DAMTP/13-53
arXiv:1312.0428[hep-th]
5 March, 2014

Constraints on RG Flow for Four Dimensional Quantum Field Theories

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The response of four dimensional quantum field theories to a Weyl rescaling of the metric in the presence of local couplings and which involve a, the coefficient of the Euler density in the energy momentum tensor trace on curved space, is reconsidered. Previous consistency conditions for the anomalous terms, which implicitly define a metric G on the space of couplings and give rise to gradient flow like equations for a, are derived taking into account the role of lower dimension operators. The results for infinitesimal Weyl rescaling are integrated to finite rescalings $e^{2\sigma}$ to a form which involves running couplings g_σ and which interpolates between IR and UV fixed points. The results are also restricted to flat space where they give rise to broken conformal Ward identities.

Expressions for the three loop Yukawa β-functions for a general scalar/fermion theory are obtained and the three loop contribution to the metric G for this theory are also calculated. These results are used to check the gradient flow equations to higher order than previously. It is shown that these are only valid when $\beta \to B$, a modified β-function, and that the equations provide strong constraints on the detailed form of the three loop Yukawa β-function. $\mathcal{N} = 1$ supersymmetric Wess-Zumino theories are also considered as a special case. It is shown that the metric for the complex couplings in such theories may be restricted to a hermitian form.

Keywords: RG flow, a-theorem, Weyl scaling

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1. Introduction

The paradigm shift in our understanding of quantum field theories due to Wilson in the 1970's led to the understanding that quantum field theories are not isolated objects but may be regarded as points on a manifold, with coordinates given by the couplings $\{g^I\}$, where there is a natural flow under changes of the cut off scale realising the renormalisation group. The perturbative RG flow equations are just first order equations determined by the β-functions $\beta^I(g)$, which are vector fields on the space of couplings. Even in this context the global topology of such flows has been less certain, the simplest scenario arises when the flows link fixed points in the UV short distance limit to other fixed points in the large distance IR limit. At the fixed points the quantum field theory is scale invariant and moreover is naturally expected to become a conformal field theory. However more complicated behaviours under RG flow, such as limit cycles or the flow becoming chaotic, are also feasible. As was first suggested by Cardy [1] there may be additional constraints for unitary quantum field theories in four dimensions due to the existence of a function $a(g)$ which has monotonic behaviour under RG flow, or more minimally a may be defined at fixed points so that $a_{\rm UV} - a_{\rm IR} > 0$. These two scenarios are here described as the strong and weak a-theorem, such a distinction was made in [2]. If valid a strong a-theorem constrains the RG flow without assuming any UV completion although it requires the RG flow to be described by linear equations involving β-functions.

The proposal of Cardy was for a four dimensional generalisation of the Zamolodchikov c-theorem, [3]. This constrains the structure of two dimensional quantum field theories and has a simple elegant proof depending just on the properties of the two point correlation function of the energy tensor. The crucial positivity constraint arises from unitarity conditions applied to the two point function. No such approach works in four dimensions [4], [5] but it was soon clear that only a, which is determined by the topological term in the trace of the energy momentum tensor on curved space, is a viable candidate for a monotonic flow between fixed points. The energy momentum tensor two point function in conformal theories is determined by c, the coefficient of the square of the Weyl tensor in the energy momentum tensor trace on curved space.

Much more recently Komargodski and Schwimmer [6] have described a proof of the four dimensional weak a-theorem which has been further analysed in [7] with possible extensions to higher dimensions discussed in [8]. This rests on coupling the theory to a dilaton and constructing an effective low energy field theory for the dilaton. The essential positivity requirement depends on positivity conditions arising from unitarity for the four dilaton scattering amplitude. The starting point of the discussion in [6] is the response of a conformal theory to a Weyl rescaling of the flat metric. The resulting expression determines the couplings of the dilaton introduced as a compensator for the local anomalous terms which arise under a Weyl rescaling and which have a coefficient proportional to a. The basic argument of Komargodski and Schwimmer is that coupling to a dilaton ensures a matching of these anomalies between the UV and IR fixed points.

However the results of [6] and also [7] do not immediately extend away from conformal fixed points. There is also no obvious connection with a perturbative version of the strong a-theorem for four dimensional renormalisable quantum field theories. This was based on an analysis in terms of dimensional regularisation [9] and also from Wess-Zumino consistency conditions for the response of the theory on curved background to a Weyl rescaling of the metric [10]. Instead of a dilaton as in [6] the usual linear RG equations describing the response to a variation in the RG scale μ were extended to a local infinitesimal Weyl rescaling $\sigma(x)$ by allowing the couplings also to be local $g^I(x)$, with an arbitrary dependence on x. Local RG equations for variations of $\sigma(x)$ reduce to the conventional linear differential constraints for σ and g^I constant but contain additional local contributions depending on the derivatives of g^I, as well as the curvature. The consistency conditions arise from the abelian nature of the group of Weyl scale transformations. Such an approach has also been extended to six dimensions in [11] and three in [12].

In this paper we revisit some of the results in [10], with an hopefully improved notation (although we apologise for alphabetical profligacy) and extensions. The essential result is that there is a scalar function of the couplings $\tilde{A}(g)$ such that

$$\mathrm{d}_g \tilde{A}(g) = \mathrm{d}g^I T_{IJ}(g)\beta^J(g) \,, \tag{1.1}$$

where at a fixed point $\beta^I(g_*) = 0$, $\frac{1}{4}\tilde{A}(g_*) = a$. The symmetric part of T_{IJ} defies a natural metric G_{IJ} so that under RG flow

$$\beta^I \partial_I \tilde{A} = G_{IJ}\beta^I\beta^J \,, \tag{1.2}$$

Away from fixed points $\tilde{A}(g)$ is arbitrary up to

$$\tilde{A}(g) \to \tilde{A}(g) + g_{IJ}(g)\beta^I(g)\beta^J(g) \,, \tag{1.3}$$

while correspondingly

$$G_{IJ} \to G_{IJ} + \mathcal{L}_\beta g_{IJ} \,, \quad \mathcal{L}_\beta g_{IJ} = \beta^K\partial_K g_{IJ} + \partial_I\beta^K g_{KJ} + \partial_J\beta^K g_{IK} \,. \tag{1.4}$$

It is then sufficient in order to demonstrate the strong version of the a-theorem that $G_{IJ} + \mathcal{L}_\beta g_{IJ}$ is positive definite just for some particular g_{IJ}.

In two dimensions positivity of the metric, up to the freedom in (1.4), flows from showing [10] that $G_{IJ} + \mathcal{L}_\beta g_{IJ}$, for suitable g_{IJ}, becomes the Zamolodchikov metric determined by the two point function $G_{IJ}(\mu^2x^2)_{\mathrm{Zam}} = (x^2)^2\langle \mathcal{O}_I(x)\, \mathcal{O}_J(0)\rangle$, for $\{\mathcal{O}_I\}$ scalar operators dual to $\{g^I\}$, [3]. Variation of x^2 in $G_{IJ\,\mathrm{Zam}}$ is equivalent to (1.4). However the original analysis demonstrates (1.2) and does not directly imply (1.1), see also [13].

In four dimensions G_{IJ} is related to $\langle \mathcal{O}_I \, T_{\mu\nu} \, \mathcal{O}_J \rangle$, although the precise connection is not fully clear and positivity, except at weak coupling when G_{IJ} can be calculated or at a conformal fixed point, is however less apparent from such a series expression.

The consistency conditions such as (1.1), obtained previously in [10] and discussed further in this work, are derived by considering the response to infinitesimal Weyl rescalings of the metric. We also consider the response of the theory to finite Weyl rescalings of the metric $\gamma_{\mu\nu} \to e^{2\sigma}\gamma_{\mu\nu}$. The result is also expressed in terms of running couplings $g_\sigma{}^I$ together with additional contributions also depending explicitly on σ, involving derivatives up to $\mathrm{O}(\sigma^4)$, and containing G_{IJ} and related functions as well as derivatives of the couplings. For four dimensional theories the final expression is quite involved but it extends the result at a fixed point used as a starting point for the introduction of a dilaton field in [6] and [7].

For four dimensional theories the local RG equations, from which (1.1) is derived, are essentially equivalent to expressing the energy momentum tensor trace in terms of a basis of scalar operators as well as contributions involving the curvature, defining c and a, but also scalars formed from derivatives of g^I. However even on flat space with constant g^I there may be derivative terms so that

$$\eta^{\mu\nu} T_{\mu\nu} = \beta^I(g)\mathcal{O}_I + \partial_\mu J_v{}^\mu \,. \tag{1.5}$$

Here $J_v{}^\mu$ is a current associated with an element v of the Lie algebra of the symmetry group G_K of the kinetic terms of the theory. Such terms may arise at three loops in perturbative calculations for scalar fermion theories [14], [15]. A fixed point $\beta^I(g_*) = 0$ would apparently give rise to scale but not conformally invariant theories if there is no redefinition of $T_{\mu\nu}$ which removes $\partial_\mu J_v{}^\mu$. However the β-functions have an arbitrariness related to the freedom to make transformations under G_K at the expense of a redefinition of the couplings. This freedom cancels in (1.5) so that it can be rewritten as

$$\eta^{\mu\nu} T_{\mu\nu} = B^I(g)\mathcal{O}_I \,, \tag{1.6}$$

where

$$B^I(g) = \beta^I(g) - (vg)^I \,, \tag{1.7}$$

so that if the couplings are not all invariant under G_K there may be a difference between β^I and B^I. If this possibility arises (1.1) holds for $\beta^I \to B^I$ and hence the potential strong a-theorem discussed here applies to the RG flow generated by B^I, and its vanishing, $B^I(g_*) = 0$, at a fixed point defines a CFT. The transformation from (1.5) to (1.6), in terms of the modified β-functions as in (1.7), assumes there are no anomalies in $\partial_\mu J_{\upsilon}{}^\mu$. This should be the case in parity conserving theories when $J_{\upsilon}{}^\mu$ is a vector current.

The existence of $\tilde{A}(g)$ satisfying (1.1) also requires integrability conditions which constrain the form of β-functions. This was explored in [9] and is investigated further in this paper, see also [16]. The conditions require relations between the coefficients appearing in β-functions at different loop orders and which correspond to graphs of very different topologies.

As an application of the results obtained and for the analysis of the integrability constraints on β-functions we consider here a model renormalisable scalar fermion theory with Yukawa and quartic scalar couplings. Previously [9] the various quantities appearing in the consistency conditions were calculated to lowest perturbative order for general theories including gauge fields. To go beyond this requires three loop calculations. For complex scalars coupled to Weyl fermions imposing a $U(1)$ symmetry ensures that the number of graphs necessary is O(10) rather than O(100), or more, for a completely general scalar/fermion theory. We obtain results for three loop anomalous dimensions and Yukawa β-functions without calculating more than a couple of graphs by reducing this theory to one describing the standard model top/Higgs coupling, recently obtained by Chetyrkin and Zoller [17], and also a general $\mathcal{N}=1$ supersymmetric scalar fermion theory when the relevant results have been known for some time [18]. The consistency conditions obtained here allow calculations for T_{IJ}, initially defined in terms of a curved background, to be reduced to flat space calculations and we determine the three loop contributions depending on the Yukawa couplings in the specific model theory for which the three loop β-functions were obtained. The result requires extracting the local divergences for two three-loop vacuum diagrams. The results can be checked by reducing to supersymmetry as a special case when much simpler superspace methods are possible. As usual we use dimensional regularisation which may be problematic at higher loop orders. These issues are discussed in [17], but in the absence of gauge fields here such problems appear to be irrelevant to the order considered here.

We consider in detail the application of these results to $\mathcal{N}=1$ Wess Zumino supersymmetric theories, extending the discussion in [19]. For such theories the space of couplings is naturally a complex manifold since they may be extended to chiral or anti-chiral superfields. We show that three loop calculations demonstrate that the metric is hermitian to this order. Furthermore when redefinitions as in (1.4) are extended to the supersymmetric case the assumption of a hermitian metric is preserved. There is no all orders proof of hermiticity in the context of this paper, although for superconformal theories related results have been obtained by Papadodimas [20] and Asnin [21]. The results for the metric can also be expressed in Kähler form if allowance is made for potential redefinitions of the couplings.

Although this paper is quite lengthy each section is substantially independent. In section 2 we rederive the local RG equations and associated integrability conditions which follow by considering the response to infinitesimal Weyl rescalings of the metric in theories in which the couplings are allowed to be local. In section 3 the infinitesimal transformations are integrated to obtain the change in the vacuum energy functional W under finite rescalings. The results depend on running couplings ${g_\sigma}^I$ and provide an interpolation between UV and IR fixed points. In section 4 we restrict the equations to flat space and broken conformal symmetry. This context is sufficient to allow the metric G_{IJ}, which is initially defined for curved space backgrounds, to be recovered just from flat space calculations.

The scalar fermion theory used as an illustration is introduced in section 5 and the various β-functions and anomalous dimensions listed. In particular three loop results for the Yukawa β-functions and also the anomalous dimensions for this theory are obtained, primarily using previous calculations and also the restriction to the supersymmetric case. In section 6 we analyse the RG equations for this theory. It is shown how they impose non trivial consistency conditions on the coefficients which are present in the general expansions for the β-functions and associated anomalous dimensions. In particular it is shown that at three loop order it is necessary to take account of (1.7) for (1.1) to be valid. The result for v at this order is in agreement with the detailed three loop calculations of Fortin *et al* [15] for scalar fermion theories. In section 7 we restrict to supersymmetric theories and demonstrate the consistency of a hermitian metric. The results are compared with expressions when a-maximization is extended away from superconformal fixed points by introducing Lagrange multipliers and also the possibility of a Kähler form for the metric is discussed. Sections 8 and 9 describe how the metric and related quantities can be determined by flat space calculations using dimensional regularisation. Section 8 discusses the general formalism for renormalisable theories with local couplings and sets up the required RG equations. Section 9 applies these methods to the scalar/fermion theory and determines the additional necessary field independent counterterms to three loops. These determine the metric and, specialised to the supersymmetric case, show that it is hermitian to this order.

There are four appendices containing further calculational details. Appendix A analyses how particular contributions to the anomalous dimensions in supersymmetric theories which are proportional to transcendental numbers can be extended to determine the related contributions to the metric and also a. Appendix B contains further details on the derivation of local RG equations in the context of dimensional regularisation. The RG equations are extended to allow for special conformal transformations as well as the usual variations of scale. The methods used here to obtain the three loop counterterms for

Yukawa theories with dimensional regularisation are described in appendix C and are also extended to four loops for scalar theories in appendix D.

2. Local RG Equations and Integrability Conditions

As was demonstrated in [10], and more recently in [6], non trivial constraints on the RG flow in quantum field theories can be obtained by considering the response to infinitesimal local Weyl rescalings of the metric of the form

$$\delta_\sigma \gamma_{\mu\nu} = 2\sigma\, \gamma_{\mu\nu} \,, \tag{2.1}$$

when the theory is extended to an arbitrary curved space background. Conformally invariant theories are invariant under such rescalings up to local conformal anomalies induced by the non vanishing of the energy momentum tensor on curved space. Equations for the response to such Weyl rescalings for quantum field theories not at conformal fixed points may be obtained if the couplings are extended to arbitrary local functions and at the same time as (2.1) there is a flow in the space of local couplings. The resulting equations are then an extension of the standard linear equations which determine the RG flow in terms of the usual β-functions and are realised by restricting to constant σ as well as constant couplings. Choosing couplings $\{g^I\}$, which are coordinates for a manifold $\mathcal{M}_g$, the local RG equations obtained in [10] by assuming the quantum field theories are extended to arbitrary $g^I(x)$ as well as $\gamma_{\mu\nu}(x)$ are then generated in four dimensions by the functional differential operator

$$\Delta_\sigma = \int \mathrm{d}^4x\, \sigma \left(2\gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} + \beta^I \frac{\delta}{\delta g^I} \right), \tag{2.2}$$

where the β-functions, which are contravariant vectors on $\mathcal{M}_g$, have in general a linear contribution

$$\beta^I(g) = -(d-\Delta_I)g^I + \mathrm{O}(g^2) \,. \tag{2.3}$$

In (2.3), in the present context, the spatial dimension $d=4$ and Δ_I is the scale dimension of the operator $\mathcal{O}_I$, which is dual to g^I, at the critical point when all $g^J \to 0$. Initially we restrict for simplicity to just marginal operators with $\Delta_I = 4$, as for renormalisable theories when $g^J = 0$ is the free theory.

Acting on the vacuum energy functional $W[\gamma_{\mu\nu}, g^I]$, Δ_σ gives zero up to a residual local contribution, depending just on $\gamma_{\mu\nu}, g^I$ and their derivatives at x, so that

$$\begin{aligned} \Delta_\sigma\, 16\pi^2 W = & - \int d^4x \sqrt{-\gamma}\, \sigma \Big(- C\, F + \tfrac{1}{4} A\, G + \tfrac{1}{72} B\, R^2 + E^{\mu\nu}\, G_{IJ} \partial_\mu g^I \partial_\nu g^J \\ & \qquad\qquad + \tfrac{1}{6} R \left(E_I \nabla^2 g^I + F_{IJ} \partial^\mu g^I \partial_\mu g^J \right) - X \Big) \\ & - 2 \int d^4x \sqrt{-\gamma}\, \partial_\mu \sigma \left(E^{\mu\nu}\, W_I \partial_\nu g^I + \tfrac{1}{6} R\, H_I \partial^\mu g^I + Y^\mu \right) \\ & - \int d^4x \sqrt{-\gamma}\, \nabla^2 \sigma \left(\tfrac{1}{6} R\, D + Z \right), \end{aligned} \tag{2.4}$$

where the curvature terms, apart from the Ricci scalar R, are

$$F = C^{\mu\nu\sigma\rho} C_{\mu\nu\sigma\rho}\,, \quad G = \tfrac{1}{4} \epsilon_{\mu\nu\sigma\rho} \epsilon^{\alpha\beta\gamma\delta} R^{\mu\nu}{}_{\alpha\beta} R^{\sigma\rho}{}_{\gamma\delta}\,, \quad E^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} \gamma^{\mu\nu}\, R\,, \tag{2.5}$$

so that G is the Euler density and $E^{\mu\nu}$ is the Einstein tensor. With the normalisations in (2.4)

$$\begin{aligned} C_{\text{free}} &= \tfrac{1}{40} \left(\tfrac{1}{3} n_S + n_W + 4 n_V \right), \\ A_{\text{free}} &= \tfrac{1}{90} \left(n_S + \tfrac{11}{2} n_W + 62 n_V \right), \end{aligned} \tag{2.6}$$

for n_S real scalars, n_W Weyl fermions and n_V vectors. The remaining terms in (2.4), X, Y^μ, Z, are independent of the curvature and involve just the local couplings g^I and their derivatives. X, Y^μ, Z therefore remain on restriction to flat space and can be decomposed in the form

$$\begin{aligned} X(g) &= \tfrac{1}{2}\, A_{IJ}\, \nabla^2 g^I \nabla^2 g^J + B_{IJK}\, \nabla^2 g^I \partial^\mu g^J \partial_\mu g^K + \tfrac{1}{2}\, C_{IJKL}\, \partial^\mu g^I \partial_\mu g^J\, \partial^\nu g^K \partial_\nu g^L\,, \\ Y^\mu(g) &= S_{IJ}\, \partial^\mu g^I \nabla^2 g^J + T_{IJK}\, \partial^\mu g^I\, \partial^\nu g^J \partial_\nu g^K\,, \\ Z(g) &= U_I\, \nabla^2 g^I + V_{IJ}\, \partial^\mu g^I \partial_\mu g^J\,. \end{aligned} \tag{2.7}$$

Clearly $G_{IJ}, F_{IJ}, F_{IJ}, V_{IJ}$ are symmetric while $B_{IJK} = B_{I(JK)}$, $T_{IJK} = T_{I(JK)}$ and $C_{IJKL} = C_{(IJ)(KL)} = C_{(KL)(IJ)}$. The notation in (2.4) and (2.7) is an adaptation of that in [10], with suitable modifications to ensure later simplifications. G_{IJ}, A_{IJ}, S_{IJ} are covariant tensors under a redefinition of the couplings $g^I \to h^I(g)$ while E_I, W_I, H_I, U_I are vectors. Since $\nabla^2 g^I \to \partial_J h^I \nabla^2 g^J + \partial_J \partial_K h^I \partial^\mu g^J \partial_\mu g^K$, the transformation of B_{IJK}, C_{IJKL} under such a change in the couplings contains additional inhomogeneous terms. If A_{IJ} is invertible X may be written as

$$X = \tfrac{1}{2}\, A_{IJ}\, \mathcal{D}^2 g^I \mathcal{D}^2 g^J + \tfrac{1}{2}\, \hat{C}_{IJKL}\, \partial^\mu g^I \partial_\mu g^J\, \partial^\nu g^K \partial_\nu g^L\,, \tag{2.8}$$

where $\mathcal{D}^2 g^I$ is defined by

$$\mathcal{D}^2 g^I = \nabla^2 g^I + B^I{}_{JK}\, \partial^\mu g^J \partial_\mu g^K\,, \quad B^I{}_{JK} = (A^{-1})^{IL} B_{LJK}\,, \tag{2.9}$$

with $B^I{}_{JK}$ acting as a connection on $\mathcal{M}_g$. In (2.8) $\hat{C}_{IJKL} = C_{IJKL} - B^M{}_{IJ} B_{MKL}$ which then also transforms as a tensor under redefinitions of the couplings.

Defining the energy momentum tensor and local operators $\mathcal{O}_I$ by

$$2\frac{\delta}{\delta\gamma^{\mu\nu}(x)}W = -\sqrt{-\gamma(x)}\,\langle T_{\mu\nu}(x)\rangle\,, \qquad \frac{\delta}{\delta g^I(x)}W = -\sqrt{-\gamma(x)}\,\langle \mathcal{O}_I(x)\rangle\,, \tag{2.10}$$

the result (2.4) then encodes the standard form for the trace anomaly

$$16\pi^2\big(\gamma^{\mu\nu}\langle T_{\mu\nu}\rangle - \beta^I\langle\mathcal{O}_I\rangle\big)\big|_{\partial g=0} = C\,F - \tfrac{1}{4}A\,G - \tfrac{1}{72}B\,R^2 - \tfrac{1}{6}D\,\nabla^2 R\,. \tag{2.11}$$

The crucial consistency conditions arise from the fact that the group of local Weyl transformations is abelian so that

$$\big[\Delta_\sigma, \Delta_{\sigma'}\big] = 0\,. \tag{2.12}$$

Using, under Weyl rescalings of the metric as in (2.1),

$$\begin{aligned} \delta_\sigma F =&\ -4\sigma\,F\,, \quad \delta_\sigma G = -4\sigma\,G + 8\,E^{\mu\nu}\nabla_\mu\nabla_\nu\sigma\,, \quad \delta_\sigma R = -2\sigma\,R - 6\nabla^2\sigma\,, \\ \delta_\sigma E^{\mu\nu} =&\ -4\sigma\,E^{\mu\nu} - 2(\nabla^\mu\nabla^\nu - \gamma^{\mu\nu}\nabla^2)\sigma\,, \qquad \delta_\sigma\nabla^2 = -2\sigma\,\nabla^2 + 2\partial_\mu\sigma\,\nabla^\mu\,, \end{aligned} \tag{2.13}$$

then the curvature dependent terms arising from imposing (2.12) give the integrability condition

$$\partial_I A = G_{IJ}\beta^J - \mathcal{L}_\beta W_I\,, \tag{2.14}$$

and relations which determine the R dependent terms

$$B = E_I\beta^I - \mathcal{L}_\beta D\,, \tag{2.15}$$

and

$$\begin{aligned} E_I =&\ -A_{IJ}\beta^J - \mathcal{L}_\beta U_I\,, \\ F_{IJ} =&\ G_{IJ} - B_{KIJ}\beta^K - U_K\,\partial_I\partial_J\beta^K - \mathcal{L}_\beta V_{IJ}\,, \\ H_I =&\ S_{IJ}\beta^J - \tilde{U}_I\,, \qquad \tilde{U}_I \equiv U_I + \partial_I\beta^J U_J + V_{IJ}\beta^J\,, \end{aligned} \tag{2.16}$$

together with the condition

$$\tilde{E}_I \equiv E_I + \partial_I\beta^J E_J + F_{IJ}\beta^J = \mathcal{L}_\beta H_I\,. \tag{2.17}$$

Further relations which constrain W_I, G_{IJ} are

$$\partial_{[I}W_{J]} = -\tilde{S}_{[IJ]}\,, \tag{2.18}$$

defining

$$\tilde{S}_{IJ} \equiv S_{IJ} + \partial_J \beta^K S_{IK} + T_{IJK}\beta^K \,, \tag{2.19}$$

and

$$G_{IJ} - \mathcal{L}_\beta S_{IJ} = \tilde{A}_{IJ} \equiv A_{IJ} + \partial_I \beta^K A_{KJ} + B_{JIK}\beta^K \,, \tag{2.20}$$

and also a consistency relation involving the derivative of G_{IJ} which can be simplified to

$$\Gamma^{(G)}{}_{IJK} - \mathcal{L}'_\beta T_{IJK} = \tilde{B}_{IJK} \equiv B_{IJK} + \partial_I \beta^L B_{LJK} + C_{ILJK}\beta^L \,, \tag{2.21}$$

for

$$\Gamma^{(G)}{}_{IJK} = \tfrac{1}{2}\big(\partial_J G_{IK} + \partial_K G_{IJ} - \partial_I G_{JK}\big) \,, \tag{2.22}$$

the Christoffel connection formed from G_{IJ}. From (2.19) and (2.20) we may obtain

$$\Gamma^{(G)}{}_{IJK}\beta^K + G_{IJ} + \partial_J \beta^K G_{IK} - \mathcal{L}_\beta \tilde{S}_{IJ} = \tilde{A}_{IJ} + \partial_J \beta^K \tilde{A}_{IK} + \tilde{B}_{IJK}\beta^K \,. \tag{2.23}$$

In the above relations $\mathcal{L}_\beta$ is the Lie derivative determined by β^I so that

$$\mathcal{L}_\beta W_I = \beta^J \partial_J W_I + \partial_I \beta^J W_J \,, \qquad \mathcal{L}_\beta D = \beta^J \partial_J D \,, \tag{2.24}$$

with obvious extensions for $\mathcal{L}_\beta S_{IJ}, \mathcal{L}_\beta V_{IJ}$, analogous to $\mathcal{L}_\beta g_{IJ}$ in (1.4) and we also define

$$\mathcal{L}'_\beta T_{IJK} = \mathcal{L}_\beta T_{IJK} + S_{IL}\partial_J \partial_K \beta^L \,. \tag{2.25}$$

The constraint (2.17) follows by combining (2.16) with (2.20). The Lie derivative preserves tensorial properties under redefinitions of the couplings $g^I \to h^I(g)$. $\tilde{U}_I, \tilde{E}_I, \tilde{S}_{IJ}, \tilde{A}_{IJ}$ are also tensors. The relation for F_{IJ} in (2.16) and also (2.21) are not manifestly invariant under such redefinitions but covariance can be verified by combining different identities. The result for F_{IJ} is thus equivalent to

$$G_{IJ} = \big(F_{IJ} - \partial_{(I}E_{J)}\big) + \big(B_{KIJ} - \Gamma^{(A)}{}_{KIJ}\big)\beta^K + \mathcal{L}_\beta\big(V_{IJ} - \partial_{(I}U_{J)} - A_{IJ}\big) \,, \tag{2.26}$$

where the three terms each transform as a tensor. (2.20) determines G_{IJ}, which is later used as a metric on $\mathcal{M}_g$, in terms of flat space results. It may be recast as

$$G_{IJ} = A_{IJ} - \tfrac{1}{2}\beta^K \mathcal{D}_K A_{IJ} + \mathcal{L}_\beta\big(S_{(IJ)} + \tfrac{1}{2}A_{IJ}\big) \,, \tag{2.27}$$

where

$$\mathcal{D}_K A_{IJ} = \partial_K A_{IJ} - B_{JKI} - B_{IKJ} \,. \tag{2.28}$$

The essential variation and RG equations (1.1) and (1.2) follow directly from (2.14) for

$$\tilde{A} = A + W_I \beta^I \,, \qquad T_{IJ} = G_{IJ} + \partial_I W_J - \partial_J W_I \,. \tag{2.29}$$

The coefficients in (2.4) have an intrinsic arbitrariness induced by adding to W local terms of the same form as in (2.4) for σ a constant. This freedom gives an equivalence

$$
\begin{aligned}
W_I &\sim W_I - \partial_I a + g_{IJ}\beta^J \,, \\
H_I &\sim H_I + e_I + \partial_I\beta^J e_J + f_{IJ}\beta^J \,, \\
S_{IJ} &\sim S_{IJ} + g_{IJ} - a_{IJ} - \partial_I\beta^K a_{JK} - b_{JIK}\beta^K \,, \\
T_{IJK} &\sim T_{IJK} + \Gamma^{(g)}{}_{IJK} - b_{IJK} - \partial_I\beta^L b_{LJK} - c_{ILJK}\beta^L \,, \\
D &\sim D - b + e_I\beta^I \,, \\
U_I &\sim U_I - e_I - a_{IJ}\beta^J \,, \\
V_{IJ} &\sim V_{IJ} + g_{IJ} - f_{IJ} - b_{KIJ}\beta^K \,, \\
F_{IJ} &\sim F_{IJ} + \mathcal{L}_\beta f_{IJ} + \partial_I\partial_J\beta^K e_K \,, \\
B_{IJK} &\sim B_{IJK} + \mathcal{L}_\beta b_{IJK} + \partial_J\partial_K\beta^L a_{IL} \,, \\
C_{IJKL} &\sim C_{IJKL} + \mathcal{L}_\beta c_{IJKL} + \partial_I\partial_J\beta^M\, b_{MKL} + \partial_K\partial_L\beta^M b_{MIJ} \,,
\end{aligned}
\tag{2.30}
$$

as well as

$$
(A,B,C,E_I,G_{IJ},A_{IJ}) \sim (A,B,C,E_I,G_{IJ},A_{IJ}) + \mathcal{L}_\beta(a,b,c,e_I,g_{IJ},a_{IJ}) \,. \tag{2.31}
$$

With the definition (2.19) then from (2.30)

$$
\begin{aligned}
\tilde S_{IJ} \sim \tilde S_{IJ} &- \partial_{[I}\big(g_{J]K}\beta^K\big) + g_{IJ} + \tfrac12\,\mathcal{L}_\beta g_{IJ} \\
&- (\delta_I{}^K + \partial_I\beta^K)(\delta_J{}^L + \partial_J\beta^L)a_{KL} - 2\,b_{(IJ)K}\beta^K - 2\,\partial_{(I}\beta^L b_{LJ)K}\beta^K \,.
\end{aligned}
\tag{2.32}
$$

As a consequence of (2.30) we may set, if $\delta_I{}^J + \partial_I\beta^J$ is invertible,

$$
S_{(IJ)} = T_{IJK} = D = U_I = V_{IJ} = 0 \,. \tag{2.33}
$$

To describe the RG flow of four dimensional quantum field theories it is necessary to take into account contributions to the basic equations corresponding to relevant operators, in addition to just the marginal operators with couplings $\{g^I\}$. These may induce modifications of the consistency conditions obtained above for the RG flow. We first consider vector operators. A general analysis may be obtained by extending the global symmetry group of the kinetic terms G_K to a local symmetry by introducing background gauge fields $a_\mu(x) \in \mathfrak{g}_K$, the Lie algebra corresponding to G_K, and extending all derivatives to covariant derivatives $D_\mu = \partial_\mu + a_\mu$. The symmetry extends to the full quantum field theory if, for any $\omega \in \mathfrak{g}_K$, the couplings g^I and a_μ transform as

$$
\begin{aligned}
\delta_\omega g^I(x) &= -\,(\omega g)^I(x) = -\omega^I{}_J(x) g^J(x) \,, \\
\delta_\omega a_\mu(x) &= D_\mu\omega(x) = \partial_\mu\omega(x) + [a_\mu(x),\omega(x)] \,.
\end{aligned}
\tag{2.34}
$$

where $\omega^I{}_J$ belongs to the appropriate representation of $\mathfrak{g}_K$ acting on the couplings $\{g^I\}$. Under such variations $\delta_\omega \beta^I(g) = \omega^I{}_J \beta^J(g)$. The corresponding covariant derivative acting on the couplings is then

$$D_\mu g^I = \partial_\mu g^I + (a_\mu g)^I \,, \qquad (a_\mu g)^I = a_\mu{}^I{}_J g^J \,, \tag{2.35}$$

with the curvature as usual

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] \,. \tag{2.36}$$

The generator of local G_K transformations as in (2.34) is then

$$\Delta_\omega = \int \mathrm{d}^4 x \left(D_\mu \omega \cdot \frac{\delta}{\delta a_\mu} - (\omega g)^I \frac{\delta}{\delta g^I} \right) , \qquad \big[\Delta_\omega , \Delta_{\omega'} \big] = \Delta_{[\omega,\omega']} \,. \tag{2.37}$$

The introduction of background gauge fields a_μ, so that now we take $W[\gamma_{\mu\nu}, g^I, a_\mu]$, allows (2.10) to be extended to define local vector currents by

$$\frac{\delta}{\delta a_\mu(x)} W = - \sqrt{-\gamma(x)} \, \langle J^\mu(x) \rangle \,, \qquad J^\mu \in \mathfrak{g}_K \,. \tag{2.38}$$

For this paper we assume manifest background gauge invariance so that

$$\Delta_\omega W = 0 \,, \tag{2.39}$$

although in general there can be anomalies which involve ϵ-tensor contributions. If present there would be additional consistency conditions. If (2.39) holds then from the definition (2.38) the current J^μ satisfies the conservation equation

$$\omega \cdot D_\mu \langle J^\mu \rangle = -(\omega g)^I \langle \mathcal{O}_I \rangle \,, \qquad \omega \in \mathfrak{g}_K \,. \tag{2.40}$$

Under Weyl rescalings of the metric there are additional contributions to the functional differential operator in (2.2) involving a_μ given by

$$\Delta_{\sigma,a} = \int \mathrm{d}^4 x \, \big(\sigma \, \rho_I D_\mu g^I - \partial_\mu \sigma \, v \big) \cdot \frac{\delta}{\delta a_\mu} \,, \quad \rho_I(g), v(g) \in \mathfrak{g}_K \,, \tag{2.41}$$

with $\cdot$ denoting an invariant scalar product on $\mathfrak{g}_K$. Assuming (2.38) then (2.41) implies (1.5). We assume that manifest covariance under G_K is maintained so that, for all $\omega \in \mathfrak{g}_K$,

$$\big[\Delta_\omega , \Delta_\sigma \big] = \big[\Delta_\omega , \Delta_{\sigma,a} \big] = 0 \,, \tag{2.42}$$

which implies

$$(\omega g)^J \partial_J \beta^I = (\omega\beta)^I \,, \quad (\omega g)^J \partial_J \rho_I + \rho_J\, \omega^J{}_I = [\omega, \rho_I] \,, \quad (\omega g)^J \partial_J \upsilon = [\omega, \upsilon] \,, \tag{2.43}$$

In this case (2.41) can be equivalently expressed as

$$\Delta_{\sigma,a} = \int \mathrm{d}^4 x \, \big(\sigma\, \tilde\rho_I D_\mu g^I - D_\mu(\sigma\, \upsilon)\big) \cdot \frac{\delta}{\delta a_\mu} \,, \qquad \tilde\rho_I = \rho_I + \partial_I \upsilon \,, \tag{2.44}$$

since, using (2.43),

$$D_\mu \upsilon = \partial_\mu \upsilon + [a_\mu, \upsilon] = \partial_I \upsilon\, D_\mu g^I \,. \tag{2.45}$$

For general quantum theories it is also necessary to consider the extra contributions arising from operators $\{\mathcal{O}_M\}$ with canonical dimension two. The associated couplings $\{M\}$ are mass terms belonging to the dual space V_M. The vacuum self energy now extends to a functional $W[\gamma_{\mu\nu}, g^I, a_\mu, M]$. The action of gauge transformations in (2.34) now extends also to $\delta_\omega M(x) = M(x)\omega_M(x) - \bar\omega_M(x) M(x)$ for $\omega_M, \bar\omega_M$ belonging to appropriate representations of $\mathfrak{g}_K$. There is also a corresponding additional term in Δ_ω in (2.37) which requires that (2.40) is extended to

$$\omega \cdot D_\mu \langle J^\mu \rangle = -(\omega g)^I \langle \mathcal{O}_I \rangle - (M\omega_M - \bar\omega_M M) \cdot \langle \mathcal{O}_M \rangle \,. \tag{2.46}$$

for $\frac{\delta}{\delta M} W = -\sqrt{-\gamma}\, \langle \mathcal{O}_M \rangle$ and $\cdot$ also denoting the natural scalar product on $V_M \times V_M{}^*$

As for g^I local RG equations require extension to arbitrary $M(x) \in V_M$. In (2.4) describing the response to Weyl rescalings of the metric, besides $\Delta_{\sigma,a}$, it is necessary also to include the additional term

$$\begin{aligned} \Delta_{\sigma,M} = - \int \mathrm{d}^4 x \, \Big(\sigma \big((2-\gamma_M) M + \tfrac{1}{6} R\, \eta + \delta_I\, D^2 g^I + \epsilon_{IJ}\, D^\mu g^I D_\mu g^J\big) \\ + 2\, \partial_\mu \sigma\, \theta_I D^\mu g^I + \nabla^2 \sigma\, \tau \Big) \cdot \frac{\delta}{\delta M} \,, \end{aligned} \tag{2.47}$$

where $\eta, \delta_I, \epsilon_{IJ} = \epsilon_{JI}, \theta_I, \tau \in V_M$ and $\gamma_M : V_M \to V_M$. (2.42) is extended to $[\Delta_\omega, \Delta_{\sigma,M}] = 0$.

The requirement that

$$\big[\Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}, \Delta_{\sigma'} + \Delta_{\sigma',a} + \Delta_{\sigma',M}\big] = 0 \,, \tag{2.48}$$

imposes further consistency conditions which follow by using

$$\begin{aligned} (\Delta_\sigma + \Delta_{\sigma,a}) D_\mu g^I &= \partial_\mu \sigma\, B^I + \sigma\, D_\mu g^J \big(\partial_J B^I + (\tilde\rho_J g)^I\big) + \sigma\, (\upsilon D_\mu g)^I \,, \\ (\Delta_\sigma + \Delta_{\sigma,a}) D^2 g^I &= \nabla^2 \sigma\, B^I + 2\, \partial_\mu \sigma\, D_\mu g^J \Psi_J{}^I + \sigma\, D^\mu g^J D_\mu g^K\, \Omega_{JK}{}^I \\ &\quad + \sigma\big(-2\, D^2 g^I + D^2 g^J \big(\partial_J B^I + (\tilde\rho_J g)^I\big) + (\upsilon D^2 g)^I\big) \,, \end{aligned} \tag{2.49}$$

with B^I the modified β-function defined in (1.7), $\tilde{\rho}_I$ as in (2.44), and

$$\begin{aligned} \Psi_J{}^I &= \delta_J{}^I + \partial_J B^I + \tfrac{1}{2}(\tilde{\rho}_J g)^I \,, \\ \Omega_{JK}{}^I &= \partial_J \partial_K B^I + (\partial_{(J}\tilde{\rho}_{K)} g)^I + 2\,(\tilde{\rho}_{(J})^I{}_{K)} \,. \end{aligned} \tag{2.50}$$

The Lie derivative defined by (2.24) is also extended to ensure that it transforms covariantly under G_K rotations so that $\mathcal{L}_\beta W_I \to \widetilde{\mathcal{L}}_{B,\tilde{\rho}} W_I$ where

$$\widetilde{\mathcal{L}}_{B,\tilde{\rho}} W_I = \mathcal{L}_B W_I + (\tilde{\rho}_I g)^J W_J \,. \tag{2.51}$$

With these results, and $\mathcal{L}_{\upsilon g} \upsilon = 0$, the condition (2.48) requires

$$\tilde{\rho}_I B^I = 0 \,, \tag{2.52}$$

and

$$\eta = \delta_I B^I - (\mathcal{L}_B - \gamma_M)\tau \,. \tag{2.53}$$

which determines η, and

$$\Psi_I{}^J \delta_J + \epsilon_{IJ} B^J = \big(\widetilde{\mathcal{L}}_{B,\tilde{\rho}} - \gamma_M\big)\theta_I \,. \tag{2.54}$$

The property (2.52) ensures that the extended Lie derivative commutes with contraction with B^I so that in (2.51) $B^I\, \widetilde{\mathcal{L}}_{B,\tilde{\rho}} W_I = \mathcal{L}_B(B^I W_I)$. Furthermore we then, with the definitions in (2.50),

$$\big[\widetilde{\mathcal{L}}_{B,\tilde{\rho}}\,,\, \Psi_I{}^J\big] = \Omega_{IK}{}^J\, B^K \,. \tag{2.55}$$

The functional differential operators in (2.41) and (2.47) are essentially arbitrary up to variations arising from purely local contributions which automatically maintain the consistency conditions (2.52) and (2.53). Such variations can be generated by

$$\begin{aligned} \delta\,\big(\Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}\big) &= \big[\mathcal{D}, \Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}\big] \,, \\ \delta\,\big(\Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}\big) W &= \mathcal{D}\,\big(\Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}\big) W \,, \end{aligned} \tag{2.56}$$

for any local functional differential operator $\mathcal{D}$. Choosing

$$\mathcal{D} = \int \mathrm{d}^4x\; r_I D_\mu g^I \cdot \frac{\delta}{\delta a_\mu} \,, \qquad r_I(g) \in \mathfrak{g}_K \,, \tag{2.57}$$

gives

$$\begin{aligned} &\delta\tilde{\rho}_I = (r_I g)^J \tilde{\rho}_J - (\tilde{\rho}_I g)^J r_J + (\partial_I r_J - \partial_J r_I) B^J \,, \quad \delta\upsilon = r_I B^I \,, \\ &\delta B^I = - B^J (r_J g)^I \,, \qquad \delta\delta_I = (r_I g)^J \delta_J \,, \qquad \delta\theta_I = (r_I g)^J \theta_J \,, \\ &\delta\epsilon_{IJ} = (r_I g)^K \epsilon_{KJ} + (r_J g)^K \epsilon_{IK} + (\partial_{(I} r_{J)} g)^K \delta_K + 2\,\delta_K (r_{(I})^K{}_{J)} \,. \end{aligned} \tag{2.58}$$

From this it follows that $\delta\big(\widetilde{\mathcal{L}}_{B,\tilde\rho}\,\theta_I\big) = (r_I g)^J\,\widetilde{\mathcal{L}}_{B,\tilde\rho}\,\theta_J$. In a similar fashion we may obtain

$$\begin{aligned} &\delta\eta = (\mathcal{L}_B - \gamma_M)h\,, \qquad \delta\tau = -h + d_I B^I\,, \qquad \delta\theta_I = \Psi_I{}^J d_J + e_{IJ}B^J\,,\\ &\delta\delta_I = \big(\widetilde{\mathcal{L}}_{B,\tilde\rho} - \gamma_M\big)d_I\,, \qquad \delta\epsilon_{IJ} = \big(\widetilde{\mathcal{L}}_{B,\tilde\rho} - \gamma_M\big)e_{IJ} + \Omega_{IJ}{}^K\,d_K\,, \end{aligned} \tag{2.59}$$

for $h, d_I, e_{IJ} \in V_M$. In consequence we may set $\tau = 0$.

The essential equation (2.4) is modified so that

$$\begin{aligned} &(\Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M})\,16\pi^2 W\\ &\quad = -\int d^4x\sqrt{-\gamma}\;\sigma\,\Big(- C\,F + \tfrac14 A\,G + \tfrac{1}{72}B\,R^2 + E^{\mu\nu}\,G_{IJ}D_\mu g^I D_\nu g^J\\ &\qquad\qquad + \tfrac16 R\,\big(E_I D^2 g^I + F_{IJ}D^\mu g^I D_\mu g^J + I\cdot M\big) - X\Big)\\ &\qquad - 2\int d^4x\sqrt{-\gamma}\;\partial_\mu\sigma\,\Big(E^{\mu\nu}\,W_I D_\nu g^I + \tfrac16 R\,H_I D^\mu g^I + Y^\mu\Big)\,, \end{aligned} \tag{2.60}$$

where for simplicity the part involving $\nabla^2\sigma$ is dropped since the relevant terms can be set to zero by adding local contributions to W. In (2.60) $I \in V_M{}^*$ and X, Y now have additional terms involving f and M,

$$\begin{aligned} X(g,a,M) &= \tfrac12 A_{IJ}\,D^2g^I D^2 g^J + B_{IJK}\,D^2g^I\,D^\mu g^J D_\mu g^K\\ &\quad + \tfrac12 C_{IJKL}\,D^\mu g^I D_\mu g^J\,D^\nu g^K D_\nu g^L\\ &\quad + \tfrac14\,f^{\mu\nu}\cdot\beta_f\cdot f_{\mu\nu} + \tfrac12\,M\cdot\beta_M\cdot M + f^{\mu\nu}\cdot P_{IJ}\,D_\mu g^I D_\nu g^J\\ &\quad + J_I\cdot M\,D^2 g^I + K_{IJ}\cdot M\,D^\mu g^I D_\mu g^J\,,\\ Y^\mu(g,a,M) &= S_{IJ}\,D^\mu g^I D^2 g^J + T_{IJK}\,D^\mu g^I\,D^\nu g^J D_\nu g^K\\ &\quad + f^{\mu\nu}\cdot Q_I D_\nu g^I + L_I\cdot M D^\mu g^I\,, \end{aligned} \tag{2.61}$$

for $P_{IJ} = -P_{JI}, Q_I \in \mathfrak{g}_K$, $J_I, K_{IJ} = K_{JI}, L_I \in V_M{}^*$.

The presence of the additional terms in (2.61), together with the extension $\Delta_\sigma \to \Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}$ leads to modifications of the previous consistency conditions together with some further necessary relations. In general $\beta^I \to B^I$, assuming G_K-covariance as in (2.43) with additionally

$$(\omega g)^K\partial_K G_{IJ} + G_{KJ}\omega^K{}_I + G_{IK}\omega^K{}_J = 0\,, \tag{2.62}$$

etc, but there are further required changes. To avoid too much complication we focus on the results related to the variation of A. The basic equation (2.14) becomes

$$\partial_I A = G_{IJ}B^J - \widetilde{\mathcal{L}}_{B,\tilde\rho}\,W_I\,. \tag{2.63}$$

Taking into account

$$\begin{aligned}\Delta_{\sigma,a} f_{\mu\nu} &= \sigma\big([v, f_{\mu\nu}] + (f_{\mu\nu}g)^I \tilde\rho_I + (\partial_I \tilde\rho_J - \partial_J \tilde\rho_I) D_\mu g^I D_\nu g^J\big) \\ &\quad + \partial_\mu \sigma\, \tilde\rho_I\, D_\nu g^I - \partial_\nu \sigma\, \tilde\rho_I\, D_\mu g^I\,,\end{aligned} \tag{2.64}$$

then instead of (2.18), with now $\tilde S_{IJ} = \Psi_J{}^K S_{IK} + T_{IJK} B^K$,

$$\partial_{[I} W_{J]} = -\tilde S_{[IJ]} + \tfrac12\, \tilde\rho_{[I} \cdot Q_{J]} + L_{[I} \cdot \theta_{J]}\,. \tag{2.65}$$

There are also extra relations from terms involving $f_{\mu\nu}$ which give, for any $\omega \in \mathfrak{g}_K$,

$$(\omega g)^I W_I = \, - \omega \cdot Q_I B^I\,, \tag{2.66a}$$

$$(\omega g)^J G_{IJ} = \, - \omega \cdot \widetilde{\mathcal{L}}_{B,\tilde\rho}\, Q_I - (\omega g)^J \tilde\rho_J \cdot Q_I + \omega \cdot P_{IJ} B^J - \tfrac12\, \omega \cdot \beta_f \cdot \tilde\rho_I\,. \tag{2.66b}$$

In (2.66b) the second term on the right hand side may be naturally absorbed in a extension of $\widetilde{\mathcal{L}}_{B,\tilde\rho}$ [22]. From (2.66a)

$$(\tilde\rho_I g)^J W_J = -\tilde\rho_I \cdot Q_J B^J\,, \tag{2.67}$$

so that the essential result (2.63) can still be rewritten in the succinct form (1.1)

$$\partial_I \tilde A = T_{IJ} B^J\,, \tag{2.68}$$

where $\tilde A, T_{IJ}$ are now defined, using (2.52), by an extension of (2.29) to

$$\tilde A = A + W_I B^I\,, \qquad T_{IJ} = G_{IJ} + 2\,\partial_{[I} W_{J]} + 2\,\tilde\rho_{[I} \cdot Q_{J]}\,. \tag{2.69}$$

Furthermore from (2.66b), in conjunction with (2.66a),

$$\begin{aligned}(\omega g)^I G_{IJ} B^J &= \, - \omega \cdot B^I \partial_I (Q_J B^J) - (\omega g)^I \tilde\rho_I \cdot Q_J B^J \\ &= (\omega g)^I \big(B^J \partial_J W_I + (\tilde\rho_I g)^J W_J\big) + (\omega B^I) W_I = (\omega g)^I \widetilde{\mathcal{L}}_{B,\tilde\rho}\, W_I\,,\end{aligned} \tag{2.70}$$

which ensures that (2.63) implies $(\omega g)^I \partial_I A = 0$.

The consistency conditions also generate additional relations for the terms in (2.60), (2.61) containing M which take the form.

$$I + J_I B^I = 0\,, \tag{2.71}$$

and

$$\tilde J_I + \widetilde{\mathcal{L}}_{B,\tilde\rho}\, L_I + L_I \cdot \gamma_M = \theta_I \cdot \beta_M\,, \qquad \tilde J_I \equiv \Psi_I{}^J J_J + K_{IJ} B^J\,. \tag{2.72}$$

The relation (2.20), determining G_{IJ}, now becomes

$$G_{IJ} = \tilde{A}_{IJ} + \widetilde{\mathcal{L}}_{B,\tilde{\rho}}\, S_{IJ} - J_J \cdot \theta_I - L_I \cdot \delta_J \,, \quad \tilde{A}_{IJ} = \Psi_I{}^K A_{KJ} + B_{JIK} B^K \,, \tag{2.73}$$

which gives rise to a modification of (2.27),

$$\begin{aligned} G_{IJ} = A_{IJ} - \tfrac{1}{2}\big((\tilde{\rho}_{(I}g)^K A_{J)K} + B^K \mathcal{D}_K A_{IJ}\big) \\ + \widetilde{\mathcal{L}}_{B,\tilde{\rho}}\big(S_{(IJ)} + \tfrac{1}{2}A_{IJ}\big) - J_{(I} \cdot \theta_{J)} - L_{(I} \cdot \delta_{J)} \,, \end{aligned} \tag{2.74}$$

with the definition of $\mathcal{D}_K A_{IJ}$ unchanged from (2.28). Also (2.21) becomes

$$\begin{aligned} \Gamma^{(G)}{}_{IJK} = \Psi_I{}^L B_{LJK} + C_{ILJK} B^L + \Omega_{JK}{}^L\, S_{IL} + \widetilde{\mathcal{L}}_{B,\tilde{\rho}}\, T_{IJK} \\ + \big(\partial_I \tilde{\rho}_{(J} - \partial_{(J}\tilde{\rho}_I\big) \cdot Q_{K)} - \tilde{\rho}_{(J} \cdot P_{K)I} - K_{JK} \cdot \theta_I - L_I \cdot \epsilon_{JK} \,. \end{aligned} \tag{2.75}$$

The equivalence relations (2.30), (2.31) also extend to the more general case with additional terms stemming from the presence of a_μ, M. In particular the essential equation (2.68) is arbitrary up to the equivalence relations given by

$$\begin{aligned} \tilde{A} \sim \tilde{A} + g_{IJ} B^I B^J \,, \quad G_{IJ} \sim G_{IJ} + \widetilde{\mathcal{L}}_{\beta,\rho}\, g_{IJ} = G_{IJ} + \widetilde{\mathcal{L}}_{B,\tilde{\rho}}\, g_{IJ} \,, \\ W_I \sim W_I + g_{IJ} B^J \,, \qquad \omega \cdot Q_I \sim \omega \cdot Q_I - g_{IJ} (\omega g)^J \,, \ \ \omega \in \mathfrak{g}_K \,. \end{aligned} \tag{2.76}$$

There are also extra relations arising from local contributions to W involving $f_{\mu\nu}$ such as

$$\begin{aligned} Q_I \sim Q_I + p_{IJ} B^J \,, \quad \omega \cdot P_{IJ} \equiv \omega \cdot P_{IJ} + \omega \cdot \widetilde{\mathcal{L}}_{B,\tilde{\rho}}\, p_{IJ} + (\omega g)^K \tilde{\rho}_K \cdot p_{IJ} \,, \\ C_{ILJK} \sim C_{ILJK} + (\partial_L \tilde{\rho}_J - \partial_{(J}\tilde{\rho}_L) \cdot p_{IK)} + (\partial_I \tilde{\rho}_J - \partial_{(J}\tilde{\rho}_I) \cdot p_{LK)} \,, \\ T_{IJK} \sim T_{IJK} - \tilde{\rho}_{(J} \cdot p_{IK)} \,, \qquad p_{IJ} = -p_{JI} \in \mathfrak{g}_K \,. \end{aligned} \tag{2.77}$$

This gives in (2.69) $T_{IJ} \sim T_{IJ} + 2\,\tilde{\rho}_{[I} \cdot p_{J]K} B^K$ so that $T_{IJ} B^J$ is invariant. From local terms containing M

$$\begin{aligned} J_I \sim J_I + \widetilde{\mathcal{L}}_{B,\tilde{\rho}}\, j_I + j_I \cdot \gamma_M \,, \quad L_I \sim L_I - \Psi_I{}^J j_J \,, \\ E_I \sim E_I + j_I \cdot \eta \,, \quad A_{IJ} \sim A_{IJ} - 2\, j_{(I} \cdot \delta_{J)} \,, \quad B_{IJK} \sim B_{IJK} - j_I \cdot \epsilon_{JK} \,, \\ S_{IJ} \sim S_{IJ} + j_J \cdot \theta_I \,, \qquad j_I \in V_M{}^* \,. \end{aligned} \tag{2.78}$$

For consistency with omitting $\nabla^2 \sigma$ terms in (2.60) it is necessary to impose $j_I B^I = 0$.

3. Integration of Weyl Scaling

The consistency conditions obtained in the previous section are obtained as integrability conditions for the response to local Weyl rescalings of the metric. Here we describe how results for the vacuum energy functional $W[\gamma_{\mu\nu}, g^I]$ for finite rescalings of the metric can be obtained.

For simplicity we focus initially on two dimensional quantum field theories. With the functional differential operator Δ_σ given by the corresponding form to (2.2) in two dimensions the basic equation (2.4) becomes

$$\Delta_\sigma \, 2\pi W = \int \mathrm{d}^2 x \sqrt{-\gamma} \, \Big(\sigma \left(C\, R - G_{IJ} \, \partial^\mu g^I \partial_\mu g^J \right) - 2\partial_\mu \sigma \, W_I \, \partial^\mu g^I \Big) \,, \tag{3.1}$$

for $C(g), G_{IJ}(g), W_I(g)$ depending on the couplings g^I. The consistency conditions flowing from (2.12) are just [10]

$$\partial_I C = G_{IJ} \beta^J - \mathcal{L}_\beta W_I \,, \tag{3.2}$$

which is essentially identical to the four dimensional result given in (2.14).

To integrate (3.1) we define g_σ^I by

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} g_\sigma^I = \beta^I(g_\sigma) \,, \qquad g_0{}^I = g^I \,, \tag{3.3}$$

where such running couplings depending on $\sigma(x)$ were discussed in [14]. With this definition (3.1) directly implies, for arbitrary $\delta\sigma(x)$,

$$\begin{aligned} &\delta_\sigma \, 2\pi W[e^{2\sigma}\gamma_{\mu\nu}, g_\sigma^I] \\ &= \int \mathrm{d}^2 x \sqrt{-\gamma} \, \Big(\delta\sigma \big(C(g_\sigma) \, (R - 2\nabla^2 \sigma) - G_{IJ}(g_\sigma) \, \partial^\mu g_\sigma^I \partial_\mu g_\sigma^J \big) - 2\partial_\mu \sigma \, W_I(g_\sigma) \, \partial^\mu g_\sigma^I \Big) \,, \end{aligned} \tag{3.4}$$

where on the right hand side the dependence on σ is explicit. To integrate this we first define $\breve{C}(\sigma)$ by

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \breve{C}(\sigma) = C(g_\sigma) \,, \qquad \breve{C}(0) = 0 \,, \tag{3.5}$$

and then (3.4), using (3.2) with the condition $G_{IJ} = G_{JI}$, gives

$$\begin{aligned} \delta_\sigma \bigg(2\pi W[e^{2\sigma}\gamma_{\mu\nu}, g_\sigma^I] - \int \mathrm{d}^2 x \sqrt{-\gamma} \, \Big(\breve{C}(\sigma) \, R + \big(C(g_\sigma) - W_I(g_\sigma) \beta^I(g_\sigma) \big) \, \partial^\mu \sigma \partial_\mu \sigma \Big) \bigg) \\ = - \int \mathrm{d}^2 x \sqrt{-\gamma} \, \delta\sigma \, G_{IJ}(g_\sigma) \, \bar{\partial}^\mu g_\sigma^I \bar{\partial}_\mu g_\sigma^J \\ - 2 \int \mathrm{d}^2 x \sqrt{-\gamma} \, \big(\partial_\mu \delta\sigma \, W_I(g_\sigma) + \delta\sigma \, \partial_\mu \sigma \, \mathcal{L}_\beta W_I(g_\sigma) \big) \, \bar{\partial}^\mu g_\sigma^I \,, \end{aligned} \tag{3.6}$$

where we define

$$\bar{\partial}_\mu g_\sigma^I = \partial_\mu g_\sigma^I - \beta^I(g_\sigma) \, \partial_\mu \sigma \,. \tag{3.7}$$

Noting that

$$\delta_\sigma \bar\partial_\mu g_\sigma{}^I = \delta\sigma\, \partial_J \beta^I(g_\sigma)\, \bar\partial_\mu g_\sigma{}^J \,, \tag{3.8}$$

does not involve $\partial_\mu \delta\sigma$ and defining $\breve G_{IJ}(\sigma)$ by the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \breve G_{IJ}(\sigma) + \partial_I \beta^K(g_\sigma) \breve G_{KJ}(\sigma) + \partial_J \beta^K(g_\sigma) \breve G_{IK}(\sigma) = G_{IJ}(g_\sigma) \,, \qquad \breve G_{IJ}(0) = 0 \,, \tag{3.9}$$

then we may finally obtain

$$2\pi \big(W[e^{2\sigma}\gamma_{\mu\nu}, g_\sigma{}^I] - W[\gamma_{\mu\nu}, g^I] \big) = \int \mathrm{d}^2 x \sqrt{-\gamma}\, \mathcal{W} \,, \tag{3.10}$$

where

$$\mathcal{W} = \breve C(\sigma)\, R + \tilde C(g_\sigma)\, \partial^\mu \sigma \partial_\mu \sigma - \breve G_{IJ}(\sigma)\, \bar\partial^\mu g_\sigma{}^I \bar\partial_\mu g_\sigma{}^J - 2\, W_I(g_\sigma)\, \partial^\mu g_\sigma{}^I \partial_\mu \sigma \,. \tag{3.11}$$

for $\tilde C = C + W_I \beta^I$.

The differential equations (3.5) and (3.9) may be formally solved as an expansion in σ, noting that $f(g_\sigma) = \exp(\sigma \mathcal{L}_\beta) f(g)$, in the form

$$\breve C(\sigma) = \big(\exp(\sigma \mathcal{L}_\beta) - 1 \big) \mathcal{L}_\beta{}^{-1} C(g) \,, \quad \breve G_{IJ}(\sigma) = \big(\exp(\sigma \mathcal{L}_\beta) - 1 \big) \mathcal{L}_\beta{}^{-1} G_{IJ}(g) \,, \tag{3.12}$$

which gives rise to results corresponding to those in [22]. The behaviour for large σ is less apparent in this expression.

The result (3.10) with (3.11) provides an interpolation of the anomalous contributions to the self energy functional W between UV fixed points as $\sigma \to \infty$ and IR fixed points as $\sigma \to -\infty$ assuming $g_\sigma{}^I$ is on a RG trajectory linking to fixed points satisfying $\beta^I(g_*) = 0$. If this holds then asymptotically $\breve C(\sigma) \sim C(g_*)\sigma$ and if the fixed point is a surface $\mathcal{M}_{g_*}$ in the space of couplings, corresponding to exactly marginal operators, then on $\mathcal{M}_{g_*}$ $\partial_I C(g_*) = 0$ since then $\partial_I \beta^J(g_*) = 0$.

It is also of interest to rewrite (3.10) to determine the response to just a Weyl rescaling of the metric whichcan be achieved by letting $\gamma_{\mu\nu} \to e^{-2\sigma}\gamma_{\mu\nu}$. Apart from anomalous terms arising from $\mathcal{W}$ the Weyl rescaling is realised by introducing the running couplings $g_\sigma{}^I$ since (3.10) and (3.11) give

$$2\pi \big(W[e^{-2\sigma}\gamma_{\mu\nu}, g^I] - W[\gamma_{\mu\nu}, g_\sigma{}^I] \big) = \int \mathrm{d}^2 x \sqrt{-\gamma}\, \mathcal{W}' \,, \quad \mathcal{W}' = 2\, \partial^\mu \sigma \partial_\mu \breve C - \mathcal{W} \,. \tag{3.13}$$

To complete this result it is necessary to determine $\partial_\mu \breve C$. In general $\breve C(\sigma)$, determined by (3.5), depends also the initial g^I. It is convenient to let $g^I \to g_\sigma{}^I$, $\breve C = \breve C(\sigma, g_\sigma)$ and then

$\frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{\partial}{\partial\sigma} + \beta^I(g_\sigma)\partial_I$. Hence $\partial_\mu \breve C = \frac{\partial}{\partial\sigma}\breve C\,\partial_\mu\sigma + \partial_I\breve C\,\partial_\mu {g_\sigma}^I = C(g_\sigma)\,\partial_\mu\sigma + \partial_I\breve C\,\bar\partial_\mu {g_\sigma}^I$. From (3.5), (3.9) with (3.2) we may obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\sigma}\big(\partial_I\breve C(\sigma) - \breve G_{IJ}(\sigma)\beta^J(g_\sigma) + W_I(g_\sigma)\big)\\
&\qquad + \partial_I\beta^K(g_\sigma)\big(\partial_K\breve C(\sigma) - \breve G_{KJ}(\sigma)\beta^J(g_\sigma) + W_K(g_\sigma)\big) = 0\,.
\end{aligned}
\tag{3.14}
$$

This has the solution, with the necessary boundary conditions at $\sigma = 0$,

$$
\partial_I\breve C(\sigma) - \breve G_{IJ}(\sigma)\beta^J(g_\sigma) + W_I(g_\sigma) = \breve W_I(\sigma) \tag{3.15}
$$

so long as

$$
\frac{\mathrm{d}}{\mathrm{d}\sigma}\breve W_I(\sigma) + \partial_I\beta^J(g_\sigma)\breve W_J(\sigma) = 0\,, \qquad \breve W_I(0) = W_I(g)\,. \tag{3.16}
$$

It is easy to check that $\breve W_I(\sigma)\bar\partial_\mu {g_\sigma}^I = W_I(g)\partial_\mu g^I$, $\breve W_I(\sigma)\beta^I(g_\sigma) = W_I(g)\beta^I(g)$. With these results (3.15) gives, since $\partial_\mu = \partial_\mu\sigma\frac{\mathrm{d}}{\mathrm{d}\sigma} + \bar\partial_\mu {g_\sigma}^I\partial_I$,

$$
\partial_\mu\breve C(\sigma) = C(g_\sigma)\,\partial_\mu\sigma - W_I(g_\sigma)\,\bar\partial_\mu {g_\sigma}^I + \breve G_{IJ}(\sigma)\,\bar\partial_\mu {g_\sigma}^I\beta^J + W_I(g)\,\partial_\mu g^I\,. \tag{3.17}
$$

Subject to (3.17), (3.10) and (3.11) then entail in (3.13)

$$
\mathcal{W}' = -\,\breve C(\sigma)\,R + \tilde C(g)\,\partial^\mu\sigma\partial_\mu\sigma + \breve G_{IJ}(\sigma)\,\partial^\mu g_\sigma^I\partial_\mu g_\sigma^J + 2\,W_I(g)\,\partial^\mu g^I\partial_\mu\sigma\,, \tag{3.18}
$$

where the result has been simplified by using

$$
\tilde C(g_\sigma) - \tilde C(g) = \breve G_{IJ}(\sigma)\beta^I(g_\sigma)\beta^J(g_\sigma) = \int_0^\sigma \mathrm{d}t\; G_{IJ}(g_t)\beta^I(g_t)\beta^J(g_t)\,. \tag{3.19}
$$

This follows from $\beta^I\partial_I\tilde C = G_{IJ}\beta^I\beta^J$ which may be integrated, with the definition (3.9), to give (3.19). Assuming $G_{IJ}(g')\beta^I(g')\beta^J(g') > 0$ for all $g'^I \in (g^I, g_\sigma^I)$ then from (3.19) $\tilde C(g_\sigma) < \tilde C(g)$ for $\sigma < 0$.

A similar analysis may be extended to four dimensions starting from (2.4). For simplicity we impose $D = U_I = V_{IJ} = 0$, as in (2.33), although S_{IJ}, T_{IJK} are not restricted initially. The integrability conditions (2.15) and (2.16) then become

$$
B = E_I\beta^I\,, \quad E_I = -A_{IJ}\beta^J\,, \quad F_{IJ} = G_{IJ} - B_{KIJ}\beta^K\,, \quad H_I = S_{IJ}\beta^J\,. \tag{3.20}
$$

(2.13) extends to finite Weyl rescalings of the metric to give in four dimensions

$$
\begin{aligned}
F_\sigma &= e^{-4\sigma}\,F\,,\\
G_\sigma &= e^{-4\sigma}\big(G + 8\,E^{\mu\nu}\nabla_\mu\nabla_\nu\sigma\\
&\qquad - 4\,\nabla^2(\partial^\mu\sigma\,\partial_\mu\sigma) + 8\,\nabla^\mu(\partial_\mu\sigma\,\nabla^2\sigma) + 8\,\nabla^\mu(\partial_\mu\sigma\,\partial^\nu\sigma\,\partial_\nu\sigma)\big)\,,\\
{E_\sigma}^{\mu\nu} &= e^{-4\sigma}\big(E^{\mu\nu} - 2(\nabla^\mu\nabla^\nu - \gamma^{\mu\nu}\nabla^2)\,\sigma + 2\,\partial^\mu\sigma\,\partial^\nu\sigma + \gamma^{\mu\nu}\,\partial^\lambda\sigma\,\partial_\lambda\sigma\big)\,,\\
R_\sigma &= e^{-2\sigma}\big(R - 6\,\nabla^2\sigma - 6\,\partial^\mu\sigma\,\partial_\mu\sigma\big)\,, \qquad {\nabla_\sigma}^2 = e^{-2\sigma}\big(\nabla^2 + 2\,\partial^\mu\sigma\,\partial_\mu\big)\,.
\end{aligned}
\tag{3.21}
$$

It is also important in this case to extend (3.7) defining

$$\Delta g_\sigma^I = \nabla^2 g_\sigma^I - \beta^I(g_\sigma)\,\nabla^2\sigma - 2\,\partial_J\beta^I(g_\sigma)\,\partial^\mu g_\sigma^J\partial_\mu\sigma + \beta^J(g_\sigma)\,\partial_J\beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\,, \qquad (3.22)$$

such that, analogous to (3.8),

$$\delta_\sigma \Delta g_\sigma^I = \delta\sigma\,\partial_J\beta^I(g_\sigma)\,\Delta g_\sigma^J + \delta\sigma\,\partial_J\partial_K\beta^I(g_\sigma)\,\bar\partial^\mu g_\sigma{}^J\bar\partial_\mu g_\sigma{}^K\,, \qquad (3.23)$$

and for $g^I \to h^I$, $\Delta g_\sigma^I \to \partial_J h^I \Delta g_\sigma^J + \partial_J\partial_K h^I \bar\partial^\mu g_\sigma{}^J\bar\partial_\mu g_\sigma{}^K$.

Using (2.4) it follows that the local anomalous response to Weyl rescaling can be written as

$$\delta_\sigma\,16\pi^2 W[e^{2\sigma}\gamma_{\mu\nu}, g_\sigma^I] = \int \mathrm{d}^4x\sqrt{-\gamma}\,\mathcal{A}\,, \qquad (3.24)$$

where $\mathcal{A}$ is determined by (2.4) in conjunction with (3.21). Even with (2.33) the general form is lengthy. Only the final expression is of possible interest but we include below some intermediate steps in case of any desire to verify the calculational details. For the curvature dependent terms, using (2.14), (2.17) as well as $B = E_I\beta^I$,

$$\begin{aligned}\mathcal{A}_{\text{curvature}} = {}& \delta\sigma\big(C(g_\sigma)\,F - \tfrac14 A(g_\sigma)\,G - \tfrac{1}{72}E_I(g_\sigma)\beta^I(g_\sigma)R^2 - E^{\mu\nu}\,G_{IJ}(g_\sigma)\,\bar\partial_\mu g_\sigma{}^I\bar\partial_\nu g_\sigma{}^J\big)\\ & + E^{\mu\nu}\,\delta_\sigma\Big(\big(A(g_\sigma) - W_I(g_\sigma)\beta^I(g_\sigma)\big)\partial_\mu\sigma\partial_\nu\sigma - 2\,W_I(g_\sigma)\,\bar\partial_\mu g_\sigma^I\partial_\nu\sigma\Big)\\ & - \tfrac16 R\,\delta_\sigma\Big(H_I(g_\sigma)\big(2\,\bar\partial^\mu g_\sigma^I\partial_\mu\sigma + \beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\big)\Big)\\ & - \tfrac16 R\,\delta\sigma\big(E_I(g_\sigma)\Delta g_\sigma^I + F_{IJ}(g_\sigma)\,\bar\partial^\mu g_\sigma{}^I\bar\partial_\mu g_\sigma{}^J\big)\,. \end{aligned} \qquad (3.25)$$

There are also contributions which remain on flat space and are independent of $\bar\partial g_\sigma$

$$\begin{aligned}\mathcal{A}_A = {}& \delta_\sigma\Big(A(g_\sigma)\,\big(\nabla^2\sigma + \tfrac12\,\partial^\mu\sigma\partial_\mu\sigma\big)\partial^\nu\sigma\partial_\nu\sigma\Big)\\ & + \delta\sigma\,\partial_I A(g_\sigma)\big(\nabla^2 g_\sigma^I + 2\,\partial^\mu g_\sigma^I\partial_\mu\sigma(\nabla^2\sigma + \partial^\nu\sigma\partial_\mu\sigma)\big)\\ & + \delta\sigma\,\partial_I\partial_J A(g_\sigma)\,\partial^\mu g_\sigma^I\partial_\mu g_\sigma^I - \delta\sigma\,\mathcal{L}_\beta A(g_\sigma)(\nabla^2\sigma + \tfrac12\,\partial^\nu\sigma\partial_\mu\sigma)\partial^\nu\sigma\partial_\nu\sigma\\ & + 2\,\partial_\mu\delta\sigma\,\partial_I A(g_\sigma)\,\partial^\mu g_\sigma^I\,\partial^\nu\sigma\partial_\nu\sigma\,. \end{aligned} \qquad (3.26)$$

The remaining contributions in (3.24) are also curvature independent but involve $\bar\partial g_\sigma$ in a non trivial fashion. From the R dependent terms

$$\begin{aligned}\mathcal{A}_{EFH} = {}& \delta\sigma\Big(E_I(g_\sigma)\big(\Delta g_\sigma^I + \tfrac12\,\beta^I(g_\sigma)(\nabla^2\sigma + \partial^\mu\sigma\partial_\mu\sigma)\big) + F_{IJ}(g_\sigma)\,\bar\partial^\mu g_\sigma^I\bar\partial_\mu g_\sigma^J\\ & \quad + \mathcal{L}_\beta H_I(g_\sigma)\,\big(2\,\bar\partial^\mu g_\sigma^I\partial_\mu\sigma + \beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\big)\Big)\,(\nabla^2\sigma + \partial^\nu\sigma\partial_\nu\sigma)\\ & + 2\,\partial_\mu\delta\sigma\,H_I(g_\sigma)\big(\bar\partial^\mu g_\sigma^I + \beta^I(g_\sigma)\partial^\mu\sigma\big)\,(\nabla^2\sigma + \partial^\nu\sigma\partial_\nu\sigma)\,. \end{aligned} \qquad (3.27)$$

For the terms involving W_I, including those arising from A using (2.14) in (3.26),

$$
\begin{aligned}
\mathcal{A}_W = & - \delta_\sigma \Big(W_I(g_\sigma)\big(\Delta g_\sigma^I\, \partial^\nu\sigma\partial_\nu\sigma + 2\,\bar\partial^\mu g_\sigma^I \partial_\mu\sigma\,(\nabla^2\sigma + \partial^\nu\sigma\partial_\nu\sigma)\big) \\
& \qquad + \partial_J W_I(g_\sigma)\,\bar\partial^\mu g_\sigma^I \bar\partial_\mu g_\sigma^J\, \partial^\nu\sigma\partial_\nu\sigma + W_I(g_\sigma)\beta^I(g_\sigma)\big(2\,\nabla^2\sigma + \tfrac{3}{2}\,\partial^\mu\sigma\partial_\mu\sigma\big)\partial^\nu\sigma\partial_\nu\sigma \\
& \qquad + 2\,\partial_{[I}W_{J]}(g_\sigma)\,\bar\partial^\mu g_\sigma^I \beta^J(g_\sigma)\,\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma \\
& \qquad + \mathcal{L}_\beta W_I(g_\sigma)\big(2\,\bar\partial^\mu g_\sigma^I + \beta^I(g_\sigma)\partial^\mu\sigma\big)\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma\Big) \\
& - 4\,\partial_\mu\delta\sigma\,\partial_{[I}W_{J]}(g_\sigma)\big(\bar\partial^\mu g_\sigma^I \bar\partial^\nu g_\sigma^J + \beta^I(g_\sigma)\partial^\mu\sigma\bar\partial^\nu g_\sigma^J + \tfrac{1}{2}\bar\partial^\mu g_\sigma^I \beta^J(g_\sigma)\partial^\nu\sigma\big)\partial_\nu\sigma. \quad (3.28)
\end{aligned}
$$

In a similar fashion the corresponding contributions containing G_{IJ}, including contributions from F_{IJ} in (3.27) and from (3.26) with (2.14) may be written, noting that $\partial_{(I}\beta^K G_{J)K} + \Gamma_{(IJ)K}\beta^K = \frac{1}{2}\mathcal{L}_\beta G_{IJ}$, as

$$
\begin{aligned}
\mathcal{A}_G = & - \delta_\sigma \Big(G_{IJ}(g_\sigma)\,\big(\bar\partial^\mu g_\sigma^I \bar\partial^\nu g_\sigma^J\,\partial_\mu\sigma\partial_\nu\sigma - \tfrac{1}{2}\,\bar\partial^\mu g_\sigma^I \bar\partial_\mu g_\sigma^J \partial_\nu\sigma\partial_\nu\sigma\big) \\
& \qquad\qquad - \tfrac{1}{4}\,G_{IJ}(g_\sigma)\,\beta^I(g_\sigma)\beta^J(g_\sigma)\partial^\mu\sigma\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma\Big) \\
& - \delta\sigma\,\Big(G_{IJ}(g_\sigma)\big(\Delta g_\sigma^J + \beta^J(g_\sigma)(\nabla^2\sigma + \partial^\mu\sigma\partial_\mu\sigma)\big) + \Gamma^{(G)}{}_{IJK}(g_\sigma)\bar\partial^\mu g_\sigma^J \bar\partial_\mu g_\sigma^K\Big) \\
& \qquad\qquad\qquad \times\,\big(2\,\bar\partial^\nu g_\sigma^I \partial_\nu\sigma + \beta^I(g_\sigma)\,\partial^\nu\sigma\partial_\nu\sigma\big) \\
& - \delta\sigma\,\tfrac{1}{2}\big(G_{IJ}(g_\sigma) + \partial_J\beta^K(g_\sigma)G_{IK}(g_\sigma) + \Gamma^{(G)}{}_{IJK}(g_\sigma)\beta^J(g_\sigma)\big) \\
& \qquad \times\,\big(2\,\bar\partial^\mu g_\sigma^I \partial_\mu\sigma + \beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\big)\big(2\,\bar\partial^\nu g_\sigma^J \partial_\nu\sigma + \beta^J(g_\sigma)\,\partial^\nu\sigma\partial_\nu\sigma\big)\,. \quad (3.29)
\end{aligned}
$$

For the corresponding result containing S_{IJ}, T_{IJK} we include also the terms arising from from H_I in (3.27) and from $\partial_{[I}W_{J]}$ in (3.28) using (2.18) we obtain, with $\tilde S_{IJ}$ given by (2.19),

$$
\begin{aligned}
\mathcal{A}_S = & - \delta_\sigma\Big(\big(S_{IJ}(g_\sigma)\,\Delta g_\sigma^J + T_{IJK}(g_\sigma)\,\bar\partial^\mu g_\sigma^J \bar\partial_\mu g_\sigma^K\big)\big(2\bar\partial^\nu g_\sigma^I \partial_\nu\sigma + \beta^I(g_\sigma)\,\partial^\nu\sigma\partial_\nu\sigma\big) \\
& \qquad + \tfrac{1}{2}\,\tilde S_{IJ}(g_\sigma)\,\big(2\,\bar\partial^\mu g_\sigma^I \partial_\mu\sigma + \beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\big)\big(2\,\bar\partial^\nu g_\sigma^J \partial_\nu\sigma + \beta^J(g_\sigma)\,\partial^\nu\sigma\partial_\nu\sigma\big)\Big) \\
& + \delta\sigma\,\Big(\mathcal{L}_\beta S_{IJ}(g_\sigma)\big(\Delta g_\sigma^J + \beta^J(g_\sigma)(\nabla^2\sigma + \partial^\mu\sigma\partial_\mu\sigma)\big) + \mathcal{L}'_\beta T_{IJK}(g_\sigma)\bar\partial^\mu g_\sigma^J \bar\partial_\mu g_\sigma^K\Big) \\
& \qquad\qquad\qquad \times\,\big(2\,\bar\partial^\mu g_\sigma^I \partial_\mu\sigma + \beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\big) \\
& + \delta\sigma\,\tfrac{1}{2}\mathcal{L}_\beta\tilde S_{IJ}(g_\sigma)\,\big(2\,\bar\partial^\mu g_\sigma^I \partial_\mu\sigma + \beta^I(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\big)\big(2\,\bar\partial^\nu g_\sigma^J \partial_\nu\sigma + \beta^J(g_\sigma)\,\partial^\nu\sigma\partial_\nu\sigma\big)\,.
\end{aligned}
\quad (3.30)
$$

The expressions (3.29) and (3.30) combine so that we may use (2.20) and (2.21) and also (2.23) so that the remaining terms, with the results in (3.27) applying (3.20), become

$$
\begin{aligned}
\mathcal{A}_{ABC} = \delta\sigma\Big(& \tfrac{1}{2}\,A_{IJ}(g_\sigma)\,\Delta g_\sigma^I \Delta g_\sigma^J + B_{IJK}(g_\sigma)\,\Delta g_\sigma^I\,\bar\partial^\mu g_\sigma^J \bar\partial_\mu g_\sigma^K \\
& + \tfrac{1}{2}\,C_{IJKL}(g_\sigma)\,\bar\partial^\mu g_\sigma^I \bar\partial_\mu g_\sigma^J\,\bar\partial^\nu g_\sigma^K \bar\partial_\nu g_\sigma^L\Big)\,.
\end{aligned}
\quad (3.31)
$$

With these results it is then possible to extend (3.10) to four dimensions in the form

$$16\pi^2\big(W[e^{2\sigma}\gamma_{\mu\nu},g_\sigma{}^I]-W[\gamma_{\mu\nu},g^I]\big)=\int \mathrm{d}^4x\sqrt{-\gamma}\,\mathcal{W}\,, \tag{3.32}$$

with $\mathcal{W}$ a local function expressible as sum of contributions $\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3$. The curvature dependent terms are contained in

$$\begin{aligned}\mathcal{W}_1 &= \breve{C}(\sigma)\,F-\tfrac14\breve{A}(\sigma)\,G\\ &\quad+\tilde{A}(g_\sigma)\big(E^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma+\nabla^2\sigma\,\partial^\mu\sigma\partial_\mu\sigma+\tfrac12\,\partial^\mu\sigma\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma\big)\\ &\quad-\breve{G}_{IJ}(\sigma)\,(E^{\mu\nu}+\gamma^{\mu\nu}\tfrac16R)\,\bar\partial_\mu g_\sigma{}^I\bar\partial_\nu g_\sigma{}^J-2\,W_I(g_\sigma)\,E^{\mu\nu}\partial_\mu g_\sigma{}^I\partial_\nu\sigma\,,\end{aligned} \tag{3.33}$$

where $\breve{C}(\sigma),\breve{A}(\sigma)$ are defined analogously to (3.5) and $\breve{G}_{IJ}(\sigma)$ is again given by (3.9), is an evident extension of the two dimensional result in (3.11) with $\gamma^{\mu\nu}\to E^{\mu\nu}$. The additional terms involving G,W, after some simplification, are given by

$$\begin{aligned}\mathcal{W}_2 = &-\tfrac14\,G_{IJ}(g_\sigma)\big(2\partial^\mu g_\sigma{}^I\partial_\mu\sigma-\ \beta^I(g_\sigma)\partial^\mu\sigma\partial_\mu\sigma\big)\big(2\partial^\nu g_\sigma{}^J\partial_\nu\sigma-\beta^J(g_\sigma)\partial^\nu\sigma\partial_\nu\sigma\big)\\ &+\tfrac12\,G_{IJ}(g_\sigma)\partial^\mu g_\sigma{}^I\partial_\mu g_\sigma{}^J\,\partial^\nu\sigma\partial_\nu\sigma\\ &-\big(W_I(g_\sigma)\,\nabla^2g_\sigma{}^I+\partial_IW_J(g_\sigma)\,\partial^\mu g_\sigma{}^I\partial_\mu g_\sigma{}^J\big)\,\partial^\nu\sigma\partial_\nu\sigma\\ &-2\,W_I(g_\sigma)\,\partial^\mu g_\sigma{}^I\partial_\mu\sigma\,(\nabla^2\sigma+\partial^\nu\sigma\partial_\nu\sigma)\,.\end{aligned} \tag{3.34}$$

The remaining contribution to $\mathcal{W}$ imposing, by a choice of a_{IJ} in (2.32),

$$\tilde{S}_{(IJ)}(g)=0\,, \tag{3.35}$$

then reduce to

$$\begin{aligned}\mathcal{W}_3 = &-\big(S_{IJ}(g_\sigma)\big(\nabla^2g_\sigma{}^J+2\,\partial^\mu g_\sigma{}^J\partial_\mu\sigma+(\tfrac16R-\nabla^2\sigma-\partial^\mu\sigma\partial_\mu\sigma)\beta^J(g_\sigma)\big)\\ &\qquad\qquad+T_{IJK}(g_\sigma)\,\partial^\mu g_\sigma{}^J\partial_\mu g_\sigma{}^K\big)\big(2\,\partial^\nu g_\sigma{}^I\partial_\nu\sigma-\beta^I(g_\sigma)\,\partial^\nu\sigma\partial_\nu\sigma\big)\\ &+\tfrac12\,\breve{A}_{IJ}(\sigma)\,\hat\Delta g_\sigma{}^I\hat\Delta g_\sigma{}^J+\breve{B}_{IJK}(\sigma)\,\hat\Delta g_\sigma{}^I\,\bar\partial^\mu g_\sigma{}^J\bar\partial_\mu g_\sigma{}^K\\ &+\tfrac12\,\breve{C}_{IJKL}(\sigma)\,\bar\partial^\mu g_\sigma{}^I\bar\partial_\mu g_\sigma{}^J\,\bar\partial^\nu g_\sigma{}^K\bar\partial_\nu g_\sigma{}^L\,.\end{aligned} \tag{3.36}$$

In (3.36) R dependent terms have been absorbed in a redefinition of Δg_σ,

$$\hat\Delta g_\sigma{}^I=\Delta g_\sigma{}^I+\beta^I(g_\sigma)\,\tfrac16R\,, \tag{3.37}$$

which satisfies the corresponding equation to (3.23).

In (3.36) $\breve{A}_{IJ}(\sigma)$ is defined similarly to $\breve{G}_{IJ}(\sigma)$ in (3.9) while $\breve{B}_{IJK}(\sigma)$ is determined by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\sigma}\breve{B}_{IJK}(\sigma)+\partial_I\beta^L(g_\sigma)\breve{B}_{LJK}(\sigma)+\partial_J\beta^L(g_\sigma)\breve{B}_{ILK}(\sigma)\\ &\quad+\partial_K\beta^L(g_\sigma)\breve{B}_{IJL}(\sigma)+\partial_J\partial_K\beta^L(g_\sigma)\breve{A}_{IL}(\sigma)=B_{IJK}(g_\sigma)\,,\quad \breve{B}_{IJK}(0)=0\,,\end{aligned} \tag{3.38}$$

with a corresponding equation for $\breve{C}_{IJKL}(\sigma)$. Just as in (3.12) there is a formal solution

$$
\begin{aligned}
\breve{B}_{IJK}(\sigma) = {}& \big(\exp(\sigma\mathcal{L}_\beta) - 1\big)\mathcal{L}_\beta{}^{-1}\big(B_{IJK}(g) - \partial_J\partial_K\beta^L(g)\,\mathcal{L}_\beta{}^{-1}A_{IL}(g)\big) \\
& + \big(\exp(\sigma\mathcal{L}_\beta) - 1\big)\mathcal{L}_\beta{}^{-1}\big(\partial_J\partial_K\beta^L(g)\big)\,\mathcal{L}_\beta{}^{-1}A_{IL}(g)\,.
\end{aligned}
\tag{3.39}
$$

By obtaining analogous equations to (3.14) the relations (2.20) and (2.21) imply

$$
\begin{aligned}
\breve{G}_{IJ}(\sigma) - S_{IJ}(g_\sigma) + \breve{S}_{IJ}(\sigma) &= \Psi_I{}^K(g_\sigma)\breve{A}_{KJ}(\sigma) + \breve{B}_{JIK}(\sigma)\beta^K(g_\sigma)\,, \\
\breve{\Gamma}_{IJK}(\sigma) - T_{IJK}(g_\sigma) + \breve{T}_{IJK}(\sigma) &= \Psi_I{}^L(g_\sigma)\breve{B}_{LJK}(\sigma) + \breve{C}_{ILJK}(\sigma)\beta^L(g_\sigma)\,,
\end{aligned}
\tag{3.40}
$$

with $\breve{S}_{IJ}, \breve{T}_{IJK}$ defined similarly to $\breve{W}_I$ in (3.16) and for $\Psi_I{}^J(g) = \delta_I{}^J + \partial_I\beta^J(g)$. $\breve{\Gamma}_{IJK}$ satisfies (3.38) with $\breve{B}_{IJK} \to \breve{\Gamma}_{IJK}$, $B_{IJK} \to \Gamma^{(G)}{}_{IJK}$ and $\breve{A}_{IL} \to \breve{G}_{IL}$ and as a consequence

$$
\breve{\Gamma}_{IJK} = \Gamma^{(\breve{G})}{}_{IJK}\,,
\tag{3.41}
$$

with $\Gamma^{(\breve{G})}$ defined in terms of $\breve{G}_{IJ}$ as in (2.22). As a consequence of (3.35) we have from (3.40)

$$
\begin{aligned}
\breve{G}_{IJ}(\sigma) + \tfrac{1}{2}\mathcal{L}_\beta\breve{G}_{IJ}(\sigma) = {}& \Psi_I{}^K(g_\sigma)\Psi_J{}^L(g_\sigma)\breve{A}_{IJ}(\sigma) \\
& + \Psi_I{}^L(g_\sigma)\breve{B}_{LJK}(\sigma)\beta^K(g_\sigma) + \Psi_I{}^L(g_\sigma)\breve{B}_{LJK}(\sigma)\beta^K(g_\sigma) \\
& + \breve{C}_{ILJK}(\sigma)\beta^K(g_\sigma)\beta^L(g_\sigma)\,.
\end{aligned}
\tag{3.42}
$$

Applying (3.40) in (3.34) we may use

$$
\begin{aligned}
\big(\breve{S}_{IJ}(\sigma)\,\Delta g_\sigma{}^J + \breve{T}_{IJK}(\sigma)\,\bar{\partial}^\mu g_\sigma{}^J\bar{\partial}_\mu g_\sigma{}^K\big)\bar{\partial}_\nu g_\sigma{}^I &= \big(S_{IJ}(g)\,\nabla^2 g^J + T_{IJK}(g)\,\partial^\mu g^J\partial_\mu g^K\big)\partial_\nu g^I\,, \\
\breve{S}_{IJ}(\sigma)\,\bar{\partial}_\nu g_\sigma{}^I\beta^J(g_\sigma) &= S_{IJ}(g)\,\partial_\nu g^I\beta^J(g)\,,
\end{aligned}
\tag{3.43}
$$

and similarly for $\bar{\partial}_\nu g_\sigma{}^I \to \beta^I(g_\sigma)$. By applying $\frac{\partial}{\partial\sigma}$ to (3.9) so that it becomes a homogeneous equation, we may obtain

$$
\mathcal{L}_\beta\breve{G}_{IJ}(\sigma)\,\bar{\partial}^\mu g_\sigma{}^I\bar{\partial}_\mu g_\sigma{}^J = G_{IJ}(g_\sigma)\,\bar{\partial}^\mu g_\sigma{}^I\bar{\partial}_\mu g_\sigma{}^J - G_{IJ}(g)\,\partial^\mu g^I\partial_\mu g^J\,,
\tag{3.44}
$$

and also, as in (3.19),

$$
\tilde{A}(g_\sigma) - \tilde{A}(g) = \breve{G}_{IJ}(\sigma)\,\beta^I(g_\sigma)\beta^J(g_\sigma)\,.
\tag{3.45}
$$

Starting from (3.32), with (3.33), (3.34), (3.36), and letting $\gamma_{\mu\nu} \to e^{-2\sigma}\gamma_{\mu\nu}$ then, similarly to (3.13),

$$
16\pi^2\big(W[e^{-2\sigma}\gamma_{\mu\nu}, g^I] - W[\gamma_{\mu\nu}, g_\sigma{}^I]\big) = \int \mathrm{d}^4x\sqrt{-\gamma}\,\mathcal{W}'\,,
\tag{3.46}
$$

For an IR fixed point so that $g_\sigma^I \to g_*^I$ as $\sigma \to -\infty$ then, assuming $W[\gamma_{\mu\nu}, g_\sigma^I] \to W[\gamma_{\mu\nu}, g_*^I]$ smoothly, $\mathcal{W}'$ determines the dependence on σ in the neighbourhood of the fixed point.

To determine $\mathcal{W}'$ we use (3.21) for $\sigma \to -\sigma$ and the corresponding equation to (3.17) and discard total derivatives as appropriate. Writing $\mathcal{W}' = \mathcal{W}'_1 + \mathcal{W}'_2 + \mathcal{W}'_3$ the result, using (3.40), (3.42), (3.43), (3.44), (3.45), is

$$\begin{aligned} \mathcal{W}'_1 = & -\breve{C}(\sigma)\, F + \tfrac{1}{4}\breve{A}(\sigma)\, G \\ & + \tilde{A}(g)\big(E^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \nabla^2\sigma\,\partial^\mu\sigma\partial_\mu\sigma + \tfrac{1}{2}\,\partial^\mu\sigma\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma\big) \\ & + \breve{G}_{IJ}(\sigma)\,(E^{\mu\nu} + \gamma^{\mu\nu}\tfrac{1}{6}R)\,\partial_\mu g_\sigma^I\partial_\nu g_\sigma^J + 2\,W_I(g)\,E^{\mu\nu}\partial_\mu g^I\partial_\nu\sigma\,, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \mathcal{W}'_2 = & -\tfrac{1}{4}\,G_{IJ}(g)\big(2\partial^\mu g^I\partial_\mu\sigma + \beta^I(g)\partial^\mu\sigma\partial_\mu\sigma\big)\big(2\partial^\nu g^J\partial_\nu\sigma + \beta^J(g)\partial^\nu\sigma\partial_\nu\sigma\big) \\ & + \tfrac{1}{2}\,G_{IJ}(g)\partial^\mu g^I\partial_\mu g^J\,\partial^\nu\sigma\partial_\nu\sigma \\ & - \big(W_I(g)\,\nabla^2 g^I + \partial_I W_J(g)\,\partial^\mu g^I\partial_\mu g^J\big)\,\partial^\nu\sigma\partial_\nu\sigma \\ & - 2\,W_I(g)\,\partial^\mu g^I\partial_\mu\sigma\,(\nabla^2\sigma - \partial^\nu\sigma\partial_\nu\sigma)\,, \end{aligned} \tag{3.48}$$

and,

$$\begin{aligned} \mathcal{W}'_3 = & \big(S_{IJ}(g)\big(\nabla^2 g^J - 2\,\partial^\mu g^J\partial_\mu\sigma + (\tfrac{1}{6}R + \nabla^2\sigma - \partial^\mu\sigma\partial_\mu\sigma)\beta^J(g)\big) \\ & \qquad\qquad\qquad + T_{IJK}(g)\partial^\mu g^J\partial_\mu g^K\big)\big(2\partial^\nu g^I\partial_\nu\sigma + \beta^I(g)\partial^\nu\sigma\partial_\nu\sigma\big) \\ & - \tfrac{1}{2}\,\breve{A}_{IJ}(\sigma)\,\big(\nabla^2 g_\sigma^I + \tfrac{1}{6}R\,\beta^I(g_\sigma)\big)\big(\nabla^2 g_\sigma^J + \tfrac{1}{6}R\,\beta^J(g_\sigma)\big) \\ & - \breve{B}_{IJK}(\sigma)\,\big(\nabla^2 g_\sigma^I + \tfrac{1}{6}R\,\beta^I(g_\sigma)\big)\,\partial^\mu g_\sigma^J\partial_\mu g_\sigma^K \\ & - \tfrac{1}{2}\,\breve{C}_{IJKL}(\sigma)\,\partial^\mu g_\sigma^I\partial_\mu g_\sigma^J\,\partial^\nu g_\sigma^K\partial_\nu g_\sigma^L\,. \end{aligned} \tag{3.49}$$

$\mathcal{W}'_1, \mathcal{W}'_2, \mathcal{W}'_3$ may also be obtained from $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ by letting $g_\sigma \to g$ and then $\sigma \to -\sigma$. The contributions involving $\breve{G}_{IJ}$, as well as $\breve{A}_{IJ}, \breve{B}_{IJK}, \breve{C}_{IJKL}$ depend on the RG trajectory linking g and g_σ, for variations arising from (2.30), (2.31) the associated freedom becomes a difference of contributions from the end points of the RG flow.

These expressions simplify if we assume that the x-dependence in g_σ arises only from σ, so that in solving (3.3) g^I is a constant. In this case we may take $\partial_\mu g_\sigma^I = \beta^I(g_\sigma)\partial_\mu\sigma$, $\nabla^2 g_\sigma^I + \frac{1}{6}R\,\beta^I(g_\sigma) = \beta^I(g_\sigma)\frac{1}{6}\breve{R} + \Psi_J{}^I(g_\sigma)\beta^J(g_\sigma)\partial^\mu\sigma\partial_\mu\sigma$ for $\frac{1}{6}\breve{R} = \frac{1}{6}R + \nabla^2\sigma - \partial^\mu\sigma\partial_\mu\sigma$ and then

$$\begin{aligned} \mathcal{W}' = & -\breve{C}(\sigma)\, F + \tfrac{1}{4}\breve{A}(\sigma)\, G \\ & + \tilde{A}(g_\sigma)\,(E^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \nabla^2\sigma\,\partial^\mu\sigma\partial_\mu\sigma + \tfrac{1}{2}\,\partial^\mu\sigma\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma) \\ & - \tfrac{1}{4}\,G_{IJ}(g_\sigma)\beta^I(g_\sigma)\beta^J(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma \\ & + \tfrac{1}{6}\breve{R}\,S_{IJ}(g_\sigma)\beta^I(g_\sigma)\beta^J(g_\sigma)\,\partial^\mu\sigma\partial_\mu\sigma - \tfrac{1}{2}(\tfrac{1}{6}\breve{R})^2\,\breve{A}_{IJ}(\sigma)\beta^I(g_\sigma)\beta^J(g_\sigma)\,. \end{aligned} \tag{3.50}$$

Of course at a fixed point with a vanishing beta function this coincides with the result used in [7] for $\sigma \to \tau$ and a similar expression was obtained in [22].

Although lengthy, and tedious to obtain, the extended result (3.46), with (3.47), (3.48), (3.49), is still relatively simple and potentially allows for the analysis of dilaton couplings away from conformal fixed points.[1] Setting the curvature terms to zero and $\sigma \to \tau$ (3.47) becomes part of the lagrangian determining couplings of scalar fields $\mathcal{O}_I$ to the dilaton τ in the dilaton effective action. The results used in [6] and [7] depend also on imposing additional boundary conditions whose generalisation is less apparent.

4. Broken Conformal Symmetry

The results obtained in section 2 depend on extending the quantum field theory to a curved space background. In this section we show how a subset of the consistency relation

[1] If all β terms are set to zero in (2.4) and the various conditions for integrability are implemented along with (2.33) then (2.4) becomes

$$\begin{aligned}\Delta_\sigma\, 16\pi^2 W = \int d^4x\sqrt{-\gamma}\;\sigma\Big(&C\,F - \tfrac{1}{4}\,A\,G + \tfrac{1}{2}\,G_{IJ}(\mathcal{D}^2 g^I \mathcal{D}^2 g^J - 2\,(E^{\mu\nu} + \tfrac{1}{6}R\,\gamma^{\mu\nu})\,\partial_\mu g^I \partial_\nu g^J)\\ &+ \tfrac{1}{2}\,\hat{C}_{IJKL}\,\partial^\mu g^I \partial_\mu g^J\,\partial^\nu g^K \partial_\nu g^L\Big)\\ - 2\int d^4x\sqrt{-\gamma}\;&\partial_\mu\sigma\,\Big(E^{\mu\nu} W_I \partial_\nu g^I - \partial_{[I} W_{J]}\,\partial^\mu g^I \nabla^2 g^J\Big)\,,\end{aligned}$$

where $\mathcal{D}^2 g^I$ is defined as in (2.9) with $A_{IJ} \to G_{IJ}$, $B_{IJK} \to \Gamma^{(G)}{}_{IJK}$ and we must also impose $\partial_I A = 0$. This may be integrated straightforwardly to give $16\pi^2(W[e^{2\sigma}\gamma_{\mu\nu}] - W[\gamma_{\mu\nu}]) = \int \mathrm{d}^4x\sqrt{-\gamma}\,\mathcal{W}_{\mathrm{FP}}$ where

$$\begin{aligned}\mathcal{W}_{\mathrm{FP}} = \sigma\Big(&C\,F - \tfrac{1}{4}\,A\,G + \tfrac{1}{2}\,G_{IJ}(\mathcal{D}^2 g^I \mathcal{D}^2 g^J - 2\,(E^{\mu\nu} + \tfrac{1}{6}R\,\gamma^{\mu\nu})\,\partial_\mu g^I \partial_\nu g^J)\\ &+ \tfrac{1}{2}\,\hat{C}_{IJKL}\,\partial^\mu g^I \partial_\mu g^J\,\partial^\nu g^K \partial_\nu g^L\Big)\\ &+ A\,(E^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \nabla^2\sigma\,\partial^\mu\sigma\partial_\mu\sigma + \tfrac{1}{2}\,\partial^\mu\sigma\partial_\mu\sigma\,\partial^\nu\sigma\partial_\nu\sigma)\\ &- G_{IJ}(\partial^\mu g^I \partial^\nu g^J\,\partial_\mu\sigma\partial_\nu\sigma - \tfrac{1}{2}\,\partial^\mu g^I \partial_\mu g^J\,\partial^\nu\sigma\partial_\nu\sigma)\\ &- 2\,W_I\,E^{\mu\nu}\partial_\mu g^I \partial_\nu\sigma + 2\,\partial_{[I} W_{J]}\,\partial^\mu g^I \nabla^2 g^J\,\partial_\mu\sigma\\ &- 2\,W_I\,\partial^\mu g^I \partial_\mu\sigma\,(\nabla^2\sigma + \partial^\nu\sigma\partial_\nu\sigma) - (W_I \nabla^2 g^I + \partial_I W_J \partial^\mu g^I \partial_\mu g^J)\,\partial^\nu\sigma\partial_\nu\sigma\,.\end{aligned}$$

This result is relevant at a fixed point when $\{g^I\}$ are the couplings for exactly marginal operators and so parameterise the moduli space. The terms proportional to G_{IJ} can be expressed in terms of the Riegert operator, a conformally covariant 4th order differential operator acting on dimensionless scalars. On the moduli space A is constant, whereas C may vary, and we expect, since $(\omega g)^I W_I = 0$, $W_I = \partial_I f$ for some scalar f, and so by virtue of the freedom in (2.30) we may then set $W_I = 0$.

equations can be defined by restricting to flat space and considering broken conformal symmetry. These are derived by considering diffeomorphisms, as well as Weyl rescalings. Their intersection defines the conformal group

In general quantum field theories on curved space, within appropriate regularisation schemes, are invariant under diffeomorphisms. This may be expressed, for arbitrary smooth $v^\mu(x)$, as

$$\int \mathrm{d}^4x \left(\mathcal{L}_v\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} + v^\mu\partial_\mu g^I\frac{\delta}{\delta g^I}\right)W = 0\,, \tag{4.1}$$

where

$$\mathcal{L}_v\gamma_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu\,. \tag{4.2}$$

Conformal Killing vectors satisfy

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = 2\,\sigma_v\,\gamma_{\mu\nu}\,, \tag{4.3}$$

and for any such conformal Killing vector acting on W we may take from (4.1) and (2.2)

$$\Delta_{\sigma_v} \to \Delta_v = \int \mathrm{d}^4x\,\big(-v^\mu\partial_\mu g^I + \sigma_v\,\beta^I\big)\frac{\delta}{\delta g^I}\,. \tag{4.4}$$

Defining the commutator of two diffeomorphisms by

$$[v,v']^\mu = v^\nu\partial_\nu\, v'^\mu - v'^\nu\partial_\nu\, v^\mu\,, \tag{4.5}$$

(4.3) implies

$$v^\mu\partial_\mu\,\sigma_{v'} - v'^\mu\partial_\mu\,\sigma_v = \sigma_{[v,v']}\,. \tag{4.6}$$

It is then easy to verify that, from the definition (4.4),

$$\big[\Delta_v,\Delta_{v'}\big] = \Delta_{[v,v']}\,. \tag{4.7}$$

On flat space the solutions of (4.3), for $\nabla_\mu \to \partial_\mu$, $\gamma_{\mu\nu} \to \eta_{\mu\nu}$, are of course the usual conformal Killing vectors

$$v^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2\,x^\nu b_\nu\,x^\mu\,, \qquad \sigma_v(x) = \lambda - 2\,x^\mu b_\mu\,, \tag{4.8}$$

for $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Combining (4.1) with (2.4) gives a condition on the flat space vacuum energy functional $W[g^I]$ which reduces, since $\partial^2\sigma_v = 0$, to

$$\Delta_v\,16\pi^2 W = \int \mathrm{d}^4x\,\big(\sigma_v\,X - 2\,\partial_\mu\sigma_v\,Y^\mu\big)\,, \tag{4.9}$$

for $X(g), Y^\mu(g)$ given by (2.7), albeit $\nabla^2 \to \partial^2$. In Y^μ, since $S_{(IJ)}\,\partial^\mu g^I \partial^2 g^J = S_{(IJ)}\big(\partial^\nu(\partial^\mu g^I \partial_\nu g^J) - \frac{1}{2}\,\partial^\mu(\partial^\nu g^I \partial_\nu g^J)\big)$ and $\partial_\nu\partial_\mu\sigma_v = 0$, the symmetric part of S_{IJ} may be dropped. (4.9) expresses broken conformal symmetry[2], valid so long as the couplings are local functions of x.

Linear conditions on correlation functions for the operators $\mathcal{O}_I$, which reduce to standard RG equations for $v^\mu(x) = \lambda x^\mu$ and g^I constant, can be obtained from

$$\left[\Delta_v, \frac{\delta}{\delta g^I(x)}\right] = -v^\mu(x)\,\partial_\mu \frac{\delta}{\delta g^I(x)} - \sigma_v(x)\Big(d\,\delta_I{}^J + \partial_I\beta^J\big(g(x)\big)\Big)\frac{\delta}{\delta g^J(x)}\,, \tag{4.10}$$

with $d = 4$ here again. With the definition (2.10) then

$$16\pi^2\big(\Delta_v\langle\mathcal{O}_I\rangle + 4\,\sigma_v\langle\mathcal{O}_I\rangle + \sigma_v\,\partial_I\beta^J\langle\mathcal{O}_J\rangle + v^\mu\partial_\mu\langle\mathcal{O}_I\rangle\big) = \mathcal{A}_I\,, \tag{4.11}$$

for

$$\mathcal{A}_I = -\frac{\delta}{\delta g^I}\int \mathrm{d}^4x\ \big(\sigma_v\,X - 2\,\partial_\mu\sigma_v\,Y^\mu\big)\,. \tag{4.12}$$

To impose (4.7) making use of (4.6) we note that

$$\begin{aligned}
&\Delta_v\int \mathrm{d}^4x\ \big(\sigma_{v'}\,X - 2\,\partial_\mu\sigma_{v'}\,Y^\mu\big) - \Delta_{v'}\int \mathrm{d}^4x\ \big(\sigma_v\,X - 2\,\partial_\mu\sigma_v\,Y^\mu\big)\\
&= \int \mathrm{d}^4x\ \big(\sigma_{[v,v']}\,X - 2\,\partial_\mu\sigma_{[v,v']}\,Y^\mu\big) + \int \mathrm{d}^4x\ \big(2\,k_\mu\,K^\mu + 4\,l_{\mu\nu}\,L^{\mu\nu}\big)\,,\\
&\quad k_\mu = \sigma_{v'}\,\partial_\mu\sigma_v - \sigma_v\,\partial_\mu\sigma_{v'}\,, \qquad l_{\mu\nu} = \partial_\mu\sigma_{v'}\,\partial_\nu\sigma_v - \partial_\mu\sigma_v\,\partial_\nu\sigma_{v'} = 8\,b'_{[\mu}\,b_{\nu]}\,,
\end{aligned} \tag{4.13}$$

for

$$\begin{aligned}
K^\mu &= \big(A_{IJ} + \partial_I\beta^K A_{JK} + B_{JIK}\beta^K + \mathcal{L}_\beta S_{IJ}\big)\partial^\mu g^I \partial^2 g^J\\
&\qquad + \big(B_{IJK} + \partial_I\beta^L B_{LJK} + C_{ILJK}\beta^L + S_{IL}\partial_J\partial_K\beta^L + \mathcal{L}_\beta T_{IJK}\big)\partial^\mu g^I \partial^\nu g^J \partial_\nu g^K\,,\\
L^{\mu\nu} &= -\,\big(S_{[IJ]} - \partial_{[I}\beta^K S_{J]K} + T_{[IJ]K}\beta^K\big)\partial^\mu g^I \partial^\nu g^J\,.
\end{aligned} \tag{4.14}$$

Hence (4.7) is satisfied, assuming (4.9), if the terms involving K^μ and $L^{\mu\nu}$ in (4.13) vanish. As σ_v is just linear in x the conditions in this case do not require either K^μ or $L^{\mu\nu}$ to be zero. For the term involving $l_{\mu\nu}$, since this is a constant, it is necessary and sufficient only that $L^{\mu\nu}$ is a total derivative so that we require

$$L^{\mu\nu} = \partial^{[\mu}\big(W_I\partial^{\nu]}g^I\big)\,, \tag{4.15}$$

[2] Broken conformal Ward identities were first discussed at the same time as the usual RG equations [23] but in [24] 'appear to be useless'. For other approaches see [25].

for some W_I, which is then equivalent to the result (2.18) for $\partial_{[I}W_{J]}$ and hence W_I is determined in terms of S_{IJ}, T_{IJK} up to the freedom $W_I \sim W_I - \partial_I a$. For the term containing k_μ in (4.13) then since

$$\partial_{(\nu}k_{\mu)} = 0\,, \tag{4.16}$$

it is sufficient to require

$$K^\mu = \partial_\nu\big(G_{IJ}\partial^\mu g^I \partial^\nu g^J\big) - \tfrac{1}{2}\,\partial^\mu\big(G_{IJ}\partial^\nu g^I \partial_\nu g^J\big)\,, \quad G_{IJ} = G_{JI}\,, \tag{4.17}$$

choosing the relative coefficients to match the form of K^μ in (4.14). Combining (4.14) and (4.17) is equivalent to (2.20) and (2.21) with the definition (2.22).

Although restricting to broken conformal symmetry on flat space does not directly determine A, which plays the role of a c-function, the relations defining W_I and G_{IJ} are sufficient to reconstruct the critical result (2.14). Using (2.20) and (2.21)

$$\begin{aligned}\partial_{[I}\big(G_{J]K}\beta^K\big) &= \partial_{[I}\beta^K G_{J]K} - \Gamma^{(G)}{}_{[IJ]K}\beta^K \\ &= \partial_{[I}\beta^K A_{J]K} - B_{[IJ]K}\beta^K \\ &\quad + \partial_{[I}\beta^K \mathcal{L}_\beta S_{J]K} - S_{[I\,L}\partial_{J]}\partial_K\beta^L\beta^K - \mathcal{L}_\beta T_{[IJ]K}\beta^K \\ &= G_{[IJ]} - A_{[IJ]} - \mathcal{L}_\beta\big(S_{[IJ]} + T_{[IJ]K}\beta^K\big) + \partial_{[I}\beta^K\mathcal{L}_\beta S_{J]K} - S_{[I\,L}\partial_{J]}\partial_K\beta^L\beta^K \\ &= \mathcal{L}_\beta\big(\partial_{[I}W_{J]}\big) = \partial_{[I}\mathcal{L}_\beta W_{J]}\,,\end{aligned} \tag{4.18}$$

using (2.18) and $G_{[IJ]} = A_{[IJ]} = 0$. (4.18) is the necessary condition for the integrability of (2.14) so that A may be calculated in terms of the flat space quantities $A_{IJ}, B_{IJK}, S_{IJ}, T_{IJK}$ up to a g-independent constant.

If in (4.14)

$$\Delta S_{IJ} = g_{IJ}\,, \qquad \Delta T_{IJK} = \Gamma^{(g)}{}_{IJK}\,, \tag{4.19}$$

then

$$\begin{aligned}\Delta K^\mu &= \mathcal{L}_\beta g_{IJ}\,\partial^\mu g^I \partial^2 g^J + \big(\partial_K\mathcal{L}_\beta g_{IJ} - \tfrac{1}{2}\,\partial_I\mathcal{L}_\beta g_{JK}\big)\,\partial^\mu g^I\partial^\nu g^J\partial_\nu g^K\,, \\ \Delta L^{\mu\nu} &= \partial_{[I}\big(g_{J]K}\beta^K\big)\,\partial^\mu g^I\partial^\nu g^J\,,\end{aligned} \tag{4.20}$$

and it is easy to see that this implies

$$\Delta G_{IJ} = \mathcal{L}_\beta g_{IJ}\,, \qquad \Delta W_I = g_{IJ}\beta^J\,, \tag{4.21}$$

in accord with (2.30) and (2.31).

At a fixed point, assuming

$$\partial_I\beta^J\big|_{g=g_*} = -(4-\Delta_I)\delta_I{}^J\,, \tag{4.22}$$

then with (2.10) the identity (4.9) requires, by considering $\frac{\delta}{\delta g^I(x)}\frac{\delta}{\delta g^J(y)}$ and then restricting to constant couplings,

$$\begin{aligned}\big(v^\mu(x)\partial_{\mu x} + \Delta_I\,\sigma_v(x) + v^\mu(y)\partial_{\mu y} + \Delta_J\,\sigma_v(y)\big)\big\langle \mathcal{O}_I(x)\,\mathcal{O}_J(y)\big\rangle \\ = \frac{1}{16\pi^2}\,A_{IJ}\,\partial_x{}^2\partial_y{}^2\big(\sigma_v(x)\delta^4(x-y)\big)\,.\end{aligned} \tag{4.23}$$

There is a potential term involving S_{IJ} but this cancels for $S_{IJ}=-S_{JI}$. The conformal identity (4.23) has a solution only for $\Delta_I=\Delta_J=\Delta$ when

$$\begin{aligned}\big\langle \mathcal{O}_I(x)\,\mathcal{O}_J(y)\big\rangle &= \frac{C_{IJ}}{((x-y)^2)^\Delta} - \frac{1}{16\pi^2}\,\frac{A_{IJ}}{2(4-\Delta)}\,\partial_x{}^2\partial_y{}^2\delta^4(x-y) \\ &= \frac{1}{\Delta-4}\,(\partial^2)^3\bigg(\frac{C_{IJ}}{64(\Delta-3)^2(\Delta-2)^2(\Delta-1)}\,\frac{1}{((x-y)^2)^{\Delta-3}} - \frac{A_{IJ}}{2(8\pi^2)^2}\,\frac{1}{(x-y)^2}\bigg)\,.\end{aligned} \tag{4.24}$$

For this to be well defined for $x\approx y$ we must have $(2\pi^2)^2C_{IJ}=24\,A_{IJ}+\mathrm{O}(\Delta-4)$.

With the definition (2.11) and restricting to flat space then $\langle T^{\mu\nu}\rangle$ satisfies

$$\partial_\mu\langle T^{\mu\nu}\rangle + \partial^\nu g^I\,\langle\mathcal{O}_I\rangle = 0\,, \tag{4.25a}$$

$$16\pi^2\big(\eta_{\mu\nu}\langle T^{\mu\nu}\rangle - \beta^I\langle\mathcal{O}_I\rangle\big) = X + 2\,\partial_\mu Y^\mu\,, \tag{4.25b}$$

and also, with Δ_v as in (4.4), a corresponding broken conformal identity

$$\begin{aligned}16\pi^2\big(\Delta_v\langle T^{\mu\nu}\rangle + 6\,\sigma_v\,\langle T^{\mu\nu}\rangle + \mathcal{L}_v\langle T^{\mu\nu}\rangle\big) &= \mathcal{A}^{\mu\nu}\,, \\ \mathcal{L}_v\langle T^{\mu\nu}\rangle = v^\rho\partial_\rho\langle T^{\mu\nu}\rangle - \partial_\rho v^\mu\,\langle T^{\rho\nu}\rangle &- \partial_\rho v^\nu\,\langle T^{\mu\rho}\rangle\,,\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}\mathcal{A}^{\mu\nu} = {}& 2\,\frac{\delta}{\delta\gamma_{\mu\nu}}\int \mathrm{d}^4x\,\sqrt{-\gamma}\,\big(\sigma\,X - 2\,\partial_\alpha\sigma\,Y^\alpha\big)\Big|_{\gamma_{\mu\nu}\to\eta_{\mu\nu},\sigma\to\sigma_v} \\ &+ \tfrac{1}{3}(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\big(\sigma_v(E_I\partial^2 g^I + F_{IJ}\partial^\alpha g^I\partial_\alpha g^J) + 2\,\partial_\alpha\sigma_v\,H_I\partial^\alpha g^I\big) \\ &+ \mathcal{D}^{\mu\nu\sigma\rho}\big(\sigma_v\,G_{IJ}\partial_\sigma g^I\partial_\rho g^J + 2\,\partial_\sigma\sigma_v\,W_I\partial_\rho g^I\big)\,,\end{aligned} \tag{4.27}$$

with $\mathcal{D}^{\mu\nu\sigma\rho}$ defined so that

$$\mathcal{D}^{\mu\nu\sigma\rho}f_{\sigma\rho} = \partial^2 f^{\mu\nu} + \eta^{\mu\nu}\,\partial^\sigma\partial^\rho f_{\sigma\rho} - 2\,\partial^{(\mu}\partial_\sigma f^{\nu)\sigma} + (\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2)\eta^{\sigma\rho}f_{\sigma\rho}\,, \tag{4.28}$$

for any $f_{\sigma\rho}=f_{\rho\sigma}$.

The form for $\mathcal{A}^{\mu\nu}$ in (4.26) is constrained by (4.25a,b) in conjunction with (4.11). Using $\partial_\mu(\mathcal{L}_v+6\sigma_v)T^{\mu\nu} = (\mathcal{L}_v+6\sigma_v)\partial_\mu T^{\mu\nu} + \partial^\nu\sigma_v\,\eta_{\sigma\rho}T^{\sigma\rho}$ we may obtain from (4.25a)

$$\partial_\mu\mathcal{A}^{\mu\nu} + \partial^\nu g^I\,\mathcal{A}_I = \partial^\nu\sigma_v\,(X + 2\,\partial_\mu Y^\mu)\,, \tag{4.29}$$

and from $\eta_{\mu\nu}(\mathcal{L}_v + 6\sigma_v)T^{\mu\nu} = (v^\rho \partial_\rho + 4\sigma_v)\eta_{\mu\nu}T^{\mu\nu}$ from (4.25b)

$$\eta_{\mu\nu}\mathcal{A}^{\mu\nu} - \beta^I \mathcal{A}_I = \big(\Delta_v + 4\,\sigma_v + v^\mu \partial_\mu\big)(X + 2\,\partial_\nu Y^\nu)\,. \tag{4.30}$$

(4.30) constrains the additional derivative terms in (4.27) as it reduces to

$$\begin{aligned}
&\partial^2\big(\sigma_v(E_I\partial^2 g^I + F_{IJ}\partial^\mu g^I \partial_\mu g^J) + 2\,\partial_\mu\sigma_v\, H_I \partial^\mu g^I\big) \\
&- 2(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\big(\sigma_v\, G_{IJ}\partial_\mu g^I \partial_\nu g^J + 2\,\partial_\mu \sigma_v\, W_I \partial_\nu g^I\big) \\
&= -\partial^2\big(\sigma_v(A_{IJ}\partial^2 g^I \beta^J + B_{KIJ}\partial^\mu g^I \partial_\mu g^J \beta^K) - 2\,\partial_\mu\sigma_v\, S_{IJ}\partial^\mu g^I \beta^J\big) \\
&\quad + 8\,\partial_\mu\big(\partial_\nu\sigma_v\, \tilde{S}_{[IJ]}\,\partial^\mu g^I \partial^\nu g^J\big) \\
&\quad + 2\,\big(\partial_\mu\,\sigma_v + (\partial_\mu\sigma_v)\big)\big((\tilde{A}_{IJ} + \mathcal{L}_\beta S_{IJ})\partial^\mu g^I \partial^2 g^J \\
&\qquad\qquad + (\tilde{B}_{IJK} + \mathcal{L}_\beta T_{IJK} + \partial_J\partial_K \beta^L S_{IL})\partial^\mu g^I \partial^\nu g^J \partial_\nu g^K\big)\,,
\end{aligned} \tag{4.31}$$

for $\tilde{S}_{IJ}, \tilde{A}_{IJ}, \tilde{B}_{IJK}$ as in (2.19), (2.20), (2.21). Since

$$\begin{aligned}
&(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\big(\sigma_v\, G_{IJ}\partial_\mu g^I \partial_\nu g^J + 2\,\partial_\mu\sigma_v\, W_I\partial_\nu g^I\big) \\
&\quad = \tfrac{1}{2}\,\partial^2\big(\sigma_v\, G_{IJ}\partial^\mu g^I \partial_\mu g^J\big) + 4\,\partial_\mu\big(\partial_\nu\sigma_v\,\partial_{[I}W_{J]}\partial^\mu g^I \partial^\nu g^J\big) \\
&\qquad - \big(\sigma_v\partial_\mu + 2(\partial_\mu\sigma_v)\big)\big(G_{IJ}\partial^\mu g^I \partial^2 g^J + \Gamma^{(G)}{}_{IJK}\partial^\mu g^I \partial^\nu g^J \partial_\nu g^K\big)\,,
\end{aligned} \tag{4.32}$$

(4.31) reduces to the consistency relations (2.16), (2.18), (2.20) and (2.21). Hence the broken conformal identity (4.26), with (4.27) may be used to define G_{IJ}, W_I and also E_I, F_{IJ}, H_I just in terms of correlation functions involving the energy momentum tensor on flat space.

The relations (4.25a,b) and (4.26) which are expressed in terms of local couplings can be translated into equivalent constraints on various correlation functions involving the energy momentum tensor and with g^I constant. We describe here the simplest results for the three point function $\langle T^{\mu\nu}(x)\,\mathcal{O}_J(y)\,\mathcal{O}_K(z)\rangle$ in the conformal limit assuming (4.22) with $\Delta_J = \Delta_K = \Delta$. In this case we can drop contributions arising from H_I, S_{IJ}, W_I. Suppressing the argument x the conformal Ward identity becomes

$$\begin{aligned}
16\pi^2\big(\mathcal{L}_v + 6\,\sigma_v + v^\mu(y)\partial_{\mu y} + \Delta\,\sigma_v(y) + v^\mu(z)\partial_{\mu z} + \Delta\,\sigma_v(z)\big)\big\langle T^{\mu\nu}\,\mathcal{O}_J(y)\,\mathcal{O}_K(z)\big\rangle& \\
= \mathcal{A}^{\mu\nu}_{JK}(y,z)\,,&
\end{aligned} \tag{4.33}$$

with

$$\begin{aligned}
\mathcal{A}^{\mu\nu}_{JK}(y,z) = A_{JK}\big(&2\,\partial^{(\mu}\delta_y\,\partial^{\nu)}(\sigma_v\,\partial^2\delta_z) + 2\,\partial^{(\mu}(\sigma_v\,\partial^2\delta_y)\,\partial^{\nu)}\delta_z \\
&- \eta^{\mu\nu}\,\big(\partial^\rho\delta_y\,\partial_\rho(\sigma_v\,\partial^2\delta_z) + \partial^\rho(\sigma_v\,\partial^2\delta_y)\,\partial_\rho\delta_z + \sigma_v\,\partial^2\delta_y\,\partial^2\delta_z\big)\big) \\
&+ \tfrac{1}{3}(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\big(\sigma_v\big((4-\Delta)\,A_{JK}(\partial^2\delta_y\,\delta_z + \delta_y\,\partial^2\delta_z) + 2\,G_{JK}\,\partial^\rho\delta_y\,\partial_\rho\delta_z\big)\big) \\
&+ 2\,G_{JK}\,\mathcal{D}^{\mu\nu\sigma\rho}\big(\sigma_v\,\partial_{(\sigma}\delta_y\,\partial_{\rho)}\delta_z\big)\big)\,,
\end{aligned} \tag{4.34}$$

for $\delta_y \equiv \delta^4(x-y)$, $\delta_z \equiv \delta^4(x-z)$ and where we have let $E_I \to -A_{IJ}\beta^J$, $F_{IJ} \to G_{IJ}$. Corresponding to $(4.25a,b)$ we have

$$
\begin{aligned}
&\partial_\mu \big\langle T^{\mu\nu}\, \mathcal{O}_J(y)\, \mathcal{O}_K(z)\big\rangle - \partial^\nu \delta_y \, \big\langle \mathcal{O}_J\, \mathcal{O}_K(z)\big\rangle - \partial^\nu \delta_z \, \big\langle \mathcal{O}_J(y)\, \mathcal{O}_K\big\rangle = 0\,, \\
&16\pi^2 \big(\eta_{\mu\nu} \big\langle T^{\mu\nu}\, \mathcal{O}_J(y)\, \mathcal{O}_K(z)\big\rangle \\
&\quad + (\Delta-4)\, \delta_y \, \big\langle \mathcal{O}_J(y)\, \mathcal{O}_K(z)\big\rangle + (\Delta-4)\, \delta_z \, \big\langle \mathcal{O}_J(y)\, \mathcal{O}_K(z)\big\rangle \big) = A_{JK}\, \partial^2 \delta_y \, \partial^2 \delta_z \, .
\end{aligned}
\tag{4.35}
$$

It is again somewhat non trivial to check consistency of (4.33) and (4.35), the necessary condition reduces to

$$
G_{JK} = (\Delta - 3)\, A_{JK}\,, \tag{4.36}
$$

which is equivalent to (2.20) in the conformal limit.

5. Beta functions for Scalar Fermion Theory

We consider as an example for the application of the general consistency relations a general scalar fermion field theory involving n_ψ, n_χ two component chiral spinor fermion fields ψ, χ, of opposite chirality, and n_ϕ complex scalars ϕ_i, $i=1,\dots n_\phi$, with a Lagrangian of the form

$$
\mathcal{L} = -\partial\bar\phi^i \cdot \partial\phi_i - \bar\psi\, i\sigma\cdot\partial\,\psi - \bar\chi\, i\bar\sigma\cdot\partial\,\chi - \bar\chi\, m(\phi)\,\psi - \bar\psi\, \bar m(\bar\phi)\,\chi - V(\bar\phi,\phi)\,, \tag{5.1}
$$

where $\sigma{\cdot}a\,\bar\sigma{\cdot}a = -a^2\,1$, $\mathrm{tr}_\sigma(\sigma{\cdot}a\,\bar\sigma{\cdot}b) = -2\,a{\cdot}b$ with $\cdot$ in this context denoting contraction of Lorentz indices. In (5.1) we assume

$$
m(\phi) = y^i\phi_i + \mu\,, \quad \bar m(\bar\phi) = \bar\phi^i\,\bar y_i + \bar\mu\,, \quad V(\bar\phi,\phi) = \tfrac14\,\lambda_{ij}{}^{kl}\,\bar\phi^i\bar\phi^j\,\phi_k\phi_l + \mathrm{O}(\phi^2\bar\phi, \phi\bar\phi^2)\,. \tag{5.2}
$$

The Yukawa coupling y^i is a $n_\chi \times n_\psi$ matrix and $\bar y_i = (y^i)^\dagger$. Also $(\lambda_{ij}{}^{kl})^* = \lambda_{kl}{}^{ij}$. For $n_\chi = n_\psi$ (5.1) can be re-expressed in terms of four component Dirac fermions. The Lagrangian (5.1) has a $U(1)\times U(1)$ symmetry for the dimension four interactions under

$$
\psi \to e^{i\theta}\psi\,, \qquad \chi \to e^{i\tau}\chi\,, \qquad \phi_i \to e^{i(\tau-\theta)}\phi_i\,. \tag{5.3}
$$

This is sufficient to significantly reduce the number of Feynman diagrams at each loop order.

The β-functions associated with the couplings y, λ in $\mathcal{L}$ can be expressed as

$$
\begin{aligned}
\beta_y{}^i &= \tilde\beta_y{}^i + \gamma_\chi\, y^i + y^i\,\gamma_\psi + y^j\gamma_{\phi,j}{}^i\,, \\
\beta_V &= \tilde\beta_V + V^j\gamma_{\phi\,j}{}^i\phi_i + \bar\phi^i\gamma_{\phi\,i}{}^j V_j\,,
\end{aligned}
\tag{5.4}
$$

for

$$V^j = \frac{\partial}{\partial \phi_j} V \, , \qquad V_j = \frac{\partial}{\partial \bar\phi^j} V \, . \tag{5.5}$$

In addition

$$\beta_{\bar y i} = (\beta_y{}^i)^\dagger \, . \tag{5.6}$$

In giving results for β and related functions it is convenient to rescale

$$\lambda_{ij}{}^{kl} \to 16\pi^2 \lambda_{ij}{}^{kl} \, , \quad y^i \to 4\pi \, y^i \, , \quad \bar y_i \to 4\pi \, \bar y_i \, , \tag{5.7}$$

thereby removing factors of $1/16\pi^2$ which arise at each loop order. The anomalous dimension matrices at one and two loops are given by

$$\gamma_\chi{}^{(1)} = \tfrac{1}{2} \, y^j \bar y_j \, , \qquad \gamma_\psi{}^{(1)} = \tfrac{1}{2} \, \bar y_j y^j \, , \qquad \gamma_\phi{}^{(1)}{}_j{}^i = \mathrm{tr}(\bar y_j y^i) \, , \tag{5.8}$$

and

$$\begin{aligned} \gamma_\chi{}^{(2)} &= - \tfrac{1}{8} \, y^i \bar y_j \, y^j \bar y_i - \tfrac{3}{4} \, \mathrm{tr}(y^j \bar y_i) \, y^i \bar y_j \, , \\ \gamma_\psi{}^{(2)} &= - \tfrac{1}{8} \, \bar y_i \, y^j \bar y_j \, y^i - \tfrac{3}{4} \, \mathrm{tr}(y^j \bar y_i) \, \bar y_j y^i \, , \\ \gamma_\phi{}^{(2)}{}_j{}^i &= \tfrac{1}{4} \, \lambda_{jk}{}^{mn} \lambda_{mn}{}^{ki} - \tfrac{3}{4} \big(\mathrm{tr}(\bar y_j \, y^k \bar y_k \, y^i) + \mathrm{tr}(\bar y_k \, y^k \, \bar y_j \, y^i) \big) \, . \end{aligned} \tag{5.9}$$

The β-functions are then given by (5.4) with [26]

$$\begin{aligned} \tilde\beta_y^{(1)i} &= 0 \, , \qquad \tilde\beta_y^{(2)i} = 2 \, y^j \bar y_k \, y^i \bar y_j \, y^k - 2 \, \lambda_{jk}{}^{li} \, y^j \bar y_l \, y^k \, , \\ \tilde\beta_V^{(1)} &= \tfrac{1}{2} \, V_{rs} V^{rs} - 2 \, \mathrm{tr}(m \, \bar m \, m \, \bar m) \, , \\ \tilde\beta_V^{(2)} &= - \tfrac{1}{2} \, V_{rst} V^{rs}{}_u V^{tu} - 2 \, \mathrm{tr}(\bar y_i \, y^j) \big(V^{ik} V_{kj} + V_j{}^k V_k{}^i \big) \\ &\quad + 2 \, \mathrm{tr}(y^k \, \bar m \, y^l \bar m) \, V_{kl} + 2 \, \mathrm{tr}(\bar y_k \, m \, \bar y_l \, m) \, V^{kl} \\ &\quad + 2 \big(\mathrm{tr}(y^k \bar y_k \, m \, \bar m \, m \, \bar m) + \mathrm{tr}(\bar y_k \, y^k \bar m \, m \, \bar m \, m) + 2 \, \mathrm{tr}(y^k \bar m \, m \, \bar y_k \, m \, \bar m) \big) \, , \end{aligned} \tag{5.10}$$

where $a_r b^r = a^i b_i + a_i b^i$ and V_{rs}, V_{rst} are defined by obvious extensions of (5.5). In consequence $\frac{1}{2} \, V_{rs} V^{rs} = V_{ij} V^{ij} + V_i{}^j V_j{}^i$.

Two special cases are of particular interest. Assuming $n_\chi = r$, $n_\psi = rn$, $n_\phi = n$ we require

$$m(\phi)\psi = y \, \phi_i \psi^i \, , \quad \bar\psi \, m(\bar\phi) = \bar y \, \bar\psi_i \bar\phi^i \, , \quad V(\bar\phi, \phi) = \tfrac{1}{2} \lambda \, (\bar\phi^i \phi_i)^2 \, , \tag{5.11}$$

and there is then a manifest $U(n)$ symmetry (for the scalar couplings the symmetry extends to $O(2n)$), with $\chi, \bar\chi$ singlets, and the couplings reduce to just $\lambda, y, \bar y$. In the above formulae

$$\lambda_{ij}{}^{kl} \to \lambda(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) \, , \qquad y^i \, \bar y_j \to \bar y y \, \delta_j{}^i \, , \qquad \bar y_i y^i \to \bar y y \, 1_n \, . \tag{5.12}$$

The anomalous dimensions are no longer matrices and from the above we get

$$\begin{aligned}
&\gamma_\psi{}^{(1)} = \tfrac{1}{2}\,\bar y y\,, \qquad \gamma_\chi{}^{(1)} = \tfrac{1}{2} n\,\bar y y\,, \qquad \gamma_\phi{}^{(1)} = r\,\bar y y\,, \\
&\gamma_\psi{}^{(2)} = -\tfrac{1}{8}(6r+n)\,(\bar y y)^2\,, \qquad \gamma_\chi{}^{(2)} = -\tfrac{1}{8}(6r+1)n\,(\bar y y)^2\,, \\
&\gamma_\phi{}^{(2)} = (n+1)\big(\tfrac{1}{2}\lambda^2 - \tfrac{3}{4} r\,(\bar y y)^2\big)\,,
\end{aligned} \tag{5.13}$$

with r arising from the trace due to additional fermion degrees of freedom. Furthermore from (5.10)

$$\begin{aligned}
&\tilde\beta_y{}^{(1)} = 0\,, \qquad \tilde\beta_y{}^{(2)} = 2\big((\bar y y)^2 - (n+1)\,\bar y y\,\lambda\big)y\,, \qquad \tilde\beta_\lambda{}^{(1)} = 2(n+4)\,\lambda^2 - 4r\,(\bar y y)^2\,, \\
&\tilde\beta_\lambda{}^{(2)} = -4(5n+11)\lambda^3 - 4(n+4)\,\lambda^2 \bar y y + 8r\,\lambda(\bar y y)^2 + 4(n+3)r\,(\bar y y)^3\,,
\end{aligned} \tag{5.14}$$

where now

$$\beta_y = \tilde\beta_y + (\gamma_\chi + \gamma_\psi + \gamma_\phi)\,y\,, \qquad \beta_\lambda = \tilde\beta_\lambda + 4\gamma_\phi\,\lambda\,. \tag{5.15}$$

Combining (5.14) and (5.13) for $n=2$ reproduces standard model results in [17].[3]

The other special case corresponds to $\mathcal{N}=1$ supersymmetry. This is achieved by letting $n_\psi = n_\phi = n_C$ and imposing

$$\bar\chi \to \tilde\psi = \psi^T C\,, \qquad \chi \to \tilde{\bar\psi} = -C^{-1}\bar\psi^T\,, \tag{5.16}$$

with $C^T = -C\ C\bar\sigma C^{-1} = -\sigma^T$, and then rescaling $\psi, \bar\psi$ to achieve a canonical kinetic term. ϕ_i, ψ_i and $\bar\phi^i, \bar\psi^i$ form n_C chiral supermultiplets and a general renormalisable $\mathcal{N}=1$ supersymmetric Lagrangian is achieved by letting

$$\begin{aligned}
&V(\bar\phi,\phi) = u^i(\phi)\,\bar u_i(\bar\phi)\,, \quad m^{ij}(\phi) = u^{i,j}(\phi) = m^{ji}(\phi)\,, \quad \bar m_{ij}(\bar\phi) = \bar u_{i,j}(\bar\phi) = \bar m_{ji}(\bar\phi)\,, \\
&Y^{ijk} = u^{i,jk} = Y^{(ijk)}\,, \qquad \bar Y_{ijk} = \bar u_{i,jk} = \bar Y_{(ijk)}\,, \qquad \lambda_{ij}{}^{kl} = \bar Y_{ijm} Y^{mkl}\,.
\end{aligned} \tag{5.17}$$

(5.16) is compatible with (5.3) if $\tau = -\theta$ so that $U(1)\times U(1) \to U(1)_R$ corresponding to the usual R-symmetry. Standard supersymmetry results based on superspace ensure that the β-functions are determined in terms of the anomalous dimension

$$\beta_Y^{ijk} = Y^{ljk}\gamma_l{}^i + Y^{ilk}\gamma_l{}^j + Y^{ijl}\gamma_l{}^k\,, \qquad \beta_{\bar Y\, ijk} = \gamma_i{}^l\,\bar Y_{ljk} + \gamma_j{}^l\,\bar Y_{ilk} + \gamma_k{}^l\,\bar Y_{ijl}\,. \tag{5.18}$$

Hence with the definitions (5.4)

$$\tilde\beta_Y = 0\,, \qquad \tilde\beta_V(\phi,\bar\phi) = 2\,u^i(\phi)\,\gamma_i{}^j\,\bar u_j(\bar\phi)\,. \tag{5.19}$$

[3] Assuming (5.12) the detailed relation with the results of [17] at each loop order ℓ is given by $\beta_\lambda{}^{(\ell)}|_{n=2} = 4\,\beta_\lambda{}^{(\ell)}|_{\lambda\to\frac{1}{2}\lambda,\,g_s=0}$, $\beta_y{}^{(\ell)}|_{n=2} = 2\,\beta_{y_t}{}^{(\ell)}|_{\lambda\to\frac{1}{2}\lambda,\,g_s=0}$, $\gamma_\psi{}^{(\ell)}|_{n=2} = \gamma^t_{2,L}{}^{(\ell)}|_{\lambda\to\frac{1}{2}\lambda,\,g_s=0}$, $\gamma_\chi{}^{(\ell)}|_{n=2} = \gamma^t_{2,R}{}^{(\ell)}|_{\lambda\to\frac{1}{2}\lambda,\,g_s=0}$ and $\gamma_\phi{}^{(\ell)}|_{n=2} = \gamma^\Phi_{2,L}{}^{(\ell)}|_{\lambda\to\frac{1}{2}\lambda,\,g_s=0}$ where $\bar y = y = y_t$ and $r = d_R$.

The results for anomalous dimensions and beta functions for (5.1) with (5.17) reduce to the supersymmetric form so long as the coefficient of all traces, which each correspond to a fermion loop, have an additional coefficient $\frac{1}{2}$. This reflects the restriction (5.16). Then we have

$$\gamma_{\psi\, i}{}^{j} = \gamma_{\phi\, i}{}^{j} = \gamma_i{}^{j} \, , \qquad \gamma_\chi{}^{j}{}_i = \bar\gamma^{j}{}_i \, , \tag{5.20}$$

With the modification of the trace coefficients the results (5.8) and (5.9) are compatible with (5.20) for

$$\gamma^{(1)}{}_i{}^j = \tfrac{1}{2}\, (\bar Y Y)_i{}^j \, , \qquad \gamma^{(2)}{}_i{}^j = -\tfrac{1}{2}\, \bar Y_{ikl}\, (Y\bar Y)^l{}_m\, Y^{mkj} \, . \tag{5.21}$$

The results in (5.13) and (5.14) also correspond to a single field supersymmetric theory for $n = 1$, $r = \frac{1}{2}$ if $\lambda = \frac{1}{2}\, \bar y y$.

At three-loop order the general expressions for the anomalous dimensions are restricted to correspond to one particle irreducible graphs and have the form for the fermions

$$\begin{aligned}
\gamma_\chi{}^{(3)} &= a\, y^i \bar y_j\, y^j \bar y_k\, y^k \bar y_i + b\, y^i \bar y_j\, y^k \bar y_k\, y^j \bar y_i + c\, y^i \bar y_j\, y^k \bar y_i\, y^j \bar y_k \\
&\quad + d\, y^i \bar y_j\, \lambda_{im}{}^{kl} \lambda_{kl}{}^{mj} + e\, y^i \bar y_k\, y^j \bar y_l\, \lambda_{ij}{}^{kl} \\
&\quad + f\, \big(\mathrm{tr}(y^j \bar y_k\, y^k \bar y_i) + \mathrm{tr}(y^k \bar y_k\, y^j \bar y_i)\big)\, y^i \bar y_j \\
&\quad + g\, \mathrm{tr}(y^j \bar y_i)\, y^i \bar y_k\, y^k \bar y_j + h\, \mathrm{tr}(y^j \bar y_i)\, y^k \bar y_j\, y^i \bar y_k \\
&\quad + i\, \mathrm{tr}(y^j \bar y_k)\, \mathrm{tr}(y^k \bar y_i)\ y^i \bar y_j \, , \\
\gamma_\psi{}^{(3)} &= a\, \bar y_i\, y^j \bar y_j\, y^k \bar y_k\, y^i + b\, \bar y_i\, y^j \bar y_k\, y^k \bar y_j\, y^i + c\, \bar y_i\, y^j \bar y_k\, y^i \bar y_j\, y^k \\
&\quad + d\, \bar y_j\, y^j\, \lambda_{im}{}^{kl} \lambda_{kl}{}^{mj} + e\, \bar y_k\, y^i \bar y_l\, y^j\, \lambda_{ij}{}^{kl} \\
&\quad + f\, \big(\mathrm{tr}(y^j \bar y_k\, y^k \bar y_i) + \mathrm{tr}(y^k \bar y_k\, y^j \bar y_i)\big)\, \bar y_j\, y^i \\
&\quad + g\, \mathrm{tr}(y^j \bar y_i)\, \bar y_j\, y^k \bar y_k\, y^i + h\, \mathrm{tr}(y^j \bar y_i)\, \bar y_k\, y^i \bar y_j\, y^k \\
&\quad + i\, \mathrm{tr}(y^j \bar y_k)\, \mathrm{tr}(y^k \bar y_i)\ \bar y_j\, y^i \, ,
\end{aligned} \tag{5.22}$$

and for the scalar field

$$\begin{aligned}
\gamma_\phi{}^{(3)}{}_j{}^i &= a' \, \big(\lambda_{jk}{}^{mn} \lambda_{mn}{}^{pq} \lambda_{pq}{}^{ki} + 4\, \lambda_{jk}{}^{mn} \lambda_{ml}{}^{kp} \lambda_{np}{}^{li}\big) \\
&\quad + b' \, \big(\lambda_{jk}{}^{mn} \lambda_{mn}{}^{li} + 2\, \lambda_{jm}{}^{ln} \lambda_{kn}{}^{mi}\big)\, \mathrm{tr}(y^k \bar y_l) \\
&\quad + c' \, \big(\mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^m)\, \lambda_{km}{}^{li} + \lambda_{jl}{}^{km}\, \mathrm{tr}(\bar y_k\, y^l\, \bar y_m\, y^i)\big) \\
&\quad + d' \, \big(\mathrm{tr}(\bar y_j\, y^k \bar y_k\, y^l\, \bar y_l\, y^i) + \mathrm{tr}(\bar y_k\, y^k \bar y_l\, y^l\, \bar y_j\, y^i)\big) \\
&\quad + e' \, \big(\mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^l\, \bar y_k\, y^i) + \mathrm{tr}(\bar y_k\, y^l \bar y_l\, y^k\, \bar y_j\, y^i)\big) \\
&\quad + f' \, \mathrm{tr}(\bar y_k\, y^k \bar y_j\, y^l\, \bar y_l\, y^i) + g' \, \mathrm{tr}(\bar y_k\, y^l \bar y_j\, y^k \bar y_l\, y^i) \\
&\quad + h' \, \big(\mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^i) + \mathrm{tr}(\bar y_l\, y^k\, \bar y_j\, y^i)\big)\, \mathrm{tr}(y^l\, \bar y_k) \, .
\end{aligned} \tag{5.23}$$

The individual contributions in (5.22) and (5.23) are all hermitian except for those involving the coefficient c' where the two terms are hermitian conjugates. Furthermore the expressions are constrained by $\gamma_\chi \leftrightarrow \gamma_\psi$ and ${\gamma_\phi}^{(3)}{}_j{}^i \to {\gamma_\phi}^{(3)}{}_i{}^j$ for $y^i \leftrightarrow \bar{y}_i$, $\lambda_{ij}{}^{kl} \to \lambda_{kl}{}^{ij}$ everywhere.

Restricting to the $U(n)$ case given by (5.12)

$$\begin{aligned}
{\gamma_\chi}^{(3)} &= n\big(a + n\,b + c + r(n+1)\,f + r\,(g+h) + r^2\,i\big)(\bar{y}y)^3 + n(n+1)\big(2d\,\lambda^2\bar{y}y + e\,\lambda(\bar{y}y)^2\big)\,, \\
{\gamma_\psi}^{(3)} &= \big(n^2\,a + n\,b + c + r(n+1)\,f + rn\,(g+h) + r^2\,i\big)(\bar{y}y)^3 + (n+1)\big(2d\,\lambda^2\bar{y}y + e\,\lambda(\bar{y}y)^2\big)\,, \\
{\gamma_\phi}^{(3)} &= 2(n+1)\big(2(n+4)\,a'\,\lambda^3 + 3r\,b'\,\lambda^2\bar{y}y + r\,c'\,\lambda(\bar{y}y)^2\big) \\
&\quad + r\big((n^2+1)\,d' + 2n\,e' + n\,f' + g' + r(n+1)\,h'\big)(\bar{y}y)^3\,. \qquad (5.24)
\end{aligned}$$

Comparing with [17] for $n=2$ we may obtain

$$a = -\tfrac{5}{32}\,, \quad 2b+c = -\tfrac{7}{8}+\tfrac{3}{2}\zeta(3)\,, \quad d = -\tfrac{11}{32}\,, \quad e = f = 1\,, \quad g+h = \tfrac{5}{16}\,, \quad i = -\tfrac{3}{8}\,, \qquad (5.25)$$

and

$$a' = -\tfrac{1}{16}\,, \quad b' = -\tfrac{5}{16}\,, \quad c' = \tfrac{5}{4}\,, \quad h' = 2\,, \quad 5d' + 4e' + 2f' + g' = -\tfrac{25}{16} + 3\zeta(3)\,. \qquad (5.26)$$

The graphs associated with a', b', c' were calculated in [14], the numerical values given are consistent with (5.26) if an additional factor of 2 for fermion loops is supplied due to the absence of a symmetry factor here.

In the supersymmetric case given by (5.17) there are four independent terms [18] so that

$$\begin{aligned}
\gamma^{(3)}{}_i{}^j = {}& \bar{Y}_{ikl}\big(A\,(Y\bar{Y})^l{}_m(Y\bar{Y})^m{}_n + C\,Y^{lmp}\,(\bar{Y}Y)_p{}^q\,\bar{Y}_{qmn}\big)Y^{nkj} \\
&+ \bar{Y}_{ikl}\big(B\,(Y\bar{Y})^k{}_m(Y\bar{Y})^l{}_n + D\,Y^{kps}\,Y^{lqr}\,\bar{Y}_{prm}\,\bar{Y}_{qsn}\big)Y^{mnj}\,.
\end{aligned} \qquad (5.27)$$

From (5.22) and (5.23)

$$\begin{aligned}
&A = a + \tfrac{1}{4}i = a' + d'\,, && B = \tfrac{1}{2}g = \tfrac{1}{2}(b' + f')\,, \\
&C = b + d + f + \tfrac{1}{2}h = b' + e' + \tfrac{1}{2}h'\,, && D = c + e = 2a' + c' + \tfrac{1}{2}g'\,.
\end{aligned} \qquad (5.28)$$

According to [18]

$$A = -\tfrac{1}{4}\,, \qquad B = -\tfrac{1}{8}\,, \qquad C = 1\,, \qquad D = \tfrac{3}{2}\zeta(3)\,. \qquad (5.29)$$

This resolves the freedom present in (5.25) by requiring in addition

$$b = \tfrac{1}{16}\,, \qquad c = -1 + \tfrac{3}{2}\zeta(3)\,, \qquad g = -\tfrac{1}{4}\,, \qquad h = \tfrac{9}{16}\,, \qquad (5.30)$$

with two additional linear constraints on the coefficients also satisfied. If the results for a',b',c',h' in (5.26) are used in (5.28) with (5.29) then

$$d' = -\tfrac{3}{16}\,, \qquad e' = \tfrac{5}{16}\,, \qquad f' = \tfrac{1}{16}\,, \qquad g' = -2 + 3\zeta(3)\,. \tag{5.31}$$

With these values $5d' + 4e' + 2f' + g'$ is compatible with (5.26) providing a further check.

In a similar fashion we may write

$$\begin{aligned}
\tilde\beta_y{}^{(3)i} &= \alpha\, y^j \bar y_k\, y^l\, \lambda_{jl}{}^{mn}\lambda_{mn}{}^{ki} + \beta\, y^j \bar y_k\, y^l \big(\lambda_{jm}{}^{ni}\lambda_{nl}{}^{km} + \lambda_{jm}{}^{kn}\lambda_{nl}{}^{mi}\big) \\
&\quad + \gamma\,\big(\mathrm{tr}(y^j\bar y_m)\, y^m \bar y_l\, y^k + \mathrm{tr}(y^k\bar y_m)\, y^j \bar y_l y^m\big)\lambda_{jk}{}^{li} + \delta\,\mathrm{tr}(\bar y_l y^m)\, y^j \bar y_m\, y^k\, \lambda_{jk}{}^{li} \\
&\quad + \epsilon\,\big(y^k \bar y_m\, y^m \bar y_j\, y^l + y^k \bar y_j\, y^m \bar y_m y^l\big)\,\lambda_{kl}{}^{ji} + \eta\,\big(y^m \bar y_j\, y^k \bar y_m\, y^l + y^k \bar y_m\, y^l \bar y_j\, y^m\big)\,\lambda_{kl}{}^{ji} \\
&\quad + \zeta\, y^k \bar y_m\, y^i \bar y_n\, y^l\, \lambda_{kl}{}^{mn} \\
&\quad + \iota\,\big(y^j \bar y_l\, y^i \bar y_k\, y^l + y^l \bar y_k\, y^i \bar y_l\, y^j\big)\mathrm{tr}(y^k \bar y_j) + \kappa\,\big(y^j \bar y_k\, y^l + y^l \bar y_k\, y^j\big)\,\mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^i) \\
&\quad + \mu\,\big(y^k \bar y_j\, y^j \bar y_l\, y^i \bar y_k\, y^l + y^k \bar y_l\, y^i \bar y_k\, y^j \bar y_j\, y^l\big) \\
&\quad + \nu\,\big(y^k \bar y_l\, y^j \bar y_j\, y^i \bar y_k\, y^l + y^k \bar y_l\, y^i \bar y_j\, y^j \bar y_k\, y^l\big) \\
&\quad + \theta\,\big(y^j \bar y_k\, y^l \bar y_j\, y^i \bar y_l\, y^k + y^k \bar y_l\, y^i \bar y_j\, y^l \bar y_k\, y^j\big)\,.
\end{aligned} \tag{5.32}$$

This reduces to

$$\begin{aligned}
\tilde\beta_y{}^{(3)} &= (n+1)\big(2\alpha + (n+3)\beta\big)\,\lambda^2 \bar y y\, y + (n+1)(2\gamma+\delta)\, r\lambda(\bar y y)^2\, y \\
&\quad + (n+1)\big((n+1)\epsilon + 2\eta + \zeta\big)\,\lambda(\bar y y)^2\, y \\
&\quad + \big(2\iota + (n+1)\kappa\big) r\,(\bar y y)^3\, y + (n+1)(\mu+\nu+\theta)\,(\bar y y)^3\, y\,.
\end{aligned} \tag{5.33}$$

Comparing with [17]

$$2\alpha + 5\beta = 8\,, \quad 2\gamma + \delta = 5\,, \quad 3\epsilon + 2\eta + \zeta = \tfrac{15}{2}\,, \quad 2\iota + 3\kappa = -2\,, \quad \mu + \nu + \theta = -6\,. \tag{5.34}$$

In the supersymmetric case then $\tilde\beta_y{}^{(3)i} = 0$ requires

$$\alpha + \tfrac12\delta + 2\nu = 0\,, \quad \tfrac12\gamma + \epsilon + \tfrac12\iota + \mu = 0\,, \quad \beta + \eta + \tfrac12\zeta + \tfrac12\kappa + \theta = 0\,. \tag{5.35}$$

Each term in (5.32) corresponds to a particular Feynman graph. By calculating the relevant integrals corresponding to individual graphs we found

$$\alpha = \tfrac32\,, \quad \beta = \gamma = 1\,, \quad \delta = 3\,, \quad \epsilon = \tfrac12\,, \quad \eta = \zeta = 2\,, \tag{5.36}$$

which are consistent with the first three relations in (5.34). In [14] those graphs corresponding to $\alpha,\beta,\gamma,\delta,\epsilon,\eta$ were also calculated, the numbers quoted for each graph appear to be in accord with the coefficients in (5.36) up to factors of 2 which are a consequence of the different symmetry factors for the theory considered here. By using (5.34) and also (5.35) with (5.36) it is easy to obtain

$$\iota = -1\,, \quad \kappa = 0\,, \quad \mu = -\tfrac12\,, \quad \nu = -\tfrac32\,, \quad \theta = -4\,, \tag{5.37}$$

so that the three-loop Yukawa beta function for the theory described the lagrangian (5.1) is fully determined.

6. Gradient Flow Properties

Based on the results for the scalar fermion β-functions we explore at low loop order the constraints arising from the flow equation (1.1). Here we initially neglect the distinction between the standard perturbative β-function and the modified B-function given by (1.7). If $T_{IJ} = G_{IJ}$ is symmetric and G_{IJ} is positive definite then (1.1) defines a gradient flow. For purely scalar theories a gradient flow was postulated and investigated by Wallace and Zia [27], who showed how G_{IJ} may be found by diagrammatic arguments to quite high loop order. In general an antisymmetric part in T_{IJ} is necessary to ensure (1.1) remains valid under the equivalence relations (2.76) which correspond to the freedom in (1.3) and (1.4).

We assume here the lowest order results found in [9] determining G_{IJ}. Applied to the theory defined by (5.1), so that $g^I = \{y^i, \bar{y}_i, \lambda_{ij}{}^{kl}\}$, then at two-loop order

$$T_{IJ}{}^{(2)}\, \mathrm{d}g^I \mathrm{d}'g^J = G_{IJ}{}^{(2)}\, \mathrm{d}g^I \mathrm{d}'g^J = \tfrac{1}{3}\left(\mathrm{tr}(\mathrm{d}y^i\, \mathrm{d}'\bar{y}_i) + \mathrm{tr}(\mathrm{d}\bar{y}_i\, \mathrm{d}'y^i)\right), \tag{6.1}$$

for $\mathrm{d}g^I = \{\mathrm{d}y^i, \mathrm{d}\bar{y}_i, \mathrm{d}\lambda_{ij}{}^{kl}\}$, $\mathrm{d}'g^I = \{\mathrm{d}'y^i, \mathrm{d}'\bar{y}_i, \mathrm{d}'\lambda_{ij}{}^{kl}\}$. With the one-loop result for $\beta_y{}^i$ given by (5.4) and (5.8)

$$\tilde{A}^{(3)} = \tfrac{1}{12}\left(\mathrm{tr}(\bar{y}_i\, y^i \bar{y}_j\, y^j) + \mathrm{tr}(y^i \bar{y}_i\, y^j \bar{y}_j)\right) + \tfrac{1}{6}\, \mathrm{tr}(\bar{y}_i\, y^j)\, \mathrm{tr}(\bar{y}_j\, y^i)\,. \tag{6.2}$$

At the next order the three-loop contribution to T_{IJ} must be of the general form

$$\begin{aligned} T_{IJ}{}^{(3)} \mathrm{d}g^I \mathrm{d}'g^J = {} & \tfrac{1}{24}\, \mathrm{d}\lambda_{ij}{}^{kl}\, \mathrm{d}'\lambda_{kl}{}^{ij} \\ & + \Big(\bar{\alpha}\left(\mathrm{tr}(\mathrm{d}\bar{y}_i\, \mathrm{d}'y^i\, \bar{y}_j\, y^j) + \mathrm{tr}(\mathrm{d}\bar{y}_i\, y^j \bar{y}_j\, \mathrm{d}'y^i)\right) \\ & \quad + \bar{\beta}\left(\mathrm{tr}(\mathrm{d}\bar{y}_i\, \mathrm{d}'y^j\, \bar{y}_j\, y^i) + \mathrm{tr}(\mathrm{d}\bar{y}_i\, y^i \bar{y}_j\, \mathrm{d}'y^j)\right) \\ & \quad + \bar{\gamma}\left(\mathrm{tr}(\mathrm{d}\bar{y}_i\, y^i\, \mathrm{d}'\bar{y}_j\, y^j) + \mathrm{tr}(\mathrm{d}\bar{y}_i\, y^j\, \mathrm{d}'\bar{y}_j\, y^i)\right) \\ & \quad + \bar{\delta}\, \mathrm{tr}(\mathrm{d}\bar{y}_i\, \mathrm{d}'y^j)\, \mathrm{tr}(\bar{y}_j\, y^i) + \bar{\eta}\, \mathrm{tr}(\mathrm{d}\bar{y}_i\, y^j)\, \mathrm{tr}(\bar{y}_j\, \mathrm{d}'y^i) \\ & \quad + \bar{\epsilon}\, \mathrm{tr}(\mathrm{d}\bar{y}_i\, y^j)\, \mathrm{tr}(\mathrm{d}'\bar{y}_j\, y^i) \qquad\qquad + \text{conjugate}\Big)\,, \end{aligned} \tag{6.3}$$

where the first term was calculated in [9]. The remaining terms correspond to three-loop vacuum diagrams, with one and two fermion loops, with two vertices selected. The result is also required to be invariant under conjugation when $y \leftrightarrow \bar{y}$. Although this is not imposed the expression (6.3) is symmetric under $\mathrm{d}g^I \leftrightarrow \mathrm{d}'g^I$ so that at this order $T_{IJ}{}^{(3)} = G_{IJ}{}^{(3)}$.

The real coefficients $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\eta}, \bar{\epsilon}$ in (6.3) have not been determined hitherto. Without explicit determination the integrability conditions necessary for (1.1) provide constraints on these coefficients and also on the β-functions themselves, as was also demonstrated to two-loop order in [9]. The dependence of $\tilde{A}^{(4)}$ on λ is determined in terms of $\beta_\lambda{}^{(1)}$ and then this fixes the λ-dependent terms in $\beta_y{}^{(2)}$. Using the results for $\beta_\lambda{}^{(1)}$

$$\begin{aligned}\beta_\lambda{}^{(1)}{}_{ij}{}^{kl} = {}& \lambda_{ij}{}^{mn}\lambda_{mn}{}^{kl} + 4\,\lambda_{m(i}{}^{n(k}\lambda_{j)n}{}^{l)m} + 2\,\mathrm{tr}\big(\bar{y}_{(i}\,y^m\big)\,\lambda_{j)m}{}^{kl} + 2\,\lambda_{ij}{}^{m(k}\,\mathrm{tr}\big(\bar{y}_m\,y^{l)}\big) \\ & - 8\,\mathrm{tr}\big(\bar{y}_{(i}\,y^{(k}\,\bar{y}_{j)}\,y^{l)}\big)\,, \end{aligned} \tag{6.4}$$

and $\beta_y{}^{(2)}$ from (5.9) and (5.10) in (1.1), with (6.1) and (6.3), requires the three integrability conditions on $\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta},\bar{\eta},\bar{\epsilon}$

$$2(\bar{\beta}+\bar{\gamma}) = 4\bar{\alpha}+\tfrac{1}{6} = 2\bar{\alpha}+\bar{\delta}+\tfrac{1}{2} = \bar{\eta}+\bar{\epsilon}\,. \tag{6.5}$$

Subject to these conditions

$$\begin{aligned}\tilde{A}^{(4)} = {}& \tfrac{1}{72}\,\big(\lambda_{ij}{}^{kl}\lambda_{kl}{}^{mn}\lambda_{mn}{}^{ij} + 4\,\lambda_{ij}{}^{kl}\lambda_{km}{}^{in}\lambda_{ln}{}^{jm}\big) \\ & + \tfrac{1}{12}\,\lambda_{ij}{}^{kl}\,\mathrm{tr}(\bar{y}_l\,y^m)\,\lambda_{km}{}^{ij} - \tfrac{1}{3}\,\lambda_{ij}{}^{kl}\mathrm{tr}(\bar{y}_k\,y^i\bar{y}_l\,y^j) \\ & + \tfrac{2}{9}\,\mathrm{tr}(\bar{y}_i\,y^j\bar{y}_k\,y^i\bar{y}_j\,y^k) + \tfrac{1}{72}\,\big(\mathrm{tr}(\bar{y}_i\,y^i\,\bar{y}_j\,y^j\,\bar{y}_k\,y^k) + \mathrm{tr}(y^i\bar{y}_i\,y^j\bar{y}_j\,y^k\bar{y}_k)\big) \\ & - \tfrac{1}{6}\big(\mathrm{tr}(y^i\bar{y}_i\,y^k\bar{y}_j) + \mathrm{tr}(\bar{y}_i\,y^i\,\bar{y}_j\,y^k)\big)\,\mathrm{tr}(\bar{y}_k\,y^j) - \tfrac{1}{18}\,\mathrm{tr}(\bar{y}_i\,y^j)\,\mathrm{tr}(\bar{y}_j\,y^k)\,\mathrm{tr}(\bar{y}_k\,y^i) \\ & + 2\bar{\alpha}\,\mathrm{tr}\big(\beta_y{}^{(1)i}\beta_{\bar{y}}{}^{(1)}{}_i\big)\,. \end{aligned} \tag{6.6}$$

Precise results for $G^{(3)}{}_{IJ}$ can be obtained in terms of flat space calculations by applying (2.74), noting that $\mathcal{D}_K A_{IJ}$ is zero at three loops. This gives, with the aid of results from section 9,

$$\bar{\alpha} = -\tfrac{13}{72}\,,\quad \bar{\beta} = -\tfrac{5}{18}\,,\quad \bar{\gamma} = 0\,,\quad \bar{\delta} = -\tfrac{25}{36}\,,\quad \bar{\eta} = -\tfrac{7}{18}\,,\quad \bar{\epsilon} = -\tfrac{1}{6}\,. \tag{6.7}$$

These of course satisfy (6.5). The freedom associated with (2.76) corresponding to letting $\tilde{A} \to \tilde{A} + z\ \mathrm{tr}(\beta_y{}^i\beta_{\bar{y}\,i})$ is realised at this order by

$$\bar{\alpha} \sim \bar{\alpha} + \tfrac{1}{2}z\,,\quad \bar{\beta} \sim \bar{\beta} + z\,,\quad \bar{\delta} \sim \bar{\delta} + z\,,\quad \bar{\eta} \sim \bar{\eta} + 2z\,, \tag{6.8}$$

under which (6.5) is invariant. In this case we have correspondingly

$$W_I{}^{(3)}\mathrm{d}g^I \sim W_I{}^{(3)}\mathrm{d}g^I + \mathrm{d}\,\tfrac{1}{4}z\big(\mathrm{tr}(\bar{y}_i\,y^j\bar{y}_j\,y^i) + \mathrm{tr}(y^i\bar{y}_j\,y^j\bar{y}_i) + 2\,\mathrm{tr}(\bar{y}_i\,y^j)\,\mathrm{tr}(\bar{y}_j\,y^i)\big)\,. \tag{6.9}$$

Higher order results become more involved. At the next order the metric for the purely scalar couplings has the general form

$$G_{IJ}{}^{(4)}\,\mathrm{d}g^I\mathrm{d}g^J\big|_{\lambda\lambda} = \bar{G}\,\big(\lambda_{ij}{}^{mn}\,\mathrm{d}\lambda_{mn}{}^{kl}\,\mathrm{d}\lambda_{kl}{}^{ij} + 4\,\lambda_{im}{}^{kn}\,\mathrm{d}\lambda_{jn}{}^{lm}\,\mathrm{d}\lambda_{kl}{}^{ij}\big)\,, \tag{6.10}$$

where $\bar{G}$ is essentially arbitrary due to the freedom in (1.4) but has been calculated in a minimal subtraction scheme below. The λ-terms do not generate any consistency conditions, in accord with [27], giving

$$\tilde{A}^{(5)}\big|_{\lambda} = \tfrac{1}{96}\,\lambda_{ij}{}^{kl}\lambda_{kl}{}^{mn}\lambda_{mn}{}^{pq}\lambda_{pq}{}^{ij} - \tfrac{1}{12}\,\lambda_{ij}{}^{kl}\big(\lambda_{kl}{}^{mn}\lambda_{mp}{}^{iq}\lambda_{nq}{}^{jp} + \lambda_{km}{}^{jn}\lambda_{np}{}^{iq}\lambda_{lq}{}^{mp}\big) \\ + \tfrac{1}{4}\bar{G}\,\beta_{\lambda}{}^{(1)}{}_{ij}{}^{kl}\beta_{\lambda}{}^{(1)}{}_{kl}{}^{ij}\,. \tag{6.11}$$

With the results for β-functions in the previous section we may extend these results to include mixed scalar Yukawa contributions for the theory defined by (5.1). There is then an additional four loop contribution so that instead of (6.10)

$$G_{IJ}{}^{(4)}\,\mathrm{d}g^I\mathrm{d}g^J\big|_{\lambda\lambda} = \bar{G}\,\big(\lambda_{ij}{}^{mn}\,\mathrm{d}\lambda_{mn}{}^{kl}\,\mathrm{d}\lambda_{kl}{}^{ij} + 4\,\lambda_{im}{}^{kn}\,\mathrm{d}\lambda_{jn}{}^{lm}\,\mathrm{d}\lambda_{kl}{}^{ij}\big) \\ + \bar{H}\,\mathrm{d}\lambda_{ij}{}^{kl}\,\mathrm{tr}(\bar{y}_l\,y^m)\,\mathrm{d}\lambda_{km}{}^{ij}\,. \tag{6.12}$$

In addition we assume

$$T_{IJ}{}^{(4)}\,\mathrm{d}g^I\mathrm{d}'g^J\big|_{\lambda y} = \bar{A}\,\mathrm{d}\lambda_{ij}{}^{kl}\,\lambda_{kl}{}^{im}\,\mathrm{tr}(\bar{y}_m\,\mathrm{d}'y^j) + \bar{B}\,\mathrm{d}\lambda_{ij}{}^{kl}\,\mathrm{tr}(\bar{y}_l\,\mathrm{d}'y^m)\,\lambda_{km}{}^{ij} \\ + \bar{C}\,\mathrm{d}\lambda_{ij}{}^{kl}\,\mathrm{tr}(\bar{y}_k\,y^i\bar{y}_l\,\mathrm{d}'y^j)\,, \tag{6.13}$$

with a corresponding result for $T_{IJ}{}^{(4)}\,\mathrm{d}g^I\mathrm{d}'g^J\big|_{\lambda\bar{y}}$. In terms of (6.12) and (6.13), using the one and two loop β-functions from the previous section,

$$\begin{aligned}\tilde{A}^{(5)}\big|_{\lambda y\bar{y}} = {}& \tfrac{2}{3}\,\lambda_{ij}{}^{kl}\,\mathrm{tr}(\bar{y}_k\,y^m\bar{y}_l\,y^i\,\bar{y}_m\,y^j) \\ &+ (\bar{C}+\tfrac{1}{3})\,\lambda_{ij}{}^{kl}\,\big(\mathrm{tr}(\bar{y}_m\,y^m\bar{y}_k\,y^i\,\bar{y}_l\,y^j) + \mathrm{tr}(y^m\bar{y}_m\,y^i\,\bar{y}_k\,y^j\,\bar{y}_l)\big) \\ &+ (\bar{C}+\tfrac{2}{3})\,\lambda_{ij}{}^{kl}\,\big(\mathrm{tr}(\bar{y}_k\,y^m)\,\mathrm{tr}(\bar{y}_m\,y^i\,\bar{y}_l\,y^j) + \mathrm{tr}(\bar{y}_k\,y^i\,\bar{y}_l\,y^m)\,\mathrm{tr}(\bar{y}_m\,y^j)\big) \\ &+ \tfrac{1}{6}\,\lambda_{ij}{}^{mn}\lambda_{mn}{}^{kl}\,\mathrm{tr}(\bar{y}_k\,y^i\,\bar{y}_l\,y^j) \\ &+ \tfrac{1}{2}(\bar{A}+\bar{B}-\tfrac{1}{8})\,\lambda_{ik}{}^{lm}\lambda_{lm}{}^{kj}\,\big(\mathrm{tr}(\bar{y}_j\,y^i\,\bar{y}_n\,y^n) + \mathrm{tr}(\bar{y}_j\,y^n\bar{y}_n\,y^i)\big) \\ &+ (\bar{A}+\bar{B}-\tfrac{1}{12})\,\lambda_{ik}{}^{lm}\lambda_{lm}{}^{kj}\,\mathrm{tr}(\bar{y}_j\,y^n)\,\mathrm{tr}(\bar{y}_n\,y^i)\big) \\ &- \tfrac{1}{12}\,\big(\lambda_{ij}{}^{mn}\lambda_{mn}{}^{kl} + 2\,\lambda_{im}{}^{nl}\lambda_{jn}{}^{mk}\big)\,\mathrm{tr}(\bar{y}_k\,y^i)\,\mathrm{tr}(\bar{y}_l\,y^j) \\ &- \tfrac{1}{12}\,\big(\lambda_{ij}{}^{mn}\lambda_{mn}{}^{pq}\lambda_{pq}{}^{jk} + 4\,\lambda_{ij}{}^{mn}\lambda_{mp}{}^{jq}\lambda_{nq}{}^{pk}\big)\,\mathrm{tr}(\bar{y}_k\,y^i) \\ &+ \tfrac{1}{4}\bar{G}\,\beta_{\lambda}{}^{(1)}{}_{ij}{}^{kl}\beta_{\lambda}{}^{(1)}{}_{kl}{}^{ij}\big|_{\lambda y\bar{y}}\,.\end{aligned} \tag{6.14}$$

There is one integrability constraint which is used to eliminate $\bar{H}$,

$$\bar{H} = 2\,\bar{G} - \tfrac{1}{6}\,. \tag{6.15}$$

The result (6.14) may be used to constrain λ contributions to $\beta_y{}^{(3)}$ by considering $\mathrm{d}_{\bar{y}}\tilde{A}^{(5)}$. For generality we must include further possible λ-dependent terms in $T_{IJ}{}^{(4)}$ for which the relevant contributions are

$$T_{IJ}{}^{(4)}\,\mathrm{d}g^I\mathrm{d}'g^J\big|_{\bar{y}\lambda} = \bar{A}'\,\mathrm{tr}(\mathrm{d}\bar{y}_i\,y^m)\,\lambda_{mj}{}^{kl}\,\mathrm{d}'\lambda_{kl}{}^{ji} + \bar{B}'\,\mathrm{tr}(\mathrm{d}\bar{y}_k\,y^m)\,\lambda_{ij}{}^{kl}\,\mathrm{d}'\lambda_{lm}{}^{ij} \\ + \bar{C}'\,\mathrm{tr}(\mathrm{d}\bar{y}_i\,y^k\bar{y}_j\,y^l)\,\mathrm{d}'\lambda_{kl}{}^{ij}\,, \tag{6.16}$$

and

$$\begin{aligned} T_{IJ}{}^{(4)}\,\mathrm{d}g^I \mathrm{d}'g^J\big|_{\bar y y} &= \bar D\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}'y^j)\,\lambda_{jm}{}^{kl}\,\lambda_{kl}{}^{mi} \\ &\quad + \bar E\,\big(\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}'y^k\,\bar y_j\,y^l) + \mathrm{tr}(\mathrm{d}\bar y_i\,y^k \bar y_j\,\mathrm{d}'y^l)\big)\,\lambda_{kl}{}^{ij}\,, \\ T_{IJ}{}^{(4)}\,\mathrm{d}g^I \mathrm{d}'g^J\big|_{\bar y \bar y} &= \bar F\,\mathrm{tr}(\mathrm{d}\bar y_i\,y^k\,\mathrm{d}'\bar y_j\,y^l)\,\lambda_{kl}{}^{ij}\,. \end{aligned} \tag{6.17}$$

If $T_{IJ}{}^{(4)}$ is symmetric then $\bar A' = \bar A$, $\bar B' = \bar B$, $\bar C' = \bar C$.

At this order it is necessary to take into account the potential necessity of modifying the perturbative β-function as in (1.7). For the theory defined by (5.1)

$$v = -v^\dagger = \big\{ {v_\phi}_i{}^j, v_\psi, v_\chi \big\}\,, \tag{6.18}$$

and $(vg)^I$ is obtained by using, for any $v \in \mathfrak{g}_K$,

$$\begin{aligned} (v\,y)^i &= v_\chi y^i - y^i v_\psi - y^j {v_\phi}_j{}^i\,, \quad (v\,\bar y)_i = v_\psi \bar y_i - \bar y_i v_\chi + {v_\phi}_i{}^j \bar y_j\,, \\ (v\,\lambda)_{ij}{}^{kl} &= {v_\phi}_i{}^m \lambda_{mj}{}^{kl} + {v_\phi}_j{}^m \lambda_{im}{}^{kl} - \lambda_{ij}{}^{ml} {v_\phi}_m{}^k - \lambda_{ij}{}^{km} {v_\phi}_m{}^l\,. \end{aligned} \tag{6.19}$$

At three loops all contributions to ${\gamma_\phi}^{(3)}{}_j{}^i, {\gamma_\chi}^{(3)}, {\gamma_\psi}^{(3)}$ in (5.22), (5.23) are separately hermitian except the terms involving c' in (5.23). Hence there is a unique three loop possibility

$${v_\phi}^{(3)}{}_j{}^i = u\big(\mathrm{tr}(\bar y_j\,y^k \bar y_l\,y^m)\,\lambda_{km}{}^{li} - \lambda_{jl}{}^{km}\,\mathrm{tr}(\bar y_k\,y^l\,\bar y_m\,y^i)\big)\,. \tag{6.20}$$

Applying (1.1) for $\mathrm{d}_{\bar y}\tilde A^{(5)}$ given by (6.14) requires combining (6.16) with ${\beta_\lambda}^{(1)}$ and (6.17) with ${\beta_y}^{(1)}$, ${\beta_{\bar y}}^{(1)}$. Using also (6.1) in conjunction with the λ dependent contributions to the three-loop Yukawa beta functions given by (5.32), (5.22), (5.23) and (6.3) for $\bar\gamma = 0$, combined with the corresponding two loop results determined by (5.10) and (5.9), then to $\mathrm{O}(\lambda)$

$$\tfrac{1}{3}\,\eta = \tfrac{1}{3}\,\zeta = \tfrac{2}{3}\,, \tag{6.21}$$

and

$$\begin{aligned} \tfrac{1}{3}\,e - 2\,\bar\beta = {} & -2\,\bar\alpha + \tfrac{1}{2}\,\bar E = \tfrac{1}{3}\,\epsilon + \tfrac{1}{2}(\bar E + \bar F) = \bar C + \tfrac{1}{3}\,, \\ \tfrac{1}{3}\,(c' + u) - 2\,\bar\eta - 8\,\bar B' &= \tfrac{1}{3}\,\delta + \bar C' + \bar F = -2\,\bar\delta + \bar C' \\ = \tfrac{1}{3}\,(c' - u) - 2\,\bar\epsilon - 8\,\bar A' &= \tfrac{1}{3}\,\gamma + \bar C' + \bar E = \bar C + \tfrac{2}{3} - 8\,\bar G\,. \end{aligned} \tag{6.22}$$

To $\mathrm{O}(\lambda^2)$

$$\begin{aligned} \tfrac{1}{3}\,\beta + 2\,\bar C' &= -16\,\bar G\,, \\ \tfrac{1}{6}\,\alpha + \tfrac{1}{2}\,\bar C' &= \tfrac{1}{6} - 4\,\bar G\,, \\ \tfrac{1}{6}\,b' + \tfrac{1}{2}(\bar A' + \bar B') &= -\tfrac{1}{12} + \bar G\,, \\ \tfrac{2}{3}\,d + \tfrac{1}{2}\,\bar\beta &= \tfrac{1}{2}\,\bar\alpha + \bar D = \bar A + \bar B - \tfrac{1}{8}\,, \\ \tfrac{1}{4}\,\bar\eta + \bar B' + \bar D &= \tfrac{1}{4}(\bar\delta + \bar\epsilon) + \bar A' = \bar A + \bar B - \tfrac{1}{12} + \bar G\,, \end{aligned} \tag{6.23}$$

and to $O(\lambda^3)$

$$\tfrac{1}{3}\,a' + \bar{A}' + \bar{B}' = -\tfrac{1}{12} + 2\,\bar{G}\,. \tag{6.24}$$

The coefficient of $\bar{G}$ is arbitrary as expected since (6.22), (6.23), (6.24) are invariant under

$$\bar{G} \to \bar{G} + \xi\,, \quad \bar{A}' \to \bar{A}' + \xi\,, \quad \bar{B}' \to \bar{B}' + \xi\,, \quad \bar{C}' \to \bar{C}' - 8\xi\,. \tag{6.25}$$

as this corresponds to the freedom $\tilde{A} \to \tilde{A} + \tfrac{1}{4}\xi\,\beta_{\lambda\,ij}{}^{kl}\beta_{\lambda\,kl}{}^{ij}$. Furthermore

$$\begin{aligned}
&\big(\mathrm{tr}(\beta_y{}^{(1)i}\beta_{\bar{y}}{}^{(2)}{}_i) + \mathrm{tr}(\beta_y{}^{(2)i}\beta_{\bar{y}}{}^{(1)}{}_i)\big)\big|_{\lambda y\bar{y}} \\
&\quad = -\,2\,\lambda_{ij}{}^{kl}\,\big(\mathrm{tr}(\bar{y}_m\,y^m\bar{y}_k\,y^i\,\bar{y}_l\,y^j) + \mathrm{tr}(y^m\bar{y}_m\,y^i\,\bar{y}_k\,y^j\,\bar{y}_l)\big) \\
&\qquad\qquad + \mathrm{tr}(\bar{y}_k\,y^m)\,\mathrm{tr}(\bar{y}_m\,y^i\,\bar{y}_l\,y^j) + \mathrm{tr}(\bar{y}_k\,y^i\,\bar{y}_l\,y^m)\,\mathrm{tr}(\bar{y}_m\,y^j)\big) \\
&\qquad + \tfrac{1}{4}\,\lambda_{ik}{}^{lm}\lambda_{lm}{}^{kj}\,\big(\mathrm{tr}(\bar{y}_j\,y^i\,\bar{y}_n\,y^n) + \mathrm{tr}(\bar{y}_j\,y^n\bar{y}_n\,y^i) + 2\,\mathrm{tr}(\bar{y}_j\,y^n)\,\mathrm{tr}(\bar{y}_n\,y^i)\big)\,,
\end{aligned} \tag{6.26}$$

so that letting $\tilde{A} \to \tilde{A} + z\ \mathrm{tr}(\beta_y{}^i\beta_{\bar{y}\,i})$ corresponds in (6.14) to

$$\bar{A} + \bar{B} \to \bar{A} + \bar{B} + \tfrac{1}{2}z\,, \qquad \bar{C} \to \bar{C} - 2z\,. \tag{6.27}$$

The consistency constraints (6.22), (6.23), (6.24) are then invariant if, along with (6.8), at the same time

$$\bar{A}' \to \bar{A}' + \tfrac{1}{4}z\,, \quad \bar{B}' \to \bar{B}' - \tfrac{1}{4}z\,, \quad \bar{D} \to \bar{D} + \tfrac{1}{4}z\,, \quad \bar{E} \to \bar{E} - 2z\,, \quad \bar{F} \to \bar{F} - 2z\,. \tag{6.28}$$

The conditions (6.23), (6.24) entail various constraint equations for the coefficients appearing in the general expressions for the three-loop Yukawa β-function and associated anomalous dimensions. Together with (6.22) the full list is

$$\begin{gathered}
\eta = \zeta = 2\,, \quad 2\alpha - \beta = 2\,, \quad \delta + \gamma - 2\epsilon - \beta = 2\,, \quad a' - b' = \tfrac{1}{4}\,, \\
2c' - \beta + \gamma - 2e - 16d = 6\,.
\end{gathered} \tag{6.29}$$

Reassuringly these relations are in accord with the results (5.25), (5.26) and (5.36). In addition

$$u = -\tfrac{1}{2}\gamma - e - 8d = \tfrac{5}{4}\,. \tag{6.30}$$

This demonstrates that the RG equations such as (1.1) hold only for the modified β-function determined by a non zero v as in (6.20). The coefficient appears to be exactly in accord with that determined by Fortin *et al* [15] by explicit three loop calculation for a general scalar fermion theory.[4] It is interesting to note that $u = c'$. There are also

[4] They considered couplings to real scalars and there was also a purely Yukawa contribution to $v_\phi{}^{(3)}$ which is absent in the model discussed here.

constraints on the three-loop metric given by (6.3) with $\bar\gamma = 0$

$$\begin{aligned} 2\bar\alpha - \bar\delta &= \tfrac{1}{6}(-2\epsilon + \delta) = \tfrac{1}{3}\,, \\ \bar\beta - \bar\delta &= \tfrac{1}{12}(\beta + 2e + 2) = \tfrac{5}{12}\,, \\ 2\bar\beta - \bar\epsilon - \bar\eta &= \tfrac{1}{3}(e - c' - 4a') = 0\,, \end{aligned} \tag{6.31}$$

which are equivalent to (6.5), and so (6.31) provides an additional confirmatory check on the three loop results obtained in section 5.

From (6.23), (6.24)

$$\bar A + \bar B = \bar\alpha - \tfrac{1}{16}\,, \quad \bar C = -4\bar\alpha - \tfrac{1}{6} \tag{6.32}$$

so that $\tilde A^{(5)}|_{\lambda y \bar y}$ is determined in (6.14) up to the freedom of choice for $\bar G$ and that corresponding to (6.27). We also have $\bar A' + \bar B' = 2\bar G - \frac{1}{16}$, $\bar C' = -8\bar G - \frac{1}{6}$ so there is the potentiality of a symmetric $T_{IJ}{}^{(4)}$ if we take $\bar\alpha = 2\bar G$ but this need not be true in general renormalisation schemes (with dimensional regularisation $\bar\alpha = -\frac{7}{72}$, $\bar G = -\frac{7}{216}$).

7. Supersymmetry

For supersymmetric theories with just $\mathcal{N} = 1$ supersymmetry there are further constraints which simplify many details significantly. The results obtained in [9] were restricted to supersymmetric field theories previously in [19]. Here the analysis is extended to a general $\mathcal{N} = 1$ Wess-Zumino supersymmetric scalar fermion theory, which may be obtained from (5.1) by imposing (5.16), (5.17), to a higher order. Such a theory can of course be rewritten in terms of n_C chiral and corresponding conjugate anti-chiral superfields. The local couplings may also be extended so that $Y^{ijk}, \bar Y_{ijk}$ for this theory are also chiral, anti-chiral superfields. Divergences which arise in a perturbative expansion are cancelled by counterterms which are integrals of local polynomials in the fields and couplings of dimension two over full $\mathcal{N} = 1$ superspace. This restriction crucially ensures that β-functions for $Y^{ijk}, \bar Y_{ijk}$ are determined in terms of just the anomalous dimension matrix γ as in (5.18) but further conditions on the functions which are present in local RG equations also arise. The various RG functions are further constrained by assuming manifest $U(n_C)$ symmetry.

The formalism of section 2 can be adapted to this case by taking

$$g^I = (Y^{ijk}, \bar Y_{ijk})\,, \qquad (\omega g)^I = \big(- (Y * \omega)^{ijk}, (\omega * \bar Y)_{ijk} \big)\,, \quad \omega_i{}^j \in \mathfrak{gl}(n_C, \mathbb{C})\,, \tag{7.1}$$

where

$$\begin{aligned} (Y * \omega)^{ijk} &\equiv Y^{ljk}\omega_l{}^i + Y^{ilk}\omega_l{}^j + Y^{ijl}\omega_l{}^k\,, \\ (\omega * \bar Y)_{ijk} &\equiv \omega_i{}^l\, \bar Y_{ljk} + \omega_j{}^l\, \bar Y_{ilk} + \omega_k{}^l\, \bar Y_{ijl}\,. \end{aligned} \tag{7.2}$$

With this notation the result for the Yukawa supersymmetric β-functions (5.18) becomes[5]

$$\beta_Y = Y * \gamma \,, \qquad \beta_{\bar Y} = \gamma * \bar Y \,. \tag{7.3}$$

To avoid explicit indices where possible we also define, in this section and appendix A, a scalar product $\circ$ on Yukawa couplings so that for instance $Y \circ \bar Y = Y^{ijk} \bar Y_{ijk}$.

Besides the β-functions other expressions appearing in the equations of section 2 are determined in terms of the anomalous dimension matrix γ. Based on a superspace framework Fortin *et al* [28] showed that ρ_I to all orders is given by (a related result is given in appendix C of [29])

$$\big(\rho_I(g)\, \mathrm{d}g^I\big)_i{}^j = - \mathrm{d}_Y \gamma_i{}^j + \mathrm{d}_{\bar Y} \gamma_i{}^j \,, \tag{7.4}$$

for $\mathrm{d}_Y = \mathrm{d}Y \circ \partial_Y$, $\mathrm{d}_{\bar Y} = \mathrm{d}\bar Y \circ \partial_{\bar Y}$. In a similar fashion to the derivation of (7.4) we may also obtain in (2.47) results which are determined just in terms of $\gamma_i{}^j$,

$$\big(\delta_I(g)\, \mathrm{d}g^I\big)_i{}^j = 0 \,, \qquad \big(\epsilon_{IJ}(g)\, \mathrm{d}g^I \mathrm{d}g^J\big)_i{}^j = 2\, \mathrm{d}_{\bar Y} \mathrm{d}_Y \gamma_i{}^j \,, \tag{7.5}$$

The result (7.4) implies $\rho_I(g)\, g^I = 0$ which in turn ensures that in the supersymmetric case

$$\upsilon = 0 \,. \tag{7.6}$$

Thus there is no modification of the β-function as in (1.7). The necessary constraint (2.52) on ρ_I applied to (7.4) requires

$$\beta_Y \circ \partial_Y \gamma_i{}^j = \beta_{\bar Y} \circ \partial_{\bar Y} \gamma_i{}^j \,. \tag{7.7}$$

This is a special case of the identity, for any $\omega_i{}^j$,

$$\big((\omega * \bar Y) \circ \partial_{\bar Y} - (Y * \omega) \circ \partial_Y \big)\, \gamma_i{}^j = [\omega, \gamma]_i{}^j \,, \tag{7.8}$$

taking $\omega \to \gamma$. The result (7.8) was obtained in [30][6] and is a consequence of $\gamma_i{}^j(Y, \bar Y)$ transforming as a $(1,1)$ tensor under $U(n_C)$ with $\omega = -\bar\omega \in \mathfrak{u}(n_C)$, the associated Lie algebra.

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[5] More generally we may have $\beta_Y = Y * \gamma$, $\beta_{\bar Y} = \bar\gamma * \bar Y$. This form is preserved under transformations $Y^{ijk} \to Y^{lmn} G_l{}^i G_m{}^j G_n{}^k = Y'^{ijk}$, $\bar Y_{ijk} \to \bar G_i{}^l \bar G_j{}^m \bar G_k{}^n \bar Y_{lmn} = \bar Y'_{ijk}$ for $G \in Gl(n_C, \mathbb{C})$. In this case $\beta'_Y = Y' * \gamma'$, $\beta'_{\bar Y} = \bar\gamma' * \bar Y'$ with $\gamma' = G^{-1} \gamma\, G + G^{-1} \dot G$, $\bar\gamma' = \bar G\, \bar\gamma\, \bar G^{-1} + \dot{\bar G}\, \bar G^{-1}$ for $\dot G = (\beta_Y \circ \partial_Y + \beta_{\bar Y} \circ \partial_{\bar Y}) G$ and similarly for $\dot{\bar G}$. For $U(n_C)$ transformations $\bar G = G^{-1}$. Requiring then $\dot G + \frac{1}{2}(\gamma - \bar\gamma) G = 0$ ensures $\gamma' = \bar\gamma'$ so the general case can be reduced to $\gamma = \bar\gamma$ by virtue of $U(n_C)$ symmetry.

[6] See eq. (A.7).

In the supersymmetric theory the equation (1.1) is assumed to now take the form

$$\begin{aligned} \mathrm{d}_Y \tilde{A} &= \tfrac{1}{2}\big(\mathrm{d}Y \circ T \circ \beta_{\bar{Y}} + \beta_Y \circ K \circ \mathrm{d}Y\big) \,, \qquad K^T = -K \,, \\ \mathrm{d}_{\bar{Y}} \tilde{A} &= \tfrac{1}{2}\big(\beta_Y \circ \bar{T} \circ \mathrm{d}\bar{Y} + \mathrm{d}\bar{Y} \circ \bar{K} \circ \beta_{\bar{Y}}\big) \,, \qquad \bar{K}^T = -\bar{K} \,, \end{aligned} \tag{7.9}$$

so that

$$(\beta_Y \circ \partial_Y + \beta_{\bar{Y}} \circ \partial_{\bar{Y}})\tilde{A} = \beta_Y \circ G \circ \beta_{\bar{Y}} \,, \qquad G = \tfrac{1}{2}(T + \bar{T}) \,. \tag{7.10}$$

By $U(n_C)$ invariance $\big(\beta_Y \circ \partial_Y - \beta_{\bar{Y}} \circ \partial_{\bar{Y}}\big)\tilde{A} = 0$ so for consistency we should require $\beta_Y \circ T \circ \beta_{\bar{Y}} = \beta_Y \circ \bar{T} \circ \beta_{\bar{Y}}$. T, K may be determined by perturbative calculations but from the perspective of just analysing the integrability conditions flowing from (7.9) and using known results for β-functions, as is considered mainly in this section, there is an ambiguity such that T, K satisfy the equivalence relations

$$T \sim T + T' \,, \ K \sim K + K' \quad \text{if} \quad \mathrm{d}Y \circ T' \circ \beta_{\bar{Y}} + \beta_Y \circ K' \circ \mathrm{d}Y = 0 \,. \tag{7.11}$$

The result (2.69) constrains the form of K and $T - \bar{T}$. Writing

$$W_I \mathrm{d}g^I = \tfrac{1}{2}(\mathrm{d}Y \circ \bar{W} + W \circ \mathrm{d}\bar{Y}) \,, \qquad Q_I \mathrm{d}g^I = \tfrac{1}{2}(\mathrm{d}Y \circ \bar{Q} - Q \circ \mathrm{d}\bar{Y}) \in \mathfrak{u}(n_C) \,, \tag{7.12}$$

then

$$\begin{aligned} \mathrm{d}'Y \circ K \circ \mathrm{d}Y &= \mathrm{d}'Y \circ \mathrm{d}_Y \bar{W} - \operatorname{tr}\big(\mathrm{d}'Y \circ \bar{Q}\ \mathrm{d}_Y \gamma\big) - \mathrm{d}'Y \leftrightarrow \mathrm{d}Y \,, \\ \mathrm{d}Y \circ \tfrac{1}{2}(T - \bar{T}) \circ \mathrm{d}\bar{Y} &= \mathrm{d}_Y W \circ \mathrm{d}\bar{Y} + \operatorname{tr}\big(\mathrm{d}_Y \gamma\ Q \circ \mathrm{d}\bar{Y}\big) - \text{conjugate} \,. \end{aligned} \tag{7.13}$$

The relation (2.66a) requires

$$3\,(\bar{Y}W) - 3\,(\bar{W}Y) = Q \circ \beta_{\bar{Y}} - \beta_Y \circ \bar{Q} \,, \tag{7.14}$$

defining $(\bar{Y}W), (\bar{W}Y) \in \mathfrak{gl}(n_C, \mathbb{C})$ by

$$(Y * \omega) \circ \bar{W} = 3 \operatorname{tr}\big((\bar{W}Y)\,\omega\big) \,, \quad W \circ (\omega * \bar{Y}) = 3 \operatorname{tr}\big((\bar{Y}W)\,\omega\big) \,, \quad \omega \in \mathfrak{gl}(n_C, \mathbb{C}) \,. \tag{7.15}$$

If $W \circ \mathrm{d}\bar{Y}$ corresponds to a ℓ-loop vacuum graph then $(\bar{Y}W)$ may be represented by an associated $(\ell - 1)$-loop graph with two external lines. For any $Q', \bar{Q}'$ such that

$$Q' \circ \beta_{\bar{Y}} = \beta_Y \circ \bar{Q}' \,, \tag{7.16}$$

then $Q \sim Q + Q'$, $\bar{Q} \sim \bar{Q} + \bar{Q}'$ since (7.13) ensures that the corresponding T', K' satisfy (7.11). Up to this equivalence (7.14) determines $Q, \bar{Q}$ in terms of $W, \bar{W}$.

The RG flow equations (7.9) are invariant under

$$\Delta \tilde{A} = \beta_Y \circ g \circ \beta_{\bar{Y}} + (\beta_Y \circ \partial_Y + \beta_{\bar{Y}} \circ \partial_{\bar{Y}}) a \,, \qquad g = \bar{g} \,, \tag{7.17}$$

when

$$\begin{aligned}
\mathrm{d}'Y\circ\Delta K\circ \mathrm{d}Y &= 2\,\mathrm{d}'Y\circ g\circ(\mathrm{d}_Y\gamma * \bar Y) + \mathrm{d}'Y\circ \mathrm{d}_Y g\circ\beta_{\bar Y} - \mathrm{d}'Y \leftrightarrow \mathrm{d}Y\,,\\
\mathrm{d}Y\circ\Delta T\circ \mathrm{d}\bar Y &= 2\,\mathrm{d}_Y\big((Y*\gamma)\circ g\big)\circ \mathrm{d}\bar Y + 2\,\mathrm{d}Y\circ g\circ(\mathrm{d}_{\bar Y}\gamma * \bar Y)\\
&\quad + \mathrm{d}Y\circ\big(\beta_Y\circ\partial_Y + \beta_{\bar Y}\circ\partial_{\bar Y}\big)g\circ \mathrm{d}\bar Y - \mathrm{d}Y\circ \mathrm{d}_{\bar Y}g\circ\beta_{\bar Y} - \beta_Y\circ \mathrm{d}_Y g\circ \mathrm{d}\bar Y\,.
\end{aligned} \tag{7.18}$$

For $\Delta\bar K$, $\Delta\bar T$ given by the conjugate equations to (7.18) then $\Delta G = \frac{1}{2}(\Delta T + \Delta\bar T)$ is therefore

$$\begin{aligned}
\mathrm{d}Y\circ\Delta G\circ \mathrm{d}\bar Y = \mathrm{d}Y\circ\big(\beta_Y\circ\partial_Y + \beta_{\bar Y}\circ\partial_{\bar Y}\big)g\circ \mathrm{d}\bar Y + (\mathrm{d}Y*\gamma)\circ g\circ \mathrm{d}\bar Y + \mathrm{d}Y\circ g\circ(\gamma * \mathrm{d}\bar Y)\\
+ 2\,(Y*\mathrm{d}_Y\gamma)\circ g\circ \mathrm{d}\bar Y + 2\,\mathrm{d}Y\circ g\circ(\mathrm{d}_{\bar Y}\gamma * \bar Y)\,.
\end{aligned} \tag{7.19}$$

(7.18) and (7.19) correspond exactly to the freedom in (2.76) assuming (7.4) and demonstrate that it is consistent to require that G defines a hermitian metric for supersymmetric theories. Corresponding to this freedom there are associated variations in W_I, Q_I given by

$$\begin{aligned}
\Delta W\circ \mathrm{d}\bar Y &= \beta_Y\circ g\circ \mathrm{d}\bar Y - 2\,\mathrm{d}_{\bar Y}a\,, & \mathrm{d}Y\circ\Delta\bar W &= \mathrm{d}Y\circ g\circ\beta_{\bar Y} - 2\,\mathrm{d}_Y a\,,\\
\Delta Q\circ \mathrm{d}\bar Y &= -3\,(g\circ \mathrm{d}\bar Y\, Y) + \beta_Y\circ p\circ \mathrm{d}\bar Y\,, & \mathrm{d}Y\circ\Delta\bar Q &= -3\,(\bar Y\,\mathrm{d}Y\circ g) + \mathrm{d}Y\circ p\circ\beta_{\bar Y}\,,
\end{aligned} \tag{7.20}$$

with $(g\circ \mathrm{d}\bar Y\, Y), (\bar Y\,\mathrm{d}Y\circ g)$ defined similarly to (7.15) and $\mathrm{d}Y\circ p\circ \mathrm{d}\bar Y \in \mathfrak{gl}(n_C,\mathbb{C})$. These results ensure that (7.13) is compatible with (7.18), variations in $Q,\bar Q$ arising from p satisfy (7.16). We may also verify the invariance of (7.14), so long as $(Y*\omega)\circ\partial_Y a = (\omega*\bar Y)\circ\partial_{\bar Y}a$.

There is also freedom corresponding essentially to a choice of scheme. For this we consider variations

$$\delta\tilde A = -(Y*h)\circ\partial_Y\tilde A = -(h*\bar Y)\circ\partial_{\bar Y}\tilde A\,, \tag{7.21}$$

for arbitrary $h_i{}^j(Y,\bar Y)$. We assume that there is a corresponding variation in γ of the form

$$\delta\beta_Y = Y*\delta\gamma\,, \qquad \delta\beta_{\bar Y} = \delta\gamma * \bar Y\,, \tag{7.22}$$

for

$$\delta\gamma = \beta_{\bar Y}\circ\partial_{\bar Y}h - (Y*h)\circ\partial_Y\gamma\,. \tag{7.23}$$

This expression for $\delta\gamma$ may be rewritten in various equivalent forms using (7.8) for $\omega\to h$ or for $\omega\to\gamma$, $\gamma\to h$. In consequence $\delta\gamma^\dagger = \delta\gamma$ if $h^\dagger = h, \gamma^\dagger = \gamma$ and also if h corresponds to a 1PI graph then so does $\delta\gamma$ as well. Assuming (7.21) and (7.22), (7.23) the essential equations (7.9) are invariant if

$$\begin{aligned}
\mathrm{d}'Y\circ\delta K\circ \mathrm{d}Y = {} & -\mathrm{d}'Y\circ\big((Y*h)\circ\partial_Y K\big)\circ \mathrm{d}Y\\
& -\mathrm{d}'_Y(Y*h)\circ K\circ \mathrm{d}Y - \mathrm{d}'Y\circ K\circ d_Y(Y*h)\,,\\
\mathrm{d}Y\circ\delta T\circ \mathrm{d}\bar Y = {} & -\mathrm{d}Y\circ\big((Y*h)\circ\partial_Y T\big)\circ \mathrm{d}\bar Y\\
& -\mathrm{d}_Y(Y*h)\circ T\circ \mathrm{d}\bar Y - \mathrm{d}Y\circ T\circ d_{\bar Y}h*\bar Y\,.
\end{aligned} \tag{7.24}$$

since then $2\,\mathrm{d}_Y\delta\tilde{A} = \mathrm{d}Y\circ T\circ\delta\beta_{\bar{Y}} + \delta\beta_Y\circ K\circ\mathrm{d}Y + \mathrm{d}Y\circ\delta T\circ\beta_{\bar{Y}} + \beta_Y\circ\delta K\circ\mathrm{d}Y$.

The basic equations (7.9) may be verified using perturbative results. For convenience we adopt a notation where the one and two loop contributions to the anomalous dimension γ in (5.21) are given by $\gamma^{(1)} = \frac{1}{2}\,(\bar{Y}Y)$, $\gamma^{(2)} = -\frac{1}{2}\,(\bar{Y}Y\bar{Y}Y)$. Restricting the metric (6.1) to the supersymmetric case gives

$$\mathrm{d}Y\circ T^{(2)}\circ\mathrm{d}\bar{Y} = \tfrac{1}{3}\,\mathrm{d}Y\circ\mathrm{d}\bar{Y} = \tfrac{1}{3}\,\mathrm{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\big)\,, \tag{7.25}$$

and in general

$$\mathrm{d}Y\circ T^{(3)}\circ\mathrm{d}\bar{Y} = a\,\mathrm{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\,(\bar{Y}Y)\big) + b\,\mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}\mathrm{d}Y)\big)\,, \tag{7.26}$$

where we note that $\mathrm{tr}\big((\bar{Y}_1Y_2\bar{Y}_3Y_4)\big) = \mathrm{tr}\big((\bar{Y}_1Y_4)\,(\bar{Y}_3Y_2)\big)$. To this order $K,\bar{K} = 0$ and $T = \bar{T} = G$. For integrability we require

$$2a - b = -\tfrac{1}{2}\,, \tag{7.27}$$

which accords with the constraints for supersymmetric theories described in [19].[7] If we let

$$\mathrm{d}Y\circ g^{(2)}\circ\mathrm{d}\bar{Y} = z\,\mathrm{d}Y\circ\mathrm{d}\bar{Y}\,, \tag{7.28}$$

then (7.18) gives at this order $\Delta K = 0$ and ΔT is determined by

$$\Delta a = 3z\,, \qquad \Delta b = 6z\,, \tag{7.29}$$

under which (7.27) is invariant. Integration of (1.1) subject to (7.27) then gives

$$\begin{aligned}\tilde{A}^{(3)} &= \tfrac{1}{8}\,\mathrm{tr}\big((\bar{Y}Y)^2\big)\,,\\ \tilde{A}^{(4)} &= \tfrac{1}{24}\,\mathrm{tr}\big((\bar{Y}Y)^3\big) + \tfrac{1}{3}a\,\beta_Y{}^{(1)}\circ\beta_{\bar{Y}}{}^{(1)}\,.\end{aligned} \tag{7.30}$$

Reducing the results in (6.3) requires $a = \frac{1}{12} + 2\bar{\alpha} + \frac{1}{2}\bar{\delta}$, $b = 2\bar{\beta} + \frac{1}{2}\bar{\eta}$ and hence from (6.7)

$$a = -\tfrac{5}{8}\,, \qquad b = -\tfrac{3}{4}\,, \tag{7.31}$$

which of course satisfy (7.27).

This discussion can be extended to the next order using as input the form of the three-loop γ given by (5.27). It is convenient to summarise this in the form

$$\gamma^{(3)} = A\,\gamma_A + B\,\gamma_B + C\,\gamma_C + D\,\gamma_D\,, \tag{7.32}$$

[7] In terms of the parameters in [19] $\alpha = 2a, \beta = 2b, \gamma = 0$.

where the coefficients A, B, C, D are given in (5.29). However there is potential scheme dependence since if in (7.23) we take $h = v\,(\bar{Y}Y\bar{Y}Y)$ and $\beta_{\bar{Y}} \to \beta_{\bar{Y}}{}^{(1)}, \gamma \to \gamma^{(1)}$ then

$$\delta A = v\,, \qquad \delta B = \tfrac{1}{2} v\,. \tag{7.33}$$

From (5.29) it is evident that we may use this freedom to set $A = B = 0$ but C is scheme independent. At this order there are three relevant connected 1PI vacuum graphs with different topologies and to determine $T^{(4)}$ it is necessary to choose for each graph one Y vertex and one $\bar{Y}$ vertex in inequivalent ways. The number of possible terms multiply but this procedure gives the general expression

$$\begin{aligned} \mathrm{d}Y \circ T^{(4)} \circ \mathrm{d}\bar{Y} = {} & a_1 \operatorname{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\,(\bar{Y}Y\bar{Y}Y)\big) + a_2 \operatorname{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}Y\bar{Y}\mathrm{d}Y)\big) \\ & + a_3 \operatorname{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}\mathrm{d}Y\,\bar{Y}Y)\big) + a_4 \operatorname{tr}\big((\bar{Y}Y)\,(\mathrm{d}\bar{Y}\,Y\bar{Y}\mathrm{d}Y)\big) \\ & + a_5 \operatorname{tr}\big((\bar{Y}\mathrm{d}Y)\,(\mathrm{d}\bar{Y}\,Y\bar{Y}Y)\big) \\ & + b_1 \operatorname{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\,(\bar{Y}Y)^2\big) + b_2 \operatorname{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}\mathrm{d}Y)\,(\bar{Y}Y)\big) \\ & + b_3 \operatorname{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}Y)\,(\bar{Y}\mathrm{d}Y)\big) \\ & + c\, \mathrm{d}\bar{Y}_{ikl}\,\mathrm{d}Y^{kps}\,Y^{lqr}\,\bar{Y}_{prm}\,\bar{Y}_{qsn}Y^{mni}\,. \end{aligned} \tag{7.34}$$

In this case $\operatorname{tr}\big((\bar{Y}_1Y_2)\,(\bar{Y}_3Y_4\bar{Y}_5Y_6)\big) = \operatorname{tr}\big((\bar{Y}_5Y_4)\,(\bar{Y}_3Y_2\bar{Y}_1Y_6)\big)$. At four loops there may also be contributions to K in (7.9) so that, following a similar prescription as for $T^{(4)}$ but choosing two Y vertices and antisymmetrising, there are two possible terms

$$\begin{aligned} \mathrm{d}'Y \circ K^{(4)} \circ \mathrm{d}Y = {} & e \operatorname{tr}\big((\bar{Y}\mathrm{d}'Y)\,(\bar{Y}Y\bar{Y}\mathrm{d}Y)\big) + f \operatorname{tr}\big((\bar{Y}\mathrm{d}'Y)\,(\bar{Y}\mathrm{d}Y)\,(\bar{Y}Y)\big) \\ & - \mathrm{d}'Y \leftrightarrow \mathrm{d}Y\,. \end{aligned} \tag{7.35}$$

At this order T and $\bar{T}$ are also no longer necessarily equal since

$$\mathrm{d}Y \circ \bar{T}^{(4)} \circ \mathrm{d}\bar{Y} = \mathrm{d}Y \circ T^{(4)} \circ \mathrm{d}\bar{Y}\big|_{a_2 \leftrightarrow a_5}\,. \tag{7.36}$$

It is easy to see that, by virtue of (7.20), we may take $Q^{(2)}, W^{(3)} \to 0$. At the next order there may be non trivial Q, W. If we allow only contributions corresponding to connected diagrams then it is sufficient to assume

$$\mathrm{d}Y \circ \bar{W}^{(4)} = \sigma \operatorname{tr}\big((\bar{Y}Y)\,(\bar{Y}Y\bar{Y}\mathrm{d}Y)\big)\,, \qquad \mathrm{d}Y \circ \bar{Q}^{(3)} = -4\sigma\,(\bar{Y}Y\bar{Y}\mathrm{d}Y)\,, \tag{7.37}$$

where the coefficients are related by imposing (7.14). In this case (7.13) agrees with (7.34) and (7.35) if $a_2 = a_5$, $e = -4\sigma$, $f = 0$.

At five loops $\tilde{A}^{(5)}$ is determined in terms of the five connected vacuum diagrams for this theory at this order. The relevant contributions can be written in the general form

$$\begin{aligned} 2\,\tilde{A}^{(5)} = {} & X_1 \operatorname{tr}\big((\bar{Y}Y)^2\,(\bar{Y}Y\bar{Y}Y)\big) + X_2 \operatorname{tr}\big((\bar{Y}Y\bar{Y}Y)^2\big) + X_3 \operatorname{tr}\big((\bar{Y}Y)\,\gamma_B\big) \\ & + X_4 \operatorname{tr}\big((\bar{Y}Y)^4\big) + X_5 \operatorname{tr}\big((\bar{Y}Y)\,\gamma_D\big)\,, \end{aligned} \tag{7.38}$$

where γ_B, γ_D are explicitly defined by (5.27). Using (7.9) for $\mathrm{d}_Y \tilde{A}^{(5)}$ we may then obtain, for arbitrary values for A, B, C, D in (7.32),

$$\begin{aligned}
X_1 &= \tfrac{1}{2}(a_2 + e) + a_4 + b_1 = \tfrac{1}{2}\, a_5 + b_2 + f - \tfrac{1}{2} a \\
&= \tfrac{1}{2}(a_1 - e) + b_3 - f - \tfrac{1}{2} b = \tfrac{1}{2}(a_3 + a_5 - e) + A\,, \\
X_2 &= \tfrac{1}{2}(a_1 + a_2 + e - a) = \tfrac{1}{2}(a_3 - b + C)\,, \\
X_3 &= \tfrac{1}{2}\, a_4 = \tfrac{1}{6}(a_5 - e + 2B)\,, \\
X_4 &= \tfrac{1}{8}(b_1 + b_2 + b_3)\,, \qquad\qquad X_5 = \tfrac{1}{2}\, c = D\,.
\end{aligned} \tag{7.39}$$

For each term in (7.38) integrability conditions arise whenever the number of inequivalent Y vertices in the associated graph is greater than one. The equations (7.39) are invariant under

$$\begin{aligned}
&a_1 \to a_1 + \mu\,, \quad a_2 \to a_2 - \mu - \nu\,, \quad a_5 \to a_5 + \nu\,, \quad e \to e + \nu\,, \quad f \to f + \omega\,, \\
&b_1 \to b_1 + \tfrac{1}{2}\mu\,, \quad b_2 \to b_2 - \tfrac{1}{2}\nu - \omega\,, \quad b_3 \to b_3 - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \omega\,,
\end{aligned} \tag{7.40}$$

which correspond to variations satisfying (7.11) for one loop $\beta_Y, \beta_{\bar{Y}}$. The freedom in (7.40) in part can be realised by changes in $Q, \bar{Q}$ satisfying (7.16). As a consequence, even setting $K^{(4)} = 0$, $\tilde{A}^{(5)}$ does not determine $T^{(4)}$.

If we take

$$\mathrm{d}Y \circ g^{(3)} \circ \mathrm{d}\bar{Y} = x \, \mathrm{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\,(\bar{Y}Y)\big) + y \, \mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}\mathrm{d}Y)\big)\,, \tag{7.41}$$

then (7.18), with one loop results for γ, generates

$$\begin{aligned}
&\Delta a_1 = 2x\,, \quad \Delta a_2 = x + 3y\,, \quad \Delta a_3 = 4y\,, \quad \Delta a_4 = 2x\,, \quad \Delta a_5 = 3x + y\,, \\
&\Delta b_1 = 2x\,, \quad \Delta b_2 = 2x + y\,, \quad \Delta b_3 = 3y\,, \quad \Delta e = -3x + y\,, \quad \Delta f = -\tfrac{1}{2}(x - y)\,,
\end{aligned} \tag{7.42}$$

so that $\Delta X_1 = 3x + 2y$, $\Delta X_2 = 2y$, $\Delta X_3 = x$, $\Delta X_4 = \frac{1}{2}(x + y)$. Corresponding to (7.28), along with (7.29), we have in addition

$$\Delta a_1 = \Delta a_2 = \Delta a_5 = \Delta e = -3z\,, \qquad \Delta a_3 = -6z\,, \tag{7.43}$$

which entails $\Delta X_1 = -3z$, $\Delta X_2 = -6z$. There is one invariant under (7.42) and (7.43)

$$2\,X_1 - X_2 - 4\,X_3 - 4\,X_4 = \tfrac{1}{2}\,A - B - \tfrac{1}{4}\,C = -\tfrac{1}{4}\,, \tag{7.44}$$

imposing the numerical results in (5.29). The freedom in (7.42) may be used to set $\mathrm{d}'Y \circ K^{(4)} \circ \mathrm{d}Y = 0$.

The results for T in (7.25), (7.26) and (7.34) determine the metric G at each order. It is of interest to consider whether this is Kähler so that

$$\mathrm{d}Y \circ G \circ \mathrm{d}\bar{Y} = \mathrm{d}_Y \mathrm{d}_{\bar{Y}} F \,. \tag{7.45}$$

It is possible to construct F so long as the freedom due to variations as in (7.18) and (7.24), or equivalently (7.23), are allowed for. From (7.25), (7.26)

$$F^{(2)} = \tfrac{1}{3}\,\mathrm{tr}\big((\bar{Y}Y)\big)\,, \qquad F^{(3)} = -\tfrac{1}{4}\,\mathrm{tr}\big((\bar{Y}Y)^2\big)\,, \tag{7.46}$$

if we use (7.29) to set

$$a = b = -\tfrac{1}{2}\,. \tag{7.47}$$

At the next order the general expression has the form

$$F^{(4)} = \hat{a}\,\mathrm{tr}\big((\bar{Y}Y)(\bar{Y}Y\bar{Y}Y)\big) + \tfrac{1}{3}\hat{b}\,\mathrm{tr}\big((\bar{Y}Y)^3\big) + \tfrac{2}{9}D\,\mathrm{tr}\big(\gamma_D\big)\,, \tag{7.48}$$

and then (7.34) and (7.45) require

$$a_1 = a_3 = 2\hat{a}\,,\ a_4 = \hat{a}\,,\ a_2 = 2\hat{a} + \lambda\,,\ a_5 = 2\hat{a} - \lambda\,,\ b_1 = b_2 = b_3 = \hat{b}\,. \tag{7.49}$$

for arbitrary λ since $G^{(5)}$ depends only on $a_2 + a_5$. Imposing the conditions in (7.39) is possible only by choosing a scheme with $A = B = 0, C = 1$ and then $\hat{a}, \hat{b}$ as well as e, f are determined so that

$$\hat{a} = \hat{b} = 1\,, \qquad e = -1 - \lambda\,, \quad f = \tfrac{1}{4} + \tfrac{1}{2}\lambda\,, \tag{7.50}$$

giving $X_1 = \frac{5}{2}$, $X_2 = \frac{7}{4}$, $X_3 = \frac{1}{2}$, $X_4 = \frac{3}{8}$.

For $\mathcal{N} = 1$ supersymmetric theories there is, at critical points with vanishing β-functions, an exact expression for a [31] in terms of the anomalous dimension matrix γ or alternatively the R-charge $R = \frac{2}{3}(1+\gamma)$. Introducing terms linear in β-functions there is a corresponding expression which is valid away from critical points and this can then be shown to satisfy many of the properties associated with the a-theorem [32,33]. For the theory considered here, with n_C chiral scalar multiplets, these results take the form

$$\tilde{A} = \tfrac{1}{12}\, n_C - \tfrac{1}{2}\,\mathrm{tr}(\gamma^2) + \tfrac{1}{3}\,\mathrm{tr}(\gamma^3) + \Lambda \circ \beta_{\bar{Y}} + \beta_Y \circ H \circ \beta_{\bar{Y}}\,, \tag{7.51}$$

where we require[8]

$$\Lambda \circ \beta_{\bar{Y}} = \beta_Y \circ \bar{\Lambda}\,, \qquad H = \bar{H}\,. \tag{7.52}$$

[8] In [32] and [33] Λ plays the role of a Lagrange multiplier enforcing constraints on the R-charges. At lowest order the result for Λ and also the metric G obtained in [32] are equivalent, up to matters of definition and normalisation, with those obtained later here and in (7.25).

Λ is determined in (7.51) up to terms which may be absorbed in H so that $\Lambda \circ \mathrm{d}\bar{Y} \sim \Lambda \circ \mathrm{d}\bar{Y} + \beta_Y \circ g \circ \mathrm{d}\bar{Y}$. Assuming the result (7.51) for $\tilde{A}$ satisfies (7.9) then H is arbitrary as a consequence of (7.17).

However Λ is constrained by imposing (7.9). Defining $(\bar{Y}\Lambda)_i{}^j$ in a similar fashion to (7.15), then

$$\begin{aligned} \mathrm{d}_Y\big(-\tfrac{1}{2}\,\mathrm{tr}(\gamma^2) + \tfrac{1}{3}\,\mathrm{tr}(\gamma^3) + \Lambda \circ \beta_{\bar{Y}} \big) = {}& \mathrm{tr}\big(\mathrm{d}_Y \gamma \, \big(3\,(\bar{Y}\Lambda) - \gamma + \gamma^2\big)\big) \\ & + \big(\mathrm{d}_Y \Lambda\big) \circ \beta_{\bar{Y}} \, . \end{aligned} \tag{7.53}$$

Hence if Λ is required to obey[9]

$$3\,(\bar{Y}\Lambda) = \gamma - \gamma^2 + \Theta \circ \beta_{\bar{Y}} \, , \qquad \Theta \circ \mathrm{d}\bar{Y} \in \mathfrak{gl}(n_C, \mathbb{C}) \, , \tag{7.54}$$

then (7.51), excluding the H term, satisfies (7.9) if we take

$$\begin{aligned} & \tfrac{1}{2}\, \mathrm{d}Y \circ T \circ \mathrm{d}\bar{Y} = \mathrm{tr}\big(\mathrm{d}_Y \gamma \; \Theta \circ \mathrm{d}\bar{Y} \big) + \mathrm{d}_Y \Lambda \circ \mathrm{d}\bar{Y} + \tfrac{1}{2}\, \mathrm{d}Y \circ T' \circ \mathrm{d}\bar{Y} \, , \\ & \mathrm{d}'Y \circ K \circ \mathrm{d}Y = 0 \, , \qquad \mathrm{d}Y \circ T' \circ \beta_{\bar{Y}} = 0 \, . \end{aligned} \tag{7.55}$$

A related result, with effectively $\Theta = 0$, is contained in [32]. For supersymmetric theories, satisfying (7.54) is consequently essentially equivalent to requiring (7.9), although terms involving Θ are necessary at higher orders. The relations (7.54) and (7.55) are not invariant under variations of g as in (7.52) and so this freedom is no longer present.

Since γ is hermitian a corollary of (7.54) is that Λ, Θ must satisfy

$$3\,(\bar{Y}\Lambda) - 3\,(\bar{\Lambda}Y) = \Theta \circ \beta_{\bar{Y}} - \beta_Y \circ \bar{\Theta} \, . \tag{7.56}$$

This is essentially identical to (7.14) and suggests a relation between Λ, Θ and W, Q but a precise connection is as yet unclear.

For variations as in (7.21) and (7.23) then compatibility with (7.51) requires

$$\delta\Lambda \circ \mathrm{d}\bar{Y} = -(Y * h) \circ \partial_Y \Lambda \circ \mathrm{d}\bar{Y} + \delta'\Lambda \circ \mathrm{d}\bar{Y} \, , \tag{7.57}$$

where $\delta'\Lambda$ satisfies, assuming (7.54),

$$\delta'\Lambda \circ \beta_{\bar{Y}} = -\beta_{\bar{Y}} \circ S \circ \beta_{\bar{Y}} \, , \qquad \mathrm{d}\bar{Y} \circ S \circ \mathrm{d}\bar{Y} = \mathrm{tr}\big(\mathrm{d}_{\bar{Y}} h \; \Theta \circ \mathrm{d}\bar{Y} \big) \, . \tag{7.58}$$

Furthermore (7.54) is also invariant if

$$\delta\Theta \circ \mathrm{d}\bar{Y} = -(Y * h) \circ \partial_Y \Theta \circ \mathrm{d}\bar{Y} - \mathrm{d}_{\bar{Y}} h + \mathrm{d}_{\bar{Y}} h \, \gamma + \gamma \, \mathrm{d}_{\bar{Y}} h - \Theta \circ (\mathrm{d}_{\bar{Y}} h * \bar{Y}) + \delta'\Theta \circ \mathrm{d}\bar{Y} \, , \tag{7.59}$$

[9] More generally if $3\,(\bar{Y}\Lambda) = \gamma - \gamma^2 + \Theta \circ \beta_{\bar{Y}} + [\Xi, \, \beta_Y \circ \partial_Y \gamma]$, $\mathrm{d}'Y \circ K \circ \mathrm{d}Y = \mathrm{tr}(\Xi \, [\mathrm{d}'_Y \gamma, \mathrm{d}_Y \gamma])$. Such a term can be removed by considering changes as in (7.11).

so long as

$$3\,(\bar{Y}\delta'\Lambda) = \delta'\Theta\circ\beta_{\bar{Y}}\,. \tag{7.60}$$

This can be solved subject to (7.58) by taking

$$\delta'\Lambda\circ \mathrm{d}\bar{Y} = -\beta_{\bar{Y}}\circ S\circ \mathrm{d}\bar{Y}\,, \qquad \delta'\Theta\circ \mathrm{d}\bar{Y} = -3\,(\bar{Y}\,\mathrm{d}\bar{Y}\circ S)\,. \tag{7.61}$$

Using (7.57),(7.59),(7.61) in (7.55) generates variations in agreement with (7.24) up to contributions which may be absorbed in T'. Such variations generate terms in Θ which are 1PR. Also we may show $\delta(\Lambda\circ\beta_{\bar{Y}} - \beta_Y\circ\bar{\Lambda}) = 0$ subject to $(\beta_{\bar{Y}}\circ\partial_{\bar{Y}} - \beta_Y\circ\partial_Y)h = [\gamma, h]$.

The perturbative results obtained here for $\tilde{A}$ may be expressed in the form (7.51), although this can require additional constraints on γ beyond those required for integrability of (7.9). As was already shown in [19] the low order results in (7.30), with the one and two loop expressions for γ in (5.21), can be expressed in the form (7.51). At lowest order it is necessary that

$$\Lambda^{(2)}\circ \mathrm{d}\bar{Y} = \tfrac{1}{6}\,Y\circ \mathrm{d}\bar{Y} \quad \Rightarrow \quad 3\,(\bar{Y}\Lambda^{(2)}) = \gamma^{(1)}\,. \tag{7.62}$$

In general at the next order we may take

$$\Lambda^{(3)}\circ \mathrm{d}\bar{Y} = \lambda\,\mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}Y)\big)\,, \qquad \Theta^{(2)}\circ \mathrm{d}\bar{Y} = \theta\,(\mathrm{d}\bar{Y}Y)\,. \tag{7.63}$$

In this case

$$3\,(\bar{Y}\Lambda^{(3)}) - \Theta^{(2)}\circ\beta_{\bar{Y}}{}^{(1)} = (\lambda - \tfrac{1}{2}\theta)\,\big(2(\bar{Y}Y\bar{Y}Y) + (\bar{Y}Y)^2\big)\,. \tag{7.64}$$

Equating this to $\gamma^{(2)} - \gamma^{(1)2}$, in accord with (7.54), requires

$$\lambda - \tfrac{1}{2}\theta = -\tfrac{1}{4}\,, \tag{7.65}$$

and using (7.63) in (7.55) is compatible with (7.26) for

$$a = 2\lambda = -\tfrac{1}{2} + \theta\,, \qquad b = 2\lambda + \theta = -\tfrac{1}{2} + 2\theta\,. \tag{7.66}$$

For $\tilde{A}^{(4)}$ given by (7.30), (7.51) is then satisfied with $H^{(2)} = 0$.

At the next order there are several terms which may contribute to $\Lambda^{(4)}$ and $\Theta^{(3)}$ in (7.54). Assuming Λ, Θ can both be represented in terms of 1PI graphs we then take

$$\begin{aligned} \Lambda^{(4)}\circ \mathrm{d}\bar{Y} &= \alpha\,\mathrm{tr}\big((\bar{Y}Y)\,(\mathrm{d}\bar{Y}Y\bar{Y}Y)\big) + \beta\,\mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}Y\bar{Y}Y)\big) + D\,\tfrac{1}{9}d_{\bar{Y}}\mathrm{tr}\big(\gamma_D\big)\,, \\ \Theta^{(3)}\circ \mathrm{d}\bar{Y} &= \sigma\,(\mathrm{d}\bar{Y}Y\bar{Y}Y) + \tau\,(\bar{Y}Y\mathrm{d}\bar{Y}Y)\,, \end{aligned} \tag{7.67}$$

where a potential contribution $\mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,(\bar{Y}Y)^2\big)$ to $\Lambda^{(4)}\circ \mathrm{d}\bar{Y}$ is discarded as it would be necessary later to set the coefficient to zero for consistency. Imposing now

$$3\,(\bar{Y}\Lambda^{(4)}) - \Theta^{(3)}\circ\beta_{\bar{Y}}{}^{(1)} = \gamma^{(3)} - \gamma^{(1)}\gamma^{(2)} - \gamma^{(2)}\gamma^{(1)}\,, \tag{7.68}$$

requires then

$$\sigma = \tfrac{1}{4} - 2B - \tfrac{1}{2}\theta\,, \quad \tau = -\tfrac{1}{4} - 2A + 2B + \tfrac{1}{2}\theta\,, \quad \alpha = \tfrac{1}{8} - \tfrac{1}{4}\theta\,, \quad \beta = \tfrac{3}{8} - B - \tfrac{1}{4}\theta\,, \tag{7.69}$$

and the solution requires the constraint on $\gamma^{(3)}$

$$A - 2B - \tfrac{1}{2}C = -\tfrac{1}{2}\,. \tag{7.70}$$

This is satisfied by the calculated results (5.29) and the derivation remains valid if additional 1PR contributions are allowed in $\Theta^{(3)}$ in (7.67).

Using (7.67) with (7.69) in (7.55) gives contributions to $T^{(4)}, K^{(4)}$ of the form (7.34), (7.35) with

$$\begin{aligned} &a_1 = a_2 = 2\beta = \tfrac{3}{4} - 2B - \tfrac{1}{2}\theta\,, \quad a_3 = 2\beta + \tau = \tfrac{1}{2} - 2A\,, \quad a_4 = 2\alpha = \tfrac{1}{4} - \tfrac{1}{2}\theta\,, \\ &a_5 = 2\beta - \theta = \tfrac{3}{4} - 2B - \tfrac{3}{2}\theta\,, \qquad b_1 = \ b_2 = \ b_3 = 0\,, \quad e = f = 0\,, \end{aligned} \tag{7.71}$$

which is compatible with (7.39) for $X_1 = \frac{5}{8} - B - \frac{3}{4}\theta$, $X_2 = 1 - 2B - \frac{1}{2}\theta$, $X_3 = \frac{1}{8} - \frac{1}{4}\theta$ and $X_4 = 0$ so long as a, b satisfy (7.66). With these results we may check

$$\begin{aligned} \tilde{A}^{(5)} = {} & -\mathrm{tr}\big(\gamma^{(1)}\gamma^{(3)}\big) - \tfrac{1}{2}\,\mathrm{tr}\big(\gamma^{(2)2}\big) + \mathrm{tr}\big(\gamma^{(1)2}\gamma^{(2)}\big) \\ & + \Lambda^{(2)}\circ\beta_{\bar{Y}}{}^{(3)} + \Lambda^{(3)}\circ\beta_{\bar{Y}}{}^{(2)} + \Lambda^{(4)}\circ\beta_{\bar{Y}}{}^{(1)}\,, \end{aligned} \tag{7.72}$$

as required by (7.51) to this order with $H = 0$. The results for Λ may be expressed also in the form

$$\begin{aligned} &\Lambda^{(2)}\circ \mathrm{d}\bar{Y} = \mathrm{d}_{\bar{Y}}\,\tfrac{1}{6}\mathrm{tr}\big((\bar{Y}Y)\big)\,, \qquad \Lambda^{(3)}\circ \mathrm{d}\bar{Y} = \mathrm{d}_{\bar{Y}}\,\tfrac{1}{2}\lambda\,\mathrm{tr}\big((\bar{Y}Y)^2\big)\,, \\ &\Lambda^{(4)}\circ \mathrm{d}\bar{Y} = (\alpha - \tfrac{1}{2}\beta)\,\mathrm{tr}\big((\bar{Y}Y)\,(\mathrm{d}\bar{Y}Y\bar{Y}Y)\big) \\ &\qquad\qquad + \mathrm{d}_{\bar{Y}}\big(\tfrac{1}{2}\beta\,\mathrm{tr}\big((\bar{Y}Y)\,(\bar{Y}Y\bar{Y}Y)\big) + \tfrac{1}{9}D\,\mathrm{tr}\big(\gamma_D\big)\big)\,. \end{aligned} \tag{7.73}$$

At higher orders the number of potential constraints increases when the number of inequivalent lines of a $(\ell+1)$-loop vacuum graph, related to the number of terms in $\gamma^{(\ell)}$, becomes larger than the number of inequivalent vertices, which are related to possible contributions to $\Lambda^{(\ell+1)}$. The calculations of [34] for $\gamma^{(4)}$ in terms of $Y, \bar{Y}$ correspond to 11 distinct graphs which are related to 6 5-loop vacuum graphs giving 13 possible $\Lambda^{(5)}$. However the number of independent terms in $\gamma^{(4)}$ may be reduced by considering redefinitions as in (7.23) with $h \propto \gamma_A, \gamma_B, \gamma_C, \gamma_D$ and letting $\beta_{\bar{Y}} \to \beta_{\bar{Y}}{}^{(1)}$, $\gamma \to \gamma^{(1)}$. By taking $h = \frac{3}{4}\zeta(4)\,\gamma_D$ all terms, corresponding to non planar graphs which contain the γ_D subgraph, involving $\zeta(4)$ in the expression given in [34] are generated by (7.23). There are 7 planar graphs relevant for $\gamma^{(4)}$ and applying (7.54) in conjunction with lower order contributions gives one relation, which is invariant under changes of scheme and is analogous to (7.70), amongst the coefficients. This is satisfied by results of [34].

Some calculations checking the validity of the essential equations (7.9) or (7.54) at each loop order when new transcendental numbers appear are also undertaken in appendix A.

8. Renormalisation with Local Couplings

The results derived in section 2 can be specialised to renormalisable quantum field theories when the metric G_{IJ} and other quantities may be calculated in a perturbative loop expansion on a curved space background. Within the framework of dimensional regularisation with minimal subtraction on flat space there is also a precise prescription for determining quantities, such as S_{IJ} and W_I, which are initially defined in terms of contributions involving $\partial_\mu\sigma$, in terms of the σ-independent counterterms, necessary for a finite theory, which are simple poles in $\varepsilon = 4 - d$.

To demonstrate this we consider initially a generic renormalisable quantum field theory described by a Lagrangian density $\mathcal{L}$ formed from fields Φ and their conjugates $\bar{\Phi}$ depending on local couplings $\{g^I(x)\}$ for a complete set of marginal operators $\{\mathcal{O}_I(x)\}$. For renormalisability $\mathcal{L}$ must contain background gauge fields $\{a_\mu(x)\}$ and local couplings $\{M(x)\}$ for all relevant dimension two operators, corresponding to contributions to $\mathcal{L}$ of the form $\mathcal{L}_M = -\bar{\Phi}\, M\, \Phi$. In $\mathcal{L}$ the kinetic terms, which are bilinear in the scalar/fermion fields Φ and their conjugates $\bar{\Phi}$ and have the form $\mathcal{L}_K = -\bar{\Phi}\,\mathcal{K}(\partial)\,\Phi$, are invariant under a maximal symmetry group G_K where, for any $\mathrm{g} \in \mathrm{G}_K$, $\Phi \to \mathrm{g}\,\Phi$ and $\bar{\Phi} \to \bar{\Phi}\,\bar{\mathrm{g}}$ we require $\bar{\mathrm{g}}\,\mathrm{g} = 1$, $\bar{\mathrm{g}}\,\mathcal{K}(\partial)\,\mathrm{g} = \mathcal{K}(\partial)$. For infinitesimal transformations corresponding to the associated Lie algebra $\mathfrak{g}_K$ then for $\omega \in \mathfrak{g}_K$, $\omega + \bar{\omega} = 0$. In general G_K is not simple but is a product of $U(n)$'s or $O(n)$'s. The symmetry G_K extends to the complete action $\mathcal{L}$ if the couplings are also transformed appropriately, so that for any $\omega \in \mathfrak{g}_K$ then δg^I is given by (2.34). A local symmetry G_K is obtained as usual by replacing all derivatives in $\mathcal{K}(\partial)$ by appropriate covariant derivatives $D_\mu = \partial_\mu + a_\mu$ for $a_\mu(x) \in \mathfrak{g}_K$. In general then $\mathcal{L}(\Phi, \bar{\Phi}, g, a, M)$.

As usual a finite quantum field theory in a perturbative expansion obtained from $\mathcal{L}$ is achieved at each order by adding appropriate local counterterms $\mathcal{L}_{\text{c.t.}}$. As well as counterterms involving $\Phi, \bar\Phi$ with x-dependent couplings, additional local contributions independent of the fields involving contributions containing $\prod_i \partial^{m_i} g^{I_i}$ with $\sum_i m_i \le 4$ and also $f_{\mu\nu}$ as defined in (2.36), are also necessary. Assuming an invariant regularisation then all derivatives of the couplings are extended to covariant derivatives, $\partial_\mu g^I \to D_\mu g^I$, as in (2.35). RG equations are obtained by assuming that $\mathcal{L}$ is such that the bare Lagrangian generating a finite perturbation expansion order by order is

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L}(\Phi, \bar\Phi, g, a, M) + \mathcal{L}_{\text{c.t.}}(\Phi, \bar\Phi, g, a, M) \\ &= \mathcal{L}(\Phi_0, \bar\Phi_0, g_0, a_0, M_0) - \frac{1}{16\pi^2}\, \mathcal{X}(g, a, M)\,. \end{aligned} \tag{8.1}$$

$\mathcal{X}$ includes all the extra field independent counterterms and is arbitrary up to total derivatives. Assuming dimensional regularisation with minimal subtraction, then in a loop expansion

$$\mathcal{L}_{\text{c.t.}}(\Phi, \bar\Phi, g, a, M)^{(\ell)} = \sum_{r=1}^{\ell} \mathcal{L}_{\text{c.t.}}(\Phi, \bar\Phi, g, a, M)_r^{(\ell)}\, \frac{1}{\varepsilon^r}\,, \tag{8.2}$$

so that $\mathcal{X}$ contains just poles in ε.

The RG flow equations which are considered here are obtained from

$$\begin{aligned}\Big(\varepsilon\,\sigma - \mathcal{D}_\sigma - \mathcal{D}_{\sigma,\Phi,\bar\Phi} - (2-\varepsilon)\,\partial_\mu\sigma\, D^\mu g^I \frac{\partial}{\partial D^2 g^I}\Big)&\mathcal{L}(\Phi_0, \bar\Phi_0, g_0, a_0, M_0) \\ &= \partial^\mu\Big(\partial_\mu \sigma\, T\cdot \frac{\partial}{\partial M}\, \mathcal{L}(\Phi_0, \bar\Phi_0, g_0, a_0, M_0)\Big)\,,\end{aligned} \tag{8.3}$$

where σ is linear in x, of the same form as σ_v in (4.8), and the right hand side for $T \in V_M$ is a potential total derivative contribution when σ is not constant which can be neglected in the subsequent discussion. In (8.3) $\mathcal{D}_\sigma, \mathcal{D}_{\sigma,\Phi,\bar\Phi}$ are derivatives defined by

$$\begin{aligned}\mathcal{D}_\sigma &= \sigma\hat\beta^I\cdot\frac{\partial}{\partial g^I} + \big(\sigma\rho_I D_\mu g^I - \partial_\mu\sigma\, v\big)\cdot\frac{\partial}{\partial a_\mu} \\ &\quad + \big(\sigma(\gamma_M M - \delta_I\, D^2 g^I - \epsilon_{IJ}\, D^\mu g^I D_\mu g^J) - 2\,\partial_\mu\sigma\,\theta_I D^\mu g^I\big)\cdot\frac{\partial}{\partial M}\,, \\ \mathcal{D}_{\sigma,\Phi,\bar\Phi} &= \big(\sigma\,(\tfrac12\varepsilon - \gamma)\,\Phi\big)\cdot\frac{\partial}{\partial\Phi} + \big(\sigma\,\bar\Phi\,(\tfrac12\varepsilon - \bar\gamma)\big)\cdot\frac{\partial}{\partial\bar\Phi}\,.\end{aligned} \tag{8.4}$$

Here $\mathcal{D}_\sigma, \mathcal{D}_{\sigma,\Phi,\bar\Phi}$ act on local functions of $g^I, a_\mu, M, \Phi, \bar\Phi$ and their derivatives so that for instance acting on $f(g(x), \partial_\mu g(x))$, $h\cdot\frac{\partial}{\partial g} = h(x)\frac{\partial}{\partial g(x)} + \partial_\mu h(x)\frac{\partial}{\partial\partial_\mu g(x)}$. The action of $\mathcal{D}_\sigma$ is then equivalent to the corresponding contributions to the functional derivative operator $\Delta_\sigma + \Delta_{\sigma,a} + \Delta_{\sigma,M}$ defined by (2.2), (2.41) and (2.47) although $\beta^I \to \hat\beta^I$. A derivation of (8.3) is sketched in appendix B.

For the marginal couplings g^I (2.3) becomes

$$\hat{\beta}^I(g) = -\varepsilon\, k_I g^I + \beta^I(g)\,, \tag{8.5}$$

and minimal subtraction ensures that $\beta^I(g)$ is independent of ε. In a loop expansion

$$\big(1 + \textstyle\sum_I k_I g^I \cdot \partial_I - \frac{1}{2}\Phi\cdot\partial_\Phi - \frac{1}{2}\bar{\Phi}\cdot\partial_{\bar{\Phi}}\big)\mathcal{L}_{\rm c.t.}{}^{(\ell)} = \ell\,\mathcal{L}_{\rm c.t.}{}^{(\ell)}\,. \tag{8.6}$$

Amongst the counterterms in $\mathcal{L}+\mathcal{L}_{\rm c.t.}$ for constant g^I the quadratic kinetic terms are in general modified just by the introduction of an appropriate matrix $Z(g) = \bar{Z}(g) = 1+{\rm O}(g)$, $\mathcal{L}_K \to -\bar{\Phi}\, Z\, \mathcal{K}(\partial)\, \Phi$. This determines the anomalous dimension matrices $\gamma(g), \bar{\gamma}(g)$ for the fields $\Phi, \bar{\Phi}$ in (8.3) through

$$\hat{\beta}^I(g)\frac{\partial}{\partial g^I}\, Z(g) = \bar{\gamma}(g)\, Z(g) + Z(g)\,\gamma(g)\,. \tag{8.7}$$

At ℓ loops, with $Z^{(\ell)}$ expanded as in (8.2), (8.7) requires $\gamma^{(\ell)}+\bar{\gamma}^{(\ell)} = -\ell\, Z_1^{(\ell)}$. The standard prescription determines $\gamma^{(\ell)}(g)$ by assuming $\bar{\gamma}(g) = \gamma(g)$ so that the eigenvalues are real. In obtaining RG equations describing the RG flow it is necessary to factorise Z,

$$Z = \bar{\mathcal{Z}}\,\mathcal{Z}\,, \tag{8.8}$$

so that in (8.1)

$$\Phi_0 = \mathcal{Z}\Phi\,, \qquad \bar{\Phi}_0 = \bar{\Phi}\bar{\mathcal{Z}}\,. \tag{8.9}$$

The factorisation in (8.8) has an essential arbitrariness generated by infinitesimal variations $\delta\mathcal{Z} = \omega\,\mathcal{Z}$, $\delta\bar{\mathcal{Z}} = \bar{\mathcal{Z}}\,\bar{\omega} = -\bar{\mathcal{Z}}\,\omega$ for $\omega \in \mathfrak{g}_K$. The RG equations for $\mathcal{Z}$ then take the form, from (8.7),

$$\hat{\beta}^I(g)\frac{\partial}{\partial g^I}\,\mathcal{Z}(g) = \omega(g)\,\mathcal{Z}(g) + \mathcal{Z}(g)\,\gamma(g)\,, \qquad \omega(g) \in \mathfrak{g}_K\,. \tag{8.10}$$

Assuming $\bar{\gamma} = \gamma$ and taking $\mathcal{Z}^{(1)} = \frac{1}{2}Z^{(1)}$, $\mathcal{Z}^{(2)} = \frac{1}{2}Z^{(2)} - \frac{1}{8}Z^{(1)2}$ then combining (8.7) and (8.10) gives $\omega^{(2)} = \frac{1}{4}[\gamma^{(1)}, Z^{(1)}] = 0$ but $\omega^{(3)} = \frac{1}{4}[\gamma^{(2)}, Z^{(1)}] + \frac{1}{4}[\gamma^{(1)}, Z^{(2)}]$ may be non zero. It is possible to choose $\mathcal{Z}$ so that in (8.10) $\omega = 0$ but then $\bar{\gamma} \neq \gamma$ in general.

In (8.1)

$$a_{0\mu} = a_\mu + \mathfrak{r}_I D_\mu g^I\,, \qquad \mathfrak{r}_I \in \mathfrak{g}_K\,, \tag{8.11}$$

is determined so that all terms involving derivatives of Φ or $\bar{\Phi}$ in $\mathcal{L}_{\rm c.t.}$ are absorbed by letting $\Phi, \bar{\Phi} \to \Phi_0, \bar{\Phi}_0$ and $D_\mu\Phi, D_\mu\bar{\Phi} \to D_{0\mu}\Phi_0, D_{0\mu}\bar{\Phi}_0$ with $D_{0\mu} = \partial_\mu + a_{0\mu}$. Hence $\mathcal{L}_{K\,0} = -\bar{\Phi}_0\,\mathcal{K}(D_0)\,\Phi_0$ up to total derivatives. The RG equation from (8.3) then requires from (8.10)

$$\mathcal{D}_\sigma a_{0\mu} = -D_{0\mu}(\sigma\,\omega) = -\partial_\mu(\sigma\,\omega) - \sigma\,[a_{0\mu}, \omega]\,. \tag{8.12}$$

The resulting equations from the terms in (8.12) proportional to σ and $\partial_\mu\sigma$ become

$$\widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\,\mathfrak{r}_I + \tilde{\rho}_I = \partial_I(\upsilon-\omega) + [\mathfrak{r}_I\,,\,\upsilon-\omega]\,, \tag{8.13}$$

for

$$\hat{B}^I = \hat{\beta}^I - (\upsilon g)^I\,, \tag{8.14}$$

and

$$\mathfrak{r}_I\hat{B}^I = \upsilon - \omega\,. \tag{8.15}$$

Assuming $\mathfrak{r}_I,\omega$ contain only poles in ε, so that $\mathfrak{r}_I = \sum_{n\geq 1}\mathfrak{r}_{I,n}\,\varepsilon^{-n}$, the O(1) terms in (8.13) and (8.15) determine $\tilde{\rho}_I,\upsilon$

$$\tilde{\rho}_I = \textstyle\sum_J k_J g^J(\partial_J\,\mathfrak{r}_{I,1} - \partial_I\,\mathfrak{r}_{J,1})\,, \qquad \upsilon = -\sum_I \mathfrak{r}_{I,1}k_I g^I\,. \tag{8.16}$$

Since $\sum_I \tilde{\rho}_I\,k_I g^I = 0$ then contracting (8.13) with $\hat{B}^I$ and using (8.15) shows that these equations require

$$\tilde{\rho}_I\hat{B}^I = \tilde{\rho}_I B^I = 0\,, \tag{8.17}$$

in agreement with (2.52).

The counterterms contained in M_0, where $\mathcal{L}_{M\,0} = -\bar{\Phi}_0\,M_0\,\Phi_0$, have the general form

$$M_0 = Z_M\big(M - \mathfrak{d}_I D^2 g^I - \mathfrak{e}_{IJ}D^\mu g^I D_\mu g^J\big)\,, \tag{8.18}$$

with $\mathfrak{d}_I,\mathfrak{e}_{IJ}\in V_M$, $Z_M : V_M\to V_M$. (8.3) then implies

$$\bigg(\mathcal{D}_\sigma + (2-\varepsilon)\,\partial_\mu\sigma\,D^\mu g^I\frac{\partial}{\partial D^2 g^I}\bigg)M_0 = \sigma\,\big[\omega,\,M_0\big]\,. \tag{8.19}$$

This decomposes into

$$\hat{\beta}^I\frac{\partial}{\partial g^I}Z_M - [\omega,\,Z_M] = -Z_M\,\gamma_M\,, \tag{8.20}$$

which determines ${\gamma_M}^{(\ell)} = \ell\,Z^{(\ell)}_{M\,1}$, and

$$\begin{aligned} -\big(\widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}} - \gamma_M\big)\,\mathfrak{d}_I &= \delta_I\,,\\ -\big(\widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}} - \gamma_M\big)\,\mathfrak{e}_{IJ} - \hat{\Omega}_{IJ}{}^K\mathfrak{d}_K &= \epsilon_{IJ}\,,\\ -\hat{\Psi}_I{}^J\mathfrak{d}_J - \mathfrak{e}_{IJ}\,\hat{B}^J &= \theta_I\,, \end{aligned} \tag{8.21}$$

for $\hat{\Psi}_I{}^J = (1-\tfrac{1}{2}\varepsilon)\,\delta_I{}^J + \partial_I\hat{B}^J + \tfrac{1}{2}(\tilde{\rho}_I g)^J$ and $\hat{\Omega}_{IJ}{}^K$ as in (2.50) with $B\to\hat{B}$. (8.21) then determines the ε independent δ_I,ϵ_{IJ} and

$$\theta_I = (k_I+\tfrac{1}{2})\,\mathfrak{d}_{I,1} + \textstyle\sum_J \mathfrak{e}_{IJ,1}k_J g^J\,. \tag{8.22}$$

By virtue of (8.17), (2.55) also extends to $\big[\widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\,,\,\hat{\Psi}_I{}^J\big] = \hat{\Omega}_{IK}{}^J \hat{B}^K$ so that we may obtain directly from (8.21) the finite relation

$$\big(\widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}} - \gamma_M\big)\theta_I = \hat{\Psi}_I{}^J \delta_J + \epsilon_{IJ}\hat{B}^J\,, \tag{8.23}$$

for which the $\mathrm{O}(\varepsilon^0)$ contribution is identical to (2.54) while the $\mathrm{O}(\varepsilon)$ terms equivalently determine θ_I in terms of δ_I, ϵ_{IJ}.

The additional field independent local counterterms in (8.1) may be reduced, by discarding total derivatives, to the form

$$\begin{aligned}\mathcal{X}(g,a,M) = {}& \tfrac{1}{2}\mathcal{A}_{IJ}\, D^2 g^I D^2 g^J + \mathcal{B}_{IJK}\, D^2 g^I\, D^\mu g^J D_\mu g^K \\ & + \tfrac{1}{2}\,\mathcal{C}_{IJKL}\, D^\mu g^I D_\mu g^J\, D^\nu g^K D_\nu g^L \\ & + \tfrac{1}{4}\, f^{\mu\nu}\cdot \mathcal{L}_f \cdot f_{\mu\nu} + \tfrac{1}{2}\, M\cdot\mathcal{L}_M\cdot M + f^{\mu\nu}\cdot \mathcal{P}_{IJ}\, D_\mu g^I D_\nu g^J \\ & + \mathcal{J}_I\cdot M\, D^2 g^I + \mathcal{K}_{IJ}\cdot M\, D^\mu g^I D_\mu g^J\,.\end{aligned} \tag{8.24}$$

Assuming this expression the flat space contributions X, Y in (2.60) are determined through the RG equation

$$\begin{aligned}&\Big(\varepsilon\,\sigma - \mathcal{D}_\sigma - (2-\varepsilon)\,\partial_\mu\sigma\, D^\mu g^I \frac{\partial}{\partial D^2 g^I}\Big)\mathcal{X}(g,a,M) - \sigma\, X(g,a,M) + 2\,\partial_\mu\sigma\, Y^\mu(g,a,M) \\ &\quad = -\,2\,\partial_\mu\sigma\,\big(\partial_\nu(\mathcal{G}_{IJ}\, D^\mu g^I D^\nu g^J) - \tfrac{1}{2}\,\partial^\mu(\mathcal{G}_{IJ}\, D^\nu g^I D_\nu g^J)\big) \\ &\quad = -\,2\,\partial_\mu\sigma\,\big(\mathcal{G}_{IJ} D^\mu g^I D^2 g^J - \mathcal{G}_{IJ}(f^{\mu\nu}g)^I D_\nu g^J + \Gamma^{(\mathcal{G})}{}_{IJK} D^\mu g^I D^\nu g^J D_\nu g^K\big)\,,\end{aligned} \tag{8.25}$$

allowing on the right hand side a total derivative which generates terms of the same form as in $\mathcal{X}$ and X, Y^μ as given by (2.61). To obtain (8.25) we assume that $\mathcal{G}_{IJ} = \mathcal{G}_{JI}$ satisfies $(\omega g)^K \partial_K \mathcal{G}_{IJ} + \mathcal{G}_{KJ}\omega^K{}_I + \mathcal{G}_{IK}\omega^K{}_J = 0$. The contributions in (8.25) arising from $\mathcal{G}_{IJ}$ are the same form as the terms in Y^μ which involve $S_{(IJ)}, T_{IJK}, Q_I$ so ε-independent contributions to $\mathcal{G}_{IJ}$ give rise to a corresponding ambiguity in Y^μ. This freedom is removed by requiring that $\mathcal{G}_{IJ}$ contains only poles in ε.

Decomposing (8.25) we find for the M-dependent terms

$$\begin{aligned}\big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\,\mathcal{J}_I - \mathcal{J}_I\cdot\gamma_M + \delta_I\cdot\mathcal{L}_M &= J_I\,, \\ \big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\,\mathcal{K}_{IJ} - \mathcal{K}_{IJ}\cdot\gamma_M - \hat{\Omega}_{IJ}{}^K\mathcal{J}_K + \epsilon_{IJ}\cdot\mathcal{L}_M &= K_{IJ}\,, \\ \hat{\Psi}_I{}^J\mathcal{J}_J + \mathcal{K}_{IJ}\,\hat{B}^J - \theta_I\cdot\mathcal{L}_M &= L_I\,,\end{aligned} \tag{8.26}$$

which determine J_I, K_{IJ}, L_I so that

$$L_I = -(k_I + \tfrac{1}{2})\,\mathcal{J}_{I,1} - \textstyle\sum_J \mathcal{K}_{IJ,1} k_J g^J\,. \tag{8.27}$$

Using (8.23), and in a similar fashion, assuming

$$\big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\mathcal{L}_M - \gamma_M \cdot \mathcal{L}_M - \mathcal{L}_M \cdot \gamma_M = \beta_M \,, \tag{8.28}$$

(8.26) requires for consistency $(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}})L_I - L_I \cdot \gamma_M = \hat{\Psi}_I{}^J J_J + K_{IJ}\hat{B}^J - \theta_I \cdot \beta_M$ which is equivalent to (2.72). For the contributions involving $f^{\mu\nu}$ (8.25) reduces to

$$\begin{gathered}
\omega \cdot \big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\, \mathcal{P}_{IJ} - (\omega g)^K \tilde{\rho}_K \cdot \mathcal{P}_{IJ} - \tfrac{1}{2}\, \omega \cdot \mathcal{L}_f \cdot (\partial_I \tilde{\rho}_J - \partial_J \tilde{\rho}_I) = \omega \cdot P_{IJ} \,, \\
\omega \cdot \big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\, \mathcal{L}_f \cdot \omega' - \omega \cdot \mathcal{L}_f \cdot (\omega' g)^K \tilde{\rho}_K - (\omega g)^K \tilde{\rho}_K \cdot \mathcal{L}_f \cdot \omega' = \omega \cdot \beta_f \cdot \omega' \,, \\
- \omega \cdot \mathcal{P}_{IJ} \hat{B}^J + \tfrac{1}{2}\, \omega \cdot \mathcal{L}_f \cdot \tilde{\rho}_I + \mathcal{G}_{IJ} (\omega g)^J = \omega \cdot Q_I \,.
\end{gathered} \tag{8.29}$$

To obtain (8.29) we presume G_K covariance as in (2.42) to ensure $\hat{\beta}, \rho \to \hat{B}, \tilde{\rho}$ so that for instance $\omega \cdot \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\, \mathcal{P}_{IJ} = \omega \cdot \widetilde{\mathcal{L}}_{\hat{\beta},\rho}\, \mathcal{P}_{IJ} - [\omega, v] \cdot \mathcal{P}_{IJ}$. From (8.29)

$$Q_I = \textstyle\sum_J \mathcal{P}_{IJ,1} k_J g^J \,, \tag{8.30}$$

and also from (8.29) we may obtain, using $-(\partial_I \tilde{\rho}_J - \partial_J \tilde{\rho}_I)\hat{B}^J = \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\, \tilde{\rho}_I - (\tilde{\rho}_I g)^J \tilde{\rho}_J$, the finite relation

$$\omega \cdot \big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\, Q_I - (\omega g)^J \tilde{\rho}_J \cdot Q_I = -\omega \cdot P_{IJ} \hat{B}^J + \tfrac{1}{2}\, \omega \cdot \beta_f \cdot \tilde{\rho}_I + G_{IJ} (\omega g)^J \,, \tag{8.31}$$

assuming

$$\big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\, \mathcal{G}_{IJ} = G_{IJ} \,, \tag{8.32}$$

with G_{IJ} ε-independent. For $\varepsilon \to 0$ (8.31) is just (2.66b). Directly from (8.29)

$$\omega \cdot Q_I \hat{B}^I = (\omega g)^I \mathcal{G}_{IJ} \hat{B}^J \,. \tag{8.33}$$

Since (8.30) ensures that $\sum_I Q_I k_I g^I = 0$ so that $Q_I \hat{B}^I = Q_I B^I$, (2.66b) is satisfied if

$$W_I = -\mathcal{G}_{IJ} \hat{B}^J \quad \Rightarrow \quad W_I = \textstyle\sum_J \mathcal{G}_{IJ,1} k_J g^J \,. \tag{8.34}$$

The remaining equations arising from the decomposition of (8.25) are then

$$\begin{aligned}
&\big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\mathcal{A}_{IJ} + 2\, \mathcal{J}_{(I} \cdot \delta_{J)} = A_{IJ} \,, \\
&\big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\mathcal{B}_{IJK} - \hat{\Omega}_{JK}{}^L \mathcal{A}_{IL} + \mathcal{J} \cdot \epsilon_{JK} + \mathcal{K}_{JK} \cdot \delta_I = B_{IJK} \,, \\
&\big(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}\big)\mathcal{C}_{ILJK} - \hat{\Omega}_{IL}{}^M \mathcal{B}_{MJK} - \hat{\Omega}_{JK}{}^M \mathcal{B}_{MIL} + \mathcal{K}_{IL} \cdot \epsilon_{JK} + \mathcal{K}_{JK} \cdot \epsilon_{IL} \\
&\quad + (\partial_I \tilde{\rho}_{(J} - \partial_{(J} \tilde{\rho}_I) \cdot \mathcal{P}_{K)L} + (\partial_L \tilde{\rho}_{(J} - \partial_{(J} \tilde{\rho}_L) \cdot \mathcal{P}_{K)I} = C_{ILJK} \,,
\end{aligned} \tag{8.35}$$

and also for terms involving $\partial_\mu \sigma$,

$$\begin{gathered}
\hat{\Psi}_I{}^K \mathcal{A}_{KJ} + \mathcal{B}_{JIK} \hat{B}^K - \mathcal{J}_J \cdot \theta_I = S_{IJ} + \mathcal{G}_{IJ} \,, \\
\hat{\Psi}_I{}^L \mathcal{B}_{LJK} + \mathcal{C}_{ILJK} \hat{B}^L - \mathcal{K}_{JK} \cdot \theta_I - \tilde{\rho}_{(J} \cdot \mathcal{P}_{K)I} = T_{IJK} + \Gamma^{(\mathcal{G})}{}_{IJK} \,.
\end{gathered} \tag{8.36}$$

This determines

$$\begin{aligned} S_{IJ} = & -(k_I + \tfrac{1}{2})\,\mathcal{A}_{IJ,1} - \textstyle\sum_K \mathcal{B}_{JIK,1} k_K g^K \,, \\ T_{IJK} = & -(k_I + \tfrac{1}{2})\,\mathcal{B}_{IJK,1} - \textstyle\sum_L \mathcal{C}_{ILJK,1} k_L g^L \,. \end{aligned} \tag{8.37}$$

Since $(\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}})\Gamma^{(\mathcal{G})}{}_{IJK} - \hat{\Omega}_{JK}{}^L \mathcal{G}_{IL} + ((\partial_I \tilde{\rho}_{(J} - \partial_{(J}\tilde{\rho}_I)g)^L\, \mathcal{G}_{K)L} = \Gamma^{(G)}{}_{IJK}$ then applying $\varepsilon - \widetilde{\mathcal{L}}_{\hat{B},\tilde{\rho}}$ to (8.36) and using (8.35) gives finite relations which, after dropping $\mathrm{O}(\varepsilon)$ contributions, are identical to (2.74) and (2.75).

Furthermore eliminating $\mathcal{A}_{IJ}, \mathcal{B}_{IJK}, \mathcal{C}_{IJKL}$ from (8.36) gives

$$L_{[I}\cdot\theta_{J]} + \tfrac{1}{2}\,\tilde{\rho}_{[I}\cdot Q_{J]} - \tilde{S}_{IJ} = \Gamma^{(\mathcal{G})}{}_{[IJ]K}\hat{B}^K - \big(\hat{\Psi}_{[I}{}^K - \tfrac{1}{2}(\tilde{\rho}_{[I}g)^K\big)\mathcal{G}_{J]K} = \partial_{[I}W_{J]} \,, \tag{8.38}$$

where $\tilde{S}_{[IJ]} = -\hat{\Psi}_{[I}{}^K S_{J]K} + T_{[IJ]K}\hat{B}^K = -\Psi_{[I}{}^K S_{J]K} + T_{[IJ]K}B^K$ and W_I is determined by (8.34). Hence (2.65) is recovered.

9. Calculations for a Scalar Fermion Theory

For the theory defined by (5.1), where $\Phi = (\phi, \psi, \chi)$, $g^I = \{y^i, \bar{y}_i, \lambda_{ij}{}^{kl}\}$, then the kinetic symmetry group $\mathrm{G}_K = U(n_\phi)\times U(n_\psi)\times U(n_\chi)$ and for $\omega \in \mathfrak{g}_K$ then

$$\omega = -\omega^\dagger = \{\omega_{\phi\, i}{}^j, \omega_\psi, \omega_\chi\}\,, \quad \omega\cdot\omega' = \omega_{\phi\, i}{}^j \omega'_{\phi\, j}{}^i + \mathrm{tr}(\omega_\psi\, \omega'_\psi) + \mathrm{tr}(\omega_\chi\, \omega'_\chi)\,. \tag{9.1}$$

To allow application of the formalism of section 2 it is necessary to extend the theory to include background gauge fields $a_\mu = \{a_{\phi\mu\, i}{}^j, a_{\psi\mu}, a_{\chi\mu}\} = -a_\mu{}^\dagger \in \mathfrak{g}_K$ and a scalar field mass term

$$\begin{aligned} \mathcal{L} = & -D\bar{\phi}^i\cdot D\phi_i - \bar{\psi}\, i\sigma\cdot D\,\psi - \bar{\chi}\, i\bar{\sigma}\cdot D\,\chi - \bar{\chi}\, y^i\phi_i\,\psi - \bar{\psi}\,\bar{\phi}^i\,\bar{y}_i\,\chi \\ & - M_i{}^j\bar{\phi}^i\phi_j - \tfrac{1}{4}\,\lambda_{ij}{}^{kl}\,\bar{\phi}^i\bar{\phi}^j\phi_k\phi_l\,, \end{aligned} \tag{9.2}$$

where the covariant derivatives depending on the background gauge fields are

$$D_\mu\phi_i = \partial_\mu\phi_i + a_{\phi\mu\, i}{}^j\phi_j\,, \quad D_\mu\psi = \partial_\mu\psi + a_{\psi\mu}\psi\,, \quad D_\mu\chi = \partial_\mu\chi + a_{\chi\mu}\chi\,. \tag{9.3}$$

Acting on the local couplings, in accord with (2.35), the covariant derivative is determined by using (6.19) for $(a_\mu g)^I$. For this theory the minimal subtraction $\hat{\beta}$-functions are expressible as in (8.5) in the form

$$\hat{\beta}_{\lambda\, ij}{}^{kl} = -\varepsilon\,\lambda_{ij}{}^{kl} + \beta_{\lambda\, ij}{}^{kl}\,, \quad \hat{\beta}_y{}^i = -\tfrac{1}{2}\varepsilon\, y^i + \beta_y{}^i\,, \quad \hat{\beta}_{\bar{y}\, i} = -\tfrac{1}{2}\varepsilon\,\bar{y}_i + \beta_{\bar{y}\, i}\,. \tag{9.4}$$

To obtain counterterms involving derivatives of the couplings when they are x-dependent the methods described in [35,26], which avoid momentum space, may be adapted. For the theory defined by (5.1), neglecting mass terms and background gauge fields, the propagators are given by

$$\big\langle \psi(x)\,\bar\psi(y)\big\rangle = S(s) = -i\,\bar\sigma\cdot\partial G_0(s)\,, \quad \big\langle \chi(x)\,\bar\chi(y)\big\rangle = \bar S(s) = -i\,\sigma\cdot\partial G_0(s)\,, \tag{9.5}$$

and

$$\big\langle \phi_i(x)\,\bar\phi^j(y)\big\rangle = \delta_i{}^j G_0(s)\,, \tag{9.6}$$

with

$$G_0(s) = \frac{1}{(d-2)S_d}\,(s^2)^{1-\frac12 d}\,, \qquad S_d = \frac{2\pi^{\frac12 d}}{\Gamma(\frac12 d)}\,, \qquad s = x-y\,, \tag{9.7}$$

so that $-\partial^2 G_0(s) = \delta^d(s)$. For graphs involving two vertices the ε poles may be determined by using

$$G_0(s)^n \sim \frac{2}{\varepsilon}\,\frac{1}{(16\pi^2)^{n-1}}\,\frac{1}{(n-1)!^2}\,(\partial^2)^{n-2}\delta^d(s) \quad \text{for} \quad n=2,3,\dots\,, \tag{9.8}$$

and various extensions involving derivatives [35]. At one loop it is sufficient to use (9.8) for $n=2$ since

$$\mathrm{tr}_\sigma\big(S(s)\,\bar S(-s)\big) = -\partial^2 G_0(s)^2\,, \qquad S(s)G_0(s) = -\tfrac12 i\,\bar\sigma\cdot\partial\, G_0(s)^2\,. \tag{9.9}$$

This formalism may also be extended to allow for mass terms and gauge fields in a manifestly gauge covariant fashion.

With these results it is straightforward to obtain

$$\mathcal{L}^{(1)}_{\text{c.t.}} = \frac{1}{\varepsilon}\big(2\,\bar\phi^i\,\mathrm{tr}(\bar y_i\overleftarrow{\partial}\cdot\partial\,y^j)\,\phi_j + \bar\psi\,\bar y_i\,i\sigma\cdot\overleftrightarrow{\partial}y^i\,\psi + \bar\chi\,y^i\,i\bar\sigma\cdot\overleftrightarrow{\partial}\,\bar y_i\,\chi\big)\,, \tag{9.10}$$

for $\overleftrightarrow{\partial} = \frac12(\partial - \overleftarrow{\partial})$ and also rescaling the couplings as in (5.7). At two loops the corresponding contribution to $\mathcal{L}^{(2)}_{\text{c.t.}}$ involving χ is given by

$$\begin{aligned}\mathcal{L}^{(2)}_{\text{c.t.}\chi} = {}& \frac{1}{4\varepsilon^2}(1-\tfrac14\varepsilon)\big(\bar\chi\,y^i\bar y_j\,y^j i\bar\sigma\cdot\partial\,\bar y_i\,\chi - \bar\chi\,y^i\,i\bar\sigma\cdot\overleftarrow{\partial}\,\bar y_j\,y^j\bar y_i\,\chi\big)\\ &- \frac{1}{2\varepsilon^2}(1-\tfrac54\varepsilon)\,\bar\chi\,y^i\,i\bar\sigma\cdot(\bar y_j\overleftrightarrow{\partial}y^j)\,\bar y_i\,\chi\\ &+ \frac{1}{2\varepsilon^2}(1-\tfrac34\varepsilon)\big(\bar\chi\,y^i\,\mathrm{tr}(\bar y_i\,y^j)\,i\bar\sigma\cdot\partial\,\bar y_j\,\chi - \bar\chi\,y^i\,i\bar\sigma\cdot\overleftarrow{\partial}\,\mathrm{tr}(\bar y_i\,y^j)\,\bar y_j\,\chi\big)\\ &- \frac{1}{\varepsilon^2}(1+\tfrac14\varepsilon)\,\bar\chi\,y^i\,i\bar\sigma\cdot\mathrm{tr}(\bar y_i\overleftrightarrow{\partial}y^j)\,\bar y_j\,\chi\,,\end{aligned} \tag{9.11}$$

and similarly for $\mathcal{L}^{(2)}_{\text{c.t.}\psi}$ obtained from (9.11) with $\chi \to \psi$, $y \leftrightarrow \bar{y}$. Furthermore the two loop scalar field counterterm is given by

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\text{c.t.}\phi} = {} & \frac{1}{4\varepsilon}\,\bar{\phi}^i \lambda_{ik}{}^{mn} \overleftarrow{\partial} \cdot \partial \lambda_{mn}{}^{kj} \phi_j \\
& + \frac{2}{\varepsilon^2}(1 - \tfrac{1}{4}\varepsilon)\,\bar{\phi}^i \lambda_{ik}{}^{lj} \operatorname{tr}\big(\partial \bar{y}_l \cdot \partial y^k\big)\phi_j - \frac{1}{2\varepsilon}\,\bar{\phi}^i \lambda_{ik}{}^{lj}\big(\operatorname{tr}(\partial^2 \bar{y}_l\, y^k) + \operatorname{tr}(\bar{y}_l\, \partial^2 y^k)\big)\phi_j \\
& + \bigg(\frac{1}{\varepsilon^2}(1 - \tfrac{3}{4}\varepsilon)\,\bar{\phi}^i \operatorname{tr}\big(\bar{y}_i \overleftarrow{\partial} \cdot y^k \bar{y}_k\, \partial\, y^j\big)\phi_j \\
& \quad + \frac{1}{\varepsilon^2}(1 - \tfrac{1}{4}\varepsilon)\,\Big(\bar{\phi}^i \operatorname{tr}\big(\bar{y}_i\,(y^k \overleftrightarrow{\partial} \bar{y}_k)\cdot \partial\, y^j\big)\phi_j - \bar{\phi}^i \operatorname{tr}\big(\bar{y}_i \overleftarrow{\partial} \cdot (y^k \overleftrightarrow{\partial} \bar{y}_k)\, y^j\big)\phi_j\Big) \\
& \quad - \frac{1}{2\varepsilon^2}(1 - \tfrac{5}{4}\varepsilon)\,\bar{\phi}^i \operatorname{tr}\big(\bar{y}_i\, \partial^2 (y^k\, \bar{y}_k)\, y^j\big)\phi_j - \frac{1}{\varepsilon}\,\bar{\phi}^i \operatorname{tr}\big(\bar{y}_i\, \partial y^k \cdot \partial \bar{y}_k\, y^j\big)\phi_j \\
& \quad + \bar{\phi} \leftrightarrow \phi\,,\ y \leftrightarrow \bar{y}\bigg)\,.
\end{aligned}
\tag{9.12}
$$

The result (9.10) then determines

$$
\begin{aligned}
\mathcal{Z}^{(1)} = {} & -\frac{1}{\varepsilon}\big\{\operatorname{tr}(\bar{y}_i\, y^j),\ \tfrac{1}{2}\,\bar{y}_i\, y^i,\ \tfrac{1}{2}\, y^i \bar{y}_i\big\}\,, \\
a_0{}^{(1)}{}_\mu = \mathfrak{r}_I{}^{(1)} \partial_\mu g^I = {} & -\frac{1}{\varepsilon}\big\{2\operatorname{tr}\big(\bar{y}_i \overleftrightarrow{\partial}_\mu\, y^j\big),\ \bar{y}_i \overleftrightarrow{\partial}_\mu\, y^i,\ y^i \overleftrightarrow{\partial}_\mu\, \bar{y}_i\big\}\,,
\end{aligned}
\tag{9.13}
$$

as well as the required contributions to $M_0{}^{(1)}$

$$
M_0{}^{(1)}{}_i{}^j = \frac{1}{\varepsilon}\Big(2\,\lambda_{ik}{}^{jl} M_l{}^k + \operatorname{tr}(\bar{y}_i\, y^k) M_k{}^j + M_i{}^k \operatorname{tr}(\bar{y}_k\, y^j) - 2\operatorname{tr}\big(\partial^\mu \bar{y}_i\, \partial_\mu y^j\big)\Big)\,. \tag{9.14}
$$

In consequence at one loop from (9.13) using (8.16)

$$
\rho_I{}^{(1)} \mathrm{d}g^I = -\big\{\operatorname{tr}(\bar{y}_i\, \mathrm{d}y^j - \mathrm{d}\bar{y}_i\, y^j),\ \tfrac{1}{2}(\bar{y}_i\, \mathrm{d}y^i - \mathrm{d}\bar{y}_i\, y^i),\ \tfrac{1}{2}(y^i\, \mathrm{d}\bar{y}_i - \mathrm{d}y^i\, \bar{y}_i)\big\}\,. \tag{9.15}
$$

From (9.14) using (8.18) and (8.21)

$$
\delta_I{}^{(1)} \mathrm{d}g^I = 0\,, \qquad \big(\epsilon_{IJ}{}^{(1)} \mathrm{d}g^I \mathrm{d}g^J\big)_i{}^j = 2\operatorname{tr}\big(\mathrm{d}\bar{y}_i\, \mathrm{d}y^j\big)\,, \tag{9.16}
$$

and also

$$
\big(\theta_I{}^{(1)} \mathrm{d}g^I\big)_i{}^j = \tfrac{1}{2}\big(\operatorname{tr}(\bar{y}_i\, \mathrm{d}y^j) + \operatorname{tr}(\mathrm{d}\bar{y}_i\, y^j)\big)\,. \tag{9.17}
$$

From the two loop result (9.11) we may obtain, as well as $Z^{(2)}{}_\chi$,

$$
\begin{aligned}
a_0{}^{(2)}{}_{\chi\mu} = {} & -\frac{1}{4\varepsilon^2}(1 - \tfrac{1}{4}\varepsilon)\,\big(y^i \bar{y}_j\, y^j\, \partial_\mu \bar{y}_i - \partial_\mu y^i\, \bar{y}_j\, y^j \bar{y}_i\big) + \frac{1}{2\varepsilon^2}(1 - \tfrac{5}{4}\varepsilon)\, y^i\, (\bar{y}_j \overleftrightarrow{\partial}_\mu\, y^j)\, \bar{y}_i \\
& - \frac{1}{\varepsilon^2}(1 - \tfrac{3}{4}\varepsilon)\,\operatorname{tr}(\bar{y}_i\, y^j)\,(y^i \overleftrightarrow{\partial}_\mu\, \bar{y}_j) + \frac{1}{\varepsilon^2}(1 + \tfrac{1}{4}\varepsilon)\, y^i \bar{y}_j \operatorname{tr}(\bar{y}_i \overleftrightarrow{\partial}_\mu\, y^j) \\
& - \mathcal{Z}^{(1)}{}_\chi \overleftrightarrow{\partial}_\mu\, \mathcal{Z}^{(1)}{}_\chi - \mathcal{Z}^{(1)}{}_\chi\, a_0{}^{(1)}{}_{\chi\mu} - a_0{}^{(1)}{}_{\chi\mu}\, \mathcal{Z}^{(1)}{}_\chi\,.
\end{aligned}
\tag{9.18}
$$

Also from (9.12)

$$\begin{aligned}
{a_0}^{(2)}{}_{\phi\mu\, i}{}^j = & -\frac{1}{4\varepsilon}\,\lambda_{ik}{}^{mn}\overleftrightarrow{\partial}_{\mu}\,\lambda_{mn}{}^{kj} \\
& -\frac{1}{2\varepsilon^2}(1-\tfrac{3}{4}\varepsilon)\,\big(\mathrm{tr}(\bar y_i\, y^k\bar y_k\,\partial_\mu y^j) - \mathrm{tr}(\partial_\mu \bar y_i\, y^k\bar y_k\, y^j) \\
& \qquad\qquad + \mathrm{tr}(\bar y_k\, y^k\bar y_i\,\partial_\mu y^j) - \mathrm{tr}(y^k\bar y_k\,\partial_\mu\bar y_i\, y^j)\big) \qquad (9.19)\\
& + \frac{1}{\varepsilon^2}(1-\tfrac{1}{4}\varepsilon)\,\big(\mathrm{tr}(\bar y_i\,(y^k\overleftrightarrow{\partial}_{\mu}\,\bar y_k)\,y^j) - \mathrm{tr}((y^k\overleftrightarrow{\partial}_{\mu}\,\bar y_k)\,\bar y_i\, y^j)\big) \\
& - \mathcal{Z}^{(1)}{}_{\phi\, i}{}^k\,\overleftrightarrow{\partial}_{\mu}\,\mathcal{Z}^{(1)}{}_{\phi\, k}{}^j - \mathcal{Z}^{(1)}{}_{\phi\, i}{}^k\,{a_0}^{(1)}{}_{\phi\mu k}{}^j - {a_0}^{(1)}{}_{\phi\mu i}{}^k\,\mathcal{Z}^{(1)}{}_{\phi\, k}{}^j\,.
\end{aligned}$$

Furthermore

$$\begin{aligned}
{M_0}^{(2)}{}_i{}^j = & -\frac{1}{4\varepsilon}\,\partial\lambda_{ik}{}^{mn}\cdot\partial\lambda_{mn}{}^{kj} \\
& -\frac{2}{\varepsilon^2}(1-\tfrac{1}{4}\varepsilon)\,\lambda_{ik}{}^{lj}\,\mathrm{tr}\big(\partial\bar y_l\cdot\partial y^k\big) + \frac{1}{2\varepsilon}\,\lambda_{ik}{}^{lj}\big(\mathrm{tr}(\partial^2\bar y_l\, y^k) + \mathrm{tr}(\bar y_l\,\partial^2 y^k)\big) \\
& -\frac{1}{\varepsilon^2}(1-\tfrac{3}{4}\varepsilon)\,\Big(\mathrm{tr}\big(\partial\bar y_i\cdot y^k\bar y_k\,\partial y^j\big) + \mathrm{tr}\big(\bar y_k\, y^k\,\partial\bar y_i\cdot\partial y^j\big)\Big) \\
& -\frac{1}{\varepsilon^2}(1-\tfrac{1}{2}\varepsilon)\,\Big(\mathrm{tr}\big(\bar y_i\, y^k\,\partial\bar y_k\cdot\partial y^j\big) + \mathrm{tr}\big(\partial\bar y_i\cdot\partial y^k\,\bar y_k\, y^j\big) \\
& \qquad\qquad + \mathrm{tr}\big(\partial\bar y_k\cdot y^k\,\bar y_i\,\partial y^j\big) + \mathrm{tr}\big(\bar y_k\,\partial y^k\cdot\partial\bar y_i\, y^j\big)\Big) \\
& + \frac{1}{4\varepsilon}\,\Big(\mathrm{tr}\big(\bar y_i\,\partial y^k\cdot\bar y_k\,\partial y^j\big) + \mathrm{tr}\big(\partial\bar y_i\cdot y^k\,\partial\bar y_k\, y^j\big) \qquad (9.20)\\
& \qquad\qquad + \mathrm{tr}\big(\bar y_k\,\partial y^k\cdot\bar y_i\,\partial y^j\big) + \mathrm{tr}\big(\partial\bar y_k\cdot y^k\,\partial\bar y_i\, y^j\big)\Big) \\
& -\frac{1}{4\varepsilon}\Big(\mathrm{tr}\big(\bar y_i\,\partial^2(y^k\,\bar y_k)\,y^j\big) + \mathrm{tr}\big(\partial^2(\bar y_k\, y^k)\,\bar y_i\, y^j\big)\Big) \\
& + \frac{1}{\varepsilon}\Big(\mathrm{tr}\big(\bar y_i\,\partial y^k\cdot\partial\bar y_k\, y^j\big) + \mathrm{tr}\big(\partial\bar y_k\cdot\partial y^k\,\bar y_i\, y^j\big)\Big) \\
& - \big(\partial\mathcal{Z}^{(1)}{}_{\phi\, i}{}^k - {a_0}^{(1)}{}_{\phi\, i}{}^k\big)\cdot\big(\partial\mathcal{Z}^{(1)}{}_{\phi\, k}{}^j + {a_0}^{(1)}{}_{\phi\, k}{}^j\big) \\
& - \mathcal{Z}^{(1)}{}_{\phi\, i}{}^k {M_0}^{(1)}{}_k{}^j - {M_0}^{(1)}{}_i{}^k\mathcal{Z}^{(1)}{}_{\phi\, k}{}^j + \mathrm{O}(M)\,.
\end{aligned}$$

Letting in (9.13), using (6.19),

$$\begin{aligned}
y^i\overleftrightarrow{\partial}_{\mu}\,\bar y_i \to y^i\overleftrightarrow{D}_{\mu}\,\bar y_i = & \, y^i\overleftrightarrow{\partial}_{\mu}\,\bar y_i + y^i\,a_{\psi\mu}\,\bar y_i + y^i\bar y_j\,a_{\phi\mu i}{}^j - \tfrac{1}{2}\,a_{\chi\mu}\,y^i\bar y_i - \tfrac{1}{2}\,y^i\bar y_i\,a_{\chi\mu}\,, \\
\mathrm{tr}\big(\bar y_i\overleftrightarrow{\partial}_{\mu}\,y^j\big) \to \mathrm{tr}\big(\bar y_i\overleftrightarrow{D}_{\mu}\,y^j\big) = & \,\mathrm{tr}\big(\bar y_i\overleftrightarrow{\partial}_{\mu}\,y^j\big) + \mathrm{tr}\big(\bar y_i\,a_{\chi\mu}\,y^j\big) - \mathrm{tr}\big(a_{\psi\mu}\,\bar y_i\,y^j\big) \\
& - \tfrac{1}{2}\,\mathrm{tr}(\bar y_i\,y^k)\,a_{\phi\mu k}{}^j - \tfrac{1}{2}\,a_{\phi\mu i}{}^k\,\mathrm{tr}(\bar y_k\,y^j)\,, \qquad (9.21)
\end{aligned}$$

we may verify that the RG equations (8.12) are consistent with the double pole terms in (9.18) and (9.19) with $\omega = 0$ to this order. The double ε-poles in (9.20) are also determined by (8.26).

The two loop results (9.18) and (9.19) then entail

$$
\begin{aligned}
\big(\rho_I{}^{(2)} \mathrm{d}g^I\big)_\chi &= \tfrac{1}{8}\,\big(y^i\bar y_j\, y^j\, \mathrm{d}\bar y_i - \mathrm{d}y^i\, \bar y_j\, y^j \bar y_i\big) - \tfrac{5}{8}\, y^i\,(\bar y_j\, \mathrm{d}y^j - \mathrm{d}\bar y_i\, y^i)\, \bar y_i \\
&\quad + \tfrac{3}{4}\,\mathrm{tr}(\bar y_i\, y^j)\,(y^i\, \mathrm{d}\bar y_j - \mathrm{d}y^i\, \bar y_j) + \tfrac{1}{4}\, y^i \bar y_j\, \mathrm{tr}(\bar y_i\, \mathrm{d}y^j - \mathrm{d}\bar y_i\, y^j)\,, \\
\big(\rho_I{}^{(2)} \mathrm{d}g^I\big)_{\phi i}{}^j &= -\tfrac{1}{4}\,(\lambda_{ik}{}^{mn}\, \mathrm{d}\lambda_{mn}{}^{kj} - \mathrm{d}\lambda_{ik}{}^{mn}\, \lambda_{mn}{}^{kj}) \\
&\quad + \tfrac{3}{4}\,\big(\mathrm{tr}(\bar y_i\, y^k \bar y_k\, \mathrm{d}y^j) - \mathrm{tr}(\mathrm{d}\bar y_i\, y^k \bar y_k\, y^j) \\
&\qquad\quad + \mathrm{tr}(\bar y_k\, y^k\, \bar y_i\, \mathrm{d}y^j) - \mathrm{tr}(y^k \bar y_k\, \mathrm{d}\bar y_i\, y^j)\big) \\
&\quad - \tfrac{1}{4}\big(\mathrm{tr}(\bar y_i\,(y^k\, \mathrm{d}\bar y_k - \mathrm{d}y^k\, \bar y_k)\, y^j) - \mathrm{tr}((y^k\, \mathrm{d}\bar y_k - \mathrm{d}y^k\, \bar y_k)\, \bar y_i\, y^j)\big)\,.
\end{aligned}
\tag{9.22}
$$

A related calculation, which was extended to three loops, was described in [15]. Also from (8.12) we obtain

$$
\upsilon^{(1)} = \upsilon^{(2)} = 0\,. \tag{9.23}
$$

A useful check is to restrict (9.15) and (9.22) to the supersymmetric case (5.17), where, with a similar notation to that in section 7,

$$
\begin{aligned}
\big(\rho_I{}^{(1)} \mathrm{d}g^I\big)_{\mathrm{Susy}\, i}{}^j &= \tfrac{1}{2}\big(-(\bar Y \mathrm{d}Y)_i{}^j + (\mathrm{d}\bar Y Y)_i{}^j\big)\,, \\
\big(\rho_I{}^{(2)} \mathrm{d}g^I\big)_{\mathrm{Susy}\, i}{}^j &= \tfrac{1}{2}\big((\bar Y Y \bar Y \mathrm{d}Y)_i{}^j - (\mathrm{d}\bar Y Y \bar Y Y)_i{}^j + (\bar Y Y \mathrm{d}\bar Y Y)_i{}^j - (\bar Y \mathrm{d}Y \bar Y Y)_i{}^j\big)\,.
\end{aligned}
\tag{9.24}
$$

These results are in accord with (7.4), using (5.21).

The condition (2.52), which links different loop orders, provides a further verification of the results (9.15) and (9.22). It is easy to check that $\rho_I{}^{(1)}\beta^{(1)I} = 0$ and also

$$
\begin{aligned}
\big(\rho_I{}^{(2)}\beta^{I(1)}\big)_\psi &= -\big(\rho_I{}^{(1)}\beta^{I(2)}\big)_\psi = \tfrac{1}{16}\,\bar y_i\, y^j \bar y_j\, y^i \bar y_k\, y^k + \tfrac{3}{8}\,\bar y_i\, y^j \bar y_k\, y^k\, \mathrm{tr}(y^i \bar y_j) - \text{conjugate}\,, \\
\big(\rho_I{}^{(2)}\beta^{I(1)}\big)_{\phi i}{}^j &= -\big(\rho_I{}^{(1)}\beta^{I(2)}\big)_{\phi i}{}^j = 2\,\lambda_{ik}{}^{mn}\, \mathrm{tr}(\bar y_m\, y^k\, \bar y_n\, y^j) - \tfrac{1}{4}\,\lambda_{ik}{}^{mn}\lambda_{mn}{}^{kl}\, \mathrm{tr}(\bar y_l\, y^j) \\
&\qquad + \tfrac{3}{4}\big(\mathrm{tr}(\bar y_i\, y^k \bar y_k\, y^l) + \mathrm{tr}(\bar y_k\, y^k \bar y_i\, y^l)\big)\, \mathrm{tr}(\bar y_l\, y^j)) - \text{conjugate}\,.
\end{aligned}
\tag{9.25}
$$

From (9.20) we may also read off

$$
\begin{aligned}
\big(\delta_I{}^{(2)} \mathrm{d}g^I\big)_i{}^j &= -\lambda_{ik}{}^{lj}\big(\mathrm{tr}(\mathrm{d}\bar y_l\, y^k) + \mathrm{tr}(\bar y_l\, \mathrm{d}y^k)\big) \\
&\qquad + \tfrac{1}{2}\,\mathrm{tr}\big(\bar y_i\,(\mathrm{d}y^k\, \bar y_k + y^k\, \mathrm{d}\bar y_k)\, y^j\big) + \tfrac{1}{2}\,\mathrm{tr}\big((\mathrm{d}\bar y_k\, y^k + \bar y_k\, \mathrm{d}y^k)\, \bar y_i\, y^j\big)\,, \\
\big(\epsilon_{IJ}{}^{(2)} \mathrm{d}g^I \mathrm{d}g^J\big)_i{}^j &= \tfrac{1}{2}\, \mathrm{d}\lambda_{ik}{}^{mn}\, \mathrm{d}\lambda_{mn}{}^{kj} - \lambda_{ik}{}^{lj}\, \mathrm{tr}\big(\mathrm{d}\bar y_l\, \mathrm{d}y^k\big) \\
&\quad - \tfrac{3}{2}\,\big(\mathrm{tr}\big(\mathrm{d}\bar y_i\, y^k \bar y_k\, \mathrm{d}y^j\big) + \mathrm{tr}\big(\bar y_k\, y^k\, \mathrm{d}\bar y_i\, \mathrm{d}y^j\big)\big) \\
&\quad - \mathrm{tr}\big(\bar y_i\, y^k\, \mathrm{d}\bar y_k\, \mathrm{d}y^j\big) - \mathrm{tr}\big(\mathrm{d}\bar y_i\, \mathrm{d}y^k\, \bar y_k\, y^j\big) - \mathrm{tr}\big(\mathrm{d}\bar y_k\, y^k\, \bar y_i\, \mathrm{d}y^j\big) - \mathrm{tr}\big(\bar y_k\, \mathrm{d}y^k\, \mathrm{d}\bar y_i\, y^j\big) \\
&\quad - \tfrac{1}{2}\big(\mathrm{tr}\big(\bar y_i\, \mathrm{d}y^k\, \bar y_k\, \mathrm{d}y^j\big) + \mathrm{tr}\big(\mathrm{d}\bar y_i\, y^k\, \mathrm{d}\bar y_k\, y^j\big) + \mathrm{tr}\big(\bar y_k\, \mathrm{d}y^k\, \bar y_i\, \mathrm{d}y^j\big) + \mathrm{tr}\big(\mathrm{d}\bar y_k\, y^k\, \mathrm{d}\bar y_i\, y^j\big)\big) \\
&\quad - \mathrm{tr}\big(\bar y_i\, \mathrm{d}y^k\, \mathrm{d}\bar y_k\, y^j\big) - \mathrm{tr}\big(\mathrm{d}\bar y_k\, \mathrm{d}y^k\, \bar y_i\, y^j\big)\,.
\end{aligned}
\tag{9.26}
$$

Reducing (9.16) and (9.26) to the supersymmetric case

$$
\begin{aligned}
\big(\epsilon_{IJ}{}^{(1)}\mathrm{d}g^I\mathrm{d}g^J\big)_{\mathrm{Susy}\,i}{}^j &= (\mathrm{d}\bar{Y}\mathrm{d}Y)_i{}^j\,,\\
\big(\epsilon_{IJ}{}^{(2)}\mathrm{d}g^I\mathrm{d}g^J\big)_{\mathrm{Susy}\,i}{}^j &= -\,(\mathrm{d}\bar{Y}Y\bar{Y}\mathrm{d}Y)_i{}^j - (\mathrm{d}\bar{Y}\mathrm{d}Y\bar{Y}Y)_i{}^j\\
&\qquad -(\bar{Y}\mathrm{d}Y\mathrm{d}\bar{Y}Y)_i{}^j - (\bar{Y}Y\mathrm{d}\bar{Y}\mathrm{d}Y)_i{}^j\,,
\end{aligned}
\tag{9.27}
$$

which agrees with (7.5).

The results are compatible with the consistency relation (2.54) or (8.23). Assuming (9.17), and for simplicity $\mathrm{d}g^I = (\mathrm{d}y^i, 0, 0)$, then

$$
\begin{aligned}
\big(\mathcal{L}_{\beta^{(1)}}\theta_I{}^{(1)}\,\mathrm{d}g^I\big)_i{}^j &= \tfrac{1}{2}\,\mathrm{tr}(\bar{y}_i\,y^k\bar{y}_k\,\mathrm{d}y^j) + \tfrac{1}{2}\,\mathrm{tr}(\bar{y}_k\,y^k\bar{y}_i\,\mathrm{d}y^j)\\
&\quad + \tfrac{1}{4}\,\mathrm{tr}(\bar{y}_i\,\mathrm{d}y^k\bar{y}_k\,y^j) + \tfrac{1}{4}\,\mathrm{tr}(\bar{y}_k\,\mathrm{d}y^k\bar{y}_i\,y^j)\\
&\quad + \mathrm{tr}(\bar{y}_i\,y^k)\,\mathrm{tr}(\bar{y}_k\,\mathrm{d}y^j) + \tfrac{1}{2}\,\mathrm{tr}(\bar{y}_i\,\mathrm{d}y^k)\,\mathrm{tr}(\bar{y}_k\,y^j)\,,\\
\big(\gamma_M{}^{(1)}\theta_I{}^{(1)}\,\mathrm{d}g^I\big)_i{}^j &= \lambda_{ik}{}^{lj}\,\mathrm{tr}(\bar{y}_l\,\mathrm{d}y^k) + \tfrac{1}{2}\,\mathrm{tr}(\bar{y}_i\,y^k)\,\mathrm{tr}(\bar{y}_k\,\mathrm{d}y^j) + \tfrac{1}{2}\,\mathrm{tr}(\bar{y}_i\,\mathrm{d}y^k)\,\mathrm{tr}(\bar{y}_k\,y^j)\,,\\
\big((\rho_I{}^{(1)}g)^J\theta_J{}^{(1)}\,\mathrm{d}g^I\big)_i{}^j &= \tfrac{1}{4}\,\mathrm{tr}(\bar{y}_i\,\mathrm{d}y^k\bar{y}_k\,y^j) + \tfrac{1}{4}\,\mathrm{tr}(\bar{y}_k\,\mathrm{d}y^k\bar{y}_i\,y^j) + \tfrac{1}{2}\,\mathrm{tr}(\bar{y}_i\,y^k)\,\mathrm{tr}(\bar{y}_k\,\mathrm{d}y^j)\,.
\end{aligned}
\tag{9.28}
$$

The sum is then equal to $(\delta_I{}^{(2)} + \epsilon_{IJ}{}^{(1)}\beta^{J(1)})_i{}^j\,\mathrm{d}g^I$, as required by (2.54) to this order.

Similar calculations determine $\mathcal{X}$. At one loop there is no dependence on the couplings and

$$
\mathcal{X}(a,M)^{(1)} = \frac{1}{6\varepsilon}\Big(\mathrm{tr}\big(f_\phi{}^{\mu\nu}f_{\phi\mu\nu}\big) + 2\,\mathrm{tr}\big(f_\psi{}^{\mu\nu}f_{\psi\mu\nu}\big) + 2\,\mathrm{tr}\big(f_\chi{}^{\mu\nu}f_{\chi\mu\nu}\big)\Big) + \frac{1}{\varepsilon}\,M_i{}^jM_j{}^i\,, \tag{9.29}
$$

giving $\mathcal{L}_f{}^{(1)}$ and $\mathcal{L}_M{}^{(1)}$ in (8.24). Two loop contributions to $\mathcal{X}$, which determine the leading contributions to $\mathcal{A}_{IJ}, \mathcal{P}_{IJ}, \mathcal{J}_I, \mathcal{K}_{IJ}$ in (8.24), may also be undertaken within the framework of [35]. For the scalar/fermion theory determined by (9.2) there is just one two loop graph involving only the Yukawa couplings. For zero a_μ, M this gives

$$
W^{(2)} = -\int \mathrm{d}^dx\,\mathrm{d}^dy\;\mathrm{tr}\big(y^i(x)\,\bar{y}_i(y)\big)\;\mathrm{tr}_\sigma\big(S(s)\,\bar{S}(-s)\big)\,G_0(s)\,. \tag{9.30}
$$

Since $\mathrm{tr}_\sigma\big(S(s)\,\bar{S}(-s)\big)G_0(s) = -\frac{1}{3}\,\partial^2 G_0(s)^3$ the divergent part of (9.30) is determined by using (9.8) and gives, after rescaling according to (5.7), $\mathcal{X}(g)^{(2)} = \frac{1}{6\varepsilon}\,\mathrm{tr}\big(\partial^2 y^i\,\partial^2\bar{y}_i\big)$ as was obtained in [9].

Extending this two loop calculation to include the additional contributions involving the background gauge fields a_μ and also M gives

$$
\begin{aligned}
&\mathcal{X}(g,a,M)^{(2)}\\
&= \frac{1}{\varepsilon}\,\frac{1}{6}\,\mathrm{tr}\big(D^2y^i\,D^2\bar{y}_i\big) + \frac{2}{3\varepsilon^2}(1+\tfrac{5}{12}\varepsilon)\,\mathrm{tr}\big(D_\mu y^i\,D_\nu\bar{y}_j\big)\,f_\phi{}^{\mu\nu}{}_i{}^j
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3\varepsilon^2}(1-\tfrac{7}{12}\varepsilon)\Big(\mathrm{tr}\big(D_\mu y^i f_\psi{}^{\mu\nu} D_\nu \bar y_i\big)-\mathrm{tr}\big(D_\mu y^i\, D_\nu \bar y_i\, f_\chi{}^{\mu\nu}\big)\Big)\\
&-\frac{2}{\varepsilon^2}(1-\tfrac{1}{4}\varepsilon)\, M_i{}^j\,\mathrm{tr}\big(D^\mu \bar y_j\, D_\mu y^i\big)+\frac{1}{2\varepsilon}\, M_i{}^j\big(\mathrm{tr}(\bar y_j\, D^2 y^i)+\mathrm{tr}(D^2\bar y_j\, y^i)\big)\\
&-\frac{1}{4\varepsilon}\Big(\mathrm{tr}\big(y^i f_\psi{}^{\mu\nu} f_{\psi\,\mu\nu}\,\bar y_i\big)+\mathrm{tr}\big(y^i\bar y_i\, f_\chi{}^{\mu\nu} f_{\chi\,\mu\nu}\big)\Big)-\frac{1}{3\varepsilon^2}(1-\tfrac{1}{12}\varepsilon)\,\mathrm{tr}\big((f^{\mu\nu}y)^i\,(f_{\mu\nu}\bar y)_i\big)\\
&+\frac{1}{6\varepsilon}\,\mathrm{tr}\big((f_{\mu\nu}y)^i\bar y_j\big) f_\phi{}^{\mu\nu}{}_i{}^j+\frac{2}{\varepsilon^2}\,\lambda_{ij}{}^{kl}M_k{}^iM_l{}^j+\frac{2}{\varepsilon^2}(1-\tfrac{1}{2}\varepsilon)\, M_i{}^j\,\mathrm{tr}(\bar y_j\, y^k)\, M_k{}^i\,, \qquad (9.31)
\end{aligned}
$$

which is consistent with the general form (8.24). The RG equation (8.25) provides a non trivial check of the double poles in ε present in $\mathcal{X}^{(2)}$ which are determined in terms of (9.29) and the one loop results (9.15) and (9.16). In the $\mathrm{O}(f^2)$ terms it is useful to note $\mathrm{tr}\big((f^{\mu\nu}y)^i f_{\psi\,\mu\nu}\bar y_i\big) = -\mathrm{tr}\big(y^i f_{\psi\,\mu\nu}(f^{\mu\nu}\bar y)_i\big)$, with similar relations for $f_\psi \to f_\chi, f_\phi$. For $(\omega y)^i, (\omega\bar y)_i = 0$ the $\mathrm{O}(f^2)$ contributions are just the two loop Yukawa contribution to the gauge beta function [26,36].

The two loop contributions to X and Y^μ are determined as in (2.61). This gives using from (8.35) and (8.37)

$$
\begin{aligned}
G_{IJ}{}^{(2)}\mathrm{d}g^I\mathrm{d}g^J &= A_{IJ}{}^{(2)}\mathrm{d}g^I\mathrm{d}g^J = \tfrac{2}{3}\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}y^i)\,,\\
S_{IJ}{}^{(2)}\mathrm{d}g^I\mathrm{d}'g^J &= -\,\tfrac{1}{6}\,\big(\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}'y^i)+\mathrm{tr}(\mathrm{d}'\bar y_i\,\mathrm{d}y^i)\big)\,.
\end{aligned} \qquad (9.32)
$$

For terms involving M using (8.27),

$$
\begin{aligned}
&\big(J_I{}^{(2)}\mathrm{d}g^I\big)_i{}^j = \mathrm{tr}(\bar y_i\,\mathrm{d}y^j)+\mathrm{tr}(\mathrm{d}\bar y_i\, y^j)\,, \quad \big(K_{IJ}{}^{(2)}\mathrm{d}g^I\mathrm{d}g^J\big)_i{}^j = \mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}y^j)\,,\\
&\big(L_I{}^{(2)}\mathrm{d}g^I\big)_i{}^j = -\,\tfrac{5}{8}\,\big(\mathrm{tr}(\bar y_i\,\mathrm{d}y^j)+\mathrm{tr}(\mathrm{d}\bar y_i\, y^j)\big)\,,
\end{aligned} \qquad (9.33)
$$

while for the $f_{\mu\nu}$ terms, if P_{IJ}, Q_I are decomposed as in (9.1),

$$
\begin{aligned}
P_{IJ}{}^{(2)}\mathrm{d}g^I\mathrm{d}'g^J = \big\{ &-\tfrac{5}{18}\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}'y^j-\mathrm{d}'\bar y_i\,\mathrm{d}y^j),\\
&\tfrac{7}{18}\,(\mathrm{d}\bar y_i\,\mathrm{d}'y^i-\mathrm{d}'\bar y_i\,\mathrm{d}y^i),\ \tfrac{7}{18}\,(\mathrm{d}y^i\,\mathrm{d}'\bar y_i-\mathrm{d}'y^i\,\mathrm{d}\bar y_i)\big\}\,,\\
Q_I{}^{(2)}\mathrm{d}g^I = \big\{ &-\tfrac{5}{72}\,\mathrm{tr}(\bar y_i\,\mathrm{d}y^j-\mathrm{d}\bar y_i\, y^j),\ \tfrac{7}{72}\,(\bar y_i\,\mathrm{d}y^i-\mathrm{d}\bar y_i\, y^i),\ -\tfrac{7}{72}\,(\mathrm{d}y^i\,\bar y_i-y^i\,\mathrm{d}\bar y_i)\big\}\,.
\end{aligned} \qquad (9.34)
$$

It is easy to see that $J_I{}^{(2)} = 2\,\theta_I{}^{(1)} = \theta_I{}^{(1)}\cdot\beta_M{}^{(1)}$ in accord with (2.72) at this order. Also $G_{IJ}{}^{(2)}(\omega g)^J = -\tfrac{1}{2}\,\omega\cdot\beta_f{}^{(1)}\cdot\rho_I{}^{(1)}$ as required by (2.66*b*).

At three loops we determine for simplicity just contributions independent of a_μ, M. For the quartic scalar coupling there is a single vacuum graph

$$
W_a^{(3)} = \frac{1}{8}\int \mathrm{d}^d x\,\mathrm{d}^d y\ \lambda_{ij}{}^{kl}(x)\lambda_{kl}{}^{ij}(y)\, G_0(s)^4\,, \qquad (9.35)
$$

which gives, using (9.8) for $n=4$,

$$
\mathcal{X}_a(\lambda)^{(3)} = \frac{1}{\varepsilon}\,\frac{1}{144}\,\partial^2\lambda_{ij}{}^{kl}\,\partial^2\lambda_{kl}{}^{ij}\,. \qquad (9.36)
$$

At four-loop order the vacuum graphs involving just the scalar couplings also give

$$
\begin{aligned}
W^{(4)} = -\frac{1}{48}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{d}^d z\, \big(&\lambda_{ij}{}^{mn}(x)\lambda_{mn}{}^{kl}(y)\lambda_{kl}{}^{ij}(z) + \lambda_{ij}{}^{mn}(y)\lambda_{mn}{}^{kl}(x)\lambda_{kl}{}^{ij}(z) \\
&+ 8\,\lambda_{mi}{}^{nk}(x)\lambda_{nj}{}^{ml}(y)\lambda_{kl}{}^{ij}(z)\big) \\
&\times \mathcal{R}G_0(x-z)^2\, \mathcal{R}G_0(z-y)^2\, \mathcal{R}G_0(x-y)^2\,, \qquad (9.37)
\end{aligned}
$$

where

$$
\mathcal{R}G_0(s)^2 = G_0(s)^2 - \frac{1}{8\pi^2\varepsilon}\,\delta^d(s)\,. \qquad (9.38)
$$

The additional pole term in ε is necessary to ensure subtraction of one loop sub-divergences and would be generated by appropriate counterterms consistent with minimal subtraction. Using results from appendix D the divergent part of (9.37) determines

$$
\begin{aligned}
\mathcal{X}(\lambda)^{(4)} = {}& \frac{1}{\varepsilon^2}(1+\tfrac{11}{12}\varepsilon)\,\frac{1}{288}\,\partial^2\big(\lambda_{ij}{}^{mn}\,\lambda_{mn}{}^{kl} + 4\,\lambda_{im}{}^{kn}\,\lambda_{jn}{}^{lm}\big)\partial^2\lambda_{kl}{}^{ij} \\
& - \frac{1}{\varepsilon}\,\frac{1}{96}\,\big(\lambda_{ij}{}^{mn}\,\partial^2\lambda_{mn}{}^{kl} + 4\,\lambda_{im}{}^{kn}\,\partial^2\lambda_{jn}{}^{lm}\big)\partial^2\lambda_{kl}{}^{ij}\,.
\end{aligned} \qquad (9.39)
$$

It is easy to check that (8.25) determines the double poles in (9.39) using (6.4) and (9.4). (9.39) gives (6.10) with $\bar{G} = -7/216$.

At three-loop order there are also further vacuum graphs involving solely the Yukawa couplings. There are just two relevant graphs which contain two and one fermion loops giving at this order in addition to (9.35)

$$
\begin{aligned}
W_b^{(3)} = {}& \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\; \mathrm{tr}\big(\bar{y}_i(x)\,\partial_x{}^2 Y_{y^j}(x,y)\big)\; \mathrm{tr}\big(\bar{y}_j(y)\,\partial_y{}^2 Y_{y^i}(y,x)\big)\,, \\
W_c^{(3)} = {}& -\frac{1}{8}\int \mathrm{d}^d x\, \mathrm{d}^d y\; \Big(\mathrm{tr}\,\mathrm{tr}_\sigma\big(\bar{y}_i(x)\,\sigma\cdot\partial_x Y_{y^i}(x,y)\,\bar{\sigma}\cdot\overleftarrow{\partial}_y\;\bar{y}_j(y)\,\sigma\cdot\partial_y Y_{y^j}(y,x)\,\bar{\sigma}\cdot\overleftarrow{\partial}_x\big) \\
& + \mathrm{tr}\,\mathrm{tr}_\sigma\big(y^i(x)\,\bar{\sigma}\cdot\partial_x Y_{\bar{y}_i}(x,y)\,\sigma\cdot\overleftarrow{\partial}_y\; y^j(y)\,\bar{\sigma}\cdot\partial_y Y_{\bar{y}_j}(y,x)\,\sigma\cdot\overleftarrow{\partial}_x\big)\Big)\,, \qquad (9.40)
\end{aligned}
$$

using (9.9) with

$$
Y_f(x,y) = \int \mathrm{d}^d z\; \mathcal{R}G_0(x-z)^2\, f(z)\, G_0(z-y)\,. \qquad (9.41)
$$

From (9.41) it is easy to obtain

$$
Y_f(x,y)(-\overleftarrow{\partial}_y{}^2) = \mathcal{R}G_0(s)^2 f(y)\,. \qquad (9.42)
$$

The analysis of (9.40) is more involved than obtaining (9.31) or (9.36) and is described in appendix C by obtaining formulae for the local ε-poles which arise from products of Y_f

with derivatives. Thus

$$\begin{aligned}
\chi_b(y,\bar y)^{(3)} = {}& \frac{2}{9\varepsilon^2}\big(1+\tfrac{5}{12}\varepsilon\big)\,\mathrm{tr}(\bar y_i\,\partial^2 y^j)\,\mathrm{tr}(\partial^2\bar y_j\,y^i) + \frac{1}{9\varepsilon^2}\big(1-\tfrac{25}{12}\varepsilon\big)\,\mathrm{tr}(\partial^2\bar y_i\,\partial^2 y^j)\,\mathrm{tr}(\bar y_j\,y^i)\\
&+\frac{1}{36\varepsilon}\,\big(\mathrm{tr}(\partial^2\bar y_i\,y^j)\,\mathrm{tr}(\partial^2\bar y_j\,y^i) + \mathrm{tr}(\bar y_i\,\partial^2 y^j)\,\mathrm{tr}(\bar y_j\,\partial^2 y^i)\big)\\
&+\frac{2}{9\varepsilon^2}\big(1-\tfrac{7}{12}\varepsilon\big)\big(\mathrm{tr}(\partial^\mu\bar y_i\,\partial^2 y^j)\,\mathrm{tr}(\partial_\mu\bar y_j\,y^i) + \mathrm{tr}(\partial^2\bar y_i\,\partial^\mu y^j)\,\mathrm{tr}(\bar y_j\,\partial_\mu y^i)\big)\\
&-\frac{2}{9\varepsilon^2}\big(1+\tfrac{5}{12}\varepsilon\big)\,\mathrm{tr}(\partial^\mu\bar y_i\,\partial_\mu y^j)\big(\mathrm{tr}(\partial^2\bar y_j\,y^i) + \mathrm{tr}(\bar y_j\,\partial^2 y^i)\big)\\
&-\frac{1}{9\varepsilon}\big(\mathrm{tr}(\partial^\mu\bar y_i\,\partial^2 y^j)\,\mathrm{tr}(\bar y_j\,\partial_\mu y^i) + \mathrm{tr}(\partial^2\bar y_i\,\partial^\mu y^j)\,\mathrm{tr}(\partial_\mu\bar y_j\,y^i)\big)\\
&+\frac{4}{9\varepsilon^3}\big(1+\tfrac{5}{12}\varepsilon-\tfrac{35}{144}\varepsilon^2\big)\,\mathrm{tr}(\partial^\mu\bar y_i\,\partial^\nu y^j)\big(\mathrm{tr}(\partial_\mu\bar y_j\,\partial_\nu y^i) - \mathrm{tr}(\partial_\nu\bar y_j\,\partial_\mu y^i)\big)\\
&+\frac{4}{3\varepsilon^3}\big(1-\tfrac{1}{4}\varepsilon-\tfrac{19}{48}\varepsilon^2\big)\,\mathrm{tr}(\partial^\mu\bar y_i\,\partial_\mu y^j)\,\mathrm{tr}(\partial^\nu\bar y_j\,\partial_\nu y^i)\\
&+\frac{1}{18\varepsilon}\,\mathrm{tr}(\partial^\mu\bar y_i\,\partial^\nu y^j)\big(\mathrm{tr}(\partial_\mu\bar y_j\,\partial_\nu y^i) + \mathrm{tr}(\partial_\nu\bar y_j\,\partial_\mu y^i)\big)\,,
\end{aligned} \tag{9.43}$$

and

$$\begin{aligned}
\chi_c(y,\bar y)^{(3)} = {}& \frac{1}{18\varepsilon^2}\big(1-\tfrac{13}{12}\varepsilon\big)\,\mathrm{tr}\big(\partial^2\bar y_i\,\partial^2 y^i\,\bar y_j\,y^j + \partial^2 y^i\partial^2\bar y_i\,y^j\bar y_j\big)\\
&+\frac{1}{9\varepsilon^2}\big(1-\tfrac{7}{12}\varepsilon\big)\,\mathrm{tr}\big(\bar y_i\,\partial^2 y^i\,\partial^2\bar y_j\,y^j + \partial^2 y^i\,\bar y_i\,y^j\,\partial^2\bar y_j\big)\\
&-\frac{1}{72\varepsilon}\,\mathrm{tr}\big(\partial^2\bar y_i\,y^i\,\partial^2\bar y_j\,y^j + \bar y_i\,\partial^2 y^i\,\bar y_j\,\partial^2 y^j + \partial^2 y^i\,\bar y_i\,\partial^2 y^j\,\bar y_j + y^i\partial^2\bar y_i\,y^j\partial^2\bar y_j\big)\\
&+\frac{1}{9\varepsilon^2}\big(1-\tfrac{1}{12}\varepsilon\big)\,\mathrm{tr}\big(\partial^\mu\bar y_i\,\partial^2 y^i\,\partial_\mu\bar y_j\,y^j + \partial^2\bar y_i\,\partial^\mu y^i\,\bar y_j\,\partial_\mu y^j\\
&\qquad\qquad + \partial^\mu y^i\,\partial^2\bar y_i\,\partial_\mu y^j\,\bar y_j + \partial^2 y^i\,\partial^\mu\bar y_i\,y^j\,\partial_\mu\bar y_j\big)\\
&+\frac{1}{18\varepsilon^2}\big(1+\tfrac{5}{12}\varepsilon\big)\big(\partial^\mu\bar y_i\,\partial_\mu y^i\,(\partial^2\bar y_j\,y^j + \bar y_j\,\partial^2 y^j) + \partial^\mu y^i\partial_\mu\bar y_i\,(\partial^2 y^j\bar y_j + y^j\partial^2\bar y_j)\big)\\
&+\frac{2}{9\varepsilon^3}\big(1-\tfrac{7}{12}\varepsilon-\tfrac{41}{144}\varepsilon^2\big)\,\mathrm{tr}\big(\partial^\mu\bar y_i\,\partial^\nu y^i\,(\partial_\mu\bar y_j\,\partial_\nu y^j - \partial_\nu\bar y_j\,\partial_\mu y^j)\\
&\qquad\qquad + \partial^\mu y^i\,\partial^\nu\bar y_i\,(\partial_\mu y^j\,\partial_\nu\bar y_j - \partial_\nu y^j\,\partial_\mu\bar y_j)\big)\\
&-\frac{1}{18\varepsilon}\,\mathrm{tr}\big(\partial^\mu\bar y_i\,\partial_\mu y^i\,\partial^\nu\bar y_j\,\partial_\nu y^j + \partial^\mu y^i\,\partial_\mu\bar y_i\,\partial^\nu y^j\,\partial_\nu\bar y_j\big)\\
&+\frac{1}{18\varepsilon}\,\mathrm{tr}\big(\partial^\mu\bar y_i\,\partial^\nu y^i\,(\partial_\mu\bar y_j\,\partial_\nu y^j + \partial_\nu\bar y_j\,\partial_\mu y^j)\\
&\qquad\qquad + \partial^\mu y^i\,\partial^\nu\bar y_i\,(\partial_\mu y^j\,\partial_\nu\bar y_j + \partial_\nu y^j\,\partial_\mu\bar y_j)\big)\,.
\end{aligned} \tag{9.44}$$

The double and triple ε-poles are determined by (8.25) starting from (9.31) using the one loop results (9.15) and (9.16).

From (9.36), (9.43) and (9.44) we may determine $X(g)^{(3)}$ and $Y^\mu(g)^{(3)}$. In particular the $\partial^2 g^I \partial^2 g^J$ terms give the three loop contribution to A_{IJ} which involves both the scalar and Yukawa couplings

$$\begin{aligned}
A_{IJ}{}^{(3)}\mathrm{d}g^I\mathrm{d}g^J = {}& \tfrac{1}{24}\,\mathrm{d}\lambda_{ij}{}^{kl}\,\mathrm{d}\lambda_{kl}{}^{ij} \\
& - \tfrac{13}{36}\,\mathrm{tr}\big(\mathrm{d}\bar y_i\,\mathrm{d}y^i\,\bar y_j\,y^j + \mathrm{d}y^i\mathrm{d}\bar y_i\,y^j\bar y_j\big) - \tfrac{7}{18}\,\mathrm{tr}\big(\bar y_i\,\mathrm{d}y^i\,\mathrm{d}\bar y_j\,y^j + \mathrm{d}y^i\,\bar y_i\,y^j\,\mathrm{d}\bar y_j\big) \\
& - \tfrac{1}{12}\,\mathrm{tr}\big(\mathrm{d}\bar y_i\,y^i\,\mathrm{d}\bar y_j\,y^j + \mathrm{d}y^i\bar y_i\,\mathrm{d}y^j\bar y_j + \bar y_i\,\mathrm{d}y^i\,\bar y_j\,\mathrm{d}y^j + \mathrm{d}y^i\,\bar y_i\,\mathrm{d}y^j\,\bar y_j\big) \\
& - \tfrac{25}{18}\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\bar y_j\,y^i) + \tfrac{5}{9}\,\mathrm{tr}(\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\mathrm{d}\bar y_j\,y^i) \\
& + \tfrac{1}{6}\,\big(\mathrm{tr}(\mathrm{d}\bar y_i\,y^j)\,\mathrm{tr}(\mathrm{d}\bar y_j\,y^i) + \mathrm{tr}(\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\bar y_j\,\mathrm{d}y^i)\big)\,. \qquad (9.45)
\end{aligned}$$

At this order there are extra terms necessary to calculate G_{IJ}. Using results in (9.15), (9.17) and (9.32), (9.33) then, since $S_{(IJ)}{}^{(2)} + \frac{1}{2}A_{IJ}{}^{(2)} = 0$, (2.74) gives

$$\begin{aligned}
G_{IJ}{}^{(3)}\mathrm{d}g^I\mathrm{d}g^J = {}& \big(A_{IJ}{}^{(3)} - \tfrac{1}{2}\,(\rho_I{}^{(1)}g)^K A_{KJ}{}^{(2)} - J_I{}^{(2)}\cdot\theta_J{}^{(1)}\big)\,\mathrm{d}g^I\mathrm{d}g^J \\
= {}& \tfrac{1}{24}\,\mathrm{d}\lambda_{ij}{}^{kl}\,\mathrm{d}\lambda_{kl}{}^{ij} \\
& - \tfrac{13}{36}\,\mathrm{tr}\big(\mathrm{d}\bar y_i\,\mathrm{d}y^i\,\bar y_j\,y^j + \mathrm{d}y^i\mathrm{d}\bar y_i\,y^j\bar y_j\big) - \tfrac{5}{9}\,\mathrm{tr}\big(\bar y_i\,\mathrm{d}y^i\,\mathrm{d}\bar y_j\,y^j + \mathrm{d}y^i\,\bar y_i\,y^j\,\mathrm{d}\bar y_j\big) \\
& - \tfrac{25}{18}\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\bar y_j\,y^i) - \tfrac{7}{9}\,\mathrm{tr}(\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\mathrm{d}\bar y_j\,y^i) \\
& - \tfrac{1}{6}\,\big(\mathrm{tr}(\mathrm{d}\bar y_i\,y^j)\,\mathrm{tr}(\mathrm{d}\bar y_j\,y^i) + \mathrm{tr}(\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\bar y_j\,\mathrm{d}y^i)\big)\,. \qquad (9.46)
\end{aligned}$$

This gives the results in (6.7). The additional contributions are crucial in ensuring that the metric satisfies the necessary consistency conditions.

For

$$g_{IJ}\,\mathrm{d}g^I\mathrm{d}g^J = \mathrm{tr}(\bar y_i\,y^i)\,, \qquad (9.47)$$

then

$$\begin{aligned}
\big(\widetilde{\mathcal{L}}_{\beta^{(1)},\rho^{(1)}}\,g_{IJ}\big)\,\mathrm{d}g^I\mathrm{d}g^J = {}& \mathrm{tr}\big(\mathrm{d}\bar y_i\,\mathrm{d}y^i\,\bar y_j\,y^j + \mathrm{d}y^i\mathrm{d}\bar y_i\,y^j\bar y_j\big) \\
& + 2\,\mathrm{tr}\big(\bar y_i\,\mathrm{d}y^i\,\mathrm{d}\bar y_j\,y^j + \mathrm{d}y^i\,\bar y_i\,y^j\,\mathrm{d}\bar y_j\big) \qquad (9.48) \\
& + 2\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\bar y_j\,y^i) + 4\,\mathrm{tr}(\bar y_i\,\mathrm{d}y^j)\,\mathrm{tr}(\mathrm{d}\bar y_j\,y^i)\,,
\end{aligned}$$

which determines the possible freedom in $G_{IJ}{}^{(3)}$ shown in (6.8).

Using (8.34) we may determine from (9.32) and (9.46) the two and three loop contributions to W_I. This gives

$$\begin{aligned}
W_I{}^{(2)}\mathrm{d}g^I = {}& \mathrm{d}\,\tfrac{1}{12}\,\mathrm{tr}(\bar y_i\,y^i)\,, \\
W_I{}^{(3)}\mathrm{d}g^I = {}& \mathrm{d}\big(\tfrac{1}{144}\,\lambda_{ij}{}^{kl}\lambda_{kl}{}^{ij} - \tfrac{5}{48}\,\mathrm{tr}(\bar y_i\,y^j)\,\mathrm{tr}(\bar y_j\,y^i) \qquad (9.49) \\
& - \tfrac{11}{288}\,\mathrm{tr}(\bar y_i\,y^i\,\bar y_j\,y^j + y^i\bar y_i\,y^j\bar y_j)\big)\,.
\end{aligned}$$

Restricting to the supersymmetric case according to the prescription described in section 5 then (9.31), neglecting gauge fields and M terms, gives

$$X(Y,\bar Y)^{(2)}_{\mathrm{Susy}} = \tfrac{1}{6}\,(\partial^2\bar Y\,\partial^2 Y)_i{}^i\,, \qquad (9.50)$$

while from (9.36), (9.43) and (9.44)

$$
\begin{aligned}
X(Y,\bar Y)^{(3)}_{\rm Susy} = {} & -\tfrac{5}{16}\,(\bar Y Y)_i{}^j\,(\partial^2\bar Y\,\partial^2 Y)_j{}^i - \tfrac{1}{8}\,(\bar Y\,\partial^2 Y)_i{}^j\,(\partial^2\bar Y\,Y)_j{}^i \\
& - \tfrac{1}{8}\big((\bar Y\,\partial^\mu Y)_i{}^j\,(\partial^2\bar Y\,\partial_\mu Y)_j{}^i + (\partial^\mu\bar Y\,Y)_i{}^j\,(\partial_\mu\bar Y\,\partial^2 Y)_j{}^i\big) \\
& - \tfrac{1}{16}\,(\partial^\mu\bar Y\partial^\nu Y)_i{}^j\,(\partial_\mu\bar Y\partial_\nu Y)_j{}^i + \tfrac{9}{16}\,(\partial^\mu\bar Y\partial^\nu Y)_i{}^j\,(\partial_\nu\bar Y\partial_\mu Y)_j{}^i \\
& - \tfrac{9}{16}\,(\partial^\mu\bar Y\partial_\mu Y)_i{}^j\,(\partial^\nu\bar Y\partial_\nu Y)_j{}^i\,.
\end{aligned}
\tag{9.51}
$$

This three loop result has been verified by an independent superspace calculation.

In the supersymmetric case the gauge field contributions at two loops may be obtained from the calculations in the scalar/fermion model by letting $a_\psi = -a_\chi{}^T = a_\phi$ so that the results in (9.34) may be added to give

$$
\begin{aligned}
\big(P_{IJ}{}^{(2)}{\rm d}g^I{\rm d}'g^J\big)_{{\rm Susy}\,i}{}^j &= \tfrac14({\rm d}\bar Y{\rm d}'Y - {\rm d}'\bar Y{\rm d}Y)_i{}^j\,, \\
\big(Q_I{}^{(2)}{\rm d}g^I\big)_{{\rm Susy}\,i}{}^j &= \tfrac{1}{16}({\rm d}\bar Y Y - \bar Y{\rm d}Y)_i{}^j\,.
\end{aligned}
\tag{9.52}
$$

Assuming (7.28) then (7.20) gives $(\Delta Q_I{}^{(2)}{\rm d}g^I)_{\rm Susy} = \frac32 z({\rm d}\bar Y Y - \bar Y{\rm d}Y)$ so that $Q^{(2)}\to 0$ if $z=-\frac{1}{24}$. If this is done, from (7.29) and (7.31), $a\to-\frac34$, $b\to -1$. Furthermore from the ${\rm O}(f^2)$ terms in (9.31)

$$
\begin{aligned}
\big(\omega\cdot\beta_f{}^{(1)}\cdot\omega\big)_{\rm Susy} &= 2\,{\rm tr}(\omega^2)\,, \\
\big(\omega\cdot\beta_f{}^{(2)}\cdot\omega\big)_{\rm Susy} &= -\,2\,{\rm tr}\big(\omega^2(\bar Y Y)\big) - \tfrac13\,(Y*\omega)\circ(\omega*\bar Y)\,.
\end{aligned}
\tag{9.53}
$$

These results (9.53) together with (9.52) are sufficient to check (2.66b) with the three loop G_{IJ} given by (7.26) and (7.31). The one and two loop expressions for β_f are compatible with an extension of the NSVZ formula for the matter contributions to $\mathcal{N}=1$ gauge β-function of the form $(\omega\cdot\beta_f\cdot\omega)_{\rm Susy} = 2\,{\rm tr}\big(\omega^2(1-2\gamma)\big) - (Y*\omega)\circ\tilde G\circ(\omega*\bar Y)$.

10. Conclusion

In this paper we have endeavoured to show that the existence of a metric on the space of couplings, for renormalisable theories at least, and the associated equations, which are related to gradient flow, provide significant constraints on β-functions and anomalous dimensions. These results are applicable in the context of the standard model in that their application here provides a partial check of the three-loop Yukawa β-function in [17]. For supersymmetric theories there are additional constraints such as the metric being hermitian and Kähler which might follow from an extension of the present discussion to superspace.

A critical issue which has not been analysed in any detail here is the role of anomalies which render the assumption of invariance under arbitrary gauge transformations G_K invalid. This is crucial for a more complete analysis of supersymmetric theories where careful analysis of anomaly matching links IR and UV limits under RG flow [31].

In this paper we have avoided perturbative calculations on curved space backgrounds. Nevertheless the techniques described here for three loop calculations of vacuum graphs with local couplings should allow an extension to arbitrary metrics following [37] although as always the calculational details are non trivial.

Acknowledgements

We are grateful to Konstantin Chetyrkin and and Max Zoller for informing about some details of their work on three loop Yukawa β-functions. HO is grateful to Boaz Keren-Zur for several discussions about the details of their approach described in [22] and its relation to results in this paper. This allowed some simplifications in the expressions obtained in sections 2 and 3 and later correction of some errors. IJ is grateful to John Gracey for showing him how to apply MINCER [38], [39] and its relevance in three loop calculations. HO would also like to express his thanks for hospitality at the University of Crete, CERN and Syddansk Universitet during the slow progress of this work and also to KITP, UCSB for the final revision.

Appendix A. Higher Loops in the Wess-Zumino Supersymmetric Theory

In higher loop calculations of the anomalous dimension γ transcendental numbers, such as $\zeta(3)$ in three or more loops, arise. These numbers are associated to diagrams with particular topologies which are possible initially only at some minimal loop order ℓ. The connection between particular transcendental numbers and a particular graph topology is valid only up to scheme dependent contributions to γ and these need to be considered separately. For each such non scheme dependent term γ_ζ contributing to $\gamma^{(\ell)} = \mathrm{O}(Y^\ell \bar{Y}^\ell)$, which is proportional to a transcendental number ζ and corresponds to diagrams involving a topology which are not present at lower loop orders, the equations simplify. It is only necessary then to consider the lowest order $T^{(2)}$ and also $T^{(\ell+1)}$ to determine the associated contribution to $\tilde{A}^{(\ell+2)}$.

The simplest case is when $\mathrm{tr}(\gamma_\zeta)$ corresponds to a connected symmetric graph with $\ell+1$ loops and ℓ Y-vertices linked to ℓ $\bar{Y}$-vertices. Such graphs are edge transitive so that all 3ℓ lines are related by an automorphism and are therefore equivalent. In this case γ_ζ may be recovered from $\mathrm{tr}(\gamma_\zeta)$ by cutting any line. This implies the identity, for any $\omega_i{}^j$ and with notation as in section 7,

$$\mathrm{tr}(\omega\, \gamma_\zeta) = \frac{1}{3\ell}\,(\omega * \bar{Y}) \circ \partial_{\bar{Y}}\ \mathrm{tr}(\gamma_\zeta) = \frac{1}{3\ell}\,(Y * \omega) \circ \partial_Y\ \mathrm{tr}(\gamma_\zeta)\,. \tag{A.1}$$

With $\beta_{\zeta\bar{Y}} = (\gamma_\zeta * \bar{Y})$ and $T^{(2)}$ given by (7.25)

$$\mathrm{d}Y \circ T^{(2)} \circ \beta_{\zeta\bar{Y}} = \mathrm{tr}\big((\bar{Y}\mathrm{d}Y)\,\gamma_\zeta\big)\,. \tag{A.2}$$

To ensure integrability in (7.9) it is necessary to assume $T^{(\ell+1)}$ contains a term proportional to ζ of the form

$$\mathrm{d}Y \circ T_\zeta \circ \mathrm{d}\bar{Y} = \frac{2}{3\ell}\,\mathrm{d}_{\bar{Y}}\,\mathrm{d}_Y\ \mathrm{tr}(\gamma_\zeta)\,, \tag{A.3}$$

since then

$$\mathrm{d}Y \circ T^{(2)} \circ \beta_{\zeta\bar{Y}} + \mathrm{d}Y \circ T_\zeta \circ \beta_{\bar{Y}}{}^{(1)} = \mathrm{d}_Y\ \mathrm{tr}\big((\bar{Y}Y)\,\gamma_\zeta\big)\,. \tag{A.4}$$

In consequence there is an associated contribution $\tilde{A}_\zeta$ to $\tilde{A}^{(\ell+2)}$ given by

$$\tilde{A}_\zeta = \tfrac{1}{2}\,\mathrm{tr}\big((\bar{Y}Y)\,\gamma_\zeta\big) = \frac{1}{3\ell}\,\beta_{\bar{Y}}{}^{(1)} \circ \partial_{\bar{Y}}\ \mathrm{tr}(\gamma_\zeta) = \frac{1}{3\ell}\,\beta_Y{}^{(1)} \circ \partial_Y\ \mathrm{tr}(\gamma_\zeta)\,. \tag{A.5}$$

For this case

$$-\mathrm{tr}(\gamma^{(1)}\gamma_\zeta) + \Lambda^{(2)} \circ \beta_{\zeta\bar{Y}} = 0\,, \tag{A.6}$$

so that we must take in (7.51)

$$\tilde{A}_\zeta = \Lambda_\zeta \circ \beta_{\bar{Y}}{}^{(1)}\,, \tag{A.7}$$

where Λ_ζ is part of $\Lambda^{(\ell+1)}$. Hence from (A.5)

$$\Lambda_\zeta \circ \mathrm{d}\bar{Y} = \frac{1}{3\ell}\, \mathrm{d}_{\bar{Y}}\, \mathrm{tr}(\gamma_\zeta) \quad \Rightarrow \quad 3\,(\bar{Y}\Lambda_\zeta) = \gamma_\zeta\,, \tag{A.8}$$

in accord with (7.54). In this case the metric $G_\zeta = T_\zeta$ so that (A.3) ensures (7.45) is satisfied in this case with

$$F_\zeta = \frac{2}{3\ell}\, \mathrm{tr}(\gamma_\zeta)\,. \tag{A.9}$$

These results apply when $\ell = 3$ for the term proportional to $\zeta(3)$, $\gamma_{\zeta(3)} = D\,\gamma_D$ where $\mathrm{tr}(\gamma_{\zeta(3)}) = \frac{3}{2}\zeta(3)\,(Y^3\bar{Y}^3)_{K_{3,3}}$, and also when $\ell = 4$, according to [34], for the term proportional to $\zeta(5)$, which satisfies $\mathrm{tr}(\gamma_{\zeta(5)}) = -10\zeta(5)\,(Y^4\bar{Y}^4)_{M_8}$, with the vertices contracted as in the symmetric non planar graphs $K_{3,3}$, with 6 vertices 9 edges, and M_8 forming a cube respectively.

At the next order there are additional non planar contributions to $\gamma^{(4)}$ which are proportional to $\zeta(3)$. These are determined by the corresponding term at three loops. To show this we consider a contribution to γ, in addition to the ℓ-loop γ_ζ satisfying (A.1), at $\ell+1$ loops which is expressed in terms of γ_ζ. It is sufficient to assume that the relevant term in $\gamma^{(\ell+1)}$ has the form

$$\gamma'_{\zeta}{}_i{}^j = A\,\bar{Y}_{ikm}Y^{lmj}\gamma_{\zeta\,l}{}^k\,, \tag{A.10}$$

with an undetermined coefficient A. As usual γ'_ζ determines $\beta'_{\zeta\bar{Y}} = (\gamma'_\zeta * \bar{Y})$ and hence we may obtain $\mathrm{d}Y \circ T^{(2)} \circ \beta'_{\zeta\bar{Y}} + \mathrm{d}Y \circ T^{(3)} \circ \beta_{\zeta\bar{Y}}$ which is part of $\mathrm{d}_Y\tilde{A}'_\zeta$. There are also contributions $\mathrm{d}Y \circ T_\zeta \circ \beta_{\bar{Y}}{}^{(2)}$, determined by (A.3), but it is necessary also to allow corresponding terms in $T^{(\ell+1)}$ and $K^{(\ell+1)}$. Assuming these must contain the subgraph associated with γ_ζ they can have the general form proportional to γ_ζ

$$\begin{aligned} \mathrm{d}Y \circ T'_\zeta \circ \mathrm{d}\bar{Y} &= \alpha_1\,\mathrm{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\,\gamma_\zeta\big) + \alpha_2\,\mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,\mathrm{d}_Y\gamma_\zeta\big) + \alpha_3\,\mathrm{tr}\big((\bar{Y}\mathrm{d}Y)\,\mathrm{d}_{\bar{Y}}\gamma_\zeta\big) \\ &\quad + \alpha_4\,\mathrm{tr}\big((\bar{Y}Y)\,\mathrm{d}_{\bar{Y}}\mathrm{d}_Y\gamma_\zeta\big)\,, \\ \mathrm{d}'Y \circ K'_\zeta \circ \mathrm{d}Y &= \beta\,\mathrm{tr}\big((\bar{Y}\mathrm{d}'Y)\,\mathrm{d}_Y\gamma_\zeta\big) - \mathrm{d}'Y \leftrightarrow \mathrm{d}Y\,. \end{aligned} \tag{A.11}$$

To calculate $\mathrm{d}Y \circ T'_\zeta \circ \beta_{\bar{Y}}{}^{(1)} + \beta_Y{}^{(1)} \circ K'_\zeta \circ \mathrm{d}Y$ we use the identities

$$\big(\beta_{\bar{Y}}{}^{(1)} \circ \partial_{\bar{Y}} - \beta_Y{}^{(1)} \circ \partial_Y\big)\,\gamma_\zeta = \tfrac{1}{2}\,\big[(\bar{Y}Y), \gamma_\zeta\big]\,, \tag{A.12}$$

a special case of (7.8) valid for any γ_ζ, and

$$\mathrm{tr}\big((\bar{Y}\mathrm{d}Y)\,\beta_{\bar{Y}}{}^{(1)} \circ \partial_{\bar{Y}}\,\gamma_\zeta\big) = \mathrm{tr}\big((\bar{Y}Y)\,(\mathrm{d}_Y\beta_{\bar{Y}}{}^{(1)}) \circ \partial_{\bar{Y}}\,\gamma_\zeta\big) + \tfrac{1}{2}\,\mathrm{tr}\big(\big[(\bar{Y}\mathrm{d}Y), (\bar{Y}Y)\big]\,\gamma_\zeta\big)\,, \tag{A.13}$$

which may be derived from (A.1) and reflects that all lines in the graph for $\mathrm{tr}(\gamma_\zeta)$ are equivalent. Combining all contributions to $\mathrm{d}_Y\tilde{A}'_\zeta$ gives finally

$$2\,\tilde{A}'_\zeta = Y_1\,\mathrm{tr}\big((\bar{Y}Y\bar{Y}Y)\,\gamma_\zeta\big) + Y_2\,\mathrm{tr}\big((\bar{Y}Y)^2\,\gamma_\zeta\big) + Y_3\,\mathrm{tr}\big((\bar{Y}Y)\,\beta_{\bar{Y}}{}^{(1)} \circ \partial_{\bar{Y}}\gamma_\zeta\big)\,, \tag{A.14}$$

where

$$\begin{aligned} Y_1 &= 2a + \alpha_1 = 2b + A = -1 + \alpha_2 + \beta \,, \\ Y_2 &= a + \tfrac{1}{4}(\alpha_3 + \beta) = b - \tfrac{1}{4}(\alpha_3+\beta) + \tfrac{1}{2}\alpha_1 = \tfrac{1}{2}(\alpha_2 + \beta)\,, \\ Y_3 &= \alpha_4 = \tfrac{1}{2}(\alpha_3 - \beta)\,. \end{aligned} \tag{A.15}$$

The equations for Y_1, Y_2 give rise to integrability conditions once more so that we may eliminate $\alpha_1, \alpha_2+\beta, \alpha_3+\beta$ and then determine

$$A = -2\,, \tag{A.16}$$

independent of a, b. Remarkably this agrees with the non planar $\zeta(3)$ term in $\gamma^{(4)}$, after subtracting scheme dependent terms, obtained in [34]. Subject to (A.16)

$$Y_1 = 4a - 1\,, \qquad Y_2 = 2a\,. \tag{A.17}$$

At this order there is the freedom due to (7.18) arising from taking $g = w\, g_\zeta$ for

$$\mathrm{d}Y \circ g_\zeta \circ \mathrm{d}\bar{Y} = \frac{1}{3\ell}\, \mathrm{d}_Y \mathrm{d}_{\bar{Y}} \mathrm{tr}(\gamma_\zeta)\,, \tag{A.18}$$

which leads to an arbitrariness under variations

$$\Delta\alpha_2 = \Delta\alpha_3 = \Delta\alpha_4 = -\Delta\beta = w\,, \tag{A.19}$$

giving $\Delta Y_1 = \Delta Y_2 = 0$, $\Delta Y_3 = w$. The corresponding variation in $\tilde{A}'_\zeta$ follows from

$$\beta_Y{}^{(1)} \circ g_\zeta \circ \beta_{\bar{Y}}{}^{(1)} = \tfrac{1}{2}\, \mathrm{tr}\big((\bar{Y}Y)\, \beta_{\bar{Y}}{}^{(1)} \circ \partial_{\bar{Y}} \gamma_\zeta\big)\,. \tag{A.20}$$

For (7.28), $\mathrm{d}Y \circ g \circ \mathrm{d}\bar{Y} = z\, \mathrm{d}Y \circ \mathrm{d}\bar{Y} = z\, \mathrm{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\big)$, leading to (7.29) $\Delta a = 3z,\ \Delta b = 6z$ then also it is necessary that

$$\Delta\alpha_1 = \Delta\alpha_2 = \Delta\alpha_3 = \Delta\beta = 6z\,, \tag{A.21}$$

so that $\Delta Y_1 = 12z,\ \Delta Y_2 = 6z,\ \Delta Y_3 = 0$. In this case

$$\tfrac{1}{3}\big(\beta_Y{}^{(1)} \circ \beta_{\zeta\,\bar{Y}} + \beta_{\zeta\,Y} \circ \beta_{\bar{Y}}{}^{(1)}\big) = 2\, \mathrm{tr}\big((\bar{Y}Y\bar{Y}Y)\,\gamma_\zeta\big) + \mathrm{tr}\big((\bar{Y}Y)^2\,\gamma_\zeta\big)\,. \tag{A.22}$$

If $a = b = -\frac{1}{2}$, as in (7.47), (A.15) has the solution

$$\alpha_1 = \alpha_2 + \beta = \alpha_3 + \beta = -2\,, \tag{A.23}$$

but the metric G'_ζ obtained then from (A.11) cannot be written in the Kähler form (7.45) for any choice of $\alpha_4 = \frac{1}{2}(\alpha_3 - \beta)$ making use of the the freedom under (A.19). However if we also allow a change of scheme as in (7.24) with $T \to T^{(1)}$ and $h \to -\gamma_\zeta$ so that

$$\mathrm{d}Y \circ \delta T'_\zeta \circ \mathrm{d}\bar{Y} = \mathrm{tr}\big((\mathrm{d}\bar{Y}\mathrm{d}Y)\,\gamma_\zeta\big) + \mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,\mathrm{d}_Y\gamma_\zeta\big) + \mathrm{tr}\big((\bar{Y}\mathrm{d}Y)\,\mathrm{d}_{\bar{Y}}\gamma_\zeta\big)\,, \tag{A.24}$$

then, taking $\beta = 0, \alpha_4 = -1$, $\mathrm{d}Y \circ (T'_\zeta + \delta T'_\zeta) \circ \mathrm{d}\bar{Y} = \mathrm{d}Y \circ G'_\zeta \circ \mathrm{d}\bar{Y} = \mathrm{d}_Y \mathrm{d}_{\bar{Y}} F'_\zeta$ with

$$F'_\zeta = -\mathrm{tr}\big((\bar{Y}Y)\,\gamma_\zeta\big)\,. \tag{A.25}$$

The result (A.14) with (A.17) may also be expressed in the form (7.51). To solve (7.54) it is sufficient to take

$$\Lambda'_\zeta \circ \mathrm{d}\bar{Y} = u\,\mathrm{tr}\big((\mathrm{d}\bar{Y}Y)\,\gamma_\zeta\big) + v\,\mathrm{tr}\big((\bar{Y}Y)\,\mathrm{d}_{\bar{Y}}\gamma_\zeta\big)\,. \tag{A.26}$$

Using $\big[(\omega' * \bar{Y}) \circ \partial_{\bar{Y}}, (\omega * \bar{Y}) \circ \partial_{\bar{Y}}\big] = ([\omega, \omega'] * \bar{Y}) \circ \partial_{\bar{Y}}$ then from (A.1) we may derive

$$\mathrm{tr}\big(\omega\,(\omega' * \bar{Y}) \circ \partial_{\bar{Y}}\gamma_\zeta\big) = \mathrm{tr}\big(\omega'\,(\omega * \bar{Y}) \circ \partial_{\bar{Y}}\gamma_\zeta\big) + \mathrm{tr}\big([\omega, \omega']\,\gamma_\zeta\big)\,, \tag{A.27}$$

and hence obtain, with $\Theta^{(2)}$ as in (7.63),

$$3\,(\bar{Y}\Lambda'_\zeta) - \Theta^{(2)} \circ \beta_{\zeta\bar{Y}} = \gamma'_\zeta + (u - v)\,(\bar{Y}Y)\,\gamma_\zeta + (v - \theta)\,\gamma_\zeta\,(\bar{Y}Y) + v\,(\bar{Y}Y) * \partial_{\bar{Y}}\,\gamma_\zeta\,, \tag{A.28}$$

for γ'_ζ as in (A.10) so long as

$$2(u - \theta) = A\,. \tag{A.29}$$

Hence $3\,(\bar{Y}\Lambda'_\zeta) - \Theta^{(2)} \circ \beta_{\zeta\bar{Y}} - \Theta_\zeta \circ \beta_{\bar{Y}}{}^{(1)} = \gamma'_\zeta - \gamma^{(1)}\gamma_\zeta - \gamma_\zeta\gamma^{(1)}$ if we take

$$\Theta_\zeta \circ \mathrm{d}\bar{Y} = 2v\,\mathrm{d}_{\bar{Y}}\gamma_\zeta\,, \tag{A.30}$$

and

$$u - v = v - \theta = -\tfrac{1}{2}\,. \tag{A.31}$$

Applying (A.31) in (A.29) gives (A.16) once more.

Using (A.26) and (A.30) in (7.55) gives a metric of the form (A.11) with

$$\alpha_1 = 2u = -2 + 2\theta\,, \quad \alpha_2 = \alpha_3 = 4v = -2 + 4\theta\,, \quad \alpha_4 = 2v = -1 + 2\theta\,, \quad \beta = 0\,. \tag{A.32}$$

These results satisfy (A.15) for a, b given by (7.66) so that

$$Y_1 = -3 + 4\theta\,, \qquad Y_2 = Y_3 = -1 + 2\theta\,. \tag{A.33}$$

Since, with $\Lambda^{(2)}, \Lambda^{(3)}$ given in (7.73),

$$\begin{aligned}
& - \operatorname{tr}\big(\gamma^{(1)}\,\gamma'_\zeta\big) + \Lambda^{(2)} \circ \beta'_{\zeta\bar Y} = -\operatorname{tr}\big(\gamma^{(2)}\,\gamma_\zeta\big) + \Lambda_\zeta \circ \beta_{\bar Y}{}^{(2)} = 0\,, \\
& \operatorname{tr}\big(\gamma^{(1)2}\,\gamma_\zeta\big) = \tfrac14 \operatorname{tr}\big((\bar Y Y)^2\,\gamma_\zeta\big)\,, \\
& \Lambda^{(3)} \circ \beta_{\zeta\bar Y} = 2\lambda \operatorname{tr}\big((\bar Y Y \bar Y Y)\,\gamma_\zeta\big) + \lambda \operatorname{tr}\big((\bar Y Y)^2\,\gamma_\zeta\big)\,, \\
& \Lambda'_\zeta \circ \beta_{\bar Y}{}^{(1)} = u \operatorname{tr}\big((\bar Y Y \bar Y Y)\,\gamma_\zeta\big) + \tfrac12 u \operatorname{tr}\big((\bar Y Y)^2\,\gamma_\zeta)\big) + v \operatorname{tr}\big((\bar Y Y)\,\beta_{\bar Y}{}^{(1)} \circ \partial_{\bar Y}\gamma_\zeta\big)\,,
\end{aligned} \tag{A.34}$$

we may verify

$$\begin{aligned}
\tilde A'_\zeta = {} & - \operatorname{tr}\big(\gamma^{(1)}\,\gamma'_\zeta\big) - \operatorname{tr}\big(\gamma^{(2)}\,\gamma_\zeta\big) + \operatorname{tr}\big(\gamma^{(1)2}\gamma_\zeta\big) \\
& + \Lambda^{(2)} \circ \beta'_{\zeta\bar Y} + \Lambda^{(3)} \circ \beta_{\zeta\bar Y} + \Lambda_\zeta \circ \beta_{\bar Y}{}^{(2)} + \Lambda'_\zeta \circ \beta_{\bar Y}{}^{(1)}\,,
\end{aligned} \tag{A.35}$$

as required by (7.51) with $H = 0$.

Appendix B. Derivation of Local RG Equations

Usually RG equations are derived by considering the response to a change of cut off scale or using dimensional regularisation variations in the arbitrary mass scale μ which is necessary for dimensions $d \neq 4$. For the equations in section 8, which are related to broken conformal symmetry, a slightly different approach is required. For renormalisable scalar fermion theories in d dimensions $\mathcal{L}(\Phi_0, \bar\Phi_0, g_0, a_0, M_0)$ can be chosen to be conformal primary under conformal transformations so long as g_0, a_0, M_0 transform appropriately as well as $\Phi_0, \bar\Phi_0$. The generator of conformal transformations for this theory is then, for any conformal Killing vector v^μ,

$$\begin{aligned}
-\mathcal{D}_{0,v} = {} & \big(\mathcal{L}_v \Phi_0 - \tfrac12 \varepsilon\, \sigma_v\, \Phi_0\big) \cdot \frac{\partial}{\partial \Phi_0} + \big(\mathcal{L}_v \bar\Phi_0 - \tfrac12 \varepsilon\, \sigma_v\, \bar\Phi_0\big) \cdot \frac{\partial}{\partial \bar\Phi_0} \\
& + \big(v^\mu \partial_\mu\, g_0{}^I + \varepsilon\, \sigma_v\, k_I g_0{}^I\big) \cdot \frac{\partial}{\partial g_0{}^I} + \mathcal{L}_v a_{0\mu} \cdot \frac{\partial}{\partial a_{0\mu}} + \mathcal{L}_v M_0 \cdot \frac{\partial}{\partial M_0}\,,
\end{aligned} \tag{B.1}$$

for

$$\begin{aligned}
\mathcal{L}_v \Phi_0 &= \big(v^\mu \partial_\mu - \tfrac12\, \omega_v{}^{\mu\nu} s_{\Phi\mu\nu} + \sigma_v\, \Delta_\Phi\big) \Phi_0\,, \\
\mathcal{L}_v \bar\Phi_0 &= \ v^\mu \partial_\mu\, \bar\Phi_0 + \bar\Phi_0 \big(\tfrac12\, \omega_v{}^{\mu\nu} s_{\bar\Phi\mu\nu} + \sigma_v\, \Delta_{\bar\Phi}\big)\,, \\
\mathcal{L}_v a_{0\mu} &= v^\nu \partial_\nu\, a_{0\mu} + \partial_\mu v^\nu a_{0\nu}\,, \\
\mathcal{L}_v M_0 &= v^\mu \partial_\mu\, M_0 + \sigma_v \big(4\, M_0 - \Delta_{\bar\Phi} M_0 - M_0 \Delta_\Phi\big)\,,
\end{aligned} \tag{B.2}$$

where $\omega_v{}^{\mu\nu} = \partial^{[\mu} v^{\nu]}$ and $s_{\Phi\mu\nu}, s_{\bar\Phi\mu\nu}$ are the appropriate spin matrices. $\Delta_\Phi, \Delta_{\bar\Phi}$ are the canonical dimension matrices for $\Phi, \bar\Phi$ when $d = 4$ and in consequence $\mathcal{L}_v$ has no explicit dependence on ε for each case in (B.2). It is easy to verify

$$\big[\mathcal{D}_{0,v}, \mathcal{D}_{0,v'}\big] = \mathcal{D}_{0,[v,v']}\,. \tag{B.3}$$

The crucial assumption is then that $\mathcal{L}$ satisfies[10]

$$-\mathcal{D}_{0,v}\,\mathcal{L}(\Phi_0,\bar\Phi_0,g_0,a_0,M_0) = \big(v^\mu\partial_\mu + d\,\sigma_v\big)\mathcal{L}(\Phi_0,\bar\Phi_0,g_0,a_0,M_0)\,. \tag{B.4}$$

The derivation of finite local RG equations depends on the detailed form of the relation between $\Phi_0,\bar\Phi_0,{g_0}^I,a_{0\mu},M_0$ and the corresponding finite $\Phi,\bar\Phi,g^I,a_\mu,M$ implicitly defined by (8.1). Defining

$$\mathcal{D}_v = -v^\mu\partial_\mu\, g^I\cdot\frac{\partial}{\partial g^I} - \mathcal{L}_v a_\mu\cdot\frac{\partial}{\partial a_\mu} - \mathcal{L}_v M\cdot\frac{\partial}{\partial M} - \mathcal{L}_v\Phi\cdot\frac{\partial}{\partial\Phi} - \mathcal{L}_v\bar\Phi\cdot\frac{\partial}{\partial\bar\Phi}\,, \tag{B.5}$$

then $\mathcal{D}_{0,v}$ may be expressed in terms of $g^I,a_\mu,M,\Phi,\bar\Phi$ in the form

$$\mathcal{D}_{0,v} = \mathcal{D}_v + \mathcal{D}_{\sigma_v} + \mathcal{D}_{\sigma_v,\Phi,\bar\Phi}\,, \tag{B.6}$$

with $\mathcal{D}_\sigma,\mathcal{D}_{\sigma,\Phi,\bar\Phi}$ as in (8.4). The commutation relation (B.3) ensures that the coefficients in $\mathcal{D}_\sigma$ obey the required consistency conditions.

Since $-\mathcal{D}_v\mathcal{Z} = v^\mu\partial_\mu\mathcal{Z}$, $-\mathcal{D}_v D_\mu g^I = \mathcal{L}_v D_\mu g^I$ then

$$-\mathcal{D}_v\,\Phi_0 = \mathcal{L}_v\Phi_0\,, \qquad -\mathcal{D}_v\,\bar\Phi_0 = \mathcal{L}_v\bar\Phi_0\,, \qquad -\mathcal{D}_v\,a_{0\mu} = \mathcal{L}_v a_{0\mu}\,. \tag{B.7}$$

However

$$-\mathcal{D}_v\,D^2g^I = (v^\mu\partial_\mu + 2\,\sigma_v)D^2g^I + \partial^2 v_\mu\,D^\mu g^I\,, \qquad \partial^2 v_\mu = -(d-2)\,\partial_\mu\sigma_v\,. \tag{B.8}$$

As M_0 may contain counterterms involving D^2g^I in general $-\mathcal{D}_v M_0 \neq \mathcal{L}_v M_0$ but taking this into account

$$\begin{aligned}&-\Big(\mathcal{D}_v + (d-2)\,\partial_\mu\sigma_v\,D^\mu g^I\frac{\partial}{\partial D^2g^I}\Big)\mathcal{L}(\Phi_0,\bar\Phi_0,g_0,a_0,M_0)\\ &\qquad\sim \big(v^\mu\partial_\mu + 4\,\sigma_v\big)\mathcal{L}(\Phi_0,\bar\Phi_0,g_0,a_0,M_0)\,,\end{aligned} \tag{B.9}$$

where $\sim$ denotes equality up to total derivatives. Subtracting (B.9) from (B.4) then gives

$$\Big(\varepsilon\,\sigma - \mathcal{D}_\sigma - \mathcal{D}_{\sigma,\Phi,\bar\Phi} - (2-\varepsilon)\,\partial_\mu\sigma\,D^\mu g^I\frac{\partial}{\partial D^2g^I}\Big)\mathcal{L}(\Phi_0,\bar\Phi_0,g_0,a_0,M_0)\sim 0\,, \tag{B.10}$$

for σ linear in x, which is identical to (8.3) for a suitable choice of total derivative contributions. As shown in section 8 (B.10) is sufficient to determine the various contributions to $\mathcal{D}_\sigma$, in particular

$$\hat\beta^J\frac{\partial}{\partial g^J}\,{g_0}^I = -\varepsilon\,k_I{g_0}^I\,. \tag{B.11}$$

[10] This is the condition for $\mathcal{L}$ to be a conformal primary, it dictates the form of the scalar kinetic term so that $\mathcal{L}_{K0} = -\partial\bar\phi_0\cdot\partial\phi_0 + \frac{1}{2}\,\partial^2(\bar\phi_0\,\phi_0)$.

This is equivalent to the standard definition $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g^I|_{g_0} = \hat\beta^I$ when ${g_0}^I = \mu^{k_I\varepsilon}(g^I + L^I(g))$, with L^I containing just poles in ε and gives the standard form (8.5).

We assume also that (B.4) with (B.6) extend also to $\mathcal{L}_0$ including also the field independent counterterms so that

$$\big(\mathcal{D}_v + \mathcal{D}_{\sigma_v} + \mathcal{D}_{\sigma_v,\Phi,\bar\Phi} + v^\mu\partial_\mu + d\,\sigma_v\big)\mathcal{L}_0 \sim \frac{1}{16\pi^2}\big(\sigma_v X - 2\,\partial_\mu\sigma_v\, Y\big)\,, \tag{B.12}$$

In a similar fashion to the above this leads to (8.25). (B.12) directly implies the broken conformal Ward identities discussed in section 4.

Appendix C. Calculations

Assuming continuation to a Euclidean metric the short distance divergent parts in (9.40) may be obtained by using the integral formula

$$\begin{aligned}\frac{1}{\pi^{\frac12 d}}\int &\mathrm{d}^d z\, \big((x-z)^2\big)^{-\lambda}\big((y-z)^2\big)^{-\mu} f(z)\\ &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\sum_{n\ge0}\frac{1}{n!}\,\Gamma(\lambda+\mu-\tfrac12 d-n)\,(s^2)^{\frac12 d-\lambda-\mu+n}\, b_n(x,y)\\ &\quad + \text{terms analytic in } s\,,\end{aligned} \tag{C.1}$$

for

$$b_n(x,y) = \int_0^1 \mathrm{d}t\, t^{\frac12 d-\lambda+n-1}(1-t)^{\frac12 d-\mu+n-1}\,(\tfrac14\partial^2)^n f(x-ts)\,. \tag{C.2}$$

To verify (C.1) it is sufficient to consider Fourier transforms with respect to x, y where

$$\int \mathrm{d}^d x\, e^{ik\cdot x}\,(x^2)^{-\lambda} = \pi^{\frac12 d}\,\frac{\Gamma(\frac12 d-\lambda)}{\Gamma(\lambda)}\,\big(\tfrac14 k^2\big)^{\lambda-\frac12 d}\,, \tag{C.3}$$

and on the right hand side the sum over n reproduces the left hand side within an appropriate region of convergence. For generic λ, μ, b_n satisfies

$$(s\cdot\partial_x + d-\lambda-\mu+n-1)b_n(x,y) - n\,\tfrac14{\partial_x}^2 b_{n-1}(x,y) = (\tfrac12 d-\lambda-1)b_n(x,y)\big|_{\lambda\to\lambda+1}\,, \tag{C.4}$$

as well as the similar equation obtained by $x \leftrightarrow y, \lambda \leftrightarrow \mu$. The t-integration in b_n is convergent when $\lambda, \mu < \frac12 d + n$ but it may be extended by analytic continuation. $b_n(x,y)$ are smooth functions for y in the neighbourhood of x but there are poles for $\lambda, \mu = \frac12 d + n + p$, $p = 0, 1, \ldots$, which reflect short distance sub-divergences. The poles present in the expansion (C.1) at $\lambda + \mu = \frac12 d + n$ are generated by divergences for large z which should be cancelled by the analytic terms assuming $f(z)$ falls off sufficiently fast as $z \to \infty$.

For calculations here the divergent ε-poles are obtained by using, for μ an arbitrary scale mass,

$$\sum_{i=0}^{p} \alpha_i \,(s^2)^{-\frac{1}{2}d-n+\frac{1}{2}\delta_i} = \sum_{i=0}^{p} \alpha_i \,\frac{\mu^{\delta_i}}{\delta_i}\, S_d \,\frac{1}{(\frac{1}{2}d)_n\, n!}\,(\tfrac{1}{4}\partial^2)^n \delta^d(s) + \mathrm{O}(1)\,, \tag{C.5}$$

as $\varepsilon \to 0$ where α_i, δ_i are assumed to depend on ε such that in this limit

$$\delta_i = \mathrm{O}(\varepsilon)\,, \quad \alpha_i = \mathrm{O}(\varepsilon^{-p})\,, \quad \sum_{i=0}^{p} \alpha_i\, \delta_i^r = \mathrm{O}(1)\,, \ r = 0, \dots, p-1\,. \tag{C.6}$$

The conditions (C.6) are necessary and sufficient for the left hand side of (C.5) to have a finite limit as $\varepsilon \to 0$ and also ensure that the pole terms on the right hand side, of $\mathrm{O}(\varepsilon^{-r})$, $r = 1, \dots, p$, have no μ dependence. The result (9.8) is a special case of (C.5).

The results given in (C.1) and (C.5) may be used to obtain the ε-poles reflecting short distance divergences in products involving $Y_f(x,y)$, as defined in (9.41), and also

$$\tilde{Y}_f(x,y) = \int \mathrm{d}^d z\; G_0(x-z)\, f(z)\, G_0(z-y)\,. \tag{C.7}$$

(C.1) gives the expansion, up to terms which are regular as $s \to 0$,

$$\begin{aligned} Y_f(x,y) \sim{}& \frac{1}{4S_d^2}\,\frac{1}{(d-2)^2(d-3)}\, b_{f,0}(x,y)\,(s^2)^{3-d} - \frac{1}{\varepsilon}\,\frac{1}{4S_dS_4}\,\frac{1}{d-2}\, f(x)\,(s^2)^{1-\frac{1}{2}d} \\ &- \frac{1}{4S_d^2}\,\frac{1}{(d-2)^2(d-3)}\, b_{f,1}(x,y)\,\frac{1}{\varepsilon}\big((s^2)^{4-d}-1\big)\,, \end{aligned} \tag{C.8}$$

where

$$b_{f,n}(x,y) = \int_0^1 \mathrm{d}t\; t^{n+1-\frac{1}{2}d}\,(1-t)^n\,(\tfrac{1}{4}\partial^2)^n f(x-ts)\,, \tag{C.9}$$

and also

$$\tilde{Y}_f(x,y) \sim -\frac{1}{4S_d}\,\frac{1}{d-2}\,\frac{2}{\varepsilon}\Big(\tilde{b}_{f,0}(x,y) - \frac{1}{3-\frac{1}{2}d}\,\tilde{b}_{f,1}(x,y)\, s^2\Big)\,\big((s^2)^{2-\frac{1}{2}d}-1\big)\,, \tag{C.10}$$

for

$$\tilde{b}_{f,n}(x,y) = \int_0^1 \mathrm{d}t\; t^n(1-t)^n\,(\tfrac{1}{4}\partial^2)^n f(x-ts)\,. \tag{C.11}$$

In both (C.8) and (C.10) terms which are regular as $s \to 0$ have been subtracted to cancel an IR divergence at $\varepsilon = 0$. The terms omitted in (C.8), (C.10) are then without any ε-poles. In consequence

$$\begin{aligned} Y_f(x,x) &\sim \frac{1}{\varepsilon}\,\frac{1}{S_d^2}\,\frac{1}{(d-2)^2(d-3)(6-d)(8-d)}\,\tfrac{1}{4}\partial^2 f(x)\,, \\ \tilde{Y}_f(x,x) &\sim \frac{1}{\varepsilon}\,\frac{1}{2S_d}\,\frac{1}{d-2}\, f(x)\,. \end{aligned} \tag{C.12}$$

There is also a UV sub-divergence present in $b_{f,0}$ since

$$b_{f,0}(x,y) \sim \frac{2}{\varepsilon} f(x) \, . \tag{C.13}$$

The various results in the text can be obtained from analysing the singularities in products involving $\tilde{Y}_f, Y_f$ using (C.5). For two loop graphs relevant for calculating (9.18), (9.19) we used

$$\begin{aligned}(16\pi^2)^2 \tilde{Y}_f(x,y) \overleftarrow{\partial}_{\mu y} \, Y_g(y,x) \overleftarrow{\partial}_{\nu x} &\sim -(16\pi^2)^2 \partial_{\nu x} \tilde{Y}_f(x,y) \overleftarrow{\partial}_{\mu y} \, Y_g(y,x) \\ &= \frac{1}{2\varepsilon^2} (1 - \tfrac{1}{4}\varepsilon) \, f(x) g(x) \, \delta_{\mu\nu} \delta^d(s) \, ,\end{aligned} \tag{C.14}$$

and

$$\begin{aligned}(16\pi^2)^2 Y_f(x,y) \, G_0(s) &\sim -\frac{2}{\varepsilon^2} (1 - \tfrac{1}{2}\varepsilon) \, f(x) \, \delta^d(s) \, , \\ (16\pi^2)^2 Y_f(x,y) \overleftarrow{\partial}_{\mu y} \, G_0(s) &\sim \frac{1}{\varepsilon^2} (1 - \tfrac{1}{4}\varepsilon) \, f(x) \, \partial_\mu \delta^d(s) + \frac{1}{2\varepsilon} \, \partial_\mu f(x) \, \delta^d(s) \, , \\ (16\pi^2)^2 G_0(s) \overleftarrow{\partial}_{\mu y} \, Y_f(y,x) &\sim \frac{1}{\varepsilon^2} (1 - \tfrac{3}{4}\varepsilon) \, f(y) \, \partial_\mu \delta^d(s) + \frac{1}{2\varepsilon} \, \partial_\mu f(x) \, \delta^d(s) \, .\end{aligned} \tag{C.15}$$

For the three loop integrals in (9.40) it is necessary to determine the ε-poles in various products involving Y_f with Y_g or $G_0{}^2$. These can be reduced to

$$(16\pi^2)^3 \, Y_f(x,y) \, Y_g(y,x) \sim \frac{8}{3\varepsilon^3} \big(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{4}\varepsilon^2\big) \, f(x) g(x) \, \delta^d(s) \, , \tag{C.16a}$$

$$\begin{aligned}(16\pi^2)^3 \, Y_f(x,y) \, \mathcal{R} G_0(s)^2 \sim &-\frac{1}{3\varepsilon^2} \big(1 - \tfrac{1}{4}\varepsilon\big) \big(f(x) \; \partial^2 \delta^d(s) + \partial^2 f(x) \, \delta^d(s) \big) \\ &+ \frac{1}{3\varepsilon} \, f(y) \, \partial^2 \delta^d(s) \, ,\end{aligned} \tag{C.16b}$$

and with one derivative

$$\begin{aligned}(16\pi^2)^3 \, Y_f(x,y) \overleftarrow{\partial}_{\mu y} \, Y_g(y,x) \sim &-\frac{4}{3\varepsilon^3} \big(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{4}\varepsilon^2\big) \, f(x) g(y) \, \partial_\mu \delta^d(s) \\ &-\frac{1}{3\varepsilon^2} \big(1 - \tfrac{1}{4}\varepsilon\big) \, \partial_\mu \big(f(x) g(x) \big) \, \delta^d(s) \, ,\end{aligned} \tag{C.17a}$$

$$\begin{aligned}(16\pi^2)^3 \, Y_f(x,y) \overleftarrow{\partial}_{\mu y} \, \mathcal{R} G_0(s)^2 \sim &\frac{1}{9\varepsilon^2} \big(1 - \tfrac{7}{12}\varepsilon\big) \big(f(x) \, \partial_\mu \partial^2 \delta^d(s) - \partial_\mu \partial^2 f(x) \, \delta^d(s) \big) \\ &+ \frac{2}{9\varepsilon} \big(\partial_\mu f(y) \, \partial^2 \delta^d(s) - \partial_\nu f(y) \, \partial_\mu \partial_\nu \delta^d(s) \big) \, ,\end{aligned} \tag{C.17b}$$

and with two derivatives

$$
\begin{aligned}
&(16\pi^2)^3\, Y_f(x,y)\overleftarrow{\partial}_{\mu y}\, Y_g(y,x)\overleftarrow{\partial}_{\nu x} \\
&\sim -\Big(\frac{2}{9\varepsilon^3}\big(1-\tfrac{1}{12}\varepsilon-\tfrac{83}{144}\varepsilon^2\big)\, f(x)g(y) - \frac{1}{18\varepsilon^2}\big(1-\tfrac{13}{12}\varepsilon\big)\,\big(f(x)g(x)+f(y)g(y)\big)\Big) \\
&\qquad \times\ \big(2\partial_\mu\partial_\nu+\delta_{\mu\nu}\partial^2\big)\delta^d(s) \\
&\quad + \frac{1}{3\varepsilon^2}\big(1-\tfrac{1}{4}\varepsilon\big)\Big(f(x)g(y)-\tfrac{1}{2}\big(f(x)g(x)+f(y)g(y)\big)\Big)\,\partial_\mu\partial_\nu\delta^d(s) \\
&\quad + \frac{1}{18\varepsilon^2}\big(1-\tfrac{1}{12}\varepsilon\big)\,\delta_{\mu\nu}\,\big(f(x)\,\partial^2 g(x)+\partial^2 f(x)\, g(x)\big)\,\delta^d(s) \\
&\quad - \frac{1}{18\varepsilon^2}\big(1-\tfrac{7}{12}\varepsilon\big)\,\big(f(x)\,\partial_\mu\partial_\nu g(x)+\partial_\mu\partial_\nu f(x)\, g(x)\big)\,\delta^d(s) \\
&\quad + \frac{1}{3\varepsilon^2}\big(1-\tfrac{1}{4}\varepsilon\big)\,\partial_{[\mu}\delta^d(s)\,\big(\partial_{\nu]}f(x)\, g(y)+f(x)\,\partial_{\nu]}g(y)\big) \\
&\quad + \frac{1}{9\varepsilon}\big(\delta_{\mu\nu}\,\partial f(x)\cdot\partial g(x)+\partial_\mu f(x)\,\partial_\nu g(x)+\partial_\nu f(x)\,\partial_\mu g(x)\big)\delta^d(s)\,.
\end{aligned}
\tag{C.18}
$$

By integrating $\partial_x{}^2, \partial_y{}^2$ by parts and using (9.42) with (C.16a,b), (C.17a,b) and (C.18) we may obtain

$$
\begin{aligned}
&(16\pi^2)^3\,\frac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\; h(x)\,\partial_x{}^2 Y_f(x,y)\, k(y)\,\partial_y{}^2 Y_g(y,x) \\
&\sim \int\!\mathrm{d}^d x\,\Bigg(\frac{4}{3\varepsilon^3}\big(1-\tfrac{1}{4}\varepsilon-\tfrac{19}{48}\varepsilon^2\big)\,\partial_\mu h\,\partial_\mu f\,\partial_\nu k\,\partial_\nu g+\frac{1}{18\varepsilon}\,\partial_\mu h\,\partial_\nu f\,\big(\partial_\mu k\,\partial_\nu g+\partial_\nu k\,\partial_\mu g\big) \\
&\quad + \frac{4}{9\varepsilon^3}\big(1+\tfrac{5}{12}\varepsilon-\tfrac{35}{144}\varepsilon^2\big)\,\partial_\mu h\,\partial_\nu f\,\big(\partial_\mu k\,\partial_\nu g-\partial_\nu k\,\partial_\mu g\big) \\
&\quad + \frac{1}{18\varepsilon^2}\big(1-\tfrac{25}{12}\varepsilon\big)\big(\partial^2 h\,\partial^2 f\, k\, g+h\, f\,\partial^2 k\,\partial^2 g\big) \\
&\quad + \frac{1}{9\varepsilon^2}\big(1+\tfrac{5}{12}\varepsilon\big)\big(h\,\partial^2 f\,\partial^2 k\, g+\partial^2 h\, f\, k\,\partial^2 g\big)+\frac{1}{36\varepsilon}\big(\partial^2 h\, f\,\partial^2 k\, g+h\,\partial^2 f\, k\,\partial^2 g\big) \\
&\quad + \frac{1}{9\varepsilon^2}\big(1-\tfrac{7}{12}\varepsilon\big)\big(\partial_\mu h\,\partial^2 f\,\partial_\mu k\, g+\partial^2 h\,\partial_\mu f\, k\,\partial_\mu g+h\,\partial_\mu f\,\partial^2 k\,\partial_\mu g+\partial_\mu h\, f\,\partial_\mu k\,\partial^2 g\big) \\
&\quad - \frac{1}{9\varepsilon^2}\big(1+\tfrac{5}{12}\varepsilon\big)\big(\partial_\mu h\,\partial_\mu f\,\partial^2 k\, g+\partial_\mu h\,\partial_\mu f\, k\,\partial^2 g+\partial^2 h\, f\,\partial_\mu k\,\partial_\mu g+h\,\partial^2 f\,\partial_\mu k\,\partial_\mu g\big) \\
&\quad - \frac{1}{18\varepsilon}\big(\partial^2 h\,\partial_\mu f\,\partial_\mu k\, g+\partial_\mu h\, f\,\partial^2 k\,\partial_\mu g+\partial_\mu h\,\partial^2 f\, k\,\partial_\mu g+h\,\partial_\mu f\,\partial_\mu k\,\partial^2 g\big)\Bigg)\,.
\end{aligned}
\tag{C.19}
$$

In a similar fashion, neglecting possible ϵ-tensor contributions,

$$
\begin{aligned}
&-(16\pi^2)^3\,\frac{1}{4}\int \mathrm{d}^d x\,\mathrm{d}^d y\;\mathrm{tr}_\sigma\big(h(x)\,\sigma\cdot\partial_x Y_f(x,y)\,\bar\sigma\cdot\overleftarrow{\partial}_y\, k(y)\,\sigma\cdot\partial_y Y_g(y,x)\,\bar\sigma\cdot\overleftarrow{\partial}_x\big) \\
&\sim \int\!\mathrm{d}^d x\,\Bigg(\frac{4}{9\varepsilon^3}\big(1-\tfrac{7}{12}\varepsilon-\tfrac{41}{144}\varepsilon^2\big)\,\partial_\mu h\,\partial_\nu f\,\big(\partial_\mu k\,\partial_\nu g-\partial_\nu k\,\partial_\mu g\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{9\varepsilon}\, \partial_\mu h\, \partial_\mu f\, \partial_\nu k\, \partial_\nu g + \frac{1}{9\varepsilon}\, \partial_\mu h\, \partial_\nu f\, \big(\partial_\mu k\, \partial_\nu g + \partial_\nu k\, \partial_\mu g\big) \\
&+ \frac{1}{18\varepsilon^2}\big(1 - \tfrac{13}{12}\varepsilon\big)\big(\partial^2 h\, \partial^2 f\, k\, g + h\, f\, \partial^2 k\, \partial^2 g\big) \\
&+ \frac{1}{9\varepsilon^2}\big(1 - \tfrac{7}{12}\varepsilon\big)\big(h\, \partial^2 f\, \partial^2 k\, g + \partial^2 h\, f\, k\, \partial^2 g\big) - \frac{1}{36\varepsilon}\big(\partial^2 h\, f\, \partial^2 k\, g + h\, \partial^2 f\, k\, \partial^2 g\big) \\
&+ \frac{1}{9\varepsilon^2}\big(1 - \tfrac{1}{12}\varepsilon\big)\big(\partial_\mu h\, \partial^2 f\, \partial_\mu k\, g + \partial^2 h\, \partial_\mu f\, k\, \partial_\mu g + h\, \partial_\mu f\, \partial^2 k\, \partial_\mu g + \partial_\mu h\, f\, \partial_\mu k\, \partial^2 g\big) \\
&+ \frac{1}{18\varepsilon^2}\big(1 + \tfrac{5}{12}\varepsilon\big)\big(\partial_\mu h\, \partial_\mu f\, \partial^2 k\, g + \partial_\mu h\, \partial_\mu f\, k\, \partial^2 g + \partial^2 h\, f\, \partial_\mu k\, \partial_\mu g + h\, \partial^2 f\, \partial_\mu k\, \partial_\mu g\big)\bigg)\,.
\end{aligned}
\tag{C.20}
$$

Appendix D. Four Loop Calculations for Scalar Fields

The additional counterterms necessary for x-dependent couplings may be extended to four loops for purely scalar field theories. For simplicity we assume here a single component real scalar field ϕ with interaction $V(\phi) = \frac{1}{24}\lambda\phi^4$. For arbitrary $\lambda(x)$ the first relevant vacuum graph is at three loops giving

$$
W^{(3)} = \frac{1}{48}\int \mathrm{d}^d x\, \mathrm{d}^d y\; \lambda(x)\lambda(y)\, G_0(s)^4 \sim \frac{1}{(16\pi^2)^3}\,\frac{1}{\varepsilon}\,\frac{1}{864}\int \mathrm{d}^d x\; \partial^2\lambda\, \partial^2\lambda\,. \tag{D.1}
$$

At four loops there is also just one vacuum graph which generates simple poles in ε and therefore contributes to A_{IJ} and other terms in (8.24),

$$
W^{(4)} = -\frac{1}{48}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{d}^d z\; \lambda(x)\lambda(y)\lambda(z)\, \mathcal{R}G_0(x-z)^2\, \mathcal{R}G_0(z-y)^2\, \mathcal{R}G_0(x-y)^2\,, \tag{D.2}
$$

for $\mathcal{R}{G_0}^2$ as in (9.38). Letting

$$
Y(x,y) = \int \mathrm{d}^d z\; \mathcal{R}G_0(x-z)^2\, \lambda(z)\, \mathcal{R}G_0(z-y)^2\,, \tag{D.3}
$$

then using (C.1), in order to determine just the contributions containing poles in ε it is sufficient to replace

$$
Y(x,y) \to Y_0(x,y) + Y_1(x,y) + Y_2(x,y)\,, \tag{D.4}
$$

where

$$
\begin{aligned}
Y_0(x,y) = {}& \frac{1}{2(d-2)^4 S_d{}^3}\, \frac{\Gamma(\frac{1}{2}d)\Gamma(\frac{3}{2}d-4)}{\Gamma(d-2)^2}\, b_0(x,y)\, (s^2)^{4-\frac{3}{2}d} \\
& - \frac{1}{\varepsilon}\, \frac{1}{4(d-2)^2 S_d{}^2 S_4}\, \big(\lambda(x)+\lambda(y)\big)\, (s^2)^{2-d} + \frac{1}{\varepsilon^2}\, \frac{1}{16 S_4{}^2}\, \lambda(x)\, \delta^d(s)\,, \\
Y_1(x,y) = {}& \frac{1}{2(d-2)^4 S_d{}^3}\, \frac{\Gamma(\frac{1}{2}d)\Gamma(\frac{3}{2}d-5)}{\Gamma(d-2)^2}\, b_1(x,y)\, (s^2)^{5-\frac{3}{2}d}\,, \\
Y_2(x,y) = {}& \frac{1}{4(d-2)^4 S_d{}^3}\, \frac{\Gamma(\frac{1}{2}d)\Gamma(\frac{3}{2}d-6)}{\Gamma(d-2)^2}\, b_2(x,y)\, \big((s^2)^{6-\frac{3}{2}d} - 1\big)\,,
\end{aligned}
\tag{D.5}
$$

where now

$$b_n(x,y) = \int_0^1 \mathrm{d}t\, t^{1-\frac{1}{2}d+n}(1-t)^{1-\frac{1}{2}d+n}\,(\tfrac{1}{4}\partial^2)^n\lambda(x-ts)\,. \tag{D.6}$$

b_0 has the expansion

$$\begin{aligned} b_0(x,y) = \frac{\Gamma(3-\frac{1}{2}d)^2}{\Gamma(5-d)}\bigg(&\frac{2}{\varepsilon}\big(\lambda(x)+\lambda(y)\big) - \frac{1}{5-d}\,\frac{1}{4}\big((s\cdot\partial)^2\lambda(x)+(s\cdot\partial)^2\lambda(y)\big) \\ &+ \frac{34-5d}{(5-d)(7-d)}\,\frac{1}{192}\big((s\cdot\partial)^4\lambda(x)+(s\cdot\partial)^4\lambda(y)\big) + \mathrm{O}(s^6)\bigg)\,. \end{aligned} \tag{D.7}$$

Applying (C.5) gives

$$\begin{aligned} (16\pi^2)^4\, Y_0(x,y)G_0(s)^2 \sim{}& -\frac{1}{36\varepsilon^2}\big(1-\tfrac{7}{12}\varepsilon\big)\big(\lambda(x)+\lambda(y)\big)\,\partial^2\partial^2\delta^d(s) \\ &- \frac{1}{24\varepsilon}\big(\partial_\mu\partial_\nu\lambda(x)+\partial_\mu\partial_\nu\lambda(y)\big)\big(2\partial_\mu\partial_\nu+\delta_{\mu\nu}\partial^2\big)\delta^d(s) \\ &+ \frac{7}{144\varepsilon}\,\partial^2\partial^2\lambda(x)\,\delta^d(s)\,, \\ (16\pi^2)^4\, Y_1(x,y)G_0(s)^2 \sim{}& \frac{1}{8\varepsilon}\Big(\big(\partial^2\lambda(x)+\partial^2\lambda(y)\big)\,\partial^2\delta^d(s) - \tfrac{1}{3}\,\partial^2\partial^2\lambda(x)\,\delta^d(s)\Big)\,, \\ (16\pi^2)^4\, Y_2(x,y)\mathcal{R}G_0(s)^2 \sim{}& -\frac{1}{36\varepsilon^2}\big(1-\tfrac{1}{3}\varepsilon\big)\,\partial^2\partial^2\lambda(x)\,\delta^d(s)\,, \end{aligned} \tag{D.8}$$

and hence

$$\begin{aligned} (16\pi^2)^4\, Y(x,y)\mathcal{R}G_0(s)^2 \sim{}& -\frac{1}{36\varepsilon^2}\big(1+\tfrac{11}{12}\varepsilon\big)\Big(\big(\lambda(x)+\lambda(y)\big)\,\partial^2\partial^2\delta^d(s) + \partial^2\partial^2\lambda(x)\,\delta^d(s)\Big) \\ &+ \frac{1}{12\varepsilon}\Big(\big(\partial^2\lambda(x)+\partial^2\lambda(y)\big)\,\partial^2\delta^d(s) + \partial_x{}^2\partial_y{}^2\big(\lambda(x)\,\delta^d(s)\big)\Big)\,. \end{aligned} \tag{D.9}$$

This gives

$$W^{(4)} \sim \frac{1}{(16\pi^2)^4}\,\frac{1}{\varepsilon^2}\,\frac{1}{576}\int \mathrm{d}^d x\,\Big(\big(1+\tfrac{11}{12}\varepsilon\big)\,\lambda^2\,\partial^2\partial^2\lambda - 3\varepsilon\,\lambda\big(\partial^2\lambda\big)^2\Big)\,. \tag{D.10}$$

Using (D.4) with (D.5) we may further find

$$(16\pi^2)^3\, Y(x,y)G_0(s) \sim -\frac{1}{3\varepsilon^2}\big(1-\tfrac{1}{4}\varepsilon\big)\,\big(\lambda(x)+\lambda(y)\big)\,\partial^2\delta^d(s) + \frac{1}{3\varepsilon}\,\partial^2\lambda(x)\,\delta^d(s)\,, \tag{D.11}$$

which is equivalent to (C.16b), and to a result obtained in [9], and also

$$(16\pi^2)^3\, \tilde{Y}_f(x,y)\mathcal{R}G_0(s)^3 \sim -\frac{1}{3\varepsilon^2}\big(1-\tfrac{3}{8}\varepsilon\big)\,\big(f(x)+f(y)\big)\,\partial^2\delta^d(s) + \frac{1}{3\varepsilon^2}\big(1-\tfrac{7}{8}\varepsilon\big)\,\partial^2 f(x)\,\delta^d(s)\,. \tag{D.12}$$

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