Risk Lovers, Mixed Risk Loving And The Preference To Combine Good With Good

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Abstract

This paper examines the concept of 'risk loving' (that is risk seeking, intemperance, edginess, etc.), which can be characterized by preferences over simple lotteries. This paper analyzes the notion of preferring to combine good with good, and bad with bad, as opposed to combining good with bad as usual. The significance of such preferences have implications on utility functions and are analyzed in the paper. This paper extends Eeckhoudt and Schlesinger (2006) results to risk lovers, the results from Crainich, Eeckhoudt and Trannoy (2012) are also generalized to higher orders. We also generalize to higher orders the concept of bivariate risk seeking, introduced by Richard (1975) and called correlation loving by Epstein and Tanny (1980). In the expected utility framework, risk loving of order (N, M) coincides with the non-negativity of the $(N, M)^{th}$ partial derivative of the utility function. In dealing with mixed risk loving utility functions, we give several useful properties, for example, mixed risk loving is consistent with the mixture of positive exponential utilities and with non-increasing coefficients of absolute risk aversion at any order.

Keywords: bivariate utility function, correlation loving, cross-prudence, intemperance, mixed risk loving, risk seeking.

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1. Introduction

Several studies have demonstrated that if the utility of the individual has one argument, its successive derivatives are brought to alternate in sign, the first being positive. Recently, the analysis of the behavior of utility functions with several variables (generally two) made it possible to characterise the behavior with respect to the risk of the risk-averse individuals, by alternating the signs of the partial derivatives with the first cross partial derivative being negative. Very little attention was given to attitude with respect to the risk of the individuals attracted to risk, yet the presence of this is necessary in the financial markets and it is especially important that their existence is proven. It is thus relevant to be interested in a whole fringe of the population which was put aside, namely the risk-loving individuals.

The first steps in this direction, as with the case of utility functions with one variable and within the framework of the expected utility, indicate that the behavior of risk loving individuals with respect to risk would be compatible with the preference for the combination of good with good, and bad with bad. The importance of the sign of the derivatives of the univariate utility function and the partial derivatives of the bivariate utility function is well known. With respect to the univariate framework, several papers document the alternation of the signs of the successive derivatives, usually at least to the fourth order.

The concepts of risk aversion, prudence, temperance and edginess are related to the first four derivatives (Deck and Schlesinger (2014)). They have several applications in economics, finance and health, amongst other domains. It is well recognized that the signs of the successive derivatives of the utility function play an important role in decision making. Eeckhoudt and Schlesinger (2006) present a context-free interpretation of these signs in the univariate framework. To do so, they introduced the concept of risk apportionment, which is equivalent to the preference for combining good with bad. Jokung (2011) extends their result to the bivariate case and these two papers rely on the preference for combining good with bad, which corresponds to risk apportionment and the decision maker is always supposed to be risk averse.

In the case that the decision maker is a risk lover, this is considered by Crainich, Eeckhoudt and Trannoy (2012), who consider a risk loving individual and the preference for combining good with good. They restrict their analysis to the expected utility framework and to the first four derivatives of the univariate utility function; they find out the non-negativity of the first four successive derivatives of the utility function. They are the first, to our knowledge, to consider risk lovers instead of risk averters. In this study we generalize their results to higher orders and to the non-expected utility framework.

Caballé and Pomansky (1996) introduce a mixed risk averse utility function related to a completely monotonic function. An individual is said to be mixed risk averse if all the successive derivatives of his or her utility function alternate in sign, the first one being positive. This is equivalent to the decrease of all the coefficients of absolute risk aversion at any order (absolute risk aversion, absolute prudence, absolute temperance, absolute edginess, etc.). In dealing with risk lovers and assuming a preference for combining good with good, and bad with bad, we end up with non-negative successive derivatives of the utility function.

As far as we are concerned with risk lovers, we consider the class of absolutely monotonic functions that define mixed risk loving utility functions. An individual is said to be mixed risk loving if he or she is risk loving at any order. We define the concept of univariate risk loving with preference over simple lotteries. Multivariate risk (see Ebert and Van de Kuilen (2015)) and multivariate risk seeking was first defined by Richard (1975) as a choice between two lotteries. The risk-seeking individual prefers taking a chance on all the ‘good’ or all the ‘bad’, rather than having some of the ‘good’ and some of the ‘bad’.

In this paper, we introduce the concept of bivariate risk loving which behaves similarly to bivariate risk seeking. Our concept allows us to characterize bivariate risk loving of higher orders by preferences among bivariate lotteries in the spirit of Eeckhoudt and Schlesinger (2006) in the univariate approach, and in the spirit of Jokung (2011) in the bivariate case. Risk loving of order (N, M) encompasses univariate risk loving and in the univariate framework, risk loving indicates a preference to pool risks in the same states of nature. In the bivariate framework, risk loving will combine this property and the implications of preferring the bivariate lottery with two ‘good’ and two ‘bad’ attributes to the bivariate lottery with one ‘bad’ attribute and one ‘good’ attribute. We connect risk loving of order (N, M) with the non-negativity of the partial derivative of order (N, M) of the bivariate utility function and (univariate) risk loving of order N with the non-negativity of the derivative of order N of the univariate utility function. We refer to an individual that is risk loving at any order as a mixed risk loving individual. We show that his or her utility function is a mixture of positive exponential utilities and we give some important properties in the univariate case. For example, the coefficients of absolute risk aversion of any order (absolute risk aversion, prudence, temperance, edginess,…) are non-increasing.

Our paper contributes to the research literature in several ways. Firstly, we generalize the results from Crainich, Eeckhoudt and Trannoy (2012) to higher orders and to a non-expected utility framework, thereby extending Eeckhoudt and Schlesinger (2006) to risk lovers. Secondly, we characterize concepts such as risk-seeking, prudence, intemperance and edginess with a preference relation over simple lotteries. Thirdly, we define the concept of risk loving by preferences over lotteries. In the expected utility framework, the direction of preference for our lotteries is equivalent to signing the derivatives of the univariate utility function. Fourth, we generalize our results to bivariate utility functions. Fifth, preference for combining good with good at any order leads to the concept of mixed risk loving. We relate mixed risk loving with absolutely monotonic functions and with a mixture of non-negative exponentials. We show that an individual is mixed risk loving (he or she prefers to combine good with good at any level) if and only if all his or her coefficients of absolute risk aversion of any order are non-increasing.

Decision making in the face of the uncertainty, combined with varying levels of risk preference, have many applications in applied management science areas. For example, Subulan et al. (2015) take into account financial risks in manufacturing applications. In Crainich et al. (2017) they examine how health impacts the portfolio choice of risky assets, and the relation to differing risk preferences. In Outreville (2013) the relation between risk preference and the impact on financial and insurance products is investigated, as well as the relation to socioeconomic factors. Consequently, the analysis of risk preferences and decision under uncertainty will be beneficial to many areas of applied management science.

The paper is organized as follows: section 2 introduces the utility implications of preferring to combine good lotteries with good ones and bad lotteries with bad ones. In the next section we present the concept of risk loving and extend the result from Eeckhoudt and Schlesinger (2006) to risk lovers, on the one hand, and on the other hand this section generalizes to higher orders the results from Crainich, Eeckhoudt and Trannoy (2012). This section also extends the bivariate framework concept of risk loving and recovers the concepts of correlation loving, prudence and intemperance. In section 4 we examine the mixed risk loving individual (univariate or bivariate). Finally, we end with a conclusion.

2. Univariate And Bivariate Models: Preference For Combining 'Good' With 'Good' And 'Bad' With 'Bad'

In this section, we discuss the notion of preferring to combine 'good' with 'good' and 'bad' with 'bad', and its implications on the utility function and the behavior toward risk. We consider two cases: univariate and bivariate models.

2.1 Univariate Risk Model

In this sub-section we define the univariate risk framework, risk seeking, prudence and intemperance. When we consider the expected utility framework, we deal with univariate utility functions. We denote the univariate utility function as v, and the utility function should be increasing.

Risk seeking

Assume we have a binary lottery with equally likely outcomes $[x; y]$, such that $x < y$. Here y is a good outcome and x is bad outcome. If we now combine this lottery with $-k$ (a sure loss) and 0 (no loss) by adding them to the outcomes, then this will give the following lotteries:

- $[x - k; y + 0]$, or
- $[x + 0; y - k]$.

In the first lottery, we combine good (good outcome) with good (no loss) and bad (bad outcome) with bad (the sure loss), whereas in the second lottery we combine good (good outcome and no loss respectively) with bad (the loss and the bad outcome respectively). The individual who prefers to combine good with good and bad with bad will prefer the first lottery. That is:

$$[x; y-k] \preccurlyeq [x-k; y].$$

Hence this individual prefers to face a sure reduction in wealth with the bad outcome and nothing with the good outcome. In the expected utility framework, let v be the univariate utility function, the last condition is equivalent to:

$$v(x) + v(y-k) \leq v(x-k) + v(y), \forall x < y \Leftrightarrow v(x) - v(x-k) \leq v(y) - v(y-k), \forall x < y .$$

This means that

$$\varphi(z) = v(z) - v(z-k),$$

is an increasing function of z, or equivalently

$$v'(z) \geq v'(z-k).$$

This last condition means that v' increases, that is $v'' \geq 0$, which corresponds to the notion of risk seeking.

Prudence

Consider a random variable $\tilde{\epsilon}$ such that $E(\tilde{\epsilon}) = 0$. We know that for risk lovers $\tilde{\epsilon}$ is better than 0 due to risk seeking, so $Ev(x+\tilde{\epsilon}) \geq v(x)$ meaning that $\tilde{\epsilon}$ is good and 0 is bad. Let us now combine $[x; y]$ with $\tilde{\epsilon}$ (a zero-mean random variable) and 0 (no loss). Now, if the individual prefers to combine good with good and bad with bad, we must have

$$[x+\tilde{\epsilon}; y+0] \preccurlyeq [x+0; y+\tilde{\epsilon}].$$

In the expected utility framework, this condition is equivalent to:

$$v(y) + Ev(x+\tilde{\epsilon}) \leq v(x) + Ev(y+\tilde{\epsilon}), \forall x < y,$$

$$\Leftrightarrow Ev(x+\tilde{\epsilon}) - v(x) \leq Ev(y+\tilde{\epsilon}) - v(y), \forall x < y.$$

This means that $\gamma(z) = Ev(z + \tilde{\epsilon}) - v(z)$ is an increasing function of z, that is,

$$Ev'(z + \tilde{\epsilon}) \geq v'(z).$$

Also, with Jensen's inequality the marginal utility is convex, that is $v''' \geq 0$. We note in passing that the notion of prudence was introduced by Kimball (1990).

Intemperance

Consider two independent zero-mean random variables $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$. In the lottery $[x; x + \tilde{\epsilon}_1]$, we can combine it with $\tilde{\epsilon}_2$ (good) and 0 (bad). If the individual prefers to combine good with good and bad with bad then, we must have

$$[x + \tilde{\epsilon}_2; (x + \tilde{\epsilon}_1) + 0] \preccurlyeq [x + 0; (x + \tilde{\epsilon}_1) + \tilde{\epsilon}_2].$$

In the expected utility framework, the condition is equivalent to :

$$Ev(x + \tilde{\epsilon}_2) + Ev(x + \tilde{\epsilon}_1) \leq v(x) + Ev(x + \tilde{\epsilon}_1 + \tilde{\epsilon}_2)$$

$$\Leftrightarrow Ev(x + \tilde{\epsilon}_2) - v(x) \leq Ev(x + \tilde{\epsilon}_1 + \tilde{\epsilon}_2) - Ev(x + \tilde{\epsilon}_1)$$

$$\Leftrightarrow \delta(x) \leq E\delta(x + \tilde{\epsilon}_1),$$

where $\delta(x) = Ev(x + \tilde{\epsilon}_2) - v(x)$.

The last condition means that δ is convex, that is

$$Ev''(x + \tilde{\epsilon}_2) \geq v''(x).$$

This in turn means that v'' is convex, that is $(v'')'' = v^{(4)} \geq 0$. This is intemperance, the opposite of the feeling of temperance introduced by Kimball (1993). If we continued deriving further, we would find that a preference for combining good with good and bad with bad is consistent with $v^{(5)} \geq 0$, which is called edginess and presented by Lajeri-Chaherli (2004).

Our recursive approach shows that a preference for combining good with good and bad with bad is consistent with the non-negativity of the signs of the successive derivatives of the utility function. We recall that Eeckhoudt and Shlesinger (2006) show that a preference for combining good with bad is consistent with alternating signs for successive derivatives of the utility function.

2.2 Bivariate Risk Model

We now consider the bivariate case and introduce multivariate risk seeking (or correlation loving), cross prudence and intemperance. In the expected utility framework, we deal with the bivariate utility function $u(x, y)$. For example, the first attribute could be wealth and the second one could be health. In all that follows, we assume that more is preferred to less on each attribute.

Multivariate Risk Seeking Or Correlation Loving

Let us start with the couple (x, y) and assume that we can combine the first attribute of the utility function, namely x, with $-k$ (a sure loss) and 0 (no loss); and the second attribute, namely y, with $-c$ (a sure loss) and 0 (no loss). In other words, we can add to (x, y) one of the four following combinations:

- $(0,0)$: good with the first attribute and good with the second attribute;
- $(-k, 0)$: bad with the first attribute and good with the second attribute;
- $(0, -c)$: good with the first attribute and bad with the second attribute;
- $(-k, -c)$: bad with the first attribute and bad with the second attribute.

A preference for combining good with good and bad with bad will mean that the lottery

$$[(x, y) + (-k, -c); (x, y) + (0,0))],$$

is preferred to the lottery

$$[(x, y) + (-k, 0); (x, y) + (0, -c)], \forall x, y.$$

Or equivalently the lottery $[(x, y); (x - k, y - c)]$ is preferred to the lottery $[(x, y - c); (x - k, y)], \forall x, y$. This is exactly the definition of multivariate risk seeking given by Richard (1975), Epstein and Tanny (1980) also named this as correlation loving. If we express the former preference in terms of the utility function, we have:

$$u(x, y - c) + u(x - k, y) \leq u(x, y) + u(x - k, y - c), \forall x, y, k.$$

This inequality corresponds to the definition of multivariate risk seeking in the expected utility framework.

Cross-prudence

Let us consider now a zero-mean random variable $\tilde{\epsilon}$. Let us combine the first attribute of the utility function with $\tilde{\epsilon}$ and 0 (no loss); and the second attribute with $-c$ (a sure loss) and 0 (no loss). In other words, we add to (x, y) one of the four following combinations:

- $(\tilde{\epsilon}, 0)$: good with the first attribute and good with the second attribute;
- $(0,0)$: bad with the first attribute and good with the second attribute;
- $(\tilde{\epsilon}, -c)$: good with the first attribute and bad with the second attribute;
- $(0, -c)$: bad with the first attribute and bad with the second attribute.

A preference for combining good with good and bad with bad means that the lottery

$$[(x, y) + (\tilde{\epsilon}, 0); (x, y) + (0, -c)],$$

is preferred to the lottery

$$[(x, y) + (\tilde{\epsilon}, -c); (x, y) + (0,0)], \forall x, y.$$

Or equivalently the lottery

$$[(x + \tilde{\epsilon}, y); (x, y - c)],$$

is preferred to the lottery

$$[(x + \tilde{\epsilon}, y - c); (x, y)], \forall x, y.$$

This is exactly the definition of cross-prudence with respect to the first attribute of the utility function given by Eeckhoudt, Rey and Schlesinger (2007). If one combines the first with $-c$ (a sure loss) and 0; and the second attribute with $\tilde{\epsilon}$ and 0; you end up with cross-prudence with respect to the second attribute of the utility function.

Intemperance

We now consider two independent zero-mean random variables $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ such that $\tilde{\epsilon}_1$ is concerned with the first attribute and $\tilde{\epsilon}_2$ is concerned with the second attribute of the utility function. Let us combine the first attribute with $\tilde{\epsilon}_1$ and 0; and the second attribute with $\tilde{\epsilon}_2$ and 0. In other words, we add to (x, y) one of the four following combinations:

- $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$: good with the first attribute and good with the second attribute;
- $(0, \tilde{\epsilon}_2)$: bad with the first attribute and good with the second attribute;
- $(\tilde{\epsilon}_1, 0)$: good with the first attribute and bad with the second attribute;
- $(0,0)$: bad with the first attribute and bad with the second attribute.

A preference for combining good with good and bad with bad implies that the lottery

$$[(x,y) + (\tilde{\epsilon}_1, \tilde{\epsilon}_2) ; (x,y) + (0,0)],$$

is preferred to the lottery

$$[(x,y) + (\tilde{\epsilon}_1, 0); (x,y) + (0, \tilde{\epsilon}_2)], \forall x, y.$$

Or equivalently the lottery

$$[(x + \tilde{\epsilon}_1 \;, y + \tilde{\epsilon}_2); (x,y)]$$

is preferred to the lottery

$$[(x + \tilde{\epsilon}_1, y); (x, y + \tilde{\epsilon}_2)], \forall x, y.$$

This is the opposite of the definition of temperance given by Eeckhoudt, Rey and Schlesinger (2007), that is intemperance and the individual is intemperate. The two risks are mutually ameliorating for the individual who prefers to combine good with good, and bad with bad.

3. Univariate And Bivariate Risk Loving

In this section we examine the risk loving theory within the univariate and bivariate models. We first consider the univariate framework and we generalize the result from Crainich, Eeckhoudt and Trannoy (2012) by presenting a unified concept of risk loving. In the bivariate framework, the tradeoff between the attributes is no longer obvious. Also, the attitude towards risk in the bivariate framework is not obtained in a straightforward manner from that of the univariate framework. The interaction between the two attributes must be taken into account. We extend to the bivariate framework the concept of risk loving such that the univariate case is a special case of the bivariate one.

3.1 *Univariate Risk Loving Model*

In this section, we extend the result from Eeckhoudt and Schlesinger (2006) concerning risk averse individuals to risk lovers. The class of simple lotteries such that the direction of preference between these lotteries is equivalent to signing the N^{th} derivative of the utility function. Finally, we show that risk seeking, prudence, intemperance and edginess are special cases of risk loving.

Let $\tilde{z}$ and $\tilde{t}$ denote two possibly degenerate independent random variables; $[\tilde{z}\,;\,\tilde{t}\,]$ denotes a lottery that takes the values $\tilde{z}$ and $\tilde{t}$ with the same probabilities; $[\tilde{z}]$ corresponds to the lottery that takes the value $\tilde{z}$ with certainty. Consider the following sequence of lotteries:

$$A_1 = [-k],\, B_1 = [0];$$

$$A_2 = [0],\, B_2 = [\tilde{\epsilon}_1];$$

$$A_N = \left[B_{N-2}; A_{N-2} + \tilde{\epsilon}_{Int(\frac{N}{2})}\right],\, B_N = \left[A_{N-2}; B_{N-2} + \tilde{\epsilon}_{Int(\frac{N}{2})}\right],\ \forall N \geq 3.$$

The random variables $\tilde{\epsilon}_i$ are assumed to be independent and they are defined in the same way as $\tilde{\epsilon}_1$. In what follows, all the non-degenerate lotteries are 50-50 lotteries. This sequence of lotteries is similar to the one used by Eeckhoudt and Schlesinger (2006), except that A_2 and B_2 play the opposite roles. Let us now define the concept of risk loving of different orders by assuming that the individual prefers to combine good with good, and bad with bad.

3.1.1 *Risk Loving Of Order 1*

Definition 1: *An individual is risk loving of order 1 if the lottery* $B_1 = [0]$ *is preferred to the lottery* $A_1 = [-k]$, *for all initial wealth levels and for all k.*

In this definition, $B_1 = [0]$ is better than $A_1 = [-k]$, in other words more is preferred to less. In the expected utility framework, this preference corresponds to the monotonicity of the utility function.

3.1.2 *Risk Loving Of Order 2*

Definition 2: *An individual is said to be risk loving of order 2 if the lottery* $B_2 = [\tilde{\epsilon}_1]$ *is preferred to the lottery* $A_2 = [0]$, *for all initial wealth levels and for all zero-mean random variable* $\tilde{\epsilon}_1$.

In this definition the fact that $B_2 = [\tilde{\epsilon}_1]$ is preferred to $A_2 = [0]$ means that the individual is risk seeking. The risk loving of order 2 corresponds to risk seeking, and in the expected utility framework, the utility function is convex.

3.1.3 Risk Loving Of Order 3

Definition 3: *An individual is said to be risk loving of order 3 if the lottery* $B_3 = [-k, \tilde{\epsilon}_1]$ *is preferred to the lottery* $A_3 = [0, \tilde{\epsilon}_1 - k]$, *for all initial wealth levels, for all k and for all zero-mean random variable* $\tilde{\epsilon}_1$.

This definition is exactly the same as in Eeckhoudt and Schlesinger (2006) and it corresponds to the definition of prudence. Furthermore, we can notice that having $A_3 \preccurlyeq B_3$ in the expected utility framework is equivalent to:

$$v(x-k) + Ev(x + \tilde{\epsilon}_1) \geq v(x) + Ev(x - k + \tilde{\epsilon}_1)$$

$$\Leftrightarrow Ev(x - k + \tilde{\epsilon}_1) - v(x-k) \leq Ev(x + \tilde{\epsilon}_1) - v(x)$$

This means that $\gamma(z) = Ev(z + \tilde{\epsilon}_1) - v(z)$ is an increasing function of z , that is

$$Ev'(z + \tilde{\epsilon}_1) \geq v'(z).$$

Also, with Jensen’s inequality the marginal utility is convex, that is $v''' \geq 0$. The preference to combine good with good of order 3 is equivalent to prudence.

3.1.4 Risk Loving Of Order 4

Definition 4: *An individual is said to be risk loving of order 4 if the lottery* $B_4 = [0, \tilde{\epsilon}_1 + \tilde{\epsilon}_2]$ *is preferred to the lottery* $A_4 = [\tilde{\epsilon}_1, \tilde{\epsilon}_2]$, *for all initial wealth levels and for all zero-mean independent random variables* $\tilde{\epsilon}_1$ *and* $\tilde{\epsilon}_2$.

This definition is the opposite definition of Eeckhoudt and Schlesinger (2006) concerning the concept of temperance. Therefore, our definition corresponds to intemperance.

3.1.5 Risk Loving Of Order N

In relation to risk loving of higher orders, we have the following definition:

Definition 5: *Preferences are said to satisfy risk loving of order N if the individual prefers the lottery B_N* to the lottery A_N.

This definition is not concerned by risk loving of other orders. For example, the individual can be risk seeking and prudent simultaneously. We can notice that risk loving of odd orders correspond to risk apportionment whereas risk loving of even orders coincide with anti-risk apportionment.

3.1.6 Theorem: Risk Loving Of Order N

Risk loving coincides with some conditions on the univariate utility function in the expected utility framework. We now have the following theorem:

Theorem 1: *In the expected utility framework with differentiable univariate utility function v, risk loving of order N is equivalent to $v^{(N)}(w) \geq 0, \forall w$.*

Proof. We want to prove that

$$A_N \preccurlyeq B_N \text{ , if and only if } v^{(N)} \geq 0 \text{ .}$$

We remark that the claim holds for $N = 1$, and we assume that it holds for $k = 1, \ldots, N$. We want to show that it holds for $N + 1$ and we proceed by induction.

$$Ev(x + A_{N+1}) \leq Ev(x + B_{N+1})$$

$$\Leftrightarrow Ev(x + B_{N-1}) + Ev(x + A_{N-1} + \widetilde{\epsilon}) \leq Ev(x + A_{N-1}) + Ev(x + B_{N+1} + \widetilde{\epsilon})$$

$$\Leftrightarrow Ev(x + B_{N-1}) - Ev(x + B_{N-1} + \tilde{\epsilon}) \leq Ev(x + A_{N-1}) - Ev(x + A_{N-1} + \tilde{\epsilon})$$

$$\Leftrightarrow \phi(x + A_{N-1}) \leq \phi(x + B_{N-1}),$$

where $\phi(z) = Ev\left(z + \tilde{\epsilon}\right) - v(z)$.

The last condition is equivalent to having $\phi^{(N-1)} \geq 0$, that is

$$Ev^{(N-1)}\left(z + \tilde{\epsilon}\right) \geq v^{(N-1)}(z).$$

This means that $v^{(N-1)}$ is convex, or equivalently

$$\left(v^{(N-1)}\right)'' = v^{(N+1)} \geq 0.$$

Q.E.D. ■

3.2 Bivariate Risk Loving Model

In this section, we extend the former results to the bivariate model and define bivariate risk loving of different orders. We illustrate the manner in which our framework incorporates, via the choice between two lotteries, the bivariate risk loving of any order. We present the class of bivariate lotteries such that the direction of preference between these lotteries is equivalent to signing the $(N, M)^{th}$ partial derivative of the bivariate utility function. Finally, we show that correlation loving, cross-prudence and intemperance are special cases of bivariate risk loving.

3.2.1 Bivariate Risk Loving Of Order (N, M) Definition And Theorem

Risk loving means that the individual prefers to put all the risks in the same state of nature. In addition, it is supposed that the individual prefers taking a chance on all the ‘good’ and all the ‘bad’, rather than to getting some of the ‘good’ and some of the ‘bad’ when we add a couple of outcomes. Therefore, we have the following definition for bivariate risk loving:

Definition 6: *Preferences, in the bivariate framework, are said to satisfy risk loving of order* (N, M) *if the individual always prefers*

$$[(x + B_N, y + B_M); (x + A_N, y + A_M)],$$

to

$$[(x + B_N, y + A_M); (x + A_N, y + B_M)],\ \forall x, y.$$

The individual prefers to combine good with good (the B's lotteries), and bad with bad (the A's lotteries). This definition expresses preference for the lottery over the 'outer' bivariate risks as opposed to that of the 'inner' bivariate risks, namely:

$$[(B_N, A_M); (A_N, B_M)] \preccurlyeq [(B_N, B_M); (A_N, A_M)].$$

We recall that $A_N \preccurlyeq B_N$ and $A_M \preccurlyeq B_M$, therefore B_N and B_M are good whereas A_N and A_M are bad.

The direction of preference for our pairs of bivariate lotteries is equivalent to signing the $(N, M)^{th}$ partial derivative of the bivariate utility function. We assume that the $(N, M)^{th}$ partial derivative of the utility function exists, it is continuous, and is given by

$$u^{(N,M)}(x, y) = \frac{\partial^{N+M} u}{\partial x^N \partial y^M}(x, y)\ .$$

We have the following result:

Theorem 2: *In the expected utility framework with differentiable bivariate utility function* u*, risk loving of order* (N, M) *is equivalent to* $u^{(N,M)}(x, y) \geq 0, \forall x, y$.

Proof. Consider the couple (N, M) where N and M are two positive integers.

$$[(x + B_N, y + A_M); (x + A_N, y + B_M)] \preccurlyeq [(x + B_N, y + B_M); (x + A_N, y + A_M)]$$

In the expected utility framework, the preference condition becomes:

$Eu(x+B_N,y+B_M)+Eu(x+A_N,y+A_M)\geq Eu(x+B_N,y+A_M)+Eu(x+A_N,y+B_M)$,

$Eu(x+B_N,y+B_M)-Eu(x+B_N,y+A_M)\geq Eu(x+A_N,y+B_M)-Eu(x+A_N,y+A_M)$

$\Leftrightarrow E\eta(x+B_N)\geq E\eta(x+A_N)$,

with $\eta(z)=Eu(z,y+B_M)-Eu(z,y+A_M)$.

The last inequality is true if the univariate function η exhibits risk loving of order N. That is:

$$\eta^{(N)}(z)=E\left[u^{(N,0)}(z,y+B_M)\right]-E\left[u^{(N,0)}(z,y+A_M)\right]\geq 0$$

$$\Leftrightarrow E\left[u^{(N,0)}(z,y+B_M)\right]\geq E\left[u^{(N,0)}(z,y+A_M)\right]$$

This last inequality is verified if the univariate function $\xi(w)=u^{(N,0)}(z,w)$ exhibits risk loving of order M, meaning that:

$$\xi^{(M)}(w)=u^{(N,M)}(z,w)\geq 0,\forall z,w.$$

Q.E.D.■

3.2.2 Risk Loving Of Orders Less Than **2**

In this sub-section, we show that the concept of risk loving of orders less than 2 coincides with correlation loving ($(N,M)=(1,1)$), cross-prudence ($(N,M)=(1,2)$ or $(N,M)=(2,1)$) and intemperance ($(N,M)=(2,2)$) .

3.2.2.1 Risk Loving Of Order (1,1)

Let us consider Definition 6 with $(N,M)=(1,1)$, we get:

$$[(x+B_1,y+B_1);(x+A_1,y+A_1)]$$

is preferred to

$$[(x+B_1,y+A_1);(x+A_1,y+B_1)],\forall x,y.$$

Equivalently:

$[(x,y);(x-k,y-k)]$ is preferred to $[(x,y-k);(x-k,y)],\forall x,y$. So

$$u(x,y-k)+u(x-k,y)\leq u(x,y)+u(x-k,y-k),\forall x,y,k,$$

$$\Delta_{1,1}(x)\leq \Delta_{1,1}(x-k),\forall x,y,k,$$

with

$$\Delta_{1,1}(x)=u(x,y-k)-u(x,y).$$

The former inequality becomes:

$$\frac{\partial \Delta_{1,1}}{\partial x}(x)=u^{(1,0)}(x,y-k)-u^{(1,0)}(x,y)\leq 0$$

meaning that

$$u^{(1,0)}(x,y-k)\leq u^{(1,0)}(x,y).$$

Therefore $u^{(1,0)}(x,y)$ increases with respect to y, meaning that $u^{(1,1)}(x,y)\geq 0$. The correlation loving introduced by Epstein and Tanny (1980) or multivariate risk seeking introduced by Richard (1975) corresponds to risk loving of order $(1,1)$. In other words, a type of preference for putting together two sure losses each concerning one of the two attributes of the utility function.

3.2.2.2 Risk Loving Of Orders (1,2) And (2,1)

Consider Definition 6 with $(N,M)=(1,2)$, we have:

$$[(x+B_1,y+B_2);(x+A_1,y+A_2)],$$

is preferred to

$$[(x+B_1,y+A_2);(x+A_1,y+B_2)],\forall x,y.$$

Equivalently:

$$Eu\left(x,y+\tilde{\epsilon}_1\right)+u(x-k,y)\geq u(x,y)+Eu\left(x-k,y+\tilde{\epsilon}_1\right),\forall x,y,k,$$

$$\Leftrightarrow Eu\left(x, y+\tilde{\epsilon}_1\right)-u(x,y) \geq Eu\left(x-k, y+\tilde{\epsilon}_1\right)-u(x-k,y), \forall x,y,k,$$

$$\Leftrightarrow \Delta_{(1,2)}(x)=Eu\left(x, y+\tilde{\epsilon}_1\right)-u(x,y) \; increases\; with\; x,\ \forall y,$$

$$\Leftrightarrow Eu^{(1,0)}\left(x, y+\tilde{\epsilon}_1\right)-u^{(1,0)}(x,y) \geq 0,\ \forall y,$$

$$\Leftrightarrow Eu^{(1,0)}\left(x, y+\tilde{\epsilon}_1\right) \geq u^{(1,0)}(x,y),\ \forall y,$$

$$\Leftrightarrow u^{(1,0)}(x,y) \text{ is convex in y},$$

$$\Leftrightarrow u^{(1,2)}(x,y) \geq 0.$$

Cross-prudence with respect to the first attribute of the utility function corresponds to risk loving of order $(1,2)$. That is, a type of preference for separating a sure loss and a zero-mean random variable in the two attributes of the utility function knowing that the first attribute is concerned with the sure loss and the second attribute is concerned with the zero-mean random variable. By the same argument, cross-prudence with respect to the second attribute of the utility function corresponds to risk loving of order $(2,1)$.

3.2.2.3 Risk Loving Of Order (2,2)

When we consider Definition 6 with $(N,M)=(2,2)$, we have:

$$[(x+B_2, y+B_2);(x+A_2, y+A_2)]$$

is preferred to

$$[(x+B_2, y+A_2);(x+A_2, y+B_2)],\ \forall x,y.$$

Equivalently:

$$Eu\left(x, y+\tilde{\epsilon}_2\right)+u\left(x+\tilde{\epsilon}_1, y\right) \leq u(x,y)+Eu\left(x+\tilde{\epsilon}_1, y+\tilde{\epsilon}_2\right), \forall x,y$$

$$\Leftrightarrow \Delta_{(2,2)}(x) \leq E\Delta_{(2,2)}(x+\tilde{\epsilon}_1)), \forall x,y$$

where $\Delta_{(2,2)}(x)= \ Eu\left(x, y+\tilde{\epsilon}_2\right)-u(x,y)$.

The former condition is equivalent to the convexity of $\Delta_{(2,2)}(x)$. Equivalently, we have

$$Eu^{(2,0)}\left(x, y+\widetilde{\epsilon}_2\right) \geq u^{(2,0)}(x,y)$$

This means that $u^{(2,0)}(x,y)$ is convex in y, therefore $u^{(2,2)}(x,y) \geq 0$.

Risk loving of order $(2,2)$ corresponds to intemperance. We are in the presence of a type of preference for putting together two independent zero-mean random variables in each attribute of the utility function. The risks are loss-ameliorating.

4. Mixed Risk Loving Model

The objective of this section is to analyze the attitude of an individual who prefers to combine good with good, and bad with bad at any level. We first examine the univariate mixed risk loving model, defining terms within the univariate mixed risk loving model and deriving a useful Theorem and properties. Similarly, we examine the bivariate mixed risk loving model, defining terms within this model and deriving another beneficial Theorem and properties.

4.1 Univariate Mixed Risk Loving Model

An individual will be mixed risk loving if he or she is risk loving at any order. Therefore, we have the following definition:

Definition 7: *An individual is mixed risk loving if he prefers lottery B_N to lottery A_N*, $\forall N = 1,2 \ldots$.

We also have the following definition concerning a special class of real-valued functions.

Definition 8: *A real-valued function f defined on $(0, +\infty)$ is absolutely monotonic if and only if its derivatives $f^{(N)}$ of all orders exist and $f^{(N)}(w) \geq 0$*, $\forall w, \forall N = 1,2 \ldots$

Berstein (1926) shows the following result:

Theorem 3 (Bernstein; 1926): *A function f defined on $(0, +\infty)$ is absolutely monotonic if and only if it is a Laplace transform of a distribution function F on $(0, +\infty)$.*

That is, a utility function v is absolutely monotonic if and only if $v(w) = \int_0^{+\infty} e^{s\,w} dF(s)$ up to a positive linear transformation.

An individual who prefers to combine good with good and bad with bad at any level should have a utility function with non-negative successive derivatives. Therefore, mixed risk loving is consistent with absolutely monotonic utility function. We now have the following definition:

Definition 9: *A real-valued continuous utility function v defined on $(0, +\infty)$ exhibits mixed risk loving if and only if it has an absolutely monotonic first derivative on $(0, +\infty)$ and $v(0) = 0$.*

Property 1: *v defined on $(0, +\infty)$ exhibits mixed risk loving if and only if its first derivative can be written as $v'(w) = \int_0^{+\infty} e^{s(w-a)} dF(s)$ for some distribution function F on $(0, +\infty)$.*

In other words, preference for combining good with good and bad with bad is consistent with a positive exponential function.

Property 2: *Let v be a mixed risk loving utility function which is analytic at the point τ with interval of convergence $[\tau-\in; \tau+\in]$ where $0 \leq \in \leq \tau$. Then v can be expressed as a power series:*

$$v(w) = \sum_{N=0}^{\infty} p_N (w-\tau)^N \ \ \forall w \in [\tau-\in; \tau+\in]$$

where

$$p_0 = \int_0^{+\infty} \frac{e^{s\,\tau}}{s} dF(s),$$

$$p_N = \frac{1}{N!} \int_0^{+\infty} s^{N-1}\, e^{s\,\tau}\, dF(s),\ \forall\, N \geq 1,$$

and F is a distribution function on $(0,+\infty)$.

This property comes directly from Bernstein's result. A direct consequence of the former property is that we know the direction of change in the expected utility thanks to a marginal change on one moment of a small risk.

Property 3: *Let v be a mixed risk loving utility function which is analytic at the point τ with interval of convergence $[\tau-\in;\tau+\in]$ where $0 \leq \in \leq \tau$. Assume that $\widetilde{w}$ is a random variable whose distribution support is included in the interval $[\tau-\in;\tau+\in]$. Then $Ev(\widetilde{w})$ has non-negative derivatives with respect to the moments of $\widetilde{w}$. That is, $\frac{\partial Ev(\widetilde{w})}{\partial \mu_i} \geq 0, \forall i$, where $\mu_i = E(\widetilde{w}^i)$.*

When a mixed risk loving individual faces a choice between two small risks that differ only in the N^{th} moment, he or she will prefer the one with the higher moment. Recall that when the individual was mixed risk averse, he or she prefers the risk with the higher moment when N is odd and the risk with the lower moment when N is even.

Let us denote by $ARA_N(w) = -\frac{v^{(N)}}{v^{(N-1)}}(w)$ the coefficient of absolute risk aversion of order N. $ARS_N(w) = \frac{v^{(N)}}{v^{(N-1)}}(w)$ represents the coefficient of absolute risk seeking of order N. With $N = 2$ we have absolute risk aversion whereas with $N = 3$ we deal with absolute prudence. For the mixed risk loving utility function, we have the following characterization:

Theorem 4: *Let the continuous utility function v defined on $(0,+\infty)$ be increasing, convex and smooth on $(0,+\infty)$ with $v(0) = 0$ and $v^{(N)}(w) \neq 0$ for all positive wealth and $N = 1,2\ 3, \ldots$ Then v is mixed risk loving if and only if ARS_N (respectively ARA_N) is non-decreasing (respectively non-increasing) with wealth for $N = 1,2\ 3, \ldots$*

Proof. Assume that ARS_N is non-decreasing with wealth for $N = 1,2\ 3, \ldots$

Then

$$\frac{dARS_N(w)}{dw} = ARS_N'(w) \geq 0.$$

This means that

$$v^{(N+1)}v^{(N-1)} \geq v^{(N)}v^{(N)}$$

For $N = 1$ this gives $v''v \geq (v')^2 \geq 0$. The non-negativity of the utility function implies that the marginal utility function is increasing, meaning that the second derivative of the utility function is non-negative.

For $N = 2$ this gives $v'''v' \geq (v'')^2 \geq 0$. The monotony of the utility function implies that the third derivative of the utility function is non-negative.

For $N = 3$ this gives $v^{(4)}v^{(2)} \geq (v^{(3)})^2 \geq 0$. The convexity of the utility function implies that the fourth derivative of the utility function is non-negative.

Therefore by induction we obtain $v^{(N)} \geq 0, \forall\, N = 1, 2, \ldots$, meaning that the utility function exhibits mixed risk loving.

Assume now that the utility function v is mixed risk loving, we know that :

$$v(w) = \int_{0}^{+\infty} e^{s\,w} dF(s)$$

The Chebychev Inequality gives

$$v^{(N+1)}v^{(N-1)} \geq \left[v^{(N)}\right]^2, \forall N = 1, \ldots$$

but we know that

$$ARS'_N = \frac{v^{(N+1)}v^{(N-1)} - v^{(N)}v^{(N)}}{[v^{(N-1)}]^2}$$

Therefore $ARS'_N \geq 0$.

Also the coefficients of absolute risk seeking at any order are non-decreasing. Or equivalently, the coefficients of absolute risk aversion at any order are non-increasing.

Q.E.D. ■

This theorem shows that mixed risk loving utility functions are utility functions such that the coefficients of risk seeking $ARS_N = \frac{v^{(N)}}{v^{(N-1)}}$ are non-decreasing, meaning that the coefficients of absolute risk aversion $ARA_N = -\frac{v^{(N)}}{v^{(N-1)}}$ are non-increasing. Mixed risk lovers exhibit at least risk seeking, prudence, intemperance, edginess and those feeling are all decreasing with respect to wealth. The importance of Theorem 4 is to link the mixed risk loving attitude with the decrease of the successive non-positive coefficients of absolute risk aversion.

4.2 Bivariate Mixed Risk Loving Model

An individual will be bivariate mixed risk loving if he or she is bivariate risk loving at any order. We assert the following definition:

Definition 10: *An individual is bivariate mixed risk loving if he or she always prefers the lottery*

$$[(B_N, B_M); (A_N, A_M)]$$

to the lottery

$$[(B_N, A_M); (A_N, B_M)]\ ,\ \forall N = 1,2 \dots\ and\ \ \forall M = 1,2 \dots$$

An absolutely monotonic bivariate function is infinitely differentiable with non-negative partial derivatives.

Definition 11: *A bivariate function g defined on $(0, +\infty) \times (0, +\infty)$ is absolutely monotonic if and only if its partial derivatives $g^{(N,M)}$ of all orders exist and*

$$g^{(N,M)}(w) \geq 0, \forall w, \forall N = 1,2 \dots, \forall M = 1,2 \dots$$

An individual who prefers to combine good with good, and bad with bad, at any level should have a utility function with non-negative successive partial derivatives. Therefore, bivariate mixed risk loving is consistent with the bivariate absolutely monotonic utility function. We assert the following definition:

Definition 12: *A real-valued continuous utility function* u *defined on* $(0,+\infty)\times(0,+\infty)$ *exhibits mixed risk loving if and only if it is absolutely monotonic.*

Like in the univariate case, bivariate absolutely monotonic functions can be expressed by the help of distribution function.

Theorem 5: *A bivariate function* g *defined on* $(0,+\infty)\times(0,+\infty)$ *is absolutely monotonic if and only if it can be written as* $g(x,y)=\iint_0^{+\infty} e^{sx+ty}dF(s\,,t)$ *for some distribution function F on* $(0,+\infty)\times(0,+\infty)$.

Finally, we can say that preferring more to less and to combine good with good and bad with bad is consistent with utility function that is a mixture of positive exponential utilities.

5. Conclusion

In this paper, we analyze preferences for combining good with good. This leads to risk seeking, prudence, intemperance and edginess at lower orders. In the univariate framework, the results are the opposite of those of Eeckhoudt and Schlesinger (2005) for odd orders and are the same for even orders. In the expected utility framework, our preferences for risk loving are related to the positivity of the successive derivatives of the utility function, thus generalizing the results from Crainich, Eeckhoudt and Trannoy (2012). Our results are based on preferences over simple lotteries.

Dealing with the bivariate case, we consider a decision-maker that prefers taking a chance on all the ‘best’ or all the ‘worst’ to getting some of the ‘best’ and some of the ‘worst’ with 50-50 lotteries. We extend the former results to the case of bivariate utility function and our results are in contrast to those of Jokung (2011) who considers bivariate risk apportionment. The preferences are expressed with the help of simple bivariate lotteries (inner and outer). Risk loving of order (N, M) corresponds to the positivity of the $(N, M)^{th}$ partial derivative of the bivariate utility function. Our approach allows us to recover multivariate risk seeking or correlation loving, cross-prudence and intemperance as particular cases.

The mixed risk loving utility functions are connected with absolutely monotonic functions and presented with their main properties. They are characterized by non-increasing non-positive coefficients of absolute risk aversion. The bivariate mixed risk loving utility functions are a mixture of positive exponential functions.

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