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Scheme invariants in ϕ^4 theory in four dimensions

I. Jack[a] and C. Poole[a,b]

[a]Dept. of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK

[b]CP^3 Origins, University of Southern Denmark, Campusvej 55, 5320 Odense M, Denmark

E-mail: dij@liverpool.ac.uk, c.poole@cp3.sdu.dk

Abstract

We provide an analysis of the structure of renormalisation scheme invariants for the case of ϕ^4 theory, relevant in four dimensions. We give a complete discussion of the invariants up to four loops and include some partial results at five loops, showing that there are considerably more invariants than one might naively have expected. We also show that one-vertex reducible contributions may consistently be omitted in a well-defined class of schemes which of course includes $\overline{\text{MS}}$.

1 Introduction

Beyond leading order it is well-known that the values of β-function coefficients are scheme-dependent, i.e. depend on the renormalisation scheme. On the other hand one would expect that statements with physical meaning should be expressible in a scheme-independent way. A notable recent example is the issue of the existence of an a-function; i.e. a function which generates the β-functions through a gradient-flow equation. For this to be feasible, the β-function coefficients must satisfy a set of consistency conditions, which must clearly be scheme-invariant; as has been verified for various field theories in three [1–3], four [4] and six [5] dimensions. The number of scheme-independent combinations at each loop order would naively be expected to be given by the difference of the number of β-function coefficients and the number of independent variations of coefficients; however the number of independent invariants actually found is considerably larger. This may be understood in a pragmatic way in terms of the structure of the expressions for the scheme changes of the coefficients; however a possibly deeper insight is afforded by Hopf algebra considerations. A general discussion of scheme dependence with a particular focus on one-particle reducible (1PR) structures was recently given in Ref. [7], and here the study of scheme-invariant combinations was initiated with reference to the $\mathcal{N} = 1$ scalar-fermion theory. The present paper is to be seen as a companion to a forthcoming article [8] where the ideas of scheme invariance and the relation to Hopf algebra will be explored in general and also exemplified for the case of ϕ^3 theory in six dimensions; our purpose here is to extend the discussion to ϕ^4 theory in four dimensions. We shall summarise results of Ref. [8] where necessary to render the present discussions self-contained. An additional complication in ϕ^4 theory is due to the existence of one-vertex reducible (1VR) graphs. These are one-particle irreducible (1PI) graphs which may be separated into two distinct portions by severing a vertex. They have no simple poles when using minimal subtraction and dimensional regularisation, and hence a vanishing β-function coefficient in this scheme. It would be convenient to be able to omit these coefficients from our considerations. Indeed we shall show that although we may if desired include such coefficients, we may also consistently confine our attention to a well-defined subset of schemes in whch these coefficients are absent.

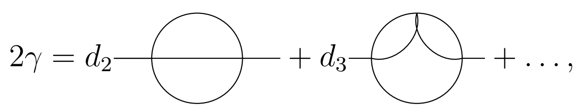
The structure of the paper is as follows: in Section 2 we introduce the ϕ^4 theory and give the results at one, two and three loops. Section 3 contains our main results, namely the full set of four-loop scheme invariants and a partial five-loop calculation. In Section 4 we show that one may straightforwardly restrict attention to a set of renormalisation schemes in which 1VR contributions are absent. In Section 5 we set our results for scheme invariants within the Hopf algebra framework. Finally we summarise our results and give pointers to future work in the Conclusion. Some general theory which is developed in detail in Ref. [8] and which underpins our work is summarised in Appendix A. Appendix B lists some Hopf algebraic cocommutative coproducts which arise in Section 5 but were too complex for inclusion in the main text. Finally, in Appendix C we show how to express scheme changes in terms of differential operators acting on the β-function coefficients.

2 One, two and three loop calculations

In this section we establish our notation and obtain the invariants up to three loop order (the first non-trivial case for ϕ^4 theory). We consider the action

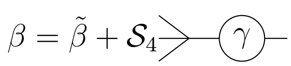
$$S = \int d^dx \left(\tfrac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \tfrac{1}{2}m^2\phi^i\phi^i - \tfrac{1}{4!}g_{ijkl}\phi^i\phi^j\phi^k\phi^l\right). \qquad (2.1)$$

for the case $d = 4$ which corresponds to a renormalisable theory. The anomalous dimension γ_{ij} may be expressed as a series of two-point 1PI diagrams with 4-point vertices connected by internal lines representing the contractions of couplings. Up to three loops we have



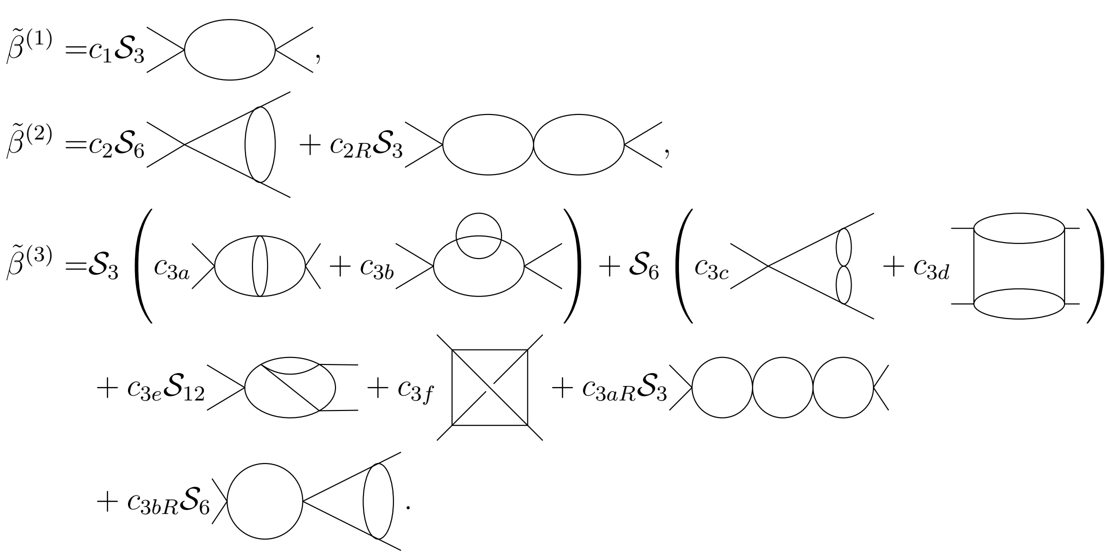
(2.2)

where here and elsewhere we suppress indices as far as possible. We consistently neglect contributions from "snail" diagrams in which a bubble is attached to a propagator. Such contributions do not arise in minimal subtraction and will not be generated by redefinitions if the redefinitions themselves do not include such diagrams. The β-function β_{ijkl} may then be decomposed into 1PI pieces together with one-particle reducible pieces determined by the anomalous dimension, in the form:



(2.3)

with $\tilde{\beta}$ denoting the 1PI contributions and $\mathcal{S}_4$ the sum over the four terms where γ is attached to each external line. Up to three loops the contributions to $\tilde{\beta}$ are given by



(2.4)

For later convenience we introduce the notation that g^λ_{3a} is the graph corresponding to c_{3a}, and g^γ_2 is the graph corresponding to d_2, etc. We note that in Eq. (2.4) the graph g^λ_{3f} is primitive in that it has no divergent subgraph.

Changes of renormalisation scheme are well-known to be equivalent to redefinitions of the coupling, which may be parametrised as [7]

$$g'_{ijkl} = (g + f(g))_{mnpq} C_{mi} C_{nj} C_{pk} C_{ql} \tag{2.5}$$

where

$$C(g) = (1 - 2c(g))^{-\frac{1}{2}}. \tag{2.6}$$

After a scheme change the β-function and anomalous dimension are represented by a similar diagrammatic series, but with modified coefficients given by

$$c_X \to c'_X = c_X + \delta c_X, \quad d_X \to d'_X = d_X + \delta d_X, \tag{2.7}$$

where c_X and d_X represent coefficients of generic diagrams in series such as Eqs. (2.4), (2.2) respectively. As explained in the Appendix (which in turn is a summary of the discussion in Ref. [8]), it is useful to parametrise the scheme change by v defined implicitly by Eq. (A.4). We assume that v is parametrised in a similar way to Eqs. (2.5), (2.6), with analogues of $f(g)$, $c(g)$ given by similar diagrammatic series to those for the β-function and anomalous dimension, but with $c_X \to \delta_X$ and $d_X \to \epsilon_X$.

At one and two loops we have

$$\delta c_1 = \delta d_1 = \delta c_2 = \delta c_{2R} = \delta d_2 = 0. \tag{2.8}$$

At three loops we find using Eqs. (A.10), (A.11)

$$\begin{aligned} \delta c_{3a} = 2X^{\lambda\lambda}_{2,1} + 2X^{\lambda\lambda}_{1,2R}, \quad \delta c_{3b} = 2X^{\gamma\lambda}_{2,1}, \quad \delta c_{3c} = 2X^{\lambda\lambda}_{1,2} + 2X^{\lambda\lambda}_{2R,1}, \\ \delta c_{3d} = 2X^{\lambda\lambda}_{1,2}, \quad \delta c_{3e} = 0, \quad \delta c_{3f} = 0, \\ \delta c_{3aR} = X^{\lambda\lambda}_{1,2R}, \quad \delta c_{3bR} = 2X^{\lambda\lambda}_{1,2R}, \quad \delta d_3 = 6X^{\lambda\gamma}_{1,2}. \end{aligned} \tag{2.9}$$

Here

$$X^{\lambda\lambda}_{X,Y} = c_X \delta_Y - \delta_X c_Y, \quad X^{\gamma\lambda}_{X,Y} = d_X \delta_Y - \epsilon_X c_Y, \tag{2.10}$$

with corresponding definitions for $X^{\lambda\gamma}_{X,Y}$, $X^{\gamma\gamma}_{X,Y}$ when needed. We see from Eq. (A.10) that the coefficients appearing in $X^{\lambda\lambda}_{2,1}$ etc should in principle be "hatted" quantities defined according to Eq. (A.11); but at this level there is no distinction between the two, i.e. $\hat{c}_1 = c_1$, $\hat{c}_2 = c_2$, $\hat{d}_2 = d_2$. Note that c_{3e} and c_{3f} are individually invariant–which in the case of c_{3f} follows immediately from the fact that it corresponds to a primitive graph. In deriving invariant combinations of coefficients it is important to note that

$$X^{\lambda\lambda}_{X,Y} = -X^{\lambda\lambda}_{Y,X}, \quad X^{\lambda\gamma}_{X,Y} = -X^{\gamma\lambda}_{Y,X}, \quad X^{\gamma\gamma}_{X,Y} = -X^{\gamma\gamma}_{Y,X}. \tag{2.11}$$

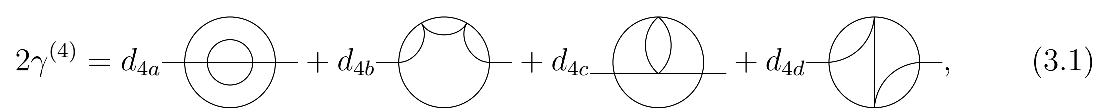
We now start the search for these invariant combinations of coefficients at lowest (three-loop) order. *A priori* since at this order there are nine three-loop coefficients and five variations δ_1, δ_1^2, δ_2, ϵ_2, δ_{2R}, one's naive expectation would be $9-5=4$ invariants. However, the variations on the right-hand side of Eq. (2.9) are expressed in terms of only three independent quantities, $X^{\lambda\lambda}_{1,2}$, $X^{\gamma\lambda}_{2,1}$ and $X^{\lambda\lambda}_{1,2R}$, and so in fact we should have $9-3=6$ independent invariant combinations of three-loop coefficients. Indeed, we easily find from Eqs. (2.9) that

$$I_1^{(3)} = c_{3a} + c_{3d} - 2c_{3aR}, \quad I_2^{(3)} = 2c_{3aR} - c_{3bR},$$
$$I_3^{(3)} = c_{3a} + c_{3c}, \quad I_4^{(3)} = 3c_{3b} + d_3, \tag{2.12}$$

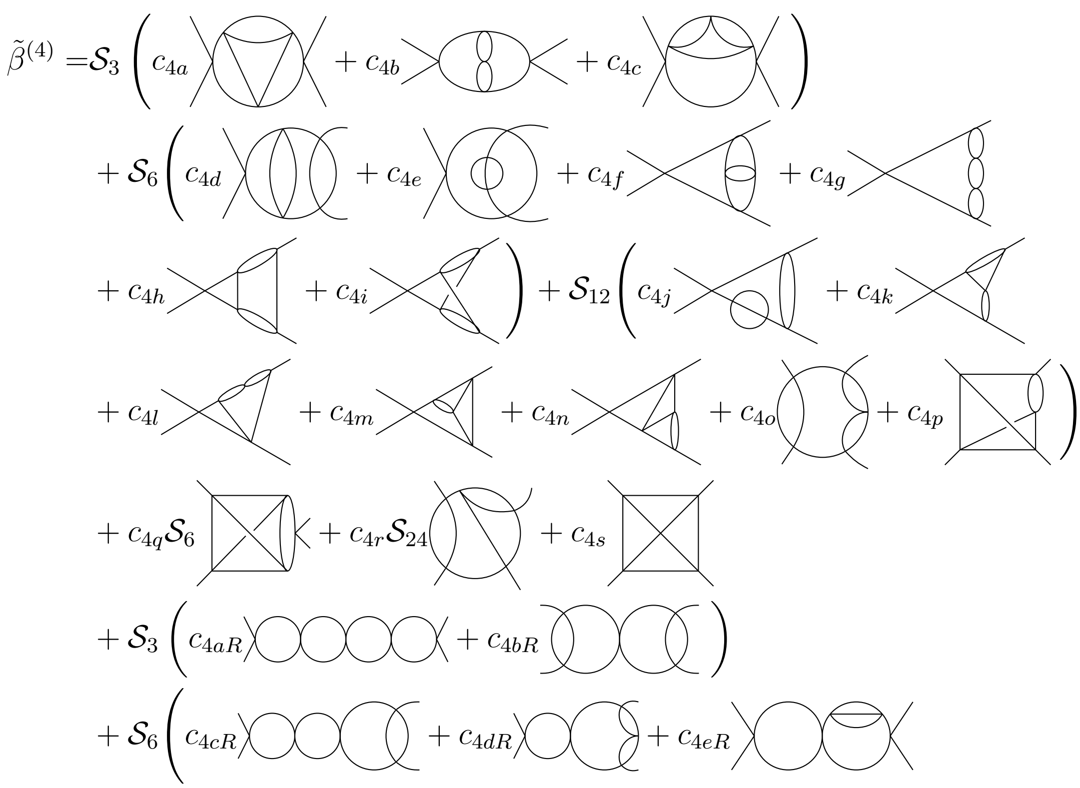
are four independent invariant combinations (making a total of six invariants with the individually invariant c_{3e} and c_{3f}).

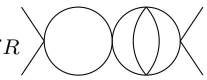
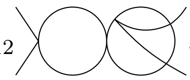
3 The four and five loop calculations

In this section we examine the issue of scheme invariants comprehensively at four loops and partially (due to increased calculational complexity) at five loops. The full list of four loop diagrams was presented in Ref. [6]. The anomalous dimension is given at this order by



while the 1PI part of the β-function will be parametrised as



$+ c_{4fR}$  $\Big) + c_{4gR}\mathcal{S}_{12}$  . (3.2)

In Eq. (3.2) the graph g_{4s}^{λ} is the only primitive one.

We find (again using Eqs. (A.10), (A.11)) variations of the four-loop cofficients given by

$$\begin{aligned}
\delta c_{4a} =& 4\hat{X}^{\lambda\lambda}_{1,3a} + 4\hat{X}^{\lambda\lambda}_{3e,1} + 4X^{\lambda\lambda}_{2,2R},\\
\delta c_{4b} =& -\delta c_{4f} = 2\hat{X}^{\lambda\lambda}_{1,3a} + 2\hat{X}^{\lambda\lambda}_{3c,1},\\
\delta c_{4c} =& 6\hat{X}^{\lambda\lambda}_{1,3b} + 2\hat{X}^{\gamma\lambda}_{3,1},\\
\delta c_{4d} =& 2\hat{X}^{\lambda\lambda}_{1,3a} + 2\hat{X}^{\lambda\lambda}_{1,3bR} + 2\hat{X}^{\lambda\lambda}_{3d,1},\\
\delta c_{4e} =& 2\hat{X}^{\lambda\lambda}_{3b,1} + 2X^{\gamma\lambda}_{2,2},\\
\delta c_{4g} =& 3\hat{X}^{\lambda\lambda}_{1,3c} + 2\hat{X}^{\lambda\lambda}_{3aR,1} + 2X^{\lambda\lambda}_{2R,2},\\
\delta c_{4h} =& \delta c_{4i} = 2\hat{X}^{\lambda\lambda}_{3d,1} + 2\hat{X}^{\lambda\lambda}_{1,3e},\\
\delta c_{4j} =& \hat{X}^{\lambda\lambda}_{1,3b} + X^{\gamma\lambda}_{2,2},\\
\delta c_{4k} =& 2\hat{X}^{\lambda\lambda}_{1,3c} + \hat{X}^{\lambda\lambda}_{1,3e} + 2\hat{X}^{\lambda\lambda}_{3bR,1},\\
\delta c_{4l} =& 2\hat{X}^{\lambda\lambda}_{1,3e} + 2\hat{X}^{\lambda\lambda}_{3c,1} + 2X^{\lambda\lambda}_{2R,2},\\
\delta c_{4m} =& \delta c_{4n} = \delta c_{4s} = 0,\\
\delta c_{4o} =& \hat{X}^{\lambda\lambda}_{1,3c} + 2\hat{X}^{\lambda\lambda}_{1,3d} + X^{\lambda\lambda}_{2R,2},\\
\delta c_{4p} =& -\delta c_{4q} = \hat{X}^{\lambda\lambda}_{1,3f},\\
\delta c_{4r} =& 2\hat{X}^{\lambda\lambda}_{1,3d} + \hat{X}^{\lambda\lambda}_{1,3e},
\end{aligned} \tag{3.3}$$

for the one-vertex irreducible coefficients,

$$\begin{aligned}
\delta c_{4aR} =& 2\hat{X}^{\lambda\lambda}_{1,3aR},\\
\delta c_{4bR} =& 4\hat{X}^{\lambda\lambda}_{1,3bR},\\
\delta c_{4cR} =& 2\hat{X}^{\lambda\lambda}_{1,3aR} + \hat{X}^{\lambda\lambda}_{1,3bR},\\
\delta c_{4dR} =& 2\hat{X}^{\lambda\lambda}_{1,3bR},\\
\delta c_{4eR} =& 2X^{\gamma\lambda}_{2,2R},\\
\delta c_{4fR} =& 2\hat{X}^{\lambda\lambda}_{1,3aR} + \hat{X}^{\lambda\lambda}_{3bR,1} + 2X^{\lambda\lambda}_{2,2R},\\
\delta c_{4gR} =& 2\hat{X}^{\lambda\lambda}_{1,3bR} + 2X^{\lambda\lambda}_{2,2R}
\end{aligned} \tag{3.4}$$

for the 1VR coefficients and

$$\delta d_{4a} = 0,$$

$$\begin{aligned}\delta d_{4b} =&3\hat{X}^{\lambda\gamma}_{1,3}+6X^{\lambda\gamma}_{2R,2},\\ \delta d_{4c} =&2\hat{X}^{\lambda\gamma}_{1,3}+6X^{\lambda\gamma}_{2,2},\\ \delta d_{4d} =&4\hat{X}^{\lambda\gamma}_{1,3}+6X^{\lambda\gamma}_{2,2}\end{aligned}\tag{3.5}$$

for the anomalous dimension coefficients. At this level, in contrast to the earlier three-loop calculation, we do need to distinguish "hatted" from "unhatted" quantities. The $\hat{X}^{\lambda\lambda}$ quantities are defined by

$$\hat{X}^{\lambda\lambda}_{X,Y}=\hat{c}_X\delta_Y-\delta_X\hat{c}_Y,\tag{3.6}$$

in other words as for $X^{\lambda\lambda}$ in Eq. (2.10) but with the β-function quantities $c_{X,Y}$ replaced by hatted quantities $\hat{c}_{X,Y}$. Similar definitions apply to $X^{\lambda\gamma}$, etc, but with $d_{X,Y}$ replaced by hatted quantities $\hat{d}_{X,Y}$ where relevant. Here again $\hat{c}_1=c_1$, $\hat{c}_2=c_2$, $\hat{d}_2=d_2$, while the quantities $\hat{c}_{3a}$ etc are defined by

$$\hat{c}_{3a}=c_{3a}+\tfrac{1}{2}\delta c_{3a}\tag{3.7}$$

with δc_{3a} as defined as in Eq. (2.9), and similar expressions for $\hat{c}_{3b}$ etc, and also $\hat{d}_3$. The additional terms in the hatted quantities derive from the first Lie derivative term on the right-hand side of Eq. (A.10).

Now again we look for invariants at this order. Note that c_{4m}, c_{4n}, c_{4s}, d_{4a} are individually invariant–which again in the case of c_{4s} follows immediately from the fact that it corresponds to a primitive graph. There are thirty four-loop coefficients whose variations are given in Eqs. (3.3), (3.4), (3.5); and eighteen variations up to the three-loop level, namely $\delta_{3a-3f,3aR,3bR}$, ϵ_3, δ_1^3, $\delta_1\delta_2$, $\delta_1\delta_{2R}$, $\delta_1\epsilon_2$, δ_2, ϵ_2, δ_{2R}, δ_1^2, δ_1. We would therefore naively expect $30-18=12$ invariants. However, the variations on the right-hand sides of Eqs. (3.3), (3.4), (3.5) are expressed in terms of only twelve independent $X/\hat{X}$combinations and therefore the correct expectation is $30-12=18$ invariants. Indeed, together with the four individually invariant coefficients c_{4m}, c_{4n}, c_{4s}, d_{4a} we find the following fourteen linear invariant combinations:

$$\begin{aligned}I^{(4)L}_1 =&c_{4h}-c_{4i},\\ I^{(4)L}_2 =&c_{4b}+c_{4f},\\ I^{(4)L}_3 =&c_{4a}+2c_{4f}+2c_{4l},\\ I^{(4)L}_4 =&c_{4l}+2c_{4o}-2c_{4r}-c_{4bR}+2c_{4gR},\\ I^{(4)L}_5 =&c_{4c}+3c_{4e}+d_{4c},\\ I^{(4)L}_6 =&c_{4d}+c_{4f}-c_{4k}+c_{4r}-c_{4bR},\\ I^{(4)L}_7 =&c_{4b}-c_{4d}+c_{4g}-c_{4o}+c_{4cR}+\tfrac{1}{2}c_{4gR},\\ I^{(4)L}_8 =&c_{4h}-c_{4k}+2c_{4o}-c_{4r}-c_{4bR}+c_{4gR},\\ I^{(4)L}_9 =&3c_{4e}+6c_{4j}+4d_{4c}-2d_{4d},\\ I^{(4)L}_{10} =&c_{4p}+c_{4q},\end{aligned}$$

$$
\begin{aligned}
I_{11}^{(4)L} =& 2d_{4b} + 3d_{4c} - 3d_{4d} + 6c_{4eR},\\
I_{12}^{(4)L} =& 4c_{4aR} - 4c_{4cR} + c_{4bR},\\
I_{13}^{(4)L} =& c_{4bR} - 2c_{4dR},\\
I_{14}^{(4)L} =& c_{4cR} - c_{4bR} + c_{4gR} - c_{4fR}.
\end{aligned}
\tag{3.8}
$$

We call these 18 invariants "linear". We also find three "quadratic" invariants

$$
\begin{aligned}
I_{1}^{(4)Q} =& c_1(2d_{4c} - d_{4d}) + 3c_2c_{3b} + 3d_2c_{3d},\\
I_{2}^{(4)Q} =& c_1c_{4eR} - c_{2R}c_{3b} - d_2c_{3bR},\\
I_{3}^{(4)Q} =& c_1(c_{4dR} - c_{4gR}) + c_2c_{3aR} - c_{2R}c_{3d},
\end{aligned}
\tag{3.9}
$$

which are a consequence of the relations

$$
\begin{aligned}
c_2X_{1,2}^{\lambda\gamma} - d_2X_{1,2}^{\lambda\lambda} =& c_1X_{2,2}^{\lambda\gamma},\\
c_{2R}X_{1,2}^{\lambda\gamma} - d_2X_{1,2R}^{\lambda\lambda} =& c_1X_{2R,2}^{\lambda\gamma},\\
c_2X_{1,2R}^{\lambda\lambda} - c_{2R}X_{1,2}^{\lambda\lambda} =& c_1X_{2,2R}^{\lambda\lambda},
\end{aligned}
\tag{3.10}
$$

respectively. Altogether we have found twenty-one invariants, considerably more than (in fact almost double) the twelve which might naively have been expected.

We note that one may derive a fourth identity

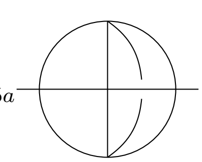
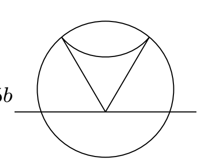
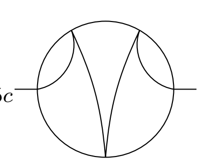
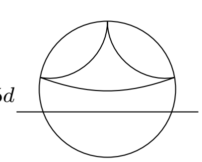
$$
c_2X_{2R,2}^{\lambda\gamma} + d_2X_{2,2R}^{\lambda\lambda} = c_{2R}X_{2,2}^{\lambda\gamma}
\tag{3.11}
$$

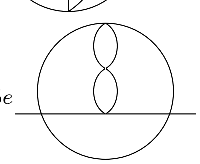
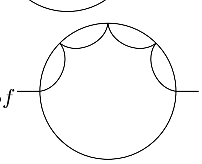
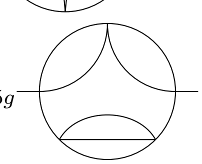
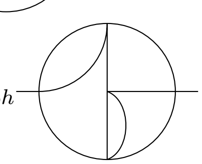
which leads to an invariant

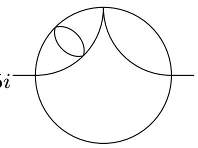
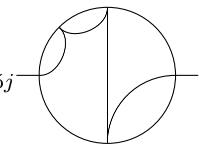
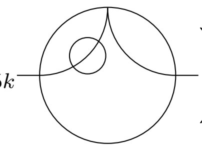
$$
I_{4}^{(4)Q} = d_2(c_{4bR} - 2c_{4gR}) + 2c_2c_{4eR} + \tfrac{2}{3}c_{2R}(2d_{4c} - d_{4d});
\tag{3.12}
$$

but in fact Eq. (3.11) may be derived from linear combinations of the identities in Eq. (3.10) and correspondingly $I_4^{(4)Q}$ is a linear combination of invariants already found in Eqs. (3.8), (3.9).

We now proceed to a very partial five-loop calculation. The number of diagrams at five loops is dauntingly high, so we have not undertaken a complete calculation of all the invariants. A natural place to start is with the five-loop anomalous dimension which has only eleven terms:

$2\gamma^{(5)} = d_{5a}$  $+\, d_{5b}$  $+\, d_{5c}$  $+\, d_{5d}$ 

$+\, d_{5e}$  $+\, d_{5f}$  $+\, d_{5g}$  $+\, \mathcal{S}_2\Big(d_{5h}$ 

$+ d_{5i}$  $+ d_{5j}$  $+ d_{5k}$  $\Bigg).$ (3.13)

We find from Eqs. (A.10), (A.11) that the variations of the coefficients in Eq. (3.13) are given by

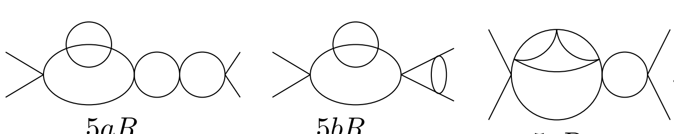
$$
\begin{aligned}
\delta d_{5a} =& 2\hat{X}^{\lambda\gamma}_{3f,2}, \\
\delta d_{5b} =& 12\hat{X}^{\lambda\gamma}_{3e,2} + 4\hat{X}^{\lambda\gamma}_{1,4c} + 4\hat{X}^{\lambda\gamma}_{2,3}, \\
\delta d_{5c} =& 6\hat{X}^{\lambda\gamma}_{3e,2} + 4\hat{X}^{\lambda\gamma}_{1,4d} + 4\hat{X}^{\lambda\gamma}_{2,3}, \\
\delta d_{5d} =& 6\hat{X}^{\lambda\gamma}_{1,4a} + 3\hat{X}^{\gamma\gamma}_{3,2}, \\
\delta d_{5e} =& 6\hat{X}^{\lambda\gamma}_{3c,2} + 2\hat{X}^{\lambda\gamma}_{1,4c} + 2\hat{X}^{\lambda\gamma}_{2R,3}, \\
\delta d_{5f} =& 6\hat{X}^{\lambda\gamma}_{3aR,2} + 4\hat{X}^{\lambda\gamma}_{1,4b} + 3\hat{X}^{\lambda\gamma}_{2R,3}, \\
\delta d_{5g} =& 2\hat{X}^{\lambda\gamma}_{1,4a} + \hat{X}^{\gamma\gamma}_{2,3}, \\
\delta d_{5h} =& 6\hat{X}^{\lambda\gamma}_{3d,2} + 3\hat{X}^{\lambda\gamma}_{3e,2} + 2\hat{X}^{\lambda\gamma}_{1,4c} + 2\hat{X}^{\lambda\gamma}_{1,4d} + 2X^{\lambda\gamma}_{2,3}, \\
\delta d_{5i} =& 3\hat{X}^{\lambda\gamma}_{3a,2} + 3\hat{X}^{\lambda\gamma}_{3bR,2} + 2\hat{X}^{\lambda\gamma}_{1,4b} + \hat{X}^{\lambda\gamma}_{1,4c} + 2\hat{X}^{\lambda\gamma}_{2,3}, \\
\delta d_{5j} =& 3\hat{X}^{\lambda\gamma}_{3c,2} + 3\hat{X}^{\lambda\gamma}_{3bR,2} + 2\hat{X}^{\lambda\gamma}_{1,4b} + 2\hat{X}^{\lambda\gamma}_{1,4d} + \hat{X}^{\lambda\gamma}_{2,3} + 2\hat{X}^{\lambda\gamma}_{2R,3}, \\
\delta d_{5k} =& 3\hat{X}^{\lambda\gamma}_{3b,2} + 2\hat{X}^{\lambda\gamma}_{1,4a} + 2\hat{X}^{\gamma\gamma}_{2,3}.
\end{aligned}
\tag{3.14}
$$

The hatted X-type terms are defined in a similar manner to Eq. (3.6), i.e. by replacing β-function quantities $c_{X,Y}$ and $d_{X,Y}$, in Eq. (2.10) by hatted quantities $\hat{c}_{X,Y}$, and $\hat{d}_{X,Y}$. The hatted coefficients are in turn defined in terms of the corresponding unhatted quantities in a manner similar to Eq. (3.7). However, in the case of four-loop anomalous dimension coefficients, we need to define

$$\hat{d}_{4b} = d_{4b} + \tfrac{1}{2}\delta' d_{4b}, \tag{3.15}$$

where $\delta' d_{4b}$ (and simlarly $\delta'_{4c,d}$) are defined as in Eq. (3.5), but with hatted replaced by unhatted quantities. This is simply a consequence of Eq. (A.11), where we see that the first-order term is of the same form as the RHS of Eq. (A.10), but with the hat removed. This feature has not been apparent in our calculations until now simply because there was no difference between the hatted and unhatted quantities appearing in the three-loop variations in Eq. (2.9).

However it proves impossible to construct an invariant combination purely of anomalous dimension coefficients and in fact we need to include some 1VR four-point contributions, depicted below:



$5aR$ $\quad$ $5bR$ $\quad$ $5cR$ $\quad$. (3.16)

The variations of the corresponding coefficients are given by

$$\begin{aligned}\delta c_{5aR} =&\hat{X}^{\lambda\lambda}_{1,4eR} + 2\hat{X}^{\gamma\lambda}_{2,3aR},\\ \delta c_{5bR} =&2\hat{X}^{\lambda\lambda}_{1,4eR} + 2\hat{X}^{\gamma\lambda}_{2,3bR},\\ \delta c_{5cR} =&6\hat{X}^{\lambda\lambda}_{1,4eR} + 2\hat{X}^{\gamma\lambda}_{3,2R},\end{aligned} \tag{3.17}$$

where the hatted quantities are again defined in a similar way to Eq. (3.6). Note that (as we see in Eq. (3.4)) the variation δc_{4eR} is expressed in terms of unhatted quantities, so there is no need to invoke the modified δ' here. Naively, no linear invariant constructed purely from the coefficients in Eqs. (3.13), (3.17) would be expected–there are 16 independent variations in Eq. (3.14) and only 14 coefficients. However, it turns out that there are three unexpected relations among the invariance conditions, resulting in just one five-loop linear invariant formed using only anomalous dimension and 1VR coefficients, namely

$$I^{(5)L}_1 = d_{5b} - 2d_{5c} - 2d_{5e} - 2d_{5f} + 4d_{5j} - 6c_{5aR} + 6c_{5bR} - c_{5cR}. \tag{3.18}$$

In addition, we also find several quadratic invariants, namely

$$\begin{aligned}I^{(5)Q}_1 =&c_1 d_{5a} + 2d_2 c_{4p} + c_{3b}c_{3f},\\ I^{(5)Q}_2 =&2c_1(d_{5g} - d_{5k}) - d_2 c_{4c} + d_3 c_{3b},\\ I^{(5)Q}_3 =&c_1(2c_{5aR} - c_{5bR}) - 2d_2(2c_{4aR} - c_{4cR}) - c_{3b}(2c_{3aR} - c_{3bR}),\\ I^{(5)Q}_4 =&c_1(d_{5d} - 3d_{5g}) - \tfrac{3}{2}d_2 J - \tfrac{1}{2}d_3^2,\\ I^{(5)Q}_5 =&c_1(3c_{5bR} - c_{5cR}) + \tfrac{1}{2}c_{2R}J + 6d_2(c_{4aR} - c_{4cR}) + d_3 c_{3bR},\\ I^{(5)Q}_6 =&c_1(d_{5c} + 2d_{5e} - 2d_{5h}) - 6d_2(c_{4b} - c_{4d} - 2c_{4aR} + 2c_{4cR}) + c_{2R}J\\ &- 2d_3(c_{3c} - c_{3d}),\\ I^{(5)Q}_7 =&c_1(d_{5b} - 2d_{5e}) + 3d_2(c_{4l} - 2c_{4o} + 2c_{4r})\\ &+ (c_2 - c_{2R})J + 6c_{3b}c_{3e} + 2c_{3c}d_3,\\ I^{(5)Q}_8 =&c_1(d_{5e} + d_{5f} - 2d_{5i}) - 3d_2(c_{4b} - 3c_{4aR} + 2c_{4cR})\\ &+ \tfrac{1}{4}(5c_{2R} - 4c_2)J + d_3(c_{3a} - c_{3c} - c_{3aR} + c_{3bR}),\end{aligned} \tag{3.19}$$

where J denotes the frequently occurring combination defined by

$$J = 2c_{4c} + 3c_{4e} - 6c_{4j}. \tag{3.20}$$

These owe their existence to relations like

$$c_1\hat{X}^{\lambda\gamma}_{3a,2} + d_2\hat{X}^{\lambda\lambda}_{1,3a} + \hat{c}_{3a}\hat{X}^{\gamma\lambda}_{2,1} = 0 \tag{3.21}$$

together with similar relations for $3b$–$3f$, $3aR$, $3bR$; together with

$$c_1\hat{X}^{\lambda\gamma}_{2,3} + \hat{d}_3\hat{X}^{\lambda\lambda}_{1,2} + c_2\hat{X}^{\gamma\lambda}_{3,1} =0,$$

$$
\begin{aligned}
c_1\hat{X}^{\lambda\gamma}_{2R,3} + \hat{d}_3\hat{X}^{\lambda\lambda}_{1,2R} + c_{2R}\hat{X}^{\gamma\lambda}_{3,1} =&0,\\
c_1\hat{X}^{\gamma\gamma}_{2,3} + \hat{d}_3\hat{X}^{\lambda\gamma}_{1,2} + d_2\hat{X}^{\gamma\lambda}_{3,1} =&0.
\end{aligned} \tag{3.22}
$$

The number of invariants is as expected, since the eleven relations of the form Eqs. (3.21), (3.22) reduce the effective number of independent variations from 16 to 5, yielding 14-5=9 invariants (both quadratic and linear).

In the absence of a complete calculation, one may estimate the total number of invariants which will be found at five loops. The five-loop β-function was calculated in Ref. [9], and contained contributions from 124 1PI 5-loop 4-point diagrams and 11 5-loop 2-point anomalous dimension diagrams, making 135 coefficients in total[1]. There are 67 independent variations at 5 loops, implying a naive expectation of 135-67=68 linear invariants. On the other hand there are 57 5-loop X-type terms (some of which of course appear in Eq. (3.14)), which following the argument explained at four loops implies an actual total of 135-57=78 linear invariants. But furthermore there are altogether 27 identities of the form Eqs. (3.21), (3.22), constructed from the one one-loop quantity, the three two-loop quantities and the nine three-loop quantities. This implies an additional 27 quadratic invariants making 105 invariants in total. As at four loops, there are considerably more invariants than might have been expected. One may also speculate on the possible existence of higher-order invariants based on higher-order Jacobi-style identities.

4 One-vertex reducible graphs

In this section we briefly discuss the issue of β-function contributions from one-particle reducible (1VR) graphs. It is well-known that no such contributions arise using minimal subtraction within dimensional regularisation ($\overline{\text{MS}}$), as may easily be established by consideration of the diagram-by-diagram subtraction process. It would be convenient if when considering scheme redefinitions one could restrict attention to schemes which have the same feature. In fact, if we start from a scheme such as $\overline{\text{MS}}$ in which the β-function coefficients corresponding to 4-point 1VR graphs G_R are zero, i.e. $c_{G_R} = 0$, it is clear from Eqs. (A.10), (A.11) that the simple conditions

$$
\delta_{G_R} = 0 \tag{4.1}
$$

will ensure that the redefined coefficients will also satisfy $c'_{G_R} = 0$.[2] This relies on the fact that for L, L' loop graphs G, G', with $L + L' \geq 3$, if (in the notation of the appendix) $\mathcal{L}_G G'$ contains 1VR graphs, then at least one of G or G' must itself be 1VR. We therefore have a simple all-orders prescription given by Eq. (4.1) for defining schemes with no 1VR contributions.

[1]The six-loop β-function was recently computed in Ref. [10]

[2]There are no 1VR 2-point graphs and therefore there is no need to impose $\epsilon_{G_R} = 0$.

The redefined coupling as given by Eqs. (2.5), (2.6) turns out to adopt a simple form when $c_{G_R} = \delta_{G_R} = 0$. We assume that $f(g)$, $c(g)$ in Eqs. (2.5), (2.6) are given by similar diagrammatic series to those for the β-function and anomalous dimension, but with $c_X \to \tilde{\delta}_X$ and $d_X \to \tilde{\epsilon}_X$. At one loop we simply find $\tilde{\delta}_1 = \delta_1$. At two loops we find

$$\begin{aligned}\tilde{\delta}_2 =&\delta_2 + \delta_1^2,\\ \tilde{\delta}_{2R} =&\delta_{2R} + \tilde{\delta}_1^2,\end{aligned} \tag{4.2}$$

so that the condition for 1VI graphs is

$$c_{2R} = 0, \quad \tilde{\delta}_{2R} = \tilde{\delta}_1^2, \tag{4.3}$$

At three loops

$$\begin{aligned}\tilde{\delta}_{3a} =&\delta_{3a} + \delta_1(\delta_2 + \delta_{2R}) + \tfrac{2}{3}\delta_1^3,\\ \tilde{\delta}_{3b} =&\delta_{3b} + \delta_1\epsilon_2,\\ \tilde{\delta}_{3c} =&\delta_{3c} + \delta_1(\delta_2 + \delta_{2R}) + \tfrac{2}{3}\delta_1^3,\\ \tilde{\delta}_{3d} =&\delta_{3d} + \delta_1\delta_2 + \tfrac{2}{3}\delta_1^3,\\ \tilde{\delta}_{3e} =&\delta_{3e} + 2\delta_1\delta_2 + \tfrac{2}{3}\delta_1^3,\\ \tilde{\delta}_{3aR} =&\delta_{3aR} + \tfrac{5}{2}\delta_1\delta_{2R} + \delta_1^3,\\ \tilde{\delta}_{3bR} =&\delta_{3bR} + \delta_1(\delta_2 + \delta_{2R}) + \delta_1^3,\\ \tilde{\epsilon}_3 =&\epsilon_3 + 3\delta_1\epsilon_2.\end{aligned} \tag{4.4}$$

It is easy to confirm using Eq. (4.2) that $\delta_{2R} = \delta_{3aR} = \delta_{3bR} = 0$ corresponds to

$$\tilde{\delta}_{3aR} = \tilde{\delta}_1^3, \quad \tilde{\delta}_{3bR} = \tilde{\delta}_1\tilde{\delta}_2. \tag{4.5}$$

The emerging pattern is clear; the value for $\tilde{\delta}_{G_R}$ is the product of the $\tilde{\delta}$s for its 1VI subgraphs. At four loops we find

$$\begin{aligned}\tilde{\delta}_{4aR} =&\delta_{4aR} + 3\delta_1\delta_{3aR} + \tfrac{3}{2}\delta_{2R}^2 + \tfrac{13}{3}\delta_1^2\delta_{2R} + \delta_1^4,\\ \tilde{\delta}_{4bR} =&\delta_{4bR} + 2\delta_1\delta_{3bR} + \delta_2^2 + \tfrac{4}{3}\delta_1^2\delta_{2R} + 2\delta_1^2\delta_2 + \delta_1^4,\\ \tilde{\delta}_{4cR} =&\delta_{4cR} + \delta_1\delta_{3aR} + \tfrac{3}{2}\delta_1\delta_{3bR} + \delta_1^2\delta_2 + \tfrac{8}{3}\delta_1^2\delta_{2R} + \delta_1^4,\\ \tilde{\delta}_{4dR} =&\delta_{4dR} + \delta_1\delta_{3c} + \delta_1\delta_{3bR} + \delta_{2R}^2 + \delta_1^2\delta_2 + \tfrac{5}{3}\delta_1^2\delta_{2R} + \tfrac{2}{3}\delta_1^4,\\ \tilde{\delta}_{4eR} =&\delta_1\delta_{3b} + \epsilon_2\delta_{2R} + \delta_1^2\epsilon_2,\\ \tilde{\delta}_{4fR} =&\delta_{4fR} + \delta_1\delta_{3a} + \delta_1\delta_{3aR} + \tfrac{1}{2}\delta_1\delta_{3bR} + \delta_1^2\delta_2 + 2\delta_1^2\delta_{2R} + \tfrac{2}{3}\delta_1^4,\\ \tilde{\delta}_{4gR} =&\delta_{4gR} + \delta_1\delta_{3e} + \delta_1\delta_{3bR} + 2\delta_1^2\delta_2 + \tfrac{2}{3}\delta_1^2\delta_{2R} + \tfrac{2}{3}\delta_1^4,\end{aligned} \tag{4.6}$$

Using Eqs. (4.4), (4.2) we find that $\delta_{G_R} = 0$ up to this level corresponds to taking

$$\tilde{\delta}_{4aR} = \tilde{\delta}_1^4, \quad \tilde{\delta}_{4bR} = \tilde{\delta}_2^2, \quad \tilde{\delta}_{4cR} = \tilde{\delta}_1^2\tilde{\delta}_2, \quad \tilde{\delta}_{4dR} = \tilde{\delta}_1\tilde{\delta}_{3c},$$

$$\tilde{\delta}_{4eR} = \tilde{\delta}_1\tilde{\delta}_{3b}, \quad \tilde{\delta}_{4fR} = \tilde{\delta}_1\tilde{\delta}_{3a}, \quad \tilde{\delta}_{4gR} = \tilde{\delta}_1\tilde{\delta}_{3e}, \tag{4.7}$$

so that each four-loop 1VR δ is the product of the δs for its 1VI subgraphs, as expected. It seems highly likely that this simple pattern persists to all orders, but we have not been able to construct a proof.

When considering the scheme invariants, we can therefore restrict ourselves to those schemes with $c_{G_R} = 0$. The counting of invariants is then slightly different. Upon setting $c_{3aR} = c_{3bR} = 0$ in Eq. (2.12), there are then just three invariant combinations, namely $I_1^{(3\prime)} = c_{3a} + c_{3d}$, $I_3^{(3)}$ and $I_4^{(3)}$. We have lost two coefficients (c_{3aR} and c_{3bR}) and one independent variation ($X_{1,2R}^{\lambda\lambda}$) and so we expect to lose $2 - 1 = 1$ invariants.

The pattern is similar at four loops; if we impose Eq. (4.5), then we have $\delta c_{4aR-4gR} = 0$ and so we can can consistently set $c_{4aR-4gR} = 0$ in Eq. (3.8). We now have 23 coefficients and the 14 variations $\tilde{\delta}_{3a-3f}$, $\tilde{\epsilon}_3$, $\tilde{\delta}_1^3$, $\tilde{\delta}_1\tilde{\delta}_2$, $\tilde{\delta}_1\tilde{\epsilon}_2$, $\tilde{\delta}_2$, $\tilde{\epsilon}_2$, $\tilde{\delta}_1^2$, $\tilde{\delta}_1$, leading to a naive expectation of 23-14=9 invariants. On the other hand, out of the original eighteen linear invariants in Eq. (3.8) we are left with eleven invariant linear combinations, plus the four individual invariants, making 15. Again this is as anticipated, since we have lost the seven coefficients $c_{4aR-4gR}$ and the four independent variations $X_{1,3aR}^{\lambda\lambda}$, $X_{1,3bR}^{\lambda\lambda}$, $X_{2,2R}^{\lambda\lambda}$ and $X_{2R,2}^{\lambda\gamma}$ so we lose $7 - 4 = 3$ invariant linear combinations. Furthermore it is clear that in the 1VI case only one of the identities in Eq. (3.10) remains, and consequently only one of the quadratic invariants in Eq. (3.9) survives. The total number of invariants is therefore 16; once again, almost double the naively expected number.

Finally we can consistently set $c_{5aR} = c_{5bR} = c_{5cR} = 0$ in Eq. (3.18), to obtain a invariant constructed solely from anomalous dimension coefficients

$$I_1^{(5)\prime} = d_{5b} - 2d_{5c} - 2d_{5e} - 2d_{5f} + 4d_{5j}. \tag{4.8}$$

5 Relation with Hopf algebra

Scheme invariants may be described graphically by adopting and extending rules described by Panzer [11] using the Hopf algebra coproduct $\Delta : \mathcal{G} \to \mathcal{G}\otimes\mathcal{G}$, where $\mathcal{G}$ is the vector space spanned by the set of connected 1PI superficially divergent graphs and the disconnected products of such graphs. The action of the coproduct Δ on a Feynman graph $g \in \mathcal{G}$ is defined by

$$\Delta g = \sum_i g_i \otimes g/g_i \quad \forall \quad \text{subgraphs} \quad g_i \subset g, g_i, g \in \mathcal{G}, \quad g_i \neq 1, g, \quad \text{otherwise} \quad \Delta g = \emptyset. \tag{5.1}$$

Here g/g_i denotes the graph obtained from g by contracting each connected 1PI graph in the subgraph to a single vertex, or a single line if the connected 1PI graph has two external lines. Further details and a general discussion will be presented in Ref. [8], but this brief overview is sufficient for our present purposes. The invariants of Eqs. (3.8), (3.9) and

(3.18) should correspond to combinations of graphs with a symmetric, or cocommutative, coproduct, following the general results of Ref. [8]. In this section we verify this by explicit calclulation. Firstly, we readily derive the following useful results: At three loops

$$\begin{aligned}
\Delta({g_\lambda}^{3a}) =& {g_\lambda}^1 \otimes {g_\lambda}^{2R} + 2{g_\lambda}^2 \otimes g_\lambda^1, \\
\Delta({g_\lambda}^{3b}) =& {g_\gamma}^2 \otimes g_\lambda^1, \\
\Delta({g_\lambda}^{3c}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^2 + {g_\lambda}^{2R} \otimes g_\lambda^1, \\
\Delta({g_\lambda}^{3d}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^2 + (g_\lambda^1)^2 \otimes g_\lambda^1, \\
\Delta({g_\lambda}^{3e}) =& {g_\lambda}^1 \otimes {g_\lambda}^2 + {g_\lambda}^2 \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{3f}) =& 0, \\
\Delta({g_\gamma}^3) =& 2{g_\lambda}^1 \otimes {g_\gamma}^2, \\
\Delta({g_\lambda}^{3aR}) =& 3{g_\lambda}^1 \otimes {g_\lambda}^{2R} + 2{g_\lambda}^{2R} \otimes {g_\lambda}^1 + (g_\lambda^1)^2 \otimes g_\lambda^1, \\
\Delta({g_\lambda}^{3bR}) =& {g_\lambda}^1 \otimes {g_\lambda}^{2R} + {g_\lambda}^1 \otimes {g_\lambda}^2 + {g_\lambda}^2 \otimes {g_\lambda}^1 + (g_\lambda^1)^2 \otimes g_\lambda^1,
\end{aligned} \tag{5.2}$$

and at four loops we have for the 4-point graphs

$$\begin{aligned}
\Delta({g_\lambda}^{4a}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3a} + 2{g_\lambda}^{3e} \otimes {g_\lambda}^1 + {g_\lambda}^2 \otimes {g_\lambda}^{2R}, \\
\Delta({g_\lambda}^{4b}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^{3a} + 2{g_\lambda}^{3c} \otimes {g_\lambda}^1 + {g_\lambda}^{2R} \otimes {g_\lambda}^{2R}, \\
\Delta({g_\lambda}^{4c}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^{3b} + {g_\gamma}^3 \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4d}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3a} + {g_\lambda}^1 \otimes {g_\lambda}^{3bR} + {g_\lambda}^{3d} \otimes {g_\lambda}^1 + {g_\lambda}^2 \otimes {g_\lambda}^2 \\
& + ({g_\lambda}^1)^2 \otimes {g_\lambda}^{2R} + {g_\lambda}^1 {g_\lambda}^2 \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4e}) =& {g_\gamma}^2 \otimes {g_\lambda}^2 + {g_\lambda}^{3b} \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4f}) =& {g_\lambda}^{3a} \otimes {g_\lambda}^1 + {g_\lambda}^1 \otimes {g_\lambda}^{3c} + 2{g_\lambda}^2 \otimes {g_\lambda}^2, \\
\Delta({g_\lambda}^{4g}) =& 3{g_\lambda}^1 \otimes {g_\lambda}^{3c} + {g_\lambda}^{3aR} \otimes {g_\lambda}^1 + 2{g_\lambda}^{2R} \otimes {g_\lambda}^2 + ({g_\lambda}^1)^2 \otimes {g_\lambda}^2, \\
\Delta({g_\lambda}^{4h}) =& \Delta({g_\lambda}^{4i}) = 2{g_\lambda}^1 \otimes {g_\lambda}^{3e} + {g_\lambda}^{3d} \otimes {g_\lambda}^1 + ({g_\lambda}^1)^2 \otimes {g_\lambda}^2, \\
\Delta({g_\lambda}^{4j}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3b} + {g_\gamma}^2 \otimes {g_\lambda}^2 + {g_\lambda}^1 {g_\gamma}^2 \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4k}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3c} + {g_\lambda}^1 \otimes {g_\lambda}^{3e} + {g_\lambda}^{3bR} \otimes {g_\lambda}^1 + {g_\lambda}^2 \otimes {g_\lambda}^2 + ({g_\lambda}^1)^2 \otimes {g_\lambda}^2, \\
\Delta({g_\lambda}^{4l}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^{3e} + {g_\lambda}^{2R} \otimes {g_\lambda}^2 + {g_\lambda}^{3c} \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4m}) =& \Delta({g_\lambda}^{4n}) = {g_\lambda}^1 \otimes {g_\lambda}^{3e} + {g_\lambda}^{3e} \otimes {g_\lambda}^1 + {g_\lambda}^2 \otimes {g_\lambda}^2, \\
\Delta({g_\lambda}^{4o}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3c} + 2{g_\lambda}^1 \otimes {g_\lambda}^{3d} + {g_\lambda}^{2R} \otimes {g_\lambda}^2 + 2({g_\lambda}^1)^2 \otimes {g_\lambda}^2 + {g_\lambda}^1 {g_\lambda}^{2R} \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4p}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3f}, \\
\Delta({g_\lambda}^{4q}) =& {g_\lambda}^{3f} \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4r}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3d} + {g_\lambda}^1 \otimes {g_\lambda}^{3e} + {g_\lambda}^2 \otimes {g_\lambda}^2 + ({g_\lambda}^1)^2 \otimes {g_\lambda}^2 + {g_\lambda}^1 {g_\lambda}^2 \otimes {g_\lambda}^1, \\
\Delta({g_\lambda}^{4s}) =& 0, \\
\Delta({g_\lambda}^{4aR}) =& 4{g_\lambda}^1 \otimes {g_\lambda}^{3aR} + 2{g_\lambda}^{3aR} \otimes {g_\lambda}^1 + 3{g_\lambda}^{2R} \otimes {g_\lambda}^{2R} \\
& + 3({g_\lambda}^1)^2 \otimes {g_\lambda}^{2R} + 2{g_\lambda}^1 {g_\lambda}^{2R} \otimes {g_\lambda}^1,
\end{aligned}$$

$$
\begin{aligned}
\Delta({g_\lambda}^{4bR}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^{3bR} + 2{g_\lambda}^2 \otimes {g_\lambda}^2 + ({g_\lambda}^1)^2 \otimes {g_\lambda}^{2R} + 2{g_\lambda}^1{g_\lambda}^2 \otimes {g_\lambda}^1,\\
\Delta({g_\lambda}^{4cR}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3aR} + 2{g_\lambda}^1 \otimes {g_\lambda}^{3bR} + {g_\lambda}^{3bR} \otimes {g_\lambda}^1 + {g_\lambda}^2 \otimes {g_\lambda}^{2R} + {g_\lambda}^{2R} \otimes {g_\lambda}^2\\
&+ 2({g_\lambda}^1)^2 \otimes {g_\lambda}^{2R} + {g_\lambda}^1{g_\lambda}^2 \otimes {g_\lambda}^1 + {g_\lambda}^1{g_\lambda}^{2R} \otimes {g_\lambda}^1,\\
\Delta({g_\lambda}^{4dR}) =& 2{g_\lambda}^1 \otimes {g_\lambda}^{3bR} + {g_\lambda}^1 \otimes {g_\lambda}^{3c} + {g_\lambda}^{3c} \otimes {g_\lambda}^1 + {g_\lambda}^{2R} \otimes {g_\lambda}^{2R}\\
&+ 2({g_\lambda}^1)^2 \otimes {g_\lambda}^2 + {g_\lambda}^1{g_\lambda}^{2R} \otimes {g_\lambda}^1,\\
\Delta({g_\lambda}^{4eR}) =& {g_\gamma}^2 \otimes {g_\lambda}^{2R} + {g_\lambda}^1 \otimes {g_\lambda}^{3b} + {g_\lambda}^{3b} \otimes {g_\lambda}^1 + {g_\lambda}^1{g_\gamma}^2 \otimes {g_\lambda}^1,\\
\Delta({g_\lambda}^{4fR}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3aR} + {g_\lambda}^{3bR} \otimes {g_\lambda}^1 + {g_\lambda}^1 \otimes {g_\lambda}^{3a} + {g_\lambda}^{3a} \otimes {g_\lambda}^1 + 2{g_\lambda}^2 \otimes {g_\lambda}^{2R}\\
&+ ({g_\lambda}^1)^2 \otimes {g_\lambda}^{2R} + {g_\lambda}^1{g_\lambda}^2 \otimes {g_\lambda}^1,\\
\Delta({g_\lambda}^{4gR}) =& {g_\lambda}^1 \otimes {g_\lambda}^{3bR} + {g_\lambda}^2 \otimes {g_\lambda}^{2R} + {g_\lambda}^1 \otimes {g_\lambda}^{3e} + {g_\lambda}^{3e} \otimes {g_\lambda}^1\\
&+ ({g_\lambda}^1)^2 \otimes {g_\lambda}^2 + {g_\lambda}^1{g_\lambda}^2 \otimes {g_\lambda}^1,
\end{aligned} \tag{5.3}
$$

and for the 2-point graphs

$$
\begin{aligned}
\Delta({g_\gamma}^{4a}) =& {g_\gamma}^2 \otimes {g_\gamma}^2,\\
\Delta({g_\gamma}^{4b}) =& 3{g_\lambda}^1 \otimes {g_\gamma}^3 + 2{g_\lambda}^{2R} \otimes {g_\gamma}^2 + ({g_\lambda}^1)^2 \otimes {g_\gamma}^2,\\
\Delta({g_\gamma}^{4c}) =& {g_\lambda}^1 \otimes {g_\gamma}^3 + 2{g_\lambda}^2 \otimes {g_\gamma}^2,\\
\Delta({g_\gamma}^{4d}) =& 2{g_\lambda}^1 \otimes {g_\gamma}^3 + 2{g_\lambda}^2 \otimes {g_\gamma}^2 + ({g_\lambda}^1)^2 \otimes {g_\gamma}^2.
\end{aligned} \tag{5.4}
$$

At five loops, the basic co-products are

$$
\begin{aligned}
\Delta(g_\gamma^{5a}) =& g_\lambda^{3f} \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5b}) =& 2g_\lambda^{3e} \otimes g_\gamma^2 + g_\lambda^2 \otimes g_\gamma^3 + g_\lambda^1 \otimes g_\gamma^{4c},\\
\Delta(g_\gamma^{5c}) =& 2g_\lambda^{3e} \otimes g_\gamma^2 + 2g_\lambda^2 \otimes g_\gamma^3 + 2g_\lambda^1 \otimes g_\gamma^{4d} + (g_\lambda^1)^2 \otimes g_\gamma^3 + 2g_\lambda^1 g_\lambda^2 \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5d}) =& g_\gamma^3 \otimes g_\gamma^2 + 2g_\lambda^1 \otimes g_\gamma^{4a},\\
\Delta(g_\gamma^{5e}) =& 2g_\lambda^{3c} \otimes g_\gamma^2 + g_\lambda^{2R} \otimes g_\gamma^3 + 2g_\lambda^1 \otimes g_\gamma^{4c},\\
\Delta(g_\gamma^{5f}) =& 2g_\lambda^{3aR} \otimes g_\gamma^2 + 3g_\lambda^{2R} \otimes g_\gamma^3 + 4g_\lambda^1 \otimes g_\gamma^{4b} + 3(g_\lambda^1)^2 \otimes g_\gamma^3 + 2g_\lambda^1 g_\lambda^{2R} \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5g}) =& g_\gamma^2 \otimes g_\gamma^3 + 2g_\lambda^1 \otimes g_\gamma^{4a} + 2g_\lambda^1 g_\gamma^2 \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5h}) =& g_\lambda^{3d} \otimes g_\gamma^2 + g_\lambda^{3e} \otimes g_\gamma^2 + g_\lambda^2 \otimes g_\gamma^3 + g_\lambda^1 \otimes g_\gamma^{4c} + g_\lambda^1 \otimes g_\gamma^{4d}\\
&+ (g_\lambda^1)^2 \otimes g_\gamma^3 + g_\lambda^1 g_\lambda^2 \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5i}) =& g_\lambda^{3a} \otimes g_\gamma^2 + g_\lambda^{3bR} \otimes g_\gamma^2 + 2g_\lambda^2 \otimes g_\gamma^3 + g_\lambda^1 \otimes g_\gamma^{4b} + g_\lambda^1 \otimes g_\gamma^{4c}\\
&+ (g_\lambda^1)^2 \otimes g_\gamma^3 + g_\lambda^1 g_\lambda^2 \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5j}) =& g_\lambda^{3c} \otimes g_\gamma^2 + g_\lambda^{3bR} \otimes g_\gamma^2 + g_\lambda^2 \otimes g_\gamma^3 + g_\lambda^{2R} \otimes g_\gamma^3 + g_\lambda^1 \otimes g_\gamma^{4b} + 2g_\lambda^1 \otimes g_\gamma^{4d}\\
&+ 2(g_\lambda^1)^2 \otimes g_\gamma^3 + g_\lambda^1 g_\lambda^2 \otimes g_\gamma^2 + g_\lambda^1 g_\lambda^{2R} \otimes g_\gamma^2,\\
\Delta(g_\gamma^{5k}) =& g_\lambda^{3b} \otimes g_\gamma^2 + g_\gamma^2 \otimes g_\gamma^3 + g_\lambda^1 \otimes g_\gamma^{4a} + g_\lambda^1 g_\gamma^2 \otimes g_\gamma^2,\\
\Delta(g_\lambda^{5aR}) =& 2g_\lambda^1 \otimes g_\lambda^{4eR} + g_\lambda^{4eR} \otimes g_\lambda^1 + g_\gamma^2 \otimes g_\lambda^{3aR} + g_\lambda^{3b} \otimes g_\lambda^{2R} + g_\lambda^{2R} \otimes g_\lambda^{3b}\\
&+ g_\lambda^1 g_\lambda^{3b} \otimes g_\lambda^1 + 2g_\lambda^1 g_\gamma^2 \otimes g_\lambda^{2R} + g_\lambda^{2R} g_\gamma^2 \otimes g_\lambda^1,
\end{aligned}
$$

$$
\begin{aligned}
\Delta(g_\lambda^{5bR}) =& g_\lambda^1 \otimes g_\lambda^{4eR} + g_\gamma^2 \otimes g_\lambda^{3bR} + g_\lambda^{3b} \otimes g_\lambda^2 + g_\lambda^2 \otimes g_\lambda^{3b} \\
&+ g_\lambda^1 g_\lambda^{3b} \otimes g_\lambda^1 + g_\lambda^1 g_\gamma^2 \otimes g_\lambda^{2R} + g_\lambda^2 g_\gamma^2 \otimes g_\lambda^1, \\
\Delta(g_\lambda^{5cR}) =& 2g_\lambda^1 \otimes g_\lambda^{4eR} + g_\gamma^3 \otimes g_\lambda^{2R} + g_\lambda^{4c} \otimes g_\lambda^1 + g_\lambda^1 \otimes g_\lambda^{4c} \\
&+ 2(g_\lambda^1)^2 \otimes g_\lambda^{3b} + g_\lambda^1 g_\gamma^3 \otimes g_\lambda^1.
\end{aligned} \tag{5.5}
$$

At three loops, the coproducts for ${g_\lambda}^{3e}$ and ${g_\lambda}^{3f}$ are cocommutative and zero respectively, corresponding to the individual invariance of c_{3e}, c_{3f}. Corresponding to the invariants in Eq. (2.12) we have the following combinations with cocommutative coproducts:

$$
\begin{aligned}
\Delta({g_\lambda}^{3a} + {g_\lambda}^{3d} - {g_\lambda}^{3aR}) =& 2{g_\lambda}^1 \otimes_s {g_\lambda}^2 - 2{g_\lambda}^1 \otimes_s {g_\lambda}^{2R}, \\
\Delta({g_\lambda}^{3aR} - {g_\lambda}^{3bR}) =& 2{g_\lambda}^1 \otimes_s {g_\lambda}^{2R} - {g_\lambda}^1 \otimes_s {g_\lambda}^2, \\
\Delta({g_\lambda}^{3a} + {g_\lambda}^{3c}) =& 2{g_\lambda}^1 \otimes_s {g_\lambda}^2 + {g_\lambda}^1 \otimes_s {g_\lambda}^{2R}, \\
\Delta(2{g_\lambda}^{3b} + {g_\gamma}^3) =& 2{g_\lambda}^1 \otimes_s {g_\gamma}^2,
\end{aligned} \tag{5.6}
$$

where

$$
G_1 \otimes_s G_2 = G_1 \otimes G_2 + G_2 \otimes G_1. \tag{5.7}
$$

The scheme-invariant combination of RG coefficients corresponding to a combination of graphs $\sum_i \alpha_i g_\lambda^i + \sum_j \tilde{\alpha}_j g_\gamma^j$ with a cocommutative coproduct is [8] $\sum_i \alpha_i S_i c_i + \sum_j \tilde{\alpha}_j S'_j d_j$ where S_i are the symmetry factors for the 4-point graphs, and S'_i those for the 2-point graphs. The relevant symmetry factors at this loop order are given by

$$
S_{3f} = 1, \quad S_{3e} = 2, \quad S'_3 = S_{3a} = S_{3c} = S_{3d} = S_{3bR} = 4, \quad S_{3b} = 6, \quad S_{3aR} = 8. \tag{5.8}
$$

So for instance

$$
{g_\lambda}^{3a} + {g_\lambda}^{3d} - {g_\lambda}^{3aR} \to 4c_{3a} + 4c_{3d} - 8c_{3aR} \tag{5.9}
$$

which agrees with $I_1^{(3)}$ in Eq. (2.12) up to an overall factor.

At four loops, the coproducts for ${g_\lambda}^{4m}$, ${g_\lambda}^{4n}$ and ${g_\gamma}^{4a}$ are cocommutative and that for ${g_\lambda}^{4s}$ is zero, corresponding to the individual invariance of c_{4m}, c_{4n}, c_{4s} and d_{4a}. Corresponding to the invariants in Eq. (3.8) we have the following combinations with cocommutative coproducts:

$$
\begin{aligned}
\Delta({g_\lambda}^{4h} - {g_\lambda}^{4i}) &= C_1^{(4)L}, \\
\Delta({g_\lambda}^{4b} + 2{g_\lambda}^{4f}) &= C_2^{(4)L}, \\
\Delta({g_\lambda}^{4a} + {g_\lambda}^{4f} + {g_\lambda}^{4l}) &= C_3^{(4)L}, \\
\Delta({g_\lambda}^{4l} + {g_\lambda}^{4o} - 2{g_\lambda}^{4r} - {g_\lambda}^{4bR} + 2{g_\lambda}^{4gR} - {g_\lambda}^1 {g_\lambda}^{3c}) &= C_4^{(4)L}, \\
\Delta({g_\lambda}^{4c} + 2{g_\lambda}^{4e} + {g_\gamma}^{4c}) &= C_5^{(4)L}, \\
\Delta({g_\lambda}^{4d} + {g_\lambda}^{4f} - {g_\lambda}^{4k} + {g_\lambda}^{4r} - {g_\lambda}^{4bR}) &= C_6^{(4)L}, \\
\Delta({g_\lambda}^{4b} - 2{g_\lambda}^{4d} + {g_\lambda}^{4g} - {g_\lambda}^{4o} + {g_\lambda}^{4cR} + {g_\lambda}^{4gR}) &= C_7^{(4)L},
\end{aligned}
$$

$$
\begin{aligned}
\Delta({g_\lambda}^{4h} - {g_\lambda}^{4k} + {g_\lambda}^{4o} - {g_\lambda}^{4r} - {g_\lambda}^{4bR} + {g_\lambda}^{4gR}& \\
-{g_\lambda}^{1}{g_\lambda}^{3d} - {g_\lambda}^{1}{g_\lambda}^{3e} + {g_\lambda}^{1}{g_\lambda}^{3bR}) &= C_8^{(4)L}, \\
\Delta({g_\lambda}^{4e} + {g_\lambda}^{4j} + 2{g_\gamma}^{4c} - {g_\gamma}^{4d} + \tfrac{1}{2}{g_\lambda}^{1}{g_\gamma}^{3}) &= C_9^{(4)L}, \\
\Delta({g_\lambda}^{4q} + {g_\lambda}^{4p}) &= C_{10}^{(4)L}, \\
\Delta({g_\gamma}^{4b} + 3{g_\gamma}^{4c} - 3{g_\gamma}^{4d} + 2{g_\lambda}^{4eR} + {g_\lambda}^{1}{g_\gamma}^{3}) &= C_{11}^{(4)L}, \\
\Delta({g_\lambda}^{4aR} + {g_\lambda}^{4bR} - 2{g_\lambda}^{4cR}) &= C_{12}^{(4)L}, \\
\Delta({g_\lambda}^{4bR} - {g_\lambda}^{4dR} + {g_\lambda}^{1}{g_\lambda}^{3c}) &= C_{13}^{(4)L}, \\
\Delta(2{g_\lambda}^{4bR} - {g_\lambda}^{4cR} + {g_\lambda}^{4fR} - 2{g_\lambda}^{4gR} - {g_\lambda}^{1}{g_\lambda}^{3a}) &= C_{14}^{(4)L}.
\end{aligned}
\tag{5.10}
$$

Here, rather than give explicit expressions on the right-hand side, we use $C_i^{(l)L} \in \mathcal{G} \otimes_s \mathcal{G}$ to denote l-loop cocommutative coproducts corresponding to linear invariants. Since their exact form is not especially significant, we relegate the full expressions to Appendix B. The noteworthy new feature here is the necessity sometimes to add quadratic terms, of course with no counterpart in the original linear invariants of Eq. (3.8), on the left-hand side in order to obtain co-commutative results. The need for this is explained in general in Ref. [8].

Corresponding to the quadratic invariants in Eq. (3.9) we have

$$
\begin{aligned}
\Delta(2{g_\lambda}^{1}{g_\gamma}^{4c} - {g_\lambda}^{1}{g_\gamma}^{4d} + {g_\gamma}^{2}{g_\lambda}^{3d} + 2{g_\lambda}^{2}{g_\lambda}^{3b} - ({g_\lambda}^{1})^2{g_\lambda}^{3b}) &= C_1^{(4)Q}, \\
\Delta({g_\lambda}^{1}{g_\lambda}^{4eR} - {g_\lambda}^{2R}{g_\lambda}^{3b} - {g_\gamma}^{2}{g_\lambda}^{3bR}) &= C_2^{(4)Q}, \\
\Delta[{g_\lambda}^{1}({g_\lambda}^{4dR} - 2{g_\lambda}^{4gR}) + 2{g_\lambda}^{2}{g_\lambda}^{3aR} - {g_\lambda}^{2R}{g_\lambda}^{3d}] &= C_3^{(4)Q}.
\end{aligned}
\tag{5.11}
$$

Here we see the need for additional cubic terms on the left-hand side, in addition to the quadratic terms corresponding to those in the invariant. The relevant graph combination corresponding to the additional invariant in Eq. (3.12) may be derived from those already given and hence is not displayed here. Here we use $C_i^{(l)Q} \in \mathcal{G} \otimes_s \mathcal{G}$ to denote l-loop cocommutative coproducts corresponding to quadratic invariants. The coefficients of the linear invariants in Eq. (3.8) may be obtained from the linear terms on the left-hand side of Eq. (5.10) by substitutions similar to those described at three loops after Eq. (5.6). Likewise, the coefficients of the quadratic invariants in Eq. (3.9) may be obtained from the quadratic terms on the left-hand side of Eq. (5.11) by similar substitutions. Here the relevant symmetry factors are given by

$$
\begin{aligned}
S_{4s} = 1, \quad S_1 = S_2 = S_{4a} =& S_{4m} = S_{4n} = S_{4p} = S_{4q} = 2, \\
S_{2R} = S_{4bR} =& S_{4gR} = S_{4c} = S_{4d} = S_{4f} = S_{4h} = S_{4i} \\
= S_{4k} = S_{4l} = S_{4r} =& S'_{4c} = S'_{4d} = 4, \\
S'_2 = S_{4e} = 6, \quad & S_{4eR} = S_{4j} = S'_{4a} = 12, \\
S_{4cR} = S_{4dR} = S_{4fR} = S_{4b} =& S_{4g} = S_{4o} = S'_{4b} = 8, \quad S_{4aR} = 16,
\end{aligned}
\tag{5.12}
$$

together with those in Eq. (5.8). We also find corresponding to Eq. (3.18)

$$\Delta(4g_\gamma{}^{5b} - 4g_\gamma{}^{5c} - 2g_\gamma{}^{5e} - g_\gamma{}^{5f} + 4g_\gamma{}^{5j} - 2g_\lambda{}^{5aR} + 4g_\lambda{}^{5bR} \\ -g_\lambda{}^{5cR} + g_\lambda{}^1 g_\lambda{}^{4c} - 4g_\lambda{}^2 g_\lambda{}^{3b} + 2g_\lambda{}^{2R} g_\lambda{}^{3b}) = C_1^{(5)L}. \tag{5.13}$$

Corresponding to the quadratic invariants in Eq. (3.19), we find

$$\begin{aligned}
\Delta[g_\lambda{}^1 g_\lambda{}^{5a} + g_\gamma{}^2 g_\lambda{}^{4p} + g_\lambda{}^{3b} g_\lambda{}^{3f}] &= C_1^{(5)Q},\\
\Delta[g_\lambda{}^1(g_\gamma{}^{5g} - 2g_\gamma{}^{5k}) - g_\gamma{}^2 g_\lambda{}^{4c} + g_\gamma{}^3 g_\lambda{}^{3b}] &= C_2^{(5)Q}\\
\Delta[g_\lambda{}^1(g_\lambda{}^{5aR} - g_\lambda{}^{5bR}) + g_\gamma{}^2(-g_\lambda{}^{4aR} + g_\lambda{}^{4cR}) + g_\lambda{}^{3b}(-g_\lambda{}^{3aR} + g_\lambda{}^{3bR})] &= C_3^{(5)Q},\\
\Delta[g_\lambda{}^1(g_\gamma{}^{5d} - g_\gamma{}^{5g}) - 2g_\gamma{}^2 G_J - \tfrac{1}{2}(g_\gamma{}^3)^2 + g_\lambda{}^1 g_\gamma{}^2 g_\gamma{}^3] &= C_4^{(5)Q},\\
\Delta[g_\lambda{}^1(2g_\lambda{}^{5bR} - g_\lambda{}^{5cR}) + g_\lambda{}^{2R} G_J + g_\gamma{}^2(g_\lambda{}^{4aR} - 2g_\lambda{}^{4cR})&\\
+g_\gamma{}^3 g_\lambda{}^{3bR} + (g_\lambda{}^1)^2(g_\lambda{}^{4j} - g_\lambda{}^{4e} - g_\lambda{}^{4eR}) + g_\lambda{}^1 g_\lambda{}^{2R} g_\lambda{}^{3b}] &= C_5^{(5)Q},\\
\Delta[g_\lambda{}^1(g_\gamma{}^{5c} + g_\gamma{}^{5e} - 2g_\gamma{}^{5h}) - g_\gamma{}^2(g_\lambda{}^{4b} - 2g_\lambda{}^{4d} - g_\lambda{}^{4aR} + 2g_\lambda{}^{4cR}) + g_\lambda{}^{2R} G_J&\\
-g_\gamma{}^3(g_\lambda{}^{3c} - g_\lambda{}^{3d}) + (g_\lambda{}^1)^2(g_\lambda{}^{4j} - g_\lambda{}^{4c} - g_\lambda{}^{4e} - g_\lambda{}^{4eR}) + g_\lambda{}^1 g_\lambda{}^{2R} g_\lambda{}^{3b}] &= C_6^{(5)Q},\\
\Delta[g_\lambda{}^1(2g_\gamma{}^{5b} - g_\gamma{}^{5e}) + g_\gamma{}^2(g_\lambda{}^{4l} - g_\lambda{}^{4o} + 2g_\lambda{}^{4r})&\\
+(2g_\lambda{}^2 - g_\lambda{}^{2R})G_J + 4g_\lambda{}^{3b} g_\lambda{}^{3e} + g_\lambda{}^{3c} g_\gamma{}^3] &= C_7^{(5)Q},\\
\Delta[g_\lambda{}^1(2g_\gamma{}^{5e} + g_\gamma{}^{5f} - 4g_\gamma{}^{5i}) - g_\gamma{}^2(2g_\lambda{}^{4b} - 3g_\lambda{}^{4aR} + 4g_\lambda{}^{4cR})&\\
+(5g_\lambda{}^{2R} - 8g_\lambda{}^2)G_J + g_\gamma{}^3(2g_\lambda{}^{3a} - 2g_\lambda{}^{3c} - g_\lambda{}^{3aR} + 2g_\lambda{}^{3bR})&\\
+(g_\lambda{}^2 - g_\lambda{}^{2R})g_\lambda{}^1 g_\gamma{}^3 - g_\lambda{}^1 g_\gamma{}^2 g_\lambda{}^{3a} + (g_\lambda{}^1)^2 g_\gamma{}^{4c}] &= C_8^{(5)Q},
\end{aligned} \tag{5.14}$$

where

$$G_J = g_\lambda{}^{4c} + g_\lambda{}^{4e} - g_\lambda{}^{4j} \tag{5.15}$$

corresponds to J defined in Eq. (3.20). The invariants of Eqs. (3.18), (3.19) may be recovered from Eqs. (5.13), (5.14) as before. Here the relevant symmetry factors (in addition to those in Eqs. (5.8), (5.12)) are

$$S'_{5a} = 1, \quad S'_{5b} = 2, \quad S'_{5c} = S'_{5h} = 4, \quad S_{5cR} = S'_{5d} = S'_{5e} = S'_{5i} = S'_{5j} = 8, \\ S_{5bR} = S'_{5k} = 12, \quad S'_{5f} = 16, \quad S_{5aR} = S'_{5g} = 24. \tag{5.16}$$

6 a-function considerations

A good deal of effort has been invested in recent years [12–15] in the search for an a-theorem, a generalisation of Zamolodchikov's two-dimensional c-theorem [16] to four dimensions (or indeed to other dimensions higher than two [17, 19–22]). From our point of view, as mentioned in the introduction, the crucial development is the demonstration that the β-functions in theories in four and six dimensions obey a gradient flow equation similar to one which plays a critical role in the derivation of the c-theorem [23–26]. These gradient flow equations often place constraints relating the β-function coefficients, as has been shown for four-dimensional gauge theories [4] and six-dimensional ϕ^3 theories [5] (similar gradient flows have been demonstrated in three dimensions [1–3] though here the theoretical underpinning has not yet been provided). Our purpose in this section is to apply the same considerations to our four-dimensional ϕ^4 theory where we are able to confirm our results using the explicit calculations available to a high loop order. We start by presenting the basic theoretical background in general notation in the interests of clarity and brevity. For a theory with couplings g^I, the corresponding β-functions are defined by

$$\beta^I = \mu \frac{d}{d\mu} g^I \tag{6.1}$$

where μ is a mass scale (in practice usually the standard dimensional regularisation mass scale). The essential conclusion of Refs. [24], [25] is the existence of a function A such that

$$\partial_I A = T_{IJ} \beta^J \tag{6.2}$$

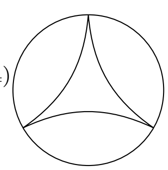
where $\partial_I \equiv \frac{\partial}{\partial g^I}$ and

$$T_{IJ} = G_{IJ} + \partial_I W_J - \partial_J W_I \tag{6.3}$$

with G_{IJ} symmetric[3]. The function A is invariant up to

$$A \to A + g_{IJ} \beta^I \beta^J, \tag{6.4}$$

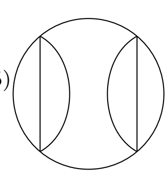
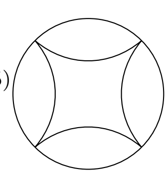
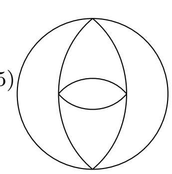
where g_{IJ} is an arbitrary symmetric matrix. At lowest order we have an a-function given by

$$A^{(4)} = A_1^{(4)}$$  (6.5)

and Eq. (6.2) simply implies

$$3A_1^{(4)} = 3c_1 \implies A_1^{(4)} = c_1 \tag{6.6}$$

(the factor of 3 on the right-hand side derives from the multiplicity factor of $\mathcal{S}_3$ for the corresponding term in the β-function). At the next order we have

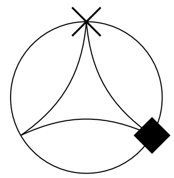
$A^{(5)} = A_1^{(5)}$  $+ A_2^{(5)}$  $+ A_3^{(5)}$  (6.7)

[3]In general for a theory with a symmetry, the β-function should be replaced by a "generalised" β-function [25]. It was shown by explicit calculation in Ref. [28] that the difference between the two becomes non-trivial at three loops for a fermion-scalar theory in four dimensions. However, for a pure scalar theory we do not expect any distinction until five loops which is beyond our interests in this section.

and now Eq. (6.2) entails

$$
\begin{aligned}
4A_1^{(5)} =& 2d_2, \\
4A_2^{(5)} =& 3c_{2R} + c_1 T^{(4)}, \\
4A_3^{(5)} =& 6c_2 + 2c_1 T^{(4)},
\end{aligned}
\tag{6.8}
$$

Here $T^{(4)}$ represents the coefficient of the single fourth-order metric term. The figure below displays this structure by showing its contraction with a dg (represented by a cross) and a $\beta^{(1)}$ (represented by a diamond).

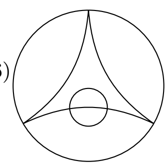
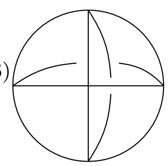
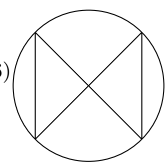
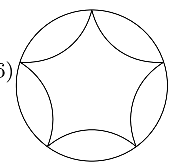


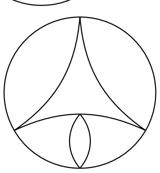
(6.9)

In Eq. (6.8) there are two equations and three unknowns resulting in one residual free parameter. This corresponds to the invariance under

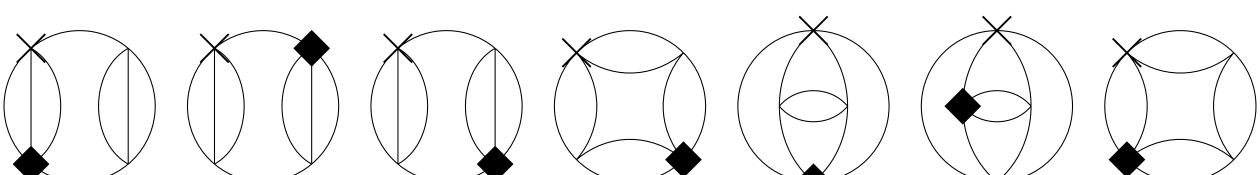
$$
A_2^{(5)} \to A_2^{(5)} + 3g^{(3)} c_1^2, \quad A_3^{(5)} \to A_3^{(5)} + 6g^{(3)} c_1^2, \quad T^{(4)} \to T^{(4)} + 12g^{(3)} c_1 \tag{6.10}
$$

reflecting the freedom described by Eq. (6.4) at lowest order (with $g_{IJ} = g^{(3)}\tilde{\delta}_{IJ}$, $g^{(3)}$ arbitrary). The six-loop a-function is given by

$A^{(6)} = A_1^{(6)}$  $+ A_2^{(6)}$  $+ A_3^{(6)}$  $+ A_4^{(6)}$ 

$+ A_5^{(6)}$  (6.11)

and the seven associated five-loop metric contributions are depicted below, with the same conventions as for $T^{(4)}$ earlier.



$T_1^{(5)}$ $T_2^{(5)}$ $T_3^{(5)}$ $T_4^{(5)}$ $T_5^{(5)}$ $T_6^{(5)}$ $T_7^{(5)}$

(6.12)

We now find from Eq. (6.2)

$$
A_1^{(6)} = 3c_{3b} + d_2 T^{(4)},
$$

$$
\begin{aligned}
2A_1^{(6)} =& 2d_3 + 3c_1(T_6^{(5)} + T_7^{(5)}),\\
2A_1^{(6)} =& 3c_1 T_5^{(5)} + d_2 T^{(4)},\\
5A_2^{(6)} =& 3c_{3f},\\
A_3^{(6)} =& 6c_{3d} + 2c_1 T_2^{(5)},\\
4A_3^{(6)} =& 12c_{3e} + 2c_1 T_3^{(5)} + 4c_2 T^{(4)},\\
5A_4^{(6)} =& 3c_{3aR} + c_1(T_4^{(5)} + T_1^{(5)}) + c_{2R} T^{(4)},\\
2A_5^{(6)} =& 6c_{3c} + c_1 T_3^{(5)} + 2c_{2R} T^{(4)},\\
2A_5^{(6)} =& 6c_{3bR} + c_1(T_2^{(5)} + 2T_4^{(5)}) + c_2 T^{(4)},\\
A_5^{(6)} =& 3c_{3a} + 2c_1 T_1^{(5)} + c_2 T^{(4)}.
\end{aligned} \tag{6.13}
$$

The values of the coefficients may be extracted from Ref. [6] and are given at one and two loops by

$$
c_1 = 1, \quad c_2 = -1, \quad c_{2R} = 0, \quad d_2 = \tfrac{1}{6} \tag{6.14}
$$

and at three loops by

$$
\begin{gathered}
c_{3a} = \tfrac{1}{2}, \quad c_{3b} = -\tfrac{3}{8}, \quad c_{3c} = c_{3d} = -\tfrac{1}{2}, \quad c_{3e} = 2, \quad c_{3f} = 12\zeta_3,\\
d_3 = -\tfrac{1}{8}, \quad c_{3aR} = c_{3bR} = 0.
\end{gathered} \tag{6.15}
$$

The solution of Eq. (6.13) is then

$$
\begin{aligned}
A_1^{(6)} =& -\tfrac{9}{8} + \tfrac{1}{6}T^{(4)},\\
A_2^{(6)} =& \tfrac{36}{5}\zeta_3,\\
A_3^{(6)} =& \tfrac{51}{5} - 2T^{(4)} + 4A_4^{(6)},\\
A_5^{(6)} =& \tfrac{27}{10} - T^{(4)} + 4A_4^{(6)},\\
T_1^{(5)} =& \tfrac{3}{5} + 2A_4^{(6)},\\
T_2^{(5)} =& \tfrac{33}{5} - T^{(4)} + 2A_4^{(6)},\\
T_3^{(5)} =& \tfrac{42}{5} - 2T^{(4)} + 8A_4^{(6)},\\
T_4^{(5)} =& -\tfrac{3}{5} + 3A_4^{(6)},\\
T_5^{(5)} =& -\tfrac{3}{4} + \tfrac{1}{18}T^{(4)},\\
T_6^{(5)} + T_7^{(5)} =& -\tfrac{2}{3} + \tfrac{1}{9}T^{(4)}.
\end{aligned} \tag{6.16}
$$

Here we have nine equations for ten unknowns, again resulting in one free parameter. This corresponds to the invariance under

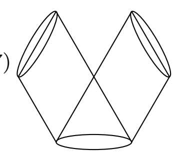
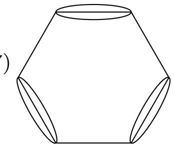
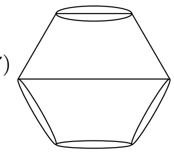
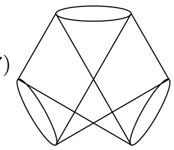
$$
\begin{gathered}
A_3^{(6)} \to A_3^{(6)} + 4g^{(4)}, \quad A_4^{(6)} \to A_4^{(6)} + g^{(4)}, \quad A_5^{(6)} \to A_5^{(6)} + 4g^{(4)},\\
T_4^{(5)} \to T_4^{(5)} + 3g^{(4)}, \quad T_1^{(5)} \to T_1^{(5)} + 2g^{(4)}, \quad T_2^{(5)} \to T_2^{(5)} + 2g^{(4)},
\end{gathered}
$$

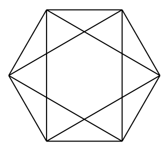
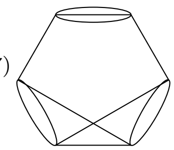
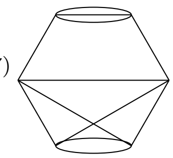
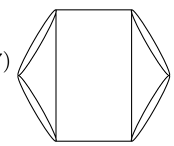
$$T_3^{(5)} \to T_3^{(5)} + 8g^{(4)}, \tag{6.17}$$

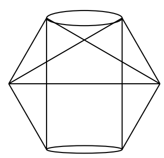
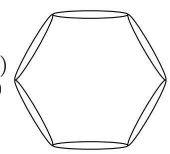
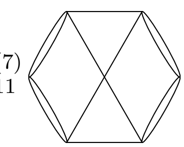
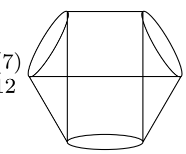
reflecting the freedom under

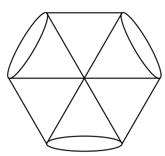
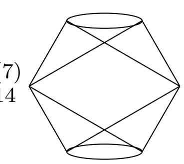
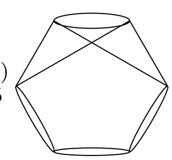
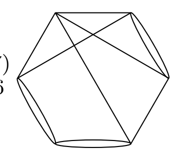
$$A \to A + g^{(4)} \beta_{ijkl}\beta_{ijmn} g_{klmn}, \tag{6.18}$$

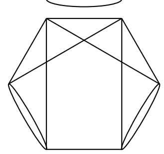
with $g^{(4)}$ arbitrary. Finally, the seven-loop a-function is parametrised as

$A^{(7)} = A_1^{(7)}$  $+ A_2^{(7)}$  $+ A_3^{(7)}$  $+ A_4^{(7)}$ 

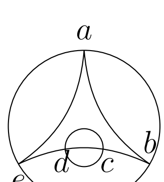
$+ A_5^{(7)}$  $+ A_6^{(7)}$  $+ A_7^{(7)}$  $+ A_8^{(7)}$ 

$+ A_9^{(7)}$  $+ A_{10}^{(7)}$  $+ A_{11}^{(7)}$  $+ A_{12}^{(7)}$ 

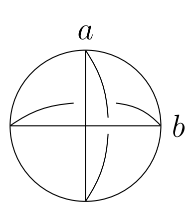
$+ A_{13}^{(7)}$  $+ A_{14}^{(7)}$  $+ A_{15}^{(7)}$  $+ A_{16}^{(7)}$ 

$+ A_{17}^{(7)}$ . (6.19)

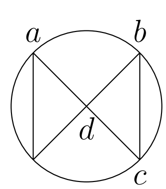
These seven-loop vacuum diagrams were given in Fig. 6 of Ref. [29] and we have retained their ordering (similarly, the five and six loop vacuum diagrams were depicted in their Figs. 4 and 5 respectively). Since there are 24 6-loop metric contributions, we have introduced a compact notation to avoid depicting them all individually. Eq. (6.20) shows the six-loop vacuum diagrams; seen already in Eq. (6.11), but now with some vertices labelled. We introduce the notation $T_{nxy}^{(6)}$ to denote a metric contribution where the vertices x, y in diagram n correspond to the I, J indices respectively of a contribution to T_{IJ}. The labellings shown are sufficient to cover all the independent possibilities.



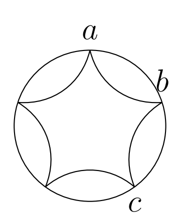
1



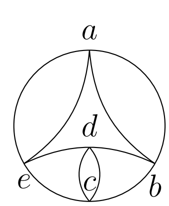
2



3



4



5

(6.20)

The number of T-type contributions is the number of distinct ways of selecting an ordered pair of vertices from the diagrams shown in (6.20), namely 24. At this order Eq. (6.2) implies

$$
\begin{aligned}
4A_1^{(7)} =& \tfrac{3}{2}d_2(T_6^{(5)} + T_7^{(5)}) + 2d_{4a} \\
2A_1^{(7)} =& \tfrac{3}{2}d_2T_5^{(5)} \\
6A_2^{(7)} =& \tfrac{1}{2}d_2(T_5^{(5)} + T_6^{(5)} + T_7^{(5)}) \\
2A_3^{(7)} =& c_1T_{1ab}^{(6)} + c_{3b}T^{(4)} + d_2(T_4^{(5)} + 2T_1^{(5)}) + 6c_{4eR} \\
2A_3^{(7)} =& c_1(T_{1cb}^{(6)} + T_{1ca}^{(6)}) + 3c_{2R}(T_6^{(5)} + T_7^{(5)}) + 2d_{4b} \\
2A_3^{(7)} =& c_1T_{1be}^{(6)} + c_1T_{1ba}^{(6)} + 3c_{2R}T_5^{(5)} + d_2T_4^{(5)} \\
6A_4^{(7)} =& c_1T_{3ad}^{(6)} + 2c_{3d}T^{(4)} + 6c_{4i} \\
6A_5^{(7)} =& c_{4s} \\
2A_6^{(7)} =& 2c_1T_{1ab}^{(6)} + \tfrac{1}{2}d_2(2T_2^{(5)} + T_3^{(5)}) + 12c_{4j} \\
2A_6^{(7)} =& 2c_1T_{1cb}^{(6)} + 3c_2(T_6^{(5)} + T_7^{(5)}) + 2d_{4d} \\
2A_6^{(7)} =& 2c_1T_{1be}^{(6)} + 3c_2T_5^{(5)} + \tfrac{1}{2}d_2T_3^{(5)} \\
2A_7^{(7)} =& 2c_{3b}T^{(4)} + d_2T_3^{(5)} + 6c_{4e} \\
2A_7^{(7)} =& 2c_1T_{1ca}^{(6)} + 3c_2(T_6^{(5)} + T_7^{(5)}) + 2d_{4c} \\
2A_7^{(7)} =& 2c_1T_{1ba}^{(6)} + 3c_2T_5^{(5)} + d_2T_2^{(5)} \\
4A_8^{(7)} =& 3c_1(T_{1cd}^{(6)} + T_{1bc}^{(6)}) + d_3T^{(4)} \\
2A_8^{(7)} =& 3c_1T_{1ac}^{(6)} + d_3T^{(4)} + 3c_{4c} \\
4A_9^{(7)} =& T_{2ab}^{(6)} + 12c_{4p} \\
2A_9^{(7)} =& 6c_{4q} \\
6A_{10}^{(7)} =& c_1(T_{4ab}^{(6)} + T_{4ac}^{(6)}) + c_{2R}(T_4^{(5)} + T_1^{(5)}) + c_{3aR}T^{(4)} + 3c_{4aR} \\
4A_{11}^{(7)} =& c_1(T_{5cd}^{(6)} + T_{5bc}^{(6)}) + 2c_{2R}T_4^{(5)} + c_{3c}T^{(4)} + 6c_{4dR} \\
2A_{11}^{(7)} =& c_1T_{5ac}^{(6)} + 2c_{2R}T_1^{(5)} + c_{3c}T^{(4)} + 3c_{4b} \\
4A_{12}^{(7)} =& c_1(2T_{5cb}^{(6)} + T_{3ad}^{(6)}) + c_2T_3^{(5)} + 4c_{3bR}T^{(4)} + 12c_{4k} \\
2A_{12}^{(7)} =& 2c_1(T_{5be}^{(6)} + T_{5ab}^{(6)}) + c_2T_2^{(5)} + 2c_{3d}T^{(4)} + 6c_{4d} \\
6A_{13}^{(7)} =& c_1T_{3ad}^{(6)} + 2c_{3d}T^{(4)} + 6c_{4h} \\
4A_{14}^{(7)} =& 2c_1T_{5ca}^{(6)} + c_2T_3^{(5)} + 2c_{3a}T^{(4)} + 6c_{4f} \\
2A_{14}^{(7)} =& 2c_1T_{5ba}^{(6)} + c_2T_2^{(5)} + 3c_{4bR} \\
2A_{15}^{(7)} =& c_1(T_{5cb}^{(6)} + T_{5ca}^{(6)}) + c_{2R}T_3^{(5)} + 2c_{3aR}T^{(4)} + 6c_{4g} \\
2A_{15}^{(7)} =& c_1(T_{5be}^{(6)} + T_{5ba}^{(6)} + 2T_{4ab}^{(6)}) + c_2T_4^{(5)} + c_{2R}T_2^{(5)} + c_{3bR}T^{(4)} + 6c_{4cR}
\end{aligned}
$$

$$
\begin{aligned}
2A^{(7)}_{15} =& c_1(T^{(6)}_{5ab} + 2T^{(6)}_{4ac}) + c_2(T^{(5)}_4 + 2T^{(5)}_1) + c_{3a}T^{(4)} + c_{3bR}T^{(4)} + 6c_{4fR}\\
2A^{(7)}_{16} =& c_1(T^{(6)}_{3ab} + T^{(6)}_{3ac}) + 2c_{2R}T^{(5)}_3 + 4c_{3c}T^{(4)} + 12c_{4l}\\
2A^{(7)}_{16} =& c_1(2T^{(6)}_{5bc} + T^{(6)}_{3bc}) + 4c_2T^{(5)}_4 + 2c_{3e}T^{(4)} + 12c_{4gR}\\
A^{(7)}_{16} =& c_1(2T^{(6)}_{5cd} + T^{(6)}_{3da}) + 2c_{2R}T^{(5)}_2 + 12c_{4o}\\
A^{(7)}_{16} =& 2c_1T^{(6)}_{5ac} + 4c_2T^{(5)}_1 + 2c_{3e}T^{(4)} + 3c_{4a}\\
2A^{(7)}_{17} =& 2c_1(T^{(6)}_{3bc} + T^{(6)}_{3da}) + 4c_2T^{(5)}_2 + 24c_{4r}\\
2A^{(7)}_{17} =& 2c_1T^{(6)}_{3ac} + 4c_{3e}T^{(4)} + 12c_{4m}\\
2A^{(7)}_{17} =& 2c_1T^{(6)}_{3ab} + 4c_2T^{(5)}_3 + 4c_{3e}T^{(4)} + 12c_{4n}
\end{aligned}
\tag{6.21}
$$

The counting of unknowns is now slightly more subtle; we shall explain in some detail since the solution of Eqs. (6.21) leads to constraints on the β-function coefficients, and we would like to be sure that we have obtained the correct number of these. There are thirty-six four-loop structures (including 1PR structures which cannot contribute to the β-function and hence must be set to zero) leading to the thirty-six equations in Eq. (6.21); and there are 17 A coefficients (as shown in Eq. (6.19)) and 24 T coefficients at this order . However, $T^{(6)}_{1cd}$ and $T^{(6)}_{1bc}$ only appear in the combination $T^{(6)}_{1cd} + T^{(6)}_{1bc}$; furthermore, there are two invariances, under shifts among $T^{(6)}_{3bc}$, $T^{(6)}_{3da}$, $T^{(6)}_{5cd}$, $T^{(6)}_{5bc}$, and among $T^{(6)}_{4ab}$, $T^{(6)}_{4ac}$, $T^{(6)}_{5ab}$, $T^{(6)}_{5be}$. Therefore there is a total of $17 + 24 - 3 = 38$ unknowns at this order. The lower-order metric coefficients $T^{(5)}_1$–$T^{(5)}_3$ get determined in Eq. (6.16) up to one unknown, resulting in 39 unknowns in total. There are seven five-loop vacuum diagrams which can contribute to the freedom in Eq. (6.4) (the diagrams appearing in (6.12) but with insertions of $\beta^{(1)}$ replacing the diamonds and crosses), but two of these give the same contribution. There is also one four-loop vacuum diagram contributing to the freedom in Eq. (6.4) (the one appearing in (6.9) but with insertions of $\beta^{(1)}$, $\beta^{(2)}$ replacing the diamond and cross respectively). Therefore the number of unknowns which are solved for is only $39 - 6 - 1 = 32$. This implies that $36 - 32 = 4$ of the 36 equations must remain as constraints. Indeed after solving the equations we find the constraints

$$
\begin{aligned}
2c_1d_{4a} - d_2I^{(3)}_4 =& 0,\\
2c_1(I^{(4)}_{11} - I^{(4)}_{15} - 3I^{(4)}_{16}) + 3d_2(I^{(3)}_2 - I^{(3)}_3 + \tfrac{1}{2}c_{3e}) + 2(c_2 - c_{2R})I^{(3)}_4 =& 0,\\
2c_1(I^{(4)}_{11} - I^{(4)}_9) + 4(c_2 - c_{2R})I^{(3)}_4 - 3d_2(2I^{(3)}_2 - c_{3e}) =& 0,\\
c_1(2I^{(4)}_2 - I^{(4)}_3 + I^{(4)}_4 + I^{(4)}_{13} + \tfrac{1}{2}c_{4n} + \tfrac{1}{2}c_{4o}) + (c_2 - c_{2R})(2I^{(3)}_2 - c_{3e}) =& 0.
\end{aligned}
\tag{6.22}
$$

We note that as is to be expected, these constraints may be expressed in terms of the invariants defined in Eqs. (2.12), (3.8) and (3.9). At four loops (again extracted from Ref. [6]) the coefficients are

$$
c_{4a} = \tfrac{1}{3}(6\zeta_3 - 11), \quad c_{4b} = 1 - \zeta_3, \quad c_{4c} = \tfrac{7}{12}, \qquad c_{4d} = \tfrac{1}{2}, \quad c_{4e} = \tfrac{121}{144},
$$

$$c_{4f} = 1 - 2\zeta_3, \quad c_{4g} = c_{4o} = \tfrac{1}{4}(2\zeta_3 - 1), \quad c_{4h} = c_{4l} = \tfrac{1}{6}(5 - 6\zeta_3),$$
$$c_{4i} = \tfrac{5}{6}, \quad c_{4j} = -\tfrac{37}{288}, \quad c_{4k} = c_{4r} = \tfrac{2}{3}, \quad c_{4m} = 4\zeta_3 - 5, \quad c_{4n} = -5,$$
$$c_{4p} = 3(\zeta_4 - 2\zeta_3), \quad c_{4q} = -3(2\zeta_3 + \zeta_4), \quad c_{4s} = -40\zeta_5,$$
$$d_{4a} = -\tfrac{5}{48}, \quad d_{4b} = -\tfrac{5}{32}, \quad d_{4c} = \tfrac{13}{48}, \quad d_{4d} = \tfrac{2}{3}, \tag{6.23}$$

with $c_{4aR} = \ldots = c_{4fR} = 0$, and we may easily check that the values in Eqs. (6.14), (6.15) and (6.23) satisfy the constraints in Eq. (6.22).

We refrain from giving the values of the a-coefficients in the general case. However an interesting special case is that of a symmetric T_{IJ}. It turns out that we can impose symmetry on T_{IJ} up to this order without needing to impose any further constraints on the β-function coefficients. The a-function coefficients are then

$$\begin{aligned}
A_1^{(7)} &= -\tfrac{3}{32} + \tfrac{1}{144}T^{(4)}, \\
A_2^{(7)} &= -\tfrac{17}{864} + \tfrac{1}{432}T^{(4)}, \\
A_3^{(7)} &= \tfrac{79}{96} - \tfrac{3}{8}T^{(4)} + \tfrac{5}{6}A_4^{(6)}, \\
A_4^{(7)} &= \tfrac{7}{10} - 3\zeta_3 + T^{(4)} - 6A_4^{(6)} + 4A_{10}^{(7)} - 2A_{11}^{(7)}, \\
A_5^{(7)} &= -\tfrac{20}{3}\zeta_5, \\
A_6^{(7)} &= \tfrac{67}{40} - \tfrac{13}{24}T^{(4)} + A_4^{(6)}, \\
A_7^{(7)} &= \tfrac{773}{240} - \tfrac{13}{24}T^{(4)} + \tfrac{2}{3}A_4^{(6)}, \\
A_8^{(7)} &= \tfrac{19}{5} - \tfrac{5}{8}T^{(4)} + A_4^{(6)}, \\
A_9^{(7)} &= -9(2\zeta_3 + \zeta_4), \\
A_{12}^{(7)} &= \tfrac{18}{5} - \tfrac{21}{2}\zeta_3 + 3T^{(4)} - 18A_4^{(6)} + 12A_{10}^{(7)} - 5A_{11}^{(7)}, \\
A_{13}^{(7)} &= \tfrac{7}{10} - 4\zeta_3 + T^{(4)} - 6A_4^{(6)} + 4A_{10}^{(7)} - 2A_{11}^{(7)}, \\
A_{14}^{(7)} &= -\tfrac{21}{10} - \tfrac{3}{2}\zeta_3 + T^{(4)} - 2A_4^{(6)} + A_{11}^{(7)}, \\
A_{15}^{(7)} &= \tfrac{33}{20} - 3\zeta_3 + T^{(4)} - 7A_4^{(6)} + 6A_{10}^{(7)} - A_{11}^{(7)}, \\
A_{16}^{(7)} &= -\tfrac{97}{5} + 12\zeta_3 + 5T^{(4)} - 8A_4^{(6)} + 4A_{11}^{(7)}, \\
A_{17}^{(7)} &= -\tfrac{314}{5} + 30\zeta_3 + 12T^{(4)} - 16A_4^{(6)} + 4A_{11}^{(7)},
\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}
T_{1cb}^{(6)} &= \tfrac{1}{120} - \tfrac{3}{8}T^{(4)} + A_4^{(6)}, \\
T_{1ac}^{(6)} &= \tfrac{39}{20} - \tfrac{3}{8}T^{(4)} + \tfrac{2}{3}A_4^{(6)}, \\
T_{1ab}^{(6)} &= \tfrac{371}{240} - \tfrac{3}{8}T^{(4)} + \tfrac{1}{2}A_4^{(6)}, \\
T_{1cd}^{(6)} &= \tfrac{607}{120} - \tfrac{5}{12}T^{(4)} + \tfrac{1}{3}A_4^{(6)}, \\
T_{1be}^{(6)} &= \tfrac{1}{5} - \tfrac{3}{8}T^{(4)} + \tfrac{2}{3}A_4^{(6)},
\end{aligned}$$

$$
\begin{aligned}
T^{(6)}_{2ab} =& -72\zeta_4,\\
T^{(6)}_{3bc} =& -\tfrac{284}{5} + 48\zeta_3 + 3T^{(4)} + 24A^{(6)}_4 - 24A^{(7)}_{10} + 16A^{(7)}_{11},\\
T^{(6)}_{3ab} =& -16 + 30\zeta_3 + 4T^{(4)} + 4A^{(7)}_{11},\\
T^{(6)}_{3ac} =& -\tfrac{164}{5} + 6\zeta_3 + 8T^{(4)} - 16A^{(6)}_4 + 4A^{(7)}_{11},\\
T^{(6)}_{3ad} =& -\tfrac{4}{5} - 18\zeta_3 + 7T^{(4)} - 36A^{(6)}_4 + 24A^{(7)}_{10} - 12A^{(7)}_{11},\\
T^{(6)}_{4ab} =& -\tfrac{27}{20} + \tfrac{9}{4}\zeta_3 - \tfrac{1}{2}T^{(4)} + 3A^{(6)}_4 + \tfrac{3}{2}A^{(7)}_{11},\\
T^{(6)}_{4ac} =& -T^{(6)}_{4ab} + 6A^{(7)}_{10},\\
T^{(6)}_{5cd} =& -\tfrac{39}{5} + 12\zeta_3 - T^{(4)} + 14A^{(6)}_4 - 12A^{(7)}_{10} + 8A^{(7)}_{11},\\
T^{(6)}_{5bc} =& \tfrac{39}{5} - 12\zeta_3 + \tfrac{3}{2}T^{(4)} - 14A^{(6)}_4 + 12A^{(7)}_{10} - 4A^{(7)}_{11},\\
T^{(6)}_{5ac} =& -3 + 3\zeta_3 + \tfrac{1}{2}T^{(4)} + 2A^{(7)}_{11},\\
T^{(6)}_{5be} =& \tfrac{21}{5} - 9\zeta_3 + \tfrac{5}{2}T^{(4)} - 16A^{(6)}_4 + 12A^{(7)}_{10} - 6A^{(7)}_{11},\\
T^{(6)}_{5ba} =& \tfrac{6}{5} - \tfrac{3}{2}\zeta_3 + \tfrac{1}{2}T^{(4)} - A^{(6)}_4 + A^{(7)}_{11},
\end{aligned}
\tag{6.25}
$$

We see that the effect of imposing symmetry has been to reduce the freedom in the a-function coefficients from the original six parameters to two.

7 Conclusions

We have shown how scheme changes in ϕ^4 theory may be analysed within a compact and efficient framework. In particular we have derived the full set of scheme invariants up to four loop order and shown that their number is consistent with general expectations, though considerably higher than might be expected from a naive counting. In particular we have identified the existence of quadratic invariants which would be missed in a naive counting. Furthermore, we have shown that in the context of the Hopf algebra approach to renormalisation, each invariant is associated with a cocommutative combination of graphs. We have also considered the construction of the a-function generating the β-functions up to four-loop order via a gradient flow equation. In particular we have analysed the consistent conditions which guarantee this construction, again showing that their number is as expected and furthermore that, as expected, they may be expressed in terms of linear combinations of the scheme invariants. Finally we have considered one-vertex reducible diagrams and shown that there is a natural family of schemes in which these do not contribute to the β-function.

Future work might explore the Hopf algebra connection further. Furthermore, at higher orders than we have yet considered there might be the possibility of cubic and higher order invariants. The extension of the analysis presented here to gauge theories might present additional challenges.

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A General results

For a theory with couplings g^I, the corresponding β-functions are defined by

$$\beta^I(g) = \mu \frac{d}{d\mu} g^I \tag{A.1}$$

and the β-functions in a new renormalisation group scheme defined by $g'^I(g)$ are given by

$$\beta'^I(g') = \beta(g)_g g'^I, \tag{A.2}$$

where for any vector V in coupling space,

$$V_g \equiv V^J \frac{\partial}{\partial g^J}. \tag{A.3}$$

We choose to parametrise the redefined coupling as

$$g' = e^{v_g} g. \tag{A.4}$$

We then find using the easily proved result

$$f(e^{v_g} h) = e^{v_g} f(h) \tag{A.5}$$

that

$$\beta'(g) = e^{-v_g} \beta_g(g) e^{v_g} g. \tag{A.6}$$

Then using

$$[v_g, V_g] = (\mathcal{L}_v V)_g, \quad \mathcal{L}_v V = v_g V - V_g v, \tag{A.7}$$

together with

$$e^A B e^{-A} = B + [A, B] + \tfrac{1}{2}[A, [A, B]] + \ldots \tag{A.8}$$

we find

$$\beta'(g) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} {\mathcal{L}_v}^n \beta(g). \tag{A.9}$$

For our purposes it is useful to use this result in the form

$$\delta\beta(g) = \beta'(g) - \beta(g) = -\mathcal{L}_v \hat{\beta}, \tag{A.10}$$

where

$$\hat{\beta} = \beta - \tfrac{1}{2!}\mathcal{L}_v \beta + \tfrac{1}{3!}{\mathcal{L}_v}^2 \beta + \ldots \tag{A.11}$$

B Symmetric Hopf co-product

In this Appendix we give the full results for the co-commutative expressions on the right-hand sides of Eqs. (5.10), (5.11), (5.13) and (5.14). For the combinations correponding to four-loop linear invariants in Eq. (5.10), we have

$$
\begin{aligned}
C_1^{(4)L} =&0,\\
C_2^{(4)L} =&2g_\lambda{}^1 \otimes_s g_\lambda{}^{3a} + 2g_\lambda{}^1 \otimes_s g_\lambda{}^{3c} + 4g_\lambda{}^2 \otimes g_\lambda{}^2 + g_\lambda{}^{2R} \otimes g_\lambda{}^{2R},\\
C_3^{(4)L} =&g_\lambda{}^1 \otimes_s g_\lambda{}^{3a} + g_\lambda{}^1 \otimes_s g_\lambda{}^{3c} + 2g_\lambda{}^1 \otimes_s g_\lambda{}^{3e} + g_\lambda{}^2 \otimes_s g_\lambda{}^{2R} + 2g_\lambda{}^2 \otimes g_\lambda{}^2,\\
C_4^{(4)L} =&2g_\lambda{}^1 \otimes_s g_\lambda{}^{3e} + 2g_\lambda{}^2 \otimes_s g_\lambda{}^{2R} - 4g_\lambda{}^2 \otimes g_\lambda{}^2 - 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^2 - (g_\lambda{}^1)^2 \otimes_s g_\lambda{}^{2R},\\
C_5^{(4)L} =&2g_\lambda{}^1 \otimes_s g_\lambda{}^{3b} + g_\lambda{}^1 \otimes_s g_\gamma{}^3 + 2g_\lambda{}^2 \otimes_s g_\gamma{}^2,\\
C_6^{(4)L} =&g_\lambda{}^1 \otimes_s g_\lambda{}^{3a} + g_\lambda{}^1 \otimes_s g_\lambda{}^{3d} - g_\lambda{}^1 \otimes_s g_\lambda{}^{3bR} + g_\lambda{}^2 \otimes g_\lambda{}^2,\\
C_7^{(4)L} =&2g_\lambda{}^1 \otimes_s g_\lambda{}^{3c} - 2g_\lambda{}^1 \otimes_s g_\lambda{}^{3d} + g_\lambda{}^1 \otimes_s g_\lambda{}^{3e} + g_\lambda{}^1 \otimes_s g_\lambda{}^{3aR} + g_\lambda{}^1 \otimes_s g_\lambda{}^{3bR}\\
&+ 2g_\lambda{}^2 \otimes_s g_\lambda{}^{2R} - 2g_\lambda{}^2 \otimes g_\lambda{}^2 + g_\lambda{}^{2R} \otimes g_\lambda{}^{2R},\\
C_8^{(4)L} =&g_\lambda{}^2 \otimes_s g_\lambda{}^{2R} - 4g_\lambda{}^2 \otimes g_\lambda{}^2 - 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^2 + g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{2R},\\
C_9^{(4)L} =&2g_\lambda{}^2 \otimes_s g_\gamma{}^2 + g_\lambda{}^1 \otimes_s g_\lambda{}^{3b} + \tfrac{1}{2} g_\lambda{}^1 \otimes_s g_\gamma{}^3 + g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\gamma{}^2,\\
C_{10}^{(4)L} =&g_\lambda{}^1 \otimes_s g_\lambda{}^{3f},\\
C_{11}^{(4)L} =&2g_\lambda{}^{2R} \otimes_s g_\gamma{}^2 + 2g_\lambda{}^1 \otimes_s g_\lambda{}^{3b} + g_\lambda{}^1 \otimes_s g_\gamma{}^3 + 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\gamma{}^2,\\
C_{12}^{(4)L} =&2g_\lambda{}^1 \otimes_s g_\lambda{}^{3aR} - 2g_\lambda{}^1 \otimes_s g_\lambda{}^{3bR} + 2g_\lambda{}^2 \otimes g_\lambda{}^2 + 3g_\lambda{}^{2R} \otimes g_\lambda{}^{2R} - 2g_\lambda{}^2 \otimes_s g_\lambda{}^{2R},\\
C_{13}^{(4)L} =&2g_\lambda{}^2 \otimes g_\lambda{}^2 - g_\lambda{}^{2R} \otimes g_\lambda{}^{2R} + 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^2 + (g_\lambda{}^1)^2 \otimes_s g_\lambda{}^{2R},\\
C_{14}^{(4)L} =&4g_\lambda{}^2 \otimes g_\lambda{}^2 - g_\lambda{}^2 \otimes_s g_\lambda{}^{2R} - 2g_\lambda{}^1 \otimes_s g_\lambda{}^{3e} - g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{2R} - 2(g_\lambda{}^1)^2 \otimes_s g_\lambda{}^2.
\end{aligned}
\tag{B.1}
$$

For the combinations correponding to four-loop quadratic invariants in Eq. (5.11), we have

$$
\begin{aligned}
C_1^{(4)Q} =&2g_\lambda{}^1 \otimes_s g_\gamma{}^{4c} - g_\lambda{}^1 \otimes_s g_\gamma{}^{4d} + g_\gamma{}^2 \otimes_s g_\lambda{}^{3d} + 2g_\lambda{}^2 \otimes_s g_\lambda{}^{3b} + 2g_\lambda{}^1 \otimes_s g_\lambda{}^2 g_\gamma{}^2\\
&+ 2g_\lambda{}^2 \otimes_s g_\lambda{}^1 g_\gamma{}^2 + 2g_\lambda{}^1 g_\lambda{}^2 \otimes_s g_\gamma{}^2 - (g_\lambda{}^1)^2 \otimes_s g_\lambda{}^{3b} - (g_\lambda{}^1)^3 \otimes_s g_\gamma{}^2,\\
C_2^{(4)Q} =&g_\lambda{}^1 \otimes_s g_\lambda{}^{4eR} - g_\lambda{}^{2R} \otimes_s g_\lambda{}^{3b} - g_\gamma{}^2 \otimes_s g_\lambda{}^{3bR} + (g_\lambda{}^1)^2 \otimes_s g_\lambda{}^{3b} - g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{3b}\\
&- g_\lambda{}^1 g_\gamma{}^2 \otimes_s g_\lambda{}^2 - g_\lambda{}^2 g_\gamma{}^2 \otimes_s g_\lambda{}^1 - g_\lambda{}^{2R} g_\gamma{}^2 \otimes_s g_\lambda{}^1 - g_\lambda{}^1 g_\gamma{}^2 \otimes_s (g_\lambda{}^1)^2,\\
C_3^{(4)Q} =&g_\lambda{}^1 \otimes_s (g_\lambda{}^{4dR} - 2g_\lambda{}^{4gR}) + 2g_\lambda{}^2 \otimes_s g_\lambda{}^{3aR} - g_\lambda{}^{2R} \otimes_s g_\lambda{}^{3d}\\
&+ (g_\lambda{}^1)^2 \otimes_s g_\lambda{}^{3c} + g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{3c} - 2(g_\lambda{}^1)^2 \otimes_s g_\lambda{}^{3e} - 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{3e}\\
&+ g_\lambda{}^{2R} \otimes_s g_\lambda{}^1 g_\lambda{}^{2R} + 4g_\lambda{}^{2R} \otimes_s g_\lambda{}^1 g_\lambda{}^2 - 2g_\lambda{}^2 \otimes_s g_\lambda{}^1 g_\lambda{}^{2R} + 4g_\lambda{}^1 \otimes_s g_\lambda{}^2 g_\lambda{}^{2R}\\
&+ 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{3d} - 2g_\lambda{}^1 \otimes_s g_\lambda{}^1 g_\lambda{}^{3aR} - 2(g_\lambda{}^1)^2 \otimes_s g_\lambda{}^1 g_\lambda{}^2 + 5(g_\lambda{}^1)^2 \otimes_s g_\lambda{}^1 g_\lambda{}^{2R}.
\end{aligned}
\tag{B.2}
$$

For the combination corresponding to the five-loop linear invariant in Eq. (5.13), we have

$$
C_1^{(5)L} = - 2g_\gamma{}^2 \otimes_s g_\lambda{}^{3aR} + 4g_\gamma{}^2 \otimes_s g_\lambda{}^{3bR} - 2g_\lambda{}^1 \otimes_s g_\lambda{}^{4eR} - g_\lambda{}^{2R} \otimes_s g_\gamma{}^3
$$

$$+\,2g_\gamma^{2}\otimes_s g_\lambda^{1}g_\lambda^{2R}-4g_\lambda^{1}g_\lambda^{2}\otimes_s g_\gamma^{2}+(g_\lambda^{1})^2\otimes_s g_\gamma^{3}+2g_\lambda^{1}\otimes_s g_\lambda^{1}g_\lambda^{3b} \tag{B.3}$$

Finally, for the combinations corresponding to five-loop quadratic invariants in Eq. (5.14), we have

$$\begin{aligned}
C_1^{(5)Q}=&g_\lambda^{1}\otimes_s g_\lambda^{5a}+g_\gamma^{2}\otimes_s g_\lambda^{4p}+g_\lambda^{3b}\otimes_s g_\lambda^{3f}\\
&+g_\gamma^{2}\otimes_s g_\lambda^{1}g_\lambda^{3f}+g_\lambda^{3f}\otimes_s g_\lambda^{1}g_\gamma^{2}+g_\lambda^{1}\otimes_s g_\gamma^{2}g_\lambda^{3f},\\
C_2^{(5)Q}=&g_\lambda^{1}\otimes_s (g_\gamma^{5g}-2g_\gamma^{5k})-g_\gamma^{2}\otimes_s g_\lambda^{4c}+g_\gamma^{3}\otimes_s g_\lambda^{3b}\\
&-(g_\gamma^{3}+2g_\lambda^{3b})\otimes_s g_\lambda^{1}g_\gamma^{2}+2g_\lambda^{1}g_\gamma^{2}\otimes g_\lambda^{1}g_\gamma^{2},\\
C_3^{(5)Q}=&g_\lambda^{1}\otimes_s (g_\lambda^{5aR}-g_\lambda^{5bR})+g_\gamma^{2}\otimes_s (-g_\lambda^{4aR}+g_\lambda^{4cR})\\
&+g_\lambda^{3b}\otimes_s (-g_\lambda^{3aR}+g_\lambda^{3bR})+(g_\lambda^{1})^2\otimes_s g_\lambda^{4eR}+g_\lambda^{1}\otimes_s g_\lambda^{1}g_\lambda^{4eR}\\
&-2g_\lambda^{1}\otimes_s g_\lambda^{3b}g_\lambda^{2R}-g_\lambda^{1}g_\lambda^{3b}\otimes_s g_\lambda^{2R}+g_\lambda^{3b}\otimes_s g_\lambda^{1}(g_\lambda^{2R}-g_\lambda^{2})\\
&+g_\lambda^{1}\otimes_s g_\lambda^{2}g_\lambda^{3b}-2g_\lambda^{1}g_\gamma^{2}\otimes_s g_\lambda^{3aR}-3g_\lambda^{1}\otimes_s g_\lambda^{3aR}g_\gamma^{2}+2g_\lambda^{1}\otimes_s g_\lambda^{3bR}g_\gamma^{2}\\
&+g_\lambda^{1}g_\gamma^{2}\otimes_s g_\lambda^{3bR}-3g_\lambda^{2R}\otimes_s g_\lambda^{2R}g_\gamma^{2}+g_\lambda^{2}\otimes_s g_\lambda^{2R}g_\gamma^{2}+g_\lambda^{2R}\otimes_s g_\lambda^{2}g_\gamma^{2}\\
&-g_\lambda^{1}g_\lambda^{2R}\otimes_s g_\lambda^{1}g_\gamma^{2}-(g_\lambda^{1})^2\otimes_s g_\lambda^{2R}g_\gamma^{2}+g_\lambda^{1}g_\lambda^{2}\otimes_s g_\lambda^{1}g_\gamma^{2},\\
C_4^{(5)Q}=&g_\lambda^{1}\otimes_s (g_\gamma^{5d}-g_\gamma^{5g})-2g_\gamma^{2}\otimes_s G_J-g_\gamma^{3}\otimes g_\gamma^{3}\\
&-g_\lambda^{1}\otimes_s g_\gamma^{2}g_\gamma^{3}-2(g_\lambda^{1}\otimes_s g_\lambda^{3b}g_\gamma^{2}+g_\lambda^{3b}\otimes_s g_\lambda^{1}g_\gamma^{2})+2g_\lambda^{1}\otimes_s g_\lambda^{1}(g_\gamma^{2})^2\\
&+2g_\lambda^{1}g_\gamma^{2}\otimes g_\lambda^{1}g_\gamma^{2},\\
C_5^{(5)Q}=&g_\lambda^{1}\otimes_s (2g_\lambda^{5bR}-g_\lambda^{5cR})+g_\lambda^{2R}\otimes_s G_J\\
&+g_\gamma^{2}\otimes_s (g_\lambda^{4aR}-2g_\lambda^{4cR})+g_\gamma^{3}\otimes_s g_\lambda^{3bR}+g_\lambda^{1}\otimes_s g_\lambda^{1}g_\lambda^{4c}-(g_\lambda^{1})^2\otimes_s g_\lambda^{4c}\\
&-(g_\lambda^{1})^2\otimes_s g_\lambda^{4e}+(g_\lambda^{1})^2\otimes_s g_\lambda^{4j}-(g_\lambda^{1})^2\otimes_s g_\lambda^{4eR}-2g_\lambda^{1}\otimes_s g_\lambda^{1}g_\lambda^{4eR}\\
&+2g_\lambda^{2}\otimes_s g_\lambda^{1}g_\lambda^{3b}+2g_\lambda^{1}g_\lambda^{2}\otimes_s g_\lambda^{3b}+2g_\lambda^{1}\otimes_s g_\lambda^{2R}g_\lambda^{3b}+g_\lambda^{2R}\otimes_s g_\lambda^{1}g_\lambda^{3b}\\
&+2g_\lambda^{3b}\otimes_s g_\lambda^{1}g_\lambda^{2R}+2g_\lambda^{3aR}\otimes_s g_\lambda^{1}g_\gamma^{2}+2g_\gamma^{2}g_\lambda^{3aR}\otimes_s g_\lambda^{1}-2g_\gamma^{2}g_\lambda^{3bR}\otimes_s g_\lambda^{1}\\
&-2g_\lambda^{3bR}\otimes_s g_\lambda^{1}g_\gamma^{2}+2g_\lambda^{1}g_\lambda^{3bR}\otimes_s g_\gamma^{2}+g_\lambda^{1}\otimes_s g_\lambda^{2R}g_\gamma^{3}+g_\lambda^{2}\otimes_s g_\lambda^{1}g_\gamma^{3}\\
&+g_\lambda^{1}\otimes_s g_\lambda^{2}g_\gamma^{3}-2g_\lambda^{2R}\otimes_s g_\lambda^{2}g_\gamma^{2}-2g_\lambda^{2}\otimes_s g_\lambda^{2R}g_\gamma^{2}\\
&+3g_\gamma^{2}g_\lambda^{2R}\otimes_s g_\lambda^{2R}-2(g_\lambda^{1})^3\otimes_s g_\lambda^{3b}+2(g_\lambda^{1})^2\otimes_s g_\lambda^{3b}g_\lambda^{1}+2g_\lambda^{1}\otimes_s g_\lambda^{3b}(g_\lambda^{1})^2\\
&+(g_\lambda^{1})^2\otimes_s g_\lambda^{1}g_\gamma^{3}+2(g_\lambda^{1})^2\otimes_s g_\lambda^{2}g_\gamma^{2}+(g_\lambda^{1})^2\otimes_s g_\lambda^{2R}g_\gamma^{2}+2(g_\lambda^{1})^3\otimes_s g_\lambda^{1}g_\gamma^{2},\\
C_6^{(5)Q}=&g_\lambda^{1}\otimes_s (g_\gamma^{5c}+g_\gamma^{5e}-2g_\gamma^{5h})-g_\gamma^{2}\otimes_s (g_\lambda^{4b}-2g_\lambda^{4d}-g_\lambda^{4aR}+2g_\lambda^{4cR})\\
&+g_\lambda^{2R}\otimes_s G_J-g_\gamma^{3}\otimes_s (g_\lambda^{3c}-g_\lambda^{3d})\\
&-(g_\lambda^{1})^2\otimes_s g_\lambda^{4c}-(g_\lambda^{1})^2\otimes_s g_\lambda^{4e}+(g_\lambda^{1})^2\otimes_s g_\lambda^{4j}-2g_\lambda^{1}\otimes_s g_\lambda^{1}g_\lambda^{4eR}\\
&-(g_\lambda^{1})^2\otimes_s g_\lambda^{4eR}+2g_\lambda^{1}\otimes_s g_\lambda^{2R}g_\lambda^{3b}+g_\lambda^{2R}\otimes_s g_\lambda^{1}g_\lambda^{3b}+2g_\lambda^{3b}\otimes_s g_\lambda^{1}g_\lambda^{2R}\\
&-2g_\lambda^{1}\otimes_s (g_\lambda^{3c}-g_\lambda^{3d}+g_\lambda^{3bR})g_\gamma^{2}+2g_\lambda^{3aR}\otimes_s g_\lambda^{1}g_\gamma^{2}+2g_\lambda^{1}\otimes_s g_\gamma^{2}g_\lambda^{3aR}\\
&-2g_\lambda^{3bR}\otimes_s g_\lambda^{1}g_\gamma^{2}+g_\gamma^{3}\otimes_s g_\lambda^{1}g_\lambda^{2R}+2g_\lambda^{2}\otimes_s g_\lambda^{2}g_\gamma^{2}-2g_\lambda^{2R}\otimes_s g_\lambda^{2}g_\gamma^{2}\\
&-2g_\lambda^{2}\otimes_s g_\lambda^{2R}g_\gamma^{2}+2g_\lambda^{2R}\otimes_s g_\lambda^{2R}g_\gamma^{2}-2g_\lambda^{1}g_\lambda^{2R}\otimes_s g_\lambda^{1}g_\gamma^{2}+(g_\lambda^{1})^2\otimes_s g_\lambda^{2R}g_\gamma^{2}\\
&-(g_\lambda^{1})^3\otimes_s (g_\gamma^{3}+2g_\lambda^{3b})+2(g_\lambda^{1})^3\otimes_s g_\lambda^{1}g_\gamma^{2},
\end{aligned}$$

$$
\begin{aligned}
C_7^{(5)Q} =& g_\lambda^{\,1} \otimes_s (2g_\gamma^{\,5b} - g_\gamma^{\,5e}) + g_\gamma^{\,2} \otimes_s (g_\lambda^{\,4l} - g_\lambda^{\,4o} + 2g_\lambda^{\,4r}) \\
& + (2g_\lambda^{\,2} - g_\lambda^{\,2R}) \otimes_s G_J + 4g_\lambda^{\,3b} \otimes_s g_\lambda^{\,3e} + g_\lambda^{\,3c} \otimes_s g_\gamma^{\,3} \\
& + 2g_\lambda^{\,3b} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2} + 4g_\lambda^{\,2} \otimes_s g_\lambda^{\,1} g_\lambda^{\,3b} + 6g_\lambda^{\,1} \otimes_s g_\lambda^{\,2} g_\lambda^{\,3b} - g_\lambda^{\,3b} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2R} \\
& - g_\lambda^{\,1} \otimes_s g_\lambda^{\,2R} g_\lambda^{\,3b} - g_\lambda^{\,3c} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} + g_\lambda^{\,1} \otimes_s g_\gamma^{\,2} g_\lambda^{\,3c} + 4g_\gamma^{\,2} \otimes_s g_\lambda^{\,1} g_\lambda^{\,3e} \\
& + 4g_\lambda^{\,3e} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} + 4g_\lambda^{\,1} \otimes_s g_\gamma^{\,2} g_\lambda^{\,3e} + 2g_\gamma^{\,3} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2} + 2g_\lambda^{\,1} \otimes_s g_\lambda^{\,2} g_\gamma^{\,3} \\
& + 2g_\lambda^{\,2} \otimes_s g_\lambda^{\,1} g_\gamma^{\,3} - g_\gamma^{\,3} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2R} + 2g_\lambda^{\,2} \otimes_s g_\lambda^{\,2} g_\gamma^{\,2} + g_\lambda^{\,1} g_\lambda^{\,2R} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} \\
& + 2g_\lambda^{\,1} g_\lambda^{\,2} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} + 4(g_\lambda^{\,1})^2 \otimes_s g_\lambda^{\,2} g_\gamma^{\,2}, \\
C_8^{(5)Q} =& g_\lambda^{\,1} \otimes_s (2g_\gamma^{\,5e} + g_\gamma^{\,5f} - 4g_\gamma^{\,5i}) - g_\gamma^{\,2} \otimes_s (2g_\lambda^{\,4b} - 3g_\lambda^{\,4aR} + 4g_\lambda^{\,4cR}) \\
& + (5g_\lambda^{\,2R} - 8g_\lambda^{\,2}) \otimes_s G_J \\
& + g_\gamma^{\,3} \otimes_s (2g_\lambda^{\,3a} - 2g_\lambda^{\,3c} - g_\lambda^{\,3aR} + 2g_\lambda^{\,3bR}) + 2g_\lambda^{\,1} \otimes_s g_\lambda^{\,1} g_\lambda^{\,4c} \\
& + 2g_\lambda^{\,1} \otimes_s g_\lambda^{\,1} g_\lambda^{\,4e} - 2g_\lambda^{\,1} \otimes_s g_\lambda^{\,1} g_\lambda^{\,4j} + (g_\lambda^{\,1})^2 \otimes_s g_\gamma^{\,4c} + 2g_\lambda^{\,1} \otimes_s g_\lambda^{\,1} g_\gamma^{\,4c} \\
& + 8g_\lambda^{\,3aR} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} + 6g_\lambda^{\,1} \otimes_s g_\gamma^{\,2} g_\lambda^{\,3aR} - 8g_\lambda^{\,3bR} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} - 4g_\lambda^{\,1} \otimes_s g_\gamma^{\,2} g_\lambda^{\,3bR} \\
& - 5g_\lambda^{\,3a} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} - g_\lambda^{\,1} \otimes_s g_\gamma^{\,2} g_\lambda^{\,3a} - g_\gamma^{\,2} \otimes_s g_\lambda^{\,1} g_\lambda^{\,3a} - 8g_\lambda^{\,1} \otimes_s g_\lambda^{\,2} g_\lambda^{\,3b} \\
& - 8g_\lambda^{\,3b} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2} + 5g_\lambda^{\,1} \otimes_s g_\lambda^{\,2R} g_\lambda^{\,3b} + 5g_\lambda^{\,3b} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2R} - 4g_\lambda^{\,1} \otimes_s g_\gamma^{\,2} g_\lambda^{\,3c} \\
& - g_\lambda^{\,1} \otimes_s g_\lambda^{\,2} g_\gamma^{\,3} - 7g_\gamma^{\,3} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2} - g_\lambda^{\,2} \otimes_s g_\lambda^{\,1} g_\gamma^{\,3} + 4g_\gamma^{\,3} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2R} \\
& - 4g_\lambda^{\,2R} \otimes_s g_\lambda^{\,2} g_\gamma^{\,2} - 4g_\lambda^{\,2} \otimes_s g_\lambda^{\,2R} g_\gamma^{\,2} + 7g_\lambda^{\,2R} \otimes_s g_\lambda^{\,2R} g_\gamma^{\,2} + (g_\lambda^{\,1})^2 \otimes_s g_\lambda^{\,1} g_\gamma^{\,3} \\
& + 2(g_\lambda^{\,1})^2 \otimes_s g_\lambda^{\,1} g_\lambda^{\,3b} - 2(g_\lambda^{\,1})^2 \otimes_s g_\lambda^{\,2} g_\gamma^{\,2} + 8g_\lambda^{\,1} g_\lambda^{\,2} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} \\
& - 6g_\lambda^{\,1} g_\lambda^{\,2R} \otimes_s g_\lambda^{\,1} g_\gamma^{\,2} + 2g_\lambda^{\,1} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2} g_\gamma^{\,2} - 3g_\lambda^{\,1} \otimes_s g_\lambda^{\,1} g_\lambda^{\,2R} g_\gamma^{\,2} \\
& - 2(g_\lambda^{\,1})^2 \otimes_s (g_\lambda^{\,1})^2 g_\gamma^{\,2}.
\end{aligned}
\tag{B.4}
$$

C Differential operators for scheme changes

Following the general considerations of Ref. [8] we may define differential operators

$$
Y = \sum_{l',s} (\delta_{l,s} Y^{\lambda l' s} + \epsilon_{l' s} Y^{\gamma l' s}), \tag{C.1}
$$

where

$$
Y^{\lambda l' s} = \sum_{l,r} (c_{lr} \mathcal{D}^{\lambda lr, \lambda l' s} + d_{lr} \mathcal{D}^{\gamma lr, \lambda l' s}), \quad Y^{\gamma l' s} = \sum_{l,r} (c_{lr} \mathcal{D}^{\lambda lr, \gamma l' s} + d_{lr} \mathcal{D}^{\gamma lr, \gamma l' s}), \tag{C.2}
$$

which generate scheme changes according to

$$
\{c_{lr}, d_{lr}\} \to \exp(Y) \{c_{lr}, d_{lr}\}. \tag{C.3}
$$

Here $\{r, s\}$ label the β or γ function coefficients at each loop order $\{l, l'\}$. The operators $\mathcal{D}^{\lambda lr, \lambda l' s}$, etc satisfy

$$
\mathcal{D}^{\lambda lr, \lambda l' s} = -\mathcal{D}^{\lambda l' s, \lambda lr}, \quad \mathcal{D}^{\lambda lr, \gamma l' s} = -\mathcal{D}^{\gamma l' s, \lambda lr}, \quad \mathcal{D}^{\gamma lr, \gamma l' s} = -\mathcal{D}^{\gamma l' s, \gamma lr}, \tag{C.4}
$$

Scheme invariants are then determined as polynomial functions $F(\{c_{lr}, d_{lr}\}$ such that

$$Y^{\lambda lr} F = Y^{\gamma lr} F = 0 \tag{C.5}$$

for all λ, r.

In the case of ϕ^4 theory we find at lowest order

$$\begin{aligned}
\mathcal{D}^{\lambda 1,\lambda 2} =& -2\frac{\partial}{\partial c_{3a}} + 2\frac{\partial}{\partial c_{3c}} + 2\frac{\partial}{\partial c_{3d}}, \\
\mathcal{D}^{\lambda 1,\lambda 2R} =& 2\frac{\partial}{\partial c_{3a}} - 2\frac{\partial}{\partial c_{3c}} + \frac{\partial}{\partial c_{3aR}} + 2\frac{\partial}{\partial c_{3bR}}, \\
\mathcal{D}^{\lambda 1,\gamma 2} =& -2\frac{\partial}{\partial c_{3b}} + 6\frac{\partial}{\partial d_3},
\end{aligned} \tag{C.6}$$

and at next-to-leading order

$$\begin{aligned}
\mathcal{D}^{\lambda 1,\lambda 3a} =& 4\frac{\partial}{\partial c_{4a}} + 2\frac{\partial}{\partial c_{4b}} + 2\frac{\partial}{\partial c_{4d}} - 2\frac{\partial}{\partial c_{4f}}, \\
\mathcal{D}^{\lambda 1,\lambda 3b} =& 6\frac{\partial}{\partial c_{4c}} - 2\frac{\partial}{\partial c_{4e}} + \frac{\partial}{\partial c_{4j}}, \\
\mathcal{D}^{\lambda 1,\lambda 3c} =& -2\frac{\partial}{\partial c_{4b}} + 2\frac{\partial}{\partial c_{4f}} + 3\frac{\partial}{\partial c_{4g}} + 2\frac{\partial}{\partial c_{4k}} + \frac{\partial}{\partial c_{4o}}, \\
\mathcal{D}^{\lambda 1,\lambda 3d} =& -2\frac{\partial}{\partial c_{4d}} - 2\frac{\partial}{\partial c_{4h}} - 2\frac{\partial}{\partial c_{4i}} + 2\frac{\partial}{\partial c_{4o}} + 2\frac{\partial}{\partial c_{4r}}, \\
\mathcal{D}^{\lambda 1,\lambda 3e} =& -4\frac{\partial}{\partial c_{4a}} + 2\frac{\partial}{\partial c_{4h}} + 2\frac{\partial}{\partial c_{4i}} + \frac{\partial}{\partial c_{4k}} + 2\frac{\partial}{\partial c_{4l}} + \frac{\partial}{\partial c_{4r}}, \\
\mathcal{D}^{\lambda 1,\lambda 3f} =& \frac{\partial}{\partial c_{4p}} - \frac{\partial}{\partial c_{4q}}, \\
\mathcal{D}^{\lambda 1,\lambda 3aR} =& -2\frac{\partial}{\partial c_{4g}} + 2\frac{\partial}{\partial c_{4aR}} + 2\frac{\partial}{\partial c_{4cR}} + 2\frac{\partial}{\partial c_{4fR}}, \\
\mathcal{D}^{\lambda 1,\lambda 3bR} =& 2\frac{\partial}{\partial c_{4d}} - 2\frac{\partial}{\partial c_{4k}} + 4\frac{\partial}{\partial c_{4bR}} + \frac{\partial}{\partial c_{4cR}} + 2\frac{\partial}{\partial c_{4dR}} + 2\frac{\partial}{\partial c_{4gR}} - \frac{\partial}{\partial c_{4fR}}, \\
\mathcal{D}^{\lambda 1,\gamma 3} =& -2\frac{\partial}{\partial c_{4c}} + 3\frac{\partial}{\partial d_{4b}} + 2\frac{\partial}{\partial d_{4c}} + 4\frac{\partial}{\partial d_{4d}}, \\
\mathcal{D}^{\lambda 2,\lambda 2R} =& 4\frac{\partial}{\partial c_{4a}} - 2\frac{\partial}{\partial c_{4g}} - 2\frac{\partial}{\partial c_{4l}} - \frac{\partial}{\partial c_{4o}} + 2\frac{\partial}{\partial c_{4gR}} + 2\frac{\partial}{\partial c_{4fR}}, \\
\mathcal{D}^{\lambda 2,\gamma 2} =& -2\frac{\partial}{\partial c_{4e}} - \frac{\partial}{\partial c_{4j}} + 6\frac{\partial}{\partial d_{4c}} + 6\frac{\partial}{\partial d_{4d}}, \\
\mathcal{D}^{\lambda 2R,\gamma 2} =& -2\frac{\partial}{\partial c_{4eR}} + 6\frac{\partial}{\partial d_{4b}},
\end{aligned} \tag{C.7}$$

Note that here we suppress the label r in the case of the one-loop β-function and the two-loop γ-function where there is only one coefficient.

The $Y^{\lambda lr}$ and $Y^{\gamma lr}$ defined according to Eq. (C.2) satisfy the commutation relations

$$\begin{aligned}
[Y^{\lambda 1}, Y^{\lambda 2}] =& -2Y^{\lambda 3a} + 2Y^{\lambda 3c} + 2Y^{\lambda 3e}, \\
[Y^{\lambda 1}, Y^{\lambda 2R}] =& 2Y^{\lambda 3a} - 2Y^{\lambda 3c} + Y^{\lambda 3aR} + 2Y^{\lambda 3bR}, \\
[Y^{\lambda 1}, Y^{\gamma 2}] =& -2Y^{\lambda 3b} + 6Y^{\gamma 3},
\end{aligned} \tag{C.8}$$

and

$$\begin{aligned}
[Y^{\lambda 1}, Y^{\lambda 3a}] =& 4Y^{\lambda 4a} + 2Y^{\lambda 4b} + 2Y^{\lambda 4d} - 2Y^{\lambda 4f}, \\
[Y^{\lambda 1}, Y^{\lambda 3b}] =& 6Y^{\lambda 4c} - 2Y^{\lambda 4e} + Y^{\lambda 4j}, \\
[Y^{\lambda 1}, Y^{\lambda 3c}] =& -2Y^{\lambda 4b} + 2Y^{\lambda 4f} + 3Y^{\lambda 4g} + 2Y^{\lambda 4k} + Y^{\lambda 4o}, \\
[Y^{\lambda 1}, Y^{\lambda 3d}] =& -2Y^{\lambda 4d} - 2Y^{\lambda 4h} - 2Y^{\lambda 4i} + 2Y^{\lambda 4o} + 2Y^{\lambda 4r}, \\
[Y^{\lambda 1}, Y^{\lambda 3e}] =& -4Y^{\lambda 4a} + 2Y^{\lambda 4h} + 2Y^{\lambda 4i} + Y^{\lambda 4k} + 2Y^{\lambda 4l} + Y^{\lambda 4r}, \\
[Y^{\lambda 1}, Y^{\lambda 3f}] =& Y^{\lambda 4p} - Y^{\lambda 4q}, \\
[Y^{\lambda 1}, Y^{\lambda 3aR}] =& -2Y^{\lambda 4g} + 2Y^{\lambda 4aR} + 2Y^{\lambda 4cR} + 2Y^{\lambda 4fR}, \\
[Y^{\lambda 1}, Y^{\lambda 3bR}] =& 2Y^{\lambda 4d} - 2Y^{\lambda 4k} + 4Y^{\lambda 4bR} + Y^{\lambda 4cR} + 2Y^{\lambda 4dR} + 2Y^{\lambda 4gR} - Y^{\lambda 4fR}, \\
[Y^{\lambda 1}, Y^{\gamma 3}] =& -4Y^{\lambda 4c} + 3Y^{\gamma 4b} + 2Y^{\gamma 4c} + 4Y^{\gamma 4d}, \\
[Y^{\lambda 2}, Y^{\lambda 2R}] =& 4Y^{\lambda 4a} - 2Y^{\lambda 4g} - 2Y^{\lambda 4l} - Y^{\lambda 4o} + 2Y^{\lambda 4gR} + 2Y^{\lambda 4fR}, \\
[Y^{\lambda 2}, Y^{\gamma 2}] =& -2Y^{\lambda 4e} - Y^{\lambda 4j} + 6Y^{\gamma 4c} + 6Y^{\gamma 4d}, \\
[Y^{\lambda 2R}, Y^{\gamma 2}] =& -2Y^{\lambda 4eR} + 6Y^{\gamma 4b},
\end{aligned} \tag{C.9}$$

Note that the structure constants appearing in Eqs. (C.8), (C.9) are the same as those in Eqs. (C.6), (C.7), which is a consequence of the Jacobi identities following from the associativity of the graph insertion process as described in Ref. [8]. At the following order we have

$$\begin{aligned}
\mathcal{D}^{\lambda 3a,\gamma 2} =& 3\frac{\partial}{\partial d_{5i}}, \\
\mathcal{D}^{\lambda 3b,\gamma 2} =& 3\frac{\partial}{\partial d_{5k}}, \\
\mathcal{D}^{\lambda 3c,\gamma 2} =& 6\frac{\partial}{\partial d_{5e}} + 3\frac{\partial}{\partial d_{5j}}, \\
\mathcal{D}^{\lambda 3d,\gamma 2} =& 6\frac{\partial}{\partial d_{5h}}, \\
\mathcal{D}^{\lambda 3e,\gamma 2} =& 12\frac{\partial}{\partial d_{5b}} + 6\frac{\partial}{\partial d_{5c}} + 3\frac{\partial}{\partial d_{5h}},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{D}^{\lambda 3f,\gamma 2} =&2\frac{\partial}{\partial d_{5a}},\\
\mathcal{D}^{\lambda 3aR,\gamma 2} =&6\frac{\partial}{\partial d_{5f}} - 2\frac{\partial}{\partial c_{5aR}},\\
\mathcal{D}^{\lambda 3bR,\gamma 2} =&3\frac{\partial}{\partial d_{5i}} + 3\frac{\partial}{\partial d_{5j}} - 2\frac{\partial}{\partial c_{5bR}},\\
\mathcal{D}^{\lambda 1,\gamma 4a} =&6\frac{\partial}{\partial d_{5d}} + 2\frac{\partial}{\partial d_{5g}} + 2\frac{\partial}{\partial d_{5k}}\\
\mathcal{D}^{\lambda 1,\gamma 4b} =&4\frac{\partial}{\partial d_{5f}} + 2\frac{\partial}{\partial d_{5i}} + 2\frac{\partial}{\partial d_{5j}},\\
\mathcal{D}^{\lambda 1,\gamma 4c} =&4\frac{\partial}{\partial d_{5b}} + 2\frac{\partial}{\partial d_{5e}} + 2\frac{\partial}{\partial d_{5h}} + \frac{\partial}{\partial d_{5i}},\\
\mathcal{D}^{\lambda 1,\gamma 4d} =&4\frac{\partial}{\partial d_{5c}} + 2\frac{\partial}{\partial d_{5h}} + 2\frac{\partial}{\partial d_{5j}},\\
\mathcal{D}^{\lambda 1,\lambda 4eR} =&\frac{\partial}{\partial c_{5aR}} + 2\frac{\partial}{\partial c_{5bR}} + 6\frac{\partial}{\partial c_{5cR}},\\
\mathcal{D}^{\lambda 2,\gamma 3} =&4\frac{\partial}{\partial d_{5b}} + 4\frac{\partial}{\partial d_{5c}} + 2\frac{\partial}{\partial d_{5h}} + 2\frac{\partial}{\partial d_{5i}} + \frac{\partial}{\partial d_{5j}},\\
\mathcal{D}^{\lambda 2R,\gamma 3} =&2\frac{\partial}{\partial d_{5e}} + 3\frac{\partial}{\partial d_{5f}} + 2\frac{\partial}{\partial d_{5j}} - 2\frac{\partial}{\partial c_{5cR}},\\
\mathcal{D}^{\gamma 2,\gamma 3} =&\frac{\partial}{\partial d_{5g}} + 2\frac{\partial}{\partial d_{5k}} - 3\frac{\partial}{\partial d_{5d}},
\end{aligned}
\tag{C.10}
$$

with, correspondingly, the commutation relations

$$
\begin{aligned}
[Y^{\lambda 3a}, Y^{\gamma 2}] =&3Y^{\gamma 5i},\\
[Y^{\lambda 3b}, Y^{\gamma 2}] =&3Y^{\gamma 5k},\\
[Y^{\lambda 3d}, Y^{\gamma 2}] =&6Y^{\gamma 5e} + 3Y^{\gamma 5j},\\
[Y^{\lambda 3d}, Y^{\gamma 2}] =&6Y^{\gamma 5h},\\
[Y^{\lambda 3e}, Y^{\gamma 2}] =&12Y^{\gamma 5b} + 6Y^{\gamma 5d} + 3Y^{\gamma 5h},\\
[Y^{\lambda 3f}, Y^{\gamma 2}] =&2Y^{\gamma 5a},\\
[Y^{\lambda 3aR}, Y^{\gamma 2}] =&6Y^{\gamma 5f} - 2Y^{\gamma 5aR},\\
[Y^{\lambda 3bR}, Y^{\gamma 2}] =&3Y^{\gamma 5i} + 3Y^{\gamma 5j} - 2Y^{\lambda 5bR},\\
[Y^{\lambda 1}, Y^{\gamma 4a}] =&6Y^{\gamma 5d} + 2Y^{\gamma 5g} + 2Y^{\gamma 5k}\\
[Y^{\lambda 1}, Y^{\gamma 4b}] =&4Y^{\gamma 5f} + 2Y^{\gamma 5i} + 2Y^{\gamma 5j},\\
[Y^{\lambda 1}, Y^{\gamma 4c}] =&4Y^{\gamma 5b} + 2Y^{\gamma 5e} + 2Y^{\gamma 5h} + 2Y^{\gamma 5i},\\
[Y^{\lambda 1}, Y^{\gamma 4d}] =&Y^{\gamma 5c} + 2Y^{\gamma 5h} + 2Y^{\gamma 5j},\\
[Y^{\lambda 1}, Y^{\lambda 4eR}] =&Y^{\lambda 5aR} + 2Y^{\lambda 5bR} + 6Y^{\lambda 5cR},
\end{aligned}
$$

$$
\begin{aligned}
[Y^{\lambda 2}, Y^{\gamma 3}] =& 4Y^{\gamma 5b} + 4Y^{\gamma 5c} + 2Y^{\gamma 5h} + 2Y^{\gamma 5i} + Y^{\gamma 5j}, \\
[Y^{\lambda 2R}, Y^{\gamma 3}] =& 2Y^{\gamma 5e} + 3Y^{\gamma 5f} + 2Y^{\gamma 5j} - 2Y^{\lambda 5cR}, \\
[Y^{\gamma 2}, Y^{\gamma 3}] =& Y^{\gamma 5g} + 2Y^{\gamma 5k} - 3Y^{\gamma 5d},
\end{aligned}
\tag{C.11}
$$

It is readily verified using Eqs. (C.2), (C.6), (C.7), (C.10) that the linear and quadratic invariants constructed in previous sections satisfy Eq. (C.5).

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