Extended Dissipativity Analysis for Discrete-Time Delayed Neural Networks Based on an Extended Reciprocally Convex Matrix Inequality

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Abstract

In this paper, the extended dissipativity analysis for discrete-time neural networks with a time-varying delay is investigated. First, a novel Lyapunov-Krasovskii functional (LKF) is constructed with a delay-product-type term introduced. Then, in the forward difference of the LKF, the sum terms are bounded via an extended reciprocally convex matrix inequality. As a result, an extended dissipativity criterion is established in terms of linear matrix inequalities. Meanwhile, this criterion is extended to the stability analysis of the counterpart system without disturbance. Finally, two numerical examples are given to demonstrate the effectiveness and improvements of the presented criterion.

Keywords: Discrete-time neural networks, Extended dissipativity, Extended reciprocally convex matrix inequality, Time-varying delay.

1. Introduction

Neural networks have attracted many researchers' attention since they have been successfully applied in various areas such as signal transmission, pattern recognition, associative memory, *etc*. [2, 29]. In engineering applications, although the discrete systems cannot present the dynamic behaviors of the continuous counterparts even for a short sampling period, it is essential to formulate discrete-time neural networks that are analogue of continuous ones [16]. Inevitably, there do exist time delays during the process of information transmission between the neurons, which often leads to undesired characteristics [30]. Therefore, it is necessary to investigate the stability and robust performance of discrete-time delayed neural networks (DNNs) so as to improve their application to practice [7].

The main method for the stability analysis of discrete-time DNNs [8, 15, 17, 22] is the Lyapunov direct method. Based on this method, many stability criteria are developed via constructing a suitable Lyapunov-Krasovskii functional (LKF) and/or tightly estimating the forward difference of the LKF [28]. Hence, this research aims to obtain less conservative stability criteria with small computation complexity. To this end, numerous approaches were presented. In the terms of LKFs, the simple LKF [15, 22], the delay segmented LKF [17] and the augmented LKF [8] have been constructed. Then, for estimating the forward difference of the LKF effectively, some bounding techniques have been proposed. For instance, the free weighting matrix (FWM) technique [27], the discrete Jensen's inequality [32], the reciprocally convex combination lemma (RCCL) [13], the zero equations [8], the discrete Wirtinger-based inequality [14] and the free-matrix-based integral inequality [6, 20], *etc.* have been presented in existing results.

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In fact, apart from the stable analysis of DNNs, their robust analysis has also played a key role in various engineering fields. The reason is that when the dynamic system encounters external perturbations, it is still required to keep a smooth operation. As a typical robust performance, the (Q, S, R)–γ–dissipativity [18, 31] has gained increased popularity recently. Based on the energy-related input and output description, the dissipativity theory could provide flexible instruments for system analysis and controller design [4]. Moreover, it covers some well-known performance indices, such as H_∞ performance [1], passivity [21], *etc.* In spite of this, the l_2–l_∞ performance [3] which is referred to as the energy-to-peak performance and extended H_2 performance, cannot be included in the (Q, S, R)–γ–dissipativity. For solving this problem, Zhang et al. proposed the concept of extended dissipativity which enables the dissipativity, H_∞ performance, passivity and l_2–l_∞ to be included as special cases [23]. Via adjusting the weighting matrices in a new performance index, these four desired performances could be obtained from the extended concept immediately.

Motivated by these work, Lee et al. started to analyze the extended dissipativity of the continuous DNNs [9], and that of discrete-time DNNs was also investigated [5] in which related stability criterion was simultaneously developed. In order to make improvement to existing results, in [5], authors considered from two aspects. First, a LKF was constructed via augmenting with the activation functions. Then, the forward difference of the LKF was bounded via the Jensen's inequality combined with the RCCL method. Despite that, there still remains space to be improved. For instance, the augmented LKF is too complex to release the heavy computational burden when it is used to obtain criterion. Then, during the estimating procedure, an equivalent treatment [11] to the quadratic term of time-varying delay also increases the number of decision variables. Moreover, either the Jensen's inequality or the RCCL method has higher conservatism in comparison with the Wirtinger-based inequality or the latest matrix inequality [24, 25].

Based on above discussions, this paper focuses on the extended dissipativity of discrete-time DNNs and aims to present improved result. Firstly, inspired by [26], a novel LKF with a delay-product term is constructed so as to make full use of the information of time-delay including its varied speed. Secondly, an extended reciprocally convex matrix inequality, which includes the popular RCCL and the very recently proposed matrix inequality [24] as special cases, is applied to achieve the tight estimation for the forward difference of the LKF. As a result, an extended dissipativity criterion with less conservatism and lower number of decision variables is established. Meanwhile, this criterion is available to the stability analysis of the counterpart system without disturbance input. Finally, two numerical examples are listed to illustrate the advantages of presented methods. Meanwhile, the simulation is carried out to verify the obtained results from reliability.

Notations: Throughout this paper, $\mathbb{R}^n$ is the n dimensional Euclidean space; $P > 0$ (≥ 0) means that P is a real symmetric and positive definite (semi-positive definite) matrix; I and 0 represent the identity matrix and zero matrix, respectively; the superscript T and -1 represents the transpose and the inverse of a matrix; diag$\{\cdot\}$ stands for a block-diagonal matrix; l_2 is the space of square summable infinite vector functions sequences; for any functions $x, y \in l_2$, matrix R, $\langle x, Ry\rangle_h = \sum_{k=0}^{h} x^T(k)Ry(k)$; $\|\cdot\|$ denotes the Euclidean norm of a vector; notation $*$ denotes the symmetric block in a symmetric matrix and Sym $Y = Y + Y^T$.

2. Problem formulation

Consider the discrete-time neural networks with an interval time-varying delay as follows

$$\begin{cases} x(k+1) = Ax(k) + Bg(x(k)) + Cg(x(k-\tau(k))) + u(k) \\ \quad y(k) = Dx(k), \quad k \in [-\tau_M, 0] \end{cases} \tag{1}$$

where $x(k) = [x_1(k), x_2(k), ..., x_n(k)]^T \in \mathbb{R}^n$ is the neural state vector; $g(x(k)) = [g_1(x_1(k)), g_2(x_2(k)), ..., g_n(x_n(k))]^T \in \mathbb{R}^n$ denotes the neural activation function; $u(k) \in \mathbb{R}^n$ is the disturbance input belonging to l_2; $y(k)$ is the output of the system; A, B, C and D are known constant matrices with appropriate dimensions and $\tau(k)$ is the time-varying delay satisfying

$$\tau_m \leq \tau(k) \leq \tau_M \tag{2}$$

and

$$\mu_1 \leq \Delta\tau(k) = \tau(k+1) - \tau(k) \leq \mu_2 \tag{3}$$

where τ_m, τ_M and μ are known integers.

Each neural activation function $g_i(x_i(k))$ with $g_i(0) = 0$ is assumed to be bounded and satisfies the following condition

$$k_i^- \leq \frac{g_i(p) - g_i(q)}{p - q} \leq k_i^+, p \neq q, i = 1, 2, ..., n \tag{4}$$

where k_i^- and k_i^+ are known constant values and the diagonal matrices are donated as K_m =diag$\{k_1^-, k_2^-, ..., k_n^-\}$ and K_p =diag$\{k_1^+, k_2^+, ..., k_n^+\}$.

Definition 1. *([5]) For given real symmetric matrices $\psi_i, i = 1, 2, 3, 4$ with $\psi_1 \leq 0, \psi_3 > 0, \psi_4 \geq 0$ and $(\|\psi_1\| + \|\psi_2\|)$ $\|\psi_4\|$ =0, any positive integer h, under zero initial state, if the following condition holds*

$$\langle y, \psi_1 y\rangle_h + 2\langle y, \psi_2 u\rangle_h + \langle u, \psi_3 u\rangle_h - \sup_{0\leq k\leq h} y^T(k)\psi_4 y(k) \geq 0. \tag{5}$$

system (1) is extendedly dissipative.

The lemmas to be used are listed as follows

Lemma 1. *(Wirtinger-based inequality [14]) For a given matrix $R > 0$, positive integers $b > a$, any sequence of discrete-time variable $y : Z[a, b] \to \mathbb{R}^n$, the following inequality holds*

$$\sum_{j=a}^{b-1} \eta^T(j)R\eta(j) \geq \frac{1}{b-a} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ * & 3\left(\frac{b-a+1}{b-a-1}\right)R \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \tag{6}$$

where $\eta(k) = y(k+1) - y(k)$, $\vartheta_1 = y(b) - y(a)$ and $\vartheta_2 = y(b) + y(a) - 2\sum\limits_{i=a}^{b} \frac{y(i)}{b-a+1}$.

To handle with time-varying delay $\tau(k)$ in the demonstrator of the sum terms in the forward difference of the LKF, the extended reciprocally convex matrix inequality is presented as Lemma 2.

Lemma 2. *For a real scalar $\alpha \in (0, 1)$, matrices $X_1 > 0$ and $X_2 > 0$, any matrices S_1 and S_2 , the following matrix inequality holds*

$$\begin{bmatrix} \frac{1}{\alpha}X_1 & 0 \\ 0 & \frac{1}{1-\alpha}X_2 \end{bmatrix} \geq \begin{bmatrix} X_1 + (1-\alpha)M_1 & (1-\alpha)S_1 + \alpha S_2 \\ * & X_2 + \alpha M_2 \end{bmatrix} \tag{7}$$

where $M_1 = X_1 - S_2 X_2^{-1} S_2^T$ and $M_2 = X_2 - S_1^T X_1^{-1} S_1$.

Remark 1. *If we let matrices $S = S_1 = S_2$ in the presented inequality (7), the following inequality in [24] will exist:*

$$\begin{bmatrix} \frac{1}{\alpha}X_1 & 0 \\ 0 & \frac{1}{1-\alpha}X_2 \end{bmatrix} \geq \begin{bmatrix} X_1 + (1-\alpha)M_3 & S \\ * & X_2 + \alpha M_4 \end{bmatrix} \tag{8}$$

where $M_3 = X_1 - SX_2^{-1}S^T$ *and* $M_4 = X_2 - S^T X_1^{-1} S$. *Since a restriction that* $S = S_1 = S_2$ *is omitted, inequality (7) is superior to inequality (8) from conservatism. Moreover, inequality (7) also improves the well-known RCCL [13]. This is mainly due to two reasons: first, inequality (8) is a particular form of inequality (7); second, compared with the RCCL [13], inequality (8) introduces the* M_3*-and* M_4*-dependent extra terms and constraint* $\begin{bmatrix} X_1 & S \\ * & X_2 \end{bmatrix} > 0$ *is removed. Hence, the RCCL is also improved by inequality (8) whose conservatism is reduced by inequality (7). That is to say, the employment of inequality (7) will shorten the estimation gaps in comparison of inequality (8) and the RCCL [13], which increases the probability of developing effective criteria.*

3. Main Results

In this section, an improved criterion is established resorting to the delay-product type LKF, the extended reciprocally convex matrix inequality and the Wirtinger-based inequality.

Theorem 1. *For given integers* τ_m, τ_M, μ *and matrices* $\psi_i, i = 1, 2, 3, 4$ *satisfied with Definition 1, system (1) is extendedly dissipative, if there exist a symmetric* $3n \times 3n$ *matrix* P, $2n \times 2n$ *matrix* P_1, $n \times n$ *matrices* $Q_1 > 0, Q_2 > 0, R_1 > 0$, $2n \times 2n$ *matrix* $R_2 > 0$, $n \times n$ *diagonal matrices* $H_i > 0, i = 1, 2, 3$, *symmetric* $n \times n$ *matrices* T_1, T_2 *and* $2n \times 2n$ *matrix* S_1, S_2, *such that the following LMIs hold*

$$\varphi(\tau_m) > 0, \qquad \varphi(\tau_M) > 0 \tag{9}$$

$$\begin{bmatrix} \Pi_{1a} & \Upsilon_4 S_2 \\ * & -W_2 \end{bmatrix} < 0, \qquad \begin{bmatrix} \Pi_{1b} & \Upsilon_4 S_2 \\ * & -W_2 \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} \Pi_{2a} & \Upsilon_5 S_1^T \\ * & -W_1 \end{bmatrix} < 0, \qquad \begin{bmatrix} \Pi_{2b} & \Upsilon_5 S_1^T \\ * & -W_1 \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} D^T \psi_4 D & 0 \\ * & 0 \end{bmatrix} \leq \varphi(\tau_m), \qquad \begin{bmatrix} D^T \psi_4 D & 0 \\ * & 0 \end{bmatrix} \leq \varphi(\tau_M) \tag{12}$$

where

$$\begin{aligned}
\varphi(\tau(k)) &= P + \tau(k) \begin{bmatrix} P_1 & 0 \\ * & 0 \end{bmatrix} \\
\Pi_{1a} &= \varphi_0(\tau_m) + \varphi_1(\mu_2) + \varphi_2 + \varphi_3 + \varphi_{4a} + \Xi - \Omega \\
\Pi_{1b} &= \varphi_0(\tau_m) + \varphi_1(\mu_1) + \varphi_2 + \varphi_3 + \varphi_{4a} + \Xi - \Omega \\
\Pi_{2a} &= \varphi_0(\tau_M) + \varphi_1(\mu_2) + \varphi_2 + \varphi_3 + \varphi_{4b} + \Xi - \Omega \\
\Pi_{2b} &= \varphi_0(\tau_M) + \varphi_1(\mu_1) + \varphi_2 + \varphi_3 + \varphi_{4b} + \Xi - \Omega \\
\varphi_0(\tau(k)) &= \Upsilon_2 \varphi(\tau(k)) \Upsilon_2^T - \Upsilon_1 \varphi(\tau(k)) \Upsilon_1^T \\
\varphi_1(\Delta\tau(k)) &= \Delta\tau(k) \Upsilon_2 \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \Upsilon_2^T \\
\Upsilon_1 &= [e_1, e_5 - e_1, e_6 + e_7] \\
\Upsilon_2 &= [e_s + e_1, e_5 - e_2, e_2 - e_4 + e_6 + e_7] \\
\varphi_2 &= e_1 Q_1 e_1^T - e_2 (Q_1 - Q_2) e_2^T - e_4 Q_2 e_4^T \\
\varphi_3 &= \tau_m^2 e_s R_1 e_s^T + \tau_{21}^2 [e_1, e_s] R_2 [e_1, e_s]^T - \Upsilon_3 W_0 \Upsilon_3^T + \tau_{21} (e_2 T_1 e_2^T - e_3 (T_1 - T_2) e_3^T - e_4 T_2 e_4^T)
\end{aligned}$$

$$\Upsilon_3 = \left[e_1 - e_2, e_1 + e_2 - \frac{2}{\tau_m + 1} e_5 \right]$$
$$W_0 = \begin{bmatrix} R_1 & 0 \\ * & \frac{3(\tau_m+1)}{\tau_m-1} R_1 \end{bmatrix}$$
$$\varphi_{4a} = -[\Upsilon_4, \Upsilon_5] \begin{bmatrix} 2W_1 & S_1 \\ * & W_2 \end{bmatrix} [\Upsilon_4, \Upsilon_5]^T$$
$$\varphi_{4b} = -[\Upsilon_4, \Upsilon_5] \begin{bmatrix} W_1 & S_2 \\ * & 2W_2 \end{bmatrix} [\Upsilon_4, \Upsilon_5]^T$$
$$\Upsilon_4 = [e_6, e_2 - e_3], \Upsilon_5 = [e_7, e_3 - e_4]$$
$$W_j = R_2 + \begin{bmatrix} 0 & T_j \\ * & T_j \end{bmatrix}, j = 1, 2$$
$$e_s{}^T = (A - I_n) e_1^T + B e_8^T + C e_9^T + e_{10}^T$$
$$\begin{aligned}\Xi = &-\mathrm{Sym}\left\{[e_8 - e_1 K_p] H_1 [e_8 - e_1 K_m]^T\right\} - \mathrm{Sym}\left\{[e_9 - e_3 K_p] H_2 [e_9 - e_3 K_m]^T\right\} \\ &- \mathrm{Sym}\left\{[(e_8 - e_9) - (e_1 - e_3) K_p] H_3 [(e_8 - e_9) - (e_1 - e_3) K_m]^T\right\}\end{aligned}$$
$$\Omega = e_1 D^T \psi_1 D e_1^T + \mathrm{Sym}(e_1 D^T \psi_2 e_{10}^T) + e_{10} \psi_3 e_{10}^T$$
$$e_i = [0_{(i-1)n\times n}, I_n, 0_{(10-i)n\times n}], i = 1, 2, ..., 10$$

And system (1) with disturbance $u(k) \equiv 0$ is globally asymptotically stable, if LMIs (9-11) hold when $\Omega = 0$, and vectors e_s and e_i are replaced as

$$\begin{aligned} e_s{}^T &= (A - I_n) e_1^T + B e_8^T + C e_9^T \\ e_i &= [0_{(i-1)n\times n}, I_n, 0_{(9-i)n\times n}], i = 1, 2, ..., 9. \end{aligned}$$

Proof. Construct the following LKF candidate

$$V(x(k)) = \sum_{i=1}^{3} V_i(x(k))$$

where

$$\begin{aligned} V_1(x(k)) &= \zeta_1^T(k) P \zeta_1(k) + \tau(k) \zeta_2^T(k) P_1 \zeta_2(k) \\ V_2(x(k)) &= \sum_{i=k-\tau_m}^{k-1} x^T(i) Q_1 x(i) + \sum_{i=k-\tau_M}^{k-\tau_m-1} x^T(i) Q_2 x(i) \\ V_3(x(k)) &= \tau_m \sum_{l=-\tau_m}^{-1} \sum_{i=k+l}^{k-1} \eta^T(i) R_1 \eta(i) + \tau_{21} \sum_{l=-\tau_M}^{-\tau_m-1} \sum_{i=k+l}^{k-1} \begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}^T R_2 \begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix} \end{aligned}$$

with $\eta(k) = x(k+1) - x(k), \tau_{21} = \tau_M - \tau_m$ and

$$\zeta_1^T(k) = \left[\zeta_2^T(k), \sum_{i=k-\tau_M}^{k-\tau_m-1} x^T(i) \right], \zeta_2^T(k) = \left[x^T(k), \sum_{i=k-\tau_m}^{k-1} x^T(i) \right]$$

Define the following notation

$$\xi^T(k) = \left[x^T(k), x^T(k-\tau_m), x^T(k-\tau(k)), x^T(k-\tau_M), \sum_{i=k-\tau_m}^{k} x^T(i), \sum_{i=k-\tau(k)}^{k-\tau_m-1} x^T(i), \sum_{i=k-\tau_M}^{k-\tau(k)-1} x^T(i), g^T(x(k)), g^T(x(k-\tau(k))), u^T(k) \right]$$

The non-summable term $V_1(k)$ can be rewritten as

$$V_1(x(k)) = \zeta_1^T(k)\varphi(\tau(k))\zeta_1(k)$$

where $\varphi(\tau(k))$ is defined in Theorem 1. Based on the convex combination technique, if LMIs (9) hold, then $\varphi(\tau(k))>0$ together with $Q_i > 0, i = 1,2$ and $R_i > 0, i = 1,2$ will contribute to the positive definite $V(k)$.

Defining forward difference $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$ and calculating along the solution of system (1) yield

$$\Delta V(x(k)) = \sum_{i=1}^{3} \Delta V_i(x(k))$$

where

$$\begin{aligned}
\Delta V_1(x(k)) &= \zeta_1^T(k+1)\varphi(\tau(k+1))\zeta_1(k+1) - \zeta_1^T(k)\varphi(\tau(k))\zeta_1(k) \\
&= \xi^T(k)(\varphi_0(\tau(k)) + \varphi_1(\Delta\tau(k)))\xi(k) \qquad (13) \\
\Delta V_2(x(k)) &= x^T(k)Q_1x(k) - x^T(k-\tau_m)(Q_1 - Q_2)x(k-\tau_m) - x^T(k-\tau_M)Q_2x(k-\tau_M) \\
&= \xi^T(k)\varphi_2\xi(k) \qquad (14) \\
\Delta V_3(x(k)) &= \tau_m^2\eta^T(k)R_1\eta(k) + \tau_{21}^2\begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix}^T R_2 \begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix} - \tau_m\sum_{i=k-\tau_m}^{k-1}\eta^T(i)R_1\eta(i) - \tau_{21}\sum_{i=k-\tau_M}^{k-\tau_m-1}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}^T R_2 \begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}. \qquad (15)
\end{aligned}$$

Then, based on Lemma 1, the R_1-dependent term is estimated as

$$\tau_m\sum_{i=k-\tau_m}^{k-1}\eta^T(i)R_1\eta(i) \geq \xi^T(k)\Upsilon_3\begin{bmatrix} R_1 & 0 \\ * & \frac{3(\tau_m+1)}{\tau_m-1}R_1 \end{bmatrix}\Upsilon_3^T\xi(k). \qquad (16)$$

Now, for any positive integers $0 \leq c_1 \leq c_2$ and symmetric matrix T_i, the following zero equation holds

$$\begin{aligned}
0 &= \tau_{21}x^T(k-c_1)T_ix(k-c_1) - \tau_{21}x^T(k-c_2)T_ix(k-c_2) - \tau_{21}\sum_{i=k-c_2}^{k-c_1-1}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}^T\begin{bmatrix} 0 & T_i \\ * & T_i \end{bmatrix}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix} \\
&= \varpi(c_1, c_2, T_i)
\end{aligned}$$

Add equations $\varpi(\tau_m, \tau(k), T_1)$ and $\varpi(\tau(k), \tau_M, T_2)$ into R_2-dependent term for any matrices T_1 and T_2, and bound it with the Jensen's inequality as

$$\begin{aligned}
&-\tau_{21}\sum_{i=k-\tau_M}^{k-\tau_m-1}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}^T R_2\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix} \\
= \;& \tau_{21}x^T(k-\tau_m)T_1x(k-\tau_m) - \tau_{21}x^T(k-\tau_M)T_2x(k-\tau_M) - \tau_{21}x^T(k-\tau(k))(T_1 - T_2)x(k-\tau(k)) \\
&-\tau_{21}\sum_{i=k-\tau(k)}^{k-\tau_m-1}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}^T\left\{R_2 + \begin{bmatrix} 0 & T_1 \\ * & T_1 \end{bmatrix}\right\}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix} - \tau_{21}\sum_{i=k-\tau_M}^{k-\tau(k)-1}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix}^T\left\{R_2 + \begin{bmatrix} 0 & T_2 \\ * & T_2 \end{bmatrix}\right\}\begin{bmatrix} x(i) \\ \eta(i) \end{bmatrix} \\
\leq \;& \xi^T(k)\{\tau_{21}[e_2T_1e_2^T - e_3(T_1 - T_2)e_3^T - e_4T_2e_4^T] - \hbar(\tau(k))\}\xi(k) \qquad (17)
\end{aligned}$$

where

$$\hbar(\tau(k)) = \frac{1}{\alpha}\Upsilon_4W_1\Upsilon_4^T + \frac{1}{1-\alpha}\Upsilon_5W_2\Upsilon_5^T, \alpha = \frac{\tau(k) - \tau_m}{\tau_{21}}.$$

For any appropriate matrix S_1 and S_2, employing Lemma 2 to deal with time-varying delay $\tau(k)$ in the denominator yields

$$\hbar(\tau(k)) \geq \tilde{\hbar}(\tau(k)) \tag{18}$$

where

$$\tilde{\hbar}(\tau(k)) = [\Upsilon_4, \Upsilon_5] \begin{bmatrix} W_1 + (1-\alpha)M_1 & (1-\alpha)S_1 + \alpha S_2 \\ * & W_2 + \alpha M_2 \end{bmatrix} [\Upsilon_4, \Upsilon_5]^T$$
$$M_1 = W_1 - S_2 W_2^{-1} S_2^T, M_2 = W_2 - S_1^T W_1^{-1} S_1$$

.

Therefore, combining the right sides of (15)-(18)yields

$$\Delta V_3(x(k)) \leq \xi^T(k)(\varphi_3 + \tilde{\hbar}(\tau(k)))\xi(k). \tag{19}$$

Considering the activation functions satisfying (4) yeilds

$$0 \leq -2\left[g(x(k)) - K_m x(k)\right]^T H_1 \left[g(x(k)) - K_p x(k)\right] \tag{20}$$
$$0 \leq -2\left[g(x(k-\tau(k))) - K_m x(k-\tau(k))\right]^T H_2 \left[g(x(k-\tau(k))) - K_p x(k-\tau(k))\right] \tag{21}$$
$$0 \leq -2\left[g(x(k)) - g(x(k-\tau(k))) - K_m(x(k) - x(k-\tau(k)))\right]^T H_3 \left[g(x(k)) - g(x(k-\tau(k))) - K_p(x(k) - x(k-\tau(k)))\right] \tag{22}$$

where

$$H_i = diag\{h_{1i}, h_{2i} ... h_{ni}\} > 0, i = 1, 2, 3$$

Define the following quadratic supply rate function for system (1)

$$J(i) = y^T(i)\psi_1 y(i) + 2y^T(i)\psi_2 u(i) + u^T(i)\psi_3 u(i) = \xi^T(i)\Omega\xi(i)$$

where matrices ψ_1, ψ_2, ψ_3 are satisfied with Definition 1 and matrix Ω is defined in Theorem 1.

Combine the right sides of (13), (14) and (19)-(22), and define the following difference

$$\Delta V(x(k)) - J(k) \leq \xi^T(k)\Theta(\tau(k))\xi(k) \tag{23}$$

where

$$\Theta(\tau(k)) = \varphi_0(\tau(k)) + \varphi_1(\Delta\tau(k)) + \varphi_2 + \varphi_3 + \tilde{\hbar}(\tau(k)) + \Xi - \Omega.$$

It follows from the Schur complement and convex combination technique that, when time-varying delay $\tau(k)$ is satisfied with (2) and (3), inequality $\Theta(\tau(k)) < 0$ holds which turns into $\Delta V(x(k)) < J(k)$ if and only if LMIs (10)-(11) are feasible. Afterwards, the extended dissipativity of system (1) and the stability of system (1) with $u(k) \equiv 0$ are proved by turns.

According to Definition 1, we aims at proving the extended dissipativity of system (1) by

$$\sup_{0 \leq k \leq h} y^T(k)\psi_4 y(k) \leq \sum_{i=0}^{h} J(i). \tag{24}$$

Under zero initial condition, $\Delta V(x(k)) < J(k)$ leads to

$$\sum_{i=0}^{k-1}(\Delta V(x(i)) - J(i)) = V(x(k)) - \sum_{i=0}^{k-1} J(i) \leq 0. \tag{25}$$

Moreover, if LMIs (12) hold, the following inequality exists

$$\xi_1^T(k)\begin{bmatrix} D^T\psi_4 D & 0 \\ * & 0 \end{bmatrix}\xi_1(k) \leq \xi_1^T(k)\varphi(\tau(k))\xi_1(k).$$

That is $V_1(x(k)) \geq y^T(k)\psi_4 y(k)$ which together with $V(x(k)) \geq V_1(x(k))$ and inequality (25) leads to

$$y^T(k)\psi_4 y(k) \leq \sum_{i=0}^{k-1} J(i). \tag{26}$$

To prove (24), two cases, $\psi_4 = 0$ and $\psi_4 > 0$, are considered respectively.

If matrix $\psi_4 = 0$, inequality $0 \leq \sum_{i=0}^{h} J(i)$ is deduced via (25) and then, (24) is implied.

When matrix $\psi_4 > 0$, Definition 1 can be available for $\psi_1 = 0, \psi_2 = 0$. Then, $J(i) = u^T(i)\psi_3 u(i) > 0$ holds for $\psi_3 > 0$. It follows from (26) that $y^T(k)\psi_4 y(k) \leq \sum_{i=0}^{k-1} J(i) \leq \sum_{i=0}^{h} J(i)$ holds for any positive integers $k \leq h$. Hence, computing the the supremum of $y^T(k)\psi_4 y(k)$ with respect to $k \in [0, h]$ leads to (24). In short, if LMIs (9)-(12) hold, system (1) is occupied with extended dissipativity .

When disturbance $u(k) \equiv 0$, $J(i) = y^T(i)\psi_1 y(i) \leq 0$ is derived with $\psi_1 \leq 0$ from Definition 1. Then, according to inequality $\Delta V(x(k)) < J(k)$, $\Delta V(x(k)) \leq -c\,\|x(k)\|^2$ is obtained for a sufficiently small scalar $c > 0$, which means that system (1) without disturbance is globally asymptotically stable. This completes the proof. □

Remark 2. *The results presented in [5, 8], whose conservatism is reduced via constructing the comlpex augmented LKFs and introducing multiple summation terms, possess heavy computation burden. On the contrary, a delay-product type LKF is constructed in this paper inspired by [26]. By this means, more information about time-varying delay $\tau(k)$ is taken into account. Moreover, matrix $P + \tau(k)P_1 > 0$ take the place of matrix $P > 0$ in all LKF, which is more unrestrained to develop conservatism-reduced criterion. Then, computing the forward difference of the LKF brings out the change rate of $\tau(k)$. Naturally, this change rate is embedded into the developed criterion, by which more system information about time delay is considered and wider feasible domain for acquired LMIs is expected to be realized.*

Remark 3. *In this paper, the extended reciprocally convex matrix inequality which is more general than the matrix inequality presented in [24] is employed to give tight estimations for the forward difference of the LKF. Meantime, the Wirtinger-based inequality [14] together with the zero equation [8] is also applied. After that, a less conservative criterion is established compared with those existing criteria obtained via the simple LKF [15, 22], the FWM technique [15] and the Jensen's inequality [5, 8, 17]. Moreover, in [5], due to the construction of the LKF and handing methods to its forward difference, term $\tau^2(k)$ is introduced. Then, for solving the criterion in LMIs based on the convex combination technique, $\tau^2(k)$ is transformed via an equivalent linearity condition [5] embedded with an appropriately dimensioned matrix Ψ, which result in high number of decision variables. Hence, we consider different manners to avoid the appearance of term $\tau^2(k)$ to lower the computation complexity of developed criterion without any conservatism increased.*

4. Numerical examples

In this section, two numerical examples are given to illustrate the superiority of our presented method over previous results considering both the conservatism and the number of decision variables (NoDVs).

Example 1. *Consider system (1) with the following parameters*

$$A=\begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, B=\begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, C=\begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, D=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, K_m = \mathrm{diag}\{0,0\}, K_p = \mathrm{diag}\{1,1\}$$

This example is used to show the advantage of Theorem 1 in comparison with the existing results. Four robust performances for system (1) with respect to various τ_M provided by different criteria, together with the NoDVs, are summarized in Table I-IV, where $\mu = -\mu_1 = \mu_2$. They can be clearly presented as follows.

i) $l_2 - l_\infty$ performance is computed via the extended dissipativity condition with matrices $\psi_1 = 0, \psi_2 = 0, \psi_3 = \gamma^2 I$ and $\psi_4 = I$. Table I reveals the allowable minimum $l_2 - l_\infty$ performance γ with respect to various τ_M and $\tau_m = 3$ calculated by Theorem 1 and existing results [5].

Table 1: The minimum $l_2 - l_\infty$ performance γ for various τ_M (EXAMPLE 1)

Criteria	τ_M					NoDVs
	8	12	16	20	24	
Th.1 [5]	1.2676	1.4265	1.6067	1.8113	2.0433	$44n^2 + 13n$
Th.1($\mu=1$)	1.2119	1.3989	1.5925	1.8045	2.0403	$19n^2 + 9n$

ii) Passivity is obtained from the extended dissipativity condition when matrices $\psi_1 = 0, \psi_2 = I, \psi_3 = \gamma I$ and $\psi_4 = 0$. Table II shows the allowable minimum passivity level γ computed from Theorem 1 and existing results [5] with various τ_M and $\tau_m = 5$.

Table 2: The minimum passivity level γ for various τ_M (EXAMPLE 1)

Criteria	τ_M					NoDVs
	7	9	11	13	15	
Th.1 [5]	1.9497	2.0501	2.1697	2.3031	2.4512	$44n^2 + 13n$
Th.1 ($\mu=1$)	1.9473	2.0471	2.1663	2.2994	2.4473	$19n^2 + 9n$

iii) H_∞ performance is available from the extended dissipativity condition with matrices $\psi_1 = -I, \psi_2 = 0, \psi_3 = \gamma^2 I$ and $\psi_4 = 0$. Table III lists the allowable minimum H_∞ performance γ obtained from Theorem 1 and previous results [5] with different τ_M when $\tau_m = 3$ is given.

Table 3: The minimum H_∞ performance γ for various τ_M (EXAMPLE 1)

Criteria	τ_M					NoDVs
	9	11	13	15	17	
Th.1 [5]	2.3658	2.6113	2.8690	3.1407	3.4268	$44n^2 + 13n$
Th.1($\mu=1$)	2.3634	2.6093	2.8651	3.1342	3.4189	$19n^2 + 9n$

iv) Dissipativity is deduced from the extended dissipativity condition when matrices $\psi_1 = -0.5I, \psi_2 = I, \psi_3 = (R - \gamma I)$ and $\psi_4 = 0$. Table IV shows the allowable maximum dissipativity level γ calculated from Theorem 1 and existing results [5] with matrix $R=\begin{bmatrix} 4.5 & 0.15 \\ 0.15 & 5 \end{bmatrix}$, variable τ_M and $\tau_m = 4$.

Table 4: The maximum dissipativity level γ for various τ_M (EXAMPLE 1)

Criteria	τ_M					NoDVs
	7	8	9	11	13	
Th.1 [5]	2.1330	2.0349	1.9077	1.5651	1.0984	$44n^2 + 13n$
Th.1($\mu = 1$)	2.1347	2.0366	1.9092	1.5652	1.0986	$19n^2 + 9n$

From Tables I-III, when τ_M becomes bigger, the $l_2 - l_\infty$ performance, passivity and H_∞ performance level γ increases. That is, the dynamic characteristics of system (1) is obviously influenced by the value of time delay. In Table IV, the dissipativity level γ decreases when τ_M increases. Hence, the dynamic behaviors of system may be slowly declined since higher value of γ means that system (1) has better anti-interference and fault tolerance capacity.

From Tables I-IV, due to the construction of the delay-product type LKF and the usage of the extended reciprocally convex matrix inequality together with the Wirtinger-based inequality, Theorem 1 has received less conservatism than the existing results [5] obtained via the Jensen's inequality and the RCCL. Simultaneously, since different estimation methods employed in this paper, the equivalent linear conversion condition used in [5] with matrix Ψ embedded is dismissed. Therefore, Theorem 1 achieves desired dynamic behaviors with the decreases of computation complexity.

Example 2. *Consider system (1) with the following parameters*

$$A=\begin{bmatrix}0.4 & 0 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0.3\end{bmatrix}, B=\begin{bmatrix}0.2 & -0.2 & 0.1\\ 0 & -0.3 & 0.2\\ -0.2 & -0.1 & -0.2\end{bmatrix}, C=\begin{bmatrix}-0.2 & 0.1 & 0\\ -0.2 & 0.3 & 0.1\\ 0.1 & -0.2 & 0.3\end{bmatrix}, K_m = \mathrm{diag}\{0, -0.4, -0.2\}, K_p = \mathrm{diag}\{0.6, 0, 0\}$$

In this example, the stability of system (1) with disturbance $u(k) \equiv 0$ is investigated. Table V lists the allowable upper bounds with respect to various lower bounds on time-varying delay, which are calculated from obtained criterion together with previous results, and $\mu = -\mu_1 = \mu_2$. Due to the delay-product type LKF, Theorem 1 realizes higher upper bounds than those developed with the simple LKF [15, 22], the delay segmented LKF [17] and the augmented LKF [5, 8]. Moreover, since the extended reciprocally convex matrix inequality includes the matrix inequality in [24] as a special case and the Wirtinger-based inequality is employed to give tight estimations for the forward difference of the LKF, Theorem 1 is less conservative than the results obtained via the FWM techniques [15] and the Jensen's inequality [5, 17] combined with the RCCL [8]. Thus, this numerical example again shows the benefits of the developed criterion.

Table 5: The maximal upper bounds of τ_M for various τ_m (EXAMPLE 2)

Criteria	τ_m						NoDVs
	2	4	6	10	15	20	
Th.1 [15]	8	10	11	15	20	24	$17.5n^2 + 4.5n$
Th.2 [22]	8	10	12	16	21	26	$15n^2 + 5n$
Th.2 [24]	11	12	14	18	22	27	$15.5n^2 + 12.5n$
Th.1 [17]	12	14	16	20	25	30	$4.5n^2 + 7.5n$
Th.1 [8]	17	19	21	25	30	35	$61.5n^2 + 17.5n$
Co.1 [5]	18	19	21	25	30	35	$44n^2 + 13n$
Th.1($\mu = 1$)	18	20	22	26	31	36	$19n^2 + 9n$

For simulation, the activation function is taken as $g(x) = [\tanh(x_1), \tanh(x_2), \tanh(x_3)]^{\mathrm{T}}$. Then, according to Table 5, the DNN is stable for the case of $\tau(k) \in [2, 18]$. This random delay could be obtained from band-limited white noise which is shown in subfigure in Figure 1. Hence, when the initial condition is given as $x(k) = [-1.5, -0.5, 1]^T, k \in [-18, 0]$, the response of the DNN is given in Figure 1. Figure 1 reveals that the curves of response states are asymptotically converging to zero which means the DNN with given parameters is stable at its equilibrium point. Therefore, the effectiveness of the developed criterion is further verified.

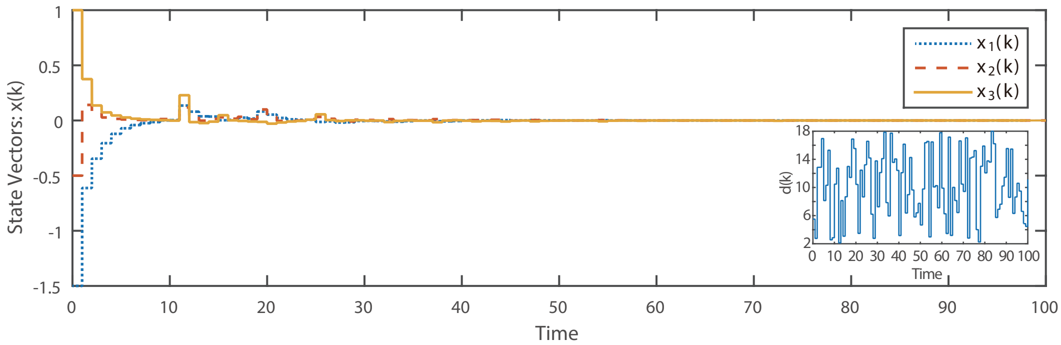

Figure 1: State trajectories of the DNN of Example 2

5. Conclusions

In this paper, the extended dissipativity analysis for the discrete-time neural networks with a time-varying delay has been investigated. An improved extended dissipativity condition has been established via constructing a novel LKF with a delay-product-term introduced and estimating the sum terms in the forward difference of the LKF with the extended reciprocally convex matrix inequality and Wirtinger-based inequality. After that, the developed dissipativity condition has been extended to the stability analysis of the counterpart system without disturbance. Finally, two numerical examples have been given to demonstrate the improvements of the obtained criterion whose effectiveness is further verified via the simulation result.

Moreover, it should be noted that, although these two examples are given with very low dimensions, either the extended dissipativity or the stability of a discrete-time neural network with higher order (closer to practical one) could be analyzed via the similar criterion. There is no technique difficulty to replace the system parameters in the numerical examples by some higher dimension matrices. In addition, the presented method in this paper can be extended to the analysis of other types neural networks such as those with Markov jump [12, 19], stochastic memristor [10], *etc.*

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[1] G.I. Bara and M. Boutayeb, "Static output feedback stabilization with H_∞ performance for linear discrete-time systems," *IEEE Trans. Automatic Cont.,* vol. 50, no. 2, pp. 250-254, 2005.

[2] J. Cao and J. Wang, "Global exponential stability and periodicity of recurrent neural networks with time delays," *IEEE Trans. Circuits Sys. I, Reg. Papers,* vol. 52, no. 5, pp. 920-931, 2005.

[3] Y. Chen and W.X. Zheng, "Stability and L_2 performance analysis of stochastic delayed neural networks," *IEEE Trans. Neural Netw.,* vol. 22, no. 10, pp. 1662-1668, 2011.

[4] Z. Feng and J. Lam, “Stability and dissipativity analysis of distributed delay cellular neural networks,” *IEEE Trans. Neural Netw.,* vol. 22, no. 6, pp. 976-981, 2011.
[5] Z.G. Feng and W.X. Zheng, “On extended dissipativity of discrete-time neural networks with time delay,” *IEEE Trans. Neural Netw. Learning Syst.,* vol. 26, no. 12, pp. 3293-3300, 2015.
[6] C.C. Hua, S.S. Wu, X. Yang and X.P. Guan, “Stability analysis of time-delay systems via free-matrix-based double integral inequality,” *Int. J. Syst. Sci.,* vol. 48, no. 2, pp. 257-263, 2017.
[7] L. Jin, Y. He and M. Wu, “Improved delay-dependent stability analysis of discrete-time neural networks with time-varying delays,” *J. Frankl. Inst.,* vol. 354, no. 4, pp. 1922-1936, 2017.
[8] O.M. Kwon, M.J. Park, J.H. Park, S.M. Lee and E.J. Cha, “New criteria on delay-dependent stability for discrete-time neural networks with time-varying delays,” *Abstr. Appl. Anal.,* vol. 121, pp. 185-194, 2013.
[9] T.H. Lee, M.J. Park, J.H. Park, O.M. Kwon and S.M. Lee, “Extended dissipative analysis for neural networks with time-varying delays,” *IEEE Trans. Neural Netw. Learning Syst.,* vol. 25, no. 10, pp. 1936-1941, 2014.
[10] H. Liu, Z.D. Wang, B. Shen, T.W. Huang and F.E. Alsaadi, “Stability analysis for discrete-time stochastic memristive neural networks with both leakage and probabilistic delays,” *Neural Networks*, vol. 102, pp. 1-9, 2018.
[11] J. Liu, J. Zhang, “Note on stability of discrete-time time-varying delay systems,” *IET Contr. Theory Appl.*, vol.6, no. 2, pp. 335-339, 2012.
[12] G. Nagamani, S. Ramasamy and A. Meyer-Baese, “Robust dissipativity and passivity based state estimation for discrete-time stochastic Markov jump neural networks with discrete and distributed time-varying delays,” *Neural Comput. Appl.*, vol. 28, no. 4, pp. 717-735, 2017.
[13] P. Park, J.W. Ko, and C.K. Jeong, “Reciprocally convex approach to stability of systems with time-varying delays,” *Automatica,* vol. 47, no. 1, pp. 235-238, 2011.
[14] A. Seuret, F. Gouaisbaut and E. Fridman, “Stability of discrete-time systems with time-varying delays via a novel summation inequality,” *IEEE Trans. Automat. Cont.,* vol. 60, no. 10, pp. 2740-2745, 2015.
[15] Q. Song and Z. Wang, “A delay-dependent LMI approach to dynamics analysis of discrete-time recurrent neural networks with time-varying delays,” *Phys. Lett. A,* vol. 368, no. 1-2, pp. 134-145, 2007.
[16] L. Wu, Z.G. Feng and J. Lam, “Stability and synchronization of discrete time neural networks with switching parameters and time-varying delays,” *IEEE Trans. Circuits Syst. II, Exp. Briefs,* vol. 24, no. 12, pp. 1957-1972, 2013.
[17] Z. Wu, H. Su, J. Chu and W. Zhou, “Improved delay-dependent stability condition of discrete recurrent neural networks with time-varying delays,” *IEEE Trans. Neural Netw.,* vol. 21, no. 4, pp. 692-697, 2010.
[18] Z.G. Wu, P. Shi, H. Su and J. Chu, “Dissipativity analysis for discrete-time stochastic neural networks with time-varying delays,” *IEEE Trans. Neural Netw. Learning Syst.,* vol. 24, no. 3, pp. 345-355, 2013.
[19] X. Yang, Z. Feng, J. Feng and J. Cao, “Synchronization of discrete-time neural networks with delays and Markov jump topologies based on tracker information,” *Neural Networks*, vol. 85, pp. 157-164, 2017.
[20] H.B. Zeng, Y. He, M. Wu and J.H. She, “Free-matrix-based integral inequality for stability analysis of systems with time-varying delay,” *IEEE Trans. Automat. Cont.,* vol. 60, no. 10, pp. 2768-2772, 2015.
[21] H.B. Zeng, Y. He, M. Wu and H.Q. Xiao, “Improved conditions for passivity of neural networks with a time-varying delay,” *IEEE Trans. Cybernetics,* vol. 44, no. 6, pp. 785-792, 2014.
[22] B. Zhang, S. Xu and Y. Zou, “Improved delay-dependent exponential stability criteria for discrete-time recurrent neural networks with time varying delays,” *Neurocomputing,* vol. 72, no. 1-3, pp. 321-330, 2008.
[23] B. Zhang, W.X. Zheng and S. Xu, “Filtering of markovian jump delay systems based on a new performance index,” *IEEE Trans. Circuits Syst. I, Reg. Papers,* vol. 60, no. 5, pp. 1250-1263, 2013.
[24] C.K. Zhang, Y. He, L. Jiang, Q.G. Wang and M. Wu, “Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality,” *IEEE Trans. Cybernetics*, vol. 47, no. 10, pp. 3040-3049, 2017.
[25] C.K. Zhang, Y. He, L. Jiang and M. Wu, “An improved summation inequality to discrete-time systems with time-varying delay,” *Automatica*, vol. 74, pp. 10-15, 2016.
[26] C.K. Zhang, Y. He, L. Jiang, M. Wu and H.B. Zeng, “Delay-variation dependent stability of delayed discrete-time systems,” *IEEE Trans. Autom. Control*, vol. 61, no. 9, pp. 2663-2669, 2016.
[27] C.K. Zhang, Y. He, L. Jiang, M. Wu and H.B. Zeng, “Summation inequalities to bounded real lemmas of discrete-time systems with time-varying delay,” *IEEE Trans. Autom. Control,* vol. 62, no. 5, pp. 2582-2588, 2017.
[28] C.K. Zhang, Y. He, L. Jiang, Q.H. Wu and M. Wu, “Delay-dependent stability criteria for generalized neural networks with two delay components,” *IEEE Trans. Neural Netw. Learning Syst.,* vol. 25, no. 7, pp. 1263-1276, 2014.
[29] X.M. Zhang, Q.L. Han, and J. Wang, “Admissible delay upper bounds for global asymptotic stability of neural networks with time-varying delays,” *IEEE Trans. Neural Netw.,* DOI: 10.1109/TNNLS.2018.2797279, in press.
[30] X.M. Zhang, Q.L. Han, Z. Wang, and B.L. Zhang, “Neuronal state estimation for neural networks with two additive time-varying delay components,” *IEEE Trans. Cybernetics,* vol. 47, no. 10, pp. 3184-3194, 2017.
[31] Y. Zhang, P. Shi, R.K. Agarwal and Y. Shi, “Dissipativity analysis for discrete time-delay fuzzy neural networks with markovian jumps,”

IEEE Trans. Fuzzy Syst., vol. 24, no. 2, pp. 432-443, 2016.
[32] X.L. Zhu and G.H. Yang, “Jensen inequality approach to stability analysis of discrete-time systems with time-varying delay,” *American Control Conference*, pp. 1644-1649, 2008.