# Model Reduction for Friction-induced Vibration of Multi-degree-of-freedom Systems and Experimental Validation

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## Abstract

Most frictional systems are composed of components that are in sliding contact. Thus they have natural contact interfaces on which normal contact forces and tangential friction forces are acting. These real systems are complicated and friction brings further complexity, which presents a challenge to their dynamic analysis and control. In this paper, a model reduction strategy for complicated frictional systems is put forward based on the idea of mode synthesis, and its applications and experimental validation are presented. Firstly, the accuracy of the reduction method and the feasibility of the reduced model in stability analysis are verified on a theoretical multi-degree-of-freedom frictional system. Then, a tribometer in the form of a pad-on-disc test rig is designed and built and its corresponding detailed finite element (FE) model is constructed. A specific reduction strategy for this real friction system which involves direct contact at the pad-disc interface, is proposed. It is found that the reduced model with 80 modes of each of the two substructures of the system can already correctly predict the mode-coupling instability of the full FE model (which has over 900,000 degree-of-freedom) of the real system, and unstable frequencies that are fairly close to the unstable frequencies of the FE full model computed by Abaqus CEA and the vibration frequencies measured in the test on the tribometer, which demonstrates the efficiency and accuracy of the model reduction strategy for dynamic analysis of real friction systems. Its computational efficiency would be even much greater when applied to nonlinear dynamic analysis of real friction systems. Model reduction for friction-induced vibration validated by experiments has not been reported in the literature.

Keywords: friction-induced vibration, model reduction, stability analysis, mode coupling, tribometer.

## Introduction

Friction is everywhere and important in our daily life as well as in industry. Systems with friction exhibit diverse behaviour, especially when it comes to friction-induced vibration problems. Friction-induced vibration is one kind of self-excited oscillations that can potentially cause problems like wear, fatigue failure, and noise, for example, brake squeal [1, 2]. Noise or growing vibration [3-7] induced by dry friction has aroused much attention of researchers over decades.

Some of the past investigations are devoted to gaining insight knowledge of friction-induced vibration based on idealised models from a theoretical point of view [8-19]. Several instability mechanisms of the self-excited friction-induced vibration have been proposed, among which the four essential ones are: the negative friction-velocity slope [11], stick-slip [10], sprag-slip [8, 14] and mode-coupling [12, 20] or mode locking [18]. A recent study [21] suggested that the number of degrees-of-freedom of theoretical models should not be too small. On the other hand, some works adopted large frictional systems [22, 23] to investigate the stability of the system [24, 25], squeal propensity [26-28], or dynamic behaviour [29, 30] for the sake of engineering applications. However, friction-induced vibration is not fully understood.

Complex eigenvalue analysis (CEA) is a widely adopted approach to investigate the stability of systems with dry friction. Although CEA on the linearised system could overestimate or underestimate the unstable vibration [31] and thus some new indexes were developed to address this issue [32] , it is a convenient tool to provide preliminary estimations for the onset of unstable motion [24, 33-35]. For theoretical models that only have a few degrees of freedom, transient dynamic transient analysis (DTA) can be easily carried out. However, for large models, the expensive computational cost associated with DTA becomes a barrier for dynamic analysis [24, 36]. Nowadays, finite element software are a widely used tool for CEA of large structures. Nevertheless, expensive computational cost is still a big issue for CEA of large finite element models with dry friction, as solving the associated asymmetric eigenvalue problem is time consuming, let alone DTA.

Frictional systems can be treated as consisting of several separate substructures, and they share frictional interfaces, which are the key parts of the systems and contain the key characteristics (the complex friction force, contact states) of the systems. Generally, systems having substructures and connections can be analysed through substructuring techniques [37, 38]. Furthermore, nonlinearities which may arise from the friction law or contact could result in even more elusive dynamic characteristics in friction-induced vibration [31, 39, 40]. Nonlinear dynamic analysis of a friction system is very time consuming, even for a low-degree-of-freedom system (for example, as in[17]), and model reduction would be even more advantageous in analysing real friction systems. Model reduction is a feasible and efficient method for both practical and theoretical studies [41].

Recently, several numerical studies on the reduction of frictional systems have been put forward [42-48]. Brizard et al. [43] applied model reduction techniques in the investigation of the stability of a disc-pad system and made suggestions for improving two popular model reduction methods. Fazio et al. [45] proposed a reduction strategy aiming to reduce the number of contact points of a brake system, by which improvements on predicting the stability of a real brake were gained. Do et al. [47] proposed a completely new strategy that coupled the control approach, the homotopy analysis and CMS (component mode synthesis) techniques to analyse the stability of a friction-induced vibration problem, which was both efficient and very accurate, as demonstrated by a numerical example. On the nonlinear friction-induced vibration, Loyer et al. [44] tested the performance of different reduction methods in the study of the nonlinear behaviour of a friction-destabilised elastic layer and concluded that both of conventional modal truncation base and the one with harmonic component base gave good accuracy. Besset and Sinou [48] developed an efficient numerical reduction technique based on complex interface modes to predict the occurrence of brake squeal of linearised systems in which the contact is modelled as cubic nonlinear springs.’

As it has been known, frictional systems have some distinct dynamic properties, for example, the bifurcation of stability with the change of the friction coefficient. Thus, a big concern on using a reduced model is whether it can preserve those important properties of the original system. Developing a model reduction method for CEA is worthwhile. The main aim of this paper is to investigate the performance of the reduced model on the stability estimation of the original complicated real system with both theoretical and experimental validations, which have not been seen in published papers. This paper reports the preliminary study of this new topic, in which contact is modelled as linear springs. More realistic contact models are available, for example, as exponential springs [39]. However, this topic is beyond the scope of the current paper and thus is not discussed any further.

This paper is organised as follows: firstly, based on the idea of CMS method, a reduction strategy for systems with friction is proposed, which is described in Section 2. Then, in Section 3, the accuracy of the method in predicting the natural frequencies of the system and mode-coupling instability is verified by comparing the results of a full 9-degree-of-freedom model and the reduced models in eigenvalue analysis. Finally, in Section 4, the reduction of a complicated finite element model of a real test structure is carried out, followed by validation via test results.

## Theoretical formulations of the reduction strategy

Friction systems, like a brake system, can be modelled as two separate multi-degree-of-freedom systems which share friction contact at the interface. Figure 1 shows the schematic diagram of a discrete model with a pad on the surface of a disc. The ‘pad’ system in this figure represents the friction material and all associated components (for example, pad back plate, calliper, etc.) in a brake; and the ‘disc’ system in this figure represents the disc and other associated components (shaft, support, etc.) in a brake. The contact between the pad and disc is modelled as linear springs.



Figure 1. The diagram of a pad-disc system

The physical displacements of the upper pad system and lower disc system are defined as vectors  and , which are in the following form:

，

in which superscript *u* denotes the tangential degrees of freedom (DoFs) on the interface; *w* denotes the normal DoFs on the interface; *o* denotes all the OTHER DoFs; Thus ,  and  are the physical displacements of the pad in *o*, *u* and *w* DoFs; ,  and  are the physical displacements of the disc in the corresponding DoFs. If the pad is an *n*-degree-of-freedom system and the disc is an *m-*degree-of-freedom system, then  is an *n*×1 vector and  is an *m*×1 vector.

Physical coordinates  and  can be expressed in the mode expansion form：

 and 

in which  and  are the modal coordinate vectors of the pad and disc,  and  are the mass-normalised mode shape matrices of the pad and the disc as separate systems. If *i* modes of the pad and *j* modes of the disc are used, The size of  and  is *n*×*i* and a *m*×*j*;  and  are a *i*×1 vector and a *j*×1 vector respectively.

The mode shape matrix  and  are partitioned into three submatrices, expressed as Eq.

, 

in which ,  and  are the components of the mode shape matrices of the pad in the *o-*，*u-* and *w-*DoFs. Similarly, ,  and  are the submatrices containing the mode shape components in the corresponding DoFs of the disc.

### Definitions of the contact force and the friction force

In this method, artificial linear springs and dampers are introduced to model the contact between the pad and the disc (see Fig. 1). Then the contact force can be obtained by Eq. :



in which  is the pre-load vector due to initial braking pressure, is the contact damping matrix,  is the contact stiffness matrix, which are taken to be in the following simple forms:

，

where  and  is the stiffness and damping of the *i*thcontact node.

Then, the friction force can be expressed as Eq. , if the friction coefficient takes a constant value .



### Mode synthesis strategy

The equation of motion of the pad and the disc in matrix form can be written as：





in which , and  are the mass, damping and stiffness matrices of the pad; ,  and  are the mass, damping and stiffness matrices of the disc. A transformation matrix **T** is introduced which allows the dimension of the force vector acting on the pad to be correctly matched with the force vector on the disc at their interface even if the number of the degrees of freedom of the pad and the disc at the interface are different; sometimes **T** = **I** (an identity matrix)when the DoFs of the pad are identical to those of the disc. (*r*, *s* = *o*, *u*, *w*) means the geometric relation between the *r* kind of DoFs on the pad to the *s* kind of DoFs on the disc.

By pre-multiplying  on both sides of Eq. (7), and  on both sides of Eq. , and according to the orthogonality properties of modes, the equations of motion of the pad and the disc in modal coordinates can be obtained, and then by collecting these equations, the equation of motion of the whole system in matrix form with respect to the modal coordinates can be expressed as:



in which

, 

, 

where  and  are thenatural frequencies of the pad and disc as separate systems, respectively,



As the reduced model is a linear system,  does not affect the eigenvalues of the system. The stability or the natural frequency of the reduced system can be investigated by implementing a conventional complex eigenvalue analysis of Eq. . The eigenvalues  of the system can be calculated by Eq.



The general process of conducting the stability analysis of a reduced friction system is illustrated in Figure 2.



Figure 2. The flow chart of the stability analysis process of the reduced model

The advantage of this strategy is that the natural frequencies and the stability of the whole system with friction can be investigated with the information of the friction coefficient and only a small amount of modal data of the substructures: (1) the mode shape components at the DoFs on the potential contact interface and natural frequencies of the substructures, when the potential contact interfaces of the substructures are set to be free; (2) the relations between the DoFs on the interfaces of the substructures. Please note that what is being presented in this section is a general approach applicable to linear systems with multiple contact interfaces.

Additionally, nonlinearities may be involved in a system. Generally, nonlinearities are present at only a relatively very small number of degrees-of-freedom, for example, at a contact interface. There are a number of algorithms that deal with structural analysis with local nonlinearities and one classic work is [49]. It is believed that nonlinearities can be included in the formulation presented in this paper though some additional work will have to be done.

## Applications to a 9-degree-of-freedom model

In this section, a 9-degree-of-freedom model is used to verify the applicability and accuracy of the preceding reduction strategy in the reduction of the multi-degree-of-freedom system.

### Introduction of the 9-degree-of-freedom model

Figure 3 shows the 9-degree-of-freedom undamped system with friction contact, which is comprised of two substructures (the upper slider part (‘pad’) and lower belt part (‘disc’)). The upper part is a 6-degree-of-freedom mass-spring system with four point masses *m*1, *m*2, *m*3 and *m*4, and linear springs. Among the springs, *k*5, *k*7 and *k*10 are three inclined springs, and the angles of them with respect to the vertical direction are *θ*1, *θ*2 and *θ*3 respectively. The lower part is a belt modelled as a rigid body denoted by *m*. The length of *m* to the left end of the belt is *lk*1 and to the right end is *lk*2. *m*2 and *m*3 are in sliding contact, which is modelled as linear springs *k*c1 and *k*c2, with the rigid moving belt. The two contact points are massless sliders. The distances of the contact points of *m*2 and *m*3 to the mass centre of the belt are *l*c1 and *l*c2 respectively. Coulomb friction with a constant coefficient of friction *μ* is considered. The definitions of the coordinates system are shown in Figure 3.  and  are the contact forces between *m*2 and the belt, and between *m*3 and the belt; and the horizontal friction forces are denoted by  and . As the coefficient of friction  is assumed as constant, the friction forces are expressed as and.



Figure 3. The 9-degree-of-freedom model

### Equations of motion established through the substructure strategy

According to the definitions of the coordinate systems of the pad and the disc, the upper part is a 6-degree-of-freedom system described by *x*1, *x*2, *y*2, *x*3, *y*3, *y*4, and the lower belt is a 3-degree-of-freedom system described by *x*5, *y*5 and *y*6. Following the definition (in Eq. ), the displacement vectors of the slider and the belt are given in Eq. (note that the lower part does not have the *o*-DoF):

, 

#### The equation of motion of the substructures

The equation of the motion of the individual slider system is expressed as:



in which

, and 

where



and



For the belt system, its equation of motion is written as:



in which

,  and 

where

 and 

in which *J* is the moment of inertia of the rigid belt, which is defined as: .

#### The contact matrix and the transformation matrix

As there is no external force acting on this 9-degree-of-freedom model, the contact force is expressed as Eq. , according to its definition in Section 2.1.



in which



As the degrees of freedom of the contact points of upper part at the contact surface do not match those of the lower belt part, then on the base of the definitions of  (Eq. ) and  (Eq. ), the transformation matrix of this problem is:



in which



As there is no damping or external force, and furthermore , and  in the transformation matrix are empty, the assembled equation of motion through the idea of the preceding section is simplified as：



in which



### The equation of motion established through the direct method

Generally, the equation of motion of the system is obtained through the Lagrange equation of the whole system, which is called the direct method here in order to distinguish this method from the model reduction method proposed in the preceding section.

Here, the displacement vector of the whole slider-belt model is defined as:



Then the equation of motion of the whole slider-belt model is expressed as:



in which





where the expression of , , , , , , , , ,  and  are given in Eqs. and . Apparently, the stiffness matrix of the system  is asymmetric.

### Numerical analysis: Case 1

The theoretical formulations of the model reduction strategy and the direct method of the 9-degree-of freedom model are presented in the preceding sections. In this subsection, numerical simulations are carried out: firstly to calculate the natural frequencies and the mode shapes of the substructures; then to implement the model reduction strategy using all the modes of the substructures and compute the natural frequencies of the system; then to compare these frequencies with those calculated by the direct method to verify the correctness of the model reduction strategy; finally, to generate different reduced models and compare them with the full model.

The parameter values used in this subsection are: *m*1=*m*2=*m*3=*m*4=*m*=1, *θ*1=*θ*2=*θ*3=45°, *k*1=*k*2=*k*3=*k*4=*k*5=*k*6=*k*7=*k*8=*k*9=*k*10=*k*11=*k*12=*k*13=*k*c1=*k*c2=1, *lk1*=*lk2*=1, *l*c1=*l*c2=0.5.

#### Eigenvalue analysis of the full model

Firstly, the natural frequencies and the mode shapes of the slider and the belt are calculated. After obtaining the modal information of the two substructures through Eqs. (12) and (15), the natural frequencies of the full system can by calculated by solving the eigenvalue problem of the equation of motion of the assembled system Eq. with full modes of the substructures. The evolution of the eigenvalues of the system with the friction coefficient, calculated by both the model reduction method and the direct method, is also examined. It turns out that the bifurcation results by the substructuring strategy and the direct method are identical. Thus, only one set of results is illustrated in Figure 4, which shows the changes of the real parts and imaginary parts of the eigenvalues with the friction coefficient. Because the system is a hypothetical model, the friction coefficient range under investigation is mathematical and thus may not reflect that of a real system. This large range allows all bifurcation points to be revealed.

|  |  |
| --- | --- |
|  |  |
| (a) | (b) |

Figure 4. Eigenvalue analysis results of the full model by direct method and the substructuring strategy. (a) The change of the real part (Growth rate) with friction coefficient *μ*; (b) The change of the imaginary (Frequency) part with friction coefficient *μ*

It can be noticed that, for this system, while the friction coefficient is growing, some frequencies of the corresponding mode become closer and two of them coalesce at *μ*=0.9, where the first bifurcation happens (the 2nd and the 3rd modes); then with the further increase of the friction coefficient, two other modes merge and the second bifurcation takes place (the 5th and the 6th modes). Then the third bifurcation takes place at *μ*=14.6 (the 6th and the 7th modes), shown in Figure 4 (a), at the same time one of the unstable mode (the 5th mode) become stable. Thus, there are still 4 unstable modes (the 2nd, the 3rd, the 6th and 7thmodes) in total.

The above results indicate that the substructuring strategy proposed in the current work is correct.

#### Model reduction

In the following, the reductions of the 9-degree-of-freedom model are carried out, and the characteristics of the eigenvalues of the reduced models are analysed and compared with the results of the full model. The first reduced model (reduced model 1) is obtained by using the first three modes of the upper slider system and 2 modes of the lower belt system. The natural frequencies of reduced model 1 with *μ*=1 are listed in Table 1. Apparently, reduced model 1 only has 5 modes. By comparing with the results of the full model in Table 1, it can be clearly seen that the first three natural frequencies of reduced model 1 are very close to the corresponding results of the full model. However, the correlations of the 4th and 5th modes to the modes of the full model are poor. Moreover, the eigenvalues of another reduced model (reduced model 2) with 4 slider modes and 2 belt modes are examined. The natural frequencies of reduced model 2 with *μ*=1 are listed in Table 1, which have a better accuracy than reduced model 1 as it is expected to be.

Table 1, Comparisons between the natural frequencies of the full model and the reduced models when *μ*=1 (rad/s)

|  |  |  |  |
| --- | --- | --- | --- |
| Mode | Full model | Reduced model 1  (3+2) | Reduced model 2  (4+2) |
| 1 | 1 | 1 | 1 |
| 2 | 1.0720 | 1.0797 | 1.0796 |
| 3 | 1.0720 | 1.0795 | 1.0796 |
| 4 | 1.3809 | 1.4529 | 1.4624 |
| 5 | 1.5698 | 2.1606 | 1.5596 |
| 6 | 1.7718 | ╳ | 2.1245 |
| 7 | 1.999 | ╳ | ╳ |
| 8 | 2.2313 | ╳ | ╳ |
| 9 | 5.4970 | ╳ | ╳ |

In the following, the change of the eigenvalues of reduced model 1 with the friction coefficient is examined, illustrated in Figure 5. The black dots represent the results of the full model (original model), and the coloured lines are for the results of reduced model 1. It can be seen that, reduced model 1 is able to simulate the first stability bifurcation of the system (mode coupling between the 2nd and 3rd mode), and the critical friction coefficient (0.9) at the bifurcation point is very close to that (0.97) of the full model. The comparison plot of the changes of the frequencies, in Figure 5 (b), show that the values and also the variation trend of natural frequencies with *μ* of the first three modes of reduced model 1 are nearly identical to the results of the full model. Furthermore, it can be seen that the 4th mode of reduced model 1 (pink line) actually corresponds to the 4th mode of the original system, but the correlation of its 5th mode to that of the full system is not very clearly shown.

|  |  |
| --- | --- |
|  |  |
| (a) | (b) |

Figure 5. Reduced model 1 of Case 1: The evolution of the eigenvalues with the friction coefficient. (a) The change of the real part with friction coefficient *μ*; (b) The change of the imaginary part with friction coefficient *μ*

Furthermore, Figure 6 shows the bifurcation plot of the eigenvalues of reduced model 2 against the friction coefficient. Reduced model 2 is able to predict all the bifurcations of the full model, of which the first mode coupling is nearly the same as that of the full model, and the tendency of the second (4th and 5th mode) and the third (5th and 6th mode) mode couplings are similar to those of the full model. Moreover, the critical friction coefficients of the second and third mode couplings of the reduced model are smaller than the corresponding critical friction coefficients of the full model, which over-estimate the instability. As the reduced model predicts the bifurcation transition point conservatively, using the reduced model to represent its original model for stability analysis is safe.

|  |  |
| --- | --- |
| comparision full and model 4plus2 real vs mu 2 |  |
| (a) | (b) |

Figure 6. Reduced model 2 of Case 1: The evolution of the eigenvalues with friction coefficient *μ*. (a) The change of the real part with friction coefficient *μ;* (b) The change of the imaginary part with friction coefficient *μ*

### Numerical analysis: Case 2

By considering the fact that the mass of the belt is larger than the mass of the slider, constraints in the belt system are stiffer than the ones in the slider system and the contact stiffness should be also larger than the elastic constraints in the system in real applications, some of the parameter values in the preceding numerical analysis are adjusted, which now become: *m*=2, *k*12=*k*13=2, *k*c1=*k*c2=5.

There are several combinations of the modes of the slider and belt system that can be used in the reduced model. In the following, the analysis of two reduced models is carried out: reduced model 1 consists of 3 slider modes and 2 belt modes; reduced model 2 is assembled by 4 slider modes and 2 belt modes.

The performance of reduced models 1 and 2 on simulating the changes of the eigenvalues of the system with the friction coefficient is investigated, which are shown in Figures. 7 and 8 respectively. With the increase of the friction coefficient, the first coalescence of the frequencies is between the 5th and the 6th mode at *μ*=0.4, and then the second one is between the 1st and the 2nd mode at *μ*=1.35. Figure 7 (b) shows that reduced model 1 is capable of predicting only the second coalescence between the 1st and 2nd modes and a quite accurate unstable frequency, but the critical friction coefficient for the coalescence is larger than the exact value. Except the natural frequencies of the first two modes, frequencies of the other modes of the reduced model are not very accurate.

|  |  |
| --- | --- |
|  |  |
| (a) | (b) |

Figure 7. Reduced model 1 of Case 2: The evolution of the eigenvalues with the friction coefficient. (a) The change of the real part with friction coefficient *μ*; (b) The change of the imaginary part with friction coefficient *μ*

As to reduced model 2 which contains 4 slider modes and 2 belt modes, shown in Figure 8, it can simulate all the mode coupling instabilities of the original system (full model). The frequencies of the first three modes of reduced model 2 are very close to the corresponding frequencies of the full model. However, the first mode coupling predicted by reduced model 2 is still about the 1st and 2nd modes, and the second mode coupling is not very accurate in terms of the critical friction coefficient and the unstable frequency value. As a whole, reduced model 2 maintains the instability characteristics of the full model and can predict quite accurately frequencies of the lower modes. Finally, for this 9-degree-of-freedom system, the smallest reduced model that can predict all the bifurcations of the full system should contain at least 4 slider modes and 2 belt modes.

|  |  |
| --- | --- |
|  |  |
| (a) | (b) |

Figure 8. Reduced model 2 of Case 2: The evolution of the eigenvalues with the friction coefficient. (a) The change of the real part with friction coefficient *μ*; (b) The change of the imaginary part with friction coefficient *μ*

Through the preceding two numerical cases, it can be concluded that although system parameters affect the performance of the reduced model, predicting key stability characteristics of the original system by the reduced model is feasible.

## Applications to a pad-on-disc system with experimental results

In this section, a laboratory test rig with a pad sliding on a rotating disc and the corresponding complicated finite element model built in Abaqus are presented. The test structure is a customised tribometer.

There are two goals in this section: (1) Model reduction of this real complicated pad-disc system is carried out using the reduction strategy proposed in Section 2 and comparison between the CEA results of the reduced model and the full FE model is made to further validate the model reduction strategy; (2) The theoretical CEA results are validated against experimental results of vibration frequencies of the test rig. These results show that the model reduction method presented in this paper is correct and efficient.

### Description of the test rig of the pad-on-disc system

Figure 9 shows the schematic of the experimental set-up for testing the friction-induced vibration of the pad-on-disc system, which contains two substructures (an upper pad substructure and a lower disc substructure). The pad is a composite friction material specimen and fixed to the pad holder. The upper pad substructure consists of several components. An acceleration sensor is attached to the pad holder, and a tangential force sensor and a normal force sensor are integrated in the holder. The horizontal motion of the upper part of the pad holder is constrained by a linear bearing. The top of the upper part is a vertical hydraulic cylinder which can apply a vertical compression force to the pad and the cylinder is attached to the support system. For the lower disc substructure, the central area of the disc is held by two disc holders on both sides of the disc by bolts and fixed to a shaft of the rotational motion device. The pad substructure below the cylinder can move in the vertical direction controlled by the hydraulic pressure in the cylinder and the disc can rotate at a constant speed.

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Figure 9. The schematic of the experimental set-up

An impact hammer modal test was carried out on the two substructures of the test rig separately, and on the whole test rig in stationary conditions (when the pad is brought into contact with the disc which stays at rest). The experimental frequency spectra of the stationary test structure are shown in Figure 10 (a)-(c).

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| (c) | |

Figure 10. Test results of the acceleration PSD of the test rig. (a) The upper part; (b) The lower part; (c) The whole machine

The vibration signals of the test rig from the above-mentioned three sensors, when the pad is under compression force of 500 N and the rotating speed of the disc is at 13 rad/s, are also recorded. During the sliding motion, the horizontal vibration of the pad is excited, and the measured vibration frequency in the low frequency range is 492 Hz and the vibration frequency in the high frequency range is 1705 Hz. The measured friction coefficient of the pad is in the range of 0.3 to 0.4.

### Description of the finite element model

Figure 11 shows the finite element model of the pad-on-disc system which is referred to as the original system with respect to the reduced model in the next few sub-sections. It includes the key components of the real test rig. The dimensions of the various components are the same as those of the test rig. Based on the real materials of the components, the material parameters are shown in Table 2. In this model, the pad surface is in direct contact with the disc surface, and there are 294 contact nodes on both contact surfaces of the pad and the disc that form 294 pairs of nodes of identical coordinates. The whole model is built in a rectangular coordinate system. Based on the test rig, the boundary conditions (BC for short in Figure 11) of the upper substructure are: the upper surface of the spring holder is fixed in the *x* and *z* directions and free in the vertical *y* direction, and part of the back surface of the bearing bracket plate is fixed in the *y* direction and elastically constrained in the *x* and *z* directions with ground springs whose stiffness is 107 N/mm. To simulate the stiffness between the spring holder and the cylinder (as shown in Figure 9), a spring is placed in the *y* direction at the centre of the spring holder and the cylinder with stiffness of 106 N/mm, and the spring in the *x, y* and *z* direction at the centre of each edge and four corners of the spring holder and the cylinder with stiffness of 107, 103 and 107 N/mm, respectively. The boundary conditions of the lower substructure are: the two interfaces between the disc holders and the disc are fully constrained in three perpendicular directions. The element type of the whole model is C3D8R (8-node linear brick element with reduced integration) and the total element number of the system is 121,864 (904,968 degrees of freedom in total). The above stiffness values are tuned so that the theoretical frequencies of the substructures match the experimental frequencies.

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Figure 11. Finite element model of the test machine

As model reduction is carried out on this finite element model, verifying the correlation between the finite element model and the test rig is very important. To do this, both of the natural frequencies of each substructure (the upper part and the lower part) having the same boundary conditions but being free at the interface, and the natural frequencies of the full model are computed by Abaqus. They are compared with measured frequencies, which indicates that there is a fairly good correlation of the finite element model with the testing machine, shown in Table 3.

Table 2 The material parameters of each component in the FE model

|  |  |  |  |
| --- | --- | --- | --- |
| Component | Density (g/cm3) | Young modulus (GPa) | Poisson’s ration |
| Disc holder | 7.7 | 200 | 0.27 |
| Disc | 7.8 | 210 | 0.27 |
| Pad holder | 7.0 | 180 | 0.3 |
| Pad | 4.5 | 20 | 0.29 |
| Force sensor | 7.7 | 190 | 0.28 |
| Sensor holder | 7.8 | 196 | 0.3 |
| Linear bearing | 7.85 | 210 | 0.28 |
| Cylinder | 7.8 | 196 | 0.3 |
| Spring holder | 7.8 | 196 | 0.3 |

Table 3 The fundamental frequencies (Hz) by Abaqus and the modal test

|  |  |  |  |
| --- | --- | --- | --- |
|  | Test | Abaqus | Error |
| Upper part | 314.8 | 331.44 | 5.02% |
| Lower part | 1301.5 | 1300.6 | 0.07% |
| Whole system | 474.2 | 510.91 | 7.58% |

### Reduction of the pad-on-disc system and stability analysis

To implement the model reduction on this pad-on-disc system, three kinds of information are required from Abaqus:

1. Natural frequencies of the upper part and the lower part, denoted by  and , when their contact interfaces are free.
2. The mode shape matrices at the interface DoFs of the pad and the disc, which are , , , ,  and  in Eq. .
3. The coordinates of the nodes on the contact interface between the pad and the disc.

#### Contact stiffness in the reduced model

As the contact in the model reduction method is described as linear springs/dampers between the nodes on the interface of the pad and disc, which cannot be identified directly from the finite element model, the surface properties must be determined by means of correlation between the reduced model and the full FE model.

Firstly, a proper contact stiffness is identified by adjusting the contact stiffness value in the model reduction method to obtain a reduced model whose natural frequencies are very close to those of the full FE model computed by Abaqus, when the system is stationary. The natural frequencies of the stationary full FE model are calculated without friction (since the disc is stationary), so the friction coefficient in the reduction process of a stationary system is zero. The specific procedure is illustrated in Figure 12.

Natural frequencies of the four reduced models in stationary (without friction) state with linear contact springs are investigated, in which reduced model 1 (RM 1), reduced model 2 (RM 2), reduced model 3 (RM 3) and reduced model 4 (RM 4) are composed of 10 modes, 40 modes, 80 modes and 245 modes of each of the two substructures (the upper part and lower part), respectively. Figure 13 shows the influence of the contact stiffness on the natural frequencies of RM 2 (40 pad modes + 40 disc modes). The vertical coordinate is the error between the natural frequencies of the reduced model and the full FE model, and zero means the results of the two models are identical. It shows that when the value of the contact stiffness is at a proper level, natural frequencies of the system are not very sensitive to the change of the contact stiffness. The results of RM 1, 3 and 4 are similar to these of RM 2 and thus are not shown here for the sake of simplicity.



Figure 12. The flow chart of calculating the contact stiffness

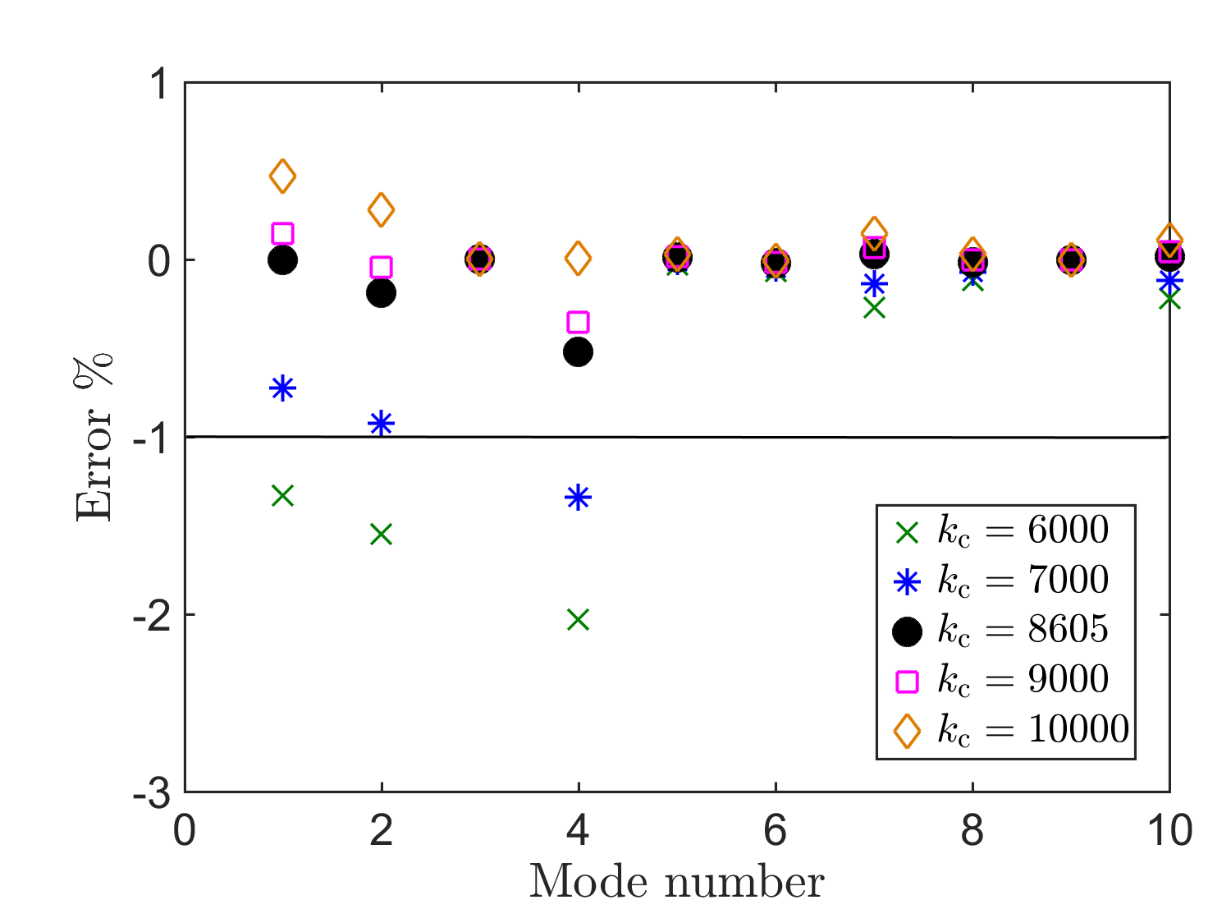


Figure 13. The errors of the natural frequencies of the reduced model with different contact stiffness with respect to the full FE model

It is found that when the system is reduced, the contact stiffness for different reduced models is different. In general，if fewer modes of the substructures are included in the reduced model, to ensure the consistency between the natural frequencies of the assembled system after reduction with those of original system, a smaller contact stiffness is needed. The respective proper contact stiffness values for RM 1, 2, 3 and 4, which make the natural frequencies of the corresponding reduced models very close to the ones of the full FE model, are found to be 3684, 8605 (black round mark in Figure 13), 10970 and 12510 N/m respectively. Table 4 shows the first 18 natural frequencies of the four reduced models and those calculated by Abaqus modal analysis. The error between the reduced model and the full model of a real structure can be very small, which indicates that the reduction strategy with the assumption of linear contact springs works well on predicting the natural frequencies of the full FE model of the stationary friction system even when only 10+10 substructure modes are used.

Table 4 Comparisons between the natural frequencies (Hz) of the FE model and reduced models

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Mode | FE  model | RM 1  (error%) | RM 2  (error%) | RM 3  (error%) | RM 4  (error%) |
| 1 | 510.91 | 510.88  (0.0059) | 510.90  (0.0019) | 510.89  (0.0039) | 510.91  (0) |
| 2 | 550.79 | 544.86  (1.0766) | 549.74  (0.1906) | 553.48  (0.4884) | 552.01  (0.2215) |
| 3 | 759.54 | 759.54  (0) | 759.54  (0) | 759.54  (0) | 759.54  (0) |
| 4 | 1027.5 | 1033.42  (0.5761) | 1022.2  (0.5158) | 1028.7  (0.1168) | 1026.8  (0.0681) |
| 5 | 1306.7 | 1305.97  (0.0559) | 1306.9  (0.0153) | 1307.2  (0.0383) | 1306.8  (0.0077) |
| 6 | 1383.8 | 1384.5  (0.0506) | 1383.5  (0.0217) | 1383.7  (0.0072) | 1383.7  (0.0072) |
| 7 | 1469.8 | 1468.8  (0.0680) | 1470.2  (0.0272) | 1470.7  (0.0612) | 1470.0  (0.0136) |
| 8 | 1517.1 | 1516.1  (0.0659) | 1516.9  (0.0132) | 1517.9  (0.0527) | 1518.0  (0.0593) |
| 9 | 1593.4 | 1593.5  (0.0063) | 1593.5  (0.0063) | 1593.5  (0.0063) | 1593.4  (0) |
| 10 | 1643.1 | 1642.1  (0.0609) | 1643.4  (0.0183) | 1644.7  (0.0974) | 1643.4  (0.0183) |
| 11 | 1687.0 | 1686.5  (0.0296) | 1686.8  (0.0119) | 1687.2  (0.0119) | 1687.1  (0.0059) |
| 12 | 1837.7 | 1837.6  (0.0054) | 1837.6  (0.0054) | 1838.1  (0.022) | 1837.9  (0.0109) |
| 13 | 2163.3 | 2162.1  (0.0554) | 2161.9  (0.0647) | 2165.8  (0.1156) | 2166.1  (0.1294) |
| 14 | 2256.6 | 2248.7  (0.3500) | 2259  (0.1063) | 2262.6  (0.2659) | 2257.7  (0.0487) |
| 15 | 2466.1 | 2472.5  (0.2595) | 2465.8  (0.012) | 2466.4  (0.012) | 2466.3  (0.0081) |
| 16 | 2975.4 | 3061.2  (2.8836) | 2961.9  (0.4537) | 3004.3  (0.9713) | 2988.3  (0.4336) |
| 17 | 3161.2 | 3222.6  (1.9423) | 3156.1  (0.1613) | 3166.9  (0.1803) | 3167.2  (0.1898) |
| 18 | 3255.8 | 3255.6  (0.0061) | 3259.9  (0.1259) | 3264.1  (0.2549) | 3257.8  (0.0614) |

#### Stability analysis of the reduced model

After obtaining the appropriate contact stiffness value, the friction force can be introduced in the theoretical reduction of the FE pad-on-disc model. In the following, the stability analysis of the reduced model with friction is carried out following the flow chart shown in Figure 2. Specifically, the bifurcation of the eigenvalues of the reduced model with the change of the friction coefficient is examined, followed by comparison with those results calculated by the Abaqus complex eigenvalue analysis of the full FE model.

Figure 14 shows the changes of the real and imaginary parts of the eigenvalues of the reduced models with the friction coefficient. Firstly, it can be seen that with the increase of the friction coefficient, some of the real part of reduced models become positive, consequently mode-coupling kind of instability can happen in the reduced models. The first mode coupling is between the 1st and the 2nd modes and the second one is between the 6th and the 7th modes. Secondly, the results of the four reduced models (RM1, RM2, RM3 and RM4) are very similar, although the number of modes used in the model reduction are quite different (10+10, 40+40, 80+80, 245+245). It should be noted here that damping is not included in the CEA analyses of the reduced models or the full FE model.

|  |  |
| --- | --- |
| Comparisons of real vs mu of 4 models second | E:\OneDrive\!Model reduction\Real model\model2 use\250 modes mass normalised\results x+-z correct!\Comparisons of Frequency vs mu of 4 models second3.tif |
| (a) | (b) |

Figure 14. Reduced model of FE model: The evolution of the eigenvalues of different reduced model with the friction coefficient. (a) The change of the real part with friction coefficient *μ* (b) The change of the imaginary part with friction coefficient *μ*

Moreover, Abaqus CEA of the full model is carried out for the real operation condition of the experiment: the normal compression force of 500 N and the rotating speed of 13 rad/s. Figure 15 illustrates the Abaqus CEA results of the full model. By comparing the results of the reduced models (Figure 14 (b)) with those in Figure 15, it can found that the reduced models can capture the main mode-coupling instability phenomena of the whole system, which include: (1) the first coupling between the 1st and the 2nd modes and the second one between the 6th and the 7th modes, (2) the critical friction coefficient for the mode coupling at acceptable accuracy, (3) a good enough unstable frequency range. Furthermore, as vibration frequencies measured in the experiment cover the range from 492 Hz (dominant) to 1705 Hz, as reported in Section 4.1, and the unstable vibration frequencies of the reduced model are 531 Hz and 1411.1 Hz (at the coalescing points of the imaginary parts). The correlation between the reduced models and the experimental results is satisfactory.

On the other hand, it can be seen that some of the features of the bifurcation plot of the full model are not captured by the reduced model, such as the unlocking phenomenon of the coupling of the 1st and the 2nd mode, the decrease of the 3rd and the 4th mode with the friction coefficient, this may be due to several reasons. Two apparent reasons are that Abaqus CEA (1) is a complex process which considers the disc-pad contact area which is influenced by the compression force and (2) uses a particular surface-to-surface contact algorithm, which are different from the conventional theoretical complex eigenvalue analysis implemented in the model reduction method presented in this chapter. However, the important unstable features of the real complex frictional system are reproduced at low cost.

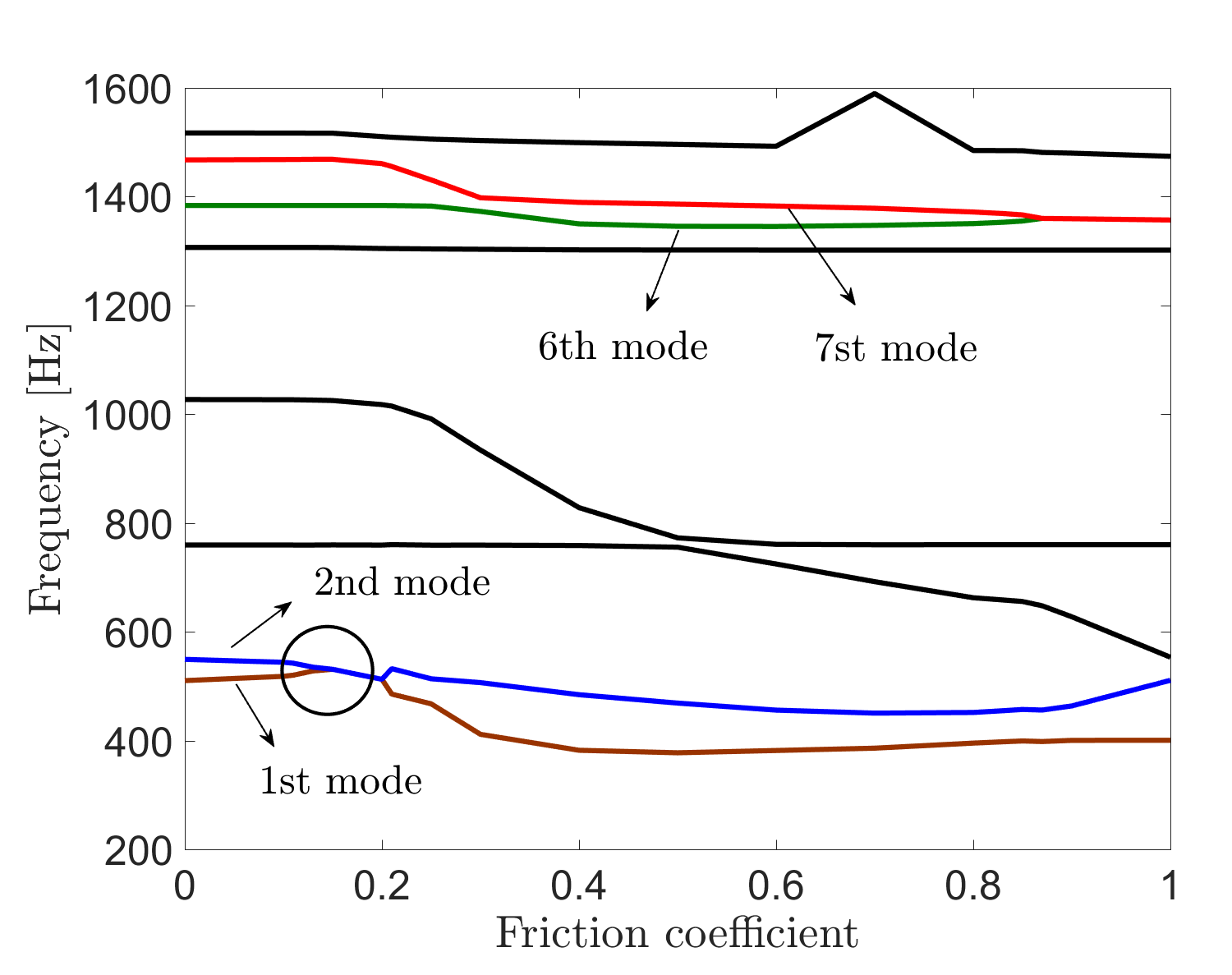


Figure 15 Abaqus CEA results of the full model

The preceding results show that the reduction strategy of this chapter is applicable for the reduction of the complex real system with friction and the stability of the real frictional system can be assessed on the reduced models. Moreover, the reduced models are able to predict the key unstable features of the real structure, even when relatively very few modes of the substructures are used, which is verified by the CEA results of full EF model and the experimental results. With the validation, the advantages of the reduced model, in terms of the computational efficiency and applicability, are demonstrated and this model reduction method is very useful in the dynamic study of the friction-induced vibration of real structures. A further application of this method is in the dynamic transient analysis of nonlinear friction-induced vibration, which should show even higher efficiency but will not be covered within this paper.

## Conclusions

A strategy especially for the model reduction of a multi-degree-of-freedom system with substructures in sliding friction contact is established. Its application to a 9-degree-of-freedom system as well as a real pad-on-disc structure is carried out. The 9-degree-of-freedom model are composed of a slider and belt substructures with linear contact springs. The real pad-on-disc test rig and the corresponding finite element model (FE model) are build up, and the correlation between them is established, which offers a trustworthy full FE model for the subsequent model reduction and ensures the reliability of the validation. The reduction strategy is shown to work well in the modal and stability analysis of the full models and agree quite well with the experimental results of the real pad-on-disc structure.

The conclusions of the model reduction work are:

1. For the 9-degree-of-freedom model, the eigenvalues of the whole system with sliding friction calculated by the model reduction method, when full modes are used, are exactly the same as the results calculated by the direct method using the mass and stiffness matrices of the whole model. Importantly, the mode-coupling properties of the frictional system can be kept in the reduced model if a proper number of substructure modes are used. A properly reduced model can reproduce frequencies of the lower modes at good accuracy. Moreover, depending on the parameter values, the reduced models may overestimate or underestimate the critical friction coefficient of the whole system, especially the critical friction coefficient of high modes.

2. The model reduction of a real pad-on-disc structure of which the pad and disc are in direct contact is carried out. Reduction can be implemented by introducing imaginary linear contact springs between the contact nodes. Results show that the reduced model with a proper contact stiffness value, gives nearly the same natural frequencies of the full finite element model. Moreover, the stability analysis results of the reduced models have good agreement with the Abaqus CEA results of the full FE model, in terms of correctly predicting the mode-coupling behaviour of the system, even when relatively only a small number of modes of the substructures are used in model reduction. Furthermore, the unstable frequencies of the reduced models are shown to have fairly good correlation with the vibration frequencies measured in the experiment. Therefore, the model reduction strategy is effective, and the reduced models can be implemented in the analysis of the real structures with friction, with advantages of high computational efficiency and low demand of modal information of the substructures.

These findings provide a promising basis for nonlinear dynamic analysis of complicated frictional systems at low cost in the future.

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