THE STABILITY SPACE OF COMPACTIFIED UNIVERSAL JACOBIANS

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Abstract. In this paper we describe compactified universal Jacobians, i.e. compactifications of the moduli space of line bundles on smooth curves obtained as moduli spaces of rank 1 torsion-free sheaves on stable curves, using an approach due to Oda–Seshadri. We focus on the combinatorics of the stability conditions used to define compactified universal Jacobians. We explicitly describe an affine space, the stability space, with a decomposition into polytopes such that each polytope corresponds to a proper Deligne–Mumford stack that compactifies the moduli space of line bundles. We apply this description to describe the set of isomorphism classes of compactified universal Jacobians (answering a question of Melo), and to resolve the indeterminacy of the Abel–Jacobi sections (a problem raised by Grushevsky–Zakharov).

1. Introduction

In this paper we study the problem of extending the universal Jacobian $\mathcal{J}^d_{g,n}$ over the moduli space of smooth n-pointed curves of genus g to a proper family over the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves. Recall that $\mathcal{J}^d_{g,n}$ is the moduli space of degree d line bundles on smooth curves. We extend it as a moduli space of sheaves. One extension of $\mathcal{J}^d_{g,n}$ is the moduli space $\mathrm{Simp}^d_{g,n}$ of all simple rank 1 torsion-free sheaves of degree d, but this extension fails to be proper. Indeed, while it satisfies the existence part of the valuative criterion of properness [Est01, Theorem 32], it is not proper because it fails to be separated and of finite type.

Rather than working directly with $\mathrm{Simp}^d_{g,n}$, we analyze extensions of $\mathcal{J}^d_{g,n}$ that are suitable proper subspaces (or substacks) of $\mathrm{Simp}^d_{g,n}$. The proper subspaces of $\mathrm{Simp}^d_{g,n}$ we describe are the subspaces defined by choosing a set of multidegrees for each curve and taking the subspace of $\mathrm{Simp}^d_{g,n}$ parameterizing the sheaves with multidegree equal to one of the chosen multidegrees. Here the multidegree of a line bundle L on a reducible curve is the vector whose components are the degrees of the restrictions of L to the irreducible components of the curve. The problem of prescribing a collection of multidegrees with the property that the resulting subspace of $\mathrm{Simp}^d_{g,n}$ is a proper extension of $\mathcal{J}^d_{g,n}$ has been studied by a large number of authors; see e.g. [OS79, AK80, Cap94, Sim94, Pan96, Est01, Cap08, Mel09, Mel11, Mel16]. In this paper, we introduce and study subspaces of $\mathrm{Simp}^d_{g,n}$ produced by generalizing to the universal family of curves an approach developed by Oda–Seshadri. With our construction, we produce the commonly studied spaces that extend $\mathcal{J}^d_{g,n}$ to a proper space. In particular, our construction recovers the moduli spaces constructed by Melo in [Mel16]. We explain the relation with Melo's work in Remark 4.6 and with other work in Remarks 5.11 and 6.8.

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In [OS79] Oda–Seshadri introduced, for a nodal curve C (and $d = 0$), the *stability space* $V^d(C)$ as the affine space of functions $\phi: \{C_i \subset C \text{ an irreducible component}\} \to \mathbb{R}$ such that $\sum \phi(C_i) = d$. For a *nondegenerate* $\phi \in V^d(C)$ (i.e. a ϕ not lying in a certain locally finite collection of hyperplanes) they proved that the moduli space of ϕ-semistable sheaves, i.e. sheaves whose multidegree is sufficiently close to ϕ (in a sense that we make precise in Definition 4.1), is a proper subspace $\overline{J}_C(\phi)$ of the space of simple sheaves on C, which we call a (fine) ϕ-compactified Jacobian.

We extend Oda–Seshadri's approach to describe moduli spaces over $\overline{\mathcal{M}}_{g,n}$. In Section 3 we construct a space $V^d_{g,n}$ of stability conditions for the universal stable pointed curve. The affine space $V^d(C)$ associated to a nodal curve C depends only on the dual graph Γ_C of C. Denoting by $\mathcal{G}_{g,n}$ the set of isomorphism classes of stable n-marked graphs of genus g, we define

$$V^d_{g,n} \subset \prod_{\Gamma \in \mathcal{G}_{g,n}} V^d(\Gamma)$$

as the subspace consisting of those vectors $\phi = (\phi(\Gamma) \in V^d(\Gamma))_{\Gamma \in \mathcal{G}_{g,n}}$ that satisfy a compatibility condition with respect to automorphisms and contractions of the dual graphs (see Definition 3.2 for details). For a *nondegenerate* $\phi \in V^d_{g,n}$, we show that

$$\{\phi\text{-stable sheaves on stable curves}\} \subset \mathrm{Simp}^d_{g,n}$$

is a proper moduli space that we call a (fine) ϕ-compactified universal Jacobian $\overline{\mathcal{J}}_{g,n}(\phi)$. We give the precise definition of $\overline{\mathcal{J}}_{g,n}(\phi)$ in Section 4. When ϕ is nondegenerate, we show in Corollary 4.4 that $\overline{\mathcal{J}}_{g,n}(\phi)$ is a proper Deligne–Mumford stack, a result we deduce from Simpson's representability result [Sim94, Theorem 1.21].

The main result about the stability space is Theorem 1, where we describe $V^d_{g,n}$ as the degree d subspace of the real relative Picard group of the universal curve. As a by-product of that theorem, in Corollary 3.6 we prove that an element $\phi \in V^d_{g,n}$ is uniquely determined by its components $\phi(\Gamma)$ for Γ the dual graph of certain stable pointed curves with two smooth irreducible components and at most two nodes, so that in particular, two extensions $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ that coincide in codimension 2 must be equal.

The main contribution of this paper is the description in Section 5 of how the moduli spaces depend on ϕ. There we define ϕ_1 to be equivalent to ϕ_2 when ϕ_1-stability coincides with ϕ_2-stability. The equivalence classes are the interiors of rational bounded convex polytopes in $V^d_{g,n}$ that we call *stability polytopes.* We then exhibit in Theorem 2 an explicit set of equations for the defining hyperplanes.

Theorem. *For $g \geq 2$ and $N = N(g,n)$ the number of boundary divisors in $\overline{\mathcal{M}}_{g,n}$, the decomposition of $V^{g-1}_{g,n} \cong \mathbb{R}^{N-1} \times \mathbb{R}^n$ into stability polytopes is the product of the decomposition of $\mathbb{R}^{N-1}$ by integer translates of coordinate hyperplanes and the decomposition of $\mathbb{R}^n$ by integer translates of the following hyperplanes*

$$\left\{ \vec{x} \in \mathbb{R}^n \colon \sum_{i \in S} x_i - \frac{\ell}{2g-2} \sum_{i=1}^{n} x_i = 0 \right\} \text{ for } \ell = 0, \ldots, 2g-3, \;\; S \subseteq \{1, \ldots, n\}. \tag{1}$$

This is Theorem 2 in the special case when $d = g-1$. When $d \neq g-1$, the decomposition of $V^d_{g,n}$ is similar but the hyperplanes are translated. The factors in the product decomposition correspond to two vector spaces $C_{g,n}$ and $D_{g,n}$ that we introduce in Definition 3.5.

The hyperplanes in (1) with $\ell = 0$ are known as the resonance hyperplanes in the literature, so (1) defines a refinement of the resonance hyperplane arrangement. The above description of the stability polytopes should be compared with a similar description in [KP16]. There we carried out the analogous program for extensions of $\mathcal{J}^{g-1}_{g,n}$ over the moduli stack of treelike curves $\mathcal{M}^{\mathrm{TL}}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ using an affine space $V^{\mathrm{TL}}_{g,n}$ analogous to $V^d_{g,n}$. As we explain in Remarks 3.11 and 4.5, $V^{\mathrm{TL}}_{g,n}$ is canonically isomorphic to $C_{g,n}$. Theorem 2 shows a considerable increase in the combinatorial complexity when passing from the problem of extending $\mathcal{J}^d_{g,n}$ over $\mathcal{M}^{\mathrm{TL}}_{g,n}$ to the problem of extending it over $\overline{\mathcal{M}}_{g,n}$ as the resonance hyperplane arrangement is more complicated than the arrangement of coordinate hyperplanes.

The difference between $\mathcal{M}^{\mathrm{TL}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ is also demonstrated by the results in Section 6.2, where we describe how $\overline{\mathcal{J}}_{g,n}(\phi)$ depends on $\phi \in V^d_{g,n}$. Over treelike curves, we showed in [KP16] that, while changing $\phi \in V^{\mathrm{TL}}_{g,n}$ changes the set of ϕ-stable sheaves, the Deligne–Mumford stack $\overline{\mathcal{J}}_{g,n}(\phi)$ does not change. The situation over $\overline{\mathcal{M}}_{g,n}$ is different. We show

Theorem. *When $\overline{\mathcal{M}}_{g,n}$ is of general type, there exists nondegenerate $\phi_1, \phi_2 \in V_{g,n}$ such that $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ are not isomorphic as Deligne–Mumford stacks.*

This is Corollary 6.17, and the result answers a question of Melo in [Mel16, Question 4.15] (see Remark 4.6 for a description of the relation of that work to this paper).

We deduce the result from Corollary 6.15 which states that, for all (g, n) with $g > 0$ except for those in the finite list (41), there exist nondegenerate ϕ_1 and ϕ_2 such that $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ are not isomorphic as Deligne–Mumford stacks over $\overline{\mathcal{M}}_{g,n}$. In fact, in Section 6.2 we show that the isomorphism classes of $\overline{\mathcal{J}}_{g,n}(\phi)$'s, considered as stacks over $\overline{\mathcal{M}}_{g,n}$, are in bijection with the quotient of the set $\mathcal{P}_{g,n}$ of stability polytopes by the action of the generalized dihedral group $\widetilde{\mathrm{PR}}_{g,n}$ of the relative Picard group of the universal curve. From this analysis we also deduce that, for fixed (g, n), there are finitely many non-isomorphic $\overline{\mathcal{J}}_{g,n}(\phi)$ for all $d \in \mathbb{Z}$ and all nondegenerate $\phi \in V^d_{g,n}$.

In Section 6.1 we give a second application of our description of $V^d_{g,n}$, namely a resolution of the indeterminacy of the Abel–Jacobi sections. Recall that, given a vector $\vec{d} = (d_1, \dots, d_n)$ of integers satisfying $d_1 + \dots + d_n = d$, the rule

$$(C, p_1, \dots, p_n) \mapsto \mathcal{O}_C(d_1 p_1 + \dots + d_n p_n) \tag{2}$$

defines a morphism $\sigma_{\vec{d}} \colon \mathcal{M}_{g,n} \to \mathcal{J}^d_{g,n}$ and hence a rational map from $\overline{\mathcal{M}}_{g,n}$ into any extension of $\mathcal{J}^d_{g,n}$. Grushevsky–Zakharov raised the problem of resolving the indeterminacy of this map in [GZ14, Remark 6.3]. In Proposition 6.4, we describe the locus of indeterminacy as

Theorem. *For ϕ nondegenerate, the locus of indeterminacy of $\sigma_{\vec{d}} \colon \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi)$ is the closure of the locus of pointed curves $(C, p_1, \dots, p_n)$ that consist of two smooth curves meeting in $k \geq 2$ nodes with the property that $\mathcal{O}_C(d_1 p_1 + \dots + d_n p_n)$ fails to be ϕ-stable.*

This result extends earlier work of Dudin. In [Dud17, Section 3], Dudin proved that, for certain ϕ, the locus of indeterminacy of $\sigma_{\vec{d}}$ is contained in the closure of the locus of pointed curves that consist of two smooth curves meeting in $k \geq 2$ nodes satisfying the above stability condition [Dud17, Proposition 3.3]. Thus the main new content of the above theorem is that the containment of the indeterminacy locus is in fact an equality. (For a detailed explanation of which $\overline{\mathcal{J}}_{g,n}(\phi)$ Dudin studies, see the discussion immediately after [KP16, Corollary 5.4] and Remark 4.6.)

The result can be described in terms of the *degenerate* vector $\phi_{\vec{d}} \in V^d_{g,n}$ that is the multidegree of $\mathcal{O}_C(d_1p_1 + \ldots + d_np_n)$. When ϕ is nondegenerate and sufficiently close to $\phi_{\vec{d}}$, our result states that the locus of indeterminacy is empty. For general ϕ, the rational map $\sigma_{\vec{d}}: \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi)$ has indeterminacy that we can resolve as follows. If ϕ_0 is nondegenerate and sufficiently close to $\phi_{\vec{d}}$, then $\overline{\mathcal{J}}_{g,n}(\phi)$ is related to $\overline{\mathcal{J}}_{g,n}(\phi_0)$ by a series of flips that correspond to the values of $t \in [0,1]$ such that $t\phi_0 + (1-t)\phi$ lies in the boundary of a stability polytope. Indeed, the moduli spaces $\overline{\mathcal{J}}_{g,n}(\phi)$ are locally constructed using GIT (through our use of [Sim94]), and the structure of these flips is described by Thaddeus in [Tha96]. The above theorem shows that the indeterminacy of $\sigma_{\vec{d}}$ is resolved by modifying $\overline{\mathcal{J}}_{g,n}(\phi)$ by these flips.

The relation of this result to the work of Grushevsky–Zakharov [GZ14] is complicated as they consider $\sigma_{\vec{d}}$ as a rational map into the extension of $\mathcal{J}^0_{g,n}$ given by Mumford's rank 1 degenerations, and this extension is different from, but related to, the $\overline{\mathcal{J}}_{g,n}(\phi)$'s. We discuss this relation in detail in Remark 6.8.

The reader is encouraged to also see David Holmes' preprint [Hol17] for another approach to analyzing the indeterminacy of the Abel–Jacobi section. Rather than modifying the target of the section (as we do in this paper), Holmes modifies $\overline{\mathcal{M}}_{g,n}$ to resolve the indeterminacy. He analyzes $\sigma_{\vec{d}}$ when $\sum d_i = 0$ and produces a morphism from an open substack of an explicit toric blowup into the separated space J parameterizing multidegree 0 line bundles on stable curves. Holmes uses this resolution to study the double ramification cycle, a topic we do not study here.

This paper is organized as follows. In Section 2 we recollect background material on the moduli spaces of curves. In Section 2.1 we fix the notation for stable graphs, in Section 2.2 we define a notion of contraction, in Section 2.3 we discuss the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type and in Section 2.4 we describe the relative Picard group of the universal curve $\pi: \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. In Section 3 we introduce the universal stability space $V^d_{g,n}$ and prove two results that describe it explicitly: Theorem 1 and Corollary 3.6. In Section 4 we define the stacks $\overline{\mathcal{J}}_{g,n}(\phi)$ and prove that they are k-smooth Deligne–Mumford stacks when ϕ is nondegenerate. In Section 5 we introduce the stability polytope decomposition $\mathcal{P}_{g,n}$ and prove Theorem 2, which gives an explicit description of the stability hyperplanes in the stability spaces $V^d_{g,n}$. In Section 6 we apply our results to resolve the indeterminacy of the Abel–Jacobi sections (Section 6.1) and to enumerate the different $\overline{\mathcal{J}}_{g,n}(\phi)$ (Section 6.2). Section 7 is the Appendix, where we recollect some algebra lemmas needed in Section 6.

1.1. **Conventions.** We denote by $[n]$ the set $\{1, \ldots, n\}$. If $S \subset [n]$, we write S^c for $[n] \smallsetminus S$. For a given subset $S \subset [n]$ and $f: S \to \mathbb{Z}$, we denote by f_S the sum $\sum_{j \in S} f(j)$.

By $\delta_{1,g}$ we denote the Kronecker delta:

$$\delta_{1,g} = \begin{cases} 1 & \text{when } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We work over a fixed algebraically closed field k of characteristic 0 throughout.

A *curve* over a field $\operatorname{Spec}(k)$ is a $\operatorname{Spec}(k)$-scheme $C/\operatorname{Spec}(k)$ that is proper over $\operatorname{Spec}(k)$, geometrically connected, and pure of dimension 1. A curve $C/\operatorname{Spec}(k)$ is a *nodal curve* if C is geometrically reduced and the completed local ring of $C \otimes \overline{k}$ at a non-regular point is isomorphic to $\overline{k}[[x,y]]/(xy)$. Here $\overline{k}$ is an algebraic closure of k.

A *family of curves* over a k-scheme T is a proper, flat morphism $C \to T$ whose fibers are curves. A family of curves $C \to T$ is a *family of nodal curves* if the fibers are nodal curves.

A *family of rank* 1 *torsion-free sheaves* over a family of curves $C \to T$ is a rank 1 sheaf F on C, flat over T, whose fibers over the geometric points are torsion-free.

If F is a rank 1 torsion-free sheaf on a nodal curve C with irreducible components C_i, we define the *multidegree* of F by $\deg(F) := (\deg(F_{C_i}))$. Here F_{C_i} is the maximal torsion-free quotient of $F \otimes \mathcal{O}_{C_i}$. We define the *(total) degree* of F to be $\deg_C(F) := \chi(F) - 1 + p_a(C)$ where $p_a(C) = h^1(C, \mathcal{O}_C)$ is the arithmetic genus of C. The total degree and the multidegree of F are related by the formula $\deg_C(F) = \sum \deg_{C_i} F - \delta_C(F)$, where $\delta_C(F)$ denotes the number of nodes of C where F fails to be locally free.

2. Background

2.1. **Graphs.** A graph Γ is a tuple $(\mathrm{Vert}, \mathrm{HalfEdge}, \mathrm{a}, \mathrm{i})$ consisting of a finite set of vertices Vert, a finite set of half-edges HalfEdge, an assignment function $\mathrm{a}\colon \mathrm{HalfEdge} \to \mathrm{Vert}$, and a fixed point free involution $\mathrm{i}\colon \mathrm{HalfEdge} \to \mathrm{HalfEdge}$. The edge set is defined as the quotient set $\mathrm{Edge} := \mathrm{HalfEdge}/\mathrm{i}$. The endpoint of a half-edge $h \in \mathrm{Edge}$ is defined to be $v = a(h)$. A loop based at v is an edge whose two endpoints coincide.

A n-marked graph is a graph Γ together with a (genus) map $g\colon \mathrm{Vert}(\Gamma) \to \mathbb{N}$ and a (markings) map $p\colon \{1, \dots, n\} \to \mathrm{Vert}(\Gamma)$. We call $g(v)$ the genus of $v \in \mathrm{Vert}(\Gamma)$. If $v = p(j)$, then we say that the j-th marking lies on the vertex v.

A subgraph Γ' of Γ is always assumed to be proper ($\mathrm{Vert}(\Gamma') \subsetneq \mathrm{Vert}(\Gamma)$) and complete (for all $v' \in \mathrm{Vert}(\Gamma')$, if $h \in \mathrm{HalfEdge}(\Gamma)$ and $a(h) = v'$, then $h \in \mathrm{HalfEdge}(\Gamma')$). A subgraph of a n-marked graph is tacitly assumed to be given the induced genus and marking maps.

We say that a n-marked graph Γ is *stable* if it is connected (in the obvious sense, a bit tedious to write down), and if for all v with $g(v) = 0$, the sum of the number of half-edges with v as an endpoint plus the number of markings lying on v is at least 3. The (arithmetic) *genus* of Γ is $g(\Gamma) := \sum_{v \in \mathrm{Vert}(\Gamma)} g(v) - \#\mathrm{Vert}(\Gamma) + \#\mathrm{Edge}(\Gamma) + 1$.

An *isomorphism* of $\Gamma = (\mathrm{Vert}, \mathrm{HalfEdge}, \mathrm{a}, \mathrm{i})$ to $\Gamma' = (\mathrm{Vert}', \mathrm{HalfEdge}', \mathrm{a}', \mathrm{i}')$ is a pair of bijections $\alpha_V\colon \mathrm{Vert} \to \mathrm{Vert}'$ and $\alpha_{\mathrm{HE}}\colon \mathrm{HalfEdge} \to \mathrm{HalfEdge}'$ that satisfy the compatibilities $\alpha_{\mathrm{HE}} \circ i = i'$ and $\alpha_V \circ a = a'$. If Γ and Γ' are endowed with structures of n-marked graphs by the maps (g,p) and by (g',p') respectively, $(\alpha_V, \alpha_{\mathrm{HE}})$ is an isomorphism of n-marked graphs if it also satisfies the compatibilities $\alpha_V \circ p = p'$ and $\alpha_V \circ g = g'$. An *automorphism* is an isomorphism of a graph to itself.

We fix once and for all a finite set $\mathcal{G}_{g,n}$ of stable n-marked dual graphs of genus g, one for each isomorphism class.

2.2. **Contractions.** We will need a notion for contractions of stable graphs. This notion is ubiquitous in the literature of moduli of curves (see, for example [GP03, Appendix], where contractions are key to giving an algorithmic description of the intersection product of tautological classes). Here we first introduce a strict contraction (of one edge), and then define a contraction to be a strict contraction followed by an isomorphism. Unlike in [GP03] and in other sources, with our terminology an isomorphism of graphs is *not* a particular case of a contraction.

If Γ is a n-marked graph and $e \in \operatorname{Edge}(\Gamma)$ is an edge, the *strict (elementary) contraction* of e in Γ is the graph Γ_e where the half-edges corresponding to e are removed, the two (possibly coinciding) endpoints v_1 and v_2 of e are replaced by a unique vertex v_e, and the genus and marking functions are extended to v_e by $p_e(j) := v_e$ whenever $p(j)$ equals v_1 or v_2, and

$$g_e(v_e) := \begin{cases} g(v_1) + g(v_2) & \text{when } e \text{ is not a loop,} \\ g(v_1) + 1 & \text{when } e \text{ is a loop.} \end{cases}$$

If Γ and Γ' are n-marked graphs, a *(elementary) contraction* $c: \Gamma \to \Gamma'$ is the choice of an edge e of Γ, and of an isomorphism of Γ_e (the strict contraction of e in Γ) to Γ'. The contraction c is completely determined by the two maps it induces $c_V: \operatorname{Vert}(\Gamma) \to \operatorname{Vert}(\Gamma')$ (on vertices) and $c_{\mathrm{HE}}: \operatorname{HalfEdge}(\Gamma) \to \operatorname{HalfEdge}(\Gamma')$ (on half-edges).

2.3. **Moduli of curves.** In this paper we always assume that g, n are natural numbers satisfying $2g - 2 + n > 0$. Under this assumption, the moduli stack $\overline{\mathcal{M}}_{g,n}$ parameterizing families of stable n-pointed curves of arithmetic genus g is a k-smooth and proper Deligne–Mumford stack. We will denote by $\pi: \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the universal curve, and by ω_π its relative dualizing sheaf.

If $(C, p_1, \dots, p_n)$ is a stable pointed curve, we define its *dual graph* Γ_C to be the n-marked graph whose vertices are the irreducible components of C, whose edges are the nodes of C, whose genus map is given by assigning the geometric genus to each vertex, and whose markings map is the assignment $p: \{1, \dots, n\} \to \operatorname{Vert}(\Gamma_C)$ such that $p(j)$ is the vertex containing p_j.

For each $\Gamma \in \mathcal{G}_{g,n}$, the locus $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{g,n}$ of stable curves whose dual graph is isomorphic to Γ is locally closed. We are now going to fix a notation for some special stable graphs Γ (and their corresponding loci $\mathcal{M}_\Gamma$), which will play an important role in this paper.

For all pairs (i, S) with $0 \le i \le g$ and $S \subset [n]$, such that if $i = 0$ then $|S| \ge 2$, and if $i = g$ then $|S| \le n - 2$, we define $\Gamma(i, S)$ to be the graph with two vertices of genera i and $g - i$ connected by one edge with markings S and S^c respectively. The closure of the locus $\mathcal{M}_{\Gamma(i,S)}$ in $\overline{\mathcal{M}}_{g,n}$ is a divisor that we we will denote by $\Delta(i, S)$. In this paper we will assume (in summation formulas etc.) that the set of indices $\{(i, S)\}$ for $0 \le i \le g$ and $S \subset [n]$ satisfies the additional requirement that

(1) if $n = 0$, then $i < g - i$,
(2) if $n \ge 1$, then $1 \in S$.

We adopt this convention so that there is a bijection between the set of indices (i,S) and the set of boundary divisors $\Delta(i,S)$ whose inverse image in the universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ consists of two components.

When $g \geq 1$ and $n \geq 1 + \delta_{1,g}$, for each $j = 1 + \delta_{1,g}, \dots, n$ we denote by Γ_j the graph with two vertices of genera 0 and $g-1$ respectively, connected by two edges, and with marking j on the first vertex and all other markings on the second vertex.

Another collection of curves that will play a crucial role in this paper, and that includes those discussed in the previous two paragraphs, consists of the so-called *generalized dollar sign curves*. These are by definition curves with two smooth irreducible components or, equivalently, curves whose dual graph has two vertices and no loops.

2.4. **The relative Picard group of the universal curve.** In this paper we will often need to work with the relative Picard group of the universal curve $\pi: \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and with its affine subspaces of elements of fixed fiberwise degree. For this reason, we introduce the following definition/notation.

Definition 2.1. We denote by

$$\mathrm{PicRel}_{g,n}(\mathbb{Z}) := \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})/\pi^*(\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}))$$

the relative Picard group of the universal curve π and by

$$\mathrm{PicRel}_{g,n}(\mathbb{R}) := \mathrm{PicRel}_{g,n}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

the relative Picard group of real line bundles.

For every $d \in \mathbb{Z}$ (resp. $\in \mathbb{R}$), we let $\mathrm{PicRel}^d_{g,n}(\mathbb{Z})$ (resp. $\mathrm{PicRel}^d_{g,n}(\mathbb{R})$) be the affine subspace of $\mathrm{PicRel}_{g,n}(\mathbb{Z})$ (resp. of $\mathrm{PicRel}_{g,n}(\mathbb{R})$) of elements of fiberwise degree d.

Let Σ_j be the j-th section of the universal curve $\pi: \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and ω_π be the relative dualizing sheaf.

We make the following (canonical) choice for a base point in $\mathrm{PicRel}^d_{g,n}(\mathbb{R})$:

$$\begin{cases} \frac{d}{2g-2} \cdot \omega_\pi & \text{when } g \geq 2, \\ d \cdot \Sigma_1 & \text{when } g \leq 1. \end{cases} \tag{3}$$

The choice of a base point makes $\mathrm{PicRel}^d_{g,n}(\mathbb{R})$ into a vector space isomorphic to $\mathrm{PicRel}^0_{g,n}(\mathbb{R})$.

We now recollect what will later be needed about the structure of the free abelian group $\mathrm{PicRel}_{g,n}(\mathbb{Z})$ and the striucture of its subgroup $\mathrm{PicRel}^0_{g,n}(\mathbb{Z})$. The results we state follow from the description of the Picard group of $\overline{\mathcal{M}}_{g,n}$ by Arbarello–Cornalba [AC87].

Definition 2.2. For each pair (i,S) satisfying the assumptions of Section 2.3, we define $C^+_{i,S}$ and $C^-_{i,S}$ to be the two components of the universal curve $\pi: \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ over the boundary divisor $\Delta(i,S)$. The component $C^+_{i,S}$ is the one that contains the first marked point, and, when $n=0$, it is the component of lowest genus. We define $W_{g,n}$ to be the subgroup of $\mathrm{PicRel}^0_{g,n}(\mathbb{Z})$ generated by the line bundles $\mathcal{O}(C^+_{i,S})$.

For $j = 1 + \delta_{1,g}, \dots, n$, define the *twisted sections*

$$T_j := \begin{cases} \mathcal{O}(\Sigma_j - \Sigma_1) & \text{if } g = 1, \\ \mathcal{O}((2g-2)\Sigma_j) \otimes \omega_\pi^{-1} & \text{if } g \geq 2. \end{cases}$$

(When $g = 0$ we have intentionally defined no twisted sections).

Fact 1. *The group* $\mathrm{PicRel}_{g,n}(\mathbb{Z})$ *is* freely *generated*

1. *by the components* $\mathcal{O}(C^+_{i,S})$ *over the boundary divisors when* $g = 0$,
2. *by the components over the boundary divisors and by the first section* Σ_1 *(or by any other section) when* $g = 1$,
3. *by the components over the boundary divisors, by the relative dualizing sheaf* ω_π *and by all sections* $\Sigma_1, \dots, \Sigma_n$ *when* $g \geq 2$.

Proof. As observed in [AC87], rational and homological equivalence coincide in the Picard group of $\overline{\mathcal{M}}_{g,n}$, so $H^2(\overline{\mathcal{M}}_{g,n})$ coincides with $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$. Identify the universal curve $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ with the map forgetting the last point and stabilizing $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. On $\overline{\mathcal{M}}_{g,n+1}$ choose the generators indicated in [AC98, Theorem 2.2]. Under this identification, the relative dualizing sheaf ω_π becomes $\kappa_1 + \psi_1 + \ldots + \psi_n$ and each section Σ_j becomes the boundary divisor $\Delta(0, \{j, n+1\})$. Then use the right hand side of the equalities in [AC98, Lemma 1.2] to eliminate redundant generators. When $g \leq 2$, use the relations indicated in (c) and (d) of [AC98, Theorem 2.2] to get rid of the relative dualizing sheaf, and of the sections. □

By singling out the degree zero elements, we deduce the following corollary.

Corollary 2.3. *The group* $\mathrm{PicRel}^0_{g,n}(\mathbb{Z})$ *is* freely *generated by the components* $\mathcal{O}(C^+_{i,S})$, *and by the twisted sections* T_j.

In particular, when either $g = 0$ or $n = 0$, the group $\mathrm{PicRel}^0_{g,n}(\mathbb{Z})$ coincides with $W_{g,n}$.

3. The universal stability space

In this section we construct and study the stability $\mathbb{R}$-vector space $V_{g,n}$, whose affine subspaces $V^d_{g,n}$ of elements of total degree $d \in \mathbb{Z}$ are the stability spaces of ϕ-compactified universal Jacobians over $\overline{\mathcal{M}}_{g,n}$, which we will construct in Section 4.

In [KP16, Section 3] we introduced a similar stability space, which we called $V^{\mathrm{TL}}_{g,n}$: the stability space of degree ϕ-compactified universal Jacobians of degree $g-1$ over moduli of treelike curves. For more details on $V^{\mathrm{TL}}_{g,n}$ and its relation to the space $V^{g-1}_{g,n}$ we introduce here, we direct the reader to Remarks 3.11 and 4.5.

Our main result is Theorem 1, which describes the stability space $V_{g,n}$ as the real relative Picard group of the universal curve $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. An important by-product is Corollary 3.6, where we show that, for fixed d, every degree d stability parameter is uniquely determined by its restriction to all stable curves with two smooth irreducible components that (1) have precisely one separating node, and (2) have two nodes and one component of genus 0 that carries a unique marked point (the curves whose dual graph is $\Gamma(i, S)$ or Γ_j respectively, see Section 2.3). In order to make the statement of our main result more transparent, in this section we allow d to be a real number.

Definition 3.1. Given $\Gamma \in \mathcal{G}_{g,n}$ a stable n-marked graph of genus g, we denote by $V(\Gamma) := \mathbb{R}^{\mathrm{Vert}(\Gamma)}$ the free real vector space generated by the vertices of Γ. For $d \in \mathbb{R}$, the affine subspace $V^d(\Gamma)$ is the set of $\phi \in V(\Gamma)$ such that $\sum_{v \in \mathrm{Vert}(\Gamma)} \phi(v) = d$.

Every automorphism α of Γ induces an automorphism of $V(\Gamma)$ defined by $\alpha(\phi)(v) = \phi(\alpha(v))$. An element $\phi \in V(\Gamma)$ is automorphism invariant if $\phi(v) = \phi(\alpha(v))$ for all

$v \in \mathrm{Vert}(\Gamma)$. A vector $\phi \in \Pi_{\Gamma \in \mathcal{G}_{g,n}} V(\Gamma)$ is *automorphism invariant* if for every $\Gamma \in \mathcal{G}_{g,n}$, the component $\phi(\Gamma)$ of ϕ along Γ is automorphism invariant in the sense just defined.

Suppose that $c\colon \Gamma_1 \to \Gamma_2$ is a contraction of stable marked graphs as defined in Section 2.2. We say that $\phi(\Gamma_1) \in V(\Gamma_1)$ is c-compatible with $\phi(\Gamma_2) \in V(\Gamma_2)$ if

$$\phi(\Gamma_2)(v_2) = \sum_{c(v_1)=v_2} \phi(\Gamma_1)(v_1) \tag{4}$$

for all vertices $v_2 \in \mathrm{Vert}(\Gamma_2)$. An element $\phi \in \Pi_{\Gamma \in \mathcal{G}_{g,n}} V(\Gamma)$ is *compatible with contractions* if its components are c-compatible for every contraction $c\colon \Gamma_1 \to \Gamma_2$.

Definition 3.2. We define $V_{g,n}$ to be the subspace of $\prod_{\Gamma \in \mathcal{G}_{g,n}} V(\Gamma)$ of vectors that are automorphism invariant and compatible with contractions. For $d \in \mathbb{R}$, we define $V^d_{g,n}$ to be the affine subspace of vectors $\phi \in V_{g,n}$ that satisfy $\sum_{v \in \mathrm{Vert}(\Gamma)} \phi(\Gamma)(v) = d$ for all $\Gamma \in \mathcal{G}_{g,n}$.

Remark 3.3. We could have equivalently defined $V^d_{g,n}$ as the subspace of vectors of $\prod_{\Gamma \in \mathcal{G}_{g,n}} V^d(\Gamma)$ that are automorphism invariant and compatible with contractions.

If $\pi\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ denotes the universal curve, there is a natural multidegree homomorphism $\deg\colon \mathrm{Pic}(\overline{\mathcal{C}}_{g,n}) \to V_{g,n}$ defined by associating to L the vector $\phi = \deg(L)$ whose Γ-component $\phi(\Gamma)$ is the multidegree of L on any stable pointed curve whose dual graph is isomorphic to Γ.

There is a natural choice of a basepoint in $V^d_{g,n}$ that mirrors the basepoint we chose in Section 2.4 for the relative Picard group.

Definition 3.4. We define the *canonical parameter* as follows

$$\phi^d_{\mathrm{can}} := \begin{cases} \frac{d}{2g-2} \cdot \deg(\omega_\pi) & \text{when } g \geq 2, \\ d \cdot \deg(\Sigma_1) & \text{when } g \leq 1. \end{cases} \tag{5}$$

In order to state the main result of this section, we first observe that for every $L \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$, the stability parameter $\deg(\pi^*(L)) \in V_{g,n}$ is trivial, so the multidegree map descends to a well-defined map $\deg\colon \mathrm{PicRel}_{g,n}(\mathbb{Z}) \to V_{g,n}$.

Theorem 1. *The multidegree homomorphism* deg *induces an isomorphism*

$$\deg\colon \mathrm{PicRel}_{g,n}(\mathbb{R}) = \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})/\pi^*(\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})) \otimes_{\mathbb{Z}} \mathbb{R} \to V_{g,n}$$

from the relative Picard group of real line bundles to the stability space.

Before we prove Theorem 1, we now give another description of the stability spaces that will be the one we will mostly use in this paper. To this end, we make the following definition.

Definition 3.5. Let

$$T_{g,n} := \bigoplus_{\substack{\#\mathrm{Vert}(\Gamma)=2 \\ \Gamma \text{ has no loops}}} V^0(\Gamma).$$

Then define:

(1) The vector space $C_{g,n}$ is the quotient space of $T_{g,n}$ obtained as the direct sum of all $V^0(\Gamma(i,S))$ (see Section 2.3 for the definition of $\Gamma(i,S)$).

(2) The vector space $D_{g,n}$ is the quotient space of $T_{g,n}$ obtained as the direct sum of all $V^0(\Gamma_j)$ for $j = 1 + \delta_{1,g}, \dots, n$ (see Section 2.3 for the definition of Γ_j. In particular, $D_{0,n} = \{0\}$).

There are natural projections $p_C: T_{g,n} \to C_{g,n}$ and $p_D: T_{g,n} \to D_{g,n}$.

There is a natural restriction map $\rho: V^0_{g,n} \to T_{g,n}$. More generally, for every $d \in \mathbb{R}$, there is a natural map $\rho^d: V^d_{g,n} \to T_{g,n}$ obtained by composing ρ with the translation $\phi \mapsto \phi - \phi^d_{\text{can}}$. Choosing the canonical parameter ϕ^d_{can} for the origin in $V^d_{g,n}$ makes ρ^d into a homomorphism of vector spaces. We have then the following alternative description of each degree d stability space.

Corollary 3.6. *The composite homomorphism*

$$(p_C \oplus p_D) \circ \rho^d: V^d_{g,n} \to C_{g,n} \oplus D_{g,n}$$

is an isomorphism.

We now aim to prove Theorem 1 and Corollary 3.6. Here is the idea of our proof when $g \geq 2$. It is not hard to reduce both results to proving that, in degree zero, both maps

$$\text{PicRel}^0_{g,n}(\mathbb{R}) \to V^0_{g,n} \to C_{g,n} \oplus D_{g,n}$$

are isomorphisms. Injectivity of $\text{PicRel}^0_{g,n}(\mathbb{Z}) \to C_{g,n} \oplus D_{g,n}$ follows by computing the bidegree of the free generators of the Picard group on curves whose dual graph is $\Gamma(i,S)$ and Γ_j (Lemma 3.7), and observing that the resulting matrix is nonsingular. From this we immediately deduce that $\text{PicRel}^0_{g,n}(\mathbb{R}) \to C_{g,n} \oplus D_{g,n}$ is an isomorphism, because the source and the target have the same dimension. Our results follow if we can prove that $\text{PicRel}^0_{g,n}(\mathbb{R}) \to V^0_{g,n}$ is surjective, or equivalently that $V^0_{g,n} \to C_{g,n} \oplus D_{g,n}$ is injective. This is the content of Proposition 3.10, which we prove inductively in n. An important intermediate step is Lemma 3.8, where we show that $V^0_{g,n} \to T_{g,n}$ is also injective. The base case of the induction $n = 0$ is settled by combining Lemma 3.8 and Lemma 3.9

We now compute the bidegree of the free generators we chose in Corollary 2.3 for the relative Picard group $\text{PicRel}^0_{g,n}(\mathbb{Z})$ on the special dollar sign curves appearing in parts (1) and (2) of Definition 3.5. When ordering the two components of the curve, we follow the same convention that we chose in Definition 2.2 to order the two components $C^+_{i,S}$ and $C^-_{i,S}$ of the inverse image of $\Delta_{i,S}$ in the universal curve.

Lemma 3.7. *The bidegrees of the components $\mathcal{O}(C^+_{i',S'})$ and of the twisted sections T_k on curves whose dual graph is $\Gamma(i,S)$ and Γ_j is given by the following formulas:*

$$\deg\left(\mathcal{O}(C^+_{i,S})|\Gamma(i',S')\right) = \begin{cases} (-1,+1) & \textit{if } (i',S') = (i,S), \\ (0,0) & \textit{if } (i',S') \neq (i,S). \end{cases} \tag{6}$$

$$\deg\left(\mathcal{O}(C^+_{i,S})|\Gamma_j\right) = (0,0) \tag{7}$$

$$\deg\left(T_k|\Gamma(i,S)\right) = \begin{cases} (2g-2i-1, 2i+1-2g) & \textit{if } k \in S, \\ (1-2i, 2i-1) & \textit{if } k \notin S; \end{cases} \tag{8}$$

$$\deg(T_k|\Gamma_j) = \begin{cases} (2g-2-\delta_{1,g}, 2-2g+\delta_{1,g}) & \text{if } j = k, \\ (0,0) & \text{if } j \neq k. \end{cases} \tag{9}$$

Proof. Straightforward. □

Combining Lemma 3.7 with Corollary 2.3 we deduce that rational and numerical equivalence are equivalent in $\mathrm{PicRel}^0_{g,n}(\mathbb{Z})$, and from this we deduce that the composite map $(p_C \oplus p_D) \circ \rho \circ \deg\colon \mathrm{PicRel}^0_{g,n}(\mathbb{Z}) \to C_{g,n} \oplus D_{g,n}$ is injective.

An important fact that we will use throughout is that every element $\phi \in V^0_{g,n}$ is completely determined by its value over all curves with two smooth irreducible components (the generalized dollar sign curves, see Section 2.3 and Definition 3.5).

Lemma 3.8. *The restriction $\rho\colon V^0_{g,n} \to T_{g,n}$ is injective.*

Proof. Let $\Gamma' \in \mathcal{G}_{g,n}$ be a stable graph. By applying compatibility with contractions we can assume without loss of generality that Γ' has no loops. Then consider a spanning tree Γ of Γ', and run the injectivity part of the proof in [KP16, Lemma 3.9], with the only difference being that the right hand side of [KP16, Equation (15)] should equal zero. □

Lemma 3.9. *Let $\phi \in V^0_{g,0}$ and $\Gamma \in \mathcal{G}_g$ be a graph with two vertices connected by ≥ 2 edges. Then $\phi(\Gamma)$ is trivial.*

Proof. We begin by fixing the notation for loopless graphs with two vertices and at least two edges, when $n = 0$. Let $\alpha, i, j \in \mathbb{N}$ such that $\alpha + i + j - 1 = g$, $\alpha \geq 2$ and subject to the stability condition $\min(i,j) = 0 \implies \alpha \geq 3$. Define the stable graph $\Gamma(\alpha, i, j) \in \mathcal{G}_g$ to consist of two vertices v_1, v_2 of genera i and j respectively, with α edges connecting v_1 to v_2. We aim to prove that $\phi(\Gamma(\alpha,i,j)) = (0,0)$.

We first prove our claim in the special case when $j = 0$ (so $3 \leq \alpha = g+1-i$). Consider the trivalent graph GSym_g that has $2g-2$ vertices $v_1, \ldots, v_{2g-2}$ of genus 0, where each vertex v_i is connected to v_{i-1}, v_{i+1} and v_{i+g-1} (here indices should be considered modulo $2g-2$). The cyclic group of order $2g-2$ acts on GSym_g and its induced action on the set of vertices $\mathrm{Vert}(\mathrm{GSym}_g)$ is transitive. By automorphism invariance, the component $\phi(\mathrm{GSym}_g)$ of $\phi \in V^0_{g,0}$ along GSym_g is trivial.

Choose $2g-\alpha$ consecutive vertices of GSym_g and contract them to one vertex, possibly contracting all loops based at that vertex in the process. Then also contract the complement set of $\alpha-2$ vertices of GSym_g to a second vertex and contract all loops based at that vertex. The resulting graph is isomorphic to $\Gamma(\alpha, i, 0)$ and by compatibility with contractions we have that the $\Gamma(\alpha,i,0)$-component of $\phi \in V^0_{g,0}$ is trivial, thus proving the claim in the case $j = 0$.

We now deduce that the $\Gamma(\alpha,i,j)$-component of ϕ is zero for all α, i, j by using contractions to relate $\Gamma(\alpha,i,j)$ to a graph of the form $\Gamma(\beta,k,0)$, specifically the graph $\Gamma(i-j+2, \alpha+2j-2, 0)$.

Assume without loss of generality that $i > j$. Consider the graph with 4 vertices w_1, w_2, w_3, w_4 where w_1 and w_2 have genus j, and w_3 and w_4 have genus 0. The vertices w_1 and w_2 are connected by $\alpha - 1$ edges; w_1 is connected to w_3 by one edge, and so is w_2 to w_4. Finally, w_3 and w_4 are connected by $i - j + 1$ edges. By invariance under automorphism, the component of $\phi \in V^0_{g,0}$ along this graph equals $(a, a, -a, -a)$ for some $a \in \mathbb{R}$.

Contracting the vertices w_1, w_2 and w_4 to a single vertex and then contracting all remaining loops on it, produces a graph isomorphic to $\Gamma(i - j + 2, \alpha + 2j - 2, 0)$. Because we have already computed that the component of ϕ along this graph is trivial, we deduce that $a = 0$.

Contracting the vertices w_1, w_3 and w_4 to a single vertex and then contracting all remaining loops on it, produces a graph isomorphic to $\Gamma(\alpha, i, j)$, so by compatibility with contractions the component $\phi(\Gamma(\alpha, i, j))$ is also trivial, and the statement is proven. □

The following is the key part of the proof of the main result of this section.

Proposition 3.10. *Both the multidegree homomorphism*

$$\deg\colon \mathrm{PicRel}^0_{g,n}(\mathbb{R}) \to V^0_{g,n} \tag{10}$$

and the composition

$$(p_C \oplus p_D) \circ \rho\colon V^0_{g,n} \to C_{g,n} \oplus D_{g,n} \tag{11}$$

are isomorphisms.

Proof. In Lemma 3.7 we computed the bidegrees of all generators of $\mathrm{PicRel}^0_{g,n}(\mathbb{Z})$ given in Corollary 2.3 against curves whose dual graph is $\Gamma(i, S)$ and Γ_j. The corresponding square matrix is nonsingular: it consists of four blocks

$$\begin{pmatrix} (6) & (7) \\ (8) & (9) \end{pmatrix},$$

where (6) is the identity, (7) is the zero matrix and (9) is $(2g-2+\delta_{1,g})$ times the identity. Combining this with Corollary 2.3, we deduce that the composite map

$$(p_C \oplus p_D) \circ \rho \circ \deg\colon \mathrm{PicRel}^0_{g,n}(\mathbb{R}) \to C_{g,n} \oplus D_{g,n} \text{ is an isomorphism.} \tag{12}$$

Because of this, both claims of this proposition follow by proving that

$$(p_C \oplus p_D) \circ \rho\colon V^0_{g,n} \to C_{g,n} \oplus D_{g,n} \text{ is injective.} \tag{13}$$

The $g = 0$ case is easily settled. We have that $T_{g,n} = C_{g,n}$, so (13) follows immediately from Lemma 3.8.

From now on we assume $g \geq 1$ and we aim for proving (13). We simplify the problem by quotienting out the images of the space $W_{g,n} \otimes \mathbb{R}$ via deg and via $(p_C \oplus p_D) \circ \rho \circ \deg$. Applying parts (6) and (7) of Lemma 3.7, we deduce that the image via $(p_C \oplus p_D) \circ \rho \circ \deg$ of $W_{g,n} \otimes \mathbb{R}$ equals $C_{g,n} \oplus \{0\}$. Call $\mathrm{Ker}_{g,n}$ the kernel of $V^0_{g,n} \to C_{g,n}$. By (12) we know that $p_D \circ \rho\colon \mathrm{Ker}_{g,n} \to D_{g,n}$ is surjective. For these reasons, to prove (13) it is enough to prove the inequality

$$\dim(\mathrm{Ker}_{g,n}) \leq n - \delta_{1,g}. \tag{14}$$

(A posteriori, (14) will be an equality.) We will prove Inequality (14) inductively in n.

When $g = n = 1$ it is straightforward to check that $V^0_{1,1} = \{0\}$. When $g \geq 2$, the $n = 0$ case of (14) follows from Lemma 3.9, which implies that $(p_C \oplus p_D) \circ \rho = p_C \circ \rho$ is injective.

From now on, we apply the induction hypothesis. Assuming (14) holds, we aim to prove it holds for $n+1$, i.e.

$$\dim(\mathrm{Ker}_{g,n+1}) \leq n+1-\delta_{1,g}. \tag{15}$$

Define the subspace $K_{g,n+1} \subseteq \mathrm{Ker}_{g,n+1}$ as the subspace of vectors $\phi \in \mathrm{Ker}_{g,n+1}$ such that $\phi(\Gamma)(v)$ equals zero for all $\Gamma \in \mathcal{G}_{g,n}$ and $v \in \mathrm{Vert}(\Gamma)$ such that v becomes unstable after forgetting the last marking $n+1$. To prove (15) it is enough to prove the two inequalities

$$\dim(\mathrm{Ker}_{g,n+1}) \leq \dim(K_{g,n+1}) + 1, \text{ and } \dim(K_{g,n+1}) \leq \dim(\mathrm{Ker}_{g,n}). \tag{16}$$

We first prove the inequality $\dim(\mathrm{Ker}_{g,n+1}) \leq \dim(K_{g,n+1})+1$. By applying Lemma 3.8 again, we identify $\mathrm{Ker}_{g,n+1}$ with a subspace of $T_{g,n+1}$, and observe that $K_{g,n+1}$ contains (a posteriori, it will coincide with) the codimension-1 subspace of $\mathrm{Ker}_{g,n+1}$ of vectors ϕ whose component $\phi(\Gamma_{n+1})$ is trivial (for Γ_{n+1} defined in Section 2.3).

We prove the inequality $\dim(K_{g,n+1}) \leq \dim(\mathrm{Ker}_{g,n})$ by showing the existence of a surjective linear map $\lambda\colon \mathrm{Ker}_{g,n} \to K_{g,n+1}$, which we define as follows. Let $\Gamma \in \mathcal{G}_{g,n+1}$ and $\Gamma' \in \mathcal{G}_{g,n}$ be obtained from Γ by forgetting the last marking and possibly by stabilizing. If $\phi \in \mathrm{Ker}_{g,n}$, then $\lambda(\phi)$ is defined to equal ϕ on all vertices of Γ that correspond bijectively to vertices of Γ', and 0 on the extra vertex (if any). Because ϕ is automorphism-invariant and compatible with contractions, so is $\lambda(\phi)$. Because $\phi \in \mathrm{Ker}_{g,n}$, and because of the very definition of λ, we have that $\lambda(\phi) \in K_{g,n+1}$. Again by its very definition, $\lambda\colon \mathrm{Ker}_{g,n} \to K_{g,n+1}$ is surjective (a posteriori, it will be an isomorphism). This concludes our proof. □

From Proposition 3.10 we easily deduce Theorem 1 and Corollary 3.6.

Proof. (Of Theorem 1) By Proposition 3.10 we have that the multidegree map

$$\deg\colon \mathrm{PicRel}^0_{g,n}(\mathbb{R}) \to V^0_{g,n}$$

is an isomorphism. Moreover, both $\mathrm{PicRel}^0_{g,n}(\mathbb{R}) \subset \mathrm{PicRel}_{g,n}(\mathbb{R})$ and $V^0_{g,n} \subset V_{g,n}$ are inclusions of codimension-1 subspaces. By definition, the multidegree map deg maps the base point of $\mathrm{PicRel}^d_{g,n}(\mathbb{R})$ we defined in Equation (3) to the base point of $V^d_{g,n}$ we defined in Equation (5). This concludes our proof. □

Proof. (Of Corollary 3.6) By Proposition 3.10 we have that the projection

$$(p_C \oplus p_D) \circ \rho\colon V^0_{g,n} \to C_{g,n} \oplus D_{g,n}$$

is an isomorphism. Translation by ϕ^d_{can} is also an isomorphism of vector spaces $V^d_{g,n} \to V^0_{g,n}$. This concludes our proof. □

We conclude the section with some remarks on our Theorem 1.

Remark 3.11. In [KP16, Definition 3.7] we introduced a vector space $V^{\mathrm{TL}}_{g,n}$ governing universal stability over moduli of treelike curves, and when $d = g-1$. (Treelike curves are stable pointed curves whose nodes are either separating or belong to a unique irreducible component). By definition, there is a quotient map $q\colon V^{g-1}_{g,n} \to V^{\mathrm{TL}}_{g,n}$, and by Corollary 3.6 the treelike stability space $V^{\mathrm{TL}}_{g,n}$ is identified with $C_{g,n}$ (after subtracting the degree $g-1$ canonical parameter $\phi^{g-1}_{\mathrm{can}}$).

By Corollary 3.6 we also have that if $\psi \in V^{\mathrm{TL}}_{g,n}$, the preimage $q^{-1}(\psi)$ can be identified with $D_{g,n}$.

Remark 3.12. By Lemma 3.8 the restriction map $V^0_{g,n} \to T_{g,n}$ is injective, and in Corollary 3.6 we have made a choice of a subset Q of the set of 2-vertices loopless graphs such that the restriction map $V^0_{g,n} \to \prod_{\Gamma \in Q} V^0(\Gamma)$ is a surjection.

We claim that if Q is any such subset of the set of 2-vertices loopless graphs, then Q must contain all $\Gamma(i,S)$. Indeed as a consequence of Equation (6) we have that $W_{g,n} \to V^0_{g,n}$ is injective. Moreover, similar to what was seen in Equation (7), the bidegree of all elements of $W_{g,n}$ on curves with two smooth components and at least two nodes is trivial. It follows that the restriction map $V^0_{g,n} \to \prod_{\Gamma \in Q} V^0(\Gamma)$ would not be surjective, were Q not to contain some of the $\Gamma(i,S)$.

In this sense, choosing all graphs $\Gamma(i,S)$ in Definition 3.5 is natural. On the other hand, choosing all graphs Γ_j in loc. cit. is arbitrary — one could have opted for another choice of $n - \delta_{1,g}$ loopless graphs with two vertices and at least two edges.

4. Compactified universal Jacobians

For all nondegenerate $\phi \in V^d_{g,n}$ we construct ϕ-compactified universal Jacobians $\overline{\mathcal{J}}_{g,n}(\phi)$ as k-smooth, proper Deligne–Mumford stacks that are flat over $\overline{\mathcal{M}}_{g,n}$. Theorem 1 implies that all such universal Jacobians can be constructed from Simpson's result [Sim94, Theorem 1.21]. We start by reviewing the notion of (Oda–Seshadri) ϕ-stability on a single curve.

Let $(C, p_1, \ldots, p_n)$ be a stable pointed curve with dual graph Γ and $C_0 \subset C$ be a subcurve (i.e. the union of some of the irreducible components of C) with dual graph $\Gamma_0 \subset \Gamma$. We write $\deg_{\Gamma_0}(F)$ for the total degree $\deg_{C_0}(F)$ of the maximal torsion-free quotient of $F \otimes \mathcal{O}_{C_0}$ and $C_0 \cap C_0^c$ or $\Gamma_0 \cap \Gamma_0^c$ for the set of edges $e \in \mathrm{Edge}(\Gamma)$ that join a vertex of Γ_0 to a vertex of its complement Γ_0^c. Given a rank 1 torsion-free sheaf F of degree d, we have $\deg_{C_0}(F) + \deg_{C_0^c}(F) = d - \delta_{\Gamma_0}(F)$ for $\delta_{\Gamma_0}(F)$ the number of nodes $p \in \Gamma_0 \cap \Gamma_0^c$ such that the stalk of F at p fails to be locally free.

Definition 4.1. Given $\phi \in V^d(\Gamma)$, we define a rank 1 torsion-free sheaf F of degree d on a nodal curve C/k over an algebraically closed field to be *ϕ-semistable* (resp. *ϕ-stable*) if

$$\left| \deg_{\Gamma_0}(F) - \sum_{v \in \mathrm{Vert}(\Gamma_0)} \phi(v) + \frac{\delta_{\Gamma_0}(F)}{2} \right| \le \frac{\#(\Gamma_0 \cap \Gamma_0^c) - \delta_{\Gamma_0}(F)}{2} \quad \text{(resp. } <\text{).} \tag{17}$$

for all proper subgraphs $\Gamma_0 \subset \Gamma$.

We define $\phi \in V^d(\Gamma)$ to be *nondegenerate* if every ϕ-semistable sheaf is ϕ-stable. We say that $\phi \in V^d_{g,n}$ is *nondegenerate* if for all $\Gamma \in \mathcal{G}_{g,n}$, the Γ-component $\phi(\Gamma)$ is nondegenerate in $V^d(\Gamma)$.

Definition 4.2. Given $\phi \in V^d_{g,n}$ we say that a family of rank 1 torsion-free sheaves of degree d on a family of nodal curves is *ϕ-(semi)stable* if Equation (17) holds on all geometric fibers. We define $\overline{\mathcal{J}}^{\mathrm{pre}}_{g,n}(\phi)$ to be the category fibered in groupoids whose objects are tuples $(C, p_1, \ldots, p_n; F)$ consisting of a family of stable n-pointed curves $(C/T, p_1, \ldots, p_n)$ of genus g, and a family of ϕ-(semi)stable rank 1 torsion-free sheaves

F of degree d on C/T. The morphisms of $\overline{\mathcal{J}}^{\mathrm{pre}}_{g,n}(\phi)$ over a k-morphism $t\colon T \to T'$ are pairs consisting of an isomorphism of pointed curves $\widetilde{t}\colon (C, p_1, \ldots, p_n) \cong (C'_T, (p'_1)_T, \ldots, (p'_n)_T)$, and an isomorphism of $\mathcal{O}_C$-modules $F \cong \widetilde{t}^*(F'_T)$.

For every object $(C, p_1, \ldots, p_n; F)$ of $\overline{\mathcal{J}}^{\mathrm{pre}}_{g,n}(\phi)(T)$ the rule that sends $g \in \mathbb{G}_m(T)$ to the automorphism of F defined by multiplication by g defines an embedding $\mathbb{G}_m(T) \to \mathrm{Aut}(C, p_1, \ldots, p_n; F)$ that is compatible with pullbacks. The image of this embedding is contained in the center of the automorphism group, so the rigidification stack is defined, and we call this stack the *ϕ-compactified universal Jacobian* $\overline{\mathcal{J}}_{g,n}(\phi)$.

Theorem 1 implies that, for all nondegenerate ϕ's, the ϕ-compactified universal Jacobian can be given the structure of a proper Deligne-Mumford stack using Simpson's formalism, as we are now going to show.

Let $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ and $A, M \in \mathrm{Pic}(C)$ with A ample, and let a (respectively m) be the total degree of A (respectively of M). In [KP16, Formula 10] we observed that if $\phi(A, M)$ is defined by the formula

$$\phi(A, M) := \frac{(d + 1 - g + m)}{a} \cdot \deg(A) + \frac{1}{2} \cdot \deg(\omega_C) - \deg(M), \tag{18}$$

and F is any rank 1 torsion-free sheaf of degree d on C, then F is ϕ-(semi)stable if and only if $F \otimes M$ is slope (semi)stable (in the sense of slope/Gieseker-stability) with respect to A.

In [CMKV15, p.10], the authors proved that for every $\phi \in V^d(\Gamma_C)$ there exist A, M as above such that $\phi = \phi(A, M)$. Reasoning in the same way, and employing Theorem 1, we can prove that the same holds over $\overline{\mathcal{M}}_{g,n}$.

Corollary 4.3. *Let $\phi \in V^d_{g,n}$. Then there exist line bundles A, M on the universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ with A ample relative to $\overline{\mathcal{M}}_{g,n}$ such that, for every stable curve $(C, p_1, \ldots, p_n)$, a rank 1 torsion-free sheaf F of degree d on C is ϕ-semistable if and only if $F \otimes M$ is A-(semi)stable.*

Proof. We first observe that we can reduce to the case when ϕ has rational coefficients. In order to do that, we claim that for all $\phi \in V^d_{g,n}$ there exists $\phi_\epsilon \in V^0_{g,n}$ such that $(\phi + \phi_\epsilon)$ has rational coefficients, and ϕ-(semi)stability is equivalent to $(\phi + \phi_\epsilon)$-(semi)stability. This follows immediately from Theorem 2, a result we prove later in Section 5, which in particular asserts that the locus of degenerate ϕ's in $V^d_{g,n}$ consists of a locally finite union of hyperplanes.

Define $M := \omega_\pi(p_1 + \ldots + p_n)^{-k}$ for $k >> 0$ a sufficiently large integer such that the inequality

$$\phi(\Gamma_C)(v) + \deg_{C_v}(M) - \frac{\deg_{C_v}(\omega_C(p_1 + \ldots + p_n))}{2} > 0 \tag{19}$$

holds for all $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ and for all vertices v of Γ_C. That such k exists follows from the fact that $\omega_\pi(p_1 + \ldots + p_n)$ is ample relative to $\overline{\mathcal{M}}_{g,n}$, from the fact that the multidegree of a line bundle on $\overline{\mathcal{C}}_{g,n}$ is the same for curves with the same dual graph, and from the fact that the set of dual graphs $\mathcal{G}_{g,n}$ is finite.

For simplicity, denote by m the total degree of M. For any integer e, the parameter $\psi \in V^e_{g,n}$ defined by

$$\psi(\Gamma_C) := \frac{e}{2d + 2m - (2g - 2 + n)} \cdot (2\phi(\Gamma_C) + 2\deg_C(M) - \deg_C(\omega_\pi(p_1 + \ldots + p_n))) \tag{20}$$

has rational coefficients (because ϕ does), and so by Theorem 1 it is equal to $\deg(A)$ for some rational line bundle $A \in \mathrm{PicRel}^e_{g,n}(\mathbb{R})$. By taking e to be sufficiently divisible, we can clear denominators so that A is an *integral* line bundle. Moreover, by possibly replacing e with $-e$, we can assume that

$$\frac{e}{2d + 2m - (2g - 2 + n)} > 0$$

holds. Because Inequality (19) holds for all geometric points of $\overline{\mathcal{M}}_{g,n}$, we deduce that A is ample relative to $\overline{\mathcal{M}}_{g,n}$.

For all $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ we have then $\phi(\Gamma_C) = \phi(A, M)(\Gamma_C)$, where the latter is defined by Formula (18), and this concludes our proof. □

By combining Corollary 4.3, [KP16, Proposition 3.30] (Simpson's representability result [Sim94, Theorem 1.21] rewritten in our language) and [KP16, Lemma 3.33], we deduce the following.

Corollary 4.4. *Let $\phi \in V^d_{g,n}$ be nondegenerate. Then $\overline{\mathcal{J}}_{g,n}(\phi)$ is a k-smooth Deligne–Mumford stack, and the forgetful morphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{M}}_{g,n}$ is representable, proper and flat.*

Following existing literature we will refer to $\overline{\mathcal{J}}_{g,n}(\phi)$ as a *fine* ϕ-compactified universal Jacobian. The authors expect that when ϕ is degenerate $\overline{\mathcal{J}}_{g,n}(\phi)$ can naturally be given the structure of an Artin stack.

Remark 4.5. Corollary 4.4 generalizes our previous [KP16, Corollary 3.35] by removing the hypothesis $n \geq 1$ and allowing for any degree d (not necessarily $d = g - 1$).

Moreover, Corollary 4.4 describes all proper extensions of ϕ-compactified universal Jacobians from moduli of treelike curves (see Remark 3.11) to moduli of stable curves. In [KP16] we have already observed ([KP16, Lemma 3.26, 3.32]) that when $n \geq 1$, for every nondegenerate $\psi \in V^{\mathrm{TL}}_{g,n}$ there exists a nondegenerate $\phi \in V^{g-1}_{g,n}$ such that $q(\phi) = \psi$. For each nondegenerate $\phi \in q^{-1}(\psi)$ we have now constructed an extension of $\overline{\mathcal{J}}_{g,n}(\psi) \to \mathcal{M}^{\mathrm{TL}}_{g,n}$ to $\overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{M}}_{g,n}$.

We observed in Remark 3.11 that the preimage $q^{-1}(\psi)$ can be identified with $D_{g,n}$. In (26) we will give explicit hyperplanes describing when a given element of $D_{g,n}$ is degenerate.

Remark 4.6. Esteves constructed in [Est01] the compactified Jacobian of a family of reduced curves over a scheme. Building on his work, Melo constructed the corresponding compactified universal Jacobians over $\overline{\mathcal{M}}_{g,n}$ in [Mel16].

In their formalism, a compactified universal Jacobian in degree d is is defined in terms of a (universal) d-polarization, which is defined to be a vector bundle $\mathcal{E}$ of rank r and degree $r(d + 1 - g)$ on the universal curve $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. We claim that, for rank 1 torsion-free sheaves of degree d, the notion of $\mathcal{E}$-(semi)stability of Esteves–Melo

[Mel16, Definition 2.9] coincides with the notion of $\phi(\mathcal{E})$-(semi)stability that we gave in Definition 4.1 after posing

$$\phi(\mathcal{E}) := \frac{\deg(\mathcal{E})}{r} + \frac{\deg(\omega_\pi)}{2} \quad \in V^d_{g,n}. \tag{21}$$

The claim follows immediately by observing that, because the total degree is fixed, the lower bound for $\deg_{C_0}(F)$ of [Mel16, Inequality 2.1] on all subcurves C_0 of C is equivalent to Inequality (17) involving $\deg_{\Gamma_{C_0}}(F)$ on all proper subgraphs $\Gamma_{C_0} \subset \Gamma_C$. As a consequence of our claim, when $\phi(\mathcal{E})$ is nondegenerate, $\overline{\mathcal{J}}_{g,n}(\phi(\mathcal{E}))$ and Melo's moduli stack $\overline{\mathcal{J}}^{\mathcal{E},\mathrm{ss}}_{g,n}$ are isomorphic as Deligne–Mumford stacks over $\overline{\mathcal{M}}_{g,n}$.

Formula (21) shows that every d-polarization $\mathcal{E}$ can be translated into a ϕ-stability condition. We deduce the converse as an easy consequence of Theorem 1. We only discuss the case when $g \geq 2$ (the remaining cases are similar and easier).

For a given $\phi' \in V^d_{g,n}$, Theorem 1 implies that there exist $L \in \operatorname{Pic}^0(\overline{\mathcal{C}}_{g,n})$ and $\mathbb{N} \ni e >> 0$ such that

$$\phi' - \phi^d_{\mathrm{can}} = \frac{\deg(L)}{e}.$$

Defining the d-polarization by

$$\mathcal{E} := \omega_\pi^{\otimes e(d+1-g)} \otimes L^{\otimes(2g-2)} \oplus \mathcal{O}^{\oplus((2g-2)e-1)}$$

we have that $\phi' = \phi(\mathcal{E})$, and this completes the proof of our claim.

5. The stability hyperplanes and polytopes

In this section we describe how ϕ-stability depends on ϕ. The stability space $V^d_{g,n}$ naturally decomposes into *stability polytopes* defined by the property that two stability parameters define the same set of stable sheaves if and only if they lie in a common polytope. The main result is Theorem 2, which explicitly describes the hyperplanes that define the stability polytopes in terms of the description of the stability space given in Corollary 3.6. One consequence of this description is that every stability polytope in $V^d_{g,n}$ is a product of a polytope in $C_{g,n}$ and of a polytope in $D_{g,n}$. To begin, we recall some notation from [KP16, Section 3.2].

We say that a subgraph $\Gamma_0 \subset \Gamma$ is *elementary* if both Γ_0 and its complement Γ_0^c are connected. (The vertex set of an elementary subgraph is an elementary cut in the sense of [OS79, page 31].) We now define the combinatorial objects that control stability of rank 1 torsion-free sheaves on a stable pointed curve whose dual graph is Γ.

Definition 5.1. Let Γ be a stable marked graph. To a subgraph $\Gamma_0 \subset \Gamma$ and an integer $k \in \mathbb{Z}$ we associate the affine linear function $\ell(\Gamma_0, k) \colon V^d(\Gamma) \to \mathbb{R}$ defined by

$$\ell(\Gamma_0, k)(\phi) := k - \sum_{v \in \mathrm{Vert}(\Gamma_0)} \phi(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2}. \tag{22}$$

When $\Gamma_0 \subset \Gamma$ is an elementary subgraph we call the hyperplane

$$H(\Gamma_0, k) := \{\phi \in V^d(\Gamma); \ell(\Gamma_0, k)(\phi) = 0\} \subset V^d(\Gamma) \tag{23}$$

a *stability hyperplane*. (An element $\phi \in V^d(\Gamma)$ is nondegenerate according to Definition 4.1 if and only if ϕ does not belong to any such hyperplane.)

A *stability polytope* in $V^d(\Gamma)$ is defined to be a connected component of the complement of all stability hyperplanes in $V^d(\Gamma)$:

$$V^d(\Gamma) - \bigcup_{\substack{\Gamma_0 \subset \Gamma \text{ elementary} \\ k \in \mathbb{Z}}} \{\phi \in V^d(\Gamma) : \ell(\Gamma_0, k)(\phi) = 0.\}$$

If $\phi_0 \in V^d(\Gamma)$ is nondegenerate, we write $\mathcal{P}(\phi_0)$ for the unique stability polytope in $V^d(\Gamma)$ that contains ϕ_0. By definition we have

$$\mathcal{P}(\phi_0) = \{\phi \in V^d(\Gamma) : \ell(\Gamma_0, k)(\phi) > 0 \text{ for all } \ell(\Gamma_0, k) \text{ s.t. } \ell(\Gamma_0, k)(\phi_0) > 0\}. \tag{24}$$

The stability polytope $\mathcal{P}(\phi_0)$ is a rational bounded convex polytope because in Equation (24) only finitely many $\ell(\Gamma_0, k)$'s are needed to define $\mathcal{P}(\phi_0)$.

Assume $\phi_1, \phi_2 \in V(\Gamma)$ are nondegenerate. As a consequence of [KP16, Lemma 3.20] we have that ϕ_1-(semi)stability coincides with ϕ_2-(semi)semistability if and only if $\mathcal{P}(\phi_1) = \mathcal{P}(\phi_2)$. (The fibers $\overline{J}_C(\phi_1)$ and $\overline{J}_C(\phi_2)$ of the moduli stacks $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ over a geometric point $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}$ whose dual graph is Γ parameterize different set of sheaves.) In [KP16, Example 3.21] we showed that this would no longer be true if in Definition 5.1 the *elementary* condition for the subgraphs was dropped.

The simplest nontrivial examples of this stability decomposition occur when C consists of two components.

Example 5.2. (ϕ-stability on generalized dollar sign curves). Suppose that C is a nodal curve consisting of two smooth irreducible components of genus i and j connected by α nodes. Its dual graph $\Gamma(\alpha, i, j)$ has two vertices of genus i and j connected by α edges. The degree d stability space $V^d(\Gamma)$ is the line in the plane $V(\Gamma)$ consisting of points with coordinates summing to d. The stability hyperplanes in $V^d(\Gamma)$ are the points whose coordinates are integers when α is even, and are the points whose coordinates are $1/2$ plus an integer when α is odd, as follows from Definition 5.1. The stability polytopes are segments. If ϕ belongs to the relative interior of one such segment, the number of bidegrees of line bundles that are ϕ-stable equals α. The stable bidegrees are those that are closest (in the obvious sense) to ϕ. If ϕ varies from the relative interior of a segment to a wall, the stable bidegree furthest away from that wall becomes strictly semistable as does the nearest unstable bidegree.

We now define the analogous objects for $V^d_{g,n}$. Similarly to what we did in Definition 5.1 for a single stable graph Γ, we introduce stability hyperplanes in $V^d_{g,n}$ such that ϕ is nondegenerate according to Definition 4.1 if and only if ϕ does not belong to one such hyperplane.

Definition 5.3. For Γ a stable marked graph, $\Gamma_0 \subset \Gamma$ an elementary subgraph, and $k \in \mathbb{Z}$ an integer, we call the subset

$$H(\Gamma, \Gamma_0, k) := \{\phi \in V^d_{g,n} : \ell(\Gamma_0, k)(\phi(\Gamma)) = 0\}$$

of $V^d_{g,n}$ a *stability hyperplane.*

A *stability polytope* in $V^d_{g,n}$ is defined to be a connected component of the complement of all stability hyperplanes in $V^d_{g,n}$, i.e. a connected component of

$$V^d_{g,n} - \bigcup_{\substack{\Gamma \text{ stable graph} \\ \Gamma_0 \subset \Gamma \text{ elementary} \\ k \in \mathbb{Z}}} H(\Gamma, \Gamma_0, k)$$

and the *stability polytope decomposition* of $V^d_{g,n}$ is defined to be the set of all stability polytopes.

We define $\mathcal{P}_{g,n}$ to be the set of stability polytopes of $V^d_{g,n}$ for all $d \in \mathbb{Z}$.

As it was the case for $V^d(\Gamma)$, the stability polytopes of $V^d_{g,n}$ are rational bounded convex polytopes.

Note that in Definition 5.3 we are abusing the term "hyperplane", as it may a priori happen that $H(\Gamma, \Gamma_0, k) = V^d_{g,n}$ or that $H(\Gamma, \Gamma_0, k) = \varnothing$. In fact, this can only happen when $n = 0$, and the examples of this phenomenon are discussed in Remark 5.9.

It follows from Lemma 3.8 that a stability parameter $\phi \in V^d_{g,n}$ is uniquely determined by its restriction to all loopless graphs with two vertices. We now prove analogous statements about stability hyperplanes and polytopes.

Lemma 5.4. *If $H(\Gamma, \Gamma_0, k) \subset V^d_{g,n}$ is a stability hyperplane, then there exists a loopless 2-vertex stable graph Γ' and an elementary subgraph $\Gamma'_0 \subset \Gamma'$ such that $H(\Gamma, \Gamma_0, k) = H(\Gamma', \Gamma'_0, k)$.*

Proof. Consider a sequence of contractions from Γ that contracts Γ_0 to a vertex w and its complement Γ_0^c to a vertex w^c, and then contracts all resulting loops. Call Γ' the resulting graph with two vertices w and w^c. By inductively applying compatibility with contractions, we find $\phi(\Gamma')(w) = \sum_{v \in \Gamma_0} \phi(\Gamma)(v)$. This implies that $H(\Gamma, \Gamma_0, k)$ equals $H(\Gamma', \Gamma'_0, k)$. □

A restatement of the lemma is that ϕ-stability can be detected by generalized dollar sign curves:

Corollary 5.5. *Let F be a family of rank 1 torsion-free sheaves of degree d on the universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and let $\phi \in V^d_{g,n}$. If the restriction of F to the locus of curves with at most two smooth irreducible components is ϕ-(semi)stable, then F is ϕ-(semi)stable.*

As another corollary of Lemma 5.4, we obtain the following partial description of the set of stability polytope decomposition of $V^d_{g,n}$.

Corollary 5.6. *Given a stability polytope $\mathcal{P}(\Gamma) \subset V^d(\Gamma)$ for every loopless 2-vertex stable graph Γ, there is* at most one *stability polytope $\mathcal{P} \subset V^d_{g,n}$ such that the Γ-component of $\mathcal{P}$ equals $\mathcal{P}(\Gamma)$ for all Γ.*

Using the results of Section 3, we will now write down the stability hyperplanes in $V^d_{g,n}$ that we defined in Definition 5.3. By Corollary 3.6, an element $\phi \in V^d_{g,n}$ is uniquely determined by its image under $(p_C \oplus p_D) \circ \rho^d$, i.e. the projection to $C_{g,n} \oplus D_{g,n}$ of the difference $\phi - \phi^d_{\text{can}}$. We will describe the stability hyperplanes in $V^d_{g,n}$ as inverse images via $(p_C \oplus p_D) \circ \rho^d$ of certain hyperplanes in $C_{g,n} \oplus D_{g,n}$.

In order to define the stability hyperplanes of $C_{g,n}$, denote by $(\alpha_{i,S}, -\alpha_{i,S})$ the component $\psi(\Gamma(i,S))$ of each element $\psi \in C_{g,n}$. For each triple (i,S,k) with (i,S) as in Section 2.3 and $k \in \mathbb{Z}$, we define the hyperplane $H(i,S,k)$ in $C_{g,n}$ by the equations

$$H(i,S,k) := \begin{cases} \alpha_{i,S} = k - \frac{(2i-1)(d+1-g)}{2g-2} & \text{when } g \geq 2 \\ \alpha_{i,S} = k - \frac{1}{2} & \text{when } g \leq 1. \end{cases} \tag{25}$$

(These are the translations, depending on d, of the integer translates of the coordinate hyperplanes).

When $g \geq 1$, to define the stability hyperplanes of $D_{g,n}$, for each element $\psi \in D_{g,n}$ we denote by $(x_j, -x_j)$ the component $\psi(\Gamma_j)$ for $j = 1 + \delta_{1,g}, \ldots, n$. For each triple (ℓ, S, k) with $0 < \ell \leq 2g - 2 + \delta_{1,g}, S \subseteq [n]$ and $k \in \mathbb{Z}$ (excluding the "unstable" case $\ell = 2g - 2 + \delta_{1,g}, S = [n]$), we define the hyperplane $H(\ell,S,k)$ in $D_{g,n}$ by the equation

$$H(\ell,S,k) := \left\{ \vec{x} : \; x_S + \frac{\ell(d+1-g-x_{[n]})}{2g-2+\delta_{1,g}} = k \right\}. \tag{26}$$

The following explicit description of the stability polytope decomposition of each degree d stability space follows from Corollary 5.6 and from a direct calculation.

Theorem 2. *The stability hyperplanes of $V^d_{g,n}$ are the pullback via $p_C \circ \rho^d$ of the hyperplanes in $C_{g,n}$ defined in* (25)*, and the pullback via $p_D \circ \rho^d$ of the hyperplanes in $D_{g,n}$ defined in* (26)*.*

Proof. By Lemma 5.4, all stability hyperplanes of $V^d_{g,n}$ are inverse images of the hyperplanes in $V^d(\Gamma)$ for each 2-vertex loopless graph Γ under the natural restriction maps. Because of this, proving Theorem 2 is reduced to explicitly compute the values $\phi(\Gamma)$ for all 2-vertices loopless graphs Γ in terms of d and of the translated components $(\phi - \phi^d_{\mathrm{can}})(\Gamma(i,S))$ and $(\phi - \phi^d_{\mathrm{can}})(\Gamma_j)$. (That these uniquely determine ϕ is the content of Corollary 3.6). In order to compute the components $(\phi - \phi^d_{\mathrm{can}})(\Gamma)$ we apply Lemma 3.7 to find the unique $L \in \mathrm{PicRel}^0_{g,n}(\mathbb{R})$ such that $\deg(L) = \phi - \phi^d_{\mathrm{can}}$, and then compute $\deg(L)(\Gamma)$.

For a given $\phi \in V^d_{g,n}$, define $\psi := \phi - \phi^d_{\mathrm{can}}$, and assume that the latter is uniquely determined by $\psi(\Gamma(i,S)) = (\alpha_{i,S}, -\alpha_{i,S})$ and $\psi(\Gamma_j) = (x_j, -x_j)$. Then the component of ϕ along $\Gamma(i,S)$ equals

$$\phi(\Gamma(i,S)) = \begin{cases} \left(\alpha_{i,S} + \frac{d}{2g-2}(2i-1), -\alpha_{i,S} + \frac{d}{2g-2}(2g-2i-1)\right) & \text{when } g \geq 2 \\ (\alpha_{i,S} + d, -\alpha_{i,S} - d) & \text{when } g \leq 1. \end{cases}$$

By Example 5.2 with $\alpha = 1$, the stability hyperplanes on curves whose dual graph is $\Gamma(i,S)$ are obtained by equating the first component to $1/2$ plus an arbitrary integer. This gives the set of stability hyperplanes of $C_{g,n}$ that we defined in (25).

For $\Gamma = \Gamma(\alpha, i, S)$ an arbitrary loopless graph with two vertices of genus i and $g - \alpha + 1 - i$ and with markings S and S^c respectively, connected by $\alpha \geq 2$ edges, we now compute the Γ component $\psi(\Gamma)$. To do so, we apply Formulas (6), (7), (8) and (9) and invert the corresponding matrix, to find out that $\psi = \deg(L)$ for $L \in \mathrm{PicRel}^0_{g,n}(\mathbb{R})$ defined by

$$L := \sum_{(i,S)} \left(\alpha_{i,S} + \sum_{j \in S} \frac{2i+1-2g}{2g-2} x_j + \sum_{j \notin S} \frac{1-2i}{2g-2} x_j \right) \cdot C^-_{i,S} + \sum_{j=1}^{n} \frac{x_j}{2g-2} \cdot T_j. \tag{27}$$

Translating back by ϕ^d_{can}, the Γ component $\phi(\Gamma)$ equals:

$$\phi(\Gamma) = \left(x_S + \frac{2i-2+\alpha}{2g-2} \cdot (d - x_{[n]}),\ x_{S^c} + \frac{2g-2i-\alpha}{2g-2} \cdot (d - x_{[n]}) \right). \tag{28}$$

By Example 5.2, the stability hyperplanes on curves whose dual graph is $\Gamma(\alpha, i, S)$ are obtained by equating to an integer k (resp. to a half-integer $k+\frac{1}{2}$) each component of (28) when α is even (resp. α is odd). Equation (26) for the hyperplanes of $D_{g,n}$ is then deduced by replacing $\ell := 2i-2+\alpha$. ☐

Remark 5.7. When $\ell = 2g-2+\delta_{1,g}$, the hyperplanes $H(\ell, S, k)$ of $D_{g,n} \cong \mathbb{R}^{n-\delta_{1,g}} \ni (x_{1+\delta_{1,g}}, \dots, x_n)$ are independent of d, and have the form

$$H = H(\ell, S, k) = \left\{ \sum_{j \in S^c} x_j = k \right\}.$$

These hyperplanes, when k equals zero, are known in the literature as the *resonance hyperplanes*, see [CJM11] and [SSV08]. For each $d \in \mathbb{Z}$, the hyperplane arrangement of $D_{g,n}$ described by Equations (26) is therefore a refinement of the integer translates of the resonance hyperplane arrangement. (These two arrangements are in fact equal when $g = 1$).

Further, we observe that for any $d \in \mathbb{Z}$, the map $x_i \mapsto x_i + (2g-2)$ respects the collection of stability hyperplanes (26) of $D_{g,n}$. A fundamental domain for these translations is any $n - \delta_{1,g}$ dimensional cube of edge length $(2g-2)$, and the hyperplane arrangement (26) naturally defines a hyperplane arrangement on the torus obtained by identifying the opposite faces of the cube.

Remark 5.8. In studying the stability polytope decomposition of $V^d_{g,n}$ for all $d \in \mathbb{Z}$, it is enough to analyze the cases $d = 0, \dots, g-1$. Indeed, tensoring with ω_π and possibly mapping $L \mapsto L^{-1}$ gives isomorphisms $\mathrm{PicRel}^d_{g,n}(\mathbb{Z}) \to \mathrm{PicRel}^{2g-2\pm d}_{g,n}(\mathbb{Z})$.

Further to that, we observe that for any $(d_1, \dots, d_n) \in \mathbb{Z}^n$, the affine endomorphism $V^e_{g,n} \to V^{e+\sum d_j}_{g,n}$ defined by $\phi \mapsto \phi + \deg(d_1\Sigma_1 + \dots + d_n\Sigma_n)$ respects the hyperplanes (25) and (26). This fact will play an important role in the next section, for example in Lemma 6.10.

In light of Remarks 5.7 and 5.8, in Figures 1 and 2 we give a picture of the 2-dimensional stability spaces $D_{g,n}$ when $g \leq 3$.

Some important concluding remarks are in order.

Remark 5.9. Here we analyze more explicitly the content of Theorem 2 when n equals zero. In this case we have $D^d_{g,0} \cong \{0\}$, and from Equation (26) we derive the condition for the trivial vector space not to coincide with a hyperplane:

$$\frac{\ell(d+1-g)}{2g-2} \notin \mathbb{Z} \text{ for all } 0 < \ell \leq g-1. \tag{29}$$

In other words, a *fine* ϕ-compactified universal Jacobian of degree d on $\overline{\mathcal{M}}_g$ exists if and only if Condition (29) holds. It is elementary to check that Condition (29) is equivalent to the condition that $d+1-g$ and $2g-2$ have no nontrivial common divisors:

$$\gcd(d+1-g, 2g-2) = 1. \tag{30}$$

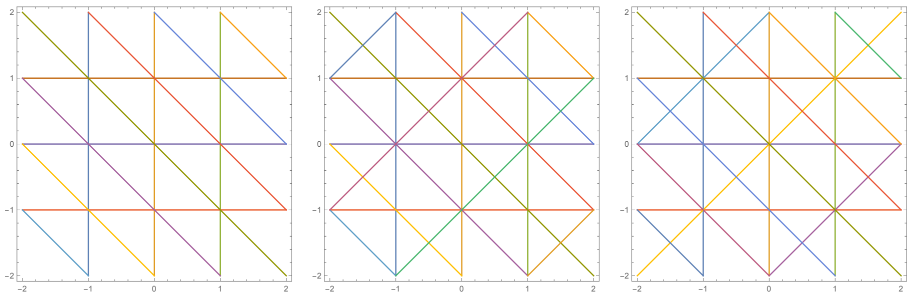

FIGURE 1. The stability space $D_{1,3}$ (any d) and the spaces $D_{2,2}$ in degrees $d = 0, 1$ respectively.

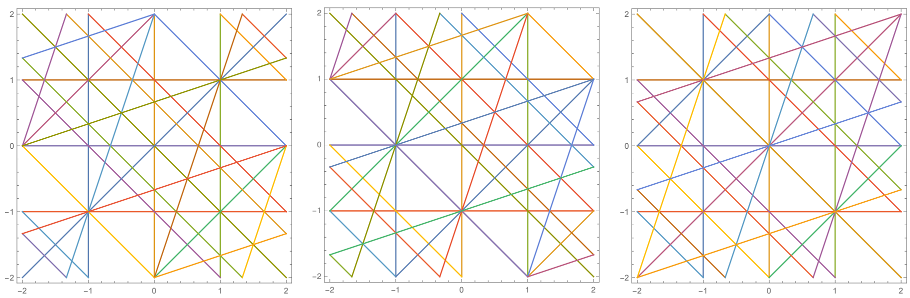

FIGURE 2. The stability spaces $D_{3,2}$ in degrees $d = 0, 1, 2$ respectively.

Remark 5.10. Here we analyze when ϕ^d_{can} belongs to a stability hyperplane. We begin by observing that this does not depend on the number n of marked points.

The projection of ϕ^d_{can} to $D_{g,n}$ does not belong to a hyperplane if and only if Condition (29) is satisfied. The projection of ϕ^d_{can} to $C_{g,n}$ belongs to a stability hyperplane if and only if $d(2i-1)$ is an odd multiple of $g-1$ for some $0 \leq i \leq g$, a condition that is included in Condition (29).

We deduce that the canonical parameter ϕ^d_{can} is in the interior of a stability polytope of $V^d_{g,n}$ if and only if Equation (30) is satisfied.

The other extreme cases occur when ϕ^d_{can} is a vertex of the polytope decomposition of $V^d_{g,n}$. By the above description, this occurs if and only if $d = g - 1 + \ell(2g-2)$, the degrees we have studied in [KP16].

Remark 5.11. The stacks $\overline{\mathcal{J}}_{g,n}(\phi)$ are related to the compactified universal Jacobians constructed by Caporaso [Cap94] and by Pandharipande [Pan96]. We will focus on Pandharipande's work as that work is closest to the present paper. Pandharipande constructed a scheme $\overline{P}^d_g$ together with a natural transformation from the functor of isomorphism classes of objects of $\overline{\mathcal{J}}_{g,0}(\phi_{\text{can}})$ to $\overline{P}^d_g$. Furthermore, the natural transformation is universal among all morphisms from the functor of isomorphisms classes to k-schemes [Pan96, Theorem 9.1.1]. Caporaso does not parameterize rank 1 torsion-free sheaves on stable curves and instead parameterizes line bundles on certain semistable curves that satisfy a condition on the multidegree (balancedness). Pandharipande constructed an isomorphism between $\overline{P}^d_g$ and Caporaso's compactification [Pan96, Theorem 10.3.1], and recently Esteves–Pacini have shown that this isomorphism is induced by an isomorphism of the corresponding moduli functors [EP16, Theorem 6.3].

In Corollary 4.4, we showed that $\overline{\mathcal{J}}_g(\phi^d_{\text{can}})$ is a Deligne–Mumford stack when $d+1-g$ is relatively prime to $2g-2$ (so that ϕ^d_{can} is nondegenerate; Remark 5.10). The condition of ϕ^d_{can}-stability is the condition considered by Pandharipande, so $\overline{P}^d_g$ is the coarse space (in the sense of [KM97]) of $\overline{\mathcal{J}}_g(\phi^d_{\text{can}})$). Under the isomorphism constructed by Pandharipande, $\overline{P}^d_g$ corresponds to the coarse space of the stack constructed by Caporaso in [Cap08, Theorem 5.9], and Esteves–Pacini's result produces an isomorphism between Caporaso's stack and $\overline{\mathcal{J}}_g(\phi^d_{\text{can}})$.

Esteves–Pacini's result also produces a natural transformation $\overline{\mathcal{J}}_{g,n}(\phi^d_{\text{can}}) \to \overline{\mathcal{J}}_{g,0}(\phi^d_{\text{can}})$ defined by taking the direct image under the stabilization map $C \to C^{\text{st}}$. To show this natural transformation is well-defined, it is sufficient to show that the direct image of a family of ϕ^d_{can}-semistable rank 1 torsion-free sheaves of degree d is again a family of ϕ_{can}-semistable rank 1 torsion-free sheaves of degree d. This follows from [EP16]. Indeed, a computation shows that every ϕ^d_{can}-semistable sheaf is admissible (in the sense of [EP16, Section 3]) with respect to $C \to C^{\text{st}}$ (i.e. the total degree of a ϕ^d_{can}-semistable sheaf on a rational chain is always -1, 0, or $+1$). We conclude from loc. cit. that the direct image of a family of ϕ^d_{can}-semistable rank 1 torsion-free sheaves is a family of rank 1 torsion-free sheaves whose formation commutes with base change. For line bundles this is [EP16, Theorem 3.1], and the general case can be deduced from the line bundle case using [EP16, Proposition 5.2]. Finally, a computation of stability conditions shows that the direct image is ϕ^d_{can}-semistable.

While the stability parameter $\phi^d_{\text{can}} \in V^d_{g,n}$ may be degenerate, we can use it to relate $\overline{P}^d_g$ to a $\overline{\mathcal{J}}_{g,n}(\phi)$ that is a Deligne–Mumford stack as follows. If ϕ_ϵ is a nondegenerate stability parameter that is sufficiently close to ϕ^d_{can} (in the sense that $\phi^d_{\text{can}} \in \overline{\mathcal{P}}(\phi_\epsilon)$), then every ϕ_ϵ-stable sheaf is ϕ^d_{can}-semistable, so there is a tautological natural transformation $\overline{\mathcal{J}}_{g,n}(\phi_\epsilon) \to \overline{\mathcal{J}}_{g,n}(\phi^d_{\text{can}})$. The stack $\overline{\mathcal{J}}_{g,n}(\phi_\epsilon)$ is Deligne–Mumford, and by composition, we get a morphism

$$\overline{\mathcal{J}}_{g,n}(\phi_\epsilon) \to \overline{P}^d_g \tag{31}$$

that lifts the forgetful map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$.

6. Applications

In this section we apply our earlier results in two ways. We study the problem of extending the sections of the forgetful map $\mathcal{J}^d_{g,n} \to \mathcal{M}_{g,n}$, and the problem of finding different isomorphism classes of fine ϕ-compactified universal Jacobians.

For fixed integers $(k, d_1, \dots, d_n)$ satisfying $k(2g-2) + d_1 + \dots + d_n = d$, we define a natural map $\sigma_{k,\vec{d}}\colon \mathcal{M}_{g,n} \to \mathcal{J}^d_{g,n}$ by the rule

$$\sigma_{k,\vec{d}}\colon (C/S, \Sigma_1, \dots, \Sigma_n) \mapsto \omega_\pi^{\otimes k} \otimes \mathcal{O}_C(d_1\Sigma_1 + \dots + d_n\Sigma_n). \tag{32}$$

This section is sometimes called an *Abel–Jacobi section.* These are the only rational sections of the forgetful map $\mathcal{J}^d_{g,n} \to \mathcal{M}_{g,n}$ from the universal Jacobian of degree d to the moduli space of smooth pointed curves, by the following result, which motivates Section 6.1.

Fact 2. *(Strong Franchetta conjecture.) Every rational section of the forgetful map* $\mathcal{J}^d_{g,n} \to \mathcal{M}_{g,n}$ *is of the form* $\sigma_{k,\vec{d}}$ *for some* k *and* $\vec{d}$ *as above. In particular, every rational section extends to a regular section* $\mathcal{M}_{g,n} \to \mathcal{J}^d_{g,n}$.

Proof. When $n \geq 1$ this is a well-known consequence of Fact 1, see [AC87, Section 4]. When $n = 0$ this was proven by Mestrano [Mes87] and then by Kouvidakis [Kou91, Theorem 2]. □

Fact 2 implies the following result on birational maps of universal Jacobians, which motivates Section 6.2.

Corollary 6.1. *Let* $\alpha: \mathcal{J}^{e_1}_{g,n} \dashrightarrow \mathcal{J}^{e_2}_{g,n}$ *be a birational map that commutes with the forgetful maps to* $\mathcal{M}_{g,n}$. *Then there exist*

$$(k, d_1, \ldots, d_n) \in \mathbb{Z}^{n+1} \text{ and } t \in \{0,1\} \quad \text{with} \quad k(2g-2) + \sum d_j = e_2 - (-1)^t e_1,$$

such that α *is defined by the rule*

$$\alpha: L \mapsto L^{(-1)^t} \otimes \omega_C^{\otimes k} \otimes \mathcal{O}_C(d_1\Sigma_1 + \ldots + d_n\Sigma_n). \tag{33}$$

In particular, α *is an isomorphism.*

Proof. The case $n = 0$ is due to Caporaso, see [BFV12, Theorem 7.2]. From now on in this proof we assume $n \geq 1$.

By applying a translation automorphism, it is enough to prove the claim when $e_1 = e_2 = 0$, so α is a birational automorphism of $\mathcal{J}^0_{g,n}$ that commutes with the forgetful map. In this case, consider the birational automorphism β of the generic Jacobian J^0_C that α induces. Because the locus of indeterminacy is covered by rational curves and a Jacobian variety cannot contain any rational curve, β is in fact an automorphism of J^0_C. Furthermore, β must preserve the principal polarization because the Néron–Severi group of J^0_C is cyclic for a very general C by [BL92, Corollary 17.5.2] and because Θ_C is the unique generator of the Néron–Severi group of J^0_C that is ample. We conclude using a version of the Torelli theorem [Mil86, Theorem 12.1] that implies that β must lie in the group generated by translations and the involution $L \mapsto L^{-1}$. By Fact 2, this group is the group of automorphisms of the form (33) with $k(2g-2) + \sum d_j = 0$. Since α coincides with an automorphism of the form (33) on the generic fiber, it must be equal to that automorphism. □

In Corollary 6.1 it is essential to assume that α commutes with the forgetful maps. The problem of characterizing arbitrary birational maps $\mathcal{J}^{e_1}_{g,n} \dashrightarrow \mathcal{J}^{e_2}_{g,n}$ is harder than the problem of classifying birational maps $\mathcal{M}_{g,n} \dashrightarrow \mathcal{M}_{g,n}$, and this second classification is not available, even for g and n large.

6.1. **Extensions of Abel–Jacobi sections.** Motivated by Fact 2, we fix integers $(k, d_1, \ldots, d_n)$ and apply the earlier results of this paper to analyze extensions of the Abel–Jacobi section $\sigma_{k,\vec{d}}$ to $\overline{\mathcal{M}}_{g,n}$. The main result is Corollary 6.7, in which we characterize the nondegenerate ϕ's such that the Abel–Jacobi section extends to a well-defined morphism $\sigma_{k,\vec{d}}: \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}(\phi)$.

Corollary 6.7 follows from the more general Proposition 6.4. That proposition describes the locus of indeterminacy of $\sigma_{k,\vec{d}}: \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi)$ as the closure of the locus of pointed curves $(C, p_1, \dots, p_n)$ with two smooth irreducible components meeting in least two nodes such that $\omega_C^k \otimes \mathcal{O}_C(d_1p_1 + \ldots + d_np_n)$ fails to be $\phi(C, p_1, \dots, p_n)$-stable. To prove this result, we first observe that all nondegenerate $\phi \in V_{g,n}^d$ that have the same projection to $D_{g,n}$ correspond to isomorphic moduli stacks $\overline{\mathcal{J}}_{g,n}(\phi)$, so we can reduce to the case where $\omega_C^k \otimes \mathcal{O}_C(d_1p_1 + \ldots + d_np_n)$ is stable on all curves with at most one node. The proof that the indeterminacy locus is not smaller than the one that we claimed essentially follows from the fact that there exists a unique rank 1 torsion free sheaf that extends to $\overline{\mathcal{C}}_{g,n}$ the restriction of $\omega_C^k \otimes \mathcal{O}_C(d_1p_1 + \ldots + d_np_n)$ to $\overline{\mathcal{C}}_{g,n}^{\leq 1}$, defined as the universal curve over $\overline{\mathcal{M}}_{g,n}^{\leq 1}$ (the moduli stack of stable curves with at most one node). The problem of resolving the indeterminacy of the Abel–Jacobi sections was raised by Grushevsky–Zakharov in [GZ14], and in Remark 6.8, we discuss how that work relates to the present paper.

In analyzing the locus of indeterminacy, the following line bundles on the universal curve play a fundamental role.

Definition 6.2. Let $\mathcal{O}(\mathcal{D})$ be the line bundle on the universal curve $\overline{\mathcal{C}}_{g,n}$ defined by

$$\mathcal{O}(\mathcal{D}) := \omega_\pi^{\otimes k} \otimes \mathcal{O}(d_1\Sigma_1 + \ldots + d_n\Sigma_n),$$

and let $\phi_{k,\vec{d}} \in V_{g,n}^d$ be its multidegree:

$$\phi_{\vec{d},k}(\Gamma(i,S)) := (d_S + (1-2i)k, d - d_S + (2i-1)k), \quad \phi_{\vec{d},k}(\Gamma_j) := (d_j, d - d_j). \tag{34}$$

(By Corollary 3.6, Equation (34) defines a unique element of $V_{g,n}^d$).

For $\phi \in V_{g,n}^d$ nondegenerate, we define the following modification of $\mathcal{O}(\mathcal{D})$ (slightly generalizing what we did for $k = 0$ in [KP16, Section 5]):

$$\mathcal{O}(\mathcal{D}(\phi)) := \mathcal{O}(\mathcal{D}) \otimes \mathcal{O}\left(\sum_{(i,S)} \left(-d_S - k(2i-1) + \left\lfloor \phi(\Gamma(i,S))(v) + \frac{1}{2} \right\rfloor\right) \cdot C_{i,S}^-\right),$$

where v is the first vertex of $\Gamma(i,S)$ according to the convention we fixed in Section 2.3, and $C_{i,S}^-$ is defined in Section 2.4.

The line bundle $\mathcal{O}(\mathcal{D}(\phi))$ is defined so that its restriction to smooth pointed curves equals the restriction of $\mathcal{O}(\mathcal{D})$, and its restriction to stable curves with at most one node is ϕ-stable. Stability follows from the following proposition, which describes the properties of $\mathcal{O}(\mathcal{D}(\phi))$ that we will use next.

Proposition 6.3. *The line bundle $\mathcal{O}(\mathcal{D}(\phi))$ satisfies the following.*

(1) The restriction $\mathcal{O}(\mathcal{D}(\phi))|_{(C,p_i)}$ to a stable pointed curve (C, p_i) with one node is $\phi(\Gamma_{(C,p_i)})$-stable.

(2) The restriction of $\mathcal{O}(\mathcal{D}(\phi))$ to a stable pointed curve with two smooth components and at least two nodes equals the restriction of $\mathcal{O}(\mathcal{D})$.

Proof. The proof of the first claim follows by computing the bidegrees. The second claim follows from the fact that the line bundles $\mathcal{O}(C_{i,S}^-)$ become trivial when they are restricted to curves with two smooth irreducible components and at least two nodes. □

We can now state and prove our first characterization of the indeterminacy locus of the Abel–Jacobi section.

Proposition 6.4. *Given a nondegenerate $\phi \in V^d_{g,n}$, the locus of indeterminacy of the rational map $\sigma_{k,\vec{d}}\colon \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi)$ defined by (32) is the closure of the locus $Z(\phi)$ of stable curves $(C, p_1, \dots, p_n)$ with two smooth irreducible components separated by at least two nodes, such that the restriction $\mathcal{O}(\mathcal{D})|_{(C,p_i)}$ fails to be $\phi(\Gamma_{(C,p_i)})$-stable.*

Proof. We first rule out the case $n = 0$. Indeed, without marked points, the total degree d of $\mathcal{O}(\mathcal{D})$ is forced to be a multiple of $2g-2$. However, as we observed in Remark 5.10, a nondegenerate $\phi \in V^d_{g,0}$ only exists when $\gcd(d+1-g, 2g-2) = 1$.

We can therefore assume $n \geq 1$. With this hypothesis we know by [KP16, Lemma 3.39] that there exists a tautological sheaf $F_{\mathrm{tau}}(\phi)$ on $\overline{\mathcal{J}}_{g,n}(\phi) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n}$. Call $\tilde{\sigma}_{k,\vec{d}}$ the map $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{J}}_{g,n}(\phi) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n}$ obtained by pulling back the section $\sigma_{k,\vec{d}}\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}(\phi)$.

We extend $\sigma_{k,\vec{d}}$ by the rule

$$(C, p_i) \mapsto \mathcal{O}(\mathcal{D}(\phi))|_{(C,p_i)}.$$

This rule makes $\sigma_{k,\vec{d}}$ into a well-defined morphism (at least) over $U(\phi)$, defined as the locus of curves $(C, p_1, \dots, p_n)$ of $\overline{\mathcal{M}}_{g,n}$ such that the restriction $\mathcal{O}(\mathcal{D}(\phi))|_{(C,p_i)}$ is $\phi(\Gamma_{(C,p_i)})$-stable. Proposition 6.3 implies the inclusion

$$U(\phi) \subseteq \overline{\mathcal{M}}_{g,n} \smallsetminus \overline{Z}(\phi). \tag{35}$$

To conclude, we need to show that the inclusion of (35) is an equality.

By definition of tautological sheaf and of $\tilde{\sigma}_{k,\vec{d}}$, over $U(\phi)$ we have an isomorphism of line bundles

$$\tilde{\sigma}^*_{k,\vec{d}}|_{\pi^{-1}(U(\phi))}(F_{\mathrm{tau}}(\phi)) \cong \mathcal{O}(\mathcal{D}(\phi))|_{\pi^{-1}(U(\phi))}. \tag{36}$$

Because $U(\phi)$ contains the locus of curves of $\overline{\mathcal{M}}_{g,n}$ with at most one node, its complement in $\overline{\mathcal{M}}_{g,n}$ has codimension at least 2. By Corollary 7.2 there exists at most one extension of (36) to a family of rank 1 torsion-free sheaves.

We are now ready to prove the reverse inclusion of (35). Assuming $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n} \smallsetminus \overline{Z}(\phi)$, we want to prove that $(C, p_1, \dots, p_n)$ belongs to $U(\phi)$. By applying the same contractions that appear in the proof of Lemma 5.4, we can assume that $(C, p_1, \dots, p_n)$ has two smooth irreducible components. By the above paragraph, Isomorphism (36) is valid at $(C, p_1, \dots, p_n)$. Because $F_{\mathrm{tau}}(\phi)|_{(C,p_i)}$ is by definition $\phi(\Gamma_{(C,p_i)})$-stable, so is $\mathcal{O}(\mathcal{D}(\phi))|_{(C,p_i)}$. By the second part of Proposition (6.3), we also have that $\mathcal{O}(\mathcal{D})|_{(C,p_i)}$ is $\phi(\Gamma_{(C,p_i)})$-stable. This proves that $(C, p_1, \dots, p_n) \in U(\phi)$, which concludes our proof. □

The proposition just proven reduces the problem of analyzing the locus of indeterminacy of $\sigma_{k,\vec{d}}$ to the problem of describing those ϕ such that $\mathcal{O}(\mathcal{D})$ is ϕ-stable. The line bundle $\mathcal{O}(\mathcal{D})$ is ϕ-stable for $\phi = \phi_{k,\vec{d}}$, but this stability parameter is degenerate (so there is not an associated Deligne–Mumford stack). We will now identify the nondegenerate parameters $\phi \in V^d_{g,n}$ such that $\mathcal{O}(\mathcal{D})$ is ϕ-stable, and those ϕ for which the Abel–Jacobi section $\sigma_{k,\vec{d}}$ extends to a regular section $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}(\phi)$.

Definition 6.5. Define the polytope $\mathcal{Q}(\phi_{k,\vec{d}})$ in $V^d_{g,n}$ through the isomorphism of Corollary 3.6 by the inequalities

$$\left|\phi(\Gamma)(v) - \phi_{k,\vec{d}}(\Gamma)(v)\right| < \frac{\alpha}{2} \tag{37}$$

for all graphs Γ with two vertices v and w of genus i and $g - \alpha + 1 - i$ connected by α edges and $S \subset [n]$ markings on the first vertex.

We remark that $\mathcal{Q}(\phi_{k,\vec{d}})$ is *not* a stability polytope in the sense of Definition 5.3.

Corollary 6.6. *For $\phi \in V^d_{g,n}$ nondegenerate, the line bundle $\mathcal{O}(\mathcal{D})$ is ϕ-stable if and only if ϕ belongs to $\overline{\mathcal{Q}}(\phi_{k,\vec{d}})$, the closure of the polytope $\mathcal{Q}(\phi_{k,\vec{d}})$.*

Proof. This follows from Lemma 3.8 (stability can be checked on curves with two components), Definition 4.1 (the definition of ϕ-stability) and the fact that, for a nondegenerate parameter, a sheaf is semistable if and only if it is stable. □

Corollary 6.7. *For $\phi \in V^d_{g,n}$ nondegenerate, the morphism $\sigma_{k,\vec{d}}$ extends to a regular section $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}(\phi)$ if and only if the projection of $\overline{\mathcal{Q}}(\phi_{k,\vec{d}})$ to $D_{g,n}$ contains the projection of ϕ.*

Proof. This follows formally from Proposition 6.4, Corollary 6.6, and Theorem 2. □

Concretely, this means that $\sigma_{k,\vec{d}}$ extends to a regular morphism on $\overline{\mathcal{M}}_{g,n}$ if and only if the Equation (37) is satisfied by ϕ for all loopless graphs Γ with 2 vertices and at least $\alpha \geq 2$ edges.

We conclude this section by comparing the result just proven with the work of Grushevsky–Zakharov in [GZ14].

Remark 6.8. For $\vec{d}$ satisfying $\sum d_i = 0$, Grushevsky–Zakharov describe the indeterminacy of the Abel–Jacobi section $\sigma_{\vec{d}} := \sigma_{0,\vec{d}}$ considered as a morphism into a stack $\mathcal{X}'_g \to \mathcal{A}'_g$ they call Mumford's partial compactification. This partial compactification is an extension of the universal family of principally polarized abelian varieties $\mathcal{X}_g \to \mathcal{A}_g$ that is constructed so that the fiber over a point of $\mathcal{A}'_g - \mathcal{A}_g$ is an explicit compactification of a semiabelian variety with 1-dimensional maximal torus called a rank 1 degeneration.

The Torelli map extends to a regular morphism $\overline{\mathcal{M}}^{t\leq 1}_g \to \mathcal{A}'_g$ out of the locus $\overline{\mathcal{M}}^{t\leq 1}_g \subseteq \overline{\mathcal{M}}_g$ of curves whose generalized Jacobian has torus rank at most 1. The pullback of $\mathcal{X}'_g$ under the composition $\overline{\mathcal{M}}^{t\leq 1}_{g,n} \to \overline{\mathcal{M}}^{t\leq 1}_g \to \mathcal{A}'_g$ is an extension of the universal Jacobian in degree 0, so we can consider $\sigma_{\vec{d}}$ as a rational morphism into $\mathcal{X}'_g$. Grushevsky–Zakharov explain that, for a general choice of $\vec{d}$, the locus of indeterminacy is the locus of stable pointed curves with at least 2 nonseparating edges [GZ14, Example 6.1, 6.2].

Grushevsky–Zakharov's description of the locus of indeterminacy of $\sigma_{\vec{d}}\colon \overline{\mathcal{M}}^{t\leq 1}_{g,n} \dashrightarrow \mathcal{X}'_g$ is similar to our description of the indeterminacy locus of $\sigma_{\vec{d}}\colon \overline{\mathcal{M}}^{t\leq 1}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi)$ in Proposition 6.4, but the pullback of $\mathcal{X}'_g$ to $\overline{\mathcal{M}}^{t\leq 1}_{g,n}$ is not one of the moduli stacks $\overline{\mathcal{J}}_{g,n}(\phi)$. Indeed, for all nondegenerate ϕ, the compactified Jacobian $\overline{J}_C(\phi)$ of a stable pointed curve C that is the union of two smooth curves meeting in two nodes has two irreducible components (corresponding to the two ϕ-stable multidegrees of line bundles, as can be shown by a computation), but the analogous fiber $\overline{X}_C$ of $\mathcal{X}'_g \to \mathcal{A}'_g$ is irreducible.

The pullback of $\mathcal{X}'_g$ is, however, related to compactified Jacobians. Every extension of $\mathcal{J}^0_g$, or a nonempty open substack of $\mathcal{J}^0_g$, maps rationally to $\mathcal{X}'_g$ because the universal abelian variety $\mathcal{X}_g$ pullsback to $\mathcal{J}^0_g$ under the Torelli map. In particular, the degree 0 Caporaso–Pandharipande family $\overline{P}^0_g$ (discussed in Remark 5.11) maps rationally $\overline{P}^0_g \dashrightarrow \mathcal{X}'_g$. The authors believe it is expected that this rational map restricts to an isomorphism over the locus of automorphism-free curves in $\overline{\mathcal{M}}^{t\leq 1}_g$. For example, Caporaso describes the fiber $\overline{P}^0_C$ of $\overline{P}^0_g \to \overline{M}_g$ over a curve with 2 nonseparating nodes as a rank 1 degeneration in [Cap94, Figure 8], and on [Nam76, page 240], Namikawa indicates a relation between families over $\mathcal{A}'_g$, and more generally toroidal compactifications of $\mathcal{A}_g$, and Oda–Seshadri's compactified Jacobians. A proof that the two families are isomorphic does not, however, seem to be available, although Alexeev has proven a parallel statement for $\overline{P}^{g-1}_g$ [Ale04, Corollary 5.4]. Note that it is necessary to restrict to the automorphism-free locus for otherwise the families have different fibers: for a smooth curve C, the relevant fiber of $\mathcal{X}'_g$ is J_C but the fiber of P^0_g is $J_C/\operatorname{Aut}(C)$.

The work in this section illuminates the indeterminacy of $\sigma_{\vec{d}}\colon \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{P}^0_g$, and hence of $\sigma_{\vec{d}}\colon \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{X}'_g$ assuming $\overline{P}^0_g \dashrightarrow \mathcal{X}'_g$ has no indeterminacy over the locus of automorphism-free curves. Recall from Remark 5.11 that, if $\phi_\epsilon \in V^0_{g,n}$ is nondegenerate and sufficiently close to ϕ_{can}, there is a morphism $\overline{\mathcal{J}}_{g,n}(\phi_\epsilon) \to \overline{P}^0_g$ that extends the forgetful morphism on generic fibers. Depending on the choice of $\vec{d}$, the rational section $\sigma_{\vec{d}}\colon \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi_\epsilon)$ may or may not have indeterminacy. We have shown, however, that we can find a second $\phi' \in V^0_{g,n}$ such that $\sigma_{\vec{d}}\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}(\phi')$ has no indeterminacy. Thus we have resolved the indeterminacy by passing from the pullback of $\overline{P}_g$ to the birational stack $\overline{\mathcal{J}}_{g,n}(\phi')$.

6.2. **Different fine compactified universal Jacobians.** The goal of this section is to enumerate the isomorphism classes of fine compactified universal Jacobians $\overline{\mathcal{J}}_{g,n}(\phi)$ and, in particular, show the existence of non-isomorphic $\overline{\mathcal{J}}_{g,n}(\phi)$'s. We do this exploiting the natural action of a group $\widetilde{\mathrm{PR}}_{g,n}$ (defined in Definition 6.9) on the set of stability polytopes $\mathcal{P}_{g,n}$. In Lemma 6.11 we prove that two compactified Jacobians $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ are isomorphic over $\overline{\mathcal{M}}_{g,n}$ if and only if $\mathcal{P}(\phi_1)$ lies in the same orbit as $\mathcal{P}(\phi_2)$. We study the property of this group action in Corollary 6.14, where we show that it fails to be transitive except in few special cases. We immediately deduce that, except for the special cases, for a given (g,n), there exists at least two $\overline{\mathcal{J}}_{g,n}(\phi)$'s that are not isomorphic over $\overline{\mathcal{M}}_{g,n}$. This is Corollary 6.15, and in Corollary 6.16, we prove the stronger statement that, provided $\overline{\mathcal{M}}_{g,n}$ is of general type, there exist $\overline{\mathcal{J}}_{g,n}(\phi)$'s that are not isomorphic as stacks (rather than as stacks over $\overline{\mathcal{M}}_{g,n}$).

The group acting on stability polytopes is the following one.

Definition 6.9. Let $\widetilde{\mathrm{PR}}_{g,n}$ be the generalized dihedral group defined by the action $L \mapsto L^{(-1)^t}$ of $t \in \mathbb{Z}/2\mathbb{Z}$ on $\mathrm{PicRel}_{g,n}(\mathbb{Z})$. In other words, $\widetilde{\mathrm{PR}}_{g,n}$ is the semi-direct product

$$\widetilde{\mathrm{PR}}_{g,n} := (\mathrm{PicRel}_{g,n}(\mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The group $\widetilde{\mathrm{PR}}_{g,n}$ acts on families of rank 1 torsion-free sheaves on families of stable pointed curves by the rule

$$\psi(L,t)\colon F \mapsto F^{(-1)^t} \otimes L. \tag{38}$$

(That $F \mapsto F^{-1}$ gives a well-defined map in families follows from Lemma 7.4 in Section 7). Similarly, the group $\widetilde{\mathrm{PR}}_{g,n}$ also acts on the stability space $V_{g,n}$ by the affine endomorphisms

$$\lambda(L,t)\colon \phi \mapsto (-1)^t \cdot \phi + \deg(L). \tag{39}$$

The following key observation relates the two actions ψ and λ.

Lemma 6.10. *Assume that F has degree d and that $\phi \in V^d_{g,n}$. Then F is ϕ-(semi)stable if and only if $\psi(L,t)(F)$ is $\lambda(L,t)(\phi)$-(semi)stable. In particular, when ϕ is nondegenerate, $\psi(L,t)$ induces a well-defined isomorphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{J}}_{g,n}(\lambda(\phi))$ that commutes with the forgetful maps to $\overline{\mathcal{M}}_{g,n}$.*

Proof. The claim follows immediately from Definition 4.1 and Definition 4.2. □

In fact, all isomorphisms that commute with the forgetful maps are defined by Rule (38) for a suitable choice of $(L,t) \in \widetilde{\mathrm{PR}}_{g,n}$, as we prove in the next lemma.

Lemma 6.11. *Let $\phi_1 \in V^{e_1}_{g,n}$ and $\phi_2 \in V^{e_2}_{g,n}$ be nondegenerate, and assume $\alpha\colon \overline{\mathcal{J}}_{g,n}(\phi_1) \to \overline{\mathcal{J}}_{g,n}(\phi_2)$ is a birational morphism that commutes with the forgetful maps to $\overline{\mathcal{M}}_{g,n}$. Then there exists $(L,t) \in \widetilde{\mathrm{PR}}_{g,n}$ such that $\alpha = \psi(L,t)$.*

Proof. By Corollary 6.1, the restriction of α to $\mathcal{J}^{e_1}_{g,n}$ is an isomorphism given by

$$L \mapsto L^{(-1)^t} \otimes \omega_\pi^{\otimes k} \otimes \mathcal{O}_C(d_1\Sigma_1 + \ldots + d_n\Sigma_n)$$

for some $(k, d_1, \ldots, d_n) \in \mathbb{Z}^{n+1}$ and $t \in \{0,1\}$. The latter can be extended to a well-defined morphism $\overline{\mathcal{J}}^{\le 1}_{g,n}(\phi_1) \to \overline{\mathcal{J}}^{\le 1}_{g,n}(\phi_2)$ (here $\overline{\mathcal{J}}^{\le 1}_{g,n}(\phi)$ denotes the restriction of $\overline{\mathcal{J}}_{g,n}(\phi)$ to $\overline{\mathcal{M}}^{\le 1}_{g,n}$, the substack parameterizing curves with at most one node) by the rule $\psi(L,t)$ as defined by Equation (38). Here we have defined

$$L := \omega_\pi^{\otimes k} \otimes \mathcal{O}_C(d_1\Sigma_1 + \ldots + d_n\Sigma_n + \sum a_{i,S} \cdot C^-_{i,S})$$

for $C^-_{i,S}$ as defined in Section 2.4, and $a_{i,S}$ defined to be the componentwise approximation to the nearest integer of the restriction of

$$\phi_2 - (-1)^t\phi_1 - \deg\left(\omega_\pi^{\otimes k} + d_1\Sigma_1 + \ldots + d_n\Sigma_n\right)$$

to a general curve of $\Delta(i,S)$.

While it is a priori not clear that this $\psi(L,t)$ extends to a morphism $\overline{\mathcal{J}}_{g,n}(\phi_1) \to \overline{\mathcal{J}}_{g,n}(\phi_2)$, the argument in the above paragraph shows that the restriction of α to $\overline{\mathcal{J}}^{\le 1}_{g,n}(\phi_1)$ coincides with $\psi(L,t)$. To conclude, we need to prove that α and $\psi(L,t)$ coincide on $\overline{\mathcal{J}}_{g,n}(\phi_1)$.

When $n \ge 1$, consider a tautological sheaf $F_{\mathrm{tau}}(\phi_2)$ on $\overline{\mathcal{J}}_{g,n}(\phi_2) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n}$ (which exists by [KP16, Lemma 3.39]). The pullback via $\alpha \times \mathrm{Id}$ and via $\psi \times \mathrm{Id}$ of $F_{\mathrm{tau}}(\phi_2)$ coincide on the locus

$$\overline{\mathcal{J}}_{g,n}(\phi_1)^{\le 1} \times_{\overline{\mathcal{M}}^{\le 1}_{g,n}} \overline{\mathcal{C}}^{\le 1}_{g,n} \tag{40}$$

where the underlying curve has at most one node. Because the locus (40) is an open substack of $\overline{\mathcal{J}}_{g,n}(\phi) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n}$ whose complement has codimension 2, by Lemma 7.3 and Corollary 7.2 the two pullbacks must coincide everywhere. This implies that $\psi(L,t) \times \mathrm{Id}$ can be extended to coincide with $\alpha \times \mathrm{Id}$ everywhere, which implies that $\alpha = \psi(L,t)$ on $\overline{\mathcal{J}}_{g,n}(\phi_1)$.

When $n = 0$ apply the same argument of the above paragraph after first passing to an étale cover $\mathcal{U} \to \overline{\mathcal{J}}_g(\phi)$ such that a tautological sheaf exists on $\mathcal{U} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}_g$. (To prove that $\alpha \times \mathrm{Id}$ and $\psi \times \mathrm{Id}$ coincide, it is enough to check that the same holds étale locally.) This concludes the proof. □

Here is a corollary of Lemma 6.11 that gives a new significance to the stability polytope decomposition of $D_{g,n}$ (see Theorem 2 and, in particular, Equation (26)).

Corollary 6.12. *Let $\phi_1, \phi_2 \in V^d_{g,n}$ be nondegenerate. Then there exists an isomorphism (or, equivalently, a birational morphism) $\overline{\mathcal{J}}_{g,n}(\phi_1) \to \overline{\mathcal{J}}_{g,n}(\phi_2)$ that extends the identity on $\mathcal{J}^d_{g,n}$ and that commutes with the forgetful maps to $\overline{\mathcal{M}}_{g,n}$ if and only if the projections of the stability polytopes $\mathcal{P}(\phi_1)$ and $\mathcal{P}(\phi_2)$ to $D_{g,n}$ coincide.*

Proof. Combine Lemma 6.10 and Lemma 6.11. □

Each affine endomorphism $\lambda(L,t)$ maps stability polytopes (from Definition 5.3) to stability polytopes. Thus λ induces an action, that we will call μ, of $\widetilde{\mathrm{PR}}_{g,n}$ on the set of stability polytopes $\mathcal{P}_{g,n}$. Lemma 6.11 reduces the problem of deciding whether $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ admit an isomorphism over $\overline{\mathcal{M}}_{g,n}$ to studying when the stability polytope $\mathcal{P}(\phi_1)$ lies in the same orbit of $\mathcal{P}(\phi_2)$ under the μ action of $\widetilde{\mathrm{PR}}_{g,n}$.

To study the transitivity of the action of $\widetilde{\mathrm{PR}}_{g,n}$ on $\mathcal{P}_{g,n}$, we now exhibit fundamental domains for the λ-action of $\widetilde{\mathrm{PR}}_{g,n}$ on $\coprod_{d \in \mathbb{Z}} V^d_{g,n}$. A fundamental domain U is a subset of $\coprod_{d \in \mathbb{Z}} V^d_{g,n}$ that contains at least one point of each orbit of $\widetilde{\mathrm{PR}}_{g,n}$, and no two points in the interior of U are equivalent. To state our result, we identify $V^0_{g,n}$ with $C_{g,n} \oplus D_{g,n}$ by means of Corollary 3.6.

Lemma 6.13. *A fundamental domain for the action of* $\mathrm{PicRel}_{g,n}(\mathbb{Z})$ *by the restriction of λ on $\coprod_{d \in \mathbb{Z}} V^d_{g,n}$ is given by*

(1) any hypercube in $C_{g,n}$ of edge length 1 when $g = 0$;

(2) the product of any hypercube in $C_{g,n}$ of edge length 1 with $\coprod_{d=1}^{2g-3} D^{(d)}_{g,n} : \gcd(d + 1 - g, 2g - 2) = 1\}$ when $g \geq 2$ and $n = 0$;

(3) the product of any hypercube in $C_{g,n}$ of edge length 1 and of any hypercube in $D_{g,n}$ of edge length $2g - 2 + \delta_{1,g}$ when $g, n \geq 1$.

In the proof we will use the free generators of the relative Picard group given in Fact 1.

Proof. A fundamental domain for the subgroup $W_{g,n}$ is given by the product of any hypercube in $C_{g,n}$ of edge length 1 with $\coprod_{d \in \mathbb{Z}} D^{(d)}_{g,n}$.

When $g = 0$ the claim follows, because the action of the subgroup $\mathrm{PicRel}_{g,n}(\mathbb{Z})$ generated by a section on the collection of points $\coprod_{d \in \mathbb{Z}} D^{(d)}_{0,n}$ is transitive.

When $g \geq 1$, a fundamental domain for the action of the subgroup generated by ω_π (or by Σ_1 when $g = 1$) on $\coprod_{d \in \mathbb{Z}} D^{(d)}_{g,n}$ is given by $\coprod_{d=0}^{2g-3} D^{(d)}_{g,n}$.

When $n = 0$ the set $\coprod_{d=0}^{2g-3} D_{g,0}^{(d)}$ consists of isolated points, and by Remark 5.9 the nondegenerate d's are those that satisfy $\gcd(d+1-g, 2g-2) = 1$.

When $n \geq 1$ we are left to study the action of the free rank $n - \delta_{1,g}$ abelian group generated by the sections (that are distinct from Σ_1 when $g = 1$). If Σ is any such section, translation by $\deg(\Sigma)$ identifies $D_{g,n}^{(d)}$ with $D_{g,n}^{(d+1)}$. Modulo ω_π (or modulo Σ_1 when $g = 1$), translation by $(2g-2+\delta_{1,g}) \cdot \Sigma_j$ identifies any point $(x_{1+\delta_{1,g}}, \dots, x_n) \in D_{g,n}^{(0)}$ with $(x_{1+\delta_{1,g}}, \dots, x_j + (2g-2+\delta_{1,g}), \dots, x_n)$. This concludes the proof. $\square$

The orbits of the μ-action of $\widetilde{\mathrm{PR}}_{g,n}$ on $\mathcal{P}_{g,n}$ can be read off from the action of $\mathbb{Z}/2\mathbb{Z} = \widetilde{\mathrm{PR}}_{g,n}/\operatorname{PicRel}_{g,n}(\mathbb{Z})$ on the collection of polytopes in the fundamental domains that we exhibited in Lemma 6.13.

Corollary 6.14. *The action μ of $\widetilde{\mathrm{PR}}_{g,n}$ on the set $\mathcal{P}_{g,n}$ of stability polytopes*

(1) has finitely many orbits;
(2) is free if and only if $g, n \geq 2$, or $g = 1$ and $n \geq 3$;
(3) is transitive if and only if $g = 0$, or (g, n) belongs to

$$\{(1,1),(1,2),(1,3),(2,0),(2,1),(3,0),(4,0)\}. \tag{41}$$

Proof. We apply Lemma 6.13 choosing the product of hypercubes to equal the union of (closed) polytopes (that this can be done follows from the equations of the stability hyperplanes of Theorem 2). This reduces our claims to studying the action of $\mathbb{Z}/2\mathbb{Z}$ on the set of polytopes in this fundamental domain. All three claims follow then from the explicit description of the stability walls given in Theorem 2. $\square$

Here is the main result of this section.

Corollary 6.15. *For fixed (g, n), there exist finitely many isomorphism classes over $\overline{\mathcal{M}}_{g,n}$ of $\overline{\mathcal{J}}_{g,n}(\phi)$ for all $d \in \mathbb{Z}$ and all nondegenerate $\phi \in V_{g,n}^d$. When $g > 0$ and (g, n) is not one of the values in* (41)*, there exist nondegenerate $\phi_1 \in V_{g,n}^{e_1}$ and $\phi_2 \in V_{g,n}^{e_2}$ such that $\overline{\mathcal{J}}_{g,n}(\phi_1)$ is not isomorphic to $\overline{\mathcal{J}}_{g,n}(\phi_2)$ over $\overline{\mathcal{M}}_{g,n}$.*

Proof. By Lemma 6.10 and Lemma 6.11, an isomorphism over $\overline{\mathcal{M}}_{g,n}$ exists if and only $\mathcal{P}(\phi_1)$ and $\mathcal{P}(\phi_2)$ belong to the same orbit of the μ-action of $\widetilde{\mathrm{PR}}_{g,n}$ on $\mathcal{P}_{g,n}$. The first claim then follows from the first part of Corollary 6.14, and the second claim follows from the third part of the same Corollary. $\square$

When the coarse moduli scheme $\overline{M}_{g,n}$ is a variety of general type, we can deduce that two ϕ-compactified universal Jacobans $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ as in Corollary 6.15 are in fact non-isomorphic as Deligne–Mumford stacks (and not just over $\overline{\mathcal{M}}_{g,n}$). To prove this, we will employ the following lemma, in which we exploit the birational uniqueness of the Iitaka fibration, arguing similarly to [BFV12, Theorem 7.3].

Lemma 6.16. *If the Kodaira dimension $\kappa(\overline{M}_{g,n})$ equals $3g-3+n$, any isomorphism $\overline{\mathcal{J}}_{g,n}(\phi_1) \to \overline{\mathcal{J}}_{g,n}(\phi_2)$ commutes with the forgetful map to $\overline{\mathcal{M}}_{g,n}$ up to an automorphism that permutes the marked points.*

Proof. We claim that the Kodaira dimension $\kappa(\overline{J}_{g,n}(\phi_i))$ equals $3g-3+n$. In view of Iitaka's easy addition inequality ([Uen75, Theorem 6.12]), we have

$$\kappa(\overline{J}_{g,n}(\phi_i)) \leq \dim(\overline{M}_{g,n}) + \kappa(\pi^{-1}([C,p_i]))$$

for a general curve $(C,p_1,\ldots,p_n)$ of $\overline{\mathcal{M}}_{g,n}$. The reverse inequality

$$\kappa(\overline{J}_{g,n}(\phi_i)) \geq \kappa(\overline{M}_{g,n}) + \kappa(\pi^{-1}([C,p_i])) \quad \text{for } (C,p_1,\ldots,p_n) \text{ general in } \overline{\mathcal{M}}_{g,n}$$

follows from the Iitaka conjecture for abelian varieties (the main result of [Uen78]). Since we are assuming that $\kappa(\overline{M}_{g,n}) = \dim(\overline{M}_{g,n})$, the claim follows.

The forgetful morphism of coarse moduli schemes $p\colon \overline{J}_{g,n}(\phi_i) \to \overline{M}_{g,n}$ is an algebraic fibration (i.e. it is surjective and with geometrically connected fibers) of normal varieties with $\kappa(\overline{J}_{g,n}(\phi_i)) = \dim(\overline{M}_{g,n})$, and the Kodaira dimension of a general fiber of p equals zero, so p is the Iitaka fibration by [Uen75, Theorem 6.11].

Since the Iitaka fibration is a birational invariant, any isomorphism $\alpha\colon \overline{J}_{g,n}(\phi_1) \to \overline{J}_{g,n}(\phi_2)$ induces a birational map β such that the diagram

$$\begin{tikzcd}
\overline{J}_{g,n}(\phi_1) \arrow[r, "\alpha"] \arrow[d] & \overline{J}_{g,n}(\phi_2) \arrow[d] \\
\overline{M}_{g,n} \arrow[r, dashed, "\beta"] & \overline{M}_{g,n}
\end{tikzcd}$$

commutes. To conclude, it is enough to show that β extends to an automorphism of $\overline{\mathcal{M}}_{g,n}$ that lifts to an automorphism of $\overline{\mathcal{J}}_{g,n}(\phi_i)$ for $i = 1,2$.

The birational map β induces a rational map $\overline{M}_g \dashrightarrow \overline{M}_g$, which is the identity by [BFV12, Lemma 7.4]. Therefore, if C is a general curve of $\overline{\mathcal{M}}_g$, the birational map β induces an automorphism of the Fulton-MacPherson compactification $C[n]$ of the configuration space of n points on C (the fiber of $[C]$ under the forgetful map). By [Mas16, Proposition 4.11], the automorphism group of $C[n]$ is the symmetric group on n elements. We deduce that β is the automorphism of $\overline{\mathcal{M}}_{g,n}$ induced by a certain permutation of the marked points, and as such it lifts to an automorphism of $\overline{\mathcal{J}}_{g,n}(\phi_i)$. □

We conclude this section with the following corollary.

Corollary 6.17. *When the pair (g,n) is such that $\overline{M}_{g,n}$ is of general type, there exist nondegenerate $\phi_1 \in V_{g,n}^{e_1}$ and $\phi_2 \in V_{g,n}^{e_2}$ such that $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ are non-isomorphic.*

Proof. It is well-known that when $g = 0$ and when (g,n) belongs to the set (41), the moduli scheme $\overline{M}_{g,n}$ is uniruled, in particular it is not of general type. By combining this observation with Corollary 6.15 and Lemma 6.16, we deduce the statement. □

Remark 6.18. We believe that the problem of determining all pairs (g,n) such that $\overline{M}_{g,n}$ is of general type is still open. A well-known sufficient condition for $\overline{M}_{g,n}$ to be of general type is that $g \geq 24$. This was proven by Eisenbud-Harris-Mumford in [EH87] and [HM82] when $n = 0$. For $n > 0$ this follows from loc. cit. and from the Iitaka conjecture for curves fibrations, which was proven by Viehweg in [Vie77].

7. Appendix: Properties of reflexive sheaves

Here we collect some results about reflexive sheaves that we use in Section 6. The conditions G_n, R_n, and S_n we discuss are taken from [Har94]

Lemma 7.1. *Let $f\colon \mathcal{X} \to \mathcal{S}$ be a family of curves over a regular Deligne–Mumford stack $\mathcal{S}$ and F a family of torsion-free sheaves on $\mathcal{X}$. If $\mathcal{X}$ satisfies conditions G_1 and S_2, then F is reflexive.*

Proof. By [Har94, Theorem 1.9], it is enough to show that F satisfies S_2. In other words, we need to show that if $x \in \mathcal{X}$, then $\operatorname{depth} F_x \geq \min(2, \dim \mathcal{O}_{\mathcal{X},x})$. Given x, set $s := f(x)$. By hypothesis, $\mathcal{O}_{\mathcal{S},s}$ is regular so its maximal ideal $\mathfrak{m}_s$ is generated by a regular sequence $a_1, \ldots, a_d$ of length $d := \dim \mathcal{O}_{S,s}$. The images $f^*(a_1), \ldots, f^*(a_d) \in \mathcal{O}_{X,x}$ are regular on F_x by flatness, and the quotient module $F_x / f^*(a_1) \cdot F_x + \ldots + f^*(a_d) \cdot F_x$ is torsion-free by hypothesis. Pick an element $b \in \mathcal{O}_{X,x}$ that acts as a nonzero divisor on this quotient module. Then $f^*(a_1), \ldots, f^*(a_d), b$ is a F_x-regular sequence, so $\operatorname{depth} F_x \geq d + 1 = \dim \mathcal{O}_{\mathcal{X},x}$. □

Corollary 7.2. *With the hypothesis of Lemma 7.1, if G is a second family of rank 1 torsion-free sheaves on $\mathcal{X}$ and $\mathcal{Y} \subset \mathcal{X}$ is a closed substack of codimension ≥ 2 such that $F|_{\mathcal{X}-\mathcal{Y}}$ is isomorphic to $G|_{\mathcal{X}-\mathcal{Y}}$, then F is isomorphic to G.*

Proof. This is a special case of [Har94, Theorem 1.12]. (The result is stated for schemes, and we deduce the statement for stacks by passing to an étale cover.) □

Lemma 7.3. *For any nondegenerate $\phi \in V^d_{g,n}$, the fiber product $\overline{\mathcal{J}}_{g,n}(\phi) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n}$ satisfies G_1 and S_2.*

Proof. We will prove the stronger result that the fiber product is Cohen–Macaulay and satisfies R_1. Certainly $\overline{\mathcal{J}}(\phi) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{C}}_{g,n}$ is regular at every pair consisting of a line bundle and a point that is not a node (since $\overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{M}}_{g,n}$ is smooth at a line bundle and $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is smooth at a point that is not a node). The locus of such pairs has codimension 1, so we conclude that the fiber product satisfies R_1.

To complete the proof, observe that the deformation theory argument in [KP16, Lemma 3.33] shows the completed local ring of the fiber product is a power series ring over a ring that is the completed tensor product of rings of the form $k[[x,y,u,v]]/xy-uv$ or $k[[t]]$. In particular, the completed local ring is a power series ring over a complete intersection ring and hence is Cohen–Macaulay, i.e. satisfies S_d for all d. □

Finally we show that, on a nodal curve, the rule sending a rank 1 torsion-free sheaf F to its dual $F^\vee$ commutes with base change and hence defines a isomorphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{J}}_{g,n}(-\phi)$.

Lemma 7.4. *Let F be a family of rank 1 torsion-free sheaves on a family $C \to S$ of nodal curves. Then the formulation of the dual $F^\vee := \operatorname{Hom}(F, \mathcal{O}_C)$ commutes with base change. In other words, if $T \to S$ is a k-morphism, then the natural map*

$$F^\vee \otimes \mathcal{O}_{C_T} \to (F \otimes \mathcal{O}_{C_T})^\vee$$

is an isomorphism.

Proof. By [AK80, Theorem 1.10], it is enough to show that $\mathrm{Ext}^1(I_s, \mathcal{O}_{X_s})$ vanishes for every point $s \in S$, and because X_s is Gorenstein, vanishing is a special case of [Har94, Proposition 6.1]. □

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References

[AC87] Enrico Arbarello and Maurizio Cornalba, *The Picard groups of the moduli spaces of curves*, Topology **26** (1987), no. 2, 153–171. MR 895568

[AC98] ———, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 88, 97–127 (1999). MR 1733327

[AK80] Allen B. Altman and Steven L. Kleiman, *Compactifying the Picard scheme*, Adv. in Math. **35** (1980), no. 1, 50–112. MR 555258

[Ale04] Valery Alexeev, *Compactified Jacobians and Torelli map*, Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1241–1265. MR 2105707

[BFV12] Gilberto Bini, Claudio Fontanari, and Filippo Viviani, *On the birational geometry of the universal Picard variety*, Int. Math. Res. Not. IMRN (2012), no. 4, 740–780. MR 2889156

[BL92] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 1992. MR 1217487

[Cap94] Lucia Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), no. 3, 589–660. MR 1254134

[Cap08] ———, *Néron models and compactified Picard schemes over the moduli stack of stable curves*, Amer. J. Math. **130** (2008), no. 1, 1–47. MR 2382140

[CJM11] Renzo Cavalieri, Paul Johnson, and Hannah Markwig, *Wall crossings for double Hurwitz numbers*, Adv. Math. **228** (2011), no. 4, 1894–1937. MR 2836109

[CMKV15] Sebastian Casalaina-Martin, Jesse Leo Kass, and Filippo Viviani, *The local structure of compactified Jacobians*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 2, 510–542. MR 3335286

[Dud17] Bashar Dudin, *Compactified universal Jacobian and the Double Ramification Cycle*, International Mathematics Research Notices (2017).

[EH87] David Eisenbud and Joe Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90** (1987), no. 2, 359–387. MR 910206

[EP16] Eduardo Esteves and Marco Pacini, *Semistable modifications of families of curves and compactified Jacobians*, Ark. Mat. **54** (2016), no. 1, 55–83. MR 3475818

[Est01] Eduardo Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3045–3095. MR 1828599

[GP03] T. Graber and R. Pandharipande, *Constructions of nontautological classes on moduli spaces of curves*, Michigan Math. J. **51** (2003), no. 1, 93–109. MR 1960923

[GZ14] Samuel Grushevsky and Dmitry Zakharov, *The zero section of the universal semiabelian variety and the double ramification cycle*, Duke Math. J. **163** (2014), no. 5, 953–982. MR 3189435

[Har94] Robin Hartshorne, *Generalized divisors on Gorenstein schemes*, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), vol. 8, 1994, pp. 287–339. MR 1291023

[HM82] Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, With an appendix by William Fulton. MR 664324

[Hol17] David Holmes, *Extending the double ramification cycle by resolving the Abel–Jacobi map*, To appear, 2017.

[KM97] Seán Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213. MR 1432041

[Kou91] Alexis Kouvidakis, *The Picard group of the universal Picard varieties over the moduli space of curves*, J. Differential Geom. **34** (1991), no. 3, 839–850. MR 1139648

[KP16] Jesse Leo Kass and Nicola Pagani, *Extensions of the universal theta divisor*, https://arxiv.org/abs/1507.03564; updated version available on http://pcwww.liv.ac.uk/ pagani/publications.htm, 2016.

[Mas16] Alex Massarenti, *On the biregular geometry of Fulton–MacPherson configuration spaces*, https://arxiv.org/abs/1603.06991, 2016.

[Mel09] Margarida Melo, *Compactified Picard stacks over $\overline{\mathcal{M}}_g$*, Math. Z. **263** (2009), no. 4, 939–957. MR 2551606

[Mel11] ———, *Compactified Picard stacks over the moduli stack of stable curves with marked points*, Adv. Math. **226** (2011), no. 1, 727–763. MR 2735773

[Mel16] ———, *Compactifications of the universal Jacobian over curves with marked points*, https://arxiv.org/abs/1509.06177v3, 2016.

[Mes87] Nicole Mestrano, *Conjecture de Franchetta forte*, Invent. Math. **87** (1987), no. 2, 365–376. MR 870734

[Mil86] J. S. Milne, *Jacobian varieties*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 167–212. MR 861976

[Nam76] Yukihiko Namikawa, *A new compactification of the Siegel space and degeneration of Abelian varieties. I*, Math. Ann. **221** (1976), no. 3, 201–241.

[OS79] Tadao Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90. MR 536936

[Pan96] Rahul Pandharipande, *A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles*, J. Amer. Math. Soc. **9** (1996), no. 2, 425–471. MR 1308406

[Sim94] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 47–129. MR 1307297

[SSV08] S. Shadrin, M. Shapiro, and A. Vainshtein, *Chamber behavior of double Hurwitz numbers in genus 0*, Adv. Math. **217** (2008), no. 1, 79–96. MR 2357323

[Tha96] Michael Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723. MR 1333296

[Uen75] Kenji Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, Vol. 439, Springer-Verlag, Berlin-New York, 1975, Notes written in collaboration with P. Cherenack. MR 0506253

[Uen78] ———, *On algebraic fibre spaces of abelian varieties*, Math. Ann. **237** (1978), no. 1, 1–22. MR 506652

[Vie77] Eckart Viehweg, *Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension one*, Compositio Math. **35** (1977), no. 2, 197–223. MR 0569690

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