LTH 1173

Fermion bilinear operator critical exponents at $O(1/N^2)$ in the QED-Gross-Neveu universality class

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September, 2018.

Abstract. We use the critical point large N formalism to calculate the critical exponents corresponding to the fermion mass operator and flavour non-singlet fermion bilinear operator in the universality class of Quantum Electrodynamics (QED) coupled to the Gross-Neveu model for an $SU(N)$ flavour symmetry in d-dimensions. The ϵ expansion of the exponents in $d = 4 - 2\epsilon$ dimensions are in agreement with recent three and four loop perturbative evaluations of both renormalization group functions of these operators. Estimates of the value of the non-singlet operator exponent in three dimensions are provided.

1 Introduction.

An interesting connection between two different areas of physics has been developing over recent years. The new material of intense theoretical and experimental interest in condensed matter physics is that of graphene which is a one atom thick sheet of Carbon atoms arranged in a hexagonal or honeycomb lattice. Under certain conditions, such as when the sheet is deformed by stretching, graphene undergoes a transition from a conducting to Mott-insulating phase [1, 2]. Remarkably it is believed that this phase transition is described by the universality class of a Yukawa-type theory which goes under the general heading of Gross-Neveu or Gross-Neveu-Yukawa theories, [3, 4, 5, 6, 7]. To understand the properties of such phase transitions better, theoretical calculations have been carried out in the various Gross-Neveu field theories introduced in [8]. The primary focus in the work on several of the classes [9, 10, 11, 12, 13, 14, 15], for example, was to determine estimates for the critical exponents in three dimensions. The main methods used in these studies include matched Padé approximants based on the ϵ-expansion of two and four dimensional field theories in the same universality class, the large N method, the functional renormalization group technique and Monte Carlo or numerical evaluations. The ϵ-expansion approach required computing the underlying renormalization group functions at high loop order. For certain specific universality classes there is a general consensus on the values of the exponents in three dimensions. See, for instance, the comprehensive analysis recently carried out in [15]. In other classes such as the chiral Heisenberg Gross-Neveu one the same level of precision has yet to be attained. Although there has been some progress computationally at four loops, [14], a matched Padé approximant estimate in three dimensions would require the same level of precision in the two dimensional theory of the universality class.

While such pure Yukawa type universality classes have been of interest for aspects of graphene physics, a second range of classes is also important. These are generally defined by adding in a $U(1)$ or Quantum Electrodynamics (QED) sector to the pure Gross-Neveu one and are termed the QED-Gross-Neveu universality classes. It is this general class which is the bridge between condensed matter problems and those in particle physics. This is because the underlying quantum field theory is structurally equivalent to that of the Standard Model. In its early historical construction the weak interactions were approximated by an effective 4-fermi operator which was later replaced by a gauge-Yukawa class of interactions. Given this there is clearly interest in refining our understanding of phase transitions in materials such as graphene since, in principle, they could be studied in experimental setups smaller than those at CERN, for example, and may increase our knowledge of phase transitions in the Standard Model or potential theories which lie beyond it. For instance, there was an indication in [7] that the graphene transition could be a laboratory for examining the intricacies of the Standard Model spontaneous symmetry mechanism. Another example of a recent development concerning phase transitions is that of emergent symmetries. In the chiral Ising and chiral XY Gross-Neveu universality classes, [16, 17, 18, 19], there is a fixed point where all the critical couplings are equal producing an emergent supersymmetric fixed point. The equivalence of critical couplings is a necessary but not sufficient condition for this. The additional fact that the anomalous dimensions of the bosonic and fermionic fields have the same value supports the emergent supersymmetry. Potentially there may be extensions of the Standard Model or indeed the Standard Model itself where this could also occur. Not all emergent symmetries correspond to supersymmetry. Instead a theory could be dual to another which becomes apparent when certain operators have the same critical exponent at the fixed points in the respective theories.

One example of such a duality arises in three dimensions between QED and a critical point version of the CP^1 sigma model, [19, 20]. More recently there has been an extension of this type of duality between the QED-Gross-Neveu universality class in three dimensions for a specific number of electron flavours and an $SU(2)$ symmetric non-compact CP^1 sigma model. Underlying this duality is an emergent $SO(5)$ symmetry at the deconfining quantum critical point, [22, 23, 24]. At present the duality is at the level of a conjecture based on numerical evidence and lacks a concrete proof. To study this conjecture further two major independent but simultaneous computations have been undertaken which involved renormalizing the underlying field theory of the QED-Gross-Neveu theory in four dimensions to high loop order. Three loop results were determined in [25] with the four loop results following later in [26]. Although the main results of both groups was the construction of all the renormalization group functions, one important operator was also renormalized which was the flavour non-singlet fermion bilinear which has been studied in [27, 28], for instance. Its anomalous dimension critical exponent is central to establishing the duality in three dimensions. Therefore the ϵ-expansion of the three and four loop exponent at the Wilson-Fisher fixed point has to be summed down to three dimensions and an estimate determined at $N = 1$. Experience from other situations has demonstrated that this is not a trivial process. What would be useful in contributing to the debate is an independent approach to compute the critical exponent of the same operator. That is one aim of this article where we will compute the exponent of the operator in the large N expansion at $O(1/N^2)$ in d-dimensions. The method used the large N critical point approach developed in [29, 30] for the universality class of $O(N)$ scalar fields which contains the two dimensional nonlinear sigma model and four dimensional ϕ^4 theory. The second aim of the article is to determine the flavour singlet fermion bilinear operator critical exponent at the same order as the non-singlet one. We will also refer to this as the fermion mass exponent as this is what the operator corresponds to. It has been given the former term partly to indicate that there is a large overlap in the computations to deduce both exponents which will become apparent at $O(1/N^2)$.

While the critical point large N method has already been applied to the QED-Gross-Neveu class at $O(1/N)$, [31, 32, 33], it transpires that an early $O(1/N^2)$ computation, [32], of the fermion anomalous dimension exponent, η, in the Landau gauge had an error. Therefore, a separate aspect of this article is designed to address this failing as the formalism needed for the $O(1/N^2)$ operator dimension relies centrally on not only the value of η but also the d-dimensional values of the amplitudes of the propagators in the universality class at this order. En route to correct this we will provide an expression for η as a function of the gauge parameter as this is now important for problems which were not manifest at the time of [32] and which we note later. One of the reasons why such an error did not come to light before was due to a lack of perturbative information to compare with at the time of the early work. By this we mean the following. Since the large N exponents are determined in the universal theory as a function of the spacetime dimension d, then the coefficients of ϵ in the power series expansion of an exponent at whatever large N order is available when $d = 4 - 2\epsilon$ should be in exact agreement with those in the perturbative exponent. That was not the case for η at $O(1/N^2)$ when [25] and [26] became available. So with the recent three and four loop results the correct value of η can be established. Equally the ϵ expansion of both operator exponents have to be in agreement with the corresponding three and four loop critical exponents of [25, 26]. Therefore our results will partly provide an independent non-trivial check on this recent perturbative work. That having been established the next aim can be addressed which is the restriction of the d-dimensional results to three dimensions *without* a resummation in ϵ. While $O(1/N^2)$ in d-dimensions represents a level beyond what is normal for usual $1/N$ analyses, to estimate a value of the flavour non-singlet operator exponent in question for the duality conjecture at $N = 1$ may be outside a region of applicability. However such an outcome cannot be pre-judged. Moreover, as was shown in the analysis of all accumulated analytic knowledge of field theory computations in the pure Gross-Neveu universality class, credible exponent estimates can emerge for low N by pooling all available data. Therefore our $O(1/N^2)$ computation for the flavour non-singlet fermion bilinear operator critical exponent should also be viewed in that larger context.

The article is organized as follows. We introduce the large N critical point formalism for the QED-Gross-Neveu universality class in section 2. There the derivation of η at $O(1/N^2)$ is given. The next section is devoted to the $O(1/N^2)$ computation of flavour non-singlet and singlet fermion bilinear operator critical exponents in d-dimensions. Having established these results the ϵ expansions are deduced in section 4 and compared with the recent explicit perturbative results. Estimates of the flavour non-singlet exponent in three dimensions are also determined for a range of values of N in order to compare with the resummation of the perturbative ϵ expansion for the same spacetime. In order to assist such an analysis the critical exponent for the same operator at $O(1/N^2)$ is also determined and studied in the pure Gross-Neveu universality class in section 5. Concluding remarks are provided in section 6.

2 Large N formalism.

The first step in applying the critical point large N formalism developed in [29, 30] for QED coupled to the Gross-Neveu model is to determine the Lagrangian for the underlying universality class at the d-dimensional Wilson-Fisher fixed point. First the pure Gross-Neveu universality class is the two dimensional quantum field theory given in [8] which corresponds to an $SU(N)$ multiplet of fermions with a quartic self-interaction. For the QED-Gross-Neveu universality class the Lagrangian of [8] is extended to include a second quartic self-interaction where this additional term has a Thirring model structure. To summarize the two dimensional Lagrangian is given by

$$L^{d=2} ~=~ i\bar{\psi}^i \partial\!\!\!/ \psi^i ~+~ \frac{g_1^2}{2} \left(\bar{\psi}^i \psi^i \right)^2 ~+~ \frac{g_2^2}{2} \left(\bar{\psi}^i \gamma^\mu \psi^i \right)^2 \tag{2.1}$$

where $1 \leq i \leq N$ and g_i are the two dimensionless coupling constants in two dimensions. This is not the only way of formulating the renormalizable two dimensional theory since two auxiliary fields, $\tilde{\sigma}$ and $\tilde{A}_\mu$, can be introduced by rewriting (2.1) since

$$L^{d=2} ~=~ i\bar{\psi}^i \partial\!\!\!/ \psi^i ~+~ g_1 \tilde{\sigma} \bar{\psi}^i \psi^i ~+~ g_2 \tilde{A}_\mu \bar{\psi}^i \gamma^\mu \psi^i ~-~ \frac{1}{2} \tilde{\sigma}^2 ~-~ \frac{1}{2} \tilde{A}_\mu \tilde{A}^\mu ~. \tag{2.2}$$

In two dimensions we regard $\tilde{A}_\mu$ as an auxiliary spin-1 field rather than a photon since there is no gauge symmetry and its propagator is unity. Beyond two dimensions this field will become the equivalent of the photon at criticality. From (2.2) the structure of the underlying universal Lagrangian can be deduced. Following the prescription apparent in [29, 30] the coupling constants of the universal interactions are rescaled into the quadratic terms of the auxiliary fields. In this case (2.2) becomes

$$L ~=~ i\bar{\psi}^i \partial\!\!\!/ \psi^i ~+~ \sigma \bar{\psi}^i \psi^i ~+~ A_\mu \bar{\psi}^i \gamma^\mu \psi^i ~-~ \frac{1}{2g_1^2} \sigma^2 ~-~ \frac{1}{2g_2^2} A_\mu A^\mu ~. \tag{2.3}$$

The first three terms are core to the Lagrangian of the underlying universal theory whereas the remaining two are the only two relevant local operators when the critical dimension is two. In other critical dimensions other local operators will be relevant. To see this a dimensional analysis of (2.3) implies ψ, σ and A_μ have canonical dimensions of $\frac{1}{2}(d-1)$, 1 and 1 respectively in d-dimensions. Therefore when the critical dimension of the universal theory is four the renormalizable Lagrangian which is equivalent to (2.1) and (2.2) is

$$L^{d=4} ~=~ i\bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} \tilde{F}_{\mu\nu}^2 - \frac{1}{2b} \left(\partial^\mu \tilde{A}_\mu \right)^2 + \bar{g}_1 \tilde{\sigma} \bar{\psi}^i \psi^i + \bar{g}_2 \tilde{A}_\mu \bar{\psi}^i \gamma^\mu \psi^i + \frac{\bar{g}_3^2}{24} \tilde{\sigma}^4 \tag{2.4}$$

where $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$, b is the gauge fixing parameter with $b = 0$ corresponding to the Landau gauge and $\bar{g}_i$ are the three coupling constants which are dimensionless in four dimensions. The coupling constants have been included with the respective interactions as it is this Lagrangian which has recently been renormalized to three and four loops in [25, 26] respectively. Since the canonical dimensions of σ and A_μ are both unity in the universal theory then these fields develop canonical propagators in four dimensions. A quartic fermion self-interaction cannot be present since that operator would have canonical dimension six in a critical dimension of four. Moreover given the structure of the non-gauge sector (2.4) is sometimes referred to as the QED-Gross-Neveu-Yukawa model.

The starting point for computing the d-dimensional critical exponents of the underlying QED-Gross-Neveu universality class in the large N expansion is to write down the asymptotic scaling forms of the propagators in the approach to the Wilson-Fisher fixed point. At this point no masses are present and the propagators have a power law behaviour. Therefore in d-dimensions the propagators of the three fields of (2.3) have the scaling form, [32, 33, 34, 35, 36],

$$\begin{aligned} \langle \psi^i(x) \bar{\psi}^j(y) \rangle & \sim \frac{(\not{x} - \not{y}) A \delta^{ij}}{((x-y)^2)^\alpha} \\ \langle A_\mu(x) A_\nu(y) \rangle & \sim \frac{B_A}{((x-y)^2)^{\beta_A}} \left[\eta_{\mu\nu} + \frac{2(1-b)\beta_A}{(2\mu - 2\beta_A - 1 + b)} \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right] \\ \langle \sigma(x) \sigma(y) \rangle & \sim \frac{B_\sigma}{((x-y)^2)^{\beta_\sigma}} \end{aligned} \tag{2.5}$$

in coordinate space. The quantities A, B_A and B_σ are coordinate independent amplitudes and α, β_A and β_σ are the full scaling dimensions of the respective fields. They are defined by

$$\alpha = \mu + \tfrac{1}{2}\eta \quad , \quad \beta_A = 1 - \eta - \chi_A \quad , \quad \beta_\sigma = 1 - \eta - \chi_\sigma \tag{2.6}$$

where

$$d = 2\mu \tag{2.7}$$

and η is the fermion anomalous dimension. The exponents χ_A and χ_σ correspond to the anomalous dimensions of the respective 3-point vertices. Given the nature of the universal theory the anomalous part of the critical exponent corresponding to the fermion singlet bilinear or mass operator is related to the anomalous part of β_σ. As we are computing directly at the Wilson-Fisher fixed point the critical exponents will depend only on μ and N and hence they can be formally expanded in a power series in $1/N$ such as

$$\eta = \sum_{n=1}^{\infty} \frac{\eta_n}{N^n} \tag{2.8}$$

and similar notation will also be used for other exponents. The particular Lorentz structure of the photon propagators is dictated by the canonical form of the propagator in momentum space which is then mapped to this coordinate space form by a Fourier transform. As the first stage of the large N method to compute exponents is to determine the fermion dimension by solving the Schwinger-Dyson equations for the 2-point functions of the fields in the approach to criticality, the asymptotic scaling forms of the 2-point functions are also required. These are deduced by first inverting the momentum space forms of the scaling functions which are

$$\begin{aligned} \langle \psi^i(p) \bar{\psi}^j(-p) \rangle & \sim \frac{\not{p} \tilde{A} \delta^{ij}}{(p^2)^{\mu - \alpha + 1}} \\ \langle A_\mu(p) A_\nu(-p) \rangle & \sim \frac{\tilde{B}_A}{(p^2)^{\mu - \beta_A}} \left[\eta_{\mu\nu} - (1-b) \frac{p_\mu p_\nu}{p^2} \right] \\ \langle \sigma(p) \sigma(-p) \rangle & \sim \frac{\tilde{B}_\sigma}{(p^2)^{\mu - \beta_\sigma}} \end{aligned} \tag{2.9}$$

where $\tilde{A}$, $\tilde{B}_A$ and $\tilde{B}_\sigma$ are the momentum space amplitudes. The inverse of these are then mapped back to coordinate space by a Fourier transform. We therefore have, [34, 35, 36],

$$
\begin{aligned}
\langle \psi^i(x)\bar{\psi}^j(y)\rangle^{-1} &\sim \frac{r(\alpha-1)(\not{x}-\not{y})\delta^{ij}}{A((x-y)^2)^{2\mu-\alpha+1}} \\
\langle A_\mu(x)A_\nu(y)\rangle^{-1} &\sim \frac{t(\beta_A)}{B_A((x-y)^2)^{2\mu-\beta_A}}\left[\eta_{\mu\nu}+\frac{2(2\mu-\beta_A)}{(2\beta_A-2\mu-1)}\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}\right] \\
\langle \sigma(x)\sigma(y)\rangle^{-1} &\sim \frac{p(\beta_\sigma)}{B_\sigma((x-y)^2)^{2\mu-\beta_\sigma}}
\end{aligned}
\tag{2.10}
$$

for our three fields. The process produces different coordinate independent amplitudes involving the functions

$$
r(\alpha) = \frac{\alpha a(\alpha-\mu)}{(\mu-\alpha)a(\alpha)} \quad , \quad p(\alpha) = \frac{a(\alpha-\mu)}{a(\alpha)} \quad , \quad t(\alpha) = \frac{[4(\mu-\alpha)^2-1]a(\alpha-\mu)}{4(\mu-\alpha)a(\alpha)} \tag{2.11}
$$

where

$$
a(\alpha) = \frac{\Gamma(\mu-\alpha)}{\Gamma(\alpha)} \tag{2.12}
$$

is used for shorthand, [29].

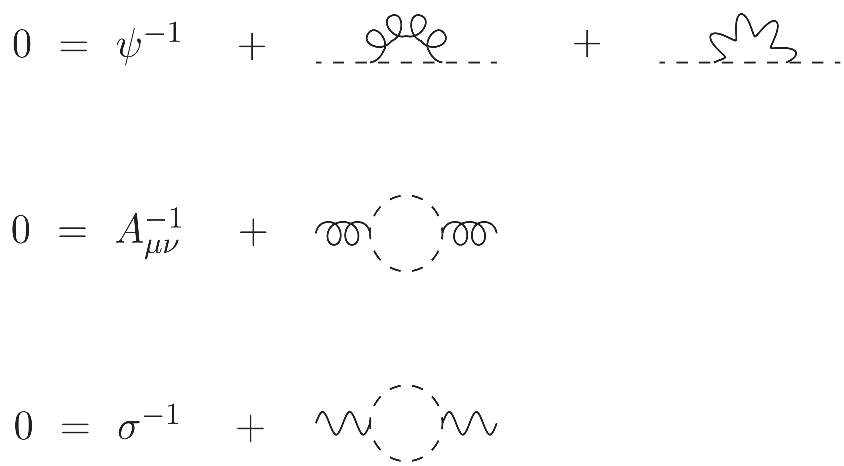

Figure 1: Leading order large N graphs contributing to the 2-point skeleton Schwinger-Dyson equations.

As we will require basic quantities such as the amplitudes and exponents for the computation of both bilinear operator exponents at $O(1/N^2)$ and since there was an error in the earlier work of [32], it is instructive to review the evaluation of η_2. The starting point is the set of Schwinger-Dyson 2-point functions in the asymptotic scaling region. The leading order contributions in the large N expansion are given in Figure 1 with the higher order corrections to each 2-point function given in Figures 2, 3 and 4. It is important to note that the graphs are ordered by powers of $1/N$ rather than the coupling constant which produces the loop expansion. The counting of powers of N is derived by noting that a closed fermion loop has a factor of N and each A_μ and σ propagator counts one power of $1/N$. Ordinarily in the large N expansion this would produce three loop graphs at the same order in $1/N$ as the graphs in Figures 3 and 4 where there are two closed fermion loops and a total of two A_μ and σ propagators. However in (2.3) while such graphs can be present in principle we have not included them as either the fermion loop contains an odd number of γ-matrices or graphs with an A_μ leg vanish by Furry's theorem. Therefore we have not included these in Figures 3 and 4. For the fermion 2-point function there are also potential three loop graphs with one fermion loop but these too are absent for similar reasons. Using (2.5) these Schwinger-Dyson equations can be represented algebraically by

$$\begin{aligned}
0 &= r(\alpha-1) ~+~ zZ_\sigma^2(x^2)^{\chi_\sigma+\Delta} \\
&\quad -~ \frac{2yZ_A^2}{(2\mu-3+b)} \left[[(2\mu-1)(\mu-2)+\mu b] + (2\mu-1+b)(1-b)(\eta+\chi_A+\Delta) \right] (x^2)^{\chi_A+\Delta} \\
&\quad +~ z^2\Sigma_1(x^2)^{2\Delta} ~+~ y^2\Sigma_2(x^2)^{2\Delta} ~+~ yz\Sigma_3(x^2)^{2\Delta} ~+~ yz\Sigma_4(x^2)^{2\Delta} ~+~ O\left(\frac{1}{N^3}\right) \\
0 &= p(\beta_\sigma) ~+~ 4NzZ_\sigma^2(x^2)^{\chi_\sigma+\Delta} ~-~ Nz^2\Gamma_1(x^2)^{2\Delta} ~-~ Nyz\Gamma_2(x^2)^{2\Delta} ~+~ O\left(\frac{1}{N^2}\right) \\
0 &= 2t(\beta_A)\left[\frac{(\mu-1)}{(2\mu-1)} + \frac{(\eta_1+\chi_{A\,1})}{(2\mu-1)^2N}\right] ~-~ 4NyZ_A^2\left[\frac{2(\mu-1)}{(2\mu-1)} + \frac{\eta_1}{(2\mu-1)^2N}\right](x^2)^{\chi_A+\Delta} \\
&\quad -~ Ny^2\left[\frac{\Pi_1}{\Delta} + \Pi'_1 + \left[\frac{\Xi_1}{\Delta} + \frac{\Xi'_1}{2(2\mu-1-\Delta)}\right]\right] \\
&\quad -~ Nyz\left[\frac{\Pi_2}{\Delta} + \Pi'_2 + \left[\frac{\Xi_2}{\Delta} + \frac{\Xi'_2}{2(2\mu-1-\Delta)}\right]\right] ~+~ O\left(\frac{1}{N^2}\right)
\end{aligned} \qquad (2.13)$$

to $O(1/N^2)$. The first two terms of each equation correspond to the three equations in Figure 1. An analytic regularization Δ has been introduced in (2.13) by the shift

$$\chi_A ~\to~ \chi_A ~+~ \Delta \quad , \quad \chi_\sigma ~\to~ \chi_\sigma ~+~ \Delta \qquad (2.14)$$

since the graphs in Figures 2, 3 and 4 are divergent. Although we are working in d-dimensional spacetime we are not using dimensional regularization. Such a regularization will not quantify the divergences in these graphs, [29, 30]. Instead this particular analytic regularization is used since in this critical point formulation of the large N expansion one is in effect carrying out perturbation theory in the vertex anomalous dimensions. The quantities Σ_i, Γ_i, Π_i and Π'_i represent the d-dependent *values* of the graphs devoid of the dimensional dependence which has been factored off into the powers of x^2 in the correction terms of (2.13). For the photon equation we have formally isolated the divergent term in Δ, Π_i, from the finite part, Π'_i, of the values of both graphs in Figure 3 because only the transverse part of the Schwinger-Dyson equation is relevant in the determination of η. As the two 3-point vertices of (2.3) each involve two fermions and one A_μ or σ field the propagator amplitudes always appear in the combinations

$$y ~=~ A^2B_A \quad , \quad z ~=~ A^2B_\sigma ~. \qquad (2.15)$$

Equally as the underlying 3-point vertices are always divergent, [29, 30, 37, 38], two vertex renormalization constants Z_A and Z_σ have been introduced. They only appear on the leading order graphs as the effect of the counterterms in the next order will not play a role until η_3 is computed if this method is used.

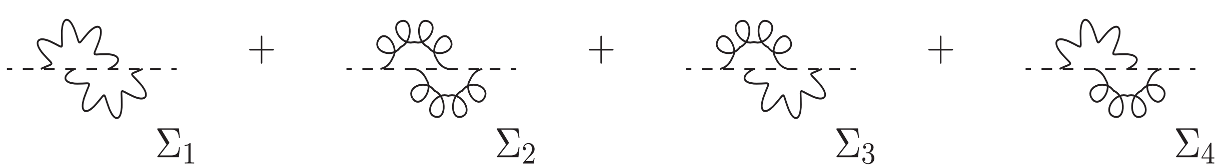

Figure 2: $O(1/N^2)$ graphs contributing to fermion 2-point function.

At leading order the three equations simplify to

$$
\begin{aligned}
0 &= r(\alpha-1) ~+~ z ~-~ \frac{2y}{(2\mu-3+b)}[(2\mu-1)(\mu-2)+\mu b] ~+~ O\left(\frac{1}{N^2}\right) \\
0 &= p(\beta_\sigma) ~+~ 4Nz ~+~ O\left(\frac{1}{N^2}\right) \\
0 &= \frac{2(\mu-1)}{(2\mu-1)}t(\beta_A) ~-~ \frac{8(\mu-1)Ny}{(2\mu-1)} ~+~ O\left(\frac{1}{N^2}\right)
\end{aligned}
\tag{2.16}
$$

which contain the three unknowns η_1, y_1 and z_1. There is no x^2 dependence since the expansion of the respective factors in the leading order terms of (2.13) are either $O(1/N)$ or $O(\Delta)$. As there are no poles in Δ the regularization can be lifted without any difficulty. This means that at leading order there is a smooth limit to the critical point as $x^2 \to 0$. Setting the canonical values of the exponents for A_μ and σ in the final two equations and $\alpha = \mu + \eta_1/N$ in the first then the three equations can be solved to produce

$$
\begin{aligned}
\eta_1 &= -~[4\mu^2 - 10\mu + 5 + (2\mu-1)b]\hat{\eta}_1 \\
y_1 &= \frac{(2\mu-1)(2\mu-3+b)\Gamma(2\mu-1)}{16[\mu-1]\Gamma(\mu)\Gamma(1-\mu)} \\
z_1 &= -~\frac{\Gamma(2\mu-1)}{4[\mu-1]\Gamma(\mu)\Gamma(1-\mu)}
\end{aligned}
\tag{2.17}
$$

where

$$
\hat{\eta}_1 ~=~ \frac{\Gamma(2\mu-1)}{4[\mu-1]\Gamma^3(\mu)\Gamma(1-\mu)} ~.
\tag{2.18}
$$

These agree with the leading order values given in [31, 32, 33].

Having established the leading order solution, including the next order terms is straightforward formally. However there are now five unknowns which are η_2, y_2, z_2, $\chi_{A\,1}$ and $\chi_{\sigma\,1}$. The former three would be expected given the three leading order equations. The latter two lurk within the $1/N$ expansion of the x-dependence in the leading order terms. Although technically there are two other unknowns which are the vertex counterterms their values are not unconnected with the values of $\chi_{A\,1}$ and $\chi_{\sigma\,1}$ and determined from the divergent part of the graphs in Figures 3 and 4. We have computed the explicit values of these graphs to the finite part in Δ using the techniques given in [29, 30, 32, 33, 34, 36]. For several graphs the method of conformal integration has been applied either directly or by writing the graph as the sum of scalar integrals after taking traces over fermion lines. The full set of values for (2.3) for the $SU(N)$ flavour symmetry and using $\mathrm{Tr}\, I = 4$ for the γ-matrix trace are

$$
\begin{aligned}
\Gamma_1 &= \frac{8}{(\mu-1)\Gamma^2(\mu)}\left[\frac{1}{\Delta} ~-~ \frac{1}{(\mu-1)}\right] \\
\Gamma_2 &= \frac{16}{(2\mu-3+b)\Gamma^2(\mu)}\left[\frac{(2\mu-1+b)}{\Delta} + \frac{3}{(\mu-1)} - 3(\mu-1)\left[\psi'(\mu-1)-\psi'(1)\right]\right. \\
&\qquad \left. -~\frac{2(2\mu-1+b)}{(2\mu-3+b)}\right]
\end{aligned}
$$

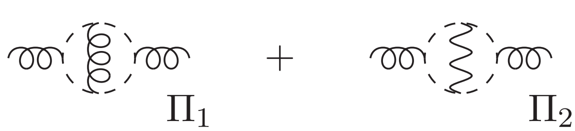

Figure 3: $O(1/N^2)$ graphs contributing to the photon 2-point function.

$$
\begin{aligned}
\Sigma_1 &= - \frac{2}{(\mu-1)\Gamma^2(\mu)} \left[\frac{1}{\Delta} - \frac{1}{2(\mu-1)} \right] \\
\Sigma_2 &= \frac{2}{\mu(2\mu-3+b)^2\Gamma^2(\mu)} \left[\frac{[(2\mu-1)(\mu-2)+\mu b]^2}{\Delta} - \frac{8[(2\mu-1)(\mu-2)+\mu b]^2}{[2\mu-3+b]} \right. \\
&\quad + 4(\mu-1)^2 \left[\frac{2(\mu-1)^2}{\mu} + \frac{1}{2}(2\mu-3) + \frac{(\mu+1)b}{2[\mu-1]} \right] \\
&\quad \left. - 4(1-b)(2\mu-1-\mu b) - (1-b)[(2\mu-1)(\mu-2)+\mu b] \right] \\
\Sigma_3 &= \Sigma_4 = \frac{1}{\Gamma^2(\mu)} \left[- \frac{4[(2\mu-1)(\mu-2)+\mu b]}{\mu(2\mu-3+b)\Delta} + \frac{2(\mu^2+\mu-1)}{\mu^2(\mu-1)} \right. \\
&\quad \left. + \frac{2(2\mu-1)(1-b)}{\mu^2(\mu-1)(2\mu-3+b)} + \frac{4(2\mu-1)(1-b)}{\mu(\mu-1)(2\mu-3+b)^2} \right] \\
\Xi_i &= - 2\Pi_i \quad , \quad \Xi'_i = - 2\Pi'_i \\
\Pi_1 &= - \frac{16[(2\mu-1)(\mu-2)+\mu b]}{\mu(2\mu-3+b)\Gamma^2(\mu)} \\
\Pi'_1 &= \frac{16}{(2\mu-3+b)\Gamma^2(\mu)} \left[3(\mu-1) \left[\psi'(\mu-1) - \psi'(1) \right] - \frac{3}{(\mu-1)} \right. \\
&\quad \left. + \frac{2[(2\mu-1)(\mu-2)+\mu b]}{\mu(2\mu-3+b)} \right] \\
\Pi_2 &= \frac{8}{\mu\Gamma^2(\mu)} \quad , \quad \Pi'_2 = - \frac{8}{\mu(\mu-1)\Gamma^2(\mu)}
\end{aligned}
\tag{2.19}
$$

where $\psi(z)$ is the Euler polygamma function. With these the vertex counterterms and values for χ_A and χ_σ can be deduced. The former are chosen in a minimal subtraction scheme by ensuring that there are no poles in Δ in the respective A_μ and σ 2-point functions. A check on the resultant values

$$
\begin{aligned}
Z_A &= 1 - \frac{\eta_1}{2\Delta N} + O\left(\frac{1}{N^2}\right) \\
Z_\sigma &= 1 - [4\mu^2 - 6\mu + 3 + (2\mu-1)b] \frac{\hat{\eta}_1}{2\Delta N} + O\left(\frac{1}{N^2}\right)
\end{aligned}
\tag{2.20}
$$

is effected by noting that these choices also render the fermion 2-point function finite where both renormalization constants are present. Otherwise if this consistency check was not satisfied we would be working with a non-renormalizable critical Lagrangian. The values of χ_A and χ_σ are determined by ensuring that there are no $\ln(x^2)$ terms in the now Δ-finite Schwinger-Dyson equations. Again the respective exponents are deduced from the A_μ and σ equations with the fermion equation used as a check. Consequently

$$
\chi_{\sigma\,1} = - \left[4\mu^2 - 6\mu + 3 + (2\mu-1)b \right] \hat{\eta}_1 \quad , \quad \chi_{A\,1} = - \eta_1
\tag{2.21}
$$

which are not unrelated to the renormalization constants. The second relation is the manifestation of the Ward-Takahashi identity in the critical point formalism for (2.3) similar to the

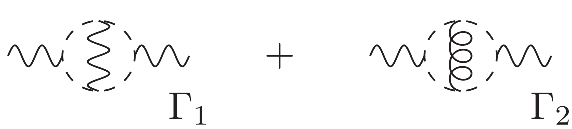

Figure 4: $O(1/N^2)$ graphs contributing to the σ 2-point function.

situation in the pure QED case, [31, 32, 35]. It implies that A_μ has a dimension of 1 for all values of N in the universal theory. We did not assume this at the outset as its emergence acts as another internal consistency check.

Once these initial leading order quantities are determined the Schwinger-Dyson equations are both finite and the $x^2 \to 0$ limit can be smoothly taken to leave the three equations analogous to (2.16) from which the three remaining variables can be determined. We find

$$\begin{aligned} y_2 &= - \left[\left[(4\mu^3 - 10\mu^2 + \mu + 2)(2\mu - 3 + b) - 4(2\mu-1)(\mu-1)^2 \right] \frac{(2\mu-1)}{4\mu[\mu-1]} \right. \\ & \left. + \frac{3}{4}(2\mu - 3 + b)(2\mu-1)^2(\mu-1) \left[\psi'(\mu-1) - \psi'(1) \right] \right] \Gamma^2(\mu) \hat{\eta}_1^2 \\ z_2 &= \left[[4\mu^2 - 14\mu + 7] + \frac{4(2\mu-1)(\mu-1)(\mu-1)}{[2\mu-3+b]} \right. \\ & - 2(2\mu-1)(\mu-1)^2 \left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1) \right] \\ & \left. + 3(2\mu-1)(\mu-1)^3 \left[\psi'(\mu-1) - \psi'(1) \right] \right] \Gamma^2(\mu) \hat{\eta}_1^2 \end{aligned} \tag{2.22}$$

for the two amplitude combinations leaving

$$\begin{aligned} \eta_2 &= \left[\frac{[8\mu^5 - 60\mu^4 + 136\mu^3 - 123\mu^2 + 50\mu - 8](2\mu-1)}{\mu^2[\mu-1]} + \frac{[4\mu^2 - 11\mu + 4](2\mu-1)^2 b}{\mu[\mu-1]} \right. \\ & - \frac{4(2\mu-1)(\mu-1)^2}{\mu} \left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1) \right] \\ & \left. + 3(2\mu-1)(\mu-1)[4\mu^2 - 10\mu + 5 + (2\mu-1)b] \left[\psi'(\mu-1) - \psi'(1) \right] \right] \hat{\eta}_1^2 \ . \end{aligned} \tag{2.23}$$

As a check on this expression and that for η_1 we have computed the ϵ expansion in $d = 4 - 2\epsilon$ dimensions to produce

$$\begin{aligned} \eta|_{d=4-2\epsilon} &= \left[\frac{1}{2}\epsilon - 3\epsilon^2 + \frac{3}{2}\epsilon^3 + [2 + \zeta_3]\epsilon^4 + \left[\frac{3}{2}\zeta_4 - 6\zeta_3 + \frac{5}{2} \right] \epsilon^5 + [3\zeta_5 + 3\zeta_3 - 9\zeta_4 + 3]\epsilon^6 \right] \frac{1}{N} \\ & + \left[\frac{3}{2}\epsilon - 10\epsilon^2 + \left[\frac{117}{8} + \frac{9}{2}\zeta_3 \right] \epsilon^3 + \left[\frac{169}{16} - \frac{45}{2}\zeta_3 + \frac{27}{2}\zeta_4 \right] \epsilon^4 \right. \\ & - \left[\frac{261}{32} - 4\zeta_3 + \frac{135}{4}\zeta_4 - 9\zeta_5 \right] \epsilon^5 \\ & \left. - \left[\frac{1535}{64} - 18\zeta_3^2 - 57\zeta_3 - 6\zeta_4 + 27\zeta_5 - \frac{45}{4}\zeta_6 \right] \epsilon^6 \right] \frac{1}{N^2} + O\left(\epsilon^7, \frac{1}{N^3} \right) \end{aligned} \tag{2.24}$$

in the Landau gauge where ζ_z is the Riemann zeta function. We recall that the Landau gauge is the fixed point of the renormalization group flow of the gauge parameter which can be regarded as a coupling. Comparing with the recent three and four loop expressions given in [25, 26] we find exact agreement. Therefore (2.23) now supersedes the $b = 0$ result of [32]. Consequently we note that in three dimensions

$$\eta|_{d=3} = - \frac{2[1-2b]}{\pi^2 N} - \frac{[(36 - 72b)\pi^2 + (672b - 256)]}{9\pi^4 N^2} + O\left(\frac{1}{N^3} \right) \tag{2.25}$$

or

$$\eta|_{d=3} = - \frac{[0.202642 - 0.405285b]}{N} + \frac{[0.044043b - 0.113275]}{N^2} + O\left(\frac{1}{N^3} \right) \tag{2.26}$$

numerically. The gauge parameter is included here as it is relevant for studies of chiral symmetry breaking in three dimensions using the large N expansion, [39]. See, for instance, recent $O(1/N^2)$

work in this area in pure QED, [40, 41, 42]. There the value of N corresponding to chiral symmetry breaking was studied using $1/N$ methods and the value shown to be independent of the gauge.

A final part of the exercise in reconstructing η_2 is to lay the foundation for determining the $O(1/N^2)$ exponents of the fermion bilinear operators. As this will be computed using the critical propagators in momentum space, (2.9), we need to record the corresponding momentum space variables to $O(1/N^2)$ which are defined in a similar way to (2.15) by

$$\tilde{y} ~=~ \tilde{A}^2 \tilde{B}_A ~~~,~~~ \tilde{z} ~=~ \tilde{A}^2 \tilde{B}_\sigma ~. \tag{2.27}$$

We found

$$\begin{aligned} \tilde{y}_1 &= -~ \frac{(2\mu-1)\Gamma(2\mu-1)}{8(\mu-1)\Gamma^2(\mu)\Gamma(1-\mu)} \\ \tilde{z}_1 &= \frac{\Gamma(2\mu-1)}{4\Gamma^2(\mu)\Gamma(1-\mu)} \end{aligned} \tag{2.28}$$

at leading order and

$$\begin{aligned} \tilde{y}_2 &= \frac{3}{2} \left[(2\mu-1)^2(\mu-1) \left[\psi^\prime(\mu-1) - \psi^\prime(1) \right] - \frac{(2\mu-1)^2}{[\mu-1]} \right] \Gamma(\mu) \hat{\eta}_1^2 \\ \tilde{z}_2 &= \left[(2\mu-1) + 2(2\mu-1)(\mu-1) \left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1) \right] \right. \\ & \left. -~ 3(2\mu-1)(\mu-1)^2 \left[\psi^\prime(\mu-1) - \psi^\prime(1) \right] \right] \Gamma(\mu) \hat{\eta}_1^2 \end{aligned} \tag{2.29}$$

at next order. These were deduced from the momentum space version of (2.13).

3 Operator critical exponents.

We now turn to the evaluation of the operator critical exponents which are those for the gauge invarianr flavour non-singlet and singlet fermion bilinears

$$\mathcal{O}_m ~=~ \bar{\psi}^i \psi^i ~~~,~~~ \mathcal{O}_{\mbox{ns}} ~=~ \bar{\psi}^i \sigma^z_{ij} \psi^j \tag{3.1}$$

where we use the notation of [25, 28] for the latter. In [26] the notation used for the non-singlet operator was $\bar{\psi}^i T^a_{ij} \psi^j$. In each case σ^z and T^a are flavour matrices and their presence is to distinguish the operator from the usual mass operator $\bar{\psi}^i \psi^i$ which is the flavour singlet partner quantity. In terms of the full critical exponent of both operators they are each comprised of two parts and defined by

$$\Delta_m ~=~ 2\mu ~-~ 1 ~+~ \eta ~+~ \eta_{\mathcal{O}_m} ~~~,~~~ \Delta_{\bar{\psi}\sigma^z\psi} ~=~ 2\mu ~-~ 1 ~+~ \eta ~+~ \eta_{\mathcal{O}_{\mbox{ns}}} \tag{3.2}$$

where $\eta_{\mathcal{O}_{\mbox{ns}}}$ is determined to $O(1/N^2)$ by computing the leading and next to leading order set of graphs given in Figures 5 and 6. For the flavour non-singlet operator there are no $O(1/N^2)$ graphs where the operator is inserted in a closed fermion loop as a trace over the flavour indices would produce either $\mbox{Tr}\, \sigma^z$ or $\mbox{Tr}\, T^a$ which vanish. For the mass operator such graphs would have to be included and these are given in Figure 7. As an aside the comparison of the mass operator dimension at criticality with the perturbative fermion mass anomalous dimension at criticality is not straightforward. This is because in the four dimensional Lagrangian the mass operator has the same canonical dimension, which is 3, as the operator is σ^3. Therefore under renormalization in the coupling constant expansion there is mixing. So in order to compare with the large N exponents one has to first compute the anomalous dimensions of the two eigen-operators of the perturbative mixing matrix. This was carried out in [26]. Evaluating these at the fixed point, one of the eigen-exponents will correspond to the fermion mass critical exponent computed in the large N expansion. By contrast when one computes using the large N critical point formalism directly at the Wilson-Fisher fixed point the fermion mass operator and σ^3 have *different* canonical dimensions which are $(2\mu-1)$ and 3 respectively. Therefore there is no mixing in the $1/N$ approach. The canonical dimensions only agree in four dimensions corresponding to the operator mixing of perturbation theory. We have mentioned this subtlety at length as there may be a potential mixing in the non-singlet situation. However, given the nature of the operator (3.1) which is a flavour vector there is no corresponding flavour non-singlet operator involving three σ fields. While we will determine the mass operator dimension given the nature of the underlying Lagrangian the computation is equivalent to finding the $O(1/N^2)$ corrections to χ. This was the method used to deduce the fermion mass in the pure Gross-Neveu model, [43].

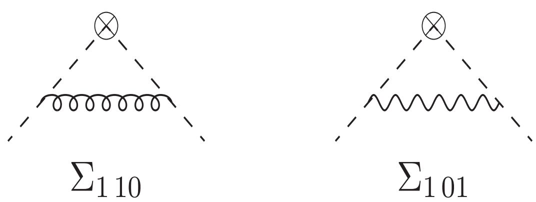

Figure 5: Leading order graphs contributing to the critical exponent of $\mathcal{O}_{\rm ns}$.

First we focus on the determination of $\eta_{\mathcal{O}_{\rm ns}}$ at $O(1/N^2)$. We have computed the graphs of Figures 5 and 6 in momentum space using the critical propagators (2.9) which required the new amplitudes (2.29) of the previous section. Throughout this section all the results will be solely in the Landau gauge as this gauge is a fixed point of the large N critical point formalism we are using. While we work in momentum space the procedure follows that outlined for determining η except that the vertex counterterms are already available when the $O(1/N^2)$ corrections to the leading graphs of Figure 5 are computed. Equally the unknown exponent $\eta_{\mathcal{O}_{\rm ns}i}$ is deduced at each order from ensuring that the $p^2 \to 0$ limit can be smoothly taken which means that there has to be no $\ln p^2$ terms. Finally as there is no mixing and there are no derivatives in the operator itself the graphs of Figures 5 and 6 can be calculated where the operator is inserted at zero momentum. This means that in effect all the graphs reduce to 2-point ones so that to evaluate the corrections the same conformal integration methods are used as those for the graphs giving η_2. At leading order the two one loop graphs are straightforward to evaluate and lead to

$$\eta_{\mathcal{O}_{\rm ns}1} ~=~ [4\mu^2 - 6\mu + 3]\hat{\eta}_1 \tag{3.3}$$

or

$$\eta_1 ~+~ \eta_{\mathcal{O}_{\rm ns}1} ~=~ 2[2\mu - 1]\hat{\eta}_1 \tag{3.4}$$

for (3.2).

To complete the $O(1/N^2)$ computation for the non-singlet operator dimension evaluation we note that the contributions from the higher order graphs are

$$\Sigma_{2\,20} ~=~ -~ \frac{4(2\mu^3-6\mu^2+4\mu-1)(2\mu-1)}{\mu^2(\mu-1)\Gamma^2(\mu)} ~~,~~ \Sigma_{2\,11a} ~=~ \frac{2(2\mu-1)^2}{\mu^2\Gamma^2(\mu)}$$
$$\Sigma_{2\,11b} ~=~ -~ \frac{4(2\mu-1)}{(\mu-1)\Gamma^2(\mu)} ~~,~~ \Sigma_{2\,02} ~=~ -~ \frac{4}{(\mu-1)\Gamma^2(\mu)}$$

$$\Sigma_{4\,20} = -\frac{2(2\mu-1)^2}{3(\mu-1)\Gamma^2(\mu)} \quad , \quad \Sigma_{4\,11a} = \Sigma_{4\,11b} = -\frac{2(2\mu-1)}{3(\mu-1)\Gamma^2(\mu)}$$
$$\Sigma_{4\,02} = -\frac{2}{3(\mu-1)\Gamma^2(\mu)} \quad , \quad \Sigma_{5\,20} = -\frac{2(2\mu-1)(2\mu-3)}{(\mu-1)\Gamma^2(\mu)}$$
$$\Sigma_{5\,11} = -\frac{2(2\mu-1)}{(\mu-1)\Gamma^2(\mu)} \quad , \quad \Sigma_{5\,02} = \frac{2}{(\mu-1)\Gamma^2(\mu)} \tag{3.5}$$

in the Landau gauge. While several of these graphs have been determined in either the pure QED or Gross-Neveu cases we have included them all here for completeness and also since they have been recalculated using with the same spinor trace conventions. Different conventions were used in the original separate cases. Equally we have written a routine in the symbolic manipulation language Form [44, 45] to evaluate the Feynman integrals. Each of the graphs is decomposed into a sum of scalar integrals after taking the trace and then computed separately. While several of these scalar integrals were already known for the pure QED and Gross-Neveu cases new ones had to be calculated such as the set $\Sigma_{i\,11}$. Equipped with (3.5) we find

$$\begin{aligned} \eta_{\mathcal{O}_{\mathrm{ns}^2}} = -\Bigg[& \frac{[8\mu^4 - 44\mu^3 + 56\mu^2 - 27\mu + 4](2\mu-1)}{\mu[\mu-1]} \\ & + 4(2\mu-1)(\mu-1)\left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1)\right] \\ & + 3[4\mu^2 - 6\mu + 3](2\mu-1)(\mu-1)\left[\psi'(\mu-1) - \psi'(1)\right]\Bigg]\hat{\eta}_1^2 \end{aligned} \tag{3.6}$$

which produces

$$\begin{aligned} \eta_2 + \eta_{\mathcal{O}_{\mathrm{ns}^2}} = -\Bigg[& \frac{2[4\mu^3 - 18\mu^2 + 15\mu - 4](2\mu-1)}{\mu^2[\mu-1]} \\ & + \frac{4(2\mu-1)^2(\mu-1)}{\mu}\left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1)\right] \\ & + 6(2\mu-1)^2(\mu-1)\left[\psi'(\mu-1) - \psi'(1)\right]\Bigg]\hat{\eta}_1^2 \end{aligned} \tag{3.7}$$

which is one of the main results of this article and we recall that $d = 2\mu$ and $\psi(z)$ is the polygamma function.

The situation for the mass operator is similar to that of the flavour non-singlet case except that the values of the additional graphs of Figure 7 have to be included. Using similar conformal integration techniques as those used in [43] produces

$$\begin{aligned} \Sigma_{6\,000} &= -\frac{4[2\mu^2 - 5\mu + 4]\Gamma(1-\mu)}{3(\mu-1)\Gamma(2\mu-1)} \\ \Sigma_{6\,110} &= \Sigma_{6\,011} = -\left[\frac{2[8\mu^3 - 28\mu^2 + 34\mu - 17]}{3(\mu-1)} + 4(\mu-1)\left[\psi'(\mu-1) - \psi'(1)\right]\right]\frac{\Gamma(1-\mu)}{\Gamma(2\mu-1)} \\ \Sigma_{6\,011} &= -\frac{[8\mu^2 - 3\mu + 3]\Gamma(1-\mu)}{3(\mu-1)\Gamma(2\mu-1)} \\ \Sigma_{7\,000} &= \left[4(\mu-1)\left[\psi'(\mu-1) - \psi'(1)\right] - \frac{4\mu}{3(\mu-1)}\right]\frac{\Gamma(1-\mu)}{\Gamma(2\mu-1)} \\ \Sigma_{7\,110} &= \Sigma_{7\,101} = \left[\frac{4[2\mu^2 - 2\mu - 1]}{3(\mu-1)} - 4(\mu-1)^2\left[\psi'(\mu-1) - \psi'(1)\right]\right]\frac{\Gamma(1-\mu)}{\Gamma(2\mu-1)} \\ \Sigma_{7\,011} &= \left[4(\mu-1)(\mu-2)\left[\psi'(\mu-1) - \psi'(1)\right] - \frac{2[4\mu^2 - 2\mu - 9]}{3(\mu-1)}\right]\frac{\Gamma(1-\mu)}{\Gamma(2\mu-1)} \end{aligned} \tag{3.8}$$

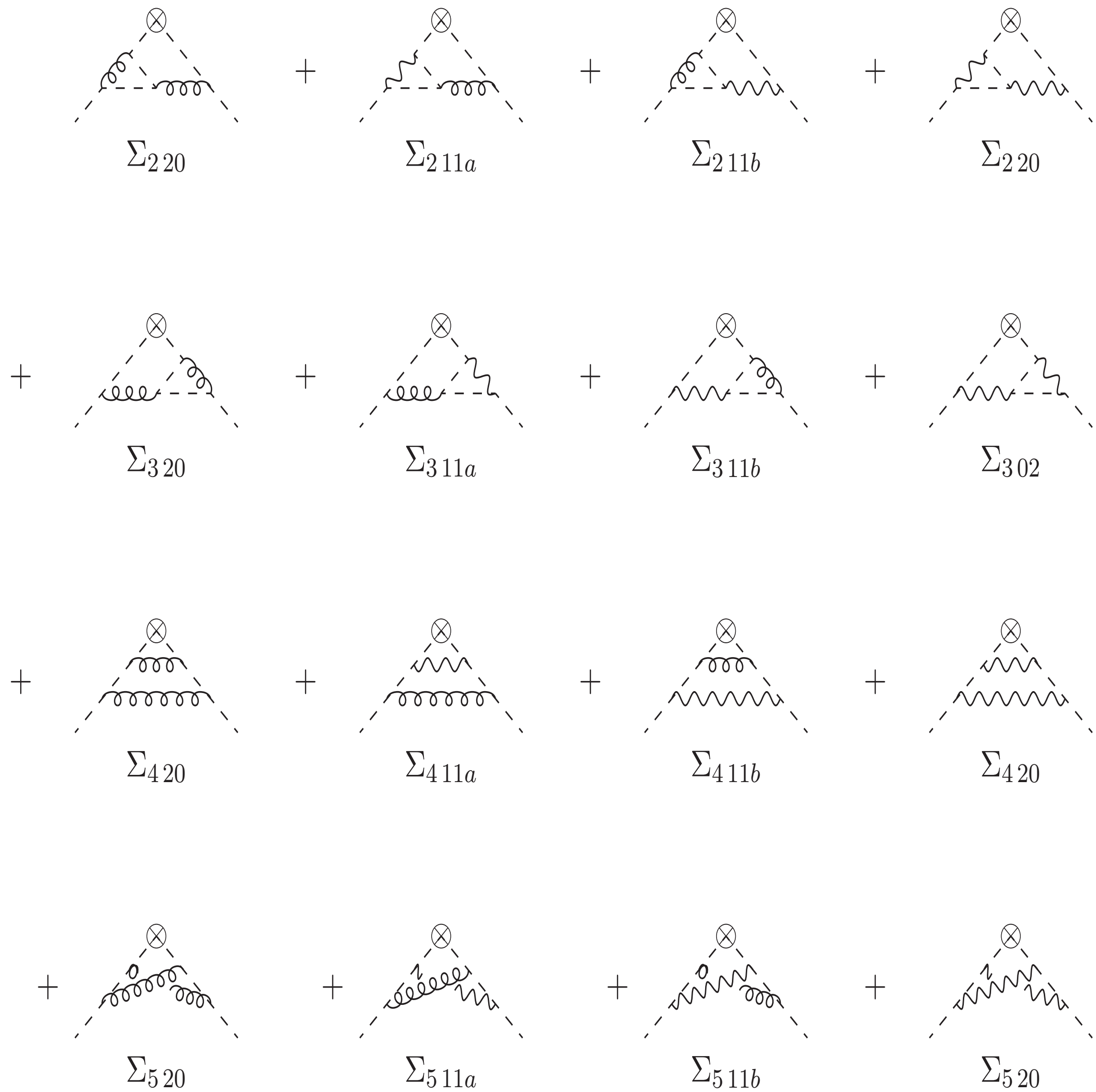

Figure 6: $O(1/N^2)$ graphs contributing to the critical exponent of $\mathcal{O}_{\mathrm{ns}}$.

in the Landau gauge. At leading order in $1/N$ since the set of graphs is the same for the extraction of the non-singlet operator exponent then

$$\eta_{\mathcal{O}_{m\,1}} ~=~ \eta_{\mathcal{O}_{\mathrm{ns}1}} ~. \tag{3.9}$$

A similar relation does not hold at next order due to the graphs of Figure 7 but these lead to

$$\begin{aligned} \eta_{\mathcal{O}_{m\,2}} ~=~ - & \left[\frac{[48\mu^6 - 208\mu^5 + 304\mu^4 - 186\mu^3 + 16\mu^2 + 23\mu - 4]}{\mu[\mu-1]} \right. \\ & +~ 4(2\mu-1)(\mu-1) \left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1) \right] \\ & \left. +~ 3[16\mu^3 - 20\mu^2 + 14\mu - 5](\mu-1) \left[\psi^\prime(\mu-1) - \psi^\prime(1) \right] \right] \hat{\eta}_1^2 \end{aligned} \tag{3.10}$$

which implies

$$\eta_2 ~+~ \eta_{\mathcal{O}_{m\,2}} ~=~ - \left[\frac{2[24\mu^7 - 112\mu^6 + 216\mu^5 - 259\mu^4 + 199\mu^3 - 100\mu^2 + 31\mu - 4]}{\mu^2[\mu-1]} \right.$$

$$+ \; \frac{4(2\mu-1)^2(\mu-1)}{\mu}\left[\psi(2\mu-1)-\psi(1)+\psi(1-\mu)-\psi(\mu-1)\right]$$
$$+ \; 6[4\mu^2+2\mu-3]\mu(\mu-1)\left[\psi'(\mu-1)-\psi'(1)\right]\Big]\,\hat{\eta}_1^2 \qquad (3.11)$$

where we note that $\chi_{\sigma\, i} = \eta_{\mathcal{O}_m\, i}$.

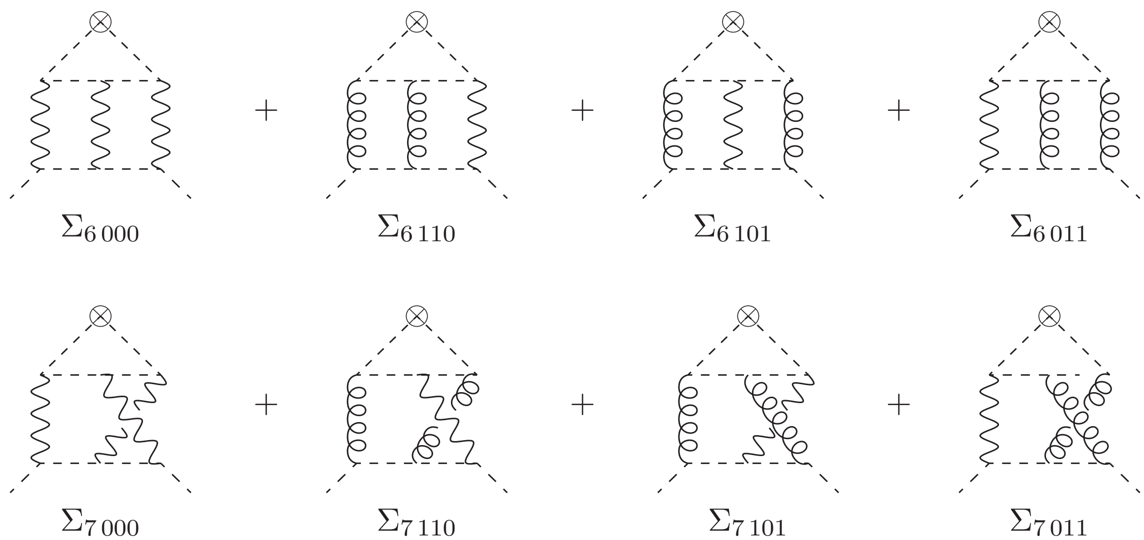

Figure 7: Additional $O(1/N^2)$ graphs for mass operator critical exponent.

4 Results.

This section is devoted to an analysis of the exponents at $O(1/N^2)$. As a first stage we note that the ϵ expansion of each of (3.7) and (3.11) near four dimensions is in total agreement with the recent three and four loop perturbative computations of [25, 26]. More explicitly we note

$$\begin{aligned}
\Delta_m|_{d=4-2\epsilon} \;=\; & 3 \;-\; 2\epsilon \\
& + \Big[-3\epsilon + 2\epsilon^2 + 3\epsilon^3 + [4-6\zeta_3]\,\epsilon^4 + [5-9\zeta_4+4\zeta_3]\,\epsilon^5 \\
& \quad + [6-18\zeta_5+6\zeta_4+6\zeta_3]\,\epsilon^6 \Big] \frac{1}{N} \\
& + \Big[\frac{9}{2}\epsilon - \frac{39}{4}\epsilon^2 + \left[\frac{413}{8} - 102\zeta_3\right]\epsilon^3 + \left[-\frac{1413}{16} - 153\zeta_4 + 297\zeta_3\right]\epsilon^4 \\
& \quad + \left[-\frac{1935}{32} - 204\zeta_5 + \frac{891}{2}\zeta_4 - 105\zeta_3\right]\epsilon^5 \\
& \quad + \left[-\frac{4017}{64} - 255\zeta_6 + 648\zeta_5 - \frac{315}{2}\zeta_4 + 162\zeta_3 - 408\zeta_3^2\right]\epsilon^6 \Big] \frac{1}{N^2} \\
& + O\left(\epsilon^7, \frac{1}{N^3}\right)
\end{aligned} \qquad (4.1)$$

for the mass operator where ζ_z is the Riemann zeta function and

$$\begin{aligned}
\Delta_{\bar{\psi}\sigma^z\psi}\Big|_{d=4-2\epsilon} \;=\; & 3 \;-\; 2\epsilon \\
& + \Big[-3\epsilon + 2\epsilon^2 + 3\epsilon^3 + [4-6\zeta_3]\,\epsilon^4 + [5+4\zeta_3-9\zeta_4]\,\epsilon^5
\end{aligned}$$

$$
\begin{aligned}
&+ \left[6 + 6\zeta_3 + 6\zeta_4 - 18\zeta_5\right]\epsilon^6\Big]\frac{1}{N} \\
&+ \Bigg[\frac{9}{2}\epsilon - \frac{15}{4}\epsilon^2 + \left[\frac{1}{8} - 27\zeta_3\right]\epsilon^3 - \left[\frac{341}{16} - 99\zeta_3 + \frac{81}{2}\zeta_4\right]\epsilon^4 \\
&\quad - \left[\frac{1103}{32} - 42\zeta_3 + \frac{297}{2}\zeta_4 - 54\zeta_5\right]\epsilon^5 \\
&\quad - \left[\frac{2161}{64} - 26\zeta_3 - 63\zeta_4 + 252\zeta_5 - \frac{135}{2}\zeta_6 - 108\zeta_3^2\right]\epsilon^6\Bigg]\frac{1}{N^2} \\
&+ O\left(\epsilon^7, \frac{1}{N^3}\right)
\end{aligned}
\tag{4.2}
$$

for the non-singlet case. In comparing both with [25, 26] allowance has to be made for the different conventions in defining ϵ. This is a highly non-trivial check for the $O(1/N^2)$ exponents and gives confidence that they are correct. Therefore we can derive expressions for the exponents in other dimensions. As an example the anomalous dimension of the non-singlet operator in two dimensions, for instance, is

$$
\begin{aligned}
\Delta_{\bar{\psi}\sigma^z\psi}\Big|_{d=2-2\epsilon} &= 1 - 2\epsilon \\
&+ \left[-\frac{1}{2} + \epsilon - \zeta_3\epsilon^3 + \left[2\zeta_3 - \frac{3}{2}\zeta_4\right]\epsilon^4 + \left[3\zeta_4 - 3\zeta_5\right]\epsilon^5 + \left[6\zeta_5 - 5\zeta_6 - \zeta_3^2\right]\epsilon^6\right]\frac{1}{N} \\
&+ \Bigg[-\frac{1}{8} + \frac{3}{8}\epsilon + \left[-\frac{5}{8} + \frac{3}{4}\zeta_3\right]\epsilon^2 + \left[\frac{7}{8} - \frac{9}{2}\zeta_3 + \frac{9}{8}\zeta_4\right]\epsilon^3 \\
&\quad + \left[\frac{3}{8} + \frac{15}{2}\zeta_3 - \frac{27}{4}\zeta_4 + \frac{3}{2}\zeta_5\right]\epsilon^4 \\
&\quad + \left[-\frac{1}{8} - \frac{7}{2}\zeta_3 + \frac{45}{4}\zeta_4 - \frac{23}{2}\zeta_5 + \frac{15}{8}\zeta_6 + 3\zeta_3^2\right]\epsilon^5 \\
&\quad + \left[-\frac{5}{8} + \frac{9}{4}\zeta_7 - \frac{35}{2}\zeta_6 + \frac{45}{2}\zeta_5 - \frac{21}{4}\zeta_4 + \frac{5}{2}\zeta_3 + 9\zeta_3\zeta_4 - 17\zeta_3^2\right]\epsilon^6\Bigg]\frac{1}{N^2} \\
&+ O\left(\epsilon^7, \frac{1}{N^3}\right) .
\end{aligned}
\tag{4.3}
$$

One novel feature of the $O(1/N)$ part of this particular exponent is the absence of rationals in the coefficients of ϵ beyond one loop or $O(\epsilon)$.

As one of the central motivations for determining the non-singlet exponent concerned three dimensions then (3.7) implies

$$
\Delta_{\bar{\psi}\sigma^z\psi}\Big|_{d=3} = 2 - \frac{8}{\pi^2 N} - \frac{16[9\pi^2 - 100]}{9\pi^4 N^2} + O\left(\frac{1}{N^3}\right)
\tag{4.4}
$$

or

$$
\Delta_{\bar{\psi}\sigma^z\psi}\Big|_{d=3} = 2 - \frac{0.810569}{N} + \frac{0.203925}{N^2} + O\left(\frac{1}{N^3}\right)
\tag{4.5}
$$

numerically. Since the focus on this operator when $N = 1$ concerns the possible duality connection with the $SU(2)$ symmetric non-compact CP^1 sigma model, setting this value in the three dimensional $O(1/N^2)$ exponent may not give a reliable estimate as it is likely to be outside the radius of convergence. What would be useful is to use resummation methods in order to see if the convergence can be improved as well as see to what extent any exponent estimate is comparable to those given in [25, 26]. While the four dimensional renormalization has produced a four loop operator dimension, extracting an estimate requires summing the ϵ expansion of the underlying critical exponent and setting $\epsilon = \frac{1}{2}$ in our ϵ conventions. Again this choice of ϵ may be near the radius of convergence for extracting an exponent estimate in three dimensions. Therefore we have used (4.5) to obtain Padé approximants as a function of N and evaluated them for various N. The numerical estimates are given in Table 1. Also included there are Padé-Borel estimates which are constructed by writing the series (4.5) as a Borel integral and then applying a Padé approximant to the integrand. The resulting integral is evaluated numerically and the results displayed in the same Table. In both the Padé and Padé-Borel cases there are no singularities either in N or the Borel integration parameter. Also included in the final column are the four loop perturbative estimates from [26]. What is evident is the close agreement of both sets of $O(1/N^2)$ estimates with four loop perturbation theory central value down to $N = 3$. This is a relatively small value of N for which large N results are similar to perturbative estimates compared to other exponents in other universality classes. Where there is a clear difference is for $N = 1$. For the large N case the exponents decrease in value as N decreases purely due to the negative sign of the $O(1/N)$ term of (3.7). The perturbative results of [26] do not decrease monotonically since the $N = 1$ estimate is larger than all the others indicated. This may be due to the alternating behaviour of the ϵ series Padé approximant with loop order and so a region where the approximant converges may not have been reached. The exponent estimate given in [25] for $N = 1$ is 2.12(50) which has a central value larger, and specifically above 2, than any of those given in Table 1. Although the Table 1 values all lie within the error quoted in [25] a value of 2.33(1) for the same exponent in the dual theory in given in [25]. In this case the $N = 1$ values of [26] and this article lie well outside its error. From the large N series (3.7) in order to obtain an overall value of the exponent above 2 would require a sizeable positive correction from the higher order terms. Overall this would suggest that before the duality can be explored more fully then higher order terms, either in large N or in four dimensional perturbation theory, should be computed. Neither of these tasks is a trivial exercise. Alternatively it may be the case that the results indicate that there is no duality.

N	$[0,1]$ P	$[1,1]$ P	$[0,2]$ P	$[1,1]$ PB	$[0,2]$ PB	[26]
1	1.423199	1.352364	1.362789	1.340835	1.418689	1.98 ± 0.08
2	1.663005	1.640000	1.641745	1.637938	1.656835	1.74 ± 0.06
3	1.761967	1.750715	1.751288	1.750017	1.757367	1.76 ± 0.05
4	1.816001	1.809349	1.809603	1.809035	1.812634	1.81 ± 0.04
5	1.850041	1.845652	1.845787	1.845491	1.847514	1.84 ± 0.03
6	1.873453	1.870342	1.870421	1.870257	1.871506	1.86 ± 0.02
10	1.922100	1.920932	1.920950	1.921027	1.921335	1.917 ± 0.007

Table 1. Estimates for $\Delta_{\bar{\psi}\sigma^z\psi}\Big|_{d=3}$ using Padé (P) and Padé-Borel (PB) approximants with comparison to [26].

Finally for the mass operator in the QED-Gross-Neveu universality class in three dimensions we note that

$$\Delta_m\big|_{d=3} ~=~ 2 ~-~ \frac{8}{\pi^2 N} ~+~ \frac{[2464 - 486\pi^2]}{9\pi^4 N^2} ~+~ O\left(\frac{1}{N^3}\right) \tag{4.6}$$

or

$$\Delta_m\big|_{d=3} ~=~ 2 ~-~ \frac{0.810569}{N} ~-~ \frac{2.660746}{N^2} ~+~ O\left(\frac{1}{N^3}\right) \tag{4.7}$$

numerically. If we follow the prescription indicated in [40] for extracting an estimate of N for which chiral symmetry breaking occurs, which we denote by N_c, we find that $N_c = 3.24$ at leading order but $N_c = 4.88$ with the $O(1/N^2)$ correction included. This should be compared with the respective values of 4.32 and 2.85, [39, 40], for the pure QED case.

5 Gross-Neveu universality class.

In this section we make a brief side step and focus on one of the constituent theories within (2.3) which is the pure Gross-Neveu model and corresponds to omitting terms involving A_μ. While the core critical exponents η and χ as well as the fermion mass dimension have been already computed to $O(1/N^3)$ and $O(1/N^2)$ respectively in [34, 43, 46, 47, 48, 49, 50] that for the non-singlet fermion bilinear operator has not been recorded. We do so now as a simple corollary of the formalism of the previous section. For completeness we first record the relevant exponents and amplitudes to the requisite orders needed to achieve this. While several of these have been recorded already we repeat them here but with the same spinor trace and flavour symmetry group conventions of this article. In previous work, for instance, two dimensional γ-matrices were used in the $O(N)$ theory, [34]. First, the exponents determining the dimensions of the fields are

$$\eta_1^{\rm GN} ~=~ -~ \frac{(\mu-1)\Gamma(2\mu-1)}{2\mu\Gamma^3(\mu)\Gamma(1-\mu)} \tag{5.1}$$

and

$$\chi_{\sigma\,1}^{\rm GN} ~=~ \frac{\mu}{(\mu-1)} \eta_1^{\rm GN} \tag{5.2}$$

where we append GN to indicate the pure Gross-Neveu universality class. Consequently the amplitude to $O(1/N^2)$ is determined from

$$\begin{aligned} \tilde{z}_1^{\rm GN} &~=~ \frac{\Gamma(2\mu-1)}{4\Gamma^2(\mu)\Gamma(1-\mu)} \\ \tilde{z}_2^{\rm GN} &~=~ -~ \left[\frac{\mu(2\mu-1)}{2(\mu-1)^2} [\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1)] \right. \\ & \qquad \left. -~ \frac{1}{(\mu-1)} \right] \frac{\mu(2\mu-1)\Gamma(\mu)}{2(\mu-1)^2} \left(\eta_1^{\rm GN} \right)^2 \end{aligned} \tag{5.3}$$

in momentum space which produces

$$\eta_2^{\rm GN} ~=~ \frac{(2\mu-1)}{(\mu-1)} \left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1) ~-~ \frac{1}{2\mu(\mu-1)} \right] \left(\eta_1^{\rm GN} \right)^2 . \tag{5.4}$$

Equipped with these basic building blocks for the large N expansion the non-singlet bilinear operator dimension is deduced from the computation of the same quantity in the QED-Gross-Neveu class by formally setting $\tilde{y} = 0$ at the outset. For example, only the graphs in Figure 5 and 6 corresponding to $\Gamma_{i\,0n}$ will contribute. At leading order we have

$$\eta_{\mathcal{O}_{\rm ns}1}^{\rm GN} ~=~ \frac{\mu}{(\mu-1)} \eta_1^{\rm GN} \tag{5.5}$$

while the next order produces

$$\eta_{\mathcal{O}_{\rm ns}2}^{\rm GN} ~=~ \frac{\mu(2\mu-1)}{(\mu-1)^2} \left[\psi(2\mu-1) - \psi(1) + \psi(1-\mu) - \psi(\mu-1) ~-~ \frac{1}{(\mu-1)} \right] \left(\eta_1^{\rm GN} \right)^2 . \tag{5.6}$$

The ϵ expansion of $\Delta_{\bar{\psi}\sigma^z\psi}$ near four dimensions is in full agreement with the corresponding three and four loop results of [25, 26] when $\bar{g}_2 = 0$ is set in the operator anomalous dimension. To assist with comparison for independent computations we note that

$$\begin{aligned} \Delta_{\bar{\psi}\sigma^z\psi}^{\rm GN} \Big|_{d=4-2\epsilon} &~=~ 3 ~-~ 2\epsilon \\ & \quad + \left[\frac{3}{2}\epsilon - \frac{7}{4}\epsilon^2 - \frac{11}{8}\epsilon^3 + \left[3\zeta_3 - \frac{19}{16} \right] \epsilon^4 + \left[\frac{9}{2}\zeta_4 - \frac{7}{2}\zeta_3 - \frac{35}{32} \right] \epsilon^5 \end{aligned}$$

$$+\left[9\zeta_5-\frac{11}{4}\zeta_3-\frac{21}{4}\zeta_4-\frac{67}{64}\right]\epsilon^6\right]\frac{1}{N}$$

$$+\left[-\frac{9}{4}\epsilon+\frac{147}{16}\epsilon^2-\frac{71}{32}\epsilon^3-\left[\frac{53}{8}+18\zeta_3\right]\epsilon^4-\left[\frac{65}{8}-\frac{231}{4}\zeta_3+27\zeta_4\right]\epsilon^5\right.$$

$$\left.-\left[\frac{2127}{256}+\frac{37}{8}\zeta_3-\frac{693}{8}\zeta_4+63\zeta_5\right]\epsilon^6\right]\frac{1}{N^2}\;+\;O\left(\epsilon^7,\frac{1}{N^3}\right) \quad (5.7)$$

near four dimensions while

$$\Delta^{\mathrm{GN}}_{\bar{\psi}\sigma^z\psi}\Big|_{d=2-2\epsilon} \;=\; 1\;-\;2\epsilon$$

$$+\left[-\frac{1}{2}\epsilon+\frac{1}{2}\epsilon^2+\frac{1}{2}\epsilon^3+\left[\frac{1}{2}-\zeta_3\right]\epsilon^4+\left[\frac{1}{2}-\frac{3}{2}\zeta_4+\zeta_3\right]\epsilon^5\right.$$

$$\left.+\left[\frac{1}{2}-3\zeta_5+\frac{3}{2}\zeta_4+\zeta_3\right]\epsilon^6\right]\frac{1}{N}$$

$$+\left[-\frac{1}{4}\epsilon+\frac{5}{8}\epsilon^2+\frac{1}{8}\epsilon^3-\left[\frac{1}{4}+2\zeta_3\right]\epsilon^4+\left[\frac{9}{2}\zeta_3-\frac{1}{2}-3\zeta_4\right]\epsilon^5\right.$$

$$\left.+\left[-\frac{5}{8}-7\zeta_5+\frac{27}{4}\zeta_4+\frac{3}{2}\zeta_3\right]\epsilon^6\right]\frac{1}{N^2}\;+\;O\left(\epsilon^7,\frac{1}{N^3}\right) \quad (5.8)$$

in the Gross-Neveu model of [8]. In three dimensions the effect of the presence or absence of the photon field can be gauged from

$$\Delta^{\mathrm{GN}}_{\bar{\psi}\sigma^z\psi}\Big|_{d=3} \;=\; 2\;+\;\frac{8}{3\pi^2N}\;+\;\frac{256}{27\pi^4N^2}\;+\;O\left(\frac{1}{N^3}\right) \quad (5.9)$$

or

$$\Delta^{\mathrm{GN}}_{\bar{\psi}\sigma^z\psi}\Big|_{d=3} \;=\; 2\;+\;\frac{0.270190}{N}\;+\;\frac{0.097337}{N^2}\;+\;O\left(\frac{1}{N^3}\right) \quad (5.10)$$

numerically. Comparing with (4.5) the first order correction is positive in contrast with (5.10). This contrasting behaviour has been quantified more concretely in Table 2 where we provide Padé estimates for the same values of N as in Table 1. However we have not recorded estimates using the Padé-Borel method as the positive sign in the leading order correction in (5.10) produces singularities in the Padé approximant of the integrand. Although we have no perturbative estimates with which to compare it appears there is reasonable agreement of the two $O(1/N^2)$ approximants down to $N=3$. While this is the same value as Table 1 a relatively low value of N for the range of validity is not unreasonable in this instance due to the relatively small $O(1/N^2)$ correction. What is interesting is that the same value should arise in the QED-Gross-Neveu case as the analogous correction is roughly three times larger. This is the main observation of this exercise which was to ascertain for how low a value of N one could garner reliable estimates from several orders in $1/N$ for this exponent.

6 Discussion.

We have completed the $O(1/N^2)$ determination of both flavour non-singlet and singlet fermion bilinear operator critical exponents at the Wilson-Fisher fixed point in the QED-Gross-Neveu universality class. Since the exponents were evaluated in d-dimensions they bridge between several theories in this class including the four dimensional QED-Gross-Neveu-Yukawa theory used in [25, 26]. As such the ϵ-expansion of both exponents provided non-trivial independent checks on the three and four loop of the operator anomalous dimension of [25, 26]. Once established to be consistent with perturbation theory we have provided numerical estimates for the three dimensional exponents. This was primarily to inform the debate on the potential

N	$[0,1]$	$[1,1]$	$[0,2]$
1	2.312392	2.422339	2.396681
2	2.144881	2.164775	2.162517
3	2.094310	2.102354	2.101749
4	2.069908	2.074233	2.073989
5	2.055539	2.058234	2.058112
6	2.046069	2.047908	2.047839
10	2.027389	2.028029	2.028014

Table 2. Estimates for $\Delta^{\mathrm{GN}}_{\bar{\psi}\sigma^z\psi}\Big|_{d=3}$ using Padé approximants for the pure Gross-Neveu universality class.

duality connection of the $N = 1$ theory with the $SU(2)$-symmetric CP^1 sigma model where the current focus is on the flavour non-singlet case. As it stands both from the perturbative and large N results it would appear that for that operator dimension more analysis needs to be carried out. From the side of the QED-Gross-Neveu universality class one way of improving the $N = 1$ estimate would be to repeat the approach of [15]. There all known data from large N and ϵ-expansions, for instance, were combined into improved matched Padé approximants up to four loops. This provided an interpolating approximation to the critical exponent across the dimensions from two to four using the perturbative renormalization group information from the critical theories in these dimensions. The form of this function for the exponents considered in [15] was not dissimilar to that obtained by functional renormalization group methods. Therefore what would be useful for the QED-Gross-Neveu universality class is the determination of the corresponding renormalization group functions in two dimensions. Although the β-functions are available at two loops, [51, 52], the field and mass anomalous dimensions remain to be determined at this and higher loop order. In addition to this operator we have provided the fermion mass dimension at $O(1/N^2)$ which also determines χ_σ as a corollary. This mass dimension is important in analyses concerning chiral symmetry breaking in the QED-Gross-Neveu universality class in three dimensions and we have provided an initial examination of this at $O(1/N^2)$. Further our mass exponent at $O(1/N^2)$ should also be useful in supplementing Monte Carlo studies of related QED-like theories considered in [53], for example, where a Lagrangian similar to (2.4) is relevant for a confinement transition.

Acknowledgements. The author thanks Prof V.P. Gusynin, Dr L. Janssen, Prof J. Maciejko, Dr M. Scherer and Prof S. Teber for valuable discussions. All the authors of [25, 26] are also thanked for sharing their results prior to publication which assisted checks of the exponents. In particular we are grateful to Prof J. Maciejko for providing the values of $\Delta_{\bar{\psi}\sigma^z\psi}\Big|_{d=3}$ for $N \geq 3$ quoted in Table 1 which were determined within the analysis of [26] and for permission to include them. The graphs were drawn with the AXODRAW package [54]. The work was carried out with the support of a DFG Mercator Fellowship.

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