# Potential Games with Aggregation in Non-cooperative General Insurance Markets

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**Abstract**

In the global insurance market, the number of product-specific policies from different companies has increased significantly, and strong market competition has boosted the demand for a competitive premium. Thus, in the present paper, by considering the competition between each pair of insurers, an N-player game is formulated to investigate the optimal pricing strategy by calculating the Nash equilibrium in an insurance market. Under that framework, each insurer is assumed to maximise its utility of wealth over the unit time interval. With the purpose of solving a game of N-players, the best-response potential game with non-linear aggregation is implemented. The existence of a Nash equilibrium is proved by finding a potential function of all insurers' payoff functions. A 12-player insurance game illustrates the theoretical findings under the framework in which the best-response selection premium strategies always provide the global maximum value of the corresponding payoff function.

**Keywords:** *Insurance Market Competition; Non-life Insurance; Potential Game with Aggregation; Pure Nash Equilibrium*

## 1 Introduction

### 1.1 Motivation

In the insurance world, determining an appropriate and attractive premium is always a highly challenging issue because of the competition among different companies. The premium loading depends critically on the price that the other insurers charge for comparable policies. Clapp (1985) was able to demonstrate it using the seminal model by Rothschild and Stiglitz (1976, 1992). Insurance pricing is a fundamental aspect that attracts the interest of both actuaries and academics. Standard actuarial approaches for non-life insurance products suggest that the premium is divided into three main components: the *actuarial price*, the *safety loading*, and the *loading for expenses*. The actuarial price is normally deduced according to different premium principles, such as the *Net Premium Principle*, the *Expected Value Premium Principle*, and others (Rolski et al., 2009; Teugels and Sundt, 2004). Classical approaches focus on determining the safety loading of each policy class proportional to the expected claim expenses or to its moment.

However, in a highly competitive insurance environment which is dominated by a relatively small number of companies (compared with the banking sector and investment funds), each insurer monitors, attempts to predict reactions, and takes advantages against the others. Thus,

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the actuarial premium might eventually be altered by the marketing and management department for several reasons, such as the customer's affordability, the market conditions, and the mutualisation across the portfolio of customers to decrease risk. What is more, the pricing cycles, which are found in different lines of insurance, appear also to be affected by market competition (Rantala, 1988; Malinovskii, 2010; Emms, 2012). These suggestions indicate that the insurance premium price should not focus only on the risk assessment. Consequently, to study the competition among insurers, a model needs to be formulated in order to investigate insurers' premium pricing interactions in the corresponding market.

## 1.2 Developments in Competitive Insurance Markets

Over the last three decades academics have been interested in how competition might affect insurance premiums and how insurers respond to changes in the premium levels that being offered by competitors.

Taylor (1986) was the first from the actuarial community who mentioned that competition is a key component in insurance premium pricing, and he used the Australian market to extract very useful remarks. The premium was priced based on unit of exposure, which is applicable in different lines of non-life insurance. Analytically, the relation between the market's behaviour and optimal response of an individual insurer was explored, with the objective of maximising the expected present value of wealth arising from a pre-defined finite time horizon. The law of demand was embedded in the modelling process to analyse the change of exposure volume through a comparison between insurer's and market average premiums. Moreover, he stated that the optimal response depends on various factors, including: a) the predicted time that will elapse before a return of market rates to profitability; b) the price elasticity of demand for the insurance product under consideration; and c) the rate of return required on the capital supporting the insurance operation. With his next paper, Taylor (1987) noted that the optimum underwriting strategies might be substantially affected by proper marginal expense rates, a concept that must be taken also into consideration.

After almost two decades of silence, Emms and his co-authors were able to extend significantly Taylor (1986, 1987)'s ideas developing a series of models in continuous time by implementing optimal control theory techniques (Emms and Haberman, 2005; Emms, 2007a; Emms et al., 2007; Emms, 2007b; Emms and Haberman, 2009; Emms, 2011). In more detail, Emms and Haberman (2005) assumed that the average premium is a positive random process with finite mean at time $t$, and left unspecified the distribution of the mean claim size process; while Emms et al. (2007) modelled the market's average premium as a geometric Brownian motion instead. Simultaneously, Emms (2007a) determined the optimal strategy for an insurer that maximises a particular objective over a fixed planning horizon and the premium by using a competitive demand model and the expected main claim size. Moreover, Emms (2007b) considered the process with different types of constrains. By assuming a deterministic control framework, the optimisation problem was solved using elements from control parametrisation. Market reactions regarding one insurer's premium were considered in (Emms, 2011).

In all previous approaches, fixed premium strategies were considered and sensitivity analysis of the parameters' involved in the model was applied. What is more, the ratio of initial market average premium to breakeven premium, the measure of the inverse elasticity of the demand function, and the non-dimensional drift of the market average premium were the most influential parameters in the optimal strategies derived.

Following previous work, Pantelous and Passalidou (2013) proposed a stochastic demand function for the volume of business in a discrete-time framework. Later on, in (Pantelous and Passalidou, 2015), the volume of business was formed as a general stochastic demand function, and making the model more pragmatic and realistic. Moreover, in (Pantelous and Passalidou, 2016), the volume of business was modelled as a non-linear function with respect to the accumulated reserves, the premium and the noise. What is more, a quadratic performance criterion concerning the utility function was implemented.

## 1.3 Game-Theoretic Approaches

However, for most of the models and approaches discussed in the previous subsection, a common assumption was made that there exists a single insurer, whose pricing strategy does not cause any reaction to the rest of the market's competitors. Thus, for each participant in the insurance market, others reaction cannot be observed, and the premium remains eventually unaffected by their actions. In reality, this situation is not often the case.

Lately, game theoretical approaches have been introduced mostly in the premium pricing processes of non-life insurance products. Competition among insurers reveals the pricing strategy of each market participant in a constructed insurance game, whereas one can only obtain a single insurer's pricing strategy through optimal control used in previous studies. However, in our approach, as it is discussed more extensively in the following subsection, a non-cooperative game model is designed for the insurance market implementing already well-defined parameters from the corresponding literature (Taylor, 1986, 1987; Emms et al., 2007; Pantelous and Passalidou, 2015).

The use of game theory in actuarial science has a long history. The first attempts go back to Borch (1962, 1974), Bühlmann (1980, 1984), and Lemaire (1984, 1991), who applied cooperative games to model insurer and reinsurer risk transfer; see also other extensions and reviews (Aase, 1993; Brockett and Xia, 1995; Tsanakas and Christofides, 2006; Boonen, 2015). Two models were applied in non-life insurance markets for non-cooperative games: a) the Bertrand oligopoly in which insurers set premiums and b) the Cournot oligopoly in which insurers choose the volume of business. See (Polborn, 1998; Rees et al., 1999; Dutang et al., 2013) for the Bertrand model and (Powers et al., 1998; Powers and Shubik, 1998) for the Cournot model.

Emms (2012) developed a model by applying a differential game-theoretic methodology for a non-cooperative market. Under his framework, each insurer's price depends on other insurers' premium strategies, assuming that each market participant chooses an optimal pricing strategy. Nevertheless, each insurer was assumed to maximize its utility of wealth at the terminal time of planning horizon. Finally, very recently, Boonen (2016) also proposed a way to optimally regulate bargaining for risk redistributions. Thus, he investigated the strategic interaction between two insurance companies that trade risk over-the-counter in a one-period model.

## 1.4 A New Approach: Potential Game with Aggregation

In our approach, a two-stage non-life insurance game is constructed in a competitive market. Numerical solutions of Nash equilibria are obtained for a large number of insurers under the two-stage framework. Moreover, instead of simply parametrizing competition through comparison between single insurer's premium and the market average premium as it has been done so far in the relevant literature (see Subsections 1.2 and 1.3), an aggregate game approach is formulated to investigate further the insurance market competition. Different from Emms (2012), the existence of Nash equilibrium is proved under our framework.

The concept of *aggregative* game, which was first proposed by Selten (1970) by considering it as the sum of the players' strategies, is applied broadly in our approach. Thus, the derived strategy for all insurers in the insurance market is presented as a single parameter, i.e., the aggregate. In greater detail, each insurer's utility (payoff) function only depends on its own pricing premium strategy and the aggregate parameter.

Also following the suggestions by Taylor (2008) and Emms (2012), market competition is measured by calculating an insurer's new volume of exposure and by summing up all of the policy flows during the competition between the insurers and the volume of exposure in a previous stage. A non-linear aggregate is obtained, which presents the strategies of all insurers in the market. Moreover, a potential game approach is further developed to prove the existence of a

Nash equilibrium in the insurance game. This approach also gives us an opportunity to simplify the problem of determining the Nash equilibrium by solving a single optimisation problem.[1]

The literature on potential games can be traced back to Monderer and Shapley (1996a,b), who created the *potential* game concept on the basis of a congestion game. Their technique did not only solve the congestion game itself but also was regarded as an equilibrium refinement tool. Following their idea, the best-response potential games were introduced and characterised by Voorneveld (2000). His paper proposed that, for any best-response potential game, if the potential has a maximum over its domain, the best-response potential game has a Nash equilibrium.

Dubey et al. (2006) were the first to embed the aggregate into potential games. By considering just a linear aggregation, they investigated a special type of best-response potential game that restricts the best-response selection to a continuously decreasing or increasing function. Then, any game with linear aggregation and a decreasing or increasing continuous best-response selection is proved to belong to a pseudo-potential game, which is pre-defined in their paper. By proving that any pseudo-potential game has a pure Nash equilibrium strategy, the existence of a Nash equilibrium was obtained in this special class of potential games irrespective of whether strategy sets were convex or payoff functions were quasi-concave.

In this paper, for the first time according to our knowledge, these two game-theoretic techniques are successfully implemented to determine the premium strategy for modelling competition in a non-life insurance market. Thus, in greater detail, a *best-response potential* game with *non-linear* aggregation is constructed and discussed. Premiums per unit of exposure are regarded as the premium strategy, which makes our game to be suitable for different lines of product-specific policies. As a new side-effect result of our approach, when it is compared with the linear aggregation limitation in Dubey et al. (2006), we still prove the existence of a pure Nash equilibrium strategy when the aggregate is non-linear. This is novel result from a game-theoretic perspective. Furthermore, from the point of view of actuarial science, the pure Nash equilibrium existence of a constructed insurance game with a non-convex strategy set is obtained.[2] That is, insurers can avoid any premium range that is not preferred to price. We solve the insurance game with respect to two distinct insurance models by calculating the best-response equations system. The numerical result for a 12-player insurance game is presented under the assumption that the best-response selection premium strategies always give the global maximum value of the corresponding payoff function.

The remainder of this paper is organised as follows. Section 2 introduces the formation of two insurance market competition models and constructs the game. In Section 3, the existence of a Nash equilibrium is proved using potential game techniques. Section 4 presents the simulation results of two models in a 12-insurer game. A conclusion can be found in Section 5.

# 2 Modelling Formulation and Preliminaries

## 2.1 Basic Notations and Assumptions

In this subsection, the necessary notation is provided and appropriate assumptions are introduced. Thus, in the next lines, the definition of key parameters is concentrated for a better understanding of the remaining paper:

[1] We won't discuss unnecessary technical details about how to introduce and solve numerically the single optimisation problem, as it is out of the scope of the present paper.

[2] It is true that since we are able to extend the results of Dubey et al. (2006) for a non-linear aggregation, the concept of our model is possible to be used in other fields of economics. For this comment, we would like cordially to thank one of our reviewers who pointed this out to us. However, further discussion falls out of the scope of this paper, since various parameters from the relevant actuarial science literature are incorporated in the construction of our insurance model (Taylor, 1986, 1987; Emms et al., 2007; Pantelous and Passalidou, 2013, 2015, 2016).

| | |
|---|---|
| $N$ | Set of *insurers* in the insurance market, $N = \{1, \dots, n\}$, $n \in \mathbb{N}$; |
| $a_i$ | *Price sensitivity* (positive) parameter of insurer $i \in N$; |
| $h_1, h_2$ | *Market presence limit factor*, which controls the amount of the flow of insurance policies attributable to the competition in the market; |
| $p_i^1$ | *Premium value* (per unit of exposure) for insurer $i \in N$ at time $t = 1$; |
| $\mathbf{P}_i$ | Set of *strategies* for insurer $i \in N$; |
| $\mathscr{P}$ | Set of *joint strategies* for all insurers in the competitive market; |
| $\mathfrak{p}$ | *Arbitrary profile* in $\mathscr{P}$; |
| $\mathfrak{p}_{-i}^1$ | *Strategy profile* of other players at time $t = 1$, $\{P_1^1, \dots, P_{i-1}^1, P_{i+1}^1, \dots, P_n^1\}$; |
| $q_i^1$ | *Exposure* (volume of business) for insurer $i \in N$ at time $t = 1$, which represents the number (quantity) of policies undertaken by $i \in N$; |
| $\Delta q_i^1$ | *Marginal difference* of exposure volume for insurer $i \in N$ at time $t = 1$; |
| $\hat{q}_i^1$ | *Actual (number of policies) volume of exposure* in the market coming to insurer $i \in N$ at time $t = 1$ from the unallocated exposure at time $t = 0$; |
| $\hat{\mathfrak{q}}_i^0$ | *Given number of policies* in the market, which is intended to flow in or away from insurer $i \in N$ at time $t = 1$ from the unallocated exposure of time $t = 0$; |
| $u_i^1$ | *Utility* of insurer $i$ at time $t = 1$, which represents the net income of insurer $i \in N$ at time $t = 1$, depends on insurer $i$'s premium and the aggregate of other players' strategies; |
| $\sigma_i$ | *Interacting function*, which represents the interaction between insurer $i$'s payoff with the others in the market; |
| $x_{-i}^1$ | Parameter indicating the *aggregation* of $\mathfrak{p}_{-i}^1$; |
| $\alpha_i$ | *Cost ratio* of holding wealth of $i \in N$, generally higher than the risk-free rate, $\alpha_i \in (0, 1)$; |
| $\pi_i^1$ | *Expected breakeven premium* (per unit of exposure) for insurer $i \in N$ at time $t = 1$, i.e., expectation of future claims plus other expenses. However, for purposes of simplicity, we skip the word "expected" when we refer to the breakeven premium in the remaining paper; |
| $k_i$ | *Breakeven ratio* for insurer $i \in N$, $k_i$ is equal to $\pi_i^1$ divided by $p_i^1$; |
| $\theta^1$ | *Market stability factor*, which is used to describe the market's condition; |
| $\beta_i$ | *Best-response correspondences* for insurer $i$ regarding all the other players' strategies; |
| $R_i$ | *Best-response correspondences* for insurer $i$ regarding $x_{-i}^1$; |
| $\hat{r}_i$ | *The maximal selections* of $R_i$; |

Before we proceed further, the following general assumption is proposed.

**Assumption 1:** In the insurance market, for any insurer $i \in N$ at time $t = 1$,

- The breakeven premium (per unit of exposure) $\pi_i^1$ is assumed to be less than the corresponding premium $p_i^1$.
- Both $\pi_i^1$ and $p_i^1$ are positive quantities.

Entries of new insurers and insurance products are not taken into consideration. Insurers avoid to set premium under cost level (Taylor, 1986, 1987; Emms et al., 2007; Pantelous and Passalidou, 2013, 2015, 2016), and see the references therein. Thus, the case that $p_i^1 \leq \pi_i^1$ is not considered in this paper.

## 2.2 Insurance Premium Pricing Model

For the proposed insurance model, every insurer must maximise its wealth. In this direction, a two-period framework: $t = 0, 1$ is investigated in a general insurance market. In line with the previous literature (see Section 1), the utility function $u_i^1$ that concerns insurer $i$ with initial wealth $u_i^0$ is formulated as follows,

$$u_i^1 = -\alpha_i u_i^0 + (1-\alpha_i)(p_i^1 - \pi_i^1)q_i^1. \tag{2.1}$$

For insurer $i$, $p_i$ is the premium value per unit of exposure; $q_i$ represents the holding exposure volume; $\pi_i$ denotes the breakeven premium per unit of exposure, which includes risk premium and other expenses. $p_i$, $q_i$, $\pi_i$ are all positive and $\alpha_i \in (0,1)$ is a given parameter that refers to the cost ratio of holding insurer $i$'s wealth. As shown in Eq. (2.1), the net income of any insurer $i$ is regarded as its utility $u_i^1$, and each insurer is assumed to receive the premium from policyholders at the beginning of time $t=1$. We also assume that the insurance market contains $N = \{1,\ldots,n\}$ insurers, and each insurer has perfect knowledge of its previous information. Moreover, $p_i^0, q_i^0, \pi_i^0, u_i^0$ are all known as constants at time $t=1$. What is more, the value of $q_i^1$ implies competition in the market and must be determined analytically. An insurer's change in the number of policies is related to the deviation in the insurer's premium which is also connected to the market's premium level (Daykin et al., 1994). With the purpose of investigating exposure changes, marginal difference of exposure volume $\Delta q_i^1$ is defined in Eq. (2.2)

$$\Delta q_i^1 = q_i^1 - q_i^0. \tag{2.2}$$

We define the total market exposure $Q_m^1 > 0$ at time $t=1$ as Emms (2012) did, which contains two components. The first part was related to the sum of the current exposure for each insurance company, i.e., $Q^1 = \sum_{i\in N} q_i^1 > 0$, and the second part had to do with the available (unallocated) exposure in the market, $\hat{Q}^1$, thus

$$Q_m^1 = Q^1 + \hat{Q}^1.$$

$\hat{Q}^1$ is allowed to be negative, and $\hat{Q}^1 \leq Q_m^1$. Policyholders may stop renewing policies at the end of time $t=0$, and new clients may buy policies at the beginning of time $t=1$ to become new policyholders. Consequently, $Q^1$ cannot be equal to $Q^0$, which causes the sum of all insurers' exposure change $\sum_{i\in N} \Delta q_i^1$ to take any value in $\mathbb{R}$. In our approach, instead of simply applying the demand function as it was the current trend (see the references in Section 1), the competition between any pair of insurers is now considered. Thus, additionally, the interaction between insurers' premiums needs to be formulated; consequently, $\Delta q_i^1$ is further analysed.

In the following two subsections, two distinct insurance models are introduced: a) the *simple exposure difference model I* ($G_I$), where $\sum_{i\in N} \Delta q_i^1$ might take any value in $\mathbb{R}$ and the available (unallocated) exposure of the insurance market $\hat{Q}^1$ is under consideration; b) the *advanced exposure difference model II* ($G_{II}$), which is used to further analyse policies for any insurer. Both models investigate the competition under the following assumption.

Let us define the transfer function $\rho$ from insurer $j$ to insurer $i$ at time $t=1$ as follows

$$\rho_{j\to i}^1 = 1 - \frac{p_j^0}{p_i^0}\frac{p_i^1}{p_j^1}. \tag{2.3}$$

The transfer function $\rho_{j\to i}^1$ in Eq. (2.3) describes that, for time $t=1$, when the quotient of insurer $i$'s premium and the previous premium $\frac{p_i^1}{p_i^0}$ is less than $j$'s quotient $\frac{p_j^1}{p_j^0}$, insurer $j$'s policies tend to flow to insurer $i$. The exposure of insurer $i$ increases in the competition with $j$, whereas the exposure of $j$ decreases. Policies flow in a reverse manner and $\frac{p_i^1}{p_i^0} > \frac{p_j^1}{p_j^0}$.

This assumption indicates that the preference of policyholders, i.e., when one insurer increases its premium and its competitor decreases its own premium, the insurer simultaneously decreases its attractiveness. When both insurers increase their premiums by different percentages, the insurer with the smaller increment becomes more attractive. Finally, in a similar manner, when both decrease their premiums, the insurer with the larger decrement becomes more attractive.

Insurer $i$ gains exposure from the competition with insurer $j$ when it offers a more attractive premium. However, policyholders sometimes choose an insurer's policies with higher premiums as the most preferable one because of a better reputation (Pantelous and Passalidou, 2015) (and the references therein). For this reason, the percentage changes in the premium are adapted in the transfer function rather than in the value of the premium itself. Note that the transfer function $\rho^1_{i\to j}$ can be either positive or negative. The policy amount of $i$ is increased when $\rho^1_{i\to j} > 0$ and reduced when $\rho^1_{i\to j} < 0$.

By investigating the flow of policies between any pair of insurers, the entire insurance market competition can be evaluated by aggregating every competition among the different pairs of insurers. This topic is the focus of discussion in the following subsections.

### 2.2.1 Simple Exposure Difference Model I ($G_I$)

Let us consider that the competition in the insurance market is formulated as follows. First, the premium levels vary over time, which might even cause a change in the total number of policies in the market. Second, potential clients consider holding insurance policies when premiums decline. In contrast, the insurance market may lose clients if the market premium level is high.

In $G_I$, we assume that for any pair of insurers $i$ and $j$, exposure $q_i$ – which is related to gain or loss – is not equal to exposure $q_j$ – which has to do with loss or gain – respectively. Thus, this assumption indicates that the available exposure joins or leaves the market because of competition between $i$ and $j$. The expected exposure to flow by insurer $i$ attributable to competition with $j$ is given by

$$\begin{aligned} q^1_{j\to i} &= h_1 a_i \rho^1_{j\to i} q^0_i \\ &\neq -q^1_{i\to j}, \quad h_1 > 0. \end{aligned} \tag{2.4}$$

The exposure gain or loss from all other insurers to $i$ is given by

$$\Delta q^1_i = \sum_{j\in N} q^1_{j\to i}. \tag{2.5}$$

Eqs. (2.4)–(2.5) are interpreted as follows. The strength (which is related to either gain or loss) of the exposure of insurer $i$ attributable to the competition with $j$ is demonstrated in Eq. (2.4). The premium $p^1_j$ is modelled as being transferred to insurer $i$'s premium by multiplying $\frac{p^0_i}{p^0_j}$ in $\rho^1_{i\to j}$ for the purpose of simultaneously comparing two insurers' premiums. Insurer $i$'s market price sensitivity parameter $a_i$ is considered as information of insurer $i$ for presenting the market power. Note that, regarding the transferred premium $\frac{p^0_i}{p^0_j}p^1_j$ as $i$'s previous premium $p^0_i$, the item $a_i\rho^1_{j\to i}q^0_i$ is just the volume of business $i$'s gain or loss when the price elasticity is $a_i$. In our case, the price elasticity of demand, $a_i$, is determined by imitating the concept of the Lerner (1934) index, i.e., the leader in the insurance market which has the larger market power has lower price sensitivity and so on and so forth. In a competitive market, $q^1_i$ depends not only on $p^1_i$ but also on other insurers' premiums. Hence, instead of comparing the previous premium $p^0_i$, the transferred premium $\frac{p^0_i}{p^0_j}p^1_j$ is adopted to characterise the change in the volume of polices. In Eq. (2.4), $h_1$ is the market presence limit factor, which is used to limit the scale of the policies' flow amount. Because different stabilities exist in various insurance markets, $h_1$ can take different positive values.

The exposure difference $\Delta q_i^1$ from the competition in the entire market is obtained by summing up all of the policies' gains or losses when competing with all insurers. Note that $\sum_{i\in N}\Delta q_i^1$ is allowed not to be equal to zero. Regarding Eqs. (2.1)–(2.5), the utility function can be deduced.

We define $u_{G_I,i}^1$ (similar for $u_{G_{II},i}^1$, see Subsection 2.2.2) be the utility functions of insurer $i$ at time $t=1$ in $G_I$ ($G_{II}$).

**Lemma 1.** *For the simple exposure difference model I, the utility function $u_{G_I,i}^1$ of insurer $i$ at time $t=1$ is given by*

$$\begin{aligned} u_{G_I,i}^1 &= -\frac{a_i h_1 q_i^0(1-\alpha_i)}{p_i^0}(\sum_{j\in N}\frac{p_j^0}{p_j^1})(p_i^1)^2 \\ &+ (1-\alpha_i)[q_i^0 + nh_1 a_i q_i^0 + \pi_i^1(\sum_{j\in N}\frac{p_j^0}{p_j^1})\frac{a_i h_1 q_i^0}{p_i^0}]p_i^1 \\ &- \alpha_i u_i^0 - \pi_i^1(1-\alpha_i)[q_i^0 + n a_i h_1 q_i^0]. \end{aligned} \tag{2.6}$$

*Proof.* By combining Eqs. (2.2)–(2.5), we obtain the exposure of $i$ considering that the competition occurred at time $t=1$.

$$\begin{aligned} q_i^1 &= q_i^0 + \Delta q_i^1 \\ &= q_i^0 + \sum_{j\in N} h_1 a_i \rho_{j\to i}^1 q_i^0 \\ &= q_i^0 + \sum_{j\in N} h_1 a_i q_i^0 (1 - \frac{p_j^0}{p_i^0}\frac{p_i^1}{p_j^1}) \\ &= q_i^0 + nh_1 a_i q_i^0 - \frac{a_i h_1 q_i^0 p_i^1}{p_i^0}\sum_{j\in N}\frac{p_j^0}{p_j^1}. \end{aligned}$$

By taking $q_i^1$ above into Eq. (2.1), we have that

$$\begin{aligned} u_{G_I,i}^1 &= -\alpha_i u_i^0 + (1-\alpha_i)(p_i^1 - \pi_i^1)(q_i^0 + nh_1 a_i q_i^0 \\ &\quad -\frac{a_i h_1 q_i^0 p_i^1}{p_i^0}\sum_{j\in N}\frac{p_j^0}{p_j^1}) \\ &= -\frac{a_i h_1 q_i^0(1-\alpha_i)}{p_i^0}(\sum_{j\in N}\frac{p_j^0}{p_j^1})(p_i^1)^2 \\ &+ (1-\alpha_i)[q_i^0 + nh_1 a_i q_i^0 + \pi_i^1(\sum_{j\in N}\frac{p_j^0}{p_j^1})\frac{a_i h_1 q_i^0}{p_i^0}]p_i^1 \\ &- \alpha_i u_i^0 - \pi_i^1(1-\alpha_i)[q_i^0 + n a_i h_1 q_i^0]. \end{aligned}$$

□

### 2.2.2 Advanced Exposure Difference Model II ($G_{II}$)

The modified exposure for insurer $i$ can be further analysed. Different from $G_I$, in $G_{II}$, we concretely characterize the two components mentioned in Subsection 2.2, i.e., a) reallocated policies of the previous market $Q^0$, and b) policies from the (unallocated) exposure $\hat{Q}^1$.

Regarding the competition between any pair of insurers $i$ and $j$, the number of exchange policies is characterised. The exposure gain or loss from $i$ to $j$ is obtained with respect to both insurers' premium strategy and market power. Given a positive market presence limit factor $h_2$, the strength of the flow of business between $i$ and $j$ is modelled as follows

$$\begin{aligned} q^1_{j\to i} &= h_2(a_i\rho^1_{j\to i}q^0_i - a_j\rho^1_{i\to j}q^0_j) \\ &= -q^1_{i\to j}, \quad h_2 > 0. \end{aligned} \tag{2.7}$$

As demonstrated in Eq. (2.7), both exposure $i$ which tended to a gain or loss, $a_i\rho^1_{j\to i}q^0_i$, and exposure $j$ which showed a potential loss or gain, $-a_j\rho^1_{i\to j}q^0_j$, represent the exchange strength from summing up the volume. The volume of the flow of exposure is further governed by a positive market presence limit factor $h_2$. Note that $\sum_{i\in N}\sum_{j\in N} q^1_{j\to i}$ equals to zero because of policies exchange between insurers in the component a). In the same way, for the b) component, the potential flow of policies, either attract or withdraw from the unallocated insurance market $\hat{Q}^1$, and it is modelled as $h_2a_i(1-\frac{p^1_i}{p^0_i}\theta^1)q^0_i$.

The flow of policies from the unallocated insurance market is modelled similarly to the concept of price elasticity: a comparison with previous premium price. Apart from the competition between pairs of insurers, they tend to lose policies to the available market when increasing their premiums and gain policies by lowering them. In addition, a positive market stability factor $\theta^1$ is adopted to describe the market condition: $\theta^1 = 1$ indicates that the market faces a general condition; the insurance industry expands when $\theta^1 < 1$ because more policies tend to flow into the industry from the unallocated market; $\theta^1 > 1$, when the market faces a situation with challenges. Overall, the exposure gain or loss for $i$ is given by

$$\Delta q^1_i = \sum_{j\in N} q^1_{j\to i} + h_2a_i(1-\frac{p^1_i}{p^0_i}\theta^1)q^0_i, \quad \theta^1 > 0. \tag{2.8}$$

Following **Assumption 1**, $k_i \in (0,1)$. Then, the objective function for the $G_{II}$ case can be deduced.

**Lemma 2.** *For the advanced exposure difference model II, the utility function $u^1_{G_{II},i}$ of insurer $i$ at time $t=1$ is given by*

$$\begin{aligned} u^1_{G_{II},i} &= -\frac{(1-k_i)(1-\alpha_i)h_2a_iq^0_i}{p^0_i}(\sum_{j\in N}\frac{p^0_j}{p^1_j}+\theta^1)(p^1_i)^2 \\ &+ (1-k_i)(1-\alpha_i)(q^0_i+(n+1)h_2a_iq^0_i - h_2\sum_{j\in N}a_jq^0_j)p^1_i \\ &+ (1-k_i)(1-\alpha_i)h_2p^0_i\sum_{j\in N}a_jq^0_j\frac{p^1_j}{p^0_j} - \alpha_iu^0_i. \end{aligned} \tag{2.9}$$

*Proof.* Using Eqs. (2.7)–(2.8) instead, Lemma 2 can be showed similarly as Lemma 1. □

In the next Subsection, the construction of the game is presented and further discussed.

## 2.3 Game Construction

### 2.3.1 Normal Form Game

Let us define an $N$-insurer game, $G$, in a two-period framework: $t = 0, 1$. Each insurer $i$'s strategy at time $t = 1$ is $p^1_i$, which stands for the action setting premium as the value of $p^1_i$, whereas $\mathbf{P}_i$ is the set of strategies. We use $\widetilde{P}^1_i$ to denote the equilibrium strategy for insurer $i$.

Insurer $i$'s payoff function is defined as $u_i^1 : \mathscr{P} \to \mathbb{R}$, where $\mathscr{P} \equiv \mathbf{P}_1 \times \cdots \times \mathbf{P}_N$ and $\mathfrak{p}$ is an arbitrary profile in $\mathscr{P}$. The notation $\mathfrak{p}_{-i}^1 \in \mathscr{P}_{-i}$ stands for $\{p_1^1, \ldots, p_{i-1}^1, p_{i+1}^1, \ldots, p_n^1\}$, which is used to represent the strategy profile of other players at time $t$. $(p_i^1, \mathfrak{p}_{-i}^1) \in \mathscr{P}$ decomposes a strategy profile in two parts, the insurer $i$'s strategy and other insurers' components. Given this game in the insurance market, instead of calculating the optimal premium that maximises a single insurer's wealth, as was the case in the previous literature (see Section 1 for further details), the calculation of the Nash equilibrium is targeted.

Generally, from a game theory perspective, the Nash equilibrium is a prediction strategy that dictates the choices that each insurer is willing to make. Given the optimal strategy profile of other insurers, the market reaches a Nash equilibrium when no insurer can increase its total payoff by changing its strategy. The Nash equilibrium is defined through the best-response correspondences. In what it follows the next definitions should be stated.

**Definition 1.** (Fudenberg and Tirole, 1991) Define $\beta_i$ by

$$\beta_i(\mathfrak{p}_{-i}^1) = \{p_i^1 \in \mathbf{P}_i : u_i^1(p_i^1, \mathfrak{p}_{-i}^1) \geq u_i^1(\acute{p}_i^1, \mathfrak{p}_{-i}^1), \forall \acute{p}_i^1 \in \mathbf{P}_i\}.$$

We call $\beta_i$ the *best-response correspondences* for insurer $i$.

For any choice $\mathfrak{p}_{-i} \in \mathscr{P}_{-i}$ of others' strategies at time $t$, the set $\beta_i(\mathfrak{p}_{-i}^1)$ of best replies of insurer $i$ is given by

$$\beta_i(\mathfrak{p}_{-i}^1) = \arg\max_{p_i^1 \in \mathbf{P}_i} u_i^1(p_i^1, \mathfrak{p}_{-i}^1).$$

Each player's predicted strategy must be a best response to the predicted strategies of the other players as the market reaches a Nash equilibrium.

**Definition 2.** (Fudenberg and Tirole, 1991) A strategy profile, $\widetilde{\mathfrak{p}}^1$, is a *Nash equilibrium* of the game (at time $t$) *if and only if* each player's strategy is a best response to the other players' strategies. That is

$$\widetilde{p}_i^1 \in \beta(\widetilde{\mathfrak{p}}_{-i}^1), \quad \forall i \in N.$$

The best-response potential game technique is further considered, which is widely used to prove the existence of Nash equilibrium.

**Definition 3.** (Voorneveld, 2000) A strategic game $\tilde{G} =< (\beta_i, \mathbf{P}_i)_{i \in N} >$ is a *best-response potential game* if there exists a function $f : \mathscr{P} \to \mathbb{R}$ such that

$$\forall i \in N, \forall \mathfrak{p}_{-i} \in \mathscr{P}_{-i} : \quad \beta_i(\mathfrak{p}_{-i}) = \arg\max_{p_i \in \mathbf{P}_i} f(p_i, \mathfrak{p}_{-i}).$$

The function $f$ is called a best-response potential function of the game $\tilde{G}$.

The potential function $f$ offers a new approach to determining the Nash equilibrium for the game $\tilde{G}$ by maximising $f$. Note that, $f$ is a function, which depends on every insurer's strategy. If $f$ has a maximum over $\mathscr{P}$, $\tilde{G}$ has a Nash equilibrium. A specific type of game, known as an *aggregate* game, is introduced to solve the Nash equilibrium for the $N$ insurers' game.

### 2.3.2 Aggregate Games

With the additional requirement that each insurer's payoff is written as a function that depends only on its own strategy and an aggregate of the full strategy profile, a normal form game can be transformed into a game with aggregation. Formally, we have the following definition.

**Definition 4.** (Martimort and Stole, 2012) An *aggregate game* in the insurance market, $G' =< (\mathbf{P}_i, u_i^1)_{i\in N}, g >$, is a normal form game with an extra condition that there exists an aggregate function, $g(\mathfrak{p}^1) : \mathscr{P} \longrightarrow \mathbf{M} \subseteq \mathbb{R}$, such that each player's payoff function can be further specialised to the aggregate form

$$\mathfrak{p}^1 \mapsto u_i^1(p_i^1, g(\mathfrak{p}^1)),$$

where $M^1 \in \mathbf{M}$, is called an aggregator of $\mathfrak{p}^1$.

The only requirement for a game to represent an aggregate game is that there exists an aggregate function (Alos-Ferrer and Ania, 2005). To construct an insurance game with aggregation, a meaningful monotone aggregate function $g$ is expected to be obtained. Here, the *Insurance Game I*, equipped with the objective function in the simple exposure difference model I, and the *Insurance Game II*, implemented with the objective function in the advanced exposure difference model II, are considered. Before we proceed further, the definitions of $G_I$ and $G_{II}$ are given as follows.

**Definition 5.** A game $G_I =< (\mathbf{P}_{G_I,i}, u^1_{G_I,i})_{i\in N} >$ has a finite set of players $N$, with compact, positive, pure strategy set $\mathbf{P}_{G_I,i}$ with respect to every $i$, whereas $u^1_{G_I,i}$ in Eq. (2.6) is the payoff function for $i$ at time $t = 1$. This type of game is called *Insurance Game I.*
Similarly, Insurance Game II is defined as $G_{II} =< (\mathbf{P}_{G_{II},i}, u^1_{G_{II},i})_{i\in N} >$, with player set $N$, compact, positive, pure strategy set $\mathbf{P}_{G_{II},i}$ and payoff function $u^1_{G_{II},i}$ in Eq. (2.9).

# 3 Main Results

In this section, the theoretical results for models $G_I$ and $G_{II}$ are presented. However, before we proceed further with the existence of a Nash equilibrium, it is necessary to show that both $G_I$ and $G_{II}$ are aggregate games.

**Lemma 3.** *Based on the definition of payoff functions stated in the previous section, both $G_I$ and $G_{II}$ are aggregate games.*

*Proof.* Denote $M^1 = \sum_{j\in N} \frac{p_j^0}{p_j^1}$ as the aggregation of $G_I$ game. Then, the payoff function in Eq. (2.6) turns out to be

$$\begin{aligned} u^1_{G_I,i} &= -\frac{a_i h_1 q_i^0(1-\alpha_i)}{p_i^0} M^1 (p_i^1)^2 + (1-\alpha_i)[q_i^0 + n h_1 a_i q_i^0 + \pi_i^1 M^1 \frac{a_i h_1 q_i^0}{p_i^0}] p_i^1 \\ &- \alpha_i u_i^0 - \pi_i^1 (1-\alpha_i)[q_i^0 + n a_i h_1 q_i^0]. \end{aligned}$$

There exists an aggregate function $g(\mathfrak{p}^1) = \sum_{j\in N} \frac{p_j^0}{p_j^1}$ in $G_I$. For $G_{II}$ game, we further denote $m^1 = \sum_{j\in N} a_j q_j^0 \frac{p_j^1}{p_j^0}$ as the other aggregation. Similarly, we obtain the payoff,

$$\begin{aligned} u^1_{G_{II},i} &= -\frac{(1-k_i)(1-\alpha_i)h_2 a_i q_i^0}{p_i^0}(M^1 + \theta^1)(p_i^1)^2 \\ &+ (1-k_i)(1-\alpha_i)(q_i^0 + (n+1)h_2 a_i q_i^0 - h_2 \sum_{j\in N} a_j q_j^0) p_i^1 \\ &+ (1-k_i)(1-\alpha_i) h_2 p_i^0 m^1 - \alpha_i u_i^0. \end{aligned}$$

Thus, the statement of the Lemma is derived. □

In aggregate games, for every player $i$, the other players in the competitive market are considered as a single player because their strategies aggregate through an interacting function

$\sigma_i : \mathscr{P}_{-i} \to \mathbf{X}_{-i} \subseteq \mathbb{R}$. Intuitively, the other players influence $i$ through the interaction function $\sigma_i(\mathfrak{p}^1_{-i})$. $\mathbf{X}_{-i} = \sigma_i(\mathscr{P}_{-i})$ is set to indicate the range of $\sigma_i$, whereas $x^1_{-i} = \sigma_i(\mathfrak{p}^1_{-i}) \in \mathbf{X}_{-i}$ for any $t$. With $x^1_{-i} = \sum_{j\neq i} \frac{p^0_j}{p^1_j}$, respectively, the $G_I$ and $G_{II}$ payoff functions are given as follows:

$$\begin{aligned} u^1_{G_I,i} &= -\frac{a_i h_1 q^0_i (1-\alpha_i)}{p^0_i} x^1_{-i} (p^1_i)^2 \\ &+ (1-\alpha_i)[q^0_i + (n-1)h_1 a_i q^0_i + \pi^1_i x^1_{-i} \frac{a_i h_1 q^0_i}{p^0_i}] p^1_i \\ &- \alpha_i u^0_i - \pi^1_i (1-\alpha_i)[q^0_i + (n-1)a_i h_1 q^0_i] \end{aligned}$$

and

$$\begin{aligned} u^1_{G_{II},i} &= -\frac{(1-k_i)(1-\alpha_i)h_2 a_i q^0_i}{p^0_i} (x^1_{-i} + \theta^1)(p^1_i)^2 \\ &+ (1-k_i)(1-\alpha_i)(q^0_i + n h_2 a_i q^0_i - h_2 \sum_{j\neq i} a_j q^0_j) p^1_i \\ &+ (1-k_i)(1-\alpha_i) h_2 p^0_i \sum_{j\neq i} a_j q^0_j \frac{p^1_j}{p^0_j} - \alpha_i u^0_i. \end{aligned}$$

To generate Nash equilibrium premium strategies, $R_i : \mathbf{X}_{-i} \to 2^{\mathbf{P}_i}$, we need to define

$$R_i(x^1_{-i}) = \operatorname*{argmax}_{p^1_i \in \mathbf{P}} u^1_i(p^1_i, x^1_{-i}),$$

which coincides with $\beta_i(\mathfrak{p}^1_{-i})$. In other words, $R_i$ describes how the interaction parameter $x^1_{-i} = \sigma_i(\mathfrak{p}^1_{-i})$ influences insurer $i$'s best-response strategy.

In the case of $G_I$, we have

$$R_{G_I,i}(x^1_{-i}) = \operatorname*{argmax}_{p^1_i \in \mathbf{P}} u^1_{G_I,i}(p^1_i, x^1_{-i}). \tag{3.1}$$

$\hat{r}_{G_I,i}$ is defined as the maximal selections of $R_{G_I,i}(x^1_{-i})$, and for $G_{II}$, we have

$$R_{G_{II},i}(x^1_{-i}) = \operatorname*{argmax}_{p^1_i \in \mathbf{P}} u^1_{G_{II},i}(p^1_i, x^1_{-i}). \tag{3.2}$$

$\hat{r}_{G_{II},i}$ is defined as the maximal selections of $R_{G_{II},i}(x^1_{-i})$.

Before we prove that both $G_I$ and $G_{II}$ are best-response potential games, we need to recall first, Lemma 4 which is proposed by Jensen (2010).

**Lemma 4.** *The game $< (\beta_i, \boldsymbol{P}_i)_{i\in N} >$ is a best-response potential game if and only if there exists a real-valued function, $f :\to \mathbb{R}$, such that:*

$$\tilde{\mathfrak{p}}^1 \succeq \mathfrak{p}^1 \Rightarrow f(\tilde{\mathfrak{p}}^1) \geq f(\mathfrak{p}^1) \tag{3.3}$$

*and*

$$\tilde{\mathfrak{p}}^1 \succ \mathfrak{p}^1 \Rightarrow f(\tilde{\mathfrak{p}}^1) > f(\mathfrak{p}^1), \tag{3.4}$$

*where the previous two binary relations are defined as:*

$$\begin{aligned} \tilde{\mathfrak{p}}^1 \succeq \mathfrak{p}^1 &\Leftrightarrow \exists i \in N, \quad s.t.\ [\tilde{\mathfrak{p}}^1_{-i} = \mathfrak{p}^1_{-i}, \text{and} \quad \tilde{\mathfrak{p}}^1_i \in R_i(x^1_{-i})] \\ \tilde{\mathfrak{p}}^1 \succ \mathfrak{p}^1 &\Leftrightarrow [\tilde{\mathfrak{p}}^1 \succeq \mathfrak{p}^1, \quad \text{and} \quad \mathfrak{p}^1_{-i} \notin R_i(x^1_{-i})] \end{aligned}$$

The next lemma is useful for the main result of our paper. Its proof is rather technical, and for better understanding, we present it using intermediate steps.

**Lemma 5.** *Both $G_I$ and $G_{II}$ are best-response potential games.*

*Proof.* Initially, $G_I$ is considered.

- **Step 1: State the best-response potential function.**

  - **Convex hull of $\mathbf{X}_{-i}$.**
    In the case that $\mathbf{P}_i$ is *not* convex, $\mathbf{X}_{-i}$ is *not* convex as well. Denote $\Sigma_{-i}$ as the convex hull of $\mathbf{X}_{-i}$, which is obviously compact.

    For $G_I$, $R_{G_I,i}$ is the best-response correspondences to $x^1_{-i}$ of $i$. We extend $R_{G_I,i}$ in a piecewise linear fashion to $\Phi_{G_I,i}$, defined on the domain $\Sigma_{-i}$. $\Phi_{G_I,i}$ coincides with $R_{G_I,i}$ on $\mathbf{X}_{-i}$. For any $s \in \Sigma_{-i} \setminus \mathbf{X}_{-i}$ define

    $$\Phi_{G_I,i}(s) = \frac{z-s}{z-y} R_{G_I,i}(y) + \frac{s-y}{z-y} R_{G_I,i}(z),$$

    with $y = \max\{v \in \mathbf{X}_{-i} | v \leq s\}$ and $z = \min\{v \in \mathbf{X}_{-i} | v \geq s\}$.

  - **For any insurer $i$, linearly enhance the best response domain to be the same as its strategy domain.**
    Let $\hat{\mathbf{P}}_{G_I,i}$ denote the range of player $i$'s best response map, and the set be $\{p^1_i \in \mathbf{P}_{G_I,i} : p^1_i \in R_i(\sigma_i(\mathfrak{p}^1_{-i}))\} \subseteq \mathbf{P}_{G_I,i}$. Denote $\phi^1_{G_I,i}$ as the selections of $\Phi_{G_I,i}$, which is continuous on $\Sigma_{-i}$. We further define a mapping $O_i(\phi^1_{G_I,i})$, which linearly enhances the domain $\hat{\mathbf{P}}_{G_I,i}$ to $\mathbf{P}_{G_I,i}$. In addition, $r^1_{G_I,i}$ is defined as the selection of $O_i(\phi^1_{G_I,i})$. In other words,

    $$\forall i, \exists \hat{x}^1_{-i} \quad s.t. \quad p^1_i \in O_i(\Phi_{G_I,i}(\hat{x}^1_{-i})).$$

    Let $\perp^1_i = \min_{\mathfrak{p}^1_{-i} \in \mathscr{P}^1_{-i}} \sigma_i(\mathfrak{p}^1_{-i})$, $\top^1_i = \max_{\mathfrak{p}^1_{-i} \in \mathscr{P}^1_{-i}} \sigma_i(\mathfrak{p}^1_{-i})$, and extend each $r^1_{G_I,i}$ to $[\perp^1_i, \top^1_i]$ along the line with Kukushkin (2004).

  - **We state that the following Eq. (3.5) is the best-response potential function of $G_I$.**

    $$f(p^1_i, \mathfrak{p}^1_{-i}) = \sum_i \left[ p^0_i \int_{\perp^1_i}^{\top^1_i} \min\{-\frac{1}{p^1_i}, -\frac{1}{r^1_{G_I,i}(\tau)}\} d\tau - \frac{p^0_i}{p^1_i} \perp_i \right] + \sum_{i<j} \frac{p^0_i p^0_j}{p^1_i p^1_j}. \tag{3.5}$$

- **Step 2: Prove that Eqs. (3.3) and (3.4) are true.**

  - **Prove that each of the correspondences $R_{G_I,i} : \mathbf{X}_{-i} \to 2^{\mathbf{P}_i}$ is a strictly decreasing selection; that is, for every $R_i$, all $x^1_{-i} \in \mathbf{X}_{-i}$ such that $R_i(\bar{x}^1_{-i}) > R_i(x^1_{-i})$ whenever $\bar{x}^1_{-i} \leq x^1_{-i}$.**
    The statement is satisfied as long as the conditions of Topkis' Theorem (see Topkis (1998) for details) are satisfied, i.e. each $\mathbf{P}_i$ is a lattice, every $u_{G_I,i}(p^1_i, x^1_{-i})$ supermodular in $p^1_i$,and has strictly decreasing differences in $p^1_i$ and $x^1_{-i}$.
    Since $p^1_i$ is one-dimensional for all $i$, the first two of these requirements are satisfied: $\mathbf{P}_i$ is a lattice for all $i$; every $u_{G_I,i}$ supermodular in $p^1_i$. In addition, because $u^1_i$ is

twice differentiable, $u_{G_I,i}(p_i^1, x_{-i}^1)$ has strictly decreasing differences in $p_i^1$ and $x_{-i}^1$ if and only if $\partial^2 u_{G_I,i}(p_i^1, x_{-i}^1)/\partial p_i^1 \partial x_{-i}^1 < 0$. In an insurance game $G_I$, we have

$$\begin{aligned}\partial^2 u_{G_I,i}(p_i^1, x_{-i}^1)/\partial x_{-i}^1 \partial p_i^1 &= \partial\{-2\frac{(1-\alpha_i)h_1 a_i q_i^0}{p_i^0} x_{-i}^1 p_i^1 + (1-\alpha_i)[q_i^0 \\ &+ (n-1)h_1 a_i q_i^0 + \pi_i^1 x_{-i}^1 \frac{a_i h_1 q_i^0}{p_i^0}]\}/\partial x_{-i}^1 \\ &= \frac{a_i h_1 q_i^0 (1-\alpha_i)}{p_i^0}(\pi_i^1 - 2p_i^1) < 0.\end{aligned}$$

According to the assumption that for any $i,\ t,\ \pi_i^1 < p_i^1$, the above item is negative. Hence, $u_i^1(p_i^1, x_{-i}^1)$ has strictly decreasing differences in $p_i^1$ and $x_{-i}^1$. Because $O_i(\phi_{G_I,i}^1)$ enhance the domain $\hat{\mathbf{P}}_{G_I,i}$ linearly, $r_{G_I,i}^1$ coincides with $\phi_{G_I,i}^1$. One can deduce that if $\hat{x}_{-i}^1 > x_{-i}^1$, we have $p_i^1 < \tilde{p}_i^1$ and vice versa.

- **The comparison between $f(\tilde{p}_i^1, \mathfrak{p}_{-i}^1)$ and $f(p_i^1, \mathfrak{p}_{-i}^1)$.**
  With equilibrium premium $\tilde{p}_i^1$ of $i$ in $\tilde{\mathfrak{p}}^1$, the difference between $f(\tilde{\mathfrak{p}}^1)$ and $f(\mathfrak{p}^1)$ is demonstrated as

$$\begin{aligned}&f(\tilde{p}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \\ = &\sum_{i\in N}\left[\int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau\right] - \sum_{i\in N}\left[\int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau\right] \\ &-\sum_{i\in N}\left[\frac{p_i^0}{\tilde{p}_i^1}\cdot \perp_i^1\right] + \sum_{i\in N}\left[\frac{p_i^0}{p_i^1}\cdot \perp_i^1\right] + \left[\frac{p_i^0}{\tilde{p}_i^1} - \frac{p_i^0}{p_i^1}\right]\cdot \sum_{j\neq i}\frac{p_j^0}{p_j^1}\end{aligned}$$

$$\begin{aligned}= &\int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\ &-\frac{p_i^0}{\tilde{p}_i^1}\cdot \perp_i^1 + \frac{p_i^0}{p_i^1}\cdot \perp_i^1 + \frac{p_i^0}{\tilde{p}_i^1}\cdot x_{-i}^1 - \frac{p_i^0}{p_i^1}\cdot x_{-i}^1 \\ = &\ p_i^0\left[\int_{\perp_i^1}^{\top_i^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\perp_i^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau\right. \\ &\left. - \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1}d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1}d\tau\right].\end{aligned}$$

When $\hat{x}_{-i}^1 > x_{-i}^1$,

$$
\begin{aligned}
&f(\tilde{P}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \\
&= \int_{\perp_i^1}^{x_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau + \int_{x_{-i}^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau \\
&+ \int_{\hat{x}_{-i}^1}^{\top_i^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau - \int_{\perp_i^1}^{x_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau \\
&- \int_{x_{-i}^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau - \int_{\hat{x}_{-i}^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \\
&= \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{x_{-i}^1}^{\hat{x}_{-i}^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau + \int_{\hat{x}_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau \\
&\qquad - \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau - \int_{x_{-i}^1}^{\hat{x}_{-i}^1} -\frac{1}{p_i^1} d\tau - \int_{\hat{x}_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau \\
&\qquad - \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \\
&\qquad = \int_{x_{-i}^1}^{\hat{x}_{-i}^1} [\frac{1}{p_i^1} - \frac{1}{r_{G_I,i}^1(\tau)}] d\tau > 0.
\end{aligned}
$$

When $\hat{x}_{-i}^1 < x_{-i}^1$,

$$
\begin{aligned}
&f(\tilde{P}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \\
&= \int_{\perp_i^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau + \int_{\hat{x}_{-i}^1}^{x_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau \\
&+ \int_{x_{-i}^1}^{\top_i^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau - \int_{\perp_i^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau \\
&- \int_{\hat{x}_{-i}^1}^{x_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau - \int_{x_{-i}^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \\
&= \int_{\perp_i^1}^{\hat{x}_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\hat{x}_{-i}^1}^{x_{-i}^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau + \int_{x_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau - \int_{x_{-i}^1}^{\hat{x}_{-i}^1} -\frac{1}{p_i^1} d\tau - \int_{\hat{x}_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \\
&= \int_{\hat{x}_{-i}^1}^{x_{-i}^1} [\frac{1}{r_{G_I,i}^1(\tau)} - \frac{1}{p_i^1}] d\tau > 0.
\end{aligned}
$$

It is obvious that if $\hat{x}_{-i}^1 = x_{-i}^1$, this item equals zero. In this case, $p_i^1, \tilde{p}_i^1 \in R_{G_I,i}(\sigma_i(\mathfrak{p}_{-i}^1))$

(i.e. if Eq. (3.3) holds but not Eq. (3.4)), $f(\tilde{P}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) = 0$. Eq. (3.3) is proved to be true in an insurance game $G_I$. If not, Eq. (3.4) is proved.

- **Step 3: Conclusion**
  We conclude that when $(p_i^1, \mathfrak{p}_{-i}^1), (\tilde{p}_i^1, \mathfrak{p}_{-i}^1) \in \mathbf{P}_i, (\tilde{p}_i^1, \mathfrak{p}_{-i}^1) \succeq (\succ)(p_i^1, \mathfrak{p}_{-i}^1) \Rightarrow f(\tilde{p}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \geq (>)0$, with respect to Lemma 5. An insurance game $G_1$ is the best-response potential game, whereas $f$ is the best-response potential function.

Similarly, in $G_{II}$,

$$\begin{aligned}
\partial^2 u_{G_{II},i}^1(p_i^1, x_{-i}^1)/\partial x_{-i}^1 \partial p_i^1 &= \partial\{-2\frac{(1-\alpha_i)(1-k_i)h_2 a_i q_i^0}{p_i^0}(x_{-i}^1 + \theta^1)p_i^1 \\
&\quad +(1-k_i)(1-\alpha_i)(q_i^0 + nh_2 a_i q_i^0 - h_2 \sum_{j \neq i} a_j q_j^0)\}/\partial x_{-i}^1 \\
&= -2\frac{(1-\alpha_i)(1-k_i)h_2 a_i q_i^0}{p_i^0} p_i^1 < 0.
\end{aligned}$$

We also obtain that $u_{G_{II},i}^1(p_i^1, x_{-i}^1)$ has strictly decreasing differences in $p_i^1$ and $x_{-i}^1$. By replacing $r_{G_I,i}^1$ by $r_{G_{II},i}^1$ in $f$ from Eq. (3.5), one obtains the best-response potential function of $u_{G_{II},i}^1$ in $G_{II}$. □

Following the discussion so far, one can deduce the useful Theorem, which is the main theoretical result of our paper.

**Theorem 1.** *The Nash equilibrium at time $t = 1$ in both $G_I$ and $G_{II}$ exists.*

*Proof.* In $G_I$, let us suppose that

$$\tilde{\mathfrak{p}}^1 \in \operatorname{argmax} f(p_i^1, \mathfrak{p}_{-i}^1).$$

Such a $\tilde{\mathfrak{p}}^1$ exists because $\mathbf{P}_i$ is compact for any $i$ and $f$ is continuous. If $\tilde{\mathfrak{p}}^1$ is not a Nash equilibrium of $G_1$, then $f(c_i^1, \tilde{\mathfrak{p}}_{-i}^1) > f(\tilde{\mathfrak{p}}^1)$ for some $c_i^1 \in \mathbf{P}_i$, contradicting that $\tilde{\mathfrak{p}}^1$ maximises $f$. Hence, the Nash equilibrium exists in $G_I$. Similarly, it can be shown that the Nash equilibrium exists in $G_{II}$.

□

# 4 Numerical Example

In this section, a numerical example with 12 major non-life insurance companies based on the number of contracts (i.e., volume of business) they have in their portfolios is proposed to illustrate the main modelling characteristics and theoretical findings of our paper. A scenario which investigates insurers with different market power is considered by consisting of a market leading insurer with 796, 139 contracts, nine almost equal insurers with around 300, 000 contracts and two followers with only around 200, 000 contracts.[3] Referring to the premium values at time $t = 0$, the pricing strategy for the entire market of insurers is derived by finding the Nash equilibrium premiums at time $t = 1$. The impact of different parameters involved in the process to the equilibrium premiums is also analysed. To generate results that are comparable to those existing in the literature of actuarial science and for simplicity in our calculations, convex premium strategy sets are considered in the numerical example.[4]

[3] We don't have here any intention to develop any type of Stackelberg leadership model. However, the Greek insurance market might be considered as an ideal case for this model. Thus, it will be considered as a future work.

[4] We recall that the theoretical results did not assume any type of convexity.

| Insurance Companies $i$ | Premium $p_i^0$ | Number of Contracts $q_i^0$ | Price sensitivity parameter $a_i$ |
|---|---|---|---|
| 1 | €269.09 | 298,269 | 2.0 |
| 2 | €282.07 | 303,673 | 2.0 |
| 3 | €377.06 | 282,224 | 2.0 |
| 4 | €371.52 | 304,609 | 2.0 |
| 5 | €281.56 | 295,769 | 2.0 |
| **6** | **€377.83** | **796,139** | **1.9** |
| 7 | €257.88 | 298,304 | 2.0 |
| **8** | **€366.99** | **200,135** | **2.1** |
| **9** | **€347.58** | **211,314** | **2.1** |
| 10 | €351.18 | 299,690 | 2.0 |
| 11 | €364.11 | 299,995 | 2.0 |
| 12 | €291.22 | 319,453 | 2.0 |

Table 1: 12 insurance companies are considered from the Greek insurance market in 2010. Premium values and number of contracts are based on data from *the Hellenic Association of Insurance Companies*. Price sensitivity parameter for every insurer demonstrated in the table is used as a benchmark.

Data is used from the Greek market, as it was presented in (Pantelous and Passalidou, 2013, 2016). Thus, the premium prices are calculated in Euros. Let us assume that the number of contracts at time $t = 0$ is demonstrated in Table 1. With respect to $t$, this dataset is adopted for a 12-player game because the insurers' premium prices and exposure in the previous period are used. With an intention to describe insurance companies' market power, the price sensitivity parameter, $a_i$, for all insurers $i$ is characterised further.

The standard values of price sensitivity parameter are set up in Table 1, and they can be used as a benchmark. As it was already demonstrated, insurer 6 is considered to be the market leader with a lower price sensitivity parameter $a_6 = 1.9$, because it occupies significant greater market weight compared with other insurers. Correspondingly, insurers 8 and 9 are regarded as market followers, which have price sensitivity parameters of value 2.1. All of the others insurers' price sensitivity parameter take the value of 2.0 in our insurance game.[5]

The diversity of the price sensitivity parameter for the insurers obviously affect the equilibrium premium profiles. Different values of $a_i$ are investigated through a simulation. However, for any $i$, $a_i^1$ are restricted in $[1.5, 2.5]$. Using the previously demonstrated market data, the Nash equilibrium premium profiles are calculated for both $G_I$ and $G_{II}$.

### 4.1 Insurance Game I Simulation Results

In Insurance Game I, $G_I$ , the Nash equilibrium premium profiles are calculated with respect to the market's data at time $t = 0$; see Table 1. Table 2 sets up also ad hoc the main parameters. Note that for any insurer $i$ in $G_1$, the breakeven premium $\pi_i^1$ is not assumed to be proportional to $p_i^1$. The percentage between $\pi_i^1$ and $p_i^1$ is used to describe the cost structure of $i$.

From Eq. (2.6), the second order condition of payoff is negative for each insurer $i$ in $G_I$. Hence, when the stationary point is in the domain $\mathbf{P}_{G_I,i}$, $i$'s payoff is maximized. What is more, one can find out the Nash equilibrium profiles by implementing the following algorithm:

[5]The values for $a_i$ have been considered ad hoc based on the concept of Lerner (1934) index. Unfortunately, we don't have access to more detailed data, and some of the model parameters are rather artificial. This is common in the corresponding literature (Emms et al., 2007; Emms and Haberman, 2009)

| | | |
|---|---|---|
| Number of market participants | $n$ | 12 |
| Market presence limit factor | $h_1$ | 0.09 |
| The breakeven ratio of every insurer | $k_i$ | 0.5 |

Table 2: Environmental parameter values in $G_I$

**Step 1**: For each insurer $i$, set the first order condition of its payoff function equal to zero as the maximum selection(s). From Eq. (3.1),

$$\hat{r}_{G_I,i} : \frac{1 + (n-1)h_1 a_i + \pi_i^1 x_{-i}^1 \frac{a_i h_1}{p_i^0}}{2h_1 a_i x_{-i}^1} p_i^0 = 0.$$

**Step 2**: Solve the system of $r_{G_I,i}^1$.
**Step 3**: Select the profile(s) corresponds (correspond) to each insurer's premium located in $\mathbf{P}_{G_I,i}$, which is (are) the Nash equilibrium premium profile(s).
Be aware that when the derived values are located outside of $\mathbf{P}_{G_I,i}$, then these are not the equilibrium premiums, as the edges of the premium domain reach a maximum instead. Furthermore, it indicates that the Nash equilibrium still exists even though the calculated premium profile have not located inside of $\mathbf{P}_{G_I,i}$. However, this case won't be analysed further here.

Let us now characterize the premium strategies set $\mathbf{P}_{G_I,i}$. For each insurer $i$, the premiums are restricted to take values between €180 and €800 during any period, i.e., $p_i^1 \in [€180, €800]$. In addition the other parameters are restrained, i.e., the market presence limit factor $h_1 \in [0.07, 0.11]$ and the breakeven premium $\pi_i^1 \in [30\% p_i^1, 70\% p_i^1]$, for any $i$, $t$. Numerical results for the system of equations $r_{G_I,i}^1$ are generated using m-file "fsolve". It should be mentioned that the Nash equilibrium premium profile might not be unique. However, among these results we chose the first positive premium profile which located in $\mathbf{P}_{G_I,i}$.[6] This result is illustrated in Figure 1. Figure 2 shows the corresponding number of contracts from insurers 1 to 12.

The ratio between insurers' equilibrium premiums at time $t = 1$ is correlative to the previous premium ratio in Figure 1. Note that the market leader insurer 6 tends to increase its premium, which leads to a reduction of its policy numbers in Figure 2. Larger market power offers insurer 6 the advantage in competition, which allows it to increase its premium until equilibrium for seeking higher profit.

Figure 3 demonstrates the effect of the increasing parameter $\pi_6^1$ in $G_I$. In Figure 3, adjustment for a single insurer's breakeven premium ratio is investigated. The market leader, insurer 6, is modelled to increase $\pi_6^1$ from 30% to 70% of $p_i^1$, whereas all other insurers keep the ratio at 50%. The increase in the breakeven premium ratio of insurer 6 is observed to cause not only an increase in its equilibrium premium but also a slight incremental increase in other insurers' premiums.

Price sensitivity parameter, $a_i$, strongly affects the equilibrium premium of each insurer $i$. The effects of modifying $a_i$ with regard to the market leading insurer 6 and the market follower 8 are illustrated in Figures 4 and 5, and all other parameters remain the same as before. Figure 4 shows that the two players' equilibrium premiums decrease as the price sensitivity parameter decreases. In Figure 5, the number of contracts is observed to increase as $a_i$ increases for both insurers 6 and 8. In addition, in both Figures 4 and 5, the slope of insurer 6 is obviously larger than that of insurer 8, indicating that parameter $a_i$ is more sensitive with respect to the market leader than the market follower.

The values of parameters $a_6$ and $h_1$ strongly affect the equilibrium premium at time $t = 1$. We give an example of insurer 6 about the sensitivity with respect to these two parameters in Figure 6.

[6] Among all the possible positive profiles, we pick up the smallest one based on the iterative algorithm of the Matlab, m-file "fsolve".

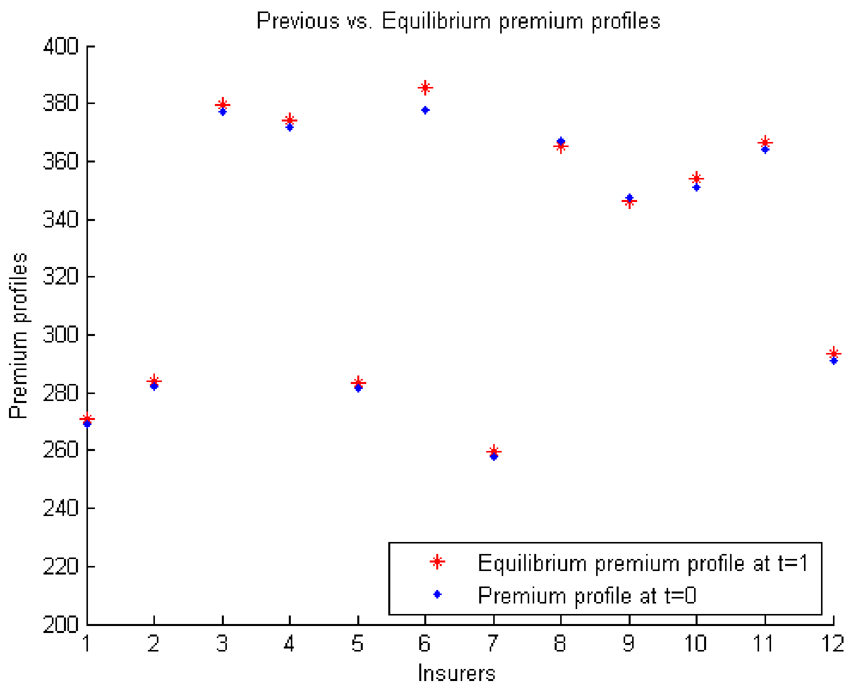


Figure 1: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) premium profiles in $G_I$. The red solid line is the equilibrium premium profile at time $t = 1$ with respect to 12 insurers, which is on the $x$-axis. Premium values are given on the $y$-axis. The blue dash line represents the previous premium profile given in the Table 1.

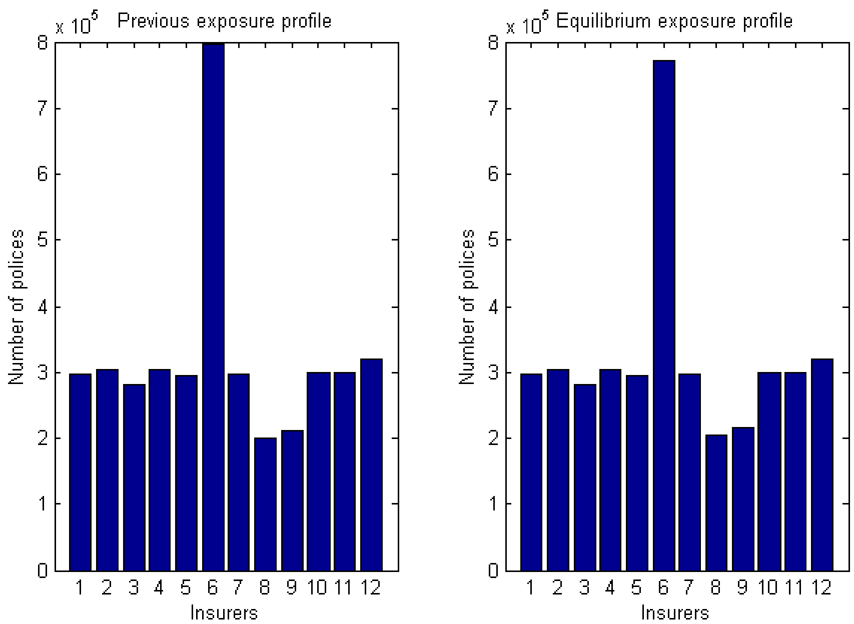


Figure 2: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) number of policies in $G_I$. The left figure illustrates the number of contracts with respect to 12 insurers at time $t = 0$, which are given in Table 1. The right figure shows the equilibrium number of contracts at time $t = 1$.

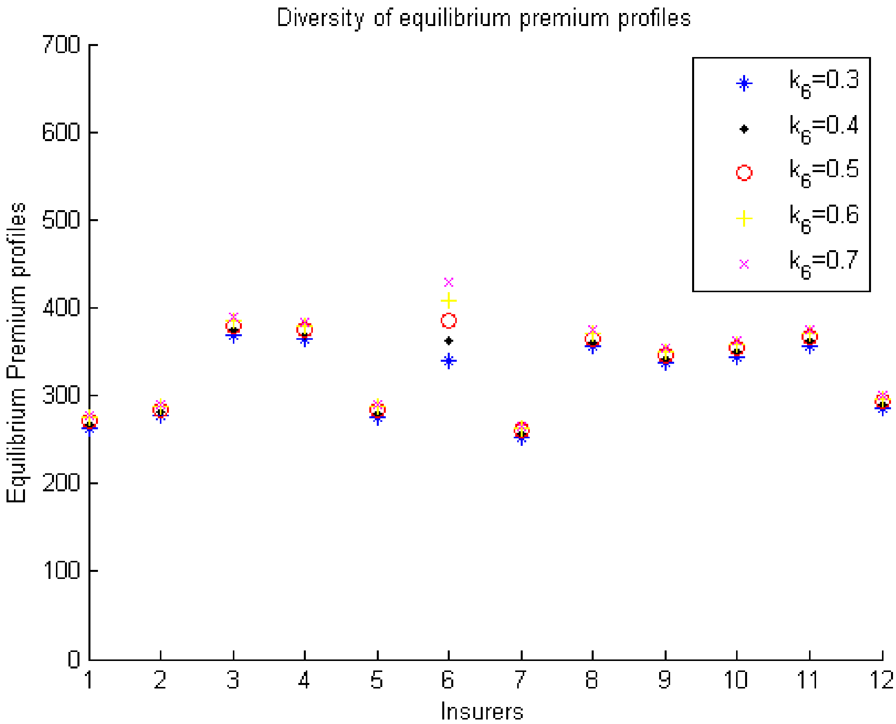


Figure 3: Diversity of equilibrium premium profiles with different $\pi_6^1$ in $G_I$. The market leader 6's breakeven premium ratio is investigated, which takes values from 30% to 70%. The corresponding 5 different equilibrium premium profiles are given.

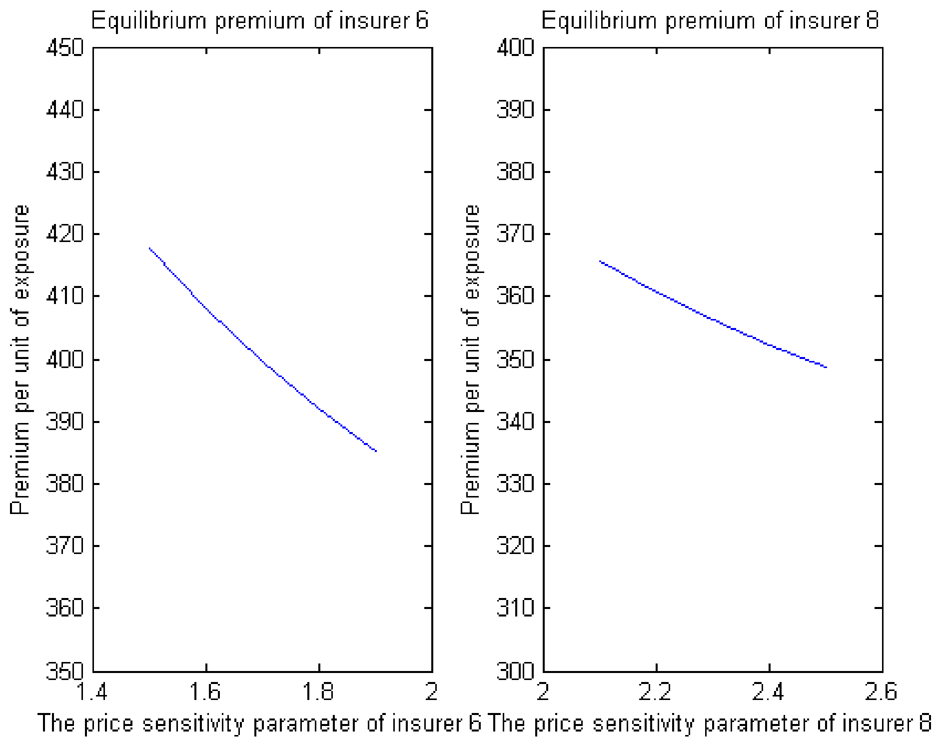


Figure 4: Equilibrium premium sensitivity test of $a_6$ and $a_8$ in $G_I$.

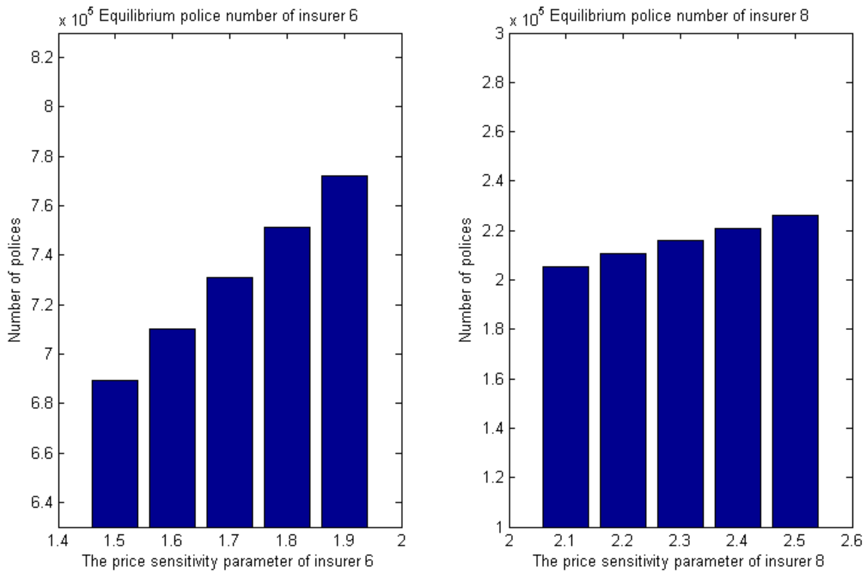


Figure 5: Equilibrium number of policies sensitivity test of $a_6$ and $a_8$ in $G_I$.

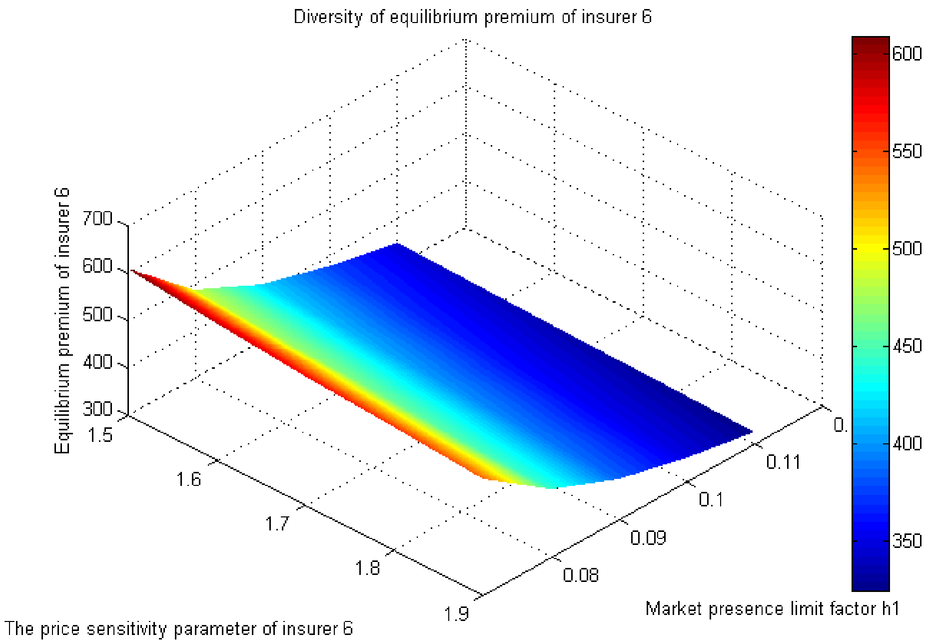


Figure 6: Diversity of insurer 6's equilibrium premium in $G_I$. Different equilibrium premium values are given, with respect to different $a_6$ and $h_1$.

## 4.2 Insurance Game II Simulation Results

Using Table 1, and the same parameters reported in Table 3, the Nash equilibrium premium profiles in $G_{II}$ are calculated. From Eq. (2.9), the second order condition of payoff is negative for each insurer $i$ in $G_{II}$. Similarly, we use the algorithm which is presented for the case $G_I$ by assuming that $\hat{r}^1_{G_{II},i}$ is defined by

$$\hat{r}^1_{G_{II},i} : \frac{q_i^0 + nh_2 a_i q_i^0 - h_2 \sum_{j \neq i} a_j q_j^0}{2h_2 a_i q_i^0 (x^1_{-i} + \theta^1)} p_i^0 = 0,$$

where $\hat{r}^1_{G_{II},i}$ is the maximal selection of $R_{G_{II},i}(x^1_{-i})$, see Eq. (3.2), for $i$ at time $t = 1$ in $G_{II}$.

Note that the breakeven ratio $k_i$ does not affect the best-reply selection in $G_{II}$. If the calculated premium for each insurer is located in $\mathbf{P}_{G_I,i}$, the Nash equilibrium is unique in $G_{II}$, since the equation of $\hat{r}^1_{G_{II},i}$ is a linear one.

| | | |
|---|---|---|
| Number of market participants | $n$ | 12 |
| Market presence limit factor | $h_2$ | 0.0205 |
| Market stability factor | $\theta^1$ | 1 |

Table 3: Environmental parameter values in $G_{II}$

In $G_{II}$, for each insurer $i$, the premiums are retained between €180 and €900 during any period, i.e., $p^1_i \in [€150, €900]$. The other parameters are also restricted, such as the market presence limit factor $h_2 \in [0.0203, 0.0207]$ and the market stability factor $\theta^1 \in [0.8, 1.2]$ for any $t$. Figures 7 and 8, respectively, show the equilibrium premium profile and number of contracts from insurers 1 to 12.

In Figure 7, similar to $G_I$, market leader insurer 6 tends to increase its premium until equilibrium. As exposure flows between insurers are enhanced, the ratio between insurers' equilibrium premium in $G_{II}$ significantly diverge from the previous. Compared with $G_I$, the market leader has a greater advantage in the competition, which generates a larger reduction in the policy numbers than in Figure 2. Market followers 8 and 9 reduce their premiums significantly to increase their exposure. As demonstrated in Figure 8, the equilibrium number of policies of insurers 8 and 9 approximately reach the other insurer's level, excluding the market leader insurer 6.

With the other parameters unaffected, the impacts of modifying $a_i$ in $G_2$ with regard to the market leading insurer 6 and the market follower 8 are illustrated in Figures 9 and 10. Similarly as $G_I$, Figures 9 and 10 indicate that both players' equilibrium premiums in $G_{II}$ decrease and the number of contracts increases as the price sensitivity parameter $a_i$ decreases. In addition, we also conclude that the parameter $a_i$ with respect to the market leader is more sensitive than the market follower in $G_{II}$. Comparing with Figures 4 and 5 in $G_I$ that $a_6$ is more sensitive than $a_8$ in $G_{II}$ is also noteworthy.

A new parameter, market stability factor $\theta^1$, significantly affect the equilibrium premium profile in $G_{II}$. Figure 11 illustrates the diversity of the equilibrium premium profiles with a varying market stability factor $\theta^1$ from 0.8 to 1.2. As $\theta^1$ represents the whole market's business condition, it is reasonable to expect the equilibrium profile entirely moves up or down with different $\theta^1$.

Similarly as in $G_I$, we test the sensitivity of $a_6$ and $h_2$ for $G_{II}$ in Figure 12. As we can observe, $h_2$ is much more sensitive than $h_1$, an tiny increase of just $10^{-4}$ in $h_2$ causes a compelling decrease in equilibrium premium for insurer 6.

Overall, we observe that insurers with larger market power take advantage in the competition, and they tend to increase their premium to reach equilibrium. On the other hand, insurers with less market power tend to decrease their premium requesting a bigger volume of exposure. The price sensitivity parameter, $a_i$, is quite sensitive. The market presence limit factor $h_1, h_2$, and the market stability factor $\theta^1$ have an impact on the market equilibrium levels, which control the exposure of volume flow among the insurers and the exposures volume flow into or away from the insurance market, respectively. Different with $G_I$, a breakeven premium for $i$ appears not to affect the insurer's equilibrium premium in $G_{II}$.

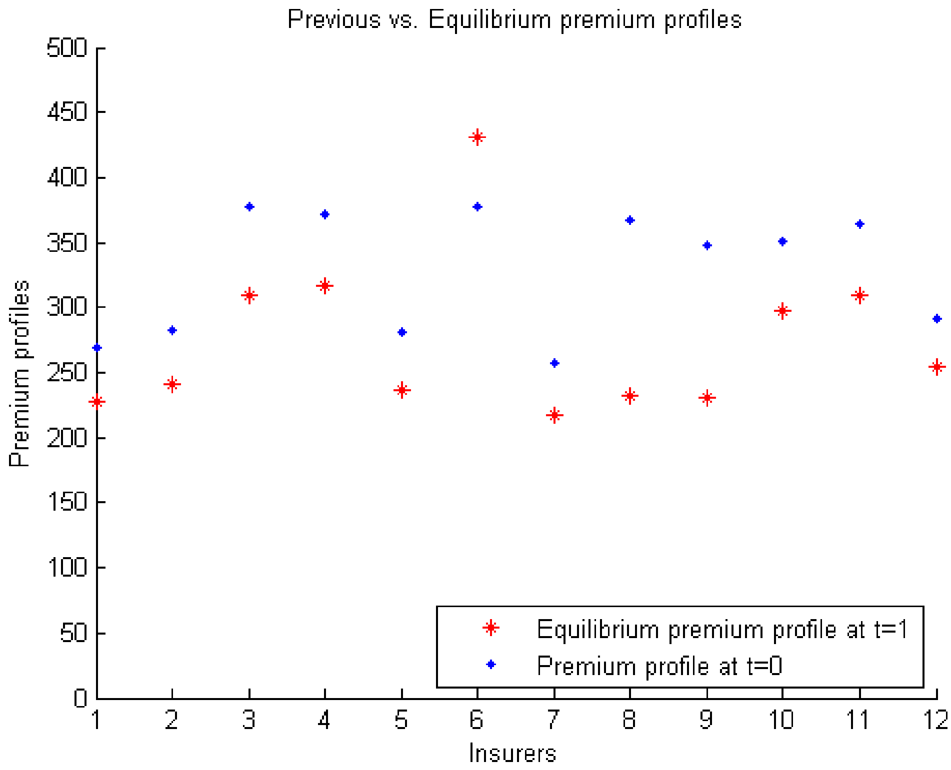


Figure 7: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) premium profiles in $G_{II}$. Similar with Figure 1.

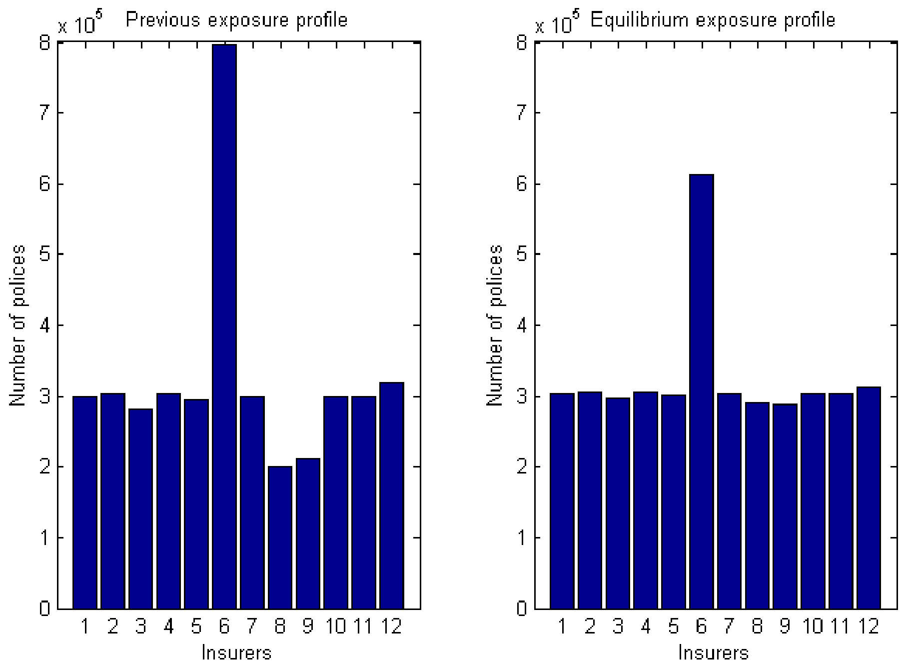


Figure 8: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) number of policies in $G_{II}$. Similar with Figure 2.

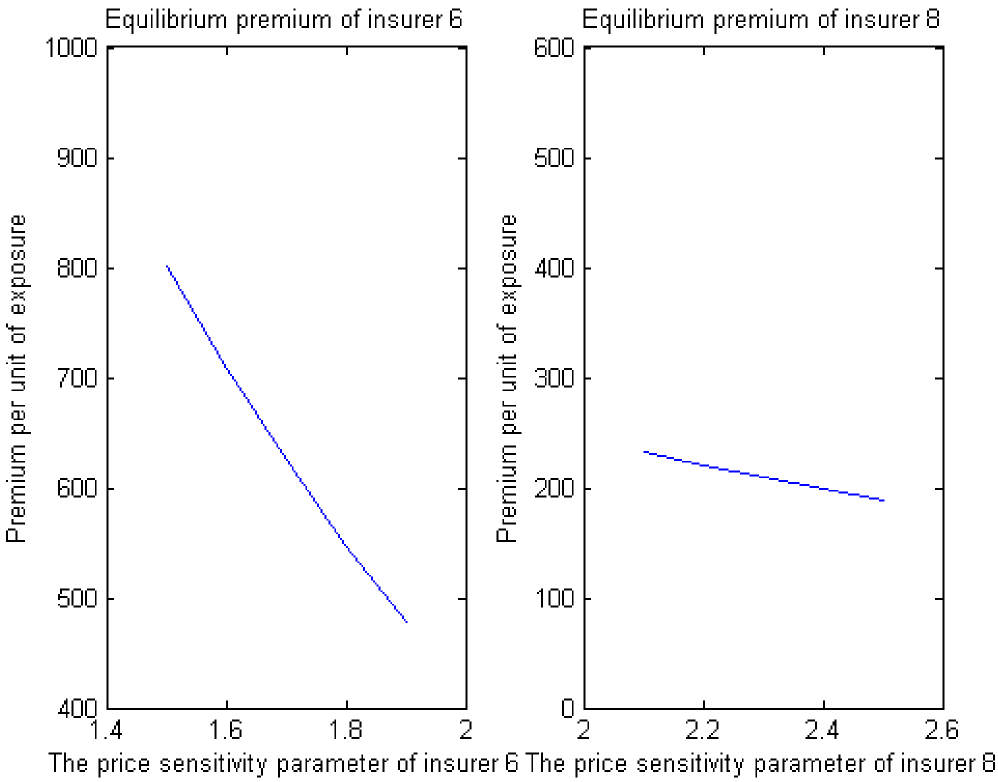


Figure 9: Equilibrium premium sensitivity test of $a_6$ and $a_8$ in $G_{II}$.

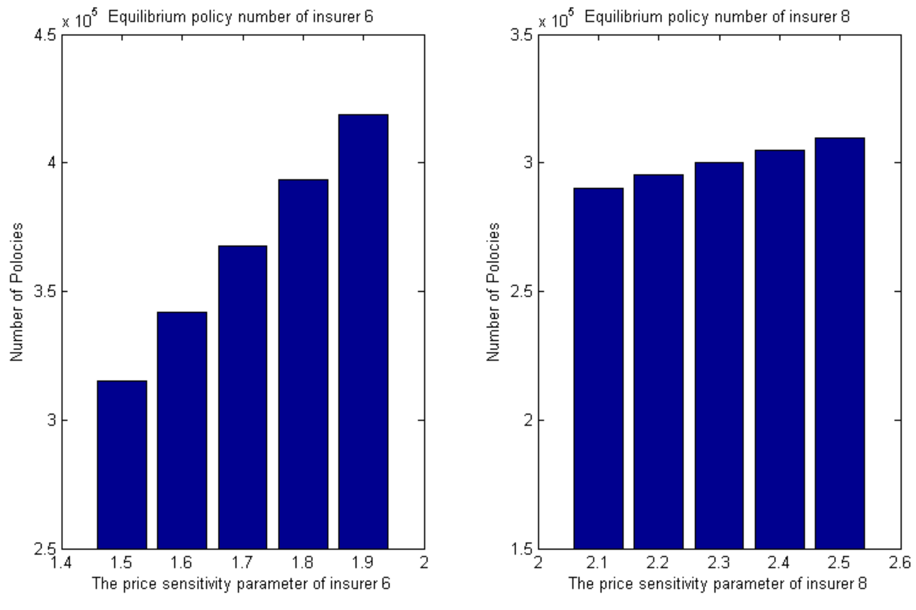


Figure 10: Equilibrium number of policies sensitivity test of $a_6$ and $a_8$ in $G_{II}$.

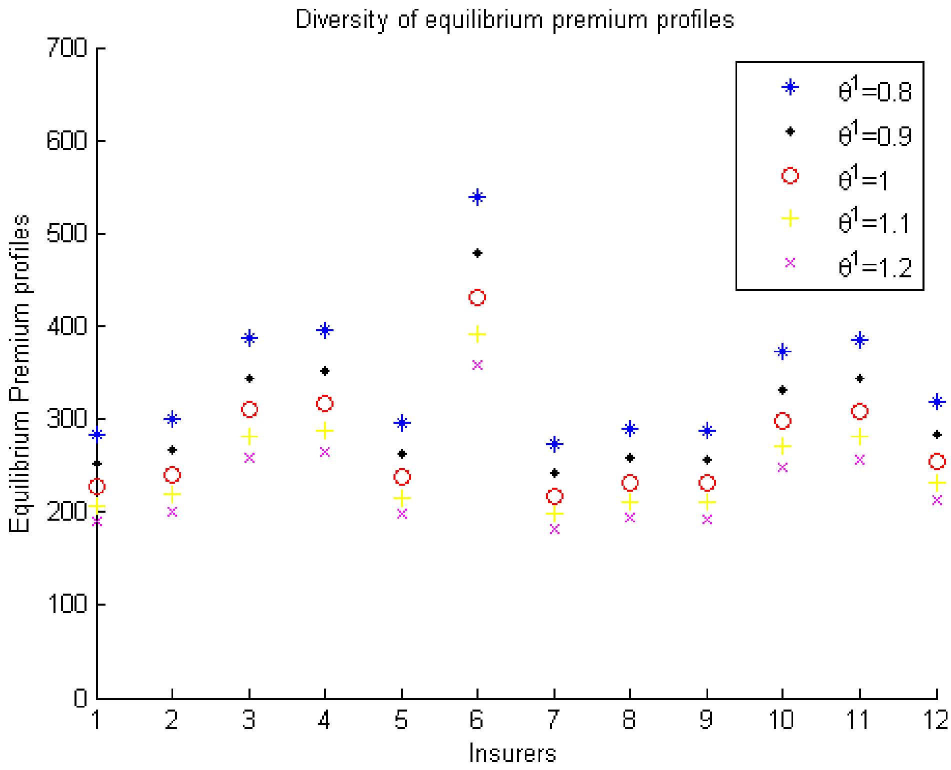


Figure 11: Diversity of equilibrium premium profiles with different $\theta^1$ in $G_{II}$. The market stability factor $\theta^1$ is investigated which takes values from 0.8 to 1.2. The corresponding 5 different equilibrium premium profiles are given.

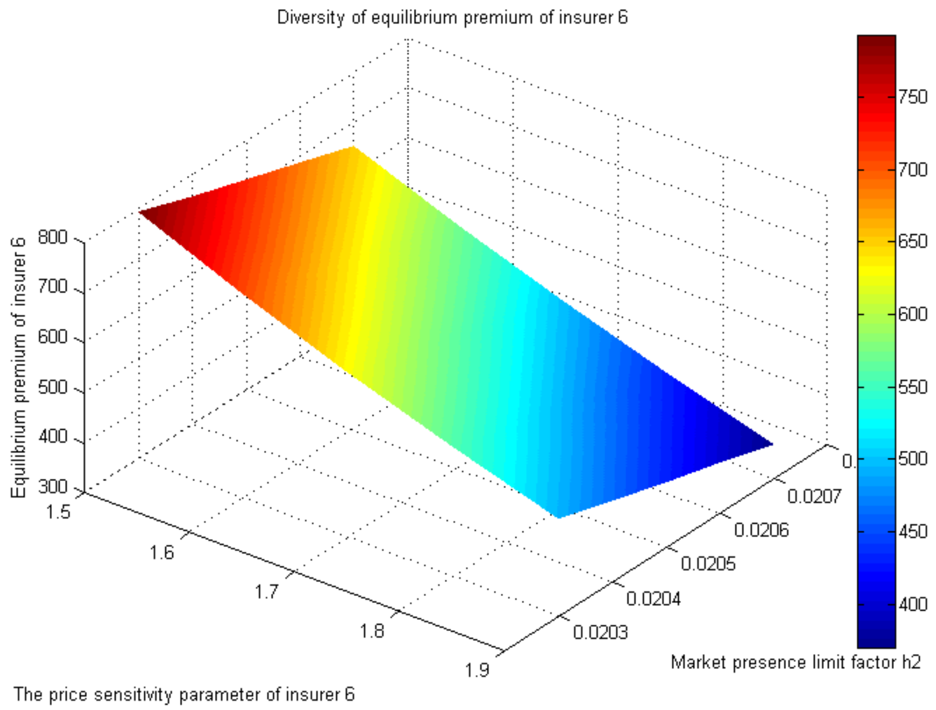


Figure 12: Diversity of insurer 6's equilibrium premium in $G_{II}$. Different equilibrium premium values are given, with respect to different $a_6$ and $h_2$.

## 5 Conclusion

This paper models two-stage non-cooperative games in an insurance market to investigate how the competition impacts the pricing process of non-life insurance products. Insurers compete to maximise their payoffs in a second stage by adjusting premium pricing strategies, which leads to diversity of the volume of exposure. We further characterise one insurer's second-stage modified volume of exposure in a way that sums up the exposure flows in or out during competitions with other insurers. The modified second volumes of exposure in any two insurers' competition are characterised by transferring one insurer's second stage premium to the other's first-stage premium and modelling the changing volume through a definition of price elasticity. Two models are discussed in detail regarding the modified volume of exposure: simple exposure difference model I ($G_I$) and advanced exposure difference model II ($G_{II}$). Using payoffs in these two models, two N-player games are constructed with non-linear aggregate and positive, compact but not necessarily convex, premium strategy sets. A potential game with an aggregation technique is applied: we prove the existence of a pure Nash equilibrium of these two games by determining the potential functions. Both games' pure Nash equilibriums can be solved by calculating the best-response equation systems. The numerical results for 12-player insurance games are presented under the framework that the best-response selection premium strategies always provide the global maximum value of the corresponding payoff function.

The insurance game can be extended in different directions. A natural next extension is to develop dynamic insurance games to observe insurance market premium pricing cycles. Applying stochastic models might be interesting in dynamic cases. Another extension would be a mixed strategy Nash equilibrium, where insurers choose a probability distribution over possible premium strategies.

**Acknowledgement:** The authors would like to thank the anonymous reviewers for their insightful comments that significantly improved the quality of this paper. The authors acknowledge the gracious support of this work by the EPSRC and ESRC Centre for Doctoral Training on Quantification and Management of Risk & Uncertainty in Complex Systems & Environments (EP/L015927/1). Furthermore, we would like to thank Yohan Pelosse in particular for the very long discussions that we had and his valuable suggestions, and his recommendation that we implement potential games in our actuarial problem. Moreover, this paper benefited from the comments of participants (Hirbod Assa, Christian Ewald, Steven Haberman, Vladimir Kaishev, Iqbal Owadally, Eudokia Passalidou, and Jukka Rantala) at the 1st Quantitative Finance and Risk Analysis Symposium, the 19th Insurance: Mathematics and Economics Congress, and seminar talks at the University of Liverpool (UK) and CASS Business School (UK). Warm thanks are due to Tim Boonen who read very carefully a preliminary version of our paper and who has afforded us considerable assistance in enhancing both the quality of the findings and the clarity of their presentation. Any remaining errors are ours.

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