Computing tight bounds of structural reliability under imprecise probabilistic information

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Abstract

In probabilistic analyses and structural reliability assessments, it is often difficult or infeasible to reliably identify the proper probabilistic models for the uncertain variables due to limited supporting databases, e.g., limited observed samples or physics-based inference. To address this difficulty, a probability-bounding approach can be utilized to model such imprecise probabilistic information, i.e., considering the bounds of the (unknown) distribution function rather than postulating a single, precisely specified distribution function. Consequently, one can only estimate the bounds of the structural reliability instead of a point estimate. Current simulation technologies, however, sacrifice precision of the bound estimate in return for numerical efficiency through numerical simplifications. Hence, they produce overly conservative results in many practical cases. This paper proposes a linear programming-based method to perform reliability assessments subjected to imprecisely known random variables. The method computes the tight bounds of structural failure probability directly without the need of constructing the probability bounds of the input random variables. The method can further be used to construct the best-possible bounds for the distribution function of a random variable with incomplete statistical information.

Keywords: Structural reliability analysis, uncertainty, probability box, Monte Carlo simulation, interval analysis, imprecise probability

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1. Introduction

The various sources of uncertainties arising from structural capacities and applied loads, as well as computational models, are at the root of the structural safety problem of civil structures. In an attempt to measure the safety of a structure, it is necessary to quantify and model these uncertainties with a probabilistic approach so as to further determine the failure probability [1–4]. In a reliability assessment, the identification of the probability distributions of the random variables is crucial. The uncertainty associated with a random variable can be classified into either aleatory or epistemic [5], with the former arising from the inherent random nature of the quantity, and the latter due to knowledge-based factors such as imperfect modelling and simplifications, and/or limited supporting database. Statistical uncertainty is an important source of the epistemic uncertainty, which accounts for the difference between the probability model of a random variable inferred from limited sampled data and the "true" one. This uncertainty may be significant if the size of available data/observations is limited. To better assess the safety of a structure, structural reliability assessment needs to consider both aleatory and epistemic uncertainties [5–9].

The result of a structural reliability assessment may be sensitive to the selection of the probability distributions of the random inputs [10]. However, in many cases, the identification of a variable's distribution function is difficult or even impossible due to limited information/data. Rather, only incomplete information such as the first- and the second-order moments (mean and variance) of the variable can be reasonably estimated. In such a case, the incompletely-informed random variable can be quantified by a *family* of candidate probability distributions rather than a single known distribution function. This is the basic concept of *imprecise probability* [11]. As a result, the structural reliability in the presence of incompletely-informed random variables can no longer be uniquely determined. A practical way to represent an imprecise probability is to use a probability bounding approach by considering the lower and upper bounds of the imprecise probability functions. Under this context, approaches of interval estimate of reliability have been used to deal with reliability problems with imprecise probabilistic information [12], including the probability-box (p-box for short) method [13], random set and Dempster-Shafer evidence theory [14–16], fuzzy random variables [17], and others. These methods are closely related to each other, and may often be used as equivalent for the purpose of reliability assessment [13, 18]. However, the bounds of structural reliability estimated using a probability bounding approach may be overly conservative in some cases, due to the fact that it only considers the bounds of the distribution function, thus some useful information inside the bounds may be lost. This fact calls for an improved approach for reliability bound estimate which can take full use of the imprecise information of the variable(s).

Over the last decade many efforts have been directed towards structural reliability assessment using imprecise probability theory. In [19], random variables and interval variables are considered simultaneously. Monte Carlo simulation was used with function approximation to reduce the total number of simulations. In [20, 21], imprecisely probability distribution functions were modeled using probability-boxes and Dempster-Shafer structures. The reliability analysis was based on the Cartesian product method and interval arithmetic. The framework was applied to environmental risk assessment. Schweiger and Peschl [22] considered stochastic finite element analyses of a deep excavation problem in which the uncertain material parameters and geometrical data were modeled as random sets. The random sets were propagated through the finite element analysis using the vertex method, under the assumption that the structural response is monotonic with respect to each random set variable. In [23], structural reliability evaluations in the presence of both random variables and interval variables were considered. The limit state functions were approximated using the response surface method to reduce the computational cost. In [24], the Tchebycheff's inequality was proposed to construct random set models of a random variable using the information of mean and standard deviation. The approach was demonstrated using two geotechnical problems. An interval Monte Carlo method was developed in [9] for structural reliability assessment under epistemic uncertainties. An imprecise cumulative distribution function with interval parameters is modeled as a probability-box. In each simulation, interval-valued samples are sampled and the range of the limit state function is computed using interval analysis. A similar approach, namely the unified interval stochastic sampling approach, was proposed in [25] to determine the statistics of the lower and upper bounds of the collapse loads of a structure involving mixture of random and interval parameters. Variance-reduction techniques have been proposed to combine with the interval Monte Carlo simulation to enhance the computational efficiency, e.g., the interval importance sampling technique [18], the interval Quasi-Monte Carlo sampling [26], and subset sampling [16, 27].

Mathematically, the use of the (complete) moment information of a random variable is equivalent to its probability distribution function since knowing one can determine the other completely through the moment generation function [28, 29]. Many previous studies have conducted reliability analysis by making use of the moment information of random variables. For instance, a second-order reliability analysis method based on an approximating paraboloid was proposed in [30]. In [31], a method for system reliability analysis was developed taking into account the moments of the system limit state function derived from point estimates. Zhao et al. [32] discussed the suitability and the monotonicity of the fourth-moment normal transformation in reliability assessment considering imprecise random inputs. Wang et al. [33] proposed an approach to estimate the time-dependent reliability of aging structures in the presence of incomplete deterioration information.

This paper considers the case of reliability assessment with imprecise probabilities in which only the low-order moments of a random variable are known, while the distribution type and distribution function are unknown. The motivation of using (limited) moment information for reliability assessment is due to the fact that in many cases only limited observations/samples of a random variable are accessible, and thus the estimation of the moments (typically the low order moments such as mean and variance) based on the limited samples is relatively straightforward and more reliable as compared with estimating the complete distribution function.

This paper proposes a linear programming-based method for solving the reliability problems in the presence of imprecise probabilistic information. The estimate of reliability bounds is transformed into finding the solution of a linear objective function, where the constraint equations are established by taking full use of the information of moments, and the range information of the random variable if available. Two types of objective functions are developed independently, which can verify the accuracy of the solutions mutually, and provide insights into the problem from different perspectives. The paper first introduces the methodology for the problems involving only one imprecise random variable; then an iterative approach is proposed to handle the problems with multiple imprecise random variables. While the proposed method computes bounds of failure probabilities directly without first constructing the probability-boxes of the imprecisely known random input variables, it can also be used to construct the best-possible cumulative distribution function (CDF) bounds for a random variable with limited statistical information. Three examples are presented to demonstrate the application of the proposed method on these two aspects.

2. Probability-box method in the presence of imprecise random variables

2.1. Impact of imprecision on reliability assessment

A typical structural reliability problem takes the form of

$$P_f = \Pr(G(\mathbf{X}) \leq 0) = \int \ldots \int_{G(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{1}$$

where Pr denotes the probability of the event in the bracket, P_f represents the failure probability of the structure, G is the limit state function in the presence of m random inputs $\mathbf{X} = \{X_1, X_2, \ldots X_m\}$, which defines structural failure if $G < 0$ and the survival of the structure otherwise, and $f_{\mathbf{X}}(\mathbf{X})$ is the joint probability density function (PDF) of $\mathbf{X}$. The failure probability in Eq. (1) is often estimated by the well-known Monte Carlo method,

$$P_f \approx \frac{1}{N} \sum_{j=1}^{N} \mathbb{I}\left[G(\mathbf{x}_j) \leq 0\right] \tag{2}$$

where N is the number of replications, $\mathbb{I}[\cdot]$ is an indicator function, which returns 1 if the statement in the bracket is true and 0 otherwise, and $\mathbf{x}_j$ is the jth simulated sample of $\mathbf{X}$. $\mathbf{x}_j$ can be generated using the inverse transform method,

$$\mathbf{x}_j = F_{\mathrm{X}}^{-1}(\mathbf{r}_j), \quad j = 1, 2, \ldots, N \tag{3}$$

with $F_{\mathrm{X}}(\,)$ being the CDF of $\mathbf{X}$, and $\mathbf{r}_j$ a sample of standard uniform random variates [1].

When the distribution function of $\mathbf{X}$ cannot be determined uniquely and one has to consider a family of all possible distribution functions, the probability of failure will vary in an interval $[\underline{P_f}, \overline{P_f}]$, which can be estimated by the interval Monte Carlo method [34]:

$$\underline{P_f} = \min\{\frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[G\left(F_X^{-1}(\mathbf{r}_j)\right) \leq 0\right], \text{ for all possible } F_X\}, \tag{4}$$

and

$$\overline{P_f} = \max\{\frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[G\left(F_X^{-1}(\mathbf{r}_j)\right) \leq 0\right], \text{ for all possible } F_X\}. \tag{5}$$

where $\underline{P_f}$ and $\overline{P_f}$ represent the lower and upper bounds of P_f, respectively.

2.2. Probability box approach

A probability-box describes a family of distribution functions by specifying the lower and upper bounds of the CDF, i.e.,

$$\underline{F}_X(x) \leq F_X(x) \leq \overline{F}_X(x), \quad x \in \mathbb{R} \tag{6}$$

where $F_X(x)$ is the (unknown) CDF of X, $\underline{F}_X$ and $\overline{F}_X$ are the lower and upper bounds of F_X respectively.

For a number of cases of imprecise probability, methods are available in the literature to construct the corresponding probability boxes. If only the mean and standard deviation of X are known, denoted by μ_X and σ_X respectively, and the distribution type is unknown, Chebyshev's inequality gives a lower and an upper bound of F_X [35], i.e.,

$$\underline{F}_X(x) = \begin{cases} 0, & x \leq \mu_X + \sigma_X \\ 1 - \dfrac{\sigma_X^2}{(x-\mu_X)^2}, & x \geq \mu_X + \sigma_X \end{cases} \tag{7a}$$

$$\overline{F}_X(x) = \begin{cases} \dfrac{\sigma_X^2}{(x-\mu_X)^2}, & x \leq \mu_X - \sigma_X \\ 1, & x \geq \mu_X - \sigma_X \end{cases} \tag{7b}$$

However, the CDF bounds as given in Eq. (7) are not the best-possible. As will be shown later in this paper, tighter CDF bounds can be constructed for this case.

In practice, the bounds of a random variable are often known, e.g., structural loads are non-negative. The range information can be utilized to tighten the bounds of F_X. Let $\underline{x}$ and

$\overline{x}$ denote the minimum and maximum of X, respectively, Ferson et al. [13] gave a tighter bounds of F_X as follows,

$$\underline{F}_X(x) = \begin{cases} 0, & x \leq \mu_X + \sigma_X^2/(\mu_X - \overline{x}) \\ 1 - [b(1+a) - c - b^2]/a, & \mu_X + \sigma_X^2/(\mu_X - \overline{x}) < x < \mu_X + \sigma_X^2/(\mu_X - \underline{x}) \\ 1/[1 + \sigma_X^2/(x - \mu_X)^2], & \mu_X + \sigma_X^2/(\mu_X - \underline{x}) \leq x < \overline{x} \\ 1, & x \geq \overline{x} \end{cases} \tag{8a}$$

$$\overline{F}_X(x) = \begin{cases} 0, & x \leq \underline{x} \\ 1/[1 + (x - \mu_X)^2/\sigma_X^2], & \underline{x} \leq x < \mu_X + \sigma_X^2/(\mu_X - \overline{x}) \\ 1 - (b^2 - ab + c)/(1 - a), & \mu_X + \sigma_X^2/(\mu_X - \overline{x}) < x < \mu_X + \sigma_X^2/(\mu_X - \underline{x}) \\ 1, & x \geq \mu_X + \sigma_X^2/(\mu_X - \underline{x}) \end{cases} \tag{8b}$$

where $a = (x - \underline{x})/(\overline{x} - \underline{x})$, $b = (\mu_X - \underline{x})/(\overline{x} - \underline{x})$, and $c = \sigma_X^2/(\overline{x} - \underline{x})^2$. Note that the CDF bounds as defined in Eq. (8) are the best possible bounds in the sense that the bounds cannot be any tighter if one only knows the min, max, mean and variance of a random variable.

A distribution function with uncertain parameters represents another common case of imprecise probabilities. As the statistical parameters of a distribution function are usually estimated by statistical inference from sample observations, uncertainties arise in the estimation of the parameters when the available data is limited. A natural way to quantify the uncertainty of the parameters is to use the confidence intervals which define interval bounds of the distribution parameters. Zhang et al. [18, 34] have considered the case in which the distribution type is known, but the distribution parameters are uncertain and modeled by intervals.

The present paper considers the imprecise probabilities in which the available information is limited to the mean and variance (either point estimates or interval estimates), and the range of the random variable (if available). The distribution type is assumed to be unknown.

2.3. Interval Monte Carlo methods to propagate p-boxes

When the reliability analysis involves probability-boxes, an interval Monte Carlo method can be used to propagate probability boxes and compute the bounds of probability of failure. The basic Monte Carlo simulation as in Eq. (2) is extended to the case where the distribution function F_X is a p-box. In the presence of the CDF envelope (c.f. Eq. (6)) for $\mathbf{X}$, for each simulation run, two samples can be generated from the lower and upper bounds of F_X, respectively, i.e.,

$$\begin{aligned} \underline{\mathbf{x}}_j &= \overline{F}_X^{-1}(\mathbf{r}_j), \\ \overline{\mathbf{x}}_j &= \underline{F}_X^{-1}(\mathbf{r}_j), \quad j = 1, \ldots, N. \end{aligned} \tag{9}$$

The interval $[\underline{\mathbf{x}}_j, \overline{\mathbf{x}}_j]$ contains all possible simulated numbers from the family of distributions contained in the p-box for a given value of $\mathbf{r}_j$.

Let $\min G(\mathbf{x}_j)$ and $\max G(\mathbf{x}_j)$ respectively denote the minimum and maximum of the limit state function $G(\mathbf{X})$ when $\underline{\mathbf{x}}_j \leq \mathbf{X} \leq \overline{\mathbf{x}}_j$. It simply follows,

$$\mathbb{I}\left[\max G(\mathbf{x}_j) \leq 0\right] \leq \mathbb{I}\left[G(\mathbf{x}_j) \leq 0\right] \leq \mathbb{I}\left[\min G(\mathbf{x}_j) \leq 0\right], \tag{10}$$

which further gives

$$\frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[\max G(\mathbf{x}_j) \leq 0\right] \leq \frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[G(\mathbf{x}_j) \leq 0\right] \leq \frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[\min G(\mathbf{x}_j) \leq 0\right]. \tag{11}$$

Thus, a lower and an upper bounds of P_f, $\underline{P_f}$ and $\overline{P_f}$, are obtained respectively as follows [34],

$$\underline{P_f} = \frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[\max G(\mathbf{x}_j) \leq 0\right], \tag{12}$$

and

$$\overline{P_f} = \frac{1}{N}\sum_{j=1}^{N}\mathbb{I}\left[\min G(\mathbf{x}_j) \leq 0\right]. \tag{13}$$

Details about interval Monte Carlo method can be found elsewhere [18, 34]. Clearly, the reliability bounds as given by Eqs. (12,13) are more conservative than the true bounds of

Eqs. (4,5).

3. Linear programming-based reliability bounds analysis

3.1. Problems involving one imprecise random variable

We first consider the case of one imprecise probability. Consider a reliability analysis problem involving the random variables $[Q, \mathbf{S}]$, in which Q is a random variable with an imprecise distribution function, and $\mathbf{S} = [S_1, S_2, \dots]$ is the remaining random vector with a known joint distribution function. Q and $\mathbf{S}$ are assumed to be statistically independent. The failure probability is given by

$$P_f = \int_{G(\mathbf{S},Q)\leq 0} f_Q(q) f_{\mathbf{S}}(\mathbf{s}) \mathrm{d}q \mathrm{d}\mathbf{s}, \tag{14}$$

in which $f_Q(q)$ and $f_{\mathbf{S}}(\mathbf{s})$ are the probability density functions of Q and $\mathbf{S}$, respectively. Eq. (14) can be rewritten as

$$P_f = \int f_Q(q) \xi_Q(q) \mathrm{d}q, \tag{15}$$

in which $\xi_Q(q)$ represents the conditional failure probability on $Q = q$, i.e.,

$$\xi_Q(q) \triangleq \Pr(G(\mathbf{S}, Q = q) \leq 0) = \int_{G(\mathbf{S},Q=q)\leq 0} f_{\mathbf{S}}(\mathbf{s}) \mathrm{d}\mathbf{s}. \tag{16}$$

Note that the conditional failure probability $\xi_Q(q)$ for a given value of $Q = q$ is customarily referred to as *fragility* in the risk analysis of natural hazards [36]. The conditional failure probability $\xi_Q(q)$ may be obtained analytically through the integration in Eq. (16), or numerically using the Monte Carlo methods.

To facilitate the derivation, Q is normalized into $[0, 1]$ by introducing a reduced random variable $X = \dfrac{Q - Q_{\min}}{Q_{\max} - Q_{\min}}$, where $Q_{\max}$ and $Q_{\min}$ are the maximum and minimum of Q, respectively. With this, Eq. (15) becomes

$$P_f = \int_0^1 f_X(x) \xi(x) \mathrm{d}x \tag{17}$$

where $f_X(x)$ is the PDF of X, and $\xi(x) = \xi_a\left((Q_{\max} - Q_{\min})x + Q_{\min}\right)$. The computation of tight bounds of Eq. (17) is discussed next, employing the algorithms of linear programming.

3.2. Objective function Type 1

As a starting point, consider the case where the only information about the imprecise probability Q is its first two moments, i.e., the mean (μ_Q) and the standard deviation (σ_Q). To apply Eq. (17), the maximum and minimum of Q need to be estimated. In practice, they can be approximated as $\mu_Q \pm k\sigma_Q$, in which k is sufficiently large (e.g., $k = 5$). Clearly, the mean and standard deviation of the reduced variable X are

$$\mu_X = \frac{\mu_Q - \min Q}{\max Q - \min Q},\ \sigma_X = \frac{\sigma_Q}{\max Q - \min Q}. \tag{18}$$

Let $\mathbb{E}(X^\tau)$ represent the τth moment of X. Lemma 1 in Appendix A states that $[\ln(\mathbb{E}(X^\tau))]'$ increases with τ for positive integer values of τ. Thus, $\dfrac{\ln(\mathbb{E}(X^{j+1})) - \ln(\mathbb{E}(X^j))}{\ln(\mathbb{E}(X^j))}$ also increases with j for $j = 1, 2, \ldots$. Fig. 1(a) illustrates the possible trajectories of $\ln(\mathbb{E}(X^j))$ as a function of j, provided that $\ln(\mathbb{E}(X)) = \ln\mu_X$ and $\ln(\mathbb{E}(X^2)) = \ln(\mu_X^2 + \sigma_X^2)$ are known. The trajectories are bounded within a circular sector with a central angle of θ_2. The upper bound of the logarithm of the jth moment is $\ln(\mu_X^2 + \sigma_X^2)$, while the lower bound is a half-line $p_0 j + q$, where

$$p_0 = \ln\frac{\mu_X^2 + \sigma_X^2}{\mu_X},\ q_0 = \ln\frac{\mu_X^2}{\mu_X^2 + \sigma_X^2}. \tag{19}$$

That is,

$$p_0 j + q_0 < \ln\left(\mathbb{E}(X^j)\right) < \ln\left(\mu_X^2 + \sigma_X^2\right) \tag{20}$$

for all integers $j > 2$. The cental angle, θ_2, equals to $|\arctan(p_0)|$. Further, if the higher-order (up to the mth) logarithmic moments of X, $\ln(\mathbb{E}(X))$, $\ln(\mathbb{E}(X^2))$, $\ldots \ln(\mathbb{E}(X^m))$ are known (see Fig. 1(b)), then the central angle for the mth order of moment, θ_m, is

$$\theta_m = \left|\arctan\left(\ln\frac{\mathbb{E}(X^{m-1})}{\mathbb{E}(X^m)}\right)\right|, \tag{21}$$

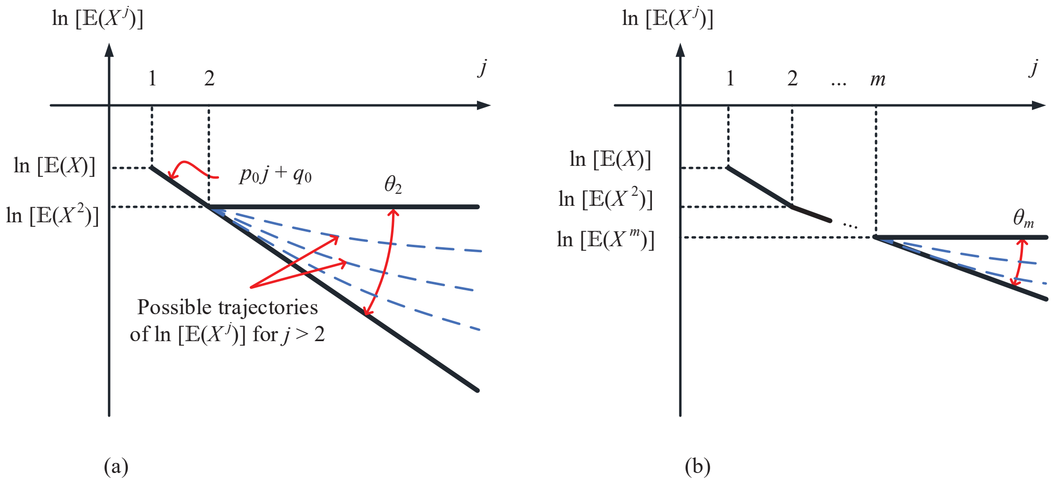


Figure 1: Schematic representation of the jth order moment of X and its bounds.

which converges to 0 when m is sufficiently large since

$$\lim_{m\to\infty} \frac{\mathbb{E}(X^{m-1})}{\mathbb{E}(X^m)} = 1. \tag{22}$$

This fact indicates that the more orders of moment are known, the more precise the probabilistic characteristics of X can be determined. Fig. 1 provides a graphical explanation of the precision of a random variable with limited orders of moments known.

In Eq. (17), as the distribution type of X is unknown, the values of $f_X(x)$ for each x cannot be uniquely determined. The domain of X ($[0,1]$) is discretized into n identical sections, $[x_0 = 0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n = 1]$, where n is sufficiently large such that $\left| f_X(x) - f_X\left(\frac{x_{i-1}+x_i}{2}\right)\right|$ is negligible for $\forall i = 1, 2, \ldots n$ and $\forall x \in [x_{i-1}, x_i]$. The sequence $f_X\left(\frac{x_{i-1}+x_i}{2}\right), \forall i = 1, 2, \ldots n$ is denoted by $\{f_1, f_2, \ldots f_n\}$ for the purpose of simplicity. With this, Eq. (17) can be approximated by

$$P_f = \int_0^1 \xi(x) f_X(x) \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} \xi\left(\frac{i-0.5}{n}\right) \frac{1}{n} \cdot f_i. \tag{23}$$

Note that the definition of the mean value and variance of X, as well as the basic characteristics of a distribution function simultaneously give

$$\begin{cases} \sum_{i=1}^{n} f_i \cdot \frac{1}{n} = 1 \\ \sum_{i=1}^{n} f_i \cdot \frac{1}{n} \cdot \frac{i}{n} = \mu_X \\ \sum_{i=1}^{n} f_i \cdot \frac{1}{n} \left(\frac{i}{n}\right)^2 = \mu_X^2 + \sigma_X^2 \\ 0 \leq f_i \leq n, \forall i = 1, 2, \ldots n. \end{cases} \tag{24}$$

Eqs. (23) and (24) indicate that the bound estimate of P_f can be converted into a classic linear programming problem, i.e., Eq. (23) is the objective function to be optimized, $\mathbf{f} = \{f_1, f_2, \ldots f_n\}$ are the vector of variables to be determined, and Eq. (24) represents the constraints. A brief introduction of linear programming is presented in Appendix B. The algorithms of linear programming-based optimization have been well studied and can be found elsewhere, e.g., [37–40].

Eqs. (23) and (24) represents a linear programming-based approach to compute the reliability bounds for imprecise probability distributions. Another useful application of Eqs. (23) and (24) is to construct the best-possible CDF bounds for a random variable with incomplete information. For an arbitrary value of τ, by setting

$$\xi(x) = \mathbb{I}(\tau \geq x) = \begin{cases} 1, & x \leq \tau \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

Eq. (23) becomes

$$\int_0^1 \xi(x) f_X(x) \mathrm{d}x = \int_0^\tau f_X(x) \mathrm{d}x = F_X(\tau). \tag{26}$$

Thus, by solving the linear programming problem defined by Eqs. (26, 24), the best-possible bounds for $F_X(\tau)$ can be obtained.

The constraints in Eq. (24) represent the case in which the only knowledge available are the point estimates of the mean and the standard deviation. The constraints can be easily modified for more generalized cases if additional information is provided. For example, if X is known to be strictly defined in the range $[\underline{x}, \overline{x}]$, where $0 \leq \underline{x} \leq \overline{x} \leq 1$, the introduction of a new variable $X' = \frac{X - \underline{x}}{\overline{x} - \underline{x}}$ enables the applicability of Eq. (24). Moreover, if the mean value of X is an interval estimate of $[\underline{\mu}_X, \overline{\mu}_X]$ rather than a point estimate, the second constraint equation in Eq. (24), $\sum_{i=1}^{n} f_i \cdot \frac{1}{n} \cdot \frac{i}{n} = \mu_X$, is modified as

$$\begin{cases} \sum_{i=1}^{n} f_i \cdot \frac{-1}{n} \cdot \frac{i}{n} \leq -\underline{\mu}_X \\ \sum_{i=1}^{n} f_i \cdot \frac{1}{n} \cdot \frac{i}{n} \leq \overline{\mu}_X. \end{cases} \tag{27}$$

A similar modification can be made to the third constraint equation in Eq. (24) if the standard deviation of X is known to have a predefined range. It should be noted that the probability-box obtained by the proposed linear programming method will be identical to the probability-box given by Eq. (8) if one knows the min, max, mean and variance of a random variable. However, the proposed linear programming-based approach represents a more general method for constructing the best-possible probability-boxes.

3.3. Objective function Type 2

While Eqs. (23) and (24) have established a straightforward approach for estimating the bounds of structural failure probability, the accuracy and efficiency of the method is yet to be investigated. An important question has been raised: have Eqs. (23) and (24) made full use of the imprecise information of X? In an attempt to address this issue, as well as to form a different insight into the problem, this section reformulates the reliability bounds-estimate problem using a different objective function, referred to as objective function Type 2.

Reconsider Eq. (17), where the variable X is assumed to have a mean value of μ_X, a standard deviation of σ_X and unknown distribution type. Fig. 1 and Lemma 1 in Appendix A have demonstrated the nonlinearity of $\ln(\mathbb{E}(X^j))$ with j. As the basis of further derivation, however, we consider a *fictitious* case where X has linear logarithmic moments, determined by a parameter pair (p_i, q_i). That is, $\ln(\mathbb{E}(X^j)) = p_i j + q_i$ for all integers $j \geq 2$. Since $\mathbb{E}(X^2) = \exp(2p_i + q_i)$, $q_i = \ln(\mu_X^2 + \sigma_X^2) - 2p_i$. The corresponding fictitious failure probability is denoted by $P_f(p_i)$. Lemma 2 in Appendix A gives the solution of $P_f(p_i)$ as a function of p_i. The choice of p_i can be arbitrary, as long as it satisfies $p_i \leq 0$.

For a sufficiently large integer n and $n-2$ different p_i's (denoted by $p_1, p_2, \ldots p_{n-2}$ respectively), let $\widetilde{E}_{ij} = \exp\left[p_j \cdot (i+1) + q_j\right]$ for $1 \leq i \leq n-2$ and $1 \leq j \leq n-2$, where

$q_j = \ln \mathbb{E}(X^2) - 2p_j$ for $\forall j$. With this,

$$\widetilde{E}_{ij} = \exp\left[p_j \cdot (i-1)\right] \cdot \mathbb{E}(X^2). \tag{28}$$

A sequence of constants $\{\gamma_i | i = 1, 2, \ldots n-2\}$ can be found such that

$$\mathcal{E} = \sum_{i=1}^{n-2} \gamma_i \hat{\boldsymbol{E}}_i \tag{29}$$

where $\mathcal{E} = \begin{bmatrix} \mathbb{E}(X^2) & \mathbb{E}(X^3) & \ldots & \mathbb{E}(X^{n-1}) \end{bmatrix}^\mathsf{T}$, and $\hat{\boldsymbol{E}}_i = \begin{bmatrix} \widetilde{E}_{1i} & \widetilde{E}_{2i} & \ldots & \widetilde{E}_{(n-2)i} \end{bmatrix}^\mathsf{T}$. The existence of sequence $\{\gamma_i\}$ in Eq. (29) is guaranteed by the fact that $\det \begin{bmatrix} \hat{\boldsymbol{E}}_1 & \hat{\boldsymbol{E}}_2 & \ldots & \hat{\boldsymbol{E}}_{m-2} \end{bmatrix} \neq 0$. According to Lemma 3 (see Appendix A),

$$P_f = \xi(0) + \begin{bmatrix} \xi(\widetilde{\beta}_1) - \xi(0) \\ \xi(\widetilde{\beta}_2) - \xi(0) \\ \vdots \\ \xi(\widetilde{\beta}_{n-2}) - \xi(0) \end{bmatrix}^\mathsf{T} \cdot \mathcal{B}^{-1} \cdot \mathcal{E} \tag{30}$$

where $\mathcal{B}$ is defined in Eq. (A.10). Substituting Eq. (29) into Eq. (30) yields

$$\begin{aligned} P_f &= \xi(0) + \sum_{i=1}^{n-2} \gamma_i \begin{bmatrix} \xi(\widetilde{\beta}_1) - \xi(0) \\ \xi(\widetilde{\beta}_2) - \xi(0) \\ \vdots \\ \xi(\widetilde{\beta}_{m-2}) - \xi(0) \end{bmatrix}^\mathsf{T} \cdot \mathcal{B}^{-1} \cdot \hat{\boldsymbol{E}}_i \\ &= \xi(0) + \sum_{i=1}^{n-2} \gamma_i (P_f(p_i) - \xi(0)) = \sum_{i=1}^{n-2} P_f(p_i) \gamma_i. \end{aligned} \tag{31}$$

Substituting Eq. (28) into Eq. (29) yields

$$\mathcal{E} = \mathbb{E}(X^2) \cdot \mathbf{P} \cdot \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n-2} \end{bmatrix}^\mathsf{T} \tag{32}$$

where $\mathbf{P} = [p_{ij}]_{(n-2)\times(n-2)}$ with $p_{ij} = \exp[p_j \cdot (i-1)]$ for $\forall i, j = 1, 2, \ldots n-2$. Note that by definition, as n is large enough, for $k = 2, 3, \dots n-1$,

$$\mathbb{E}(X^k) = \int_0^1 x^k \cdot f_X(x)\mathrm{d}x = \sum_{i=1}^{n-2} \int_{(i-1)/(n-2)}^{i/(n-2)} x^k \cdot f_X(x)\mathrm{d}x. \tag{33}$$

With the mean value theorem, there exists a sequence $\{\epsilon_i | i = 1, 2, \dots n-2, \frac{i-1}{n-2} < \epsilon_i < \frac{i}{n-2}\}$ such that

$$\mathbb{E}(X^k) = \sum_{i=1}^{n-2} \epsilon_i^k \cdot \frac{f_X(\epsilon_i)}{n-2}, \quad k = 2, 3, \dots n-1 \tag{34}$$

or equivalently,

$$\mathcal{E} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \epsilon_1^1 & \epsilon_2^1 & \cdots & \epsilon_{n-2}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_1^{n-3} & \epsilon_2^{n-3} & \cdots & \epsilon_{n-2}^{n-3} \end{bmatrix} \cdot \begin{bmatrix} \frac{f_X(\epsilon_1)}{n-2} \cdot \epsilon_1^2 \\ \frac{f_X(\epsilon_2)}{n-2} \cdot \epsilon_2^2 \\ \vdots \\ \frac{f_X(\epsilon_{n-2})}{n-2} \cdot \epsilon_{n-2}^2 \end{bmatrix}. \tag{35}$$

Comparing Eqs. (32) and (35), assigning $\exp(p_i) = \epsilon_i$ gives

$$\gamma_i = \frac{1}{\mathbb{E}(X^2)} \cdot \frac{f_X(\epsilon_i)}{n-2} \cdot \epsilon_i^2, \quad i = 1, 2, \dots n-2 \tag{36}$$

with which one has

$$\begin{cases} \sum_{i=1}^{n-2} \gamma_i = 1 \\ \sum_{i=1}^{n-2} \frac{\gamma_i}{\epsilon_i} = \frac{\mu_X}{\mu_X^2 + \sigma_X^2} \\ \sum_{i=1}^{n-2} \frac{\gamma_i}{\epsilon_i^2} = \frac{1}{\mu_X^2 + \sigma_X^2} \\ 0 \le \gamma_i \le 1, \forall i = 1, 2, \dots n-2. \end{cases} \tag{37}$$

With Eqs. (31) and (37), finding the lower and upper bounds of P_f can be formulated as a linear programming optimization, i.e., Eq. (31) is the objective function to be optimized, $\{\gamma_1, \gamma_2, \dots \gamma_{n-2}\}$ are the variable vector to be determined, and Eq. (37) is the constraints.

In the implementation, one can assign $\epsilon_i = \frac{i-0.5}{n-2}$ for $\forall i = 1, 2, \dots n-2$ since $\frac{i-1}{n-2} < \epsilon_i < \frac{i}{n-2}$ and n is sufficiently large. With this, $\epsilon_i = \exp(p_i)$ gives $p_i = \ln(\epsilon_i)$ for $\forall i$.

The new objective function in Eq. (31) as well as the constraint equations in Eq. (37) have been developed independently of those in Eqs. (23) and (24). Thus, the results from the two objective functions can be used for mutual verification. Moreover, Eqs. (31) and (37) can also be extended to the case where X has a predefined range $[\underline{x}, \overline{x}]$. As introduced in

Section 3.1, this can be handled by introducing a normalized variable $X' = \frac{X-\underline{x}}{\overline{x}-\underline{x}}$. However, Eq. (31) is not applicable to the case where the statistical parameters of X (mean or standard deviation) vary in intervals, since the statistics of X are explicitly involved in the objective function. From this point of view, objective function Type 1 is a more general approach.

3.4. Problems with multiple imprecise random variables

Sections 3.1 to 3.3 have discussed the case of only one imprecise random variable. This section discusses the reliability problems involving multiple imprecise random variables. Suppose the reliability problem involves a mixture of imprecise random variables and conventional random variables, $[\mathbf{Q}, \mathbf{S}]$, in which $\mathbf{Q} = \{Q_1, Q_2, \ldots Q_k\}$ is the vector of k imprecise random variables with unknown distribution functions, while $\mathbf{S}$ is the conventional random vector with known distribution function. Similar to Eq. (15), the failure probability is given by

$$P_f = \int_{G(\mathbf{S},\mathbf{Q})\leq 0} f_{\mathbf{Q}}(\mathbf{q}) f_{\mathbf{S}}(\mathbf{s}) \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{s} \tag{38}$$

where $f_{\mathbf{Q}}(\mathbf{q})$ is the joint distribution of $\mathbf{Q}$. It is assumed that each element in $\mathbf{Q}$, Q_1 through Q_k, is statically independent. With this, Eq. (38) becomes

$$P_f = \int \ldots \int \xi_{\mathbf{Q}}(\mathbf{q}) f_{\mathbf{S}}(\mathbf{s}) \mathrm{d}\mathbf{s} \prod_{i=1}^{k} f_{Q_i}(q_i) \mathrm{d}\mathbf{q} \tag{39}$$

where $\xi_{\mathbf{Q}}(\mathbf{q})$ is the conditional failure probability on $\mathbf{Q} = \mathbf{q}$, i.e.,

$$\xi_{\mathbf{Q}}(\mathbf{q}) \triangleq \Pr(G(\mathbf{S}, \mathbf{Q} = \mathbf{q}) \leq 0) = \int_{G(\mathbf{S},\mathbf{Q}=\mathbf{q})\leq 0} f_{\mathbf{S}}(\mathbf{s}) \mathrm{d}\mathbf{s}. \tag{40}$$

As before, in order to find the lower and upper bounds of the failure probability, the objective is to find the optimized distribution function of each element in $\mathbf{Q}$, Q_i, so as to maximize or minimize P_f in Eq. (38). To begin with, consider the case where $k = 2$ (i.e., two imprecise random variables are involved in the problem). The PDFs of Q_1 and Q_2 are written as $f_{Q_1}(x)$ and $f_{Q_2}(x)$, respectively. The failure probability P_f in Eq. (38) becomes a function of $f_{Q_1}(x)$ and $f_{Q_2}(x)$, denoted by

$$P_f = h(f_{Q_1}, f_{Q_2}). \tag{41}$$

Consider the lower bound of P_f. Note that a set of candidate distribution types exists for both $f_{Q_1}(x)$ and $f_{Q_2}(x)$, denoted by Ω_{Q_1} and Ω_{Q_2}, respectively. First, an arbitrary distribution is assigned for Q_1 and Q_2 (e.g., a normal distribution), whose PDFs are ${}_1f_{Q_1} \in \Omega_{Q_1}$ and ${}_1f_{Q_2} \in \Omega_{Q_2}$. Next, we find ${}_2f_{Q_2} \in \Omega_{Q_2}$ which minimizes $h({}_1f_{Q_1}, f_{Q_2})$ for $\forall f_{Q_2} \in \Omega_{Q_2}$, followed by determining ${}_2f_{Q_1} \in \Omega_{Q_1}$ which minimizes $h(f_{Q_1}, {}_2f_{Q_2})$ for $\forall f_{Q_1} \in \Omega_{Q_1}$. The approach to find ${}_2f_{Q_2}$ and ${}_2f_{Q_1}$ has been discussed in Section 3. As such, it is easy to see that

$$h({}_2f_{Q_1}, {}_2f_{Q_2}) \leq h({}_1f_{Q_1}, {}_2f_{Q_2}) \leq h({}_1f_{Q_1}, {}_1f_{Q_2}). \tag{42}$$

This fact implies that the pair $({}_2f_{Q_1}, {}_2f_{Q_2})$ leads to a reduced P_f compared with the pair $({}_1f_{Q_1}, {}_1f_{Q_2})$. Similarly, one can further find the subsequent sequences $({}_3f_{Q_1}, {}_3f_{Q_2})$ through $({}_nf_{Q_1}, {}_nf_{Q_2})$, in which n is a sufficiently large number of iteration. By noting that $h(f_{Q_1}, f_{Q_2})$ is bounded, according to Lemma 4 in Appendix A , it can be seen that $h({}_nf_{Q_1}, {}_nf_{Q_2})$ converges to the lower bound of P_f as n is large enough. Further, the upper bound of the failure probability can also be found using a similar procedure.

Now consider the more generalized case where $k > 2$. The failure probability in Eq. (38) is rewritten as,

$$P_f = h(f_{Q_1}, f_{Q_2}, \ldots f_{Q_k}) \tag{43}$$

where f_{Q_i} is the PDF of Q_i for $i = 1, 2, \ldots k$. Let Ω_{Q_i} denote the set of all the possible candidate distribution functions of element Q_i. In terms of the lower bound of P_f, an iteration-based approach is proposed to minimize the failure probability, as summarized in the following.

(1) Assign an arbitrary distribution for each element in **Q**, i.e., ${}_1f_{Q_1}$ through ${}_1f_{Q_k}$, and calculate $h_1 = h({}_1f_{Q_1}, {}_1f_{Q_2}, \ldots {}_1f_{Q_k})$.

(2) Find ${}_{j}f_{Q_i} \triangleq f_{Q_i} \in \Omega_{Q_i}$ which minimizes

$$h({}_{j}f_{Q_1}, {}_{j}f_{Q_2}, \dots {}_{j}f_{Q_{i-1}}, f_{Q_i}, \dots {}_{j-1}f_{Q_{i+1}}, \dots {}_{j-1}f_{Q_k})$$

for $i = 1, 2, \dots k$ and $j = 2$, and calculate $h_j = h({}_{j}f_{Q_1}, {}_{j}f_{Q_2}, \dots {}_{j}f_{Q_k})$.

(3) For each j, if $|h_j - h_{j-1}|$ is smaller than the predefined error limit (say, 10^{-5}), then h_j is found to be the lower bound of P_f; otherwise, return to step (2) with j replaced by $j+1$.

It can be seen that for each $j = 1, 2, \dots$, $h_j \leq h_{j-1}$. This observation is guaranteed by the fact that

$$\begin{aligned} &h({}_{j}f_{Q_1}, {}_{j}f_{Q_2}, \dots {}_{j}f_{Q_k}) \leq h({}_{j}f_{Q_1}, {}_{j}f_{Q_2}, \dots {}_{j-1}f_{Q_k}) \\ &\leq h({}_{j}f_{Q_1}, {}_{j}f_{Q_2}, \dots {}_{j-1}f_{Q_{k-1}}, {}_{j-1}f_{Q_k}) \leq \dots \leq h({}_{j-1}f_{Q_1}, {}_{j-1}f_{Q_2}, \dots {}_{j-1}f_{Q_k}). \end{aligned} \tag{44}$$

With Lemma 4 in Appendix A, the sequence $\{h_j\}$ converges to the lower bound of P_f as j is sufficiently large.

Finally, for the upper bound of the probability of failure, a similar procedure can be used, with the operation "minimize" replaced by "maximize".

4. Examples

In this section, three examples are presented to demonstrate the applicability and efficiency of the proposed method.

4.1. Example 1: a portal frame

The reliability of a rigid-plastic portal frame as shown in Fig. 2 is considered. The frame is subjected to a horizontal wind load W and a vertical load V. The layout and member geometry of the structure are adopted from [1]. The structure may fail due to one of the

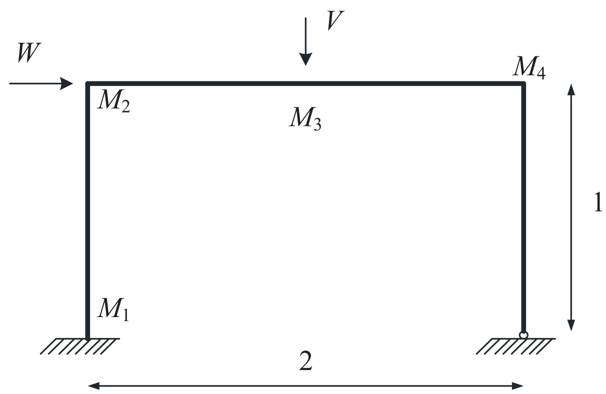


Figure 2: Example 1: a rigid-plastic portal frame (after [1]).

following three limit states,

$$\begin{aligned} G_1(\mathbf{X}) &= M_1 + 2M_3 + 2M_4 - W - V \\ G_2(\mathbf{X}) &= M_2 + 2M_3 + M_4 - V \\ G_3(\mathbf{X}) &= M_1 + M_2 + M_4 - W \end{aligned} \tag{45}$$

in which $M_1, \ldots, M_4$ are the plastic moment capacities at the joints as shown in the figure. Since the structure is a series system, the system fails if $G < 0$, where $G(\mathbf{X}) = \min\{G_1(\mathbf{X}), G_2(\mathbf{X}), G_3(\mathbf{X})\}$. The random variables considered include $\{M_1, M_2, M_3, M_4, V, W\}$. All random variables are assumed to be statistically independent with each other. The distributions of the moment capacities and the vertical load are fully known, and summarized in Table 1. However, only limited statistical information is available for the wind load W. For illustration purpose, consider the following three representative cases of the imprecise probabilistic information of W:

Case (1) W has a mean of 1.9 and a standard deviation of 0.45, with its distribution type unknown;

Case (2) W has a mean of 1.9 and a standard deviation of 0.45, and is strictly defined within $[1.0, 3.0]$, with its distribution type unknown;

Case (3) W has a mean within $[1.87, 1.93]$ and a standard deviation of 0.45, with its distribution type unknown.

Note that in Case 1 and 3, the wind load may take negative values.

Table 1: Example 1: statistics of the random variables.

Variable	Distribution type	Mean	Std. Dev.
M_1, M_2, M_3, M_4	Normal	1.0	0.3
V	Normal	1.5	0.3

4.1.1. Constructing the P-box for wind load W

The CDF bounds of the wind load W constructed from different methods are first examined. For all three cases, the p-boxes for W are determined using the proposed linear programming method using both types of objective function. As a comparison, the p-box in case (1) is also constructed using the Chebyshev's inequality (Eq. 7), and Eq. (8) for case (3).

Fig. 3 (a) compares the p-boxes for case (1) obtained from the proposed method and the Chebyshev's inequality. It can be seen that the CDF bounds obtained using the objective functions Type 1 and Type 2 (c.f. Eq. (23) and (31)) are identical, indicating that the optimization results are consistent (note that the two objective functions are linearly independent of each other). It is also evident that the p-box from the Chebyshev's inequality is significantly wider than the p-box from linear programming. This confirms that the Chebyshev's inequality does not give the best-possible bounds, thus if it is used in reliability analysis, the obtained reliability bounds may be overly conservative.

Fig. 3 (b) plots the p-boxes for case (2), obtained from the proposed linear programming, and also from Eq. (8). Again, it is shown that the two p-boxes from linear programming using objective function Type 1 and Type 2 are identical. It is also observed that the CDF bounds from the proposed method are identical to those from Eq. (8). Note that it has been proved that Eq. (8) gives the best-possible CDF bounds for this case [13]. This comparison implies that the proposed linear programming method also yields the best-possible CDF bounds.

For case (3) where the mean value of W is not deterministic but varies within an interval, there is no analytical solution in the literature for the bounds of the CDF as those in Eqs. (7) or (8). Nevertheless, the proposed optimization-based approach (Eq. 23) can be applied for constructing the best-possible CDF bounds. Fig. 4 shows the CDF bounds obtained by

Eq. (23). Note that only the objective function Type 1 can be applied to this case; objective function Type 2 cannot be used as it requires point estimates of the mean and standard deviation.

In practical reliability analyses, when the available data of a random variable is scarce, its distribution type is often assumed based on subjective judgement, e.g., assumed as one of the commonly used distribution types. This common practice is applied to the three cases, considering five candidate distribution types for W, namely normal, lognormal, Weibull, Gamma and Extreme Type 1 largest (T1Largest). Since in Case (2), W is strictly defined in the range $[1.0, 3.0]$, the bottom and the top of the candidate distributions are removed. The CDF bounds of all five candidate distributions are given by

$$\underline{F}_W(w) = \min\{F_i(w), i = 1, 2, \ldots 5\}, \tag{46a}$$

$$\overline{F}_W(w) = \max\{F_i(w), i = 1, 2, \ldots 5\}, \tag{46b}$$

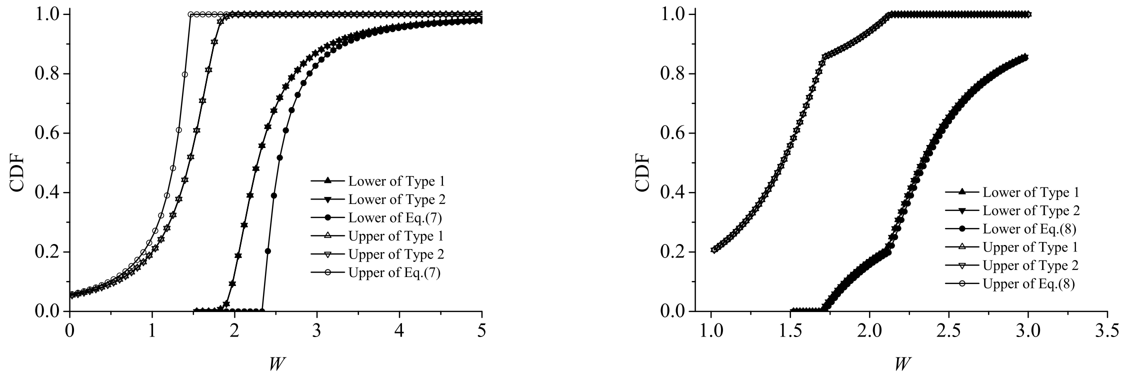
in which F_i represents the ith candidate distribution. Fig. 4 compares the CDF bounds based on Eq. (46) assuming five candidate distribution types, and from the proposed linear programming method without any assumption of the distribution type. It can be seen that in all three cases, the CDF bounds assuming five candidate distribution types are significantly narrower than those without assuming any knowledge of distribution type. This suggests that the estimate of failure probability may give a false impression of reliability if only considering a limited number of potential distribution types based on subjective judgement only.

Table 2: Example 1: bounds of failure probability.

Case No.	Interval MC (IMC1)*	Interval MC (IMC2) **	Direct optimization
(1)	$[0.0090, 0.3678]$	$[0.0184, 0.2593]$	$[0.0597, 0.1057]$
(2)	$[0.0223, 0.2490]$	$[0.0223, 0.2490]$	$[0.0831, 0.1106]$
(3)	–	$[0.0097, 0.4233]$	$[0.0523, 0.1918]$

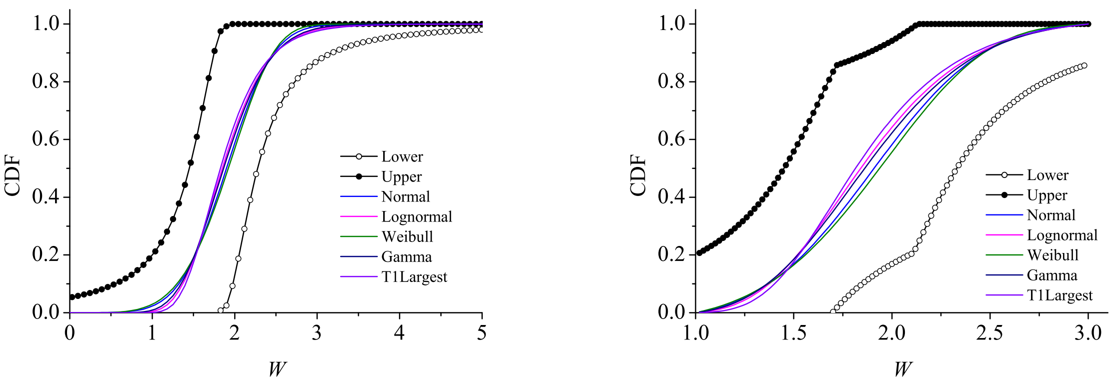
* P-box for W was obtained using Eq. (7) (case 1) and Eq. (8) (case 2)

** P-boxes for W were obtained using linear programming.

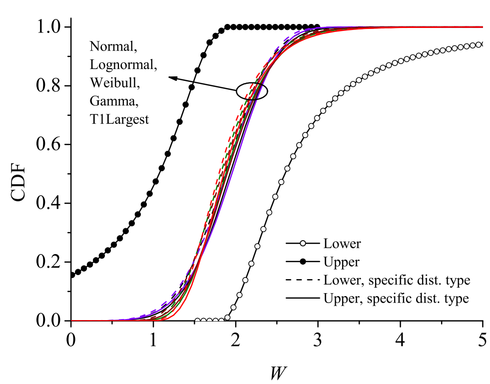


(a) Case (1) (b) Case (2)

Figure 3: Example 1: CDF bounds of W computed by the proposed method (Objective Function Type 1 and 2), and the existing methods.



(a) Case (1) (b) Case (2)



(c) Case (3)

Figure 4: Example 1: CDF bounds of W computed from Objective Function Type 1, and the CDF's of W by assuming specific distribution type.

4.1.2. Bounds of probability of failure

This section examines the bounds of failure probability for the three cases. Table 2 presents the intervals of failure probability obtained from different methods. The second column of Table 2 gives the failure probability bounds computed by the interval Monte Carlo simulation. In this method, the probability-box of W was first constructed using the existing methods, i.e., Eq. (7) for case 1 and Eq. (8) for case 2. Then the failure probability bounds were computed using the interval Monte Carlo method (Eqs. 12 and 13). This method is referred to as IMC1 in the following discussions. The results presented in the third column of Table 2 were also computed using the interval Monte Carlo method; however, the probability-boxes for W were constructed using the proposed linear programming method. This method is referred to as IMC2. The fourth column of Table 2 lists the results computed by the proposed linear programming method using objective function Type 1. In this method, it is not required to construct the probability-box of W; instead, the failure probability bounds were determined directly solving the linear programming problem. For this reason, the method is referred to as "Direct Optimization". In applying the linear programming method, the conditional failure probability function, $\xi_W(w)$, was approximated first based on 10^6 Monte Carlo simulations, and is plotted in Fig. 5. This conditional failure probability function can be fitted by an expression

$$\xi_W(w) = \Phi(0.0007w^6 - 0.0067w^5 + 0.0036w^4 + 0.133w^3 - 0.2856w^2 + 1.2389w - 3.7204) \quad (47)$$

in which $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. The R-squared of this fitted curve is 0.999. Substituting Eq. (47) into Eq. (23) yields the estimate of lower and upper bounds of P_f without the need to consider the CDF envelope of W.

The results from IMC1 and IMC2 are firstly compared. From Table 2, it can be seen that for case 1, the failure probability bounds from IMC2 is narrower than those from IMC1. This is to be expected, as the p-box for W from linear programming is tighter than that from the Chebyshev's inequality. For case 2, IMC1 and IMC2 yielded the identical results, since the p-box for W is the same in both methods. For case 3, since there is no analytical solution in the literature for constructing the CDF bounds of W, the failure probability bounds were

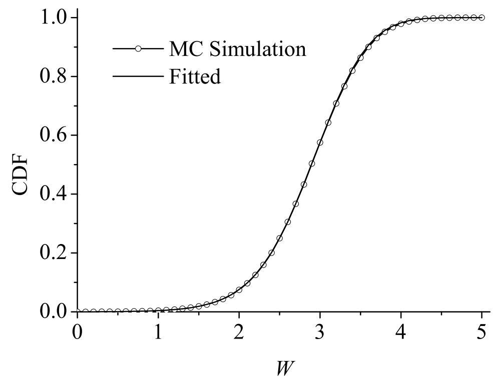


Figure 5: Example 1: conditional failure probability function $\xi_W(w)$.

not computed in IMC1. With IMC2, the failure bounds were computed as [0.0097, 0.4233].

Next, the failure probability bounds from IMC2 and the proposed method are compared. It is observed that the failure probability intervals obtained with the direct optimization method are significantly narrower than those based on interval Monte Carlo method with p-boxes. For example, the upper bound of failure probability for case 1 is 0.1057 from direct optimization, as compared to 0.2593 from IMC2. The latter is more than twice than the former. Similar observations are also made in case 2 and case 3. This comparison shows that the proposed linear programming method can better utilize the available information, and yields more informative results than the interval Monte Carlo method with p-boxes.

The improved estimate with a direction optimization than the interval Monte Carlo method propagating probability boxes can be explained by a simple example. Consider an imprecisely-known random variable X, which has two candidate CDF's as shown in Fig. 6. Note that the two candidate CDF's cross over each other. It is assumed that the failure probability is a monotonic function of X, i.e., $P_f = \mathcal{F}(X)$. Suppose that the failure probability bounds are estimated simply with two runs of simulation, generating four samples x_1, x_2, x_3 and x_4 from the two candidate distributions. With this, the interval width of the failure probability associated with a direct optimization method is

$$\mathrm{L}_1 = \left| \frac{\mathcal{F}(x_1) + \mathcal{F}(x_4)}{2} - \frac{\mathcal{F}(x_2) + \mathcal{F}(x_3)}{2} \right|, \tag{48}$$

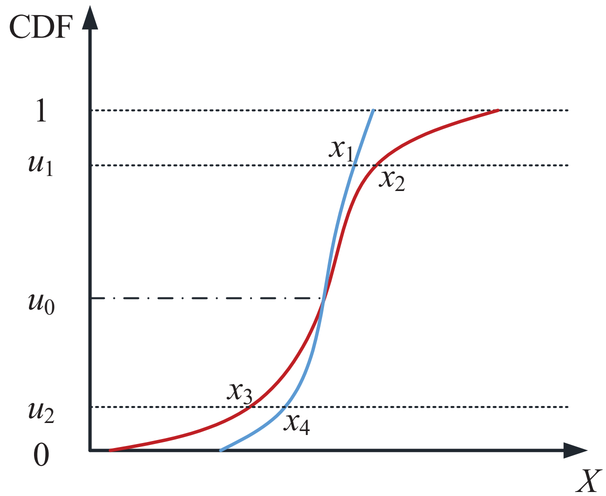


Figure 6: Schematic representation of the CDF of two random variables.

while the interval width associated with a p-box method is

$$\mathrm{L}_2 = \left| \frac{\mathcal{F}(x_1) + \mathcal{F}(x_3)}{2} - \frac{\mathcal{F}(x_2) + \mathcal{F}(x_4)}{2} \right|. \tag{49}$$

Clearly, $\mathrm{L}_1 \leq \mathrm{L}_2$, and the equality holds when either $u_1, u_2 \in [0, u_0]$ or $u_1, u_2 \in [u_0, 1]$.

4.2. Example 2: time-dependent reliability of an aging structure

Example 2 considers the time-dependent reliability of an aging structure, whose deterioration is associated with imprecise information due to the fact that the deterioration may be a multifarious process involving multiple deterioration mechanisms [2]. The example herein is adopted from Wang et al [33], where the impact of the selection of different candidate distribution types for resistance deterioration on structural reliability has been discussed. The structure was initially designed at the limit state as $0.9R_n = 1.2D_n + 1.6L_n$, in which R_n is the nominal resistance, D_n and L_n represent the nominal dead load and live load, respectively. It is assumed that $D_n = L_n$. The dead load is assumed to be deterministic and equals to D_n. The live load is modeled as a Poisson process; the magnitude of the live load follows an Extreme Type I distribution with a standard deviation of $0.12L_n$ and a time-variant mean of $(0.4 + 0.005t)L_n$ in year t. The occurrence rate of the live load is 1.0/year. The initial resistance of the structure, denoted by R_0, is assumed to be deterministic and equals to $1.05R_n$. In year t, the resistance deteriorates to $R(t)$, given by $R(t) = R_0 \cdot (1 - G(t))$, in which $G(t)$ is a *linear* degradation function. If the resistance in a particular year T, $R(T)$, can be estimated, then $G(t)$ can be readily obtained using the conditions $G(0) = 0$ and $G(T) = 1 - R(T)/R_0$. A schematic representation of the time-variant resistance and load effect of the deteriorating structure is presented in Fig. 7.

Suppose that in a particular year T, the PDF of $G(T)$ is $f_G(g)$. With this, the time-dependent reliability, $L(T)$, is given by

$$L(T) = \int_0^1 \exp\left[-\int_0^T \lambda(1 - F_S[r(t|g) - D, t])\mathrm{d}t\right] \cdot f_G(g)\mathrm{d}g \tag{50}$$

where $r(t|g)$ is the resistance at time t given that $G(T)$ equals g, λ is the occurrence rate of the load, and F_S is the CDF of each live load effect. It is noted that $G(T)$ should not be less than 0 for structures without maintenance or repair measures because the resistance process in non-increasing, nor be greater than 1 since the resistance of a structure never becomes a negative value, accounting for the integration limits of 0 and 1 in Eq. (50).

For the case where the mean of load effect increases linearly with time (i.e., $\mu_S(t) = \mu_S(0)+\kappa_m t$), while the standard deviation of load effect, σ_L, is constant, the core of Eq. (50),

$$\nu(g) = \exp\left[-\int_0^T \lambda(t)(1 - F_S[r(t|g) - D, t])\mathrm{d}t\right] \tag{51}$$

can be simplified as follows [41],

$$\nu(g) = \exp(-\lambda \cdot \Xi), \tag{52}$$

in which

$$\Xi = \exp\left(\frac{m_0 + D - r_0}{a}\right)\frac{aT}{r_0 g + \kappa_m T}\left[\exp\left(\frac{r_0 g + \kappa_m T}{a}\right) - 1\right], \tag{53}$$

where $a = \frac{\sqrt{6}\sigma_L}{\pi}$, and $m_0 = \mu_S(0) - 0.5772a$. Comparing with Eq. (17), the bound estimate of time-dependent reliability can be transformed into a standard linear programming problem, if treating $\nu(g)$ in Eq. (50) as $\xi(x)$ in Eq. (17).

Suppose that the resistance at year 40 can be estimated. The COV of $G(40)$ is 0.4; two cases of the mean of $G(40)$, denoted by $\mu_{G(40)}$, are considered, i.e., 0.2 and 0.4. Without

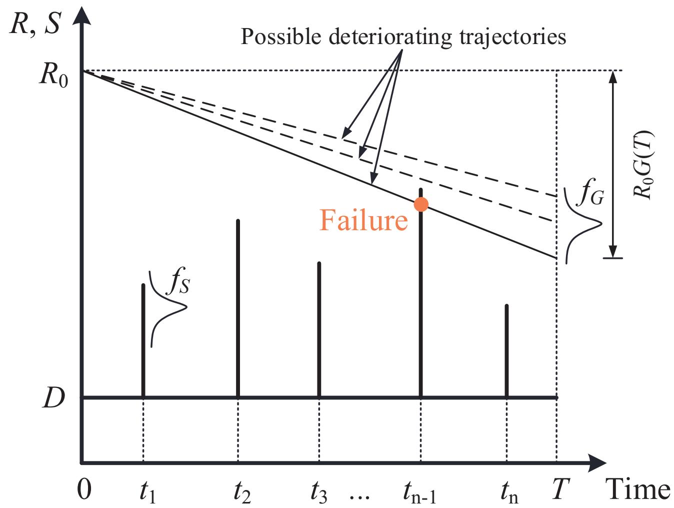
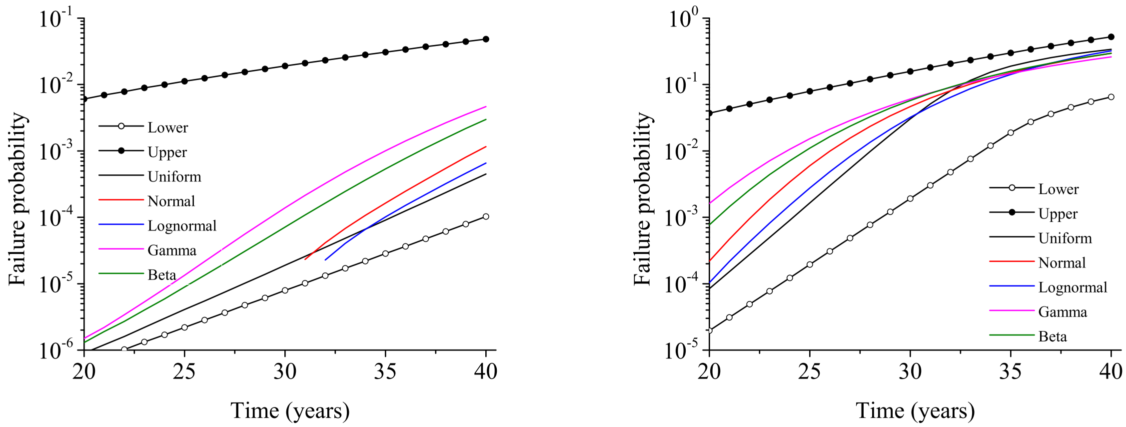


Figure 7: Schematic representation of the time-variant resistance and load effect of an aging structure.

introducing additional assumptions in regarding to the distribution type of $G(40)$, the lower and upper bounds of the time-dependent probability of failure for reference periods up to 40 years are computed using the proposed linear programming-based method, and plotted in Fig. 8. As a comparison, Fig. 8 also shows the probabilities of failure with additional assumptions of the distribution type of $G(40)$, i.e., several commonly-used distributions including normal, lognormal, Gamma, Beta and uniform distributions. The corresponding time-dependent probabilities of failure are adopted from the original literature [33]. It can be seen from Fig. 8 that for both cases of $\mu_{G(40)}$, the lower and upper bounds computed using the proposed method establish an envelope for the time-dependent reliabilities. These reliability bounds consider all possible distribution types for $G(40)$. As expected, these bounds enclose those probabilities of failure with additional assumptions for the distribution type of $G(40)$. This example clearly demonstrates that by simply assuming some common distribution types without justification, the probability of failure may be significantly underestimated.

4.3. Example 3: an oscillation system

A non-linear single degree of freedom system without damping is shown in Fig. 9. The example is adopted from [42]. The limit state function is defined by the case where the



(a) $\mu_{G(40)} = 0.2$ (b) $\mu_{G(40)} = 0.4$

Figure 8: Example 2: lower and upper bounds of the time-dependent failure probability.

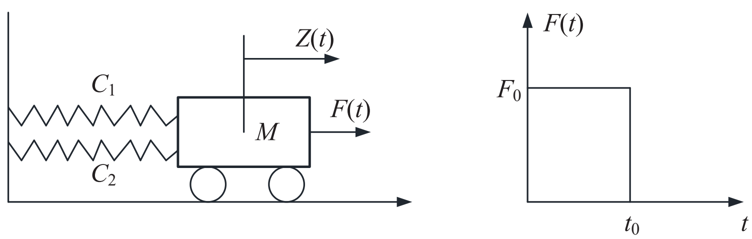


Figure 9: Example 3: schematic representation of an oscillation system.

maximum displacement response exceeds the limit, i.e.,

$$G(\mathbf{X}) = 3R - |Z_{\max}| = 3R - \left| \frac{2F_0}{M\Omega_0^2} \sin\left(\frac{\Omega_0^2 t_0}{2}\right) \right| \tag{54}$$

where $Z_{\max}$ is the maximum displacement response of the system, $\Omega_0 = \sqrt{(C_1 + C_2)/M}$, and R is the displacement when one of the two springs yields. The system is deemed to "fail" if $G(\mathbf{X}) < 0$ and "survive" otherwise. The probabilistic information regarding the six random variables in Eq. (54) is summarized in Table 3. It is assumed that the variables C_1 and C_2 are imprecise with their distribution types unknown. It is further assumed that C_1 and C_2 are statistically independent of each other.

The fragility curve of the system with respect to C_1 and C_2 is fitted through numerical simulation as follows,

$$\xi_{C_1,C_2}(c_1, c_2) = 0.072\Phi(-0.016c^6 + 0.138c^5 - 0.348c^4 + 0.182c^3 + 0.202c^2 + 1.919c - 3.656) \tag{55}$$

Table 3: Example 3: statistics of the random variables.

Variable	Distribution type	Mean	Std. Dev.
M	Normal	1	0.05
R	Normal	0.5	0.05
F_0	Normal	1	0.2
t_0	Normal	1	0.2
C_1	unknown	1	0.6
C_2	unknown	0.5	0.3

where $c = 3 - c_1 - c_2$.

Since the problem involves multiple imprecise random variables, the iteration-based approach as developed in Section 3.4 is used to find the lower and upper bounds of the system failure probability. Table 4 summarizes the bounds of P_f associated with different iteration rounds. Setting an error threshold of 10^{-4}, the bounds of failure probability are obtained with five cycles of iteration, yielding an interval of failure probability of $[0.0171, 0.0311]$. This demonstrates the applicability of the proposed method for handling multiple imprecise random variables. Furthermore, for comparison purpose, the bounds of P_f are also obtained using two different interval Monte Carlo methods, referred to as IMC1 and IMC2. The two interval Monte Carlo methods are different in that the CDF bounds of C_1 and C_2 were constructed using the existing method (Eq. 7) in IMC1, and the proposed linear programming method in IMC2.

Table 5 presents the bounds of failure probability obtained from the proposed method, IMC1 and IMC2. The interval of failure probability is found to be $[0.0171, 0.0311]$ using the proposed method, $[0.0001, 0.0655]$ for IMC1, and $[0.0020, 0.0579]$ for IMC2. The same observation as in Example 1 is made, i.e., the proposed direct-optimization method yields the tightest bounds of failure probability, followed by IMC2. IMC1 leads to the widest bounds of failure probability.

5. Conclusions

A linear programming-based method has been proposed to handle reliability analyses involving random variables with incomplete statistical information (only knowing the first two moments and possible range). The proposed method does not require the assumption

Table 4: Example 3: bounds of failure probability from the proposed iteration-based approach.

Iteration No.	Operation	Lower bound	Upper bound
1	${}_1f_{C_1}, {}_1f_{C_2} \sim$ normal distribution	0.0250	0.0250
2	${}_1f_{C_1}$ fixed, ${}_2f_{C_2}$ optimized	0.0245	0.0260
3	${}_2f_{C_2}$ fixed, ${}_2f_{C_1}$ optimized	0.0171	0.0310
4	${}_2f_{C_1}$ fixed, ${}_3f_{C_2}$ optimized	0.0171	0.0311
5	${}_3f_{C_2}$ fixed, ${}_3f_{C_1}$ optimized	0.0171	0.0311

Table 5: Example 3: bounds of failure probability from the interval MC and the proposed method.

Method	Interval
Interval MC (IMC1)*	$[0.0001, 0.0655]$
Interval MC (IMC2)**	$[0.0020, 0.0579]$
Direct optimization***	$[0.0171, 0.0311]$

* P-boxes for C_1 and C_2 were obtained using Eq. (7) .
** P-boxes for C_1 and C_2 were obtained using linear programming.
*** Iteration-based approach is used, c.f. Section 3.4.

of a distribution type; it considers all possible distribution types which are compatible with available data. The proposed method makes full use of the available information, without introducing additional assumptions.

The reliability analysis subject to imprecise probabilistic information is converted into solving a linear programming optimization problem. Two objective functions, namely Type 1 and Type 2 (c.f. Eqs. (23) and (31)), are developed independently. Three numerical examples demonstrated the efficiency and accuracy of the proposed method. The two objective functions lead to the same reliability bounds. In all three examples, the bounds on the failure probabilities obtained from the proposed method are significantly tighter than those from the interval Monte Carlo method, suggesting that more information is provided by the proposed method. The reason is that in the interval Monte Carlo method, the CDF bounds of imprecise input random variables need to be constructed first, and then are propagated through the Monte Carlo simulation. Useful information "inside" the CDF bounds of input random variables may be lost in the procedure. The proposed method, on the other hand, makes full use of available information of the imprecise random variables.

While the proposed method can compute tight bounds of failure probability directly without the need of first constructing the CDF bounds of the imprecisely known random input variables, it can also be used to construct the best-possible CDF bounds for a random variable with limited moment information. It has been shown that the proposed method can yield tighter CDF bounds than the Chebyshev's inequality when only the mean and variance of the random variable are known. In the case where the min, max, mean and variance of a random variable are known, the CDF bounds from the proposed method are the same as the best-possible bounds provided in [13]. The proposed method can also handle other general cases of imprecise probability such as interval moments, without assuming the type of distribution.

Appendix A. Some lemmas and their proofs

Lemma 1. *For any real value $\tau > 0$ and a random variable X defined in $[0, 1]$, $[\ln(\mathbb{E}(X^{\tau}))]'$ increases with τ.*

Proof. Since

$$[\ln(\mathbb{E}(X^{\tau}))]' = \lim_{\mathrm{d}\tau\to 0} \frac{\mathrm{d}\ln(\mathbb{E}(X^{\tau}))}{\mathrm{d}\tau} = \frac{1}{\mathbb{E}(X^{\tau})} \cdot \frac{\mathbb{E}(X^{\tau+\mathrm{d}\tau}) - \mathbb{E}(X^{\tau})}{\mathrm{d}\tau} \tag{A.1}$$

it is equivalent to prove that for $0 < \tau_1 < \tau_2 = \tau_1 + \mathrm{d}\tau$,

$$\frac{\mathbb{E}(X^{\tau_2})}{\mathbb{E}(X^{\tau_1})} < \frac{\mathbb{E}(X^{\tau_2+\mathrm{d}\tau})}{\mathbb{E}(X^{\tau_2})}. \tag{A.2}$$

With the Cauchy-Schwarz inequality, for two functions $\iota(x)$ and $\varrho(x)$ defined in $[0, 1]$, one has

$$\left[\int_0^1 \iota(x)\varrho(x)\mathrm{d}x\right]^2 \leq \int_0^1 \iota^2(x)\mathrm{d}x \cdot \int_0^1 \varrho^2(x)\mathrm{d}x \tag{A.3}$$

where the equality holds if and only if $\iota(x)$ is linearly proportional to $\varrho(x)$. Let

$$\iota(x) = \sqrt{x^{\tau_1} f_X(x)}, \quad \varrho(x) = \sqrt{x^{\tau_2+\mathrm{d}\tau} f_X(x)} \tag{A.4}$$

Eq. (A.3) gives

$$[\mathbb{E}(X^{\tau_2})]^2 < \mathbb{E}(X^{\tau_1}) \cdot \mathbb{E}(X^{\tau_2+\mathrm{d}\tau}) \tag{A.5}$$

which is an equivalent form of Eq. (A.2). □

Lemma 2. *For a random variable X defined in $[0, 1]$ with an unknown distribution type, if $\mathbb{E}(X^j) = \exp(pj+q)$ for $\forall j = 2, 3, \ldots$, then $P_f(p) = \int_0^1 \xi(x) f_X(x)\mathrm{d}x = (1-\mathrm{e}^q)\xi(0) + \mathrm{e}^q\xi(\mathrm{e}^p)$, where $f_X(x)$ is the PDF of X, and $q = \ln(\mathbb{E}(X^2) - 2p$.*

Proof. Since $\mathbb{E}(X^j) = \exp(pj+q)$ for $\forall j = 2, 3, \ldots$, according to [43],

$$P_f = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left[a_j + a_j \sum_{k=1}^{\infty} \frac{\exp(2pk+q)}{(2k)!} \cdot (j\pi)^{2k}(-1)^k \right] \tag{A.6}$$

where $a_j = 2\int_0^1 \xi(x)\cos(jx\pi)\mathrm{d}x$ for $j = 0, 1, 2, \ldots$. Assigning $x = \exp(p) \cdot j\pi$ in the equation $\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^{2k}$ gives

$$P_f = \frac{a_0}{2} + (1-\mathrm{e}^q)\sum_{j=1}^{\infty} a_j + \mathrm{e}^q \sum_{j=1}^{\infty} a_j \cos(\mathrm{e}^p \cdot j\pi). \tag{A.7}$$

Further, assigning $x = 0$ and $x = \mathrm{e}^p$ respectively in the Fourier expansion of $\xi(x)$, $\xi(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jx\pi)$, yields

$$\xi(0) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j, \quad \xi(\mathrm{e}^p) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(\mathrm{e}^p \cdot j\pi). \tag{A.8}$$

With Eq. (A.8), Eq. (A.7) becomes

$$P_f(p) = (1-\mathrm{e}^q)\xi(0) + \mathrm{e}^q\xi(\mathrm{e}^p) \tag{A.9}$$

which completes the proof. □

Remark 1. *A simple verification of Eq. (A.9) is that when σ_X is sufficiently small, $\mathbb{E}(X^j) \approx [\mathbb{E}(X)]^j = \mu_X^j$, thus $p = \ln \mu_X$ and $q = 0$, with which $P_f(p) = \xi(\mu_X)$. Specifically, when $\xi(0)$ is typically 0, Eq. (A.9) can be further simplified as $P_f(p) = \mathrm{e}^q\xi(\mathrm{e}^p)$.*

Remark 2. *The failure probability in Eq. (A.9) is referred to as fictitious as it is derived based on the assumption that X has linear logarithmic moments.*

Lemma 3. *For a random variable X defined in $[0,1]$, there exist two coefficient sequences $\{\widetilde{\alpha}_l, l = 1,2,\dots n-2\}$, $\{\widetilde{\beta}_l > 0, l = 1,2,\dots n-2\}$ such that $\mathbb{E}(X^j) = \sum_{l=1}^{n-2} \widetilde{\alpha}_l \cdot \widetilde{\beta}_l^j$ for $j = 2,3,\dots n-1$, and $P_f = \int_0^1 \xi(x) f_X(x) \mathrm{d}x = \xi(0) + \sum_{l=1}^{n-2} \widetilde{\alpha}_l [\xi(\widetilde{\beta}_l) - \xi(0)]$, where $f_X(x)$ is the PDF of X.*

Proof. First, the existence of sequences $\{\widetilde{\alpha}_l\}$ and $\{\widetilde{\beta}_l\}$ is guaranteed by the fact that

$$\det \mathcal{B} = \det \begin{bmatrix} \widetilde{\beta}_1^2 & \widetilde{\beta}_2^2 & \cdots & \widetilde{\beta}_{n-2}^2 \\ \widetilde{\beta}_1^3 & \widetilde{\beta}_2^3 & \cdots & \widetilde{\beta}_{n-2}^3 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\beta}_1^{n-1} & \widetilde{\beta}_2^{n-1} & \cdots & \widetilde{\beta}_{n-2}^{n-1} \end{bmatrix} = \prod_{1 \le l < k \le n-2} (\widetilde{\beta}_k - \widetilde{\beta}_l) \cdot \prod_{k=1}^{n-2} \widetilde{\beta}_k^2 \tag{A.10}$$

which is non-zero if $\widetilde{\beta}_k \neq \widetilde{\beta}_l$ for $\forall k \neq l$. Next, according to [43],

$$P_f = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left[a_j + a_j \sum_{k=1}^{\infty} \frac{\sum_{l=1}^{n-2} \widetilde{\alpha}_l \cdot \widetilde{\beta}_l^{2k}}{(2k)!} \cdot (j\pi)^{2k} (-1)^k \right] \tag{A.11}$$

where $a_j = 2\int_0^1 \xi(x) \cos(jx\pi) \mathrm{d}x$ for $j = 0,1,2,\dots$. By noting that $\cos x = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^{2k}$ holds for any x, and that $\xi(\widetilde{\beta}_l) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(\widetilde{\beta}_l \cdot j\pi)$, Eq. (A.11) becomes

$$\begin{aligned} P_f &= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j + \sum_{l=1}^{n-2} \widetilde{\alpha}_l \left\{ \sum_{j=1}^{\infty} a_j \left[\cos(\widetilde{\beta}_l \cdot j\pi) - 1 \right] \right\} \\ &= \xi(0) + \sum_{l=1}^{n-2} \widetilde{\alpha}_l [\xi(\widetilde{\beta}_l) - \xi(0)] \end{aligned} \tag{A.12}$$

which completes the proof. □

Lemma 4. *If a real sequence monotonically increases with an upper bound, then the sequence converges to the supremum.*

Proof. See, e.g., [44]. □

Appendix B. Standard form of a linear programming problem

A linear programming problem takes a standard form of

$$\min\ \mathbf{c}^{\mathsf{T}}\mathbf{x}, \quad \text{subjected to} \quad \mathbf{A}\mathbf{x} \preceq \mathbf{b} \quad \text{and} \quad \mathbf{x} \succeq \mathbf{0} \tag{B.1}$$

where $\mathbf{x}$ is a variable vector to be determined, $\mathbf{b}$ and $\mathbf{c}$ are two known vectors, $\mathbf{A}$ is a coefficient matrix, and the subscript $^{\mathsf{T}}$ denotes the transpose of a matrix. The operator $\preceq$ (or $\succeq$) in Eq. (B.1) means that each element in the left-hand vector is no more (or less) than the corresponding element in the right-hand vector. The constraints $\mathbf{A}\mathbf{x} \preceq \mathbf{b}$ and $\mathbf{x} \succeq \mathbf{0}$ simultaneously define a convex poly-tope in which the objective function, $\mathbf{c}^{\mathsf{T}}\mathbf{x}$, is to be optimized [45, 46]. The algorithms of linear programming-based optimization have been well studied and widely applied in previous works [37–40], including some useful toolboxes such as YALMIP [47].

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