Birationally rigid complete intersections of high codimension

Daniel Evans and Aleksandr Pukhlikov

We prove that a Fano complete intersection of codimension k and index 1 in the complex projective space $\mathbb{P}^{M+k}$ for $k \geqslant 20$ and $M \geqslant 8k \log k$ with at most multi-quadratic singularities is birationally superrigid. The codimension of the complement to the set of birationally superrigid complete intersections in the natural parameter space is shown to be at least $\frac{1}{2}(M-5k)(M-6k)$. The proof is based on the techniques of hypertangent divisors combined with the recently discovered $4n^2$-inequality for complete intersection singularities.

Bibliography: 23 titles.

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Introduction

0.1. Complete intersections of index one. Let $k \geqslant 20$ be a fixed integer. For any integral k-uple $\underline{d} = (d_1, \dots, d_k)$, such that $2 \leqslant d_1 \leqslant \dots \leqslant d_k$ set $M = |\underline{d}| - k$, where $|\underline{d}| = d_1 + \dots + d_k$ and let

$$\mathcal{P}(\underline{d}) = \prod_{i=1}^{k} \mathcal{P}_{d_i, M+k+1}$$

be the space of k-uples of homogeneous polynomials of degree $d_1, \dots, d_k$, respectively, on the complex projective space $\mathbb{P} = \mathbb{P}^{M+k}$. Here the symbol $\mathcal{P}_{a,N}$ stands for the linear space of homogeneous polynomials of degree a in N variables which are naturally interpreted as polynomials on $\mathbb{P}^{N-1}$. We write $\underline{f} = (f_1, \dots, f_k) \in \mathcal{P}(\underline{d})$ for an element of the space $\mathcal{P}(\underline{d})$. We set also

$$\mathcal{P}_{\text{fact}}(\underline{d}) \subset \mathcal{P}(\underline{d})$$

to be the set of k-uples $\underline{f} = (f_1, \dots, f_k)$ such that the zero set

$$V(\underline{f}) = \{f_1 = \dots = f_k = 0\} \subset \mathbb{P}$$

is an irreducible, reduced and factorial complete intersection of codimension k. Note that for any $\underline{f} \in \mathcal{P}_{\text{fact}}(\underline{d})$ the projective variety $V(\underline{f})$ is a primitive Fano variety of index 1, that is,

$$\operatorname{Cl} V(\underline{f}) = \operatorname{Pic} V(\underline{f}) = \mathbb{Z}H,$$

where H is the class of a hyperplane section (this is by the Lefschetz theorem), and $K_{V(\underline{f})} = -H$. Therefore we may ask if $V = V(\underline{f})$ is birationally rigid or superrigid (see [1, Chapter 2] for the definitions).

Theorem 0.1. *Assume that $M \geqslant 8k \log k$. Then there exists a non-empty Zariski open subset $\mathcal{P}_{\text{reg}}(\underline{d}) \subset \mathcal{P}_{\text{fact}}(\underline{d})$, such that:*

(i) *for every $\underline{f} \in \mathcal{P}_{\text{reg}}(\underline{d})$ the variety $V = V(\underline{f})$ is birationally superrigid,*

(ii) *the inequality*

$$\operatorname{codim}((\mathcal{P}(\underline{d}) \setminus \mathcal{P}_{\text{reg}}(\underline{d})) \subset \mathcal{P}(\underline{d})) \geqslant \frac{(M-5k)(M-6k)}{2} \qquad (1)$$

holds.

Birational superrigidity of generic complete intersections of index 1 (for $M \geqslant k + 7$) was shown in [2, 3], but only non-singular complete intersections were considered there, so that the complement to the set of birationally superrigid varieties could be a divisor. In this paper we include complete intersections with multi-quadratic singularities into consideration. As a result, we get a much better estimate for the codimension of the complement: when k is fixed and M is growing, the codimension is of order $\frac{1}{2}M^2$, which is quite high.

We now proceed to explicit definitions of Zariski open subsets in $\mathcal{P}(\underline{d})$.

0.2. Complete intersections with multi-quadratic singularities. Let us describe the conditions for the singularities of a complete intersection that guarantee its factoriality. Take an arbitrary k-uple $\underline{f} \in \mathcal{P}(\underline{d})$, the zero set $V = V(\underline{f})$ of which is an irreducible reduced complete intersection of codimension k. Let $o \in V$ be a point. Fix a system of affine coordinates $(z_1, \dots, z_{M+k})$ on an affine chart $\mathbb{C}^{M+k} \subset \mathbb{P}$ with the origin at the point o. Write the corresponding dehomogenized polynomials (denoted by the same symbols) in the form

$$\begin{aligned} f_1 &= q_{1,1} + q_{1,2} + \dots + q_{1,d_1}, \\ &\dots \\ f_k &= q_{k,1} + q_{k,2} + \dots + q_{k,d_k}, \end{aligned}$$

where $q_{i,j}$ is a homogeneous polynomial in z_* of degree j. For a general point $o \in V$

$$\dim\langle q_{1,1}, \dots, q_{k,1}\rangle = k,$$

that is, $o \in V$ is non-singular. Assume now that $\dim\langle q_{1,1}, \dots, q_{k,1}\rangle \leqslant k-1$, that is to say, $o \in V$ is a singular point.

Definition 0.1. The singularity $o \in V$ is a *correct multi-quadratic singularity of type* 2^l, where $l \in \{1, \dots, k\}$, if the following conditions are satisfied:

- $\dim\langle q_{1,1}, \dots, q_{k,1}\rangle = k - l$,
- for a general linear subspace $P \subset \mathbb{P}$ of dimension $\max\{2k+2, k+3l+3\}$, containing the point o, the intersection $V_P = V \cap P$ has an isolated singularity at the point o,
- for the blow up $\varphi_P\colon V_P^+ \to V_P$ of the point o the exceptional divisor $Q_P = \varphi^{-1}(o)$ is a non-singular complete intersection of type 2^l in the $\max\{k+l+1, 4l+2\}$-dimensional projective space.

Note that by Definition 0.1, the codimension of the singular locus of V near a correct multi-quadratic singularity is at least $2k+2$.

Now let us discuss the conditions of Definition 0.1 in more detail. There is a subset $I \subset \{1, \dots, k\}$ such that $|I| = k - l$ and the linear forms $q_{i,1}$, $i \in I$, are linearly independent:

$$\langle q_{1,1}, \dots, q_{k,1}\rangle = \langle q_{i,1} \mid i \in I\rangle.$$

By the genericity of P, the restrictions $q_{i,1}|_P$, $i \in I$, remain linearly independent, so that the zero set

$$V_{P,I} = \{f_i|_P = 0 \mid i \in I\}$$

near the point o is a non-singular complete intersection of codimension $k - l$. Let

$$\varphi_{P,I}\colon V_{P,I}^+ \to V_{P,I}$$

be the blow up of the point $o \in V_{P,I}$ with the exceptional divisor $E_{P,I} = \varphi_{P,I}^{-1}(o)$ being the $\max\{k+l+1, 4l+2\}$-dimensional projective space. Now we can consider the blow up φ_P as the restriction of the blow up $\varphi_{P,I}$ onto V_P, that is, V_P^+ is the strict transform of V_P on $V_{P,I}^+$. In terms of this presentation, the exceptional divisor $Q_P \subset E_{P,I}$ is given by the set of l equations

$$q_{i,2}|_{E_{P,I}} = 0, \quad i \in \{1, \dots, k\} \setminus I.$$

Definition 0.1 requires Q_P to be a non-singular complete intersection of type 2^l in the projective space $E_{P,I}$.

Definition 0.2. We say that an irreducible reduced complete intersection $V = V(\underline{f})$ has at most correct multi-quadratic singularities if every point $o \in V$ is either non-singular or a correct multi-quadratic singularity of type 2^l for some $l \in \{1, \dots, k\}$.

The set of k-uples $\underline{f} \in \mathcal{P}(\underline{d})$ such that $V(\underline{f})$ satisfies Definition 0.2 is denoted $\mathcal{P}_{\rm mq}(\underline{d})$. The subset $\mathcal{P}_{\rm mq}(\underline{d}) \subset \mathcal{P}(\underline{d})$ is obviously Zariski open. Since for $\underline{f} \in \mathcal{P}_{\rm mq}(\underline{d})$ we have

$$\mathrm{codim}\,(\mathrm{Sing}\, V(\underline{f}) \subset V(\underline{f})) \geqslant 2k+2,$$

by Grothendieck's theorem on parafactoriality of local rings (see [4]) the complete intersection $V(\underline{f})$ is a factorial variety. Therefore, $\mathcal{P}_{\rm mq}(\underline{d}) \subset \mathcal{P}_{\rm fact}(\underline{d})$.

Theorem 0.2. *The following estimate holds:*

$$\mathrm{codim}((\mathcal{P}(\underline{d}) \setminus \mathcal{P}_{\rm mq}(\underline{d})) \subset \mathcal{P}(\underline{d})) \geqslant \frac{(M-4k+1)(M-4k+2)}{2} - (k-1) \qquad (2)$$

Remark 0.1. We construct the subset $\mathcal{P}_{\rm reg}(\underline{d}) \subset \mathcal{P}_{\rm mq}(\underline{d})$ below by removing some additional closed subsets from $\mathcal{P}_{\rm mq}(\underline{d})$.

0.3. Regular complete intersections. We keep the coordinate notations of Subsection 0.2 at a point $o \in V$. For brevity and uniformity we treat the non-singular case $o \notin \mathrm{Sing}\, V$ as a multi-quadratic case of type 2^l for $l = 0$. Let us place the homogeneous polynomials

$$q_{i,1},\ i \in I, \quad q_{i,j},\ j \geqslant 2,$$

in the *standard order*, corresponding to the lexicographic order of pairs (i,j): (i_1,j_1) precedes (i_2,j_2), if $j_1 < j_2$ or $j_1 = j_2$ but $i_1 < i_2$. Thus we obtain a sequence

$$h_1, h_2, \ldots, h_{M+k-l} \qquad (3)$$

of $M+k-l$ homogeneous polynomials in z_* of non-decreasing degrees: $\deg h_{e+1} \geqslant \deg h_e$.

Definition 0.3. The point $o \in V$ is *regular* if the sequence of polynomials, which is obtained from (3) by removing the last $[2\log k] - l$ members, is regular in $\mathcal{O}_{o,\mathbb{P}}$. (Here $[\cdot]$ means the integral part of a non-negative real number; if $l > [2\log k]$, we remove no members of the sequence (3).)

In plain words, Definition 0.3 requires that the set of common zeros of the polynomials $h_e(z)$ in the sequence, obtained from (3) by removing the last $[2\log k]-l$ members, is of the correct codimension. Since the polynomials h_* are homogeneous, we may consider them as polynomials on the projective space $\mathbb{P}^{M+k-1}$ in the homogeneous coordinates $(z_1 : \cdots : z_{M+k})$ and so understand the regularity in the projective setting.

Definition 0.4. The complete intersection $V = V(\underline{f})$, for $\underline{f} \in \mathcal{P}_{\rm mq}(\underline{d})$, is *regular*, if it is regular at every point $o \in V$, singular or non-singular. If this is the case, we write $\underline{f} \in \mathcal{P}_{\rm reg}(\underline{d})$.

Theorem 0.3. *Assume that $\underline{f} \in \mathcal{P}_{\rm reg}(\underline{d})$. Then $V = V(\underline{f})$ is birationally superrigid.*

Theorem 0.4. *The following estimate holds:*

$$\mathrm{codim}((\mathcal{P}_{\rm mq}(\underline{d}) \setminus \mathcal{P}_{\rm reg}(\underline{d})) \subset \mathcal{P}(\underline{d})) \geqslant \frac{(M-5k)(M-6k)}{2}. \qquad (4)$$

Proof of Theorem 0.1. Since the right hand side of (4) is obviously higher than that of (2), Theorem 0.1 follows immediately from Theorems 0.2, 0.3 and 0.4. Q.E.D.

0.4. The structure of the paper. Our paper is organized in the following way. In Section 1 Theorem 0.3 is shown. This is done by the technique of hypertangent divisors (the constructions can be found in [2] or [1, Chapter 3] or [5]), combined with the recently discovered $4n^2$-inequality for complete intersection singularities [6]. We need to take into consideration the fact that the regularity condition holds, generally speaking, not for the whole sequence (3), but for a shorter one, so that the resulting estimates are weaker than in [2]. However, we check that they are still sufficient for birational superrigidity. By the way, the biggest deviation from the computations in [2] is for non-singular points.

In Section 2 we show Theorem 0.2. This is rather straightforward and done by induction on the codimension k of the complete intersection (here there is no need to assume that $k \geqslant 20$; the case $k = 2$ was done in [7], $k = 1$ in [8]).

In Section 3 we show Theorem 0.4. The computations needed for the proof are really hard; we did our best to make them as clear and compact as possible. The estimates for the codimension are obtained by the "projection" technique introduced in [9] and also used in [7].

0.5. Historical remarks. The first complete intersection of codimension at least 2 that was shown to be birationally rigid was the complete intersection of a quadric and a cubic $V_{2\cdot 3} \subset \mathbb{P}^5$, see [10]; for a modernized exposition, see [1, Chapter 2]. The variety $V_{2\cdot 3}$ was assumed to be general in the sense that it does not contain lines with "incorrect" normal sheaf. Singular complete intersections $V_{2\cdot 3} \subset \mathbb{P}^5$ were later studied in [11].

Generic complete intersections $V \subset \mathbb{P}^{M+k}$ of type $\underline{d}$ with $|\underline{d}| = M + k$ and $M \geqslant 2k + 1$ were proved to be birationally superrigid in [2]. In [3] superrigidity was extended to the families with $M \geqslant k + 3$, $M \geqslant 7$ and $d_k = \max\{d_i\} \geqslant 4$, and in [12] to complete intersections of k_2 quadrics and k_3 cubics such that $M \geqslant 12$ and $k_3 \geqslant 2$. Today birational superrigidity remains an open problem only for three infinite series: complete intersections of type $\underline{d}$, where $\underline{d}$ is

$$(2, \dots, 2) \quad \text{or} \quad (2, \dots, 2, 3) \quad \text{or} \quad (2, \dots, 2, 4)$$

and finitely many families with $M \leqslant 11$.

In [8] a bound for the codimension of the locus of non-superrigid hypersurfaces of index 1 was given. Such bounds are important for investigations of birational geometry of Fano fibre spaces with a higher-dimensional base, see [13, 14]. Similar bounds were obtained for complete intersections of codimension $k = 2$ in [7] and for double quadrics and cubics (which could be understood as complete intersections of codimension 2 in a weighted projective space) in [15].

For an alternative approach to proving birational superrigidity of Fano complete intersections in the projective space, see [16]. Here are also some other papers on birational geometry of Fano complete intersections and their generalizations: [17-22].

1 Proof of birational rigidity

In this section we prove Theorem 0.3. First (Subsection 1.1) we remind the definition of a maximal singularity and prove that the centre of a maximal singularity is of codimension at least 3. After that, in Subsection 1.2 we construct hypertangent divisors. The construction is standard but singular points need special attention. In Subsection 1.3 we exclude the case when the centre of the maximal singularity is not contained in the singular locus of V. In Subsection 1.4 we exclude the case when the centre of the maximal singularity is contained in the locus of multi-quadratic points of type 2^l. Since it follows that a mobile linear system can not have a maximal singularity, the variety V is shown to be birationally superrigid.

1.1. Maximal singularities. As usual, we prove that a variety $V = V(\underline{f})$, where $\underline{f} \in \mathcal{P}_{\rm reg}(\underline{d})$, is birationally superrigid by assuming the converse and obtaining a contradiction. So fix a tuple $\underline{f} \in \mathcal{P}_{\rm reg}(\underline{d})$ and the corresponding complete intersection $V = V(\underline{f})$ and assume that V is not birationally superrigid. This implies immediately that for some mobile linear system $\Sigma \subset |nH|$ and some exceptional divisor E over V the Noether-Fano inequality

$$\mathrm{ord}_E\, \Sigma > n \cdot a(E)$$

is satisfied, where $a(E)$ is the discrepancy of E with respect to V. In other words, E is a *maximal singularity* of Σ (see, for instance [1, Chapter 2]). Let $B \subset V$ be the centre of E on V, an irreducible subvariety of codimension $\geqslant 2$.

Lemma 1.1. $\mathrm{codim}(B \subset V) \geqslant 3$.

Proof. Assume the converse: $\mathrm{codim}(B \subset V) = 2$. Then $B \not\subset \mathrm{Sing}\, V$, so that the inequality

$$\mathrm{mult}_B\, \Sigma > n$$

holds. Consider the self-intersection $Z = (D_1 \circ D_2)$ of the system Σ, where $D_1, D_2 \in \Sigma$ are general divisors. Obviously, $Z = \beta B + Z_1$, where $\beta > n^2$ and the effective cycle Z_1 of codimension 2 does not contain B as a component.

Let $P \subset \mathbb{P}$ be a general $(2k+1)$-subspace. Since $\mathrm{codim}(\mathrm{Sing}\, V \subset V) \geqslant 2k+2$, the intersection $V_P = V \cap P$ is non-singular. By Lefschetz, the numerical Chow group $A^2 V_P$ of codimension 2 cycles on V_P is $\mathbb{Z}H_P^2$, where H_P is the class of a hyperplane section of V_P. Setting $Z_P = Z|_P$ and $B_P = B|_P$, we obtain the inequality

$$\deg\, (Z_P - \beta B_P) \geqslant 0.$$

As $B_P \sim mH_P^2$ for some $m \geqslant 1$, this inequality implies that

$$\deg V \cdot (n^2 - m\beta) \geqslant 0,$$

which is impossible. Q.E.D. for the lemma.

Note that if $\operatorname{codim}(B \subset V) \leqslant 2k+1$, then B is not contained in the singular locus $\operatorname{Sing} V$ of the complete intersection V.

1.2. Hypertangent divisors. In order to exclude the maximal singularity E, we need the construction of *hypertangent linear systems*. It is well known and published many times (see [2] or [1, Chapter 3] or the most recent application [5]), but some minor modifications are needed to cover the multi-quadratic case, so we sketch this construction here. We fix a point o and use the notations of Subsection 0.2 and work in the affine chart $\mathbb{C}^{M+k}$ of the space $\mathbb{P}$ with the coordinates $z_1, \dots, z_{M+k}$; the point $o \in V$ is the origin. Let $j \geqslant 2$ be an integer. Recall that for some $l \in \{0, 1, \dots, k\}$ and a subset $I \subset \{1, \dots, k\}$, such that $|I| = k - l$, the linear forms $q_{i,1}$, $i \in I$, are linearly independent, whereas the other forms $q_{i,1}$, $i \notin I$, are their linear combinations. Denote by

$$f_{i,\alpha} = q_{i,1} + \dots + q_{i,\alpha}$$

the truncated i-th equation in the tuple $\underline{f}$.

Definition 1.1. The linear system

$$\Lambda(j) = \left\{ \left(\sum_{i \in I} q_{i,1} s_{i,j-1} + \sum_{i=1}^{k} \sum_{\alpha=2}^{d_i - 1} f_{i,\alpha} s_{i,j-\alpha} \right) \Bigg|_V = 0 \right\},$$

where $s_{i,j-\alpha}$ independently run through the set of homogeneous polynomials of degree $j - \alpha$ in the variables z_* (if $j - \alpha < 0$, then $s_{j-\alpha} = 0$), is called the *j-th hypertangent system at the point o*.

For uniformity of notations, we write $\Lambda(1)$ for the *tangent linear system*:

$$\Lambda(1) = \left\{ \left(\sum_{i \in I} q_{i,1} s_{i,0} \right) \Bigg|_V = 0 \right\}.$$

The Zariski tangent space $\{q_{i,1} = 0 \,|\, i \in I\}$ will be written as T. We set $c(1) = k - l$ and for $j \geqslant 2$

$$c(j) = k - l + \sharp\{(i, \alpha) \,|\, i = 1, \dots, k,\ 1 \leqslant \alpha \leqslant \min\{j, d_i - 1\}\}.$$

Further, set $m(j) = c(j) - c(j-1)$, where $c(0) = 0$, and for $j = 1, \dots, d_k - 1$ take $m(j)$ general divisors

$$D_{j,1}, \dots, D_{j,m(j)}$$

in the linear system $\Lambda(j)$. Putting them into the standard order, corresponding to the lexicographic order of the pairs (j, α) (see Subsection 0.3 for a similar procedure), we obtain a sequence

$$R_1, \dots, R_{M-l}$$

of effective divisors on V. Set $N_l = M - l$ if $l > [2 \log k]$ and $N_l = M - [2 \log k]$, otherwise. In what follows, we will really use only the divisors $R_1, \dots, R_{N_l}$, but it is convenient to keep the entire sequence.

Proposition 1.1. *The equality*

$$\mathrm{codim}_o\left(\left(\bigcap_{j=1}^{N_l}|R_j|\right)\subset V\right)=N_l$$

holds, where $|R_j|$ *stands for the support of* R_j.

Proof. Since

$$f_{i,\alpha}|_V=(-q_{i,\alpha+1}+\dots)|_V \tag{5}$$

for $1\leqslant\alpha\leqslant d_i-1$, where the dots stand for higher order terms in z_*, the codimension of the base locus of the tangent linear system $\Lambda(1)$ near the point o is equal to $(k-l)$ and of the hypertangent linear system $\Lambda(j)$, $j\geqslant 2$, is equal to

$$(k-l)+\mathrm{codim}(\{q_{i,\alpha}|_T=0\,|\,1\leqslant i\leqslant k,\ 1\leqslant\alpha\leqslant 1+\min\{j,d_i-1\}\,\}\subset T).$$

Therefore, for a general choice of hypertangent divisors R_*, the equality

$$\mathrm{codim}_o\left(\left(\bigcap_{j=1}^{i}|R_j|\right)\subset V\right)=i$$

follows from the regularity of the subsequence

$$h_1,\dots,h_i$$

of the sequence (3). Now our claim follows immediately from the regularity condition (see Definition 0.3). Q.E.D.

For a hypertangent divisor $R_i=D_{j,\alpha}$, where $j\in\{1,\dots,d_k-1\}$ and $\alpha\in\{1,\dots,m(j)\}$, the number

$$\beta_{l,i}=\beta(R_i)=\frac{j+1}{j}$$

is its *slope*.

Set $\varphi\colon V^+\to V$ to be the blow up of the point o with $Q=\varphi^{-1}(o)$ the exceptional divisor. The symbol R_i^+ means the strict transform of R_i on V^+.

Proposition 1.2. (i) $R_i^+\sim j\varphi^*H-\gamma_iQ$, *where* $\gamma_i\geqslant j+1$.

(ii) *For any irreducible subvariety* $Y\subset V$ *of codimension* $\geqslant 2$ *such that* $Y\not\subset|R_i|$ *the algebraic cycle* $(Y\circ R_i)$ *of the scheme-theoretic intersection satisfies the inequality*

$$\frac{\mathrm{mult}_o}{\deg}(Y\circ R_i)\geqslant\beta_{l,i}\frac{\mathrm{mult}_o}{\deg}Y.$$

(Here the symbol $\mathrm{mult}_o/\deg$ means, as usual, the ratio of the multiplicity at o to the degree in $\mathbb{P}$.)

Proof. (i) follows from (5), (ii) follows from (i). Q.E.D.

1.3. The non-singular case. In the notations of Subsection 1.1, assume that $B \not\subset \operatorname{Sing} V$. We want to show that this case is impossible by obtaining a contradiction. We write N for N_0 and β_i for $\beta_{0,i}$ for simplicity of notations.

By [1, Chapter 2, Section 2] the $4n^2$-inequality is satisfied:

$$\operatorname{mult}_B Z > 4n^2,$$

where Z is the self-intersection of the mobile system $\Sigma \subset |nH|$. Take a point $o \in B$ of general position, $o \notin \operatorname{Sing} V$, and let Y_2 be an irreducible component of Z with the maximal value of the ratio $\operatorname{mult}_o / \deg$. Then

$$\frac{\operatorname{mult}_o}{\deg} Y_2 > \frac{4}{d}.$$

Take general hypertangent divisors $R_1, \dots, R_M$ as described in Subsection 1.2. The first k of them, $R_1, \dots, R_k$, are actually tangent divisors and we know that

$$\operatorname{codim}_o((|R_1| \cap \dots \cap |R_k|) \subset V) = k.$$

Proceeding as in Section 1 of [2], we construct a sequence of irreducible subvarieties

$$Y_2, \dots, Y_k,$$

such that $\operatorname{codim}(Y_i \subset V) = i$, Y_2 is an irreducible component of Z with the maximal value of the ratio $\operatorname{mult}_o / \deg$, Y_{i+1} is an irreducible component of the scheme-theoretic intersection $(Y_i \circ R_{i-1})$ with the maximal value of the ratio $\operatorname{mult}_o / \deg$ for $i = 2, \dots, k-1$. Therefore, $Y_k \subset V$ is an irreducible subvariety of codimension k, satisfying the inequality

$$\frac{\operatorname{mult}_o}{\deg} Y_k > \frac{2^k}{d},$$

where $d = d_1 \cdot \dots \cdot d_k = \deg V$.

Lemma 1.2. $Y_k \not\subset |R_{k-1}|$.

Proof. Assume the converse: $Y_k \subset |R_{k-1}|$. The hypertangent divisors being general, this implies that

$$Y_k \subset \{q_{1,1}|_V = \dots = q_{k,1}|_V = 0\}.$$

However, as $\operatorname{codim}(\operatorname{Sing} V \subset V) \geqslant 2k+2$, we can take the section V_P of V by a generic linear subspace $P \subset \mathbb{P}$ of dimension $3k+1$, which is a $(2k+1)$-dimensional non-singular complete intersection in $\mathbb{P}^{3k+1}$. By Lefschetz, the scheme-theoretic intersection of codimension k on V_P

$$(\{q_{1,1}|_{V_P} = 0\} \circ \dots \circ \{q_{k,1}|_{V_P} = 0\})$$

must be irreducible and reduced. Therefore, the scheme-theoretic intersection of codimension k on V

$$(\{q_{1,1}|_V = 0\} \circ \dots \circ \{q_{k,1}|_V = 0\})$$

is irreducible and reduced. By the regularity condition, this irreducible subvariety has multiplicity precisely 2^k at the point o and the degree d. Therefore, it cannot be equal to Y_k. We got a contradiction, proving the lemma. Q.E.D.

By the last lemma, we can proceed in exactly the same way as in [2, Section 2]: form the scheme-theoretic intersection $(Y_k \circ R_{k-1})$ and obtain an irreducible subvariety $Y_{k+1} \subset V$ of codimension $k+1$, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{k+1} > \frac{2^{k+1}}{d}.$$

After that, still following the arguments of [2, Section 2], we use the hypertangent divisors $R_{k+2}, \ldots, R_N$ to obtain a sequence of irreducible subvarieties $Y_{k+2}, \ldots, Y_N$ of codimension $\mathrm{codim}(Y_i \subset V) = i$, such that Y_i is a component of the algebraic cycle $(Y_{i-1} \circ R_i)$ of the scheme-theoretic intersection of Y_{i-1} and R_i (the regularity condition and genericity of hypertangent divisors in their linear systems guarantee that $Y_{i-1} \not\subset |R_i|$) with the maximal value of the ratio $\mathrm{mult}_o / \deg$. Therefore,

$$\frac{\mathrm{mult}_o}{\deg} Y_i \geqslant \beta_i \frac{\mathrm{mult}_o}{\deg} Y_{i-1}.$$

The last subvariety Y_N is positive-dimensional and satisfies the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma = \frac{2^{k+1}}{d} \cdot \prod_{i=k+2}^{N} \beta_i.$$

Proposition 1.3. *The inequality $\gamma \geqslant 1$ holds.*

Note that this claim provides the contradiction we need and excludes the non-singular case.

Proof. Now it is convenient to use the *whole* set $R_1, \ldots, R_M$ of hypertangent divisors, as we have the obvious identity

$$d = d_1 \cdot \cdots \cdot d_k = \prod_{j=1}^{k} \prod_{\alpha=1}^{d_j - 1} \frac{\alpha+1}{\alpha} = \prod_{i=1}^{M} \beta_i.$$

Recall that $\beta_1 = \cdots = \beta_k = 2$ and $\beta_{k+1} = \frac{3}{2}$. Therefore, γ can be re-written as

$$\gamma = \frac{4}{3d} \prod_{i=1}^{N} \beta_i = \frac{4}{3} \beta^{-1},$$

where

$$\beta = \prod_{i=N+1}^{M} \beta_i \tag{6}$$

and our proposition follows from

Lemma 1.3. *The inequality $\beta < \frac{4}{3}$ holds.*

Proof of the lemma. Note first of all that for $j \geqslant N+1$ we have $\beta_j \leqslant 1 + \frac{1}{a}$, where $a = \left[\frac{M}{k}\right]$. Indeed, assume the converse: $\beta_{N+1} > 1 + \frac{1}{a}$. This means that all homogeneous polynomials $h_{k+1}, \dots, h_{k+N}$ in the sequence (3) are some $q_{i,\alpha}$ with $\alpha < a$. Therefore,

$$N \leqslant \sharp\{q_{i,\alpha} \,|\, 1 \leqslant i \leqslant k,\, 2 \leqslant \alpha \leqslant a-1\}.$$

But the right-hand side of this inequality does not exceed $k \cdot (a-2) < M - k$. So we get:

$$M - [2 \log k] < M - k,$$

which is a contradiction.

We have shown that

$$\beta \leqslant \left(1 + \frac{1}{a}\right)^{[2 \log k]} \leqslant \left(1 + \frac{1}{a}\right)^{a/4}$$

as $M \geqslant 8k \log k$ by assumption. Therefore, $\beta < e^{1/4} < \frac{4}{3}$, as required. Q.E.D. for the lemma.

Proof of Proposition 1.3 is complete.

We have shown that the case $B \not\subset \operatorname{Sing} V$ is impossible.

1.4. The multi-quadratic case. Assume now that B is contained in the closure of the locus of multi-quadratic points of type 2^l but not in the closure of the locus of multi-quadratic points of type 2^j for $j \geqslant l+1$. In other words, a general point $o \in B$ is a singular multi-quadratic point of type 2^l. Let us fix this point.

Proposition 1.4. *The self-intersection Z satisfies the following inequality:* $\operatorname{mult}_o Z > 2^{l+2} n^2$.

Proof. This is the $4n^2$-inequality for complete intersection singularities, see [6]. Q.E.D.

Remark 1.1. The condition for a point $o \in V$ to be a correct multi-quadratic singularity (see Definition 0.1) is in fact much stronger than what is required in [6].

Now let us exclude the multi-quadratic case and thus complete the proof of Theorem 0.3.

Assume first that $1 \leqslant l \leqslant k-2$. Let

$$R_1, \dots, R_{k-l}$$

be general tangent divisors. Since by the regularity condition

$$\operatorname{codim}_o \left(\left(\bigcap_{i=1}^{k-l} |R_i| \right) \subset V \right) = k - l,$$

we may argue as in the non-singular case and construct a sequence of irreducible subvarieties

$$Y_2, \dots, Y_{k-l}$$

of codimension $\mathrm{codim}(Y_i \subset V) = i$, where Y_2 is an irreducible component of the cycle Z with the maximal value of $\mathrm{mult}_o / \deg$ and Y_{i+1} is an irreducible component of $(Y_i \circ R_{i-1})$ with the same property. Obviously,

$$\frac{\mathrm{mult}_o}{\deg} Y_{k-l} > \frac{2^k}{d}.$$

By Lefschetz, the scheme-theoretic intersection

$$(R_1 \circ R_2 \circ \cdots \circ R_{k-l})$$

is irreducible and reduced: we make this conclusion, intersecting that cycle with the section V_P of V by a generic linear subspace P of dimension $3k+1$, exactly as in the proof of Lemma 1.2 (in fact, in order to apply the Lefschetz theorem, we could take a subspace P of a smaller dimension here), we conclude that $Y_{k-l} \not\subset |R_{k-l-1}|$ and construct an irreducible subvariety Y_{k-l+1}, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{k-l+1} > \frac{2^{k+1}}{d}.$$

After that we argue exactly as in the non-singular case, producing a sequence of irreducible subvarieties $Y_{k-l+2}, \ldots, Y_N$, the last one of which satisfies the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_N > \gamma_l = \frac{4}{3}\beta(l)^{-1},$$

where

$$\beta(l) = \prod_{i=N_l+1}^{M-l} \beta_{l,i} \tag{7}$$

(recall that $N_l = M - [2 \log k] + l$ for $l \leqslant [2 \log k]$ and $N_l = M - l$, otherwise). The product (7) contains fewer terms than (6) and it is easy to see that $\beta_{l,M-l-j} = \beta_{M-j}$ for $j = 0, 1, \ldots, M - l - N_l - 1$. Therefore, $\beta(l) < \beta$ for $l \geqslant 1$ and so $\gamma_l > \gamma > 1$, which gives us the desired contradiction. The multi-quadratic case for $1 \leqslant l \leqslant k-2$ is excluded.

Finally, assume that $l \in \{k-1, k\}$. In that case the subvariety Y_2 (an irreducible component of the self-intersection Z with the maximal value of $\mathrm{mult}_o / \deg$) satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_2 > \frac{2^{k+1}}{d}$$

by Proposition 1.4. In this case we omit the part of our arguments which deals with tangent divisors and proceed straight to the second part, repeating the arguments for the case $l \leqslant k-2$ word for word.

The multi-quadratic case is excluded.

Q.E.D. for Theorem 0.3.

2 Irreducible factorial complete intersections

In this section we prove Theorem 0.2. In Section 2.1 we explain the strategy of the proof and show the case of a hypersurface. After that in Section 2.2 we start the inductive part of the proof, first looking at the easier issue of complete intersections being irreducible and reduced. Finally, in Subsection 2.3 we complete the proof, considering complete intersections with correct multi-quadratic singularities.

2.1. Complete intersections with correct multi-quadratic singularities. Set

$$\mathcal{P}^{\geqslant j} = \prod_{i=j}^{k} \mathcal{P}_{d_i, M+k+1}$$

to be the space of truncated tuples $(f_j, \ldots, f_k)$ and let $\mathcal{P}^{\geqslant j}_{\mathrm{mq}}$ be the set of tuples such that

$$V(f_j, \ldots, f_k) = \{f_j = \ldots = f_k = 0\} \subset \mathbb{P}$$

is an irreducible reduced complete intersection of codimension $k - j + 1$ with at most correct multi-quadratic singularities, in the sense of Definition 0.1 where k is replaced by $k - j + 1$. Note that $\mathcal{P}^{\geqslant 1} = \mathcal{P}(\underline{d})$ and $\mathcal{P}^{\geqslant 1}_{\mathrm{mq}} = \mathcal{P}_{\mathrm{mq}}(\underline{d})$. We will prove Theorem 0.2 by decreasing induction on $j = k, k-1, \ldots, 1$ in the following form

$$\mathrm{codim}((\mathcal{P}^{\geqslant j} \setminus \mathcal{P}^{\geqslant j}_{\mathrm{mq}}) \subset \mathcal{P}^{\geqslant j}) \geqslant \frac{(M - 4k + 1)(M - 4k + 2)}{2} - (k - 1). \qquad (8)$$

The basis of the induction is the case of a hypersurface $V(f_k) \subset \mathbb{P}$ of degree d_k. It is easy to calculate that the closed subset of reducible or non-reduced polynomials of degree d_k has codimension

$$\binom{M + k + d_k - 1}{d_k} - (M + k + 1)$$

in $\mathcal{P}_{d_k, M+k+1}$ (which corresponds to the case when f_k has a linear factor), and the closed subset of polynomials f_k such that the hypersurface $V(f_k)$ has at least one singular point, which is not a quadratic singularity of rank at least 7 has codimenison

$$\frac{(M + k - 6)(M + k - 5)}{2} + 1$$

in $\mathcal{P}_{d_k, M+k+1}$ (see a similar detailed calculation in [8] for the case of rank at least 5). Therefore, the inequality (8) is true for $k = 1$.

Now let us proceed to the inductive argument.

2.2. The step of induction: irreducibility. We assume that (8) is shown for $j + 1$. The task is, for a fixed tuple $(f_{j+1}, \ldots, f_k) \in \mathcal{P}^{\geqslant j+1}_{\mathrm{mq}}$, to estimate the codimension of the set of polynomials $f_j \in \mathcal{P}_{d_j, M+k+1}$ such that $V(f_j, \ldots, f_k)$ does not satisfy the required condition, that is, $(f_j, f_{j+1}, \ldots f_k) \notin \mathcal{P}^{\geqslant j}_{\mathrm{mq}}$.

Let us first consider the issue of irreducibility and reducedness. Since by the inductive assumption and the Grothendieck theorem [4] $V(f_{j+1},\dots,f_k)$ is a factorial complete intersection, we have the isomorphism

$$\mathrm{Cl}\,V(f_{j+1},\dots,f_k)\cong \mathrm{Pic}\,V(f_{j+1},\dots,f_k)\cong \mathbb{Z}H,$$

where H is the class of a hyperplane section, and moreover, for every $a\in\mathbb{Z}_+$ the restriction map

$$r_a:\mathrm{H}^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(a))\to \mathrm{H}^0(V_{j+1},\mathcal{O}_{V_{j+1}}(a))$$

is surjective (where for simplicity of notation we write V_{j+1} for $V(f_{j+1},\dots,f_k)$). For $a<d_{j+1}$ it is also injective, and for $a=d_{j+1}$ we have

$$\dim\mathrm{Ker}\,r_a=\#\{i\in\{j+1,\dots,k\}\mid d_i=d_{j+1}\}.$$

Now easy calculations show that the set of polynomials $f_j\in\mathcal{P}_{d_j,M+k+1}$ such that $V(f_j,f_{j+1},\dots,f_k)$ is either reducible or non-reduced, is of codimension at least

$$\binom{M+k+d_j-1}{d_j}-(M+k+1)-(k-j)$$

(again, this corresponds to the case when the divisor $\{f_j|_{V_{j+1}}=0\}$ has a hyperplane section of V_{j+1} as a component). This estimate is higher (and, in fact, much higher) than what we need so we may assume that $V(f_j,f_{j+1},\dots,f_k)$ is irreducible and reduced.

Finally, we need to consider the condition for the singularities of the complete intersection $V(f_j,f_{j+1},\dots,f_k)$ to be multi-quadratic. In order to avoid cumbersome formulae, we will consider the final case $j=1$ only, when the estimate is the weakest. For higher values of j the arguments are identically the same, just the indices and dimensions need to be adjusted appropriately.

2.3. Multi-quadratic singularities. Fix a point $o\in\mathbb{P}$ and consider a tuple $(f_1,\dots,f_k)\in\mathcal{P}^{\geqslant 1}$ with $o\in V=V(f_1,\dots,f_k)$. Fix a system of affine coordinates $(z_1,\dots,z_{M+k})$ on an affine chart $\mathbb{C}^{M+k}\subset\mathbb{P}$ with the origin at the point o. Write the corresponding dehomogenized polynomials (denoted by the same symbols) in the form

$$\begin{aligned} f_1&=q_{1,1}+q_{1,2}+\dots+q_{1,d_1},\\ &\dots\\ f_k&=q_{k,1}+q_{k,2}+\dots+q_{k,d_k},\end{aligned}$$

where $q_{i,j}$ is a homogeneous polynomial in z_* of degree j. Assume that

$$\dim\langle q_{1,1},\dots,q_{k,1}\rangle=k-l,$$

with $l\geqslant 0$. Let $I\subset\{1,\dots,k\}$ be a subset with $|I|=k-l$ such that the linear forms $\{q_{i,1}\mid i\in I\}$ are linearly independent. Set $\Pi\subset\mathbb{C}^{M+k}$ to be the subspace

$$\Pi=\{q_{i,1}=0\mid i\in I\}\cong\mathbb{C}^{M+l}.$$

By assumption, for every $j \in J = \{1, \ldots, k\} \setminus I$ there are (uniquely determined) constants $\beta_{j,i}$, $i \in I$, such that

$$q_{j,1} = \sum_{i \in I} \beta_{j,i} q_{i,1}.$$

Set for every $j \in J$

$$q^*_{j,2} = \left(q_{j,2} - \sum_{i \in I} \beta_{j,i} q_{i,2} \right) \Bigg|_{\Pi}.$$

The following statement translates the condition for the point o to be a correct multi-quadratic singularity into the language of properties of the quadratic forms $q^*_{j,2}$ introduced above.

Proposition 2.1. *Assume that for a general subspace $\Theta \subset \mathbb{P}(\Pi)$ of dimension*

$$b = \max\{k + l + 1, 4l + 2\}$$

the set of quadratic equations

$$\left\{ q^*_{j,2}|_{\Theta} = 0 \mid j \in J \right\}$$

defines a non-singular complete intersection of type 2^l. Then $o \in V$ is a correct multi-quadratic singularity of type 2^l.

Proof. Indeed, it is easy to see that the germ $o \in V$ is analytically equivalent to the closed set in Π defined by l equations

$$0 = q^*_{j,2} + \ldots, \quad j \in J,$$

where the dots stand for higher order terms. The rest is obvious. Q.E.D.

Remark 2.1. In the notations of Definition 0.1, the exceptional divisor Q_P is precisely the complete intersection of l quadrics $\{q^*_{j,2}|_{\Theta} = 0\}$, $j \in J$, in the b-dimensional space Θ. Proposition 2.1 gives a sufficient condition for the point o to be a correct multi-quadratic singularity. Now we use this criterion to estimate the codimension of the set of tuples violating the conditions of Definition 0.1 at the given point $o \in V$.

Definition 2.1. We say that an l-uple $(q^*_{j,2} \mid j \in J)$ is *correct*, if its zero set in $\mathbb{P}(\Pi)$ is an irreducible reduced complete intersection Q_{Π} satisfying the inequality

$$\operatorname{codim}(\operatorname{Sing} Q_{\Pi} \subset Q_{\Pi}) \geqslant b.$$

Corollary 2.1. *Assume that the l-uple $(q^*_{j,2} \mid j \in J)$ is correct. Then $o \in V$ is a correct multi-quadratic singularity of type 2^l.*

Since in the subsequent arguments (up to the end of this section) only the quadratic forms $q_{i,2}$ will be involved, we may assume without loss of generality that

$$J = \{1, \ldots, l\}$$

and $I = \{l+1,\dots,k\}$. Fixing the forms $q_{i,2}$ for $i \in I$, we work with the l-uples

$$(q^*_{j,2} \,|\, j = 1,\dots,l) \in \mathcal{P}^{\times l}_{2,M+l}.$$

Theorem 0.2 is obviously implied by the following proposition.

Proposition 2.2. *The codimension of the closed set $\mathcal{X} \subset \mathcal{P}^{\times l}_{2,M+l}$ of incorrect l-uples is at least*

$$\frac{(M+3-b)(M+4-b)}{2} - (l-1). \qquad (9)$$

(Recall that $b = \max\{k+l+1, 4l+2\}$.)

Proof. Elementary computations show that the codimension of the closed subset $\mathcal{X}_* \subset \mathcal{P}^{\times l}_{2,M+l}$ of linearly dependent l-uples is higher than (9), so we may assume the forms $q^*_{j,2}$, $j = 1,\dots,l$, to be linearly independent. The symbol Q_Π stands for their zero set. By the symbol $\mathrm{Sing}\, Q_\Pi$ we denote the closed set of points $p \in Q_\Pi$, such that the linear terms of dehomogenised polynomials $q^*_{j,2}$ with respect to any system of affine coordinates with the origin at p are not linearly independent. (We argue in this way in order to avoid a discussion of the zero scheme of the forms $q^*_{j,2}$, $j = 1,\dots,l$, being irreducible and reduced at this stage of the proof.) For

$$\lambda = (\lambda_1 : \dots : \lambda_l) \in \mathbb{P}^{l-1}$$

set

$$W(\lambda) = \{\lambda_1 q^*_{1,2} + \dots + \lambda_l q^*_{l,2} = 0\} \subset \mathbb{P}^{M+l-1}$$

to be the corresponding quadric hypersurface in the linear system generated by $(q^*_{j,2})$. We will use the following simple observation, which for $k = 2$ was used in [7].

Lemma 2.1. *For any point $p \in \mathrm{Sing}\, Q_\Pi$ there is $\lambda \in \mathbb{P}^{l-1}$ such that $p \in \mathrm{Sing}\, W(\lambda)$.*

Proof. Obvious computations. Q.E.D. for the lemma.

Corollary 2.2. *The following inclusion holds:*

$$\mathrm{Sing}\, Q_\Pi \subset \bigcup_{\lambda \in \mathbb{P}^{l-1}} \mathrm{Sing}\, W(\lambda).$$

Set $\mathcal{R}_{\leqslant a} \subset \mathcal{P}_{2,M+l}$ to be the closed subset of quadratic forms of rank $\leqslant a$. It is well known that

$$\mathrm{codim}(\mathcal{R}_{\leqslant a} \subset \mathcal{P}_{2,M+l}) = \frac{(M+l+1-a)(M+l+2-a)}{2}.$$

Now for every $e = 1,\dots,l$ consider the closed subset $\mathcal{X}_{e,a} \subset \mathcal{P}^{\times e}_{2,M+l}$, consisting of e-uples $(g_1,\dots,g_e)$ such that the linear span $\langle g_1,\dots,g_e\rangle$ has a positive-dimensional intersection with $\mathcal{R}_{\leqslant a}$.

Lemma 2.2. *The following estimate holds:*

$$\mathrm{codim}(\mathcal{X}_{e,a} \subset \mathcal{P}^{\times e}_{2,M+l}) \geqslant \mathrm{codim}(\mathcal{R}_{\leqslant a} \subset \mathcal{P}_{2,M+l}) - (e-1).$$

Proof. Consider the natural projections of $\mathcal{P}^{\times e}_{2,M+l} = \mathcal{P}^{\times(e-1)}_{2,M+l} \times \mathcal{P}_{2,M+l}$ onto the last factor and the direct product $\mathcal{P}^{\times(e-1)}_{2,M+l}$ of the first $e-1$ factors.

For any tuple

$$(g_1,\dots,g_e) \in \mathcal{P}^{\times e}_{2,M+l}$$

such that

$$(g_1,\dots,g_{e-1}) \notin \mathcal{X}_{e-1,a}$$

the condition $(g_1,\dots,g_e) \in \mathcal{X}_{e,a}$ implies that the quadratic form q_e belongs to the cone with the base $\mathcal{R}_{\leqslant a}$ and the vertex space $\langle g_1,\dots,g_{e-1}\rangle$, which has dimension at most $\dim \mathcal{R}_{\leqslant a} + (e-1)$. Arguing by increasing induction on $e = 1,\dots,l$, we complete the proof. Q.E.D. for the lemma.

Now we can complete the proof of Proposition 2.2. Let us consider an l-uple $(q^*_{j,2} \,|\, j = 1,\dots,l)$ such that

$$\mathrm{codim}(\mathrm{Sing}\, Q_\Pi \subset Q_\Pi) \leqslant b-1$$

or, equivalently, that

$$\dim(\mathrm{Sing}\, Q_\Pi) \geqslant M+l-b.$$

By Corollary 2.2 we conclude that the inequality

$$\max_{\lambda\in\mathbb{P}^{l-1}} \{\dim \mathrm{Sing}\, W(\lambda)\} \geqslant M+1-b$$

is satisfied, which, in its turn, implies that

$$(g_1,\dots,g_e) \notin \mathcal{X}_{l,a}$$

for $a = l+b-2$. Now by Lemma 2.2 we conclude that in the proof of Proposition 2.2 we can consider only l-uples satisfying the inequality

$$\mathrm{codim}(\mathrm{Sing}\, Q_\Pi \subset Q_\Pi) \geqslant b. \tag{10}$$

The rest is very easy. If Q_Π is an irreducible reduced complete intersection, then (10) guarantees that the tuple of quadratic forms under consideration is correct. Moreover, if for some $e \geqslant 1$ the system of quadratic equations

$$q^*_{1,2} = \dots = q^*_{e,2} = 0$$

defines an irreducible reduced complete intersection of e quadrics, then by (10) it is factorial. Now arguing as in Subsection 2.2, we can estimate the codimension of the set of tuples, the zero set of which is not an irreducible reduced complete intersections. It is easy to check that the codimension is equal to

$$\frac{(M+l-1)(M+l-2)}{2} - e.$$

This completes the proof of Proposition 2.2. Q.E.D.

This completes the proof of Theorem 0.2 as well, as the minimum of the estimate obtained in Proposition 2.2 occurs for $l = k$.

3 Regular complete intersections

In this section we prove Theorem 0.4. In Subsection 3.1 we produce the estimates for the codimension of the set of non-regular tuples of polynomials, given by the projection method. After that, the proof of Theorem 0.4 is reduced to showing a purely analytical fact: estimating the minimum of an integral sequence, consisting of certain binomial coefficients, depending on several integral parameters. The required computations are quite non-trivial. We perform them in several steps. In Subsection 3.2 a number of reductions simplifies the task. In Subsection 3.3 we employ the classical Stirling formula to approximate with good precision the expressions to be minimized by a smooth function and study that function using the standard tools of calculus. In Subsections 3.4 and 3.5 we complete the proof, showing the required estimates.

3.1. The projection method. We use the notations of Subsection 0.3. Since an elementary dimension count relates the codimension of the set of globally non-regular tuples $\underline{f}$ (which is what Theorem 0.4 estimates) to the codimension of the set of tuples $\underline{f}$ that are non-regular at a fixed point $o \in V(\underline{f})$ (see Theorem 3.1 and the comments below), we concentrate on the local problem: fix a point $o \in \mathbb{P}$, a system of affine coordinates $z_1, \dots, z_{M+k}$ with the origin at o and consider (non-homogeneous) tuples $\underline{f}$ such that $o \in V(\underline{f})$.

Next, we fix $l \in \{0, 1, \dots, k\}$ and assume that the rank of the set of linear forms $q_{i,1}$, $i = 1, \dots, k$, is equal to $k - l$, so that in the sequence (3) exactly the first $k - l$ polynomials are linear forms. We fix them, either, so that the linear subspace

$$\Pi = \{h_1 = \ldots = h_{k-l} = 0\} \cong \mathbb{C}^{M+l}$$

of the space $\mathbb{C}^{M+k}_{z_*}$ is also fixed. Recall the notation

$$N_l = M - \max\{[2 \log k], l\},$$

introduced in Subsection 1.2. Set

$$g_i = h_{k-l+i}|_{\mathbb{P}(\Pi)},$$

$i = 1, \dots, N_l$. This is a sequence of N_l homogeneous polynomials of non-decreasing degrees $m_i = \deg g_i$ on the projective space $\mathbb{P}(\Pi) \cong \mathbb{P}^{M+l-1}$. Define the space of such sequences:

$$\mathcal{G}(\underline{d}, l) = \prod_{i=1}^{N_l} \mathcal{P}_{m_i, M+l}.$$

It is obvious that the point $o \in V$ is regular (as a multi-quadratic point of type 2^l in the sense of Definition 0.3) if and only if the sequence

$$g_1, \dots, g_{N_l}$$

is regular, that is to say, if the closed algebraic set

$$\{g_1 = \cdots = g_{N_l} = 0\} \subset \mathbb{P}(\Pi)$$

has codimension N_l. Set $\mathcal{Y} = \mathcal{Y}(\underline{d}, l) \subset \mathcal{G}(\underline{d}, l)$ to be the closed set of *non*-regular tuples.

Theorem 3.1. *Assume that $M \geqslant 8k \log k$ and $k \geqslant 20$. Then*

$$\operatorname{codim}(\mathcal{Y} \subset \mathcal{G}(\underline{d}, l)) \geqslant \frac{(M-5k)(M-6k)}{2} + M + k.$$

Taking into account that the point o varies in the $M+k$-dimensional projective space $\mathbb{P}$ and the original tuple $\underline{f}$ satisfies the conditions $f_1(o) = \cdots = f_k(o) = 0$ and $\dim\langle q_{i,1} \,|\, 1 \leqslant i \leqslant k\rangle = k - l$, an elementary dimension count gives Theorem 0.4 as an immediate corollary of Theorem 3.1.

The rest of this section is the **proof of Theorem 3.1**. Our main tool is *the projection method*, developed in [9] and explained and applied to solving similar problems in [1, Chapter 3] and many papers, e.g. [7, 8]. The idea is to represent

$$\mathcal{Y} = \coprod_{e=1}^{N_l} \mathcal{Y}_e$$

as a disjoint union of constructive subsets $\mathcal{Y}_e$, consisting of tuples $(g_1, \dots, g_{N_l})$ such that the closed set

$$\{g_1 = \cdots = g_{e-1} = 0\} \subset \mathbb{P}(\Pi)$$

is of codimension $e-1$, but g_e vanishes on some irreducible component of that set (if $e = 1$, this means simply that the quadratic form g_1 is identically zero). The projection method estimates the codimension of $\mathcal{Y}_e$ in $\mathcal{G}(\underline{d}, l)$ as follows:

$$\operatorname{codim}(\mathcal{Y}_e \subset \mathcal{G}(\underline{d}, l) \geqslant \gamma(e, \underline{d}, l) = h^0(\mathbb{P}^{M+l-e}, \mathcal{O}_{\mathbb{P}^{M+l-e}}(m_e)) = \binom{M+l-e+m_e}{M+l-e},$$

where $m_e = \deg g_e$, see, for instance, [1, Chapter 3]. Therefore, in order to prove Theorem 3.1, we must show that the numbers $\gamma(e, \underline{d}, l)$ for $e = 1, \dots, N_l$ are not smaller than the right hand side of the inequality of Theorem 3.1. This is what we are going to do. The task is quite non-trivial. First, we do some preparatory work in order to simplify the inequalities to be shown and reduce the number of integral parameters, on which the numbers $\gamma(e, \underline{d}, l)$ depend.

3.2. Reductions. If the original tuple $\underline{f}$ of defining polynomials consists of k_2 quadrics, k_3 cubics, …, k_m polynomials of degree $m = d_k \geqslant 8 \log k$, then

$$k_2 + k_3 + \cdots + k_m = k$$

and

$$2k_2 + 3k_3 + \cdots + mk_m = |\underline{d}| = d_1 + \cdots + d_k.$$

It is easy to see that

$$m_e = \deg g_e = \min\left\{ j \,\middle|\, \sum_{\alpha=2}^{j} \left(\sum_{\beta=\alpha}^{m} k_\beta \right) \geqslant e \right\}.$$

This explicit presentation gives us the first reduction.

Proposition 3.1. *The following estimate holds:*

$$\gamma(e, \underline{d}, l) \geqslant \gamma(e, \underline{d}^*, l),$$

where the k-uple $\underline{d}^* = (d_1^*, \ldots, d_k^*)$ is defined by the equalities

$$d_1^* = \ldots = d_r^* = a+1, \qquad d_{r+1}^* = \ldots = d_k^* = a+2 \tag{11}$$

and $M = ka + (k - r)$, where $0 \leqslant r \leqslant k-1$.

Proof. Explicitly, the proposition states that

$$\binom{M + l - e + m_e}{M + l - e} \geqslant \binom{M + l - e + m_e^*}{M + l - e},$$

where m_e^* is calculated for the tuple $\underline{d}^*$. It is easy to see that $m_e \geqslant m_e^*$, which proves the proposition. Q.E.D.

The second reduction simplifies the situation further, allowing us to consider only the case when all degrees d_i are equal.

Proposition 3.2. *For the tuple $\underline{d}^+ = (d_1^+, \ldots, d_k^+)$ such that $d_1^+ = \cdots = d_k^+$, with $M^+ + k = |\underline{d}^+|$ and $M^+ \geqslant 8k \log k - k$ the estimate*

$$\gamma(e, \underline{d}^+, l) \geqslant \frac{(M^+ - 4k)(M^+ - 5k)}{2} + M^+ + 2k$$

holds for all $e = 1, \ldots, N_l^+ = M^+ - \max\{[2\log k], l\}$.

Let us show that Theorem 3.1 follows from Propositions 3.1 and 3.2.

Indeed, by Proposition 3.1 it is sufficient to prove the inequality

$$\gamma(e, \underline{d}^*, l) \geqslant \frac{(M - 5k)(M - 6k)}{2} + M + k$$

for $e = 1, \ldots, N_l$. Let us consider the tuple $\underline{d}^+$ with

$$d_1^+ = \cdots = d_k^+ = a + 1$$

for the constant a defined in Proposition 3.1. Set $M^+ = ka$. Obviously, $\gamma(e, \underline{d}^*, l) \geqslant \gamma(e, \underline{d}^+, l)$ for $e = 1, \ldots, N_l^+$ as $M \geqslant M^+$. If $N_l > N_l^+$, then for $i = 0, \ldots, N_l - N_l^+ - 1$ we have a similar estimate "from the other end":

$$\gamma(N_l - i, \underline{d}^*, l) \geqslant \gamma(N_l^+ - i, \underline{d}^+, l)$$

(note that $N_l - N_l^+ = M - M^+ \leqslant k$). Therefore,

$$\gamma(\underline{d}^*, l) = \min_{1 \leqslant e \leqslant N_l} \{\gamma(e, \underline{d}^*, l)\} \geqslant \gamma(\underline{d}^+, l) = \min_{1 \leqslant e \leqslant N_l^+} \{\gamma(e, \underline{d}^+, l)\},$$

and applying Proposition 3.2 and taking into account that $M^+ \geqslant M - k$, we get the claim of Theorem 3.1.

The third reduction allows us to remove the integral parameter $l \in \{0, 1, \ldots, k\}$. In order to simplify our notations, we write d_i for d_i^+, thus assuming that $d_1 = \cdots = d_k = a + 1$, so that $M = ka$. We use the notation $\gamma(\underline{d}, l)$ for the minimum of the numbers $\gamma(e, \underline{d}, l)$, $e = 1, \ldots, N_l$, introduced above.

Proposition 3.3. *The following inequality holds:*

$$\gamma(\underline{d}, l) \geqslant \gamma(\underline{d}, 0)$$

for all $l = 0, 1, \ldots, k$.

Proof. Since for $l \geqslant 1$ we have $N_0 > N_l$, it is sufficient to compare the integers $\gamma(e, \underline{d}, l)$ and $\gamma(e, \underline{d}, 0)$ for the same values of $e = 1, \ldots, N_l$. They are

$$\binom{M + l - e + m_e}{M + l - e} \quad \text{and} \quad \binom{M - e + m_e}{M - e},$$

so the claim becomes obvious. Q.E.D.

Remark 3.1. We could as well do the third reduction as the first one: show that the minimum of the integers $\gamma(e, \underline{d}, l)$ is attained for $l = 0$ (which corresponds to regular non-singular points of V), and after that prove that the worst estimates correspond to the case (11).

The last (fourth) reduction makes the computations more compact. Recall that now all degrees d_i are equal to $a + 1$. Introduce the integer-valued function $\beta\colon \{2, \ldots, a\} \to \mathbb{Z}_+$ by the formula

$$\beta(t) = \binom{k(a - t + 1) + t}{t} = \binom{kb(t) + t}{t},$$

where $b(t) = a - t + 1$. Set also

$$\alpha = \alpha(M, k) = \binom{a + 1 + [2 \log k]}{a + 1}.$$

Proposition 3.4. *The following estimate holds:*

$$\gamma(\underline{d}, 0) \geqslant \min \left\{ \min_{t \in \{2, \ldots, a\}} \{\beta(t)\}, \alpha \right\}.$$

Proof. This follows immediately from the fact that for the special tuple $\underline{d}$ of equal degrees

$$m_{ki+1} = m_{ki+2} = \cdots = m_{ki+k} = i + 2$$

for $i = 0, \dots, a-1$. Q.E.D.

Therefore, the statement of Theorem 3.1 is implied by the following facts. In the both propositions below we assume that $M \geqslant 8k \log k - k$ and $k \geqslant 20$.

Proposition 3.5. *The minimum of the function $\beta(t)$ on the set $\{2, 3, \dots, a\}$ is attained at $t = 2$.*

Proposition 3.6. *The following inequality holds:*

$$\alpha(M, k) \geqslant A(M, k) = \frac{(M - 4k)(M - 5k)}{2} + (M + 2k).$$

Remark 3.2. The proof of Proposition 3.5 only requires $k \geqslant 10$, it is Proposition 3.6 that requires $k \geqslant 20$, see a more detailed Remark 3.3.

The rest of this section is a proof of the last two propositions, which requires some (quite non-trivial) analytic arguments.

3.3. The Stirling formula. The strategy of the proof of Proposition 3.5 is as follows. Using the Stirling formula, we construct a smooth function $\varepsilon \colon \mathbb{R}_+ \to \mathbb{R}$ such that $\varepsilon(t) \leqslant \beta(t)$ for $t = 2, \dots, a$ and ε approximates β with a good precision. Then we show that the minimum of the function $\varepsilon(t)$ on the interval $[2, a]$ occurs at one of the end points, either for $t = 2$ or for $t = a$. From this we deduce the claim of Proposition 3.5.

Recall that by the Stirling formula

$$n! = \sqrt{2\pi n} n^n \exp(-n) \exp\left(\frac{\theta_n}{12n}\right)$$

for some θ_n between 0 and 1. The integral parameter e, enumerating the polynomials g_e, will never be used again in this paper, so we use the symbol e for the number $\exp(1)$. Set

$$\varepsilon(t) = \frac{\sqrt{2\pi}}{e^2} (kb(t) + t)^{(kb(t)+t+\frac{1}{2})} (kb(t))^{-(kb(t)+\frac{1}{2})} t^{-(t+\frac{1}{2})},$$

by the Stirling formula $\beta(t) \geqslant \varepsilon(t)$.

Lemma 3.1. *The smooth function $\varepsilon(t)$ for $k \geqslant 3$ has only one critical point on the interval $[2, a]$, which is a maximum, so that the minimum of that function is attained at one of the end points.*

Proof. This is shown by demonstrating that

(1) for $2 \leqslant t \leqslant \frac{M+k}{2k}$ the function $\log \varepsilon(t)$ is strictly increasing,

(2) for $\frac{M+1}{k+1} \leqslant t \leqslant a = \frac{M}{k}$ it is strictly decreasing,

(3) for $\frac{M+k}{2k} \leqslant t \leqslant \frac{M+1}{k+1}$ the second derivative of $\log \varepsilon(t)$ is strictly negative (this is where the maximum lies).

The first derivative $\frac{d}{dt}\log\varepsilon(t)$ is equal to

$$\frac{t^2 - kb(t)^2}{2b(t)t(kb(t) + t)} - k \log\left(1 + \frac{t}{kb(t)}\right) + \log\left(1 + \frac{kb(t)}{t}\right), \qquad (12)$$

the second derivative $\frac{d^2}{dt^2}\log\varepsilon(t)$ is given by the formula

$$\frac{1}{b(t)t}+\frac{(t^2-kb(t)^2)^2}{2b(t)^2t^2(kb(t)+t)^2}+\frac{(k-1)(t^2-kb(t)^2)}{b(t)t(kb(t)+t)^2}-\frac{k(t+b(t))^2}{tb(t)(kb(t)+t)}. \tag{13}$$

We present the derivatives in these forms in order to use the inequality

$$\left|\frac{t^2-kb(t)^2}{2b(t)t(kb(t)+t)}\right|\leqslant\frac{1}{2b(t)}. \tag{14}$$

Now let us consider the domains (1)-(3) separately.

(1) Assume that $2\leqslant t\leqslant\frac{M+k}{2k}$. Note that on this interval $b(t)\geqslant 2$ so that

$$\left|\frac{t^2-kb(t)^2}{2b(t)t(kb(t)+t)}\right|\leqslant\frac{1}{4}.$$

The last term in the expression (12) can be estimated as

$$\log\left(1+\frac{kb(t)}{t}\right)\geqslant\log(1+k)\geqslant\log 21>3,$$

since on the interval $[2,\frac{M+k}{2k}]$ we have $t\leqslant b(t)$. Finally, for the second term in (12) we get

$$-k\log\left(1+\frac{t}{kb(t)}\right)\geqslant-\frac{t}{b(t)}\geqslant-1.$$

Combining these estimates, we obtain the inequality

$$\frac{d}{dt}\log\varepsilon(t)\Big|_{2\leqslant t\leqslant\frac{M+k}{2k}}\geqslant-\frac{5}{4}+3>0,$$

so that indeed $\varepsilon(t)$ is increasing on the interval under consideration.

(2) Assume now that $\frac{M+1}{k+1}\leqslant t\leqslant\frac{M}{k}$. Here $t\geqslant kb(t)\geqslant k$. First of all, we have the inequality

$$\left|\frac{t^2-kb(t)^2}{2b(t)t(kb(t)+t)}\right|\leqslant\frac{1}{2}.$$

For the other two terms in the expression (12) we get the estimates

$$-k\log\left(1+\frac{t}{kb(t)}\right)\leqslant-k\log 2$$

and

$$\log\left(1+\frac{kb(t)}{t}\right)\leqslant\log 2.$$

Combining these inequalities, we see that

$$\frac{d}{dt}\log\varepsilon(t)\leqslant\frac{1}{2}-(k-1)\log 2<0$$

for $t \in [\frac{M+1}{k+1}, \frac{M}{k}]$ as we claimed above.

(3) Finally, assume that $\frac{M+k}{2k} \leqslant t \leqslant \frac{M+1}{k+1}$. On this interval $b(t) \leqslant t \leqslant kb(t)$. Let us show that the second derivative (13) is negative. Using again the inequality (14), we get that $\frac{d^2}{dt^2} \log \varepsilon(t)$ on the interval under consideration is not higher than

$$\frac{1}{b(t)t} + \frac{1}{2b(t)^2} + \frac{(k-1)}{b(t)(kb(t)+t)} - \frac{k(t+b(t))^2}{tb(t)(kb(t)+t)} =$$

$$= \frac{t^2 + kb(t)(-2t^2 - 4b(t)t + 3t - 2b(t)^2 + 2b(t))}{2tb(t)^2(kb(t)+t)}.$$

Elementary computations, together with the inequality $t \leqslant a$, show that the expression in brackets in the numerator is not higher than $-2a^2 - a$. Therefore, the whole numerator is not higher than

$$t^2 - kb(t)(2a^2 + a) \leqslant kb(t)(t - 2a^2 - a) < 0.$$

We have shown that $\frac{d^2}{dt^2} \log \varepsilon(t) < 0$ for $t \in [\frac{M+k}{2k}, \frac{M+1}{k+1}]$. This completes the proof of Lemma 3.1. Q.E.D.

3.4. Proof of Proposition 3.5. In view of the inequality $\varepsilon(t) \leqslant \beta(t)$ and Lemma 3.1, Proposition 3.5 follows from the two lemmas stated below.

Lemma 3.2. *The inequality $\beta(2) \leqslant \varepsilon(3)$ holds.*

Lemma 3.3. *The inequality $\beta(2) \leqslant \varepsilon(a)$ holds.*

Proof of Lemma 3.2. We need to estimate the error in Stirling's approximation, in order to be able to use $\beta(3)$ instead of $\varepsilon(3)$. The number $\beta(3)$ is a polynomial in M, k, which makes the task easier. From the Stirling formula we get:

$$1.126 \cdot \varepsilon(3) \leqslant \beta(3) \leqslant 1.132 \cdot \varepsilon(3).$$

The inequality of the lemma will follow if it is shown that $1.14 \cdot \beta(2) < \beta(3)$ and this is equivalent to the inequality $G_1(M, k) = 6(\beta(3) - 1.14 \cdot \beta(2)) > 0$. Here $G_1(M, k)$ is given explicitly by the expression

$$M^3 + M^2(2.58 - 6k) + M(12k^2 - 17.16k + 0.74) - 8k^3 + 20.58k^2 - 11.74k - 0.84.$$

It is easy to check that $G_1(8k \log k, k)$ and the partial derivative $\frac{\partial}{\partial M} G_1(M, k)$ are both positive for $k \geqslant 20$ and $M \geqslant 8k \log k$. This completes the proof. Q.E.D.

Proof of Lemma 3.3. The claim of the lemma is equivalent to the inequality

$$G_2(M, k) = \log \varepsilon(a) - \log \beta(2) \geqslant 0.$$

A direct calculation gives $G_2(160 \log 20 - 20, 20) > 0$. Set

$$G_3(t) = \frac{d}{dt} G_2(8t \log t - t, t).$$

Lemma 3.4. $G_3(t) > 0$ *for* $t \geqslant 20$.

Proof. Explicitly,

$$G_3(t) = \log\left(1 + \frac{8\log t - 1}{t}\right) + \frac{8}{t}\log\left(1 + \frac{t}{8\log t - 1}\right) - \frac{1}{2t} + H_1(t) + H_2(t),$$

where

$$H_1(t) = \left(\frac{8}{t} + 1\right)\frac{8\log t + t - 0.5}{8\log t + t - 1} - \left(\frac{8}{t}\right)\frac{8\log t - 0.5}{8\log t - 1} - 1,$$

$$H_2(t) = -(8\log t + 6)\left(\frac{1}{8t\log t - 2t + 2} + \frac{1}{8t\log t - 2t + 1}\right).$$

Using the power series expansion of $\log(1+x)$, we obtain the inequality

$$G_3(t) > -\frac{1}{2t} + \frac{8\log t - 1}{t} - \frac{(8\log t - 1)^2}{2t^2} + \frac{8}{t}\log\left(1 + \frac{t}{8\log t - 1}\right) + H_1(t) + H_2(t).$$

For $t \geqslant 20$ then $H_1(t) \geqslant 0$ and $H_2(t) \geqslant -\frac{4}{t}$, which can be checked directly. This gives the inequality

$$G_3(t) > \frac{16\log t - 11}{2t} - \frac{(8\log t - 1)^2}{2t^2} + \frac{8}{t}\log\left(1 + \frac{t}{8\log t - 1}\right).$$

The right hand side of the last inequality is higher than

$$\frac{1}{2t^2}(16t\log t - 11t - 64\,(\log t)^2 + \log t - 1),$$

which is positive for $t \geqslant 20$. Q.E.D. for Lemma 3.4.

We conclude that $G_2(8t\log t - t, t) > 0$ for $t \geqslant 20$. The claim of Lemma 3.3 will be proven if we show that for $k \geqslant 20$ and $M \geqslant 8k\log k - k$ the function $G_2(M, k)$ is an increasing function of M. Set

$$G_4(s,t) = \frac{\partial}{\partial s}G_2(s,t).$$

Lemma 3.5. $G_4(s,t) > 0$ *for* $t \geqslant 20, s \geqslant 8t\log t - t$.

Proof. Explicitly,

$$G_4(s,t) = \frac{1}{t}\log\left(1 + \frac{t^2}{s}\right) - \frac{t^2}{2s(t^2+s)} - \frac{2s + 3 - 2t}{s^2 + (3-2t)s + t^2 - 3t + 2}.$$

First we consider the case when $s \leqslant t^2$ and get

$$G_4|_{s\leqslant t^2} \geqslant \frac{1}{t}\log 2 - \frac{t^2}{2s(t^2+s)} - \frac{2s + 3 - 2t}{s^2 + (3-2t)s + t^2 - 3t + 2}.$$

It is easy to see that the minimum of the right hand side occurs when $s = 8t\log t - t$ is the smallest possible, so that for $s \leqslant t^2$ the function $G_4(s,t)$ is bounded from below by the expression

$$\frac{1}{t}\log 2 - \frac{1}{(16\log t - 2)(t + 8\log t - 1)} -$$
$$-\frac{16t\log t + 3 - 4t}{t^2(8\log t - 1)^2 + (3-2t)(8t\log t - t) + t^2 - 3t + 2},$$

which is positive for $t \geqslant 20$.

Now let us consider the region $s \geqslant t^2$. Here we get

$$G_4(s,t) \geqslant \frac{t}{s} - \frac{t^3}{2s^2} - \frac{t^2}{2s(t^2+s)} - \frac{2s + 3 - 2t}{s^2 + (3-2t)s + t^2 - 3t + 2}.$$

A direct check shows that for $t \geqslant 20$ the expression in the right hand side is positive. Q.E.D. for Lemmas 3.5, 3.3 and Proposition 3.5.

3.5. Proof of Proposition 3.6. This proof is obtained in the same way as that of Proposition 3.5 and we only point out the main steps of the computations, leaving the details to the reader. In order to prove the inequality $\alpha(M,k) \geqslant A(M,k)$, we use the Stirling approximation of $\alpha(M,k)$. Namely, we introduce the function $G_5(s,t,r)$ of three real variables by the formula

$$G_5(s,t,r) = \left(\frac{s}{t} + r + \frac{3}{2}\right)\log\left(\frac{s}{t} + r + 1\right) - \left(r + \frac{1}{2}\right)\log r -$$
$$-\left(\frac{s}{t} + \frac{3}{2}\right)\log\left(\frac{s}{t} + 1\right) + \log\left(\frac{\sqrt{2\pi}}{e^2}\right) - \log A(s,t).$$

By the Stirling approximation, Proposition 3.6 follows from the inequality

$$G_5(M,k,[2\log k]) \geqslant 0.$$

It is easy to see that

$$G_5(M,k,[2\log k]) \geqslant G_5(M,k,2\log k - 1),$$

so we set $G_6(s,t) = G_5(s,t,2\log t - 1)$ and prove the inequality

$$G_6(s,t) \geqslant 0$$

for $s \geqslant 8t\log t - t$, $t \geqslant 20$. First of all, explicit computations show that

$$G_6(8t\log t - t, t) \geqslant 0$$

for $t \geqslant 20$. Set

$$G_7(s,t) = \frac{\partial}{\partial s}G_6(s,t).$$

It remains to show that for $s \geqslant 8t\log t - t$ and $t \geqslant 20$ the inequality $G_7(s,t) \geqslant 0$ holds. Explicitly, $G_7(s,t)$ is given by the expression

$$\frac{1}{t}\log\left(1+\frac{2\log t-1}{\frac{s}{t}+1}\right)-\frac{2\log t-1}{2t(\frac{s}{t}+1)(\frac{s}{t}+2\log t)}-\frac{2s-9t+2}{s^2-9ts+2s+20t^2+4t}.$$

Now the inequality $G_7(s,t) \geqslant 0$ is obtained by tedious but straightforward computations, using the estimate

$$\frac{1}{t}\log\left(1+\frac{2\log t-1}{\frac{s}{t}+1}\right)>\frac{2\log t-1}{t(\frac{s}{t}+1)}-\frac{(2\log t-1)^2}{2t(\frac{s}{t}+1)^2}.$$

The details are left to the reader. Q.E.D. for Proposition 3.6 and Theorem 3.1.

Remark 3.3. (i) It is clear from the computations presented in this section and in the proof of Lemma 1.3 that in certain parts of our arguments we need much weaker lower bounds for k. For instance, Lemma 3.2 requires only that $k \geqslant 5$ and Lemmas 3.3 and 3.4 require only that $k \geqslant 10$. In the last part of Subsection 3.5, for the inequality $G_7(s,t) \geqslant 0$ only the assumption $t \geqslant 10$ is needed. However, the inequality

$$G_6(8t\log t - t, t) \geqslant 0$$

requires that $t \geqslant 20$. In order for the whole argument to work, we have to select the strongest restriction.

(ii) One more paper [23] on birational superrigidity of non-singular Fano complete intersections of index one was put on the archive when our paper was finalized. We add it on the reference list.

References

[1] Pukhlikov Aleksandr, Birationally Rigid Varieties. Mathematical Surveys and Monographs **190**, AMS, 2013.

[2] Pukhlikov A. V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angew. Math. **541** (2001), 55-79.

[3] A.V.Pukhlikov, Birationally rigid Fano complete intersections. II. Crelle J. für die reine und angew. Math. **688** (2014), 209-218.

[4] Call F. and Lyubeznik G., A simple proof of Grothendieck's theorem on the parafactoriality of local rings, Contemp. Math. **159** (1994), 15-18.

[5] Pukhlikov A. V., Birationally rigid complete intersections with a singular point of high multiplicity. To appear in Proc. Edinburgh Math. Soc. Arxiv:1704.03795, 1-20.

[6] Pukhlikov A. V., The $4n^2$-inequality for complete intersection singularities, Arnold Math. J. **3** (2017), No. 2, 187-196.

[7] Evans D. and Pukhlikov A., Birationally rigid complete intersections of codimension two, Bull. Korean Math. Soc. **54** (2017), No. 5, 1627-1654. (ArXiv:1604.00512)

[8] Eckl Th. and Pukhlikov A.V., On the locus of non-rigid hypersurfaces. In: Automorphisms in Birational and Affine Geometry, Springer Proceedings in Mathematics and Statistics **79**, 2014, 121-139.

[9] Pukhlikov A. V., Birational automorphisms of Fano hypersurfaces, Invent. Math. **134** (1998), no. 2, 401-426.

[10] Iskovskikh V. A. and Pukhlikov A. V., Birational automorphisms of multi-dimensional algebraic varieties, Journal of Math. Sciences, **82** (1996), No. 4, 3528-3613.

[11] Cheltsov I. A. and Grinenko M. M., Birational rigidity is not an open property, Bull. Korean Math. Soc. **54** (2017), No. 5, 1485-1526.

[12] Pukhlikov A. V., Birationally rigid complete intersections of quadrics and cubics. Izvestiya: Mathematics **77** (2013), No. 4, 795-845.

[13] A.V.Pukhlikov, Birationally rigid Fano fibre spaces. II. Izvestiya: Mathematics, **79** (2015), No. 4, 809-837.

[14] Pukhlikov A. V., Birational geometry of algebraic varieties, fibred into Fano double spaces. To appear in: Izvestiya: Mathematics **81** (2017), No. 3, 618-644.

[15] Johnstone E., Birationally rigid singular double quadrics and double cubics, Math. Notes **102** (2017), No. 3-4, 508-515.

[16] Suzuki F., Birational rigidity of complete intersections, Math. Z. **285** (2017), No. 1-2, 479-492.

[17] Ahmadinezhad H. and Okada T., Birationally rigid Pfaffian Fano 3-folds. ArXiv: 1508.02974, 1-33.

[18] Cheltsov I. A. and Wotzlaw L., Nonrational complete intersections. Proc. Steklov Inst. math. **246** (2004), No. 3, 303-307.

[19] Cheltsov I.A. Non-rationality of a four-dimensional smooth complete intersection of a quadric and a quartic, not containing a plane, Sbornik: Mathematics, **194** (2003), No. 11-12, 1679-1699.

[20] Okada T., Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, Proc. Lond. Math. Soc. **109** (2014), No. 6, 1549-1600.

[21] Okada T., Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections. II. ArXiv:1310.5320.

[22] Okada T., Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections. III. ArXiv:1409.1506.

[23] Zhuang Z., Birational superrigidity and K-stability of Fano complete intersections of index one. ArXiv:1802.08389, 1-16.

Department of Mathematical Sciences,
The University of Liverpool

pukh@liverpool.ac.uk, sgdevans@liverpool.ac.uk