graph) are connected by an edge if and only if the gap $G = G(X)$ has non-empty intersection with the leaf $L(v,b) \in \mathcal{L}^n_{\mathtt{w}}$.

By (**G** 2) and (**G** 1) it thus follows that the (n-th white) connection graph has kd^n edges. On the other hand it has $\#\mathcal{L}^n_{\mathtt{w}} + d^n$ vertices and is a tree. Thus it has

$\#\mathcal{L}^n_{\mathtt{w}} + d^n - 1$ edges, which implies that $\#\mathcal{L}^n_{\mathtt{w}} = (k-1)d^n + 1$. Hence

$$\sum_j (m_j - 1) = \sum_j m_j - \#\mathcal{L}^n_{\mathtt{w}} = kd^n - ((k-1)d^n + 1) = d^n - 1.$$

($\mathcal{L}^n$ 4) This follows from [Meyb, Lemma 6.15], Lemma 5.3, as well as the fact that F maps succeeding $(n+1)$-edges to succeeding n-edges and $(n+1)$-tiles to n-tiles (of the same color).

(**G** 4) Each $(n+1)$-arc is mapped by μ to an n-arc. The statement follows from (**G** 2) and ($\mathcal{L}^n$ 4).

($\mathcal{L}^n$ 5) Consider a leaf $L^n = L^n(v,b) \in \mathcal{L}^n_{\mathtt{w}}$, and an n-edge $E^n \ni v$ incident to $b \in \pi^n_{\mathtt{w}}(v)$ at v. Let $E^{n+1} \ni v'$ be an $(n+1)$-edge that is mapped by F to E^n, where $F(v') = v$. Then (by [Meyb, Lemma 6.15] and Lemma 5.3) the leaf $L^{n+1}_j = L^{n+1}(v',b') \in \mathcal{L}^n_{\mathtt{w}}$ is mapped by μ to L^n, where E^{n+1} is incident to $b' \in \pi^{n+1}_{\mathtt{w}}(v')$ at v'. If L^{n+1}_j is an ideal m_j-gon the number $d_j := m_j/m$ is the number of preimages ($(n+1)$-edges) of E^n incident to b' at v'. Thus $\sum d_j = d$.

($\mathcal{L}^n$ 1) Each n-angle $\alpha^n \in \mathbf{A}^n$ is rational by construction (see [Meyb, Sections 4.1 and Section 4.2]). The second claim follows from ($\mathcal{L}^n$ 4) and the fact that each equivalence class $[\alpha^0]_{0,w}$ ($\alpha^n \in \mathbf{A}^0$) is a singleton (recall that $\gamma^0 = \mathcal{C}$ is a Jordan curve).

The proof of (**G** 5) will be postponed to Section 5.4, the proofs of ($\mathcal{L}^n$ 6) and (**G** 6) to Section 5.5. $\square$

5.4. **Inductive construction of $\overset{n,\mathtt{w}}{\sim}$.** We come to the main result in this section, namely that $\overset{n,\mathtt{w}}{\sim}$ (or equivalently $\mathcal{L}^n_{\mathtt{w}}$) can be constructed inductively. This means that $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ together allows to recover the map F up to topological conjugacy.

The inductive construction of $\mathcal{L}^{n+1}_{\mathtt{w}}$ may be paraphrased as follows. The boundary $G \cap S^1$ of each gap $G \in \mathbf{G}^n_{\mathtt{w}}$ is mapped by μ^n onto S^1. Pulling the lamination $\mathcal{L}^1_{\mathtt{w}}$ back constructs $\mathcal{L}^{n+1}_{\mathtt{w}}$.

Recall that $\mu^n(z) = d^n t(\text{mod} 1)$ denotes the n-th iterate of $\mu\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. For any gap $G \in \mathbf{G}^n_{\mathtt{w}}$, this map is surjective on $G \cap S^1$ by (**G** 2), a homeomorphism on each n-arc, and may fail to be injective only at endpoints of n-arcs.

Given $\overset{n,\mathtt{w}}{\sim}$ we know all white n-gaps. To construct $\overset{n+1,\mathtt{w}}{\sim}$ it is enough to construct all $(n+1)$-angles that are succeeding with respect to $\overset{n+1,\mathtt{w}}{\sim}$ (Definition 5.2).

Theorem 5.10 (Inductive construction of $\overset{n,\mathtt{w}}{\sim}$)**.** *For all $\alpha^{n+1}, \tilde{\alpha}^{n+1} \in \mathbf{A}^{n+1}$ it holds*

$$\alpha^{n+1}, \tilde{\alpha}^{n+1} \textit{ are succeeding with respect to } \overset{n+1,\mathtt{w}}{\sim}$$

if and only if

$$\alpha^{n+1}, \tilde{\alpha}^{n+1} \textit{ are contained in the same white } n\textit{-gap } G^n \textit{ and}$$

$$\mu^n(\alpha^{n+1}), \mu^n(\tilde{\alpha}^{n+1}) \textit{ are succeeding with respect to } \overset{1,\mathtt{w}}{\sim}.$$

Let us emphasize that $(n+1)$-angles in disjoint n-gaps are never succeeding, though the generated equivalence relation may identify angles from different gaps.

We first need some preparation to prove Theorem 5.10. In the following lemma we consider $(n+1)$-arcs

$$a^{n+1}, b^{n+1} \subset S^1,$$

the corresponding $(n+1)$-edges

$$D^{n+1} = \gamma^{n+1}(a^{n+1}), E^{n+1} = \gamma^{n+1}(b^{n+1}),$$

and arcs $A^{n+1}, B^{n+1} \subset \bigcup \mathbf{E}^n$, satisfying

$$H_1^n(A^{n+1}) = D^{n+1}, H_1^n(B^{n+1}) = E^{n+1}.$$

Here H^n is the pseudo-isotopy from which γ^{n+1} was constructed (see Section 3.2).

Lemma 5.11. *In the setting as above*

(1)

$$a^{n+1}, b^{n+1} \textit{ are contained in the boundary of}$$
$$\textit{the } \text{same} \textit{ white } n\textit{-gap } G^n$$

if and only if

$$A^{n+1}, B^{n+1} \textit{ are contained in the boundary of}$$
$$\textit{the } \text{same} \textit{ white } n\textit{-tile } X^n.$$

(2)

$$D^{n+1}, E^{n+1} \textit{ are contained in the } \text{same} \textit{ white } (n+1)\textit{-tile,}$$

implies

$$A^{n+1}, B^{n+1} \textit{ are contained in the } \text{same} \textit{ white } n\textit{-tile.}$$

The situation is illustrated in Figure 6.

Proof. (1) Consider the n-arcs $a^n \supset a^{n+1}, b^n \supset b^{n+1}$, and the n-edges $D^n := \gamma^n(a^n), E^n := \gamma^n(b^n)$.

By construction of γ^{n+1} (see [Meyb, Definition 3.8]) it holds $D^{n+1} \subset H_1^n(D^n)$, $E^{n+1} \subset H_1^n(E^n)$. Thus $A^{n+1} \subset D^n, B^{n+1} \subset E^n$.

From (**G** 3) it follows that a^{n+1}, b^{n+1} are contained in the same white n-gap G^n if and only if $D^n \supset A^{n+1}, E^n \supset B^{n+1}$ are contained in the same white n-tile X^n.

(2) Consider the white $(n+1)$-tile $X^{n+1} \supset D^{n+1}, E^{n+1}$. Consider interiors, $U^{n+1} := \operatorname{int} X^{n+1}$ and $U^n := (H_1^n)^{-1}(U^{n+1}) \subset S^2 \setminus \bigcup \mathbf{E}^n$. Clearly $A^{n+1}, B^{n+1} \subset \partial U^n$.

The map H_1^n is a homeomorphism on U^n, this follows from [Meyb, Lemma 3.4 (H^n 3)]. Thus U^n is connected, hence in the interior of a single n-tile X^n. From [Meyb, Lemma 3.5 (H^n 5)] it follows that X^n has the same color as X^{n+1}. $\square$

We show that Property (**G** 5) follows as a corollary.

Proof of (**G** 5). We want to prove that every white $(n+1)$-gap G^{n+1} is contained in a white n-gap G^n.

Consider two $(n+1)$-arcs $a^{n+1}, b^{n+1} \subset G^{n+1} \cap S^1$. From (**G** 3) it follows that the $(n+1)$-edges $D^{n+1} := \gamma^{n+1}(a^{n+1}), E^{n+1} := \gamma^{n+1}(b^{n+1})$ are contained in the same white $(n+1)$-tile X^{n+1}. From Lemma 5.11 (2) it follows that the arcs $A^{n+1}, B^{n+1} \subset$

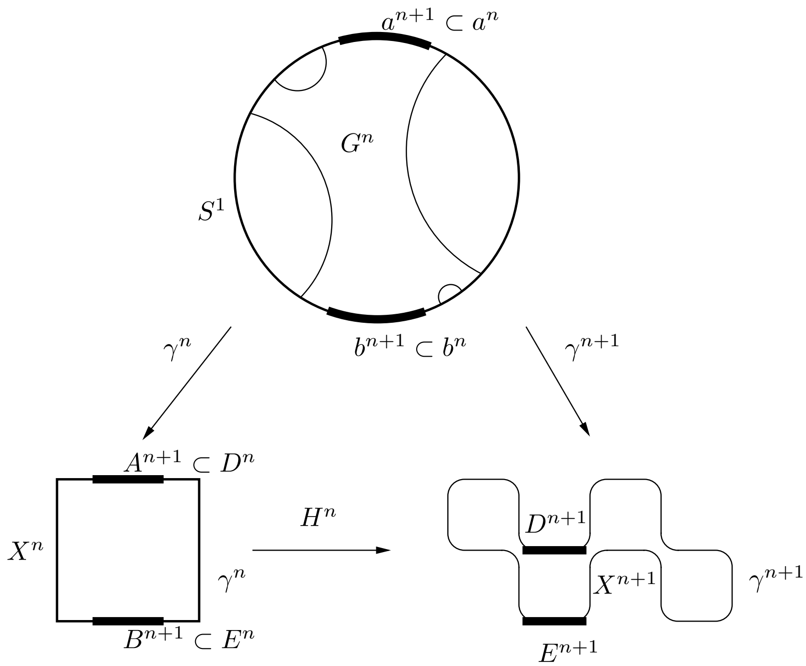


FIGURE 6. Illustration for Lemma 5.11.

$\bigcup \mathbf{E}^n$ satisfying $H_1^n(A^{n+1}) = D^{n+1}, H_1^n(B^{n+1}) = E^{n+1}$ are contained in the same white n-tile X^n. Thus, by Lemma 5.11 (1), a^{n+1}, b^{n+1} are contained in the same gap $G \in \mathbf{G}^n_{\mathtt{w}}$.

$\square$

Consider two n-arcs $a^n = [\alpha^n, \tilde{\alpha}^n]$, $b^n = [\beta^n, \tilde{\beta}^n]$ that are cyclically consecutive in ∂G^n, for a $G^n \in \mathbf{G}^n_{\mathtt{w}}$. Then we call $\beta, \tilde{\alpha}^n$ *succeeding in* G^n. Note that two n-angles are succeeding with respect to $\overset{n,\mathtt{w}}{\sim}$ if and only if they are succeeding with respect to some white n-gap.

Proof of Theorem 5.10. ($\Rightarrow$) Let $\alpha^{n+1}, \tilde{\alpha}^{n+1} \in \mathbf{A}^{n+1}$ be succeeding with respect to $\overset{n+1,\mathtt{w}}{\sim}$. Then there is a white $(n+1)$-gap G^{n+1} containing both. From (**G** 5) it follows that $\alpha^{n+1}, \tilde{\alpha}^{n+1}$ are contained in the same white n-gap G^n.

By (**G** 4) and (**G** 2) it follows that the 1-angles $\mu^n(\alpha^{n+1}), \mu(\tilde{\alpha}^{n+1})$ are succeeding with respect to $\overset{1,\mathtt{w}}{\sim}$.

($\Leftarrow$) Let $\alpha^{n+1}, \tilde{\alpha}^{n+1} \in \mathbf{A}^{n+1}$ be contained in the same gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$, such that $\mu(\alpha^{n+1}), \mu(\tilde{\alpha}^{n+1})$ are succeeding with respect to $\overset{1,\mathtt{w}}{\sim}$. Thus they are succeeding with respect to some 1-gap G^1.

From (**G** 4) and (**G** 5) it follows that there is an $(n+1)$-gap $G^{n+1} \ni \alpha^{n+1}, \tilde{\alpha}^{n+1}$ such that $\mu^n(G^{n+1} \cap S^1) = G^1 \cap S^1$. Combined with (**G** 2) it follows that $\alpha^{n+1}, \tilde{\alpha}^{n+1}$ are succeeding with respect to G^{n+1}, thus they are succeeding with respect to $\overset{n+1,\mathtt{w}}{\sim}$. $\square$

5.5. **($\mathcal{L}^n$ 6) and (G 6).** Using Theorem 5.10 we can finish the proofs of Lemma 5.7 and Lemma 5.9.

Proof of ($\mathcal{L}^n$ 6). We need to show that

$$\alpha^n \overset{n,\mathtt{w}}{\sim} \tilde{\alpha}^n \Leftrightarrow \alpha^n \overset{n+1,\mathtt{w}}{\sim} \tilde{\alpha}^n,$$

for all $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ (recall that $\mathbf{A}^n \subset \mathbf{A}^{n+1}$).

We first show the statement for $n = 0$, which is the following.

Claim. Let $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$ be distinct. Then $\alpha^0, \tilde{\alpha}^0$ are not equivalent with respect to $\overset{1,\mathtt{w}}{\sim}$.

To prove this claim consider distinct angles $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$. By definition of $\mathbf{A}^0$, $\gamma^0(\alpha^0), \gamma^0(\tilde{\alpha}^0)$ are postcritical points; in fact *different postcritical points*, since γ^0 is a Jordan curve. By construction

$$\gamma^1(\alpha^0) = \gamma^0(\alpha^0) \neq \gamma^0(\tilde{\alpha}^0) = \gamma^1(\tilde{\alpha}^0).$$

Thus $\alpha^0, \tilde{\alpha}^0$ are *not* equivalent with respect to $\overset{1,\mathtt{w}}{\sim}$, proving the claim.

After this preparation we are ready to prove the above equivalence in general.

($\Rightarrow$) Let $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ be succeeding with respect to $\overset{n,\mathtt{w}}{\sim}$. Then there is a gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$ containing both. Furthermore they are contained in the same n-leaf. Thus they are mapped by μ^n to the same point $\alpha^0 = \mu^n(\alpha^n) = \mu^n(\tilde{\alpha}^n) \in \mathbf{A}^0$ according to ($\mathcal{L}^n$ 4). Consider the elements of $[\alpha^0]_{1,w}$,

$$\alpha^0 = \alpha^1_0, \alpha^1_1, \dots, \alpha^1_N = \alpha^0.$$

Here $\alpha^1_j, \alpha^1_{j+1}$ are succeeding with respect to $\overset{1,\mathtt{w}}{\sim}$. Note that by the claim above each point α^1_j, $1 \leq j \leq N-1$ is *not* a 0-angle, thus in the *interior* of some 0-arc. Hence there is exactly *one* $(n+1)$-angle $\alpha^{n+1}_j \in G^n$, such that $\mu^n(\alpha^{n+1}_j) = \alpha^1_j$ (for $1 \leq j \leq N-1$)by (**G** 4). It follows from Theorem 5.10 that in the list

$$\alpha^n =: \alpha^{n+1}_0, \alpha^{n+1}_1, \dots, \alpha^{n+1}_{N-1}, \alpha^{n+1}_N := \tilde{\alpha}^n$$

the $(n+1)$-angles $\alpha^{n+1}_j, \alpha^{n+1}_{j+1}$ are succeeding with respect to $\overset{n+1,w}{\sim}$. Thus $\alpha^n, \tilde{\alpha}^n$ are equivalent with respect to $\overset{n+1,\mathtt{w}}{\sim}$.

($\Leftarrow$) Assume for $\alpha^n, \tilde{\alpha}^n$ it holds $\alpha^n \overset{n+1,\mathtt{w}}{\sim} \tilde{\alpha}^n$. We want to show that $\alpha^n \overset{n,\mathtt{w}}{\sim} \tilde{\alpha}^n$. Consider the sequence of succeeding $(n+1)$-angles

$$\alpha^n =: \alpha^{n+1}_0, \alpha^{n+1}_1, \dots, \alpha^{n+1}_N := \tilde{\alpha}^n.$$

Let $\alpha^{n+1}_m, m \geq 1$ be the first element (after α^n) in this sequence that is an n-angle. Since each angle $\alpha^{n+1} \in \mathbf{A}^{n+1} \setminus \mathbf{A}^n$ is contained in a single gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$ it follows that $\alpha^{n+1}_0, \dots, \alpha^{n+1}_m$ is contained in a single gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$. Note that

$$\alpha^1_0 := \mu^n(\alpha^{n+1}_0), \dots, \alpha^1_m := \mu^n(\alpha^{n+1}_m)$$

is a sequence of succeeding 1-angles, i.e., $\alpha^n \overset{1,\mathtt{w}}{\sim} \alpha^1_m$. Furthermore $\alpha^1_0, \alpha^1_k \in \mathbf{A}^0$, since α^n, α^{n+1}_m were n-angles.

From the claim above it follows that $\alpha^1_0 = \alpha^1_k \in \mathbf{A}^0$, this means that the n-angles α^n, α^{n+1}_k are contained in n-leaves of the same type. Since they are contained in the same n-gap G^n this means that they are contained in the same n-leaf (i.e., boundary component of G^n) by (**G** 2), i.e., $\alpha^n \overset{n,\mathtt{w}}{\sim} \alpha^{n+1}_m$. Continuing in this fashion, i.e., considering sequences in (α^{n+1}_j) between consecutive n-angles we conclude that $\alpha^n \overset{n,\mathtt{w}}{\sim} \tilde{\alpha}^n$ as desired.

□

Proof of (**G** 6). We prove the claim first for $n = 0$. Consider first distinct $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$. They are mapped by γ^0, hence by γ to distinct postcritical points, $p := \gamma^0(\alpha^0), \tilde{p} := \gamma^0(\tilde{\alpha}^0)$. Since F is expanding there is n_0, such that for $m \geq n_0$ no m-tile contains both $p, \tilde{p}$. Thus $\alpha^0, \tilde{\alpha}^0$ are not contained in the same white m-gap by (**G** 3).

Consider now non-equivalent (with respect to $\overset{n,\mathtt{w}}{\sim}$) $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$. Assume $\alpha^n \tilde{\alpha}^n \in G^{m+n} \in \mathbf{G}^{m+n}_{\mathtt{w}}$ for some $m \geq n_0$. Then the distinct 0-angles $\alpha^0 := \mu^n(\alpha^n), \tilde{\alpha}^0 := \mu^n(\tilde{\alpha}^n)$ are contained in the same m-gap G^m, satisfying $\mu^n(G^{m+n} \cap S^1) = G^m \cap S^1$ by (**G** 4). This contradicts the case $n = 0$ above.

□

5.6. **The critical portrait.** According to Theorem 5.10 we can recover all equivalence relations $\overset{n,\mathtt{w}}{\sim}$ from $\overset{1,\mathtt{w}}{\sim}$ (and all $\overset{n,\mathtt{b}}{\sim}$ from $\overset{1,\mathtt{b}}{\sim}$). Note that the non-trivial equivalence classes of $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ (i.e., the ones containing at least two points) are mapped by γ to critical points of F.

Definition 5.12. The sets $[\alpha^1_j]_{1,\mathtt{w}} \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1$, $j = 1, \dots, m$ satisfy the following, meaning they form a *critical portrait*.

- μ maps all points of $[\alpha^1_j]_{1,\mathtt{w}}$ to a single point,
$$\mu([\alpha^1_j]_{1,\mathtt{w}}) = \{\alpha^0_j\}.$$
- $\sum_j (\#[\alpha^1_j]_{1,\mathtt{w}} - 1) = d - 1$.
- The sets $[\alpha^1_1]_{1,\mathtt{w}}, \dots, [\alpha^1_m]_{1,\mathtt{w}}$ are non-crossing.

The orbit
$$\mathbf{A}^0 := \bigcup \{\mu^n(\alpha^1_j) \mid j = 1, \dots, m,\ n \geq 1\}$$
is a finite set.

- No set $[\alpha^1_j]_{1,\mathtt{w}}$ contains more than one point from $\mathbf{A}^0$.

The equivalence relations $\overset{n,\mathtt{w}}{\sim}$, as well as the laminations $\mathcal{L}^n_{\mathtt{w}}$ and the gaps $\mathbf{G}^n_{\mathtt{w}}$, are defined inductively as in Theorem 5.10.

- There is a constant n_0 such that the following holds. Let $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$ be distinct. Then for $m \geq n_0$ no gap $G^m \in \mathbf{G}^m_{\mathtt{w}}$ contains points from both sets $[\alpha^0]_{m,\mathtt{w}}, [\tilde{\alpha}^0]_{m,\mathtt{w}}$.

The first property follows from ($\mathcal{L}^n$ 1), the second from ($\mathcal{L}^n$ 3), the third from ($\mathcal{L}^n$ 2), the fourth from ($\mathcal{L}^n$ 6), and the last from (**G** 6).

It can be shown given any critical portrait the properties ($\mathcal{L}^n$ 1)–($\mathcal{L}^n$ 6) as well as (**G** 1), (**G** 2), (**G** 4)–(**G** 6) are then satisfied, which we do not prove here.

The equivalence classes of $\overset{1,\mathtt{b}}{\sim}$ form a critical portrait as well. This proves Theorem 1.6.

A. Poirier [Poi09] (extending work of Bielefeld-Fisher-Hubbard [BFH92]) has shown that for every critical portrait there is a polynomial "realizing it", see Section 7.4. The last condition above is actually much stronger than the corresponding condition for the general case in [Poi09]. Indeed we will show in Section 7.6 that it implies that the corresponding polynomial is of a special form. Namely closures of distinct bounded Fatou components are disjoint and do not contain any critical point.

We remark that the sets $[\alpha_j^n]_{n,\mathtt{w}}$ as well as $[\alpha^n]_{n,\mathtt{b}}$ form critical portraits as well. They will be realized by the n-th iterate of the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ realizing the critical portraits $[\alpha_j^1]_{1,\mathtt{w}}, [\alpha_j^1]_{1,\mathtt{b}}$.

5.7. **The equivalence relations $\overset{\mathtt{w}}{\sim}, \overset{\mathtt{b}}{\sim}$.** We consider the *closure of the join* of the equivalence relations $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$. We will show in Section 7 that there are polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$, such that $\overset{\mathtt{w}}{\sim}, \overset{\mathtt{b}}{\sim}$ are the equivalence relations induced by their Carathéodory semi-conjugacies. These will be the polynomials into which F unmates.

Definition 5.13 ($\overset{\mathtt{w}}{\sim}, \overset{\mathtt{b}}{\sim}$)**.** The equivalence relation $\overset{\mathtt{w}}{\sim}$ on S^1 is defined as follows,

$$\overset{\infty,\mathtt{w}}{\sim} := \bigvee \overset{n,\mathtt{w}}{\sim}, \text{ meaning } s \overset{\infty,\mathtt{w}}{\sim} t \text{ if and only if } s \overset{n,\mathtt{w}}{\sim} t \text{ for some } n;$$
$$\overset{\mathtt{w}}{\sim} \text{ is defined to be the closure of } \overset{\infty,\mathtt{w}}{\sim}.$$

The equivalence relation $\overset{\mathtt{b}}{\sim}$ is similarly defined as the closure of $\overset{\infty,\mathtt{b}}{\sim} := \bigvee \overset{n,\mathtt{b}}{\sim}$.

The equivalence relation $\sim$ induced by the invariant Peano curve γ (1.5) may be recovered from $\overset{\mathtt{w}}{\sim}, \overset{\mathtt{b}}{\sim}$ (hence F from $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ up to topological conjugacy by Theorem 1.5 and Theorem 5.10).

Lemma 5.14. *Let $\overset{\mathtt{w}}{\sim}, \overset{\mathtt{b}}{\sim}$ be defined as above, $\sim$ the equivalence relation induced by γ. Then*

(i) $\sim$ *is the* closure *of* $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$*;*
(ii) *if F has no periodic critical points, then*

$$\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} = \sim .$$

Proof. (i) Denote by $\widehat{\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}}$ the closure of $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$. The equivalence relations $\overset{n}{\sim}$ and $\overset{\infty}{\sim}$ were defined in (4.3) and (4.4).

Since $\overset{\mathtt{w}}{\sim} \geq \overset{n,\mathtt{w}}{\sim}$, $\overset{\mathtt{b}}{\sim} \geq \overset{n,\mathtt{b}}{\sim}$ for all $n \in \mathbb{N}$ it follows that

$$\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} \;\geq\; \overset{n,\mathtt{w}}{\sim} \vee \overset{n,\mathtt{b}}{\sim} \;=\; \overset{n}{\sim},$$

for all $n \in \mathbb{N}$ by (5.4). Hence $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} \;\geq\; \overset{\infty}{\sim}$ and the closure of $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ is bigger than the closure of $\overset{\infty}{\sim}$, which is $\sim$ by Theorem 4.7, i.e.,

$$\widehat{\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}} \;\geq\; \sim .$$

On the other hand we note that $\overset{n,\mathtt{w}}{\sim} \leq \overset{n}{\sim}$ for all $n \in \mathbb{N}$, thus $\overset{\infty,\mathtt{w}}{\sim} \leq \overset{\infty}{\sim}$. Taking closures yields $\overset{\mathtt{w}}{\sim} \leq \sim$ (using Theorem 4.7 again), similarly $\overset{\mathtt{b}}{\sim} \leq \sim$. Thus $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} \leq \sim$, taking closures yields

$$\widehat{\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}} \;\leq\; \sim,$$

i.e., the statement. $\square$

The proof of part (ii) of the previous lemma will be given in the next section after some preparation.

6. Equivalence classes are finite

Here we show the following.

Theorem 6.1. *Let $\sim$ be the equivalence relation induced by γ.*

- *If F has* no critical periodic cycles*, there is a number $N < \infty$ such that*
$$\#[s] \leq N \text{ for all } s \in S^1.$$
- *If F* has critical periodic cycles*, there is (at least) one finite equivalence class $[s]$, where $\gamma(s) \notin$* post.

Proof. The second statement could be proved using *Douady's lemma*, which says that given any set $A \subset S^1$, such that $\mu(A) = A$ and μ is injective on A, then A is finite. A proof of a stronger statement may be found in [Shi00, Lemma 4.2].

We give a slightly different proof, which will follow from the setup in the proof of the first claim.

Recall that $\mu(x) = \mu_d(x) = dx \,(\text{mod} 1)$ maps any $(n+1)$-arc to a n-arc. It follows that for any n-arc $a^n \subset S^1$ it holds $\operatorname{diam} a^n \leq d^{-n}$. Assume that for a given point $x \in S^2$ there is an $N \in \mathbb{N}$ such that $\gamma^{-1}(x) \subset S^1$ is contained in the union of at most N n-arcs for any $n \in \mathbb{N}$. Then it follows that $\gamma^{-1}(x)$ contains at most N points.

Let us fix a visual metric ϱ for F with expansion factor $\Lambda > 1$. All metrical properties (such as diam and the supremums norm $\|\cdot\|_\infty$) on S^2 will from now on be given in terms of ϱ. Recall from Section 3.1 that there is a constant $C > 0$ such that

$$\|\gamma - \gamma^n\|_\infty \leq C\Lambda^{-n}. \tag{6.1}$$

Thus only n-arcs a^n such that the n-edges $\gamma^n(a^n)$ have "small combinatorial distance" from x can satisfy $\gamma(a^n) \ni x$.

Let $x \in S^2$ be arbitrary, and X^n be an n-tile containing x (for any $n \in \mathbb{N}$). Consider the set of n-tiles that "can be reached in j steps from X^n", i.e.,

$$U_0^n = U_0^n(X^n) := X^n,$$
$$U_j^n = U_j^n(X^n) := \bigcup \{Z^n \in \mathbf{X}^n \mid Z^n \cap U_{j-1}^n \neq \emptyset\},$$

for all $j \geq 1$. We will show that points outside of U_j^n are "far away" from x. More precisely we show the following.

Claim. There are $\alpha > 0$, $c > 0$ (independent of n and X) such that

$$\operatorname{dist}(X^n, S^2 \setminus U_j^n) \geq c j^\alpha \Lambda^{-n},$$

for all n-tiles X^n and $j \in \mathbb{N}$.

To prove the claim let $y \in S^2 \setminus U_j^n$. Recall that $\varrho(x,y) \asymp \Lambda^{-m}$, where $m = m(x,y)$ is the smallest number such that there are disjoint m-tiles containing x, y (see Section 2.3). Thus (any) $(m-1)$-tiles X^{m-1}, Y^{m-1} containing x, y are not disjoint.

From [BM, Lemma 5.29] it follows that that there is a constant $M \in \mathbb{N}$, such that any two points in $X^{m-1} \cup Y^{m-1}$ (in particular x, y) may be joined by a chain of m-tiles of length at most M. This means there are m-tiles $Y_1, \dots, Y_M$ with $x \in Y_1, y \in Y_M$ and $Y_l \cap Y_{l+1} \neq \emptyset$ for all $1 \leq l \leq M-1$. Inductively it follows that x, y may be connected by a chain of n-tiles of length at most M^{n-m+1}. Since $y \notin U_j^n$ it follows that

$$j \leq M^{n-m+1} \quad \text{or} \quad -m \geq \frac{\log j}{\log M} - n - 1.$$

Thus

$$\varrho(x,y) \asymp \Lambda^{-m} \geq \tilde{c} j^{\alpha} \Lambda^{-n},$$

for constants $\tilde{c} = \Lambda^{-1} > 0$, $\alpha = \log \Lambda / \log M > 0$ independent of j, n. The claim follows.

From the claim together with (6.1) it follows that there is a constant $j_0 \in \mathbb{N}$ such that $x = \gamma(s) \in X^n$ and $y = \gamma^n(t) \notin U_{j_0}^n$ implies that $\gamma(t) \neq x$. Consider now the n-edges in $U_{j_0}^n$ and the n-arcs in S^1 that are mapped by γ^n into $U_{j_0}^n$,

$$\begin{aligned} E^n = E_{j_0}^n(X^n) &:= U_{j_0}^n \cap \bigcup \mathbf{E}^n \\ A^n = A_{j_0}^n(X^n) &:= (\gamma^n)^{-1}(E^n). \end{aligned}$$

From the above it follows that for all $t \in S^1$ it holds

$$s \notin A^n \Rightarrow \gamma(t) \neq x,$$

for all $n \in \mathbb{N}$.

Assume now that F has no critical periodic cycles. Then the local degree is uniformly bounded, i.e.,

$$\deg_{F^n}(v) \leq 2d,$$

for all $v \in S^2$ and $n \in \mathbb{N}$ (recall that $d = \deg F$). Recall that the number of n-tiles incident to an n-vertex n is given by $2 \deg_{F^n}(v)$. It follows that the number of n-tiles contained in $U_{j_0}^n(X^n)$ is uniformly bounded (i.e., independent of $n \in \mathbb{N}$ and the n-tile X^n). Thus the number of n-edges in $E_{j_0}^n(X^n)$, which equals the number of n-arcs in $A_{j_0}^n(X^n)$, is uniformly bounded by a number $N \in \mathbb{N}$. Thus $\gamma^{-1}(x)$ contains at most N points, i.e., each equivalence class of $\sim$ contains at most N points. The proof of the first statement of the lemma is finished.

Assume now that F has critical periodic cycles. Let X^n be a white n-tile such that $U^n = U_{j_0}^n(X^n)$ is contained in the interior of the white 0-tile $X_{\mathtt{w}}^0$ (for some sufficiently large $n \in \mathbb{N}$). The existence of such an n-tile follows easily from the expansion property of F, see [BM, Lemma 7.9]. Note that $F^n \colon X^n \to X_{\mathtt{w}}^0$ is a homeomorphism. Let $U^{2n} := (F^n|X^n)^{-1}(U^n)$. Note that U^{2n} is compactly contained in U^n. Define $U^{(i+1)n} := (F^n|X^n)^{-1}(U^{in})$ for all $i \geq 1$. Then $U^{(i+1)n}$ is compactly contained in U^{in}. Let the point $x_0 \in S^2$ be defined by

$$\{x_0\} = \bigcap_j U_{k_0}^{jn}.$$

Note that the number of in-tiles contained in U^{in} is the same for all $i \in \mathbb{N}$, since $F^n \colon U^{(i+1)n} \to U^{in}$ is a homeomorphism (for all $i \in \mathbb{N}$). Hence the number of n-edges in $E^{in} := U^{in} \cap \bigcup \mathbf{E}^{in}$, which equals the number of n-arcs in $A^{in} := \gamma^{-1}(E^{in})$, is the same for all $i \in \mathbb{N}$, i.e., uniformly bounded. This means there is a constant $N \in \mathbb{N}$, such that the number of n-arcs a^n with $x_0 \in \gamma(a^n)$ is bounded by N for all $n \in \mathbb{N}$. Thus the number of points in $\gamma^{-1}(x_0)$ (which is an equivalence class of $\sim$) is at most N. This finishes the proof of the second statement of the lemma.

$\square$

Proof of Lemma 5.14 (ii). It follows from Lemma 5.14 (i) that it is enough to show that $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ is *closed*. Consider $s, t \in S^1$ satisfying $s \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} t$. Then

$$s = s_1 \overset{\mathtt{w}}{\sim} s_2 \overset{\mathtt{b}}{\sim} \dots \overset{\mathtt{w}}{\sim} s_{M-1} \overset{\mathtt{b}}{\sim} s_M = t,$$

for some points $s_1, \dots, s_M \in S^1$. Recall from the proof of Lemma 5.14 (i) that $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} \leq \sim$. Since the size of each equivalence class with respect to $\sim$ is at most a constant N by Theorem 6.1; it follows that we can always choose $M = 2N$ independently of s, t.

Consider now convergent sequences $(s^n), (t^n) \subset S^1$, i.e., $s^n \to s, t^n \to t$ (as $n \to \infty$), such that $s^n \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} t^n$ for all $n \in \mathbb{N}$. By the above there are $s_j^n \in S^1$ ($j = 1, \dots, M$) such that

$$s^n = s_1^n \overset{\mathtt{w}}{\sim} s_2^n \overset{\mathtt{b}}{\sim} \dots \overset{\mathtt{b}}{\sim} s_M^n = t^n,$$

for all $n \in \mathbb{N}$. By taking subsequences we can assume that $s_j^n \to s_j$ as $n \to \infty$, for all j. Since $\overset{\mathtt{w}}{\sim}, \overset{\mathtt{b}}{\sim}$ are closed it follows that

$$s = s_1 \overset{\mathtt{w}}{\sim} \dots \overset{\mathtt{b}}{\sim} s_N = t,$$

meaning $s \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} t$. This means that $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ is closed as desired. □

7. F is a mating

In this section we prove Theorem 1.2. This means that we show that F is obtained as a *mating* in the case when F has no periodic critical points. The construction will however be done for the general case, in preparation to prove Theorem 1.3.

Recall from Section 1.2 the construction of mating of two polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ (which are monic and of the same degree d). Assume for now that every critical point of $P_{\mathtt{w}}, P_{\mathtt{b}}$ is *strictly preperiodic*. This means that the Fatou sets of $P_{\mathtt{w}}, P_{\mathtt{b}}$ consist both of a single (unbounded) component, i.e., their filled Julia sets equal their Julia sets $\mathcal{K}_{\mathtt{w}} = \mathcal{J}_{\mathtt{w}}$, $\mathcal{K}_{\mathtt{b}} = \mathcal{J}_{\mathtt{b}}$. Let $\sigma_{\mathtt{w}} \colon S^1 \to \mathcal{J}_{\mathtt{w}}$, $\sigma_{\mathtt{b}} \colon S^1 \to \mathcal{J}_{\mathtt{b}}$ be the Carathéodory semi-conjugacies of $\mathcal{J}_{\mathtt{w}}, \mathcal{J}_{\mathtt{b}}$. Consider the equivalence relations $\overset{\mathtt{w}}{\approx}, \overset{\mathtt{b}}{\approx}$ on S^1 induced by $\sigma_{\mathtt{w}}, \sigma_{\mathtt{b}}$,

$$\begin{aligned} z \overset{\mathtt{w}}{\approx} w &:\Leftrightarrow \sigma_{\mathtt{w}}(z) = \sigma_{\mathtt{w}}(w) \\ z \overset{\mathtt{b}}{\approx} w &:\Leftrightarrow \sigma_{\mathtt{b}}(\bar{z}) = \sigma_{\mathtt{b}}(\overline{w}), \end{aligned} \tag{7.1}$$

for all $z, w \in S^1 = \partial\mathbb{D}$. From (1.3) and Lemma 4.6 it follows that

$$\begin{aligned} z^d/\overset{\mathtt{w}}{\approx} \colon S^1/\overset{\mathtt{w}}{\approx} \;&\to\; S^1/\overset{\mathtt{w}}{\approx} \\ z^d/\overset{\mathtt{b}}{\approx} \colon S^1/\overset{\mathtt{b}}{\approx} \;&\to\; S^1/\overset{\mathtt{b}}{\approx} \end{aligned}$$

are topologically conjugate to $P_{\mathtt{w}} \colon \mathcal{J}_{\mathtt{w}} \to \mathcal{J}_{\mathtt{w}}$ and $P_{\mathtt{b}} \colon \mathcal{J}_{\mathtt{b}} \to \mathcal{J}_{\mathtt{b}}$. Let

$$\approx \;:=\; \overset{\mathtt{w}}{\approx} \vee \overset{\mathtt{b}}{\approx}.$$

From the construction of the mating of $P_{\mathtt{w}}, P_{\mathtt{b}}$ we obtain the following.

Lemma 7.1. *Let $P_{\mathtt{w}}, P_{\mathtt{b}}$ be monic polynomials of the same degree d, where every critical point is strictly preperiodic. Let $\approx$ be defined as above. Then the topological mating of $P_{\mathtt{w}}, P_{\mathtt{b}}$ is topologically conjugate to the map*

$$z^d/\!\approx\, : S^1/\!\approx\; \to\; S^1/\!\approx .$$

We will show that there are polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ as above such that

$$\stackrel{\mathtt{w}}{\approx} \;=\; \stackrel{\mathtt{w}}{\sim}, \quad \stackrel{\mathtt{b}}{\approx} \;=\; \stackrel{\mathtt{b}}{\sim} .$$

Here $\stackrel{\mathtt{w}}{\sim}, \stackrel{\mathtt{b}}{\sim}$ are the equivalence relations from Definition 5.13. Lemma 5.14 then implies that $\approx\, = \,\sim$ (i.e., the equivalence relation induced by the invariant Peano curve γ (1.5)). This will prove Theorem 1.2 using Theorem 4.7.

Proof of Lemma 7.1. The mating of $z^d/\stackrel{\mathtt{w}}{\approx}$ and $z^d/\stackrel{\mathtt{b}}{\approx}$ is given by considering the equivalence relation on the disjoint union of $S^1/\stackrel{\mathtt{w}}{\approx}$ and $S^1/\stackrel{\mathtt{b}}{\approx}$ generated by identifying $[z]_{\mathtt{w}} \in S^1/\stackrel{\mathtt{w}}{\approx}$ with $[\bar{z}]_{\mathtt{b}} \in S^1/\stackrel{\mathtt{b}}{\approx}$. The quotient is denoted by $S^1/\stackrel{\mathtt{w,b}}{\approx}$. The maps $z^d/\stackrel{\mathtt{w}}{\approx}$, $z^d/\stackrel{\mathtt{b}}{\approx}$ descend to this quotient, i.e., to a map

$$z^d/\stackrel{\mathtt{w,b}}{\approx} : S^1/\stackrel{\mathtt{w,b}}{\approx} \;\to\; S^1/\stackrel{\mathtt{w,b}}{\approx} .$$

Since $z^d/\stackrel{\mathtt{w}}{\approx}$, $z^d/\stackrel{\mathtt{b}}{\approx}$ are topologically conjugate to $P_{\mathtt{w}}, P_{\mathtt{b}}$ it follows (from the definition of the conjugacy) that $P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}$ is topologically conjugate to $z^d/\stackrel{\mathtt{w,b}}{\approx}$. Note that the map $S^1 \to S^1/\stackrel{\mathtt{w}}{\approx} \to S^1/\stackrel{\mathtt{w,b}}{\approx}$ (i.e., the composition of the quotient maps) is surjective. The equivalence relation induced by this map is $\stackrel{\mathtt{w}}{\approx} \vee \stackrel{\mathtt{b}}{\approx}$. Furthermore the following diagram commutes

$$\begin{tikzcd}
S^1 \arrow[r, "z^d"] \arrow[d] & S^1 \arrow[d] \\
S1/\stackrel{\mathtt{w}}{\approx} \arrow[r, "z^d/\stackrel{\mathtt{w}}{\approx}"] \arrow[d] & S^1/\stackrel{\mathtt{w}}{\approx} \arrow[d] \\
S^1/\stackrel{\mathtt{w,b}}{\approx} \arrow[r, "z^d/\stackrel{\mathtt{w,b}}{\approx}"] & S^1/\stackrel{\mathtt{w,b}}{\approx} .
\end{tikzcd}$$

The statement follow from Lemma 4.6. □

We now discuss the general case, where F is allowed to have periodic critical points, i.e., we outline the proof of Theorem 1.3. The Carathéodory semi-conjugacies $\sigma_{\mathtt{w}}, \sigma_{\mathtt{b}}$ (for monic, postcritically finite polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ both of degree $d \geq 2$) are defined as before.

Define equivalence relations on S^1 by

$$
\begin{aligned}
(7.2)\qquad z \overset{\mathcal{F},\mathtt{w}}{\approx} w \;:\Leftrightarrow\; & \sigma_{\mathtt{w}}(z) = \sigma_{\mathtt{w}}(w) \text{ or}\\
& \sigma_{\mathtt{w}}(z), \sigma_{\mathtt{w}}(w) \text{ are both contained in } A\\
& \text{where } A \text{ is the closure of a bounded Fatou component of } P_{\mathtt{w}}.
\end{aligned}
$$

$$
\begin{aligned}
(7.3)\qquad z \overset{\mathcal{F},\mathtt{b}}{\approx} w \;:\Leftrightarrow\; & \sigma_{\mathtt{b}}(\bar{z}) = \sigma_{\mathtt{b}}(\overline{w}) \text{ or}\\
& \sigma_{\mathtt{b}}(\bar{z}), \sigma_{\mathtt{b}}(\overline{w}) \text{ are both contained in } A\\
& \text{where } A \text{ is the closure of a bounded Fatou component of } P_{\mathtt{b}}.
\end{aligned}
$$

for all $z, w \in S^1 = \partial\mathbb{D}$. Note that in general the above does not define equivalence relations. Namely the closures of distinct bounded Fatou components A, A' may not be disjoint. We will however consider only polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ of a special type, namely where such sets A, A' are always disjoint.

We define $\widehat{\approx}$ to be the closure of $\overset{\mathcal{F},\mathtt{w}}{\approx} \vee \overset{\mathcal{F},\mathtt{b}}{\approx}$. Similarly as in the last lemma it will be shown that the quotient map

$$z^d/\widehat{\approx} : S^1/\widehat{\approx} \;\to\; S^1/\widehat{\approx}$$

is topologically conjugate to

$$P_{\mathtt{w}} \mathbin{\widehat{\perp\!\!\!\perp}} P_{\mathtt{b}} \colon \mathcal{K}_{\mathtt{w}} \mathbin{\widehat{\perp\!\!\!\perp}} \mathcal{K}_{\mathtt{b}} \to \mathcal{K}_{\mathtt{w}} \mathbin{\widehat{\perp\!\!\!\perp}} \mathcal{K}_{\mathtt{b}},$$

as defined in (1.4). We will show that there are polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ such that

$$\overset{\mathcal{F},\mathtt{w}}{\approx} = \overset{\mathtt{w}}{\sim}, \quad \overset{\mathcal{F},\mathtt{b}}{\approx} = \overset{\mathtt{b}}{\sim}.$$

Then $\widehat{\approx}\; = \sim$ by Lemma 5.14 (i). Thus Theorem 1.3 will be proved (using Theorem 1.5).

7.1. **Julia- and Fatou-type equivalence classes.** In the following S^1 is again identified with $\mathbb{R}/\mathbb{Z}$. Recall that the map $\mu = \mu_d \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is given by $\mu(s) = ds \,(\mathrm{mod} 1)$ (which is conjugate to $z^d \colon \partial\mathbb{D} \to \partial\mathbb{D}$).

The *non-trivial equivalence classes* of $\overset{1,\mathtt{w}}{\sim}$, i.e., the ones that contain at least two points, are called the (white) *critical equivalence classes*. They are mapped by γ^1 (and thus by all γ^n and γ) to critical points of F. We divide the critical equivalence classes into ones of *Fatou-type* and *Julia-type* as follows.

- If $[\alpha^1]_{1,\mathtt{w}}$ is *periodic*, i.e., if
$$\mu^n([\alpha^1]_{1,\mathtt{w}}) \subset [\alpha^1]_{1,\mathtt{w}},$$
for some $n \geq 1$, it is of *periodic Fatou-type*;
- if $[\alpha^1]_{1,\mathtt{w}}$ is the *preimage of a periodic critical cycle*, i.e., if
$$\mu^n([\alpha^1]_{1,\mathtt{w}}) \subset [\tilde{\alpha}^1]_{1,\mathtt{w}},$$
for some $n \geq 1$, where $[\tilde{\alpha}^1]_{1,\mathtt{w}}$ is of periodic Fatou-type; then $[\alpha^1]_{1,\mathtt{w}}$ is of *preperiodic Fatou-type*;
- otherwise $[\alpha^1]_{1,\mathtt{w}}$ is of *Julia-type*, i.e., the periodic cycle that $[\alpha^1]_{1,\mathtt{w}}$ eventually lands in does not contain any point of a critical equivalence class.

Every Fatou-type equivalence class is mapped by γ^1 to a point that is eventually mapped to a critical periodic cycle of F. However if $[\alpha^1]_{1,\mathtt{w}}$ is of Julia-type, and $c = \gamma^1([\alpha^1]_{1,\mathtt{w}})$, then the periodic cycle that c eventually lands in may or may not be critical. This is due to the fact that the periodic critical point of F may "come from" the black polynomial.

Consider now an equivalence class $[\alpha^n]_{n,\mathtt{w}}$ ($\alpha^n \in \mathbf{A}^n$) with respect to $\overset{n,\mathtt{w}}{\sim}$. It is defined to be of *Fatou-type/Julia-type* if $\mu^{n-1}([\alpha^n]_{n,\mathtt{w}})$ is of Fatou-type/Julia-type (recall ($\mathcal{L}^n$ 4)). We note the following (recall that $[\alpha^n]_{n,\mathtt{w}} \subset [\alpha^n]_{n+1,\mathtt{w}}$ from ($\mathcal{L}^n$ 6))

$$[\alpha^n]_{n,\mathtt{w}}, [\alpha^n]_{n+1,\mathtt{w}} \text{ are of the some type,} \tag{7.4}$$

for all $\alpha^n \in \mathbf{A}^n$.

7.2. **Sizes of equivalence classes.** The main result of this subsection is the following.

Proposition 7.2. *The expanding Thurston map F has critical periodic cycles if and only if there are Fatou-type equivalence classes of $\overset{1,\mathtt{w}}{\sim}$ or $\overset{1,\mathtt{b}}{\sim}$.*

We need some preparation. The *degree* of a critical equivalence class $[\alpha^1]_{1,\mathtt{w}}$ is its size,

$$d([\alpha^1]_{1,\mathtt{w}}) := \#[\alpha^1]_{1,\mathtt{w}}. \tag{7.5}$$

The degree of other equivalence classes will be the degree of the critical class it contains.

$$d([\alpha^n]_{n,\mathtt{w}}) := \begin{cases} \#[\alpha^1]_{1,\mathtt{w}}, & \text{if } [\alpha^1]_{1,\mathtt{w}} \subset [\alpha^n]_{n,\mathtt{w}}; \\ 1, & \text{if } [\alpha^n]_{n,\mathtt{w}} \text{ contains no critical class.} \end{cases}$$

Note that by ($\mathcal{L}^n$ 6) there can be at most one critical class contained in $[\alpha^n]_{n,\mathtt{w}}$, thus the above is well defined.

Consider now $[\alpha^n]_{n,\mathtt{w}}$, where $\alpha^n \in \mathbf{A}^n$. Let (recall ($\mathcal{L}^n$ 4))

$$[\alpha^{n-1}]_{n-1,\mathtt{w}} := \mu([\alpha^n]_{n,\mathtt{w}}),\ [\alpha^{n-2}]_{n-2,\mathtt{w}} := \mu^2([\alpha^n]_{n,\mathtt{w}}), \dots .$$

Lemma 7.3 (Size of equivalence classes)**.** *In the setting as above it holds*

$$\#[\alpha^n]_{n,\mathtt{w}} = d([\alpha^n]_{n,\mathtt{w}}) \cdot d([\alpha^{n-1}]_{n-1,\mathtt{w}}) \cdot \dots \cdot d([\alpha^1]_{1,\mathtt{w}}).$$

Proof. The statement is clear for $n = 1$. We proceed by induction. Thus we assume the statement is true for n.

Case (1). $[\alpha^{n+1}]_{n+1,\mathtt{w}}$ contains no angle $\alpha^n \in \mathbf{A}^n$.

From ($\mathcal{L}^n$ 6) and ($\mathcal{L}^n$ 2) it follows that the leaf $L([\alpha^{n+1}]_{n+1,\mathtt{w}})$ is contained in the iterior of the complement of the n-th lamination, i.e., in a white n-gap G^n. This means $[\alpha^{n+1}]_{n+1,\mathtt{w}}$ is contained in the *interior* of the n-arcs which form $G^n \cap S^1$ (see (**G** 2)). It follows that μ^n is bijective on $[\alpha^{n+1}]_{n+1,\mathtt{w}}$. Thus

$$\#[\alpha^{n+1}]_{n+1,\mathtt{w}} = \#\mu^n([\alpha^{n+1}]_{n+1,\mathtt{w}}) = \#[\alpha^1]_{1,\mathtt{w}} = d([\alpha^1]_{1,\mathtt{w}}).$$

On the other hand $[\alpha^{n+1}]_{n+1,\mathtt{w}}$ contains no $\alpha^n \in \mathbf{A}^n$, thus no $\alpha^1 \in \mathbf{A}^1$. Therefore $d([\alpha^{n+1}]_{n+1,\mathtt{w}}) = 1$.

Similarly $[\alpha^n]_{n,\mathtt{w}}$ contains no $\alpha^{n-1} \in \mathbf{A}^{n-1}$, hence $d([\alpha^n]_{n,\mathtt{w}}) = 1$. Repeating the argument yields

$$\#[\alpha^{n+1}]_{n+1,\mathtt{w}} = \underbrace{d([\alpha^{n+1}]_{n+1,\mathtt{w}}) \cdot \dots \cdot d([\alpha^2]_{2,\mathtt{w}})}_{=1} \cdot d([\alpha^1]_{1,\mathtt{w}}).$$

Case (2). $[\alpha^{n+1}]_{n+1,\mathtt{w}}$ contains some n-angles, i.e., $[\alpha^{n+1}]_{n+1,\mathtt{w}} \supset [\tilde{\alpha}^n]_{n,\mathtt{w}}$ for some $\tilde{\alpha}^n \in \mathbf{A}^n$.

Let $m := \#[\tilde{\alpha}^n]$. We want to estimate $\#[\alpha^{n+1}]_{n+1,\mathtt{w}}$. To do this we will estimate the $(n+1)$-angles between two (succeeding with respect to $\overset{n,\mathtt{w}}{\sim}$) n-angles $\tilde{\alpha}^n_1, \tilde{\alpha}^n_2 \in [\tilde{\alpha}^n]_{n,\mathtt{w}}$. For any such succeeding n-angles there is a white n-gap G^n containing these succeeding n-angles. Note that there are m succeeding n-angles in $[\tilde{\alpha}^n]$, thus there are m such n-gaps G^n.

Consider now the succeeding $(n+1)$-angles between $\tilde{\alpha}^n_1, \tilde{\alpha}^n_2$ (i.e., the $(n+1)$-angles in $[\alpha^{n+1}]_{n+1,\mathtt{w}}$ between those n-angles). By Theorem 5.10, these are exactly the $(n+1)$-angles contained in G^n that are mapped by μ^n to 1-angles which succeed $\tilde{\alpha}^0 := \mu^n(\tilde{\alpha}^n)$, i.e., mapped by μ^n to $[\tilde{\alpha}^0]_{1,\mathtt{w}}$ (note that the lower index "1" is *not* a misprint). Let $d = \#[\tilde{\alpha}^0]_{1,\mathtt{w}}$, then there are exactly $d-1$ $(n+1)$-angles in $[\alpha^{n+1}]_{n+1,\mathtt{w}}$ between (and distinct from) $\tilde{\alpha}^n_1, \tilde{\alpha}^n_2$. The same argument applies to each of the m gaps intersecting $[\tilde{\alpha}^n]$. Therefore

$$\#[\alpha^{n+1}]_{n+1} = m + m(d-1) = md.$$

By inductive hypothesis it holds

$$m = \#[\tilde{\alpha}^n]_{n,\mathtt{w}} = d([\tilde{\alpha}^n]_{n,\mathtt{w}}) \cdot d([\tilde{\alpha}^{n-1}]_{n-1,\mathtt{w}}) \cdot \dots \cdot d([\tilde{\alpha}^1])_{1,\mathtt{w}},$$

where $[\tilde{\alpha}^j]_{j,\mathtt{w}} := \mu^{n-j}([\tilde{\alpha}^n]_{n,\mathtt{w}})$ for $j = 1, \dots, n$. Note that $[\tilde{\alpha}^j]_{j,\mathtt{w}} \subset [\alpha^{j+1}]_{j+1,\mathtt{w}}$, thus $d([\tilde{\alpha}^j]_{j,\mathtt{w}}) = d([\alpha^{j+1}]_{j+1,\mathtt{w}})$ for $j \geq 1$. By the same argument $d = d([\tilde{\alpha}^0]_{1,\mathtt{w}}) = d([\alpha^1])_{1,\mathtt{w}}$. The claim follows. $\square$

Lemma 7.4. *The equivalence class* $[\alpha^n]_{n,\mathtt{w}}$ *is of* Julia-type *if and only if*

$$\lim_{m\to\infty} \#[\alpha^n]_{m,\mathtt{w}} < \infty.$$

In fact then there is an m_0 *(independent of* n *and* α^n*) such that*

$$[\alpha^n]_{m,\mathtt{w}} = [\alpha^n]_{n+m_0,\mathtt{w}},$$

for all $\alpha^n \in \mathbf{A}^n$ *and* $m \geq n + m_0$*. Furthermore*

$$\#[\alpha^n]_{n,\mathtt{w}} \leq 2^{d-1},$$

for all Julia-type equivalence classes $[\alpha^n]_{n,\mathtt{w}}$*.*

Proof. It is clear that for any Fatou-type equivalence class $[\alpha^n]_{n,\mathtt{w}}$ it holds

$$\lim_{m\to\infty} \#[\alpha^n]_{m,\mathtt{w}} = \infty$$

by Lemma 7.3.

Let $[\alpha^1]_{1,\mathtt{w}}$ be of Julia-type ($\alpha^1 \in \mathbf{A}^0$). There is an m_0 (independent of α^1), such that $\mu^j([\alpha^1]_{1,\mathtt{w}})$ is not contained in any critical equivalence class (of $\overset{1,\mathtt{w}}{\sim}$), i.e., $d(\mu^j([\alpha^1]_{1,\mathtt{w}})) = 1$, for all $j \geq m_0$.

Let $[\alpha^n]_{m,\mathtt{w}}$ be of Julia-type, $m \geq n + m_0$, $\alpha^n \in \mathbf{A}^n$. Then $d(\mu^j([\alpha^n]_{m,\mathtt{w}})) = 1$ for all $j \geq n + m_0 - 1$. This proves the first claim, using Lemma 7.3.

To estimate the maximal size of a Julia-type equivalence class, let m_j ($j = 1, \dots, N$) be the sizes of the Julia-type equivalence classes of $\overset{1,\mathtt{w}}{\sim}$. From Lemma 7.3 it follows that (for any Julia-type equivalence class) $\#[\alpha^n]_{n,\mathtt{w}} \leq \prod m_j$. Maximizing this product subject to $\sum_j (m_j - 1) = d - 1$ (see ($\mathcal{L}^n$ 3)) yields the second statement. $\square$

Proof of Proposition 7.2. Assume F has no critical periodic cycles. Then there is a constant $M < \infty$ such that $\deg_{F^n}(c) \leq M$ for all $c \in S^2$ and n. Recall that $\deg_{F^n}(c)$ is the number of white/black n-tiles attached at the n-vertex c. Let $\alpha \in S^1$ such that $\gamma^n(\alpha) = c$. Then $[\alpha]_{n,\mathtt{w}} \leq M$, $[\alpha]_{n,\mathtt{b}} \leq M$. Therefore $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ has no Fatou-type classes by Lemma 7.4.

Assume now that F has critical periodic points. Let us assume first that $c = F(c)$ is a critical point. Then there are at least two white/black 1-tiles containing c. Thus $(\gamma^1)^{-1}(c) =: [\alpha^1]_1$ contains at least two points. Let $\{\alpha^0\} := \mu([\alpha^1]_1)$. Recall from (3.1) that $F \circ \gamma^1 = \gamma^0 \circ \mu$. Furthermore μ maps $\mathbf{A}^1$ to $\mathbf{A}^0$ and $\gamma^0 = \gamma^1$ on $\mathbf{A}^0$. Thus $\gamma^0 \circ \mu = \gamma^1 \circ \mu$ on $\mathbf{A}^1$, thus $c = F \circ \gamma^1(\alpha^1) = \gamma^1(\alpha^0)$. It follows that $\alpha^0 \in [\alpha^1]_1$, or $[\alpha^0]_1 = [\alpha^1]_1$. Therefore $[\alpha^0]_{1,\mathtt{w}}$ or $[\alpha^0]_{1,\mathtt{b}}$ has to contain at least two points (since $\overset{1}{\sim} = \overset{1,\mathtt{w}}{\sim} \vee \overset{1,\mathtt{b}}{\sim}$). Note that this equivalence class is of periodic Fatou-type.

Now assume that $F^n(c) = c$ for some $n \geq 1$. The same argument as above yields that there is $[\alpha^n]_{n,\mathtt{w}}$ or $[\alpha^n]_{n,\mathtt{b}}$, without loss of generality $[\alpha^n]_{n,\mathtt{w}}$, containing at least two points, such that $\mu^n([\alpha^n]_{n,\mathtt{w}}) \subset [\alpha^n]_{n,\mathtt{w}}$. By Lemma 7.3 one of the classes $[\alpha^n]_{n,\mathtt{w}}, [\alpha^{n-1}]_{n-1,\mathtt{w}} := \mu([\alpha^n]_{n,\mathtt{w}}), \dots, [\alpha^1]_{1,\mathtt{w}} := \mu^{n-1}([\alpha^n]_{n,\mathtt{w}})$ has to contain a critical class, which is periodic with respect to μ^n. □

7.3. The equivalence relation $\overset{\mathtt{w}}{\approx}$.

Recall from Section 5.6 that the equivalence classes $[\alpha^1]_{1,\mathtt{w}}$ as well as the equivalence classes $[\alpha^1]_{1,\mathtt{b}}$ form *critical portraits* in the sense of Poirier given in [Poi09]. This means they define unique monic, centered, postcritically finite polynomials. Furthermore the equivalence relations induced by their Carathéodory semi-conjugacies may be obtained from the critical portraits.

Thus (following Poirier) we define an equivalence relation $\overset{\mathtt{w}}{\approx}$ on S^1 which by Poirier equals the one defined in (7.1) (see next section).

Recall from (**G** 5) that each white n-gap G^n is contained in (exactly) one white $(n-1)$-gap G^{n-1}. Here and in the following we will consider sequences $(G^n)_{n\in\mathbb{N}}$ of gaps $G^n \in \mathbf{G}^n_{\mathtt{w}}$ such that

$$G^1 \supset G^2 \supset \dots. \tag{7.6}$$

We write

$$[(G^n)] := \bigcap G^n \cap S^1,$$

where it is always understood that the sequence $(G^n)_{n\in\mathbb{N}}$ is as in (7.6). Define for $s,t \in S^1$

$$s \overset{G}{\sim} t \;:\Leftrightarrow\; s,t \in [(G^n)], \text{ for some sequence } (G^n)_{n\in\mathbb{N}} \text{ as above.} \tag{7.7}$$

Then we define

$$s \overset{\mathtt{w}}{\approx} t :\Leftrightarrow \text{ there are } s_1, \dots, s_N \in S^1 \text{ such that } s = s_1 \overset{G}{\sim} s_2 \overset{G}{\sim} \dots \overset{G}{\sim} s_N = t. \tag{7.8}$$

Note that $\overset{G}{\sim}$ is not an equivalence relation, but $\overset{\mathtt{w}}{\approx}$ is. Note that $\overset{G}{\sim}$ should be properly equipped with an index "$\mathtt{w}$", which we suppress. The reader should be aware that there are analogously defined relations in terms of black gaps.

Let us record the following, we set $\mathbf{A}^\infty := \bigcup \mathbf{A}^n$.

Lemma 7.5 (Properties of $[(G^n)]$)**.** *The sets $[(G^n)]$ satisfy the following.*

(1) $\#[(G^n)] \leq k$, recall that $k = \#\operatorname{post}(F)$.

(2) *If* $\alpha \in S^1 \setminus \mathbf{A}^\infty$ *then* α *is contained in a* single *set* $[(G^n)]$*;*
if $\alpha \in \mathbf{A}^\infty$ *then* α *is contained in* at most two *such sets.*

(3) *Let* $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ *be* not equivalent *with respect to* $\overset{n,\mathtt{w}}{\sim}$*. Then*

$$\alpha^n, \tilde{\alpha}^n \textit{ are } \text{not } \textit{in the same set } [(G^n)].$$

(4) *Disjoint sets* $[(G^n)]$ *are* non-crossing.

Proof. The first property follows from ($\mathbf{G}$ 2). The second from the fact that each $\alpha \notin \mathbf{A}^\infty$ is contained in a *single* n-arc for each n; each $\alpha \in \mathbf{A}^\infty$ is contained in at most two n-arcs (which can be in the same or different sets $[(G^n)]$). The third follows from ($\mathbf{G}$ 6). The last property follows from ($\mathcal{L}^n$ 2). $\square$

We now list properties of $\overset{\mathtt{w}}{\approx}$, its equivalence classes are denoted by $[\alpha]_\approx$.

Lemma 7.6 (Properties of $\overset{\mathtt{w}}{\approx}$)**.** *The equivalence relation* $\overset{\mathtt{w}}{\approx}$ *satisfies the following.*

(1) *If* $[\alpha^n]_{n,\mathtt{w}}$ *is of* Julia-type *(*$\alpha^n \in \mathbf{A}^n$*) then*

$$[\alpha^n]_\approx \cap \mathbf{A}^n = [\alpha^n]_{n,\mathtt{w}}.$$

If $[\alpha^n]_{n,\mathtt{w}}$ *is of* Fatou-type *(*$\alpha^n \in \mathbf{A}^n$*) then*

$$[\alpha^n]_\approx \cap \mathbf{A}^\infty = \{\alpha^n\}.$$

(2) *Each equivalence class of* $\overset{\mathtt{w}}{\approx}$ *is finite, in fact*

$$\#[\alpha]_\approx \leq (k-1)2^{d-1}.$$

(3) *The number of sets* $[(G^n)]$ *that may form a chain, meaning the number* N *from (7.8), is finite; more precisely*

$$N \leq 2^{d-1}.$$

Proof. (1) From Lemma 7.5 (2) it follows that distinct sets $[(G^n)]$, $[(\widetilde{G}^n)]$ may only intersect in a point $\alpha^n \in \mathbf{A}^\infty$. From Lemma 7.5 (3) it then follows that if $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ are not equivalent with respect to $\overset{n,\mathtt{w}}{\sim}$, then they are not equivalent with respect to $\overset{\mathtt{w}}{\approx}$, or

$$[\alpha^n]_\approx \cap \mathbf{A}^n \subset [\alpha^n]_{n,\mathtt{w}}.$$

Consider a Julia-type equivalence class $[\alpha^n]_{n,\mathtt{w}}$ ($\alpha^n \in \mathbf{A}^n$). Let m_0 be the constant from Lemma 7.4, thus $[\alpha^n]_{m,\mathtt{w}} = [\alpha^n]_{n+m_0}$ for all $m \geq n + m_0$. Fix a $m \geq n + m_0$. Let $\beta^m, \tilde{\beta}^m \in [\alpha^n]_{m,\mathtt{w}}$ be succeeding (with respect to $\overset{m,\mathtt{w}}{\sim}$); there are at most 2^{d-1} such succeeding m-angles (see Lemma 7.4).

There is a gap $G^m \in \mathbf{G}^m_{\mathtt{w}}$ containing $\beta^m, \tilde{\beta}^m$. Thus succeeding angles of $[\alpha^n]_{n+m_0,\mathtt{w}}$ are equivalent with respect to $\overset{\mathtt{w}}{\approx}$, meaning

$$[\alpha^n]_{n,\mathtt{w}} \subset [\alpha^n]_{n+m_0,\mathtt{w}} \subset [\alpha^n]_\approx,$$

thus $[\alpha^n]_\approx \cap \mathbf{A}^n = [\alpha^n]_{n,\mathtt{w}}$ follows as desired.

Now let $[\alpha^n]_{n,\mathtt{w}}$ be of Fatou-type ($\alpha^n \in \mathbf{A}^n$). Then $\#[\alpha^n]_{m,\mathtt{w}} \to \infty$ as $m \to \infty$ by Lemma 7.4. If $[\alpha^n]_{m,\mathtt{w}} \subsetneq [\alpha^n]_{m+1,\mathtt{w}}$, then succeeding angles in $[\alpha^n]_{m,\mathtt{w}}$ are not succeeding in $[\alpha^n]_{m+1,\mathtt{w}}$ (see Case (2) in the proof of Lemma 7.3). Thus a set $[(G^n)]$ may contain at most one point from $[\alpha^n]_{n,\mathtt{w}}$. From Lemma 7.5 (2) and Lemma 7.5 (3) it follows that

$$[\alpha^n]_\approx \cap \mathbf{A}^\infty = \{\alpha^n\}.$$

(3) From the above it follows that $N = \max \#[\alpha^n]_{n,\mathtt{w}}$, where the maximum is taken over all Julia-type classes. The claim thus follows from Lemma 7.4.

(2) Let $G^m \in \mathbf{G}^m_{\mathtt{w}}$ be the gap containing succeeding m-angles $\beta^m, \tilde{\beta}^m \in [\alpha^n]_{m,\mathtt{w}}$, as described in the proof of (1). Then $[(G^m)]$ contains at most $k-2$ points distinct from $\beta^m, \tilde{\beta}^m$ by Lemma 7.5 (1). Thus

$$\#[\alpha^n]_{\approx} \leq \#[\alpha^n]_{m,\mathtt{w}} + (k-2)\#[\alpha^n]_{m,\mathtt{w}} \leq (k-1)2^{d-1},$$

by Lemma 7.4. □

Proposition 7.7. *The equivalence relation* $\overset{\mathtt{w}}{\approx}$ *is* closed.

Proof. We first show the corresponding result for $\overset{G}{\sim}$, i.e., the following.

Claim. Let $(s_j), (t_j) \subset S^1$ be sequences such that

$$s_j \to s, t_j \to t, \text{ and } s_j \overset{G}{\sim} t_j \text{ for all } j,$$

then

$$s \overset{G}{\sim} t.$$

If $s_j = s$ is constant the claim follows, since the set $\{t_j \overset{G}{\sim} s\}$ is finite (see Lemma 7.5 (1)).

Assume now that (s_j) is not constant. Without loss of generality we can assume that (s_j) is strictly increasing, the sets $[(G^n_j)] \ni s_j, t_j$ are disjoint (for distinct lower indices $j, j' \in \mathbb{N}$), and $s_j \neq t_j$ for all j. Since disjoint sets $[(G^n_j)]$ are non-crossing (Lemma 7.5 (4)) it follows that (t_j) is strictly decreasing. Let $a^n(s) \subset S^1$ be an n-arc containing s. If s is contained in two n-arcs, i.e., if $s \in \mathbf{A}^n$, we choose $a^n(s)$ as the n-arc having s as the right endpoint. Similarly let $a^n(t) \subset S^1$ be an n-arc containing t. If $t \in \mathbf{A}^n$, let $a^n(t)$ be the n-arc with t as the left endpoint.

For each n the points s_j, t_j are in the interiors of $a^n(s), a^n(t)$ respectively for sufficiently large j. Since $s_j \overset{G}{\sim} t_j$ it follows that $a^n(s), a^n(t)$ are contained in the same gap $G^n \in \mathbf{G}^n_{\mathtt{w}}$. Thus $s, t \in [(G^n)]$ proving the claim.

Consider now sequences $(s^n), (t^n) \subset S^1$, where $s^n \to s, t^n \to t$, such that $s^n \overset{\mathtt{w}}{\approx} t^n$ for all n. Thus by Lemma 7.6 (3) there are $s^n_j \in S^1$ such that

$$s^n = s^n_1 \overset{G}{\sim} s^n_2 \overset{G}{\sim} \dots \overset{G}{\sim} s^n_N = t^n.$$

Here $N \in \mathbb{N}$ is *independent* of n. By taking subsequences we can assume that $s^n_j \to s_j$ as $n \to \infty$, for all j. From the previous claim it follows that $s_j \overset{G}{\sim} s_{j+1}$ (for $j = 1, \dots N-1$). Thus

$$s = s_1 \overset{G}{\sim} \dots \overset{G}{\sim} s_N = t,$$

meaning $s \overset{\mathtt{w}}{\approx} t$.

□

7.4. **The white polynomial.** A. Poirier [Poi09], extending work of Bielefeld-Fisher-Hubbard [BFH92], has shown that postcritically finite polynomials admit a combinatorial classification in terms of external rays. The result is paraphrased here, not in full generality, but only in the relevant case at hand.

Poirier's Theorem ([Poi09], [BFH92]). *Let the sets $[\alpha_j^1]_{1,\mathtt{w}} \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1$, $j = 1, \dots, m$ form a* critical portrait *as in Definition 5.12. Then there is a unique monic, centered, postcritically finite polynomial $P_{\mathtt{w}}$ such that*

(Poi 1) the equivalence realtion $\overset{\mathtt{w}}{\approx}$ (as defined in the last section) is the equivalence relation induced by the Carathéodory semi-conjugacy of $P_{\mathtt{w}}$, meaning that

$$s \overset{\mathtt{w}}{\approx} t \Leftrightarrow \sigma_{\mathtt{w}}(s) = \sigma_{\mathtt{w}}(t);$$

where $\sigma \colon S^1 = \mathbb{R}/\mathbb{Z} \to \partial \mathcal{K}_{\mathtt{w}} = \mathcal{J}_{\mathtt{w}}$ is defined as in Section 1.1. This is [Poi09, Theorem 1.13] *and* [Poi09, Proposition 7.7]. *In particular each set $\sigma([\alpha]_{\approx})$ is a single point.*

(Poi 2) If $[\alpha^n]_{n,\mathtt{w}}$ is of Fatou-type, then all points in $\sigma([\alpha^n]_{n,\mathtt{w}})$ are in the boundary of the same *bounded Fatou component of $P_{\mathtt{w}}$. Distinct sets $[\alpha^n]_{n,\mathtt{w}}, [\tilde{\alpha}^n]_{n,\mathtt{w}}$ are in the boundaries of* distinct *bounded Fatou components. This is* [Poi09, Proposition 8.4]. *Furthermore for each bounded Fatou component A there is a $\alpha^n \in \mathbf{A}^n$ such that $\sigma(\alpha^n) \in \operatorname{clos} A$, where $[\alpha^n]_{n,\mathtt{w}}$ is of Fatou-type.*

Since $P_{\mathtt{w}}^{-1}(\sigma(\mathbf{A}^n)) = \sigma(\mathbf{A}^{n+1})$ it follows that (for every Fatou-type class $[\alpha^n]_{n,\mathtt{w}}$)

$$\sigma([\alpha^n]_{\infty,\mathtt{w}}) \textit{ is } \text{dense}$$

in the boundary of the (bounded) Fatou component A satisfying $\sigma([\alpha^n]_{n,\mathtt{w}}) \subset \partial A$.

(Poi 3) For each gap $G^n \in \mathbf{G}_{\mathtt{w}}^n$ the set

$$\sigma(G^n) \textit{ is a } \text{connected} \textit{ subset of the Julia set of } P_{\mathtt{w}}.$$

The images of two disjoint gaps $G^n, \widetilde{G}^n \in G_{\mathtt{w}}^n$ are disjoint:

$$G^n \cap \widetilde{G}^n = \emptyset \quad \Rightarrow \quad \sigma(G^n \cap S^1) \cap \sigma(\widetilde{G}^n \cap S^1) = \emptyset;$$

see [Poi09, Section 5].

From now on the *white polynomial* $P_{\mathtt{w}}$ will be the one obtained from Poirier's Theorem from the sets $[\alpha^1]_{1,\mathtt{w}}$, $\alpha^1, \in \mathbf{A}^1$.

7.5. **Outline of the proof of Poirier's Theorem.** Poirier's definition of the critical portrait of a postcritically finite polynomial is slightly different from ours. This is due to the fact that he describes general such polynomials, not just ones with "separated Fatou set" (see Proposition 7.8) as considered here. For the convenience of the reader we give a very brief outline of the proof of the main result from Poirier's Theorem, namely the existence of the polynomial $P_{\mathtt{w}}$.

Consider a topological polynomial, i.e., a Thurston map $P \colon S^2 \to S^2$ such that $P^{-1}(\infty) = \infty$. It is well known that P is "Thurston equivalent" to a polynomial if and only if it has no "Levy cycle" (Theorem 5.4 and Theorem 5.5 in [BFH92]). A Levy cycle is a Jordan curve $\Gamma \subset S^2 \setminus \operatorname{post}(P)$ such that

- each component of $S^2 \setminus \Gamma$ contains at least two postcritical points and

- some component Γ' of $P^{-j}(\Gamma)$ is isotopic rel. post(P) to Γ for some j; and the map
$$P^j\colon \Gamma' \to \Gamma \text{ is of degree } 1.$$

To prove Poirier's Theorem (in our special case) one constructs a (postcritically finite) topological polynomial from the critical portrait. For each $\alpha^1 \in \mathbf{A}^1$ there is an "extended external ray" $R(\alpha^1)$ that is mapped by P to $R(\mu(\alpha^1))$. The extended external rays associated to the angles of one equivalence class $[\alpha^1]_{1,\mathtt{w}}$ intersect in a point. Assume there is a Levy cycle Γ (i.e., P is not equivalent to a polynomial). We can choose Γ in such a way that Γ intersects no preperiodic extended external ray (Lemma 8.7 in [BFH92]). From the last property of the critical portrait (Definition 5.12) it follows that two postcritical points are separated by some preperiodic extended external rays. Thus Γ contains at most one postcritical point in its interior, giving a contradiction.

7.6. **The Fatou set of $P_{\mathtt{w}}$.** Here we show that the Fatou set of $P_{\mathtt{w}}$ is "separated".

Proposition 7.8. *The Fatou set of $P_{\mathtt{w}}$ has the following property.*

- *The closures of two distinct bounded components A_1, A_2 of the Fatou set of $P_{\mathtt{w}}$ are disjoint,*
$$\operatorname{clos} A_1 \cap \operatorname{clos} A_2 = \emptyset.$$
- *No bounded Fatou component of $P_{\mathtt{w}}$ contains a point $\sigma([\alpha^n]_{n,\mathtt{w}})$, where $[\alpha^n]_{n,\mathtt{w}}$ is of Julia-type, in its boundary (recall from (Poi 1) and Lemma 7.6 (1) that σ maps $[\alpha^n]_{n,\mathtt{w}}$ to a single point).*

We need some preparation to prove this proposition. The key is an explicit description of the set of angles in S^1 that are mapped by σ to the boundary of a given bounded Fatou component/critical point (or more generally n-vertex).

Fix an n-angle $\alpha^n \in \mathbf{A}^n$. We will consider the equivalence class $[\alpha^n]_{m,\mathtt{w}}$ for some $m \geq n$. Let $G^m_1, \dots, G^m_{N_m} \in \mathbf{G}^m_{\mathtt{w}}$ be the gaps intersecting $[\alpha^n]_{m,\mathtt{w}}$. Here $N_m = \#[\alpha^n]_{m,\mathtt{w}}$.

Lemma 7.9. *In the setting as above*
$$\bigcup_j G^{m+n_0}_j \cap S^1 \text{ is compactly contained in } \bigcup_j G^m_j \cap S^1,$$
for all $m \geq n_0$, where n_0 is the constant from (**G** 6).

Proof. Note first that every m-angle in $[\alpha^n]_{m,\mathtt{w}}$ is contained in two m-arcs. Thus $[\alpha^n]_{m,\mathtt{w}}$ is contained in the interior of $\bigcup_j G^m_j \cap S^1$.

From Theorem 5.10 and ($\mathcal{L}^n$ 6)) it follows that $[\alpha^n]_{i,\mathtt{w}} \subset \bigcup_j G^m_j$ for all $i \geq m$ by construction

Every boundary point of $\bigcup_j G^m_j \cap S^1$ is a point $\tilde{\alpha}^m \in \mathbf{A}^m$ not equivalent (with respect to $\overset{m,\mathtt{w}}{\sim}$) to α^n. The statement follows from (**G** 6). $\square$

Lemma 7.10. *Let A be a bounded component of the Fatou set of $P_{\mathtt{w}}$, $\alpha^n \in \mathbf{A}^n$ such that $\sigma(\alpha^n) \in \partial A$. The gaps $G^m_j = G^m_j(\alpha^n) \in \mathbf{G}^m_{\mathtt{w}}$ are the ones intersecting $[\alpha^n]_{m,\mathtt{w}}$ as before. Then*
$$\sigma^{-1}(\partial A) = \bigcap_m \bigcup_j G^m_j \cap S^1.$$

Proof. The right hand side of the above expression is compact and contains all points of $[\alpha^n]_{\infty,\mathtt{w}}$. Since the set $\sigma([\alpha^n]_{\infty,\mathtt{w}})$ is dense in ∂A (Poi 2), it follows that $\sigma(\bigcap_m \bigcup_j G_j^m \cap S^1) \supset \partial A$. Note that

$$\bigcap_m \bigcup_j G_j^m = \bigcup [(G^n)],$$

where the union on the right hand side is taken over all sequences of white gaps $G^1 \supset G^2 \supset \dots$ such that $G^n \cap [\alpha^n]_{\infty,\mathtt{w}} \neq \emptyset$ for all n. For each such sequence the point $\sigma([(G^n)])$ is an accumulation point of $\sigma([\alpha^n]_{\infty,\mathtt{w}})$, thus

$$\sigma\Big(\bigcap_m \bigcup_j G_j^m \cap S^1\Big) = \partial A.$$

Consider a gap $\widetilde{G}^m \in \mathbf{G}_{\mathtt{w}}^m$ that is distinct from all gaps G_j^m. From Lemma 7.9 it follows that $\widetilde{G}^m \cap S^1$ and $G_j^{m+n_0} \cap S^1$ are disjoint. By (Poi 3) it follows that these sets are mapped to disjoint sets by σ, thus $\sigma(\widetilde{G}^m \cap S^1) \cap \partial A = \emptyset$. This proves the claim. □

The same argument as above applies to Julia-type equivalence classes (see Lemma 7.4 and Lemma 7.6 (1)).

Corollary 7.11. *Let $[\alpha^n]_{n,\mathtt{w}}$ ($\alpha^n \in \mathbf{A}^n$) be of* Julia-type, *$\sigma(\alpha^n) = v$. Then*

$$\sigma^{-1}(v) = \bigcup [(G^m)],$$

where the (finite) union is taken over all sets $[(G^m)]$ such that $G^m \cap [\alpha^n]_{m,\mathtt{w}} \neq \emptyset$ for all m.

Proposition 7.8 now follows using (**G** 6).

7.7. **The equivalence relation $\overset{\mathcal{F},\mathtt{w}}{\approx}$.** We consider the equivalence relation $\overset{\mathcal{F},\mathtt{w}}{\approx}$ obtained from the Carathéodory semi-conjugacy, together with the identification of Fatou components as in (7.2). Our main objective is to show the following.

Proposition 7.12. *We have*

$$\overset{\mathcal{F},\mathtt{w}}{\approx} = \overset{\mathtt{w}}{\sim}.$$

Recall that $\overset{\mathtt{w}}{\sim}$ is the equivalence relation from Definition 5.13. To prove this proposition some preparation is needed first. Let us first note the following, which is an immediate consequence of Proposition 7.8.

Lemma 7.13. *The relation $\overset{\mathcal{F},\mathtt{w}}{\approx}$ is an* equivalence relation*.*

We write $[s]_{\mathcal{F},\mathtt{w}} := \{t \in S^1 \mid s \overset{\mathcal{F},\mathtt{w}}{\approx} t\}$ for equivalence classes of $\overset{\mathcal{F},\mathtt{w}}{\approx}$. A description of them follows immediately from Section 7.6.

Lemma 7.14. *Consider $[s]_{\mathcal{F},\mathtt{w}}$ ($s \in S^1$). Either*

- *$[s]_{\mathcal{F},\mathtt{w}} \cap \mathbf{A}^\infty = \emptyset$, then*

  $$[s]_{\mathcal{F},\mathtt{w}} = [(G^m)],$$

  for one sequence $G^1 \supset G^2 \supset \dots$ of white gaps. Note that in this case $[s]_{\mathcal{F},\mathtt{w}}$ is compactly contained *in G^n for all n.*

- *Or there is $\alpha^n \in [s]_{\mathcal{F},\mathtt{w}}$, $\alpha^n \in \mathbf{A}^n$. Then*
$$[s]_{\mathcal{F},\mathtt{w}} = [\alpha^n]_{\mathcal{F},\mathtt{w}} = \bigcup [(G^m)],$$
where the union is taken over all sets $[(G^m)]$ (as in (7.6)), satisfying $G^m \cap [\alpha^n]_{m,\mathtt{w}} \neq \emptyset$ for all m. Again
$$[s]_{\mathcal{F},\mathtt{w}} \textit{ is } \text{compactly contained } \textit{in } \bigcup_j G_j^m,$$
for all m, where the union is taken over all white m-gaps intersecting $[\alpha^n]_{m,\mathtt{w}}$.

Lemma 7.15. *The equivalence relation $\overset{\mathcal{F},\mathtt{w}}{\approx}$ is* closed.

Proof. Consider a convergent sequence $s_n \to s$ in S^1. Let $t_n \in [s_n]_{\mathcal{F},\mathtt{w}}$ for all n, such that $t_n \to t$. We want to show that $s \overset{\mathcal{F},\mathtt{w}}{\approx} t$, i.e., $t \in [s]_{\mathcal{F},\mathtt{w}}$. This is clearly the case when $s_n = s$ is constant, since $[s]_{\mathcal{F},\mathtt{w}}$ is compact.

Thus we can assume that $s_n \neq s$ for all n. Fix an m. Since $\mathbf{A}^m$ is a finite set it follows that $[s_n]_{\mathcal{F},\mathtt{w}} \cap \mathbf{A}^m = \emptyset$ for sufficiently large n. Thus we can assume that $[s_n]_{\mathcal{F},\mathtt{w}} \cap \mathbf{A}^n = \emptyset$, for all n. It follows that
$$[s_n]_{\mathcal{F},\mathtt{w}} \text{ is contained in a } \textit{single} \text{ white } n\text{-gap } G^n.$$
Assume first that $[s]_{\mathcal{F},\mathtt{w}}$ contains $\alpha^n \in \mathbf{A}^n$. Let $G_j^m \in \mathbf{G}_{\mathtt{w}}^m$ be the gaps intersecting $[\alpha^n]_{m,\mathtt{w}}$ as in Lemma 7.14. Since $[s]_{\mathcal{F},\mathtt{w}}$ is compactly contained in $\bigcup_j G_j^m$ it follows that s_n is in the interior of $\bigcup_j G_j^m$ for sufficiently large n. Taking a subsequence if necessary as before, we can assume that
$$s_m \text{ is in the } \textit{interior} \text{ of } \bigcup_j G_j^m,$$
for all m. Thus the white m-gap $G^m \supset [s_m]_{\mathcal{F},\mathtt{w}}$ equals one of the gaps G_j^m. It follows that $t \in \bigcap_m \bigcup_j G_j^m = [s]_{\mathcal{F},\mathtt{w}}$ as desired.

The case when $[s]_{\mathcal{F},\mathtt{w}}$ contains no angle in $\mathbf{A}^\infty$ is proved by exactly the same argument. $\square$

Proof of Proposition 7.12. From the second part in Lemma 7.14 it follows that $[\alpha^n]_{n,\mathtt{w}} \subset [\alpha^n]_{\mathcal{F},\mathtt{w}}$ for all $\alpha^n \in \mathbf{A}^n$. Thus
$$\overset{n,\mathtt{w}}{\sim} \leq \overset{\mathcal{F},\mathtt{w}}{\approx}$$
for all n. It follows that $\overset{\infty,\mathtt{w}}{\sim} \leq \overset{\mathcal{F},\mathtt{w}}{\approx}$. Since $\overset{\mathcal{F},\mathtt{w}}{\approx}$ is closed by Lemma 7.15 it follows that
$$\overset{\mathtt{w}}{\sim} \leq \overset{\mathcal{F},\mathtt{w}}{\approx}.$$
To see the reverse inequality we first prove the following.

Claim. $s, t \in [(G^n)]$ implies $s \overset{\mathtt{w}}{\sim} t$.

Recall from (**G** 2) that $G^n \cap S^1$ consists of k n-arcs $[\alpha_0^n, \beta_0^n], \dots, [\alpha_{k-1}^n, \beta_{k-1}^n]$, where $\beta_j^n \overset{n,\mathtt{w}}{\sim} \alpha_{j+1}^n$ (lower index taken $\operatorname{mod} k$). Since $\lim_n \alpha_j^n = \lim_n \beta_j^n$ for all j the claim follows.

From the claim it follows using Lemma 7.14 that $\overset{\mathcal{F},\mathtt{w}}{\approx} \leq \overset{w}{\sim}$, finishing the proof. $\square$

7.8. **The black polynomial $P_{\mathtt{b}}$.** The *black polynomial* $P_{\mathtt{b}}$ is the one obtained from Poirier's Theorem from the black critical portrait, i.e., the sets $[\alpha^1]_{1,\mathtt{b}}$ (for all $\alpha^1 \in \mathbf{A}^1$). More precisely $P_{\mathtt{b}}$ is the (unique monic, centered, postcritically finite) polynomial such that the equivalence relation defined by (for all $z, w \in S^1 = \partial D$)

$$s \overset{\mathtt{b}}{\approx} t :\Leftrightarrow \sigma_{\mathtt{b}}(\bar{z}) = \sigma_{\mathtt{b}}(\overline{w})$$

is equal to the equivalence relation relation $\overset{\mathtt{b}}{\approx}$ defined in terms of the black gaps as in Section 7.3. Here $\sigma_{\mathtt{b}}$ is a Carathéodory semi-conjugacy of the Julia set of $P_{\mathtt{b}}$. The equivalence relation $\overset{\mathcal{F},\mathtt{b}}{\approx}$ on S^1 is then defined as in (7.3). As in Proposition 7.12 it follows that

$$\overset{\mathcal{F},\mathtt{b}}{\approx} = \overset{\mathtt{b}}{\sim},$$

where $\overset{\mathtt{b}}{\sim}$ was defined in Definition 5.13.

7.9. **Proof of Theorem 1.2.** We assume now that F has no periodic critical points. This means that there are no Fatou-type equivalence classes of $\overset{1,\mathtt{w}}{\sim}, \overset{1,\mathtt{b}}{\sim}$ (Proposition 7.2), hence no Fatou-type classes of $\overset{n,\mathtt{w}}{\sim}, \overset{n,\mathtt{b}}{\sim}$. The white polynomial $P_{\mathtt{w}}$ is defined as in Section 7.4, the black polynomial $P_{\mathtt{b}}$ as in Section 7.8.

From (Poi 2) it follows that the Fatou sets of $P_{\mathtt{w}}, P_{\mathtt{b}}$ have no bounded components, thus their Julia sets are dendrites. Let $\overset{\mathtt{w}}{\approx}, \overset{\mathcal{F},\mathtt{w}}{\approx}$ be the equivalence relations (on S^1) from Section 7.3 and Section 7.7. Then $\overset{\mathtt{b}}{\approx}, \overset{\mathcal{F},\mathtt{b}}{\approx}$ are defined analogously in terms of the black equivalence relations $\overset{n,\mathtt{b}}{\sim}$. Since $P_{\mathtt{w}}, P_{\mathtt{b}}$ have no bounded Fatou components it follows from Proposition 7.12 that

$$\overset{\mathtt{w}}{\approx} = \overset{\mathcal{F},\mathtt{w}}{\approx} = \overset{\mathtt{w}}{\sim} \text{ and } \overset{\mathtt{b}}{\approx} = \overset{\mathcal{F},\mathtt{b}}{\approx} = \overset{\mathtt{b}}{\sim}.$$

Recall that $\sim$ is the equivalence relation (on S^1) induced by the invariant Peano curve (1.5). From Lemma 5.14 (ii) it follows that $\sim = \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} = \overset{\mathtt{w}}{\approx} \vee \overset{\mathtt{b}}{\approx} = \approx$. Theorem 1.2 now follows using Theorem 1.5 and Lemma 7.1.

8. Proof of Theorem 1.3

We finish the proof of Theorem 1.3 here. The white/black polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are defined as in Section 7.4 and 7.8.

Recall from (7.2), (7.3) the definition of the associated equivalence relations $\overset{\mathcal{F},\mathtt{w}}{\approx}, \overset{\mathcal{F},\mathtt{b}}{\approx}$. In Proposition 7.12 it was shown that $\overset{\mathcal{F},\mathtt{w}}{\approx} = \overset{\mathtt{w}}{\sim}$ as well as $\overset{\mathcal{F},\mathtt{b}}{\approx} = \overset{\mathtt{b}}{\sim}$. Let $\sim$ be the closure of $\overset{\mathcal{F},\mathtt{w}}{\approx} \vee \overset{\mathcal{F},\mathtt{b}}{\approx} = \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$. In Lemma 5.14 it was shown that $\sim$ is the equivalence relation induced by the invariant Peano curve γ. Recall the definition of $P_{\mathtt{w}} \widehat{\perp\!\!\!\perp} P_{\mathtt{b}}$ from (1.4). In Section 8.2 we will show the following lemma.

Lemma 8.1. *The map*

$$P_{\mathtt{w}} \widehat{\perp\!\!\!\perp} P_{\mathtt{b}} \colon \mathcal{K}_{\mathtt{w}} \widehat{\perp\!\!\!\perp} \mathcal{K}_{\mathtt{b}} \to \mathcal{K}_{\mathtt{w}} \widehat{\perp\!\!\!\perp} \mathcal{K}_{\mathtt{b}}$$

is well defined and topologically conjugate to

$$z^d/\sim \colon S^1/\sim \to S^1/\sim .$$

Theorem 1.3 follows, using Theorem 1.5.

8.1. **Closures.** Here we collect some elementary lemmas that will be needed.

Let S, S' be compact metric spaces, $h\colon S \to S'$ be a continuous surjection, and $\approx$ be an equivalence relation on S'. The equivalence relation $\sim$ on S defined by

$$s \sim t :\Leftrightarrow h(s) \approx h(t), \text{ for all } s, t \in S,$$

is called the *pullback* of $\approx$ by h.

Lemma 8.2. *In the setting as above, $\sim$ is closed if and only if $\approx$ is closed.*

The proof is straightforward and left as an exercise.

We now assume that an equivalence relation $\sim$ is defined on S, we want to define a corresponding equivalence relation $\approx$ on S'.

Lemma 8.3. *Let $h\colon S \to S'$ be a surjection, $\sim$ an equivalence relation on S that is bigger than the one induced by h ($h(s) = h(t) \Rightarrow s \sim t$ for all $s, t \in S$). Define $\approx$ on S' as follows*

$$s' \approx t' :\Leftrightarrow \text{ there exists } s, t \in S \text{ such that } s' = h(s), t' = h(t),$$

for all $s', t' \in S'$. Then $\approx$ is an equivalence relation (on S') such that $\sim$ is the pullback of $\approx$ by h.

Proof. It is straightforward to check that $\approx$ is an equivalence relation, which is left as an exercise.

If $s \sim t$, then $s' = h(s) \approx \ t' = h(t)$ for all $s, t \in S$.

Now let $s' \approx t'$ for some $s', t' \in S'$. Consider $\tilde{s}, \tilde{t} \in S$ with $s' = h(\tilde{s})$, $t' = h(\tilde{t})$. We want to show that $\tilde{s} \sim \tilde{t}$. There are $s, t \in S$ with $s \sim t$ and $h(s) = s'$, $h(t) = t'$. Since $\sim$ is bigger than the equivalence relation induced by h it follows that $\tilde{s} \sim s \sim t \sim \tilde{t}$. $\square$

Lemma 8.4. *Let $h\colon S \to S'$ be a continuous surjection, where S, S' are compact metric spaces. Let $\sim, \approx$ be equivalence relations on S, S', such that $\sim$ is the pullback of $\approx$ by h. Let $\widehat{\sim}$ be the closure of $\sim$, $\widehat{\approx}$ the closure of $\approx$. Then $\widehat{\sim}$ is the pullback of $\widehat{\approx}$ by h.*

Proof. Recall from the proof of Lemma 4.5 that the closure of an equivalence relation is given by the intersection of all bigger closed equivalence relations.

The pullback of $\widehat{\approx}$ is closed by Lemma 8.2, as well as bigger than $\sim$, hence bigger than $\widehat{\sim}$.

Now consider the equivalence relation $\approx'$ on S' induced by $\widehat{\sim}$ and h as in Lemma 8.3. It is bigger than $\approx$. Furthermore $\approx'$ pulls back to $\widehat{\sim}$ by h and is closed by Lemma 8.2. It follows that $\widehat{\sim}$ is bigger than the pullback of $\widehat{\approx}$ by h. $\square$

Consider now a continuous surjection $\mu\colon S \to S$ on a compact metric space S. An equivalence relation $\sim$ on S is called *invariant* with respect to μ if

$$s \sim t \Rightarrow \mu(s) \sim \mu(t) \text{ for all } s, t \in S.$$

Lemma 8.5. *Let $\mu\colon S \to S$ be continuous, surjective; $\sim$ an equivalence relation on S invariant with respect to μ. Then the closure $\widehat{\sim}$ of $\sim$ is invariant with respect to μ.*

Proof. Consider the equivalence relation $\approx$ given by

$$s \approx t :\Leftrightarrow \mu(s) \widehat{\sim} \mu(t),$$

for all $s, t \in S$. From Lemma 8.2 it follows that $\approx$ is closed. Note that

$$s \sim t \Rightarrow \mu(s) \sim \mu(t) \Rightarrow \mu(s) \widehat{\sim} \mu(t) \Rightarrow s \approx t,$$

meaning that $\approx \, \geq \, \sim$. Note that the meet of any two closed equivalence relations is closed (see (4.2)), thus the meet

$$\begin{aligned} &\approx \wedge \widehat{\sim} \text{ is closed and bigger than } \sim\text{; which implies} \\ &\widehat{\sim} \; = \approx \wedge \widehat{\sim}. \end{aligned}$$

Thus for all $s, t \in S$

$$s \widehat{\sim} t \Rightarrow s \approx t \Rightarrow \mu(s) \widehat{\sim} \mu(t),$$

finishing the proof. $\square$

In the next lemma we "take the closure of a commutative diagram and show that everything goes well". Let $\sim, \approx$ be equivalence relations on compact metric spaces S, S'. The maps $\mu\colon S \to S$, $\varphi\colon S' \to S'$ as well as $h\colon S \to S'$ are continuous surjections such that the following diagram commutes

$$\begin{tikzcd} (S, \sim) \arrow[r, "\mu"] \arrow[d, "h"'] & (S, \sim) \arrow[d, "h"] \\ (S', \approx) \arrow[r, "\varphi"'] & (S', \approx). \end{tikzcd}$$

By this is meant that $\varphi \circ h = h \circ \mu$. The equivalence relation $\sim$ is invariant with respect to μ, and $\approx$ is invariant with respect to φ. Furthermore $\sim$ is the pullback of $\approx$ by h.

Lemma 8.6. *In the setting as above, let $\widehat{\sim}$ be the closure of $\sim$; $\widehat{\approx}$ be the closure of $\approx$. Then*

$$\mu/\widehat{\sim}\colon S/\widehat{\sim} \to S/\widehat{\sim}$$

is topologically conjugate to

$$\varphi/\widehat{\approx}\colon S'/\widehat{\approx} \to S'/\widehat{\approx}.$$

Proof. We note first that the maps $\mu/\widehat{\sim}$, $\varphi/\widehat{\approx}$ are well defined by Lemma 8.5. From Lemma 8.4 it follows that $\widehat{\sim}$ is the pullback of $\widehat{\approx}$ by h. From (CE 4) it follows that $S'/\widehat{\approx}$ is a compact Hausdorff space. Applying Lemma 4.6 to the map $S \xrightarrow{h} S' \to S'/\widehat{\approx}$ yields that $S/\widehat{\sim}$ is homeomorphic to $S'/\widehat{\approx}$, where the homeomorphism is given by $\tilde{h}([s]_{\widehat{\sim}}) = [h(s)]_{\widehat{\approx}}$. Write $\widetilde{\varphi} = \varphi/\widehat{\approx}$, $\tilde{\mu} = \mu/\widehat{\sim}$, then

$$\widetilde{\varphi} \circ \tilde{h}([s]_{\widehat{\sim}}) = \widetilde{\varphi}([h(s)]_{\widehat{\approx}}) = [\varphi \circ h(s)]_{\widehat{\approx}} = [h \circ \mu(s)]_{\widehat{\approx}} = \tilde{h}([\mu(s)]_{\widehat{\sim}}) = \tilde{h} \circ \tilde{\mu}([s])_{\widehat{\sim}}.$$

This finishes the proof. $\square$

8.2. **Proof of Lemma 8.1.** We first show that $P_{\mathtt{w}} \widehat{\perp\!\!\!\perp} P_{\mathtt{b}}$ is well defined.

Consider $\mathcal{K}_{\mathtt{w}} \sqcup \mathcal{K}_{\mathtt{b}}$, the disjoint union of $\mathcal{K}_{\mathtt{w}}, \mathcal{K}_{\mathtt{b}}$. The equivalence relation $\sim$ on $\mathcal{K}_{\mathtt{w}} \sqcup \mathcal{K}_{\mathtt{b}}$ is the one generated by

$$\begin{aligned} &\sigma_{\mathtt{w}}(z) \sim \sigma_{\mathtt{b}}(\bar{z}), \quad \text{for all } z \in S^1 = \partial\mathbb{D} \text{ and} \\ &x \sim y, \quad \text{if } x, y \in \operatorname{clos} A_{\mathtt{w}} \text{ or } x, y \in \operatorname{clos} A_{\mathtt{b}}, \end{aligned} \tag{8.1}$$

for all $x, y \in \mathcal{K}_{\mathtt{w}}$, or $x, y \in \mathcal{K}_{\mathtt{b}}$. Here $A_{\mathtt{w}}, A_{\mathtt{b}}$ are bounded components of the Fatou sets of $P_{\mathtt{w}}, P_{\mathtt{b}}$ and $\sigma_{\mathtt{w}}(z) \in \mathcal{K}_{\mathtt{w}} \subset \mathcal{K}_{\mathtt{w}} \sqcup \mathcal{K}_{\mathtt{b}}$, $\sigma_{\mathtt{b}}(\bar{z}) \in \mathcal{K}_{\mathtt{b}} \subset \mathcal{K}_{\mathtt{w}} \sqcup \mathcal{K}_{\mathtt{b}}$. The map

$$P_{\mathtt{w}} \sqcup P_{\mathtt{b}} \colon \mathcal{K}_{\mathtt{w}} \sqcup \mathcal{K}_{\mathtt{b}} \to \mathcal{K}_{\mathtt{w}} \sqcup \mathcal{K}_{\mathtt{b}},$$

is well defined. Furthermore the equivalence relation defined in (8.1) is invariant with respect to $P_{\mathtt{w}} \sqcup P_{\mathtt{b}}$. Let $\widehat{\sim}$ be the closure of $\sim$, and $\mathcal{K}_{\mathtt{w}} \widehat{\perp\!\!\!\perp} \mathcal{K}_{\mathtt{b}} = \mathcal{K}_{\mathtt{w}} \sqcup K_{\mathtt{b}}/\widehat{\sim}$. From Lemma 8.5 it follows that $P_{\mathtt{w}} \sqcup P_{\mathtt{b}}$ descends to this quotient, meaning that

$$P_{\mathtt{w}} \widehat{\perp\!\!\!\perp} P_{\mathtt{b}} \colon \mathcal{K}_{\mathtt{w}} \widehat{\perp\!\!\!\perp} \mathcal{K}_{\mathtt{b}} \to \mathcal{K}_{\mathtt{w}} \widehat{\perp\!\!\!\perp} \mathcal{K}_{\mathtt{b}},$$

is well defined.

We first show a one-sided version of Lemma 8.1. Identify the closure of each bounded Fatou component in $\mathcal{K}_{\mathtt{w}}$ to form the quotient $\mathcal{K}_{\mathtt{w}}/\mathcal{F}$. Recall that the Fatou set of $P_{\mathtt{w}}$ is separated. Since $P_{\mathtt{w}}$ maps each bounded Fatou component to a bounded Fatou component the quotient map

$$P_{\mathtt{w}}/\mathcal{F} \colon \mathcal{K}_{\mathtt{w}}/\mathcal{F} \to \mathcal{K}_{\mathtt{w}}/\mathcal{F}$$

is well defined.

Lemma 8.7. *The map $P_{\mathtt{w}}/\mathcal{F}$ as above is topologically conjugate to*

$$z^d \colon S^1/\overset{\mathcal{F},\mathtt{w}}{\approx} \to S^1/\overset{\mathcal{F},\mathtt{w}}{\approx}.$$

Proof. Consider the equivalence relation on $\mathcal{K}_{\mathtt{w}}$, defined by $(x, y \in \mathcal{K}_{\mathtt{w}})$

$$x \overset{\mathcal{F}}{\approx} y :\Leftrightarrow x, y \in \operatorname{clos} \mathcal{F},$$

where $\mathcal{F}$ is a bounded Fatou component of $P_{\mathtt{w}}$.

Claim. $\overset{\mathcal{F}}{\approx}$ is closed.

Consider two convergent sequences $x_n \to x_0$, $y_n \to y_0$ in $\mathcal{K}_{\mathtt{w}}$, satisfying $x_n \overset{\mathcal{F}}{\approx} y_n$ (for all $n \geq 1$). We need to show that $x_0 \overset{\mathcal{F}}{\approx} y_0$. This is clear when the sequence (x_n) is contained in a single equivalence class of $\overset{\mathcal{F}}{\approx}$.

Assume now that each x_n is contained in a distinct equivalence class, which we can assume to be non-trivial. This means that x_n, y_n are contained in the closure of the same bounded Fatou component A_n. From the subhyperbolicity of $P_{\mathtt{w}}$ it follows that $\operatorname{diam} A_n \to 0$, thus $x_0 = \lim x_n = \lim y_n = y_0$ proving the claim.

From the claim it follows, that $\mathcal{K}_{\mathtt{w}}/\mathcal{F}$ is a compact Hausdorff space (see (CE 4)). Clearly $\overset{\mathcal{F},\mathtt{w}}{\approx}$ is the equivalence relation (on S^1) induced by the map $S^1 \xrightarrow{\sigma_{\mathtt{w}}} \mathcal{K}_{\mathtt{w}} \to \mathcal{K}_{\mathtt{w}}/\mathcal{F}$. Thus we obtain from Lemma 4.6 that $S^1/\overset{\mathcal{F},\mathtt{w}}{\approx}$ is homeomorphic to $\mathcal{K}_{\mathtt{w}}/\mathcal{F}$. The topological conjugacy is clear, since $P_{\mathtt{w}}$ maps each point $\sigma_{\mathtt{w}}(z) \in \mathcal{K}_{\mathtt{w}}$ to $\sigma_{\mathtt{w}}(z^d) \in \mathcal{K}_{\mathtt{w}}$ (for all $z \in \partial\mathbb{D}$). $\square$

Lemma 8.8. *The equivalence relation $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ is invariant with respect to $z^d \colon S^1 \to S^1$.*

Proof. From ($\mathcal{L}^n$ 4) it follows that $\overset{\infty,\mathtt{w}}{\sim} = \bigvee \overset{n,\mathtt{w}}{\sim}$ is invariant with respect to μ. Thus $\overset{\mathtt{w}}{\sim}$ (being the closure of $\overset{\infty,\mathtt{w}}{\sim}$) is invariant with respect to $z^d \colon S^1 \to S^1$ by Lemma 8.5. Similarly $\overset{\mathtt{b}}{\sim}$ is invariant for this map. It is immediate that the join of two invariant equivalence relations is invariant. $\square$

Proof of Lemma 8.1. Let $\mathcal{K}_{\mathtt{w}}/\mathcal{F}$ be as in the last lemma, $\mathcal{K}_{\mathtt{b}}/\mathcal{F}$ the quotient obtained by identifying bounded Fatou components of $\mathcal{K}_{\mathtt{b}}$. Consider the equivalence relation $\simeq$ on $\mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F}$ generated by

$$\begin{aligned}&[\sigma_{\mathtt{w}}(z)] \simeq [\sigma_{\mathtt{b}}(\bar{z})], \text{ where}\\ &[\sigma_{\mathtt{w}}(z)] \in \mathcal{K}_{\mathtt{w}}/\mathcal{F} \subset \mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F}\\ &[\sigma_{\mathtt{b}}(\bar{z})] \in \mathcal{K}_{\mathtt{b}}/\mathcal{F} \subset \mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F},\end{aligned}$$

for all $z \in S^1 = \partial\mathbb{D}$. Clearly $\simeq$ is invariant with respect to $P_{\mathtt{w}}/\mathcal{F} \sqcup P_{\mathtt{b}}/\mathcal{F}$. Consider the map

$$S^1 \sqcup S^1 \xrightarrow{[\sigma_{\mathtt{w}}(z)],[\sigma_{\mathtt{b}}(\bar{w})]} \mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F}.$$

The pullback of $\simeq$ is $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ (on each S^1), see Proposition 7.12; it is invariant with respect to μ (Lemma 8.8). Thus we have the following commutative diagram

$$\begin{tikzcd}(S^1 \sqcup S^1, \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}) \arrow[r, "z^d"] \arrow[d] & (S^1 \sqcup S^1, \overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}) \arrow[d] \\ (\mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F}, \simeq) \arrow[r] & (\mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F}, \simeq).\end{tikzcd}$$

The quotient of $\mathcal{K}_{\mathtt{w}}/\mathcal{F} \sqcup \mathcal{K}_{\mathtt{b}}/\mathcal{F}$ with respect to the closure $\widehat{\simeq}$ is $\mathcal{K}_{\mathtt{w}} \widehat{\perp\!\!\!\perp} \mathcal{K}_{\mathtt{b}}$; the quotient of the map $P_{\mathtt{w}}/\mathcal{F} \sqcup P_{\mathtt{b}}/\mathcal{F}$ is $P_{\mathtt{w}} \widehat{\perp\!\!\!\perp} P_{\mathtt{b}}$ (see (1.4)).

The closure of $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ is $\sim$, i.e., the equivalence relation induced by γ (see Lemma 5.14. Clearly $S^1 \sqcup S^1/\sim \, = S^1/\sim$. Note that $\mathcal{K}_{\mathtt{w}}/\mathcal{F}, \mathcal{K}_{\mathtt{w}}/\mathcal{F}$ are metrizable ([Dav86, Proposition 2.2]). The claim follows from Lemma 8.6. $\square$

The author believes that in general $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim} \neq \sim$, in particular $\overset{\mathtt{w}}{\sim} \vee \overset{\mathtt{b}}{\sim}$ will not be closed in general. We do not present the examples that seem to indicate this here.

9. γ maps Lebesgue measure to measure of maximal entropy

In this section we show that γ maps Lebesgue measure on S^1 to the measure of maximal entropy on S^2, i.e., prove Theorem 1.7.

The *measure of maximal entropy* ν for $F\colon S^2 \to S^2$ may be constructed as the weak limit of $1/d^n \sum_{x \in F^{-n}(x_0)} \delta_x$, where $x_0 \in S^2$ is arbitrary ($F^{-n}(x_0)$ are the preimages of x_0 under F^n).

We denote Lebesgue measure on the circle S^1 by $|A|$, it is assumed here to be normalized ($|S^1| = 1$). This is the measure of maximal entropy of the map μ meaning it is the weak limit of $\frac{1}{d^n} \sum_{w \in \mu^{-n}(z_0)} \delta_{\mathtt{w}}$ for any $z_0 \in S^1$.

Let $x_0 \in S^2 \setminus \operatorname{post}$ be a point with the smallest number of preimages by γ; $\gamma^{-1}(x_0) = \{t_1, \dots, t_N\}$ (see Theorem 6.1). Consider preimages, $\{s_1, \dots, s_{dN}\} := \mu^{-1}(\{t_1, \dots, t_N\})$, and $\{w_1, \dots, w_d\} := F^{-1}(x_0)$. By the commutativity of the diagram from Theorem 1.4, it follows that $\gamma(\{s_1, \dots, s_{dN}\}) = \{w_1, \dots, w_d\}$ and $\gamma^{-1}(\{w_1, \dots, w_d\}) = \{s_1, \dots, s_{dN}\}$. By the minimality of x_0 it follows that $\#\gamma^{-1}(w_j) \geq N$. Thus $\#\gamma^{-1}(w_j) = N$ for all j. The same argument yields that for all $w \in F^{-n}(x_0)$ there are N points in $\mu^{-n}(\{t_1, \dots, t_N\})$ that are mapped by γ to w.

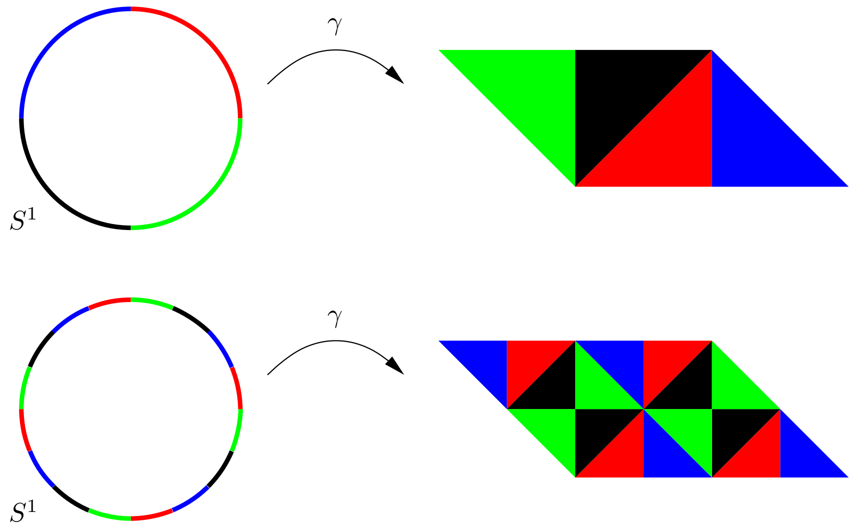


FIGURE 7. Tiling given by example from Section 5.1

Thus the (probability) measure

$$\frac{1}{Nd^n}\sum_{s\in\mu^{-n}\{t_1,\dots,t_N\}}\delta_s \quad (\text{on } S^1)$$

is mapped by γ to

$$\frac{1}{d^n}\sum_{w\in F^{-n}(x_0)}\delta_w \quad (\text{on } S^2).$$

Clearly the first measure converges weakly to Lebesgue measure on S^1, and the second measure converges weakly to the measure of maximal entropy ν (of $F\colon S^2\to S^2$). This proves the theorem.

10. Fractal tilings

From the invariant Peano curve $\gamma\colon S^1\to S^2$ one obtains *fractal tilings*. Indeed divide the circle $\mathbb{R}/\mathbb{Z}$ in d intervals $[j/d,(j+1)/d]$ $(j=0,\dots d-1)$. Since μ maps each such interval onto $\mathbb{R}/\mathbb{Z}$ it follows from Theorem 1.4 that F maps each set $\gamma([j/d,(j+1)/d])$ to the whole sphere. The tiling lifts to the *orbifold covering* (which is either the Euclidean or the hyperbolic plane). The thus obtained tiles are illustrated for the example from Section 5.1 in Figure 7.

This example is atypical however, since usually the tiles are very fractal. We show the fractal tilings obtained from the Peano curve γ for two more examples.

The first is a Lattès map whose orbifold has signature $(2,3,6)$. It is the map $R_4=1-(3z+1)^3/(9z-1)^2$ from [Mey02, Section 6.1]. The first approximation γ^1 of the Peano curve is illustrated (in the orbifold covering) in Figure 8. Tiles given by the resulting Peano curve are illustrated in Figure 9. The two critical portraits (i.e., the equivalence classes of $\overset{1,\mathtt{w}}{\sim},\overset{1,\mathtt{b}}{\sim}$) that describe R_4 according to Theorem 1.6

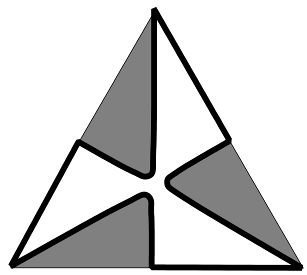
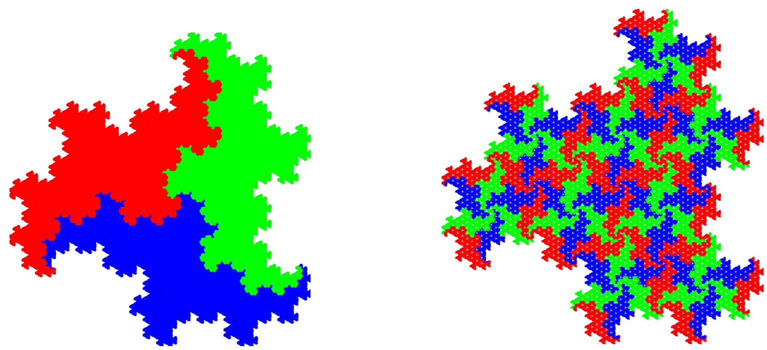


FIGURE 8. First approximation γ^1 for R_4.



(a) First order Tiles. (b) Fourth order tiles.

FIGURE 9. Tilings induced by the Peano curve of R_4.

are:

$$\text{white portrait: } \left\{\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right\},$$
$$\text{black portrait: } \left\{\frac{1}{3}, \frac{2}{3}\right\}, \left\{\frac{1}{6}, \frac{5}{6}\right\}.$$

The second example is the map $R_2 = 1 - 2(z-1)(z+3)^3/((z+1)(z-3)^3)$ (see [Mey02, Section 6.1]). It is a Lattès map whose orbifold has signature $(3,3,3)$. The first approximation γ^1 is shown (in the orbifold covering) in Figure 10. The tiles that are obtained from the resulting invariant Peano curve are shown in Figure 11. The two critical portraits describing R_2 are:

$$\text{white portrait: } \left\{\frac{7}{60}, \frac{22}{60}, \frac{37}{60}\right\}, \left\{\frac{43}{60}, \frac{58}{60}\right\},$$
$$\text{black portrait: } \left\{\frac{15}{60}, \frac{30}{60}, \frac{45}{60}\right\}, \left\{\frac{13}{60}, \frac{58}{60}\right\}.$$

All examples considered above had *parabolic orbifold*. Consider a *rational* expanding Thurston map (meaning it has no *Thurston obstruction*) with *hyperbolic orbifold*. The tiling obtained from the invariant Peano curve lifts to the *orbifold*

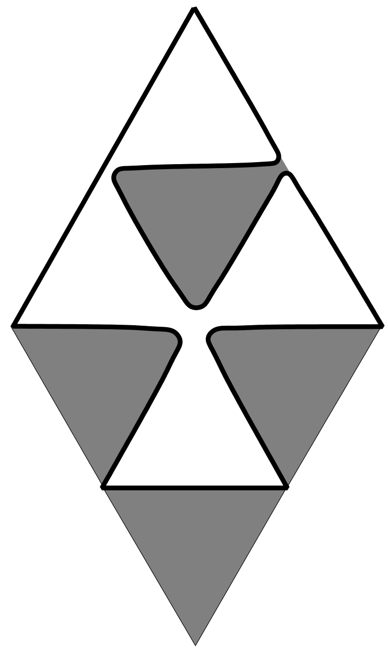
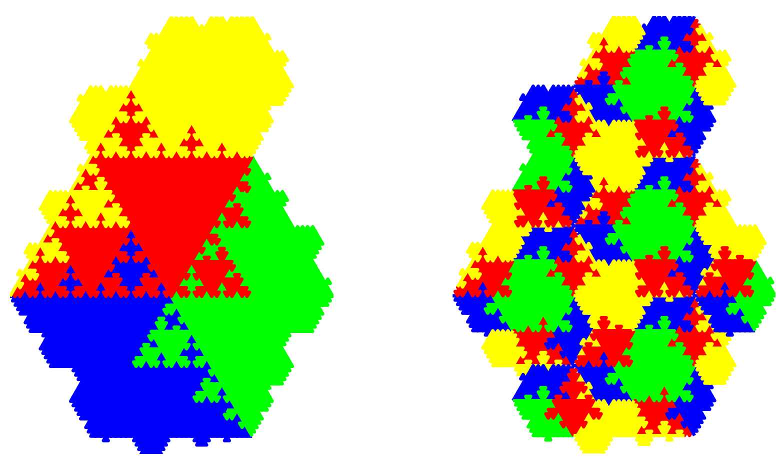


FIGURE 10. First approximation γ^1 for R_2.



(a) First order tiles. (b) Third order tiles.

FIGURE 11. Tilings induced by the Peano curve of R_2.

cover, i.e., the hyperbolic plane. Thus one obtains fractal tilings of the hyperbolic plane with interesting self-similar properties.

There are other ways to obtain fractal tilings from the invariant Peano curve γ. Instead of dividing the circle into d^n intervals of the same length, we can take the images of the n-arcs by γ. Thus we get tilings of the hyperbolic/Euclidean plane with $k(= \#\,\mathrm{post})$ different tiles. Each tile divides into tiles of the $(n+1)$-th order.

There is yet another way to obtain tilings from the invariant Peano curve γ in a natural way. Namely define tiles as the images of (either white or black) n-gaps by γ.

11. Open Questions

The construction presented here to decompose, or unmate, an expanding Thurston map into polynomials is not the most general one. In [Meya] an example of an expanding Thurston map (which is rational) was given that arises indeed as the (topological) mating of two poynomials, yet this cannot be shown with the methods presented here.

Open Problem 1. Let $f\colon S^2 \to S^2$ be a Thurston map. Give a necessary and sufficient condition that f arises as a mating.

Ideally this condition should give all shared matings, i.e., all distinct possibilities how f arises as a mating. Furthermore from one should be able to read off the polynomials into which f unmates in a combinatorial manner. For *hyperbolic* rational Thurston maps such a necessary and sufficient condition to arise as a mating is known, namely the existence of an *equator*, see [Meya, Theorem 4.2].

Several people have asked whether there is a bound on the number of points that are identified in a mating. If a mating (of strictly preperiodic polynomials) is obtained as constructed here this is answered by Theorem 6.1.

Open Problem 2. Is it possible to decide whether an expanding Thurston map F is equivalent to a rational map from the critical portraits (see Section 5.6)? By Thurston's topological characterization [DH93] this amounts to the question whether it is possible to read off *Thurston obstructions* from the *critical portraits*.

In principle this is possible. Recall that each 1-tile/1-edge has a natural corresponding 1-gap/1-arc. Thus every multicurve in $S^2 \setminus \mathrm{post}$ can be naturally represented in the "critical portrait sphere" $\widetilde{S}^2$ (i.e., the sphere whose two hemispheres are $S^2_{\mathtt{w}}$ and $S^2_{\mathtt{b}}$ as in Section 5). A multicurve Γ in this picture is just a multicurve in $\widetilde{S}^2 \setminus \mathbf{A}^0$. Since each 1-gap is mapped by μ to $S^2_{\mathtt{w}}$ or $S^2_{\mathtt{b}}$, it is possible to take the preimage of the multicurve. It is invariant if each component of the preimage is isotopic rel. $\mathbf{A}^0$ to one component of Γ. The Thurston matrix is then taken as usual.

However it is not clear whether the description above offers any advantage in finding Thurston obstructions.

Open Problem 3. Consider a postcritically finite rational map $\tilde{f}$ whose *Julia set is a Sierpiński carpet.* Identifying the closure of each Fatou component yields an expanding Thurston map f (see Section 2.1). Assume f has an invariant Peano curve γ (meaning we do not have to take an iterate in Theorem 1.4). Is it possible to construct from γ a semiconjugacy $\tilde{\gamma}\colon S^1 \to \widetilde{\mathcal{J}}$ ($\widetilde{\mathcal{J}}$ is the Julia set of $\tilde{f}$) such that $\tilde{f}(\tilde{\gamma}(z)) = \tilde{\gamma}(z^d)$ for all $z \in S^1$ (where $d = \deg \tilde{f}$)? This is false in general (see [Kam03, Section 4]), but possibly true under some additional assumptions.

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