ON ZETA FUNCTIONS IN TRIANGULATED CATEGORIES

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ABSTRACT. We prove 2-out-of-3 property for rationality of motivic zeta function in distinguished triangles in Voevodsky's category $\mathscr{DM}$. As an application, we show rationality of motivic zeta functions for all varieties whose motives are in the thick triangulated monoidal subcategory generated by motives of quasi-projective curves in $\mathscr{DM}$. Joint with a result of P.O'Sullivan it also gives an example of a variety whose motive is not finite-dimensional while the motivic zeta function is rational.

1. *Introduction*

In [1] and [2] Y. André has shown that if an object X of a pseudo-abelian rigid tensor $\mathbb{Q}$-category $\mathscr{T}$ is finite-dimensional in the sense of Kimura, [16], the corresponding motivic zeta-function $\zeta_X(t)$ is rational in the ring $K_0(\mathscr{T})[[t]]$, where $K_0(\mathscr{T})$ is the Grothendieck ring of the category $\mathscr{T}$. In particular, zeta function is rational for motives of abelian type in the category of Chow-motives $\mathscr{M}$, loc.cit. Moreover, F. Heinloth proved that for those motives zeta function satisfies a functional equation, [12].

The notion of motivic zeta function is naturally connected with many important problems in arithmetic and geometry. It was introduced by M.Kapranov in [15] wh proved its rationality and functional equation for smooth projective curves over a field with respect to any motivic measure μ satisfying the condition $\mu(\mathbb{A}^1) \neq 0$. In particular, if μ counts the number of points of a curve X defined over a finite field, then the motivic zeta function of its motive $M(X)$ is the usual Hasse-Weil zeta function associated with the curve X. The rationality of the Hasse-Weil zeta function for all varieties over a finite field was done by Dwork in [7].

The main goal of the present paper is to show that motivic zeta function is multiplicative in distinguished triangles in $\mathscr{T}$, provided $\mathscr{T}$ is a homotopy category of a simplicial model monoidal category $\mathscr{C}$. As an

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application, we show rationality of motivic zeta functions for all varieties whose motives are in the thick triangulated monoidal subcategory generated by motives of quasi-projective curves in Voevodsky's triangulated category $\mathscr{DM}$ over a field. Joint with a result of P.O'Sullivan it also gives an example of a variety whose motive is not finite-dimensional but its motivic zeta function is rational.

2. *Some recallections*

Let $k = \mathbb{F}_q$ be a finite field consisting of q elements, let X be an algebraic variety over $\mathbb{F}_q$ and let $\#X(\mathbb{F}_q)$ be the number of $\mathbb{F}_q$-rational points on X. Then the usual Hasse-Weil zeta function ζ_X associated with X can be defined by the formula

$$\zeta_X(t) = \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}\right)$$

Let $\mathrm{Sym}^n X$ be n-th symmetric power of X, i.e. a quotient of $X^{\times n}$ by the action of the symmetric group Σ_n. Points of the variety X, which are rational over $\mathbb{F}_{q^n}$, can be interpreted as points of the variety $\mathrm{Sym}^n X$, rational over $\mathbb{F}_q$. Using this and also the exponential formula coming from combinatorics, one can show that

$$\zeta_X(t) = \sum_{n=0}^{\infty} \#\mathrm{Sym}^n X(\mathbb{F}_q)t^n \ .$$

For any field k let Var be the category of quasi-projective varieties over k. Let, furthermore, $\mathbb{Z}[Var]$ be the free abelian group generated by isomorphism classes $[X]$ of quisi-projective varieties X over k. The Grothendieck group $K_0(Var)$ of the category Var is, by definition, the quotient of $\mathbb{Z}[Var]$ by the minimal subgroup containing relations of the type $[X] = [Z] + [X \backslash Z]$ where Z is a closed subvariety in X. Then $K_0(Var)$ is naturally a commutative ring with unit and with a product induced by fibered products of varieties over k. A motivic measure μ is a ring homomorphism $\mu : K_0(Var) \to A$ to any other commutative ring A. Given μ, for any variety X over k we can consider the corresponding zeta-function

$$\zeta_{X,\mu}(t) = \sum_{n=0}^{\infty} \mu[\mathrm{Sym}^n X]t^n \ ,$$

see [15]. For example, if μ counts the number of points over a finite field, then $\zeta_{X,\mu}(t)$ is the above Hasse-Weil zeta function of X. In that case $\zeta_X(t)$ is rational by Dwork's result, [7].

Kapranov proved, [15, 1.1.9], that the motivic zeta function $\zeta_{X,\mu}(t)$ is rational when X is a smooth projective curve carrying a divisor of degree one, A is a field and the motivic measure $\mu(\mathbb{A}^1)$ of the affine line is not zero. On the other hand, Larsen and Lunts have shown in [18] that there exists a measure μ built on the base of Hodge numbers $h^{i,0}$, such that $\zeta_{X,\mu}$ is not rational for a surface X with $h^{2,0} \neq 0$.

Now let $\mathscr{M}$ be the category of Chow motives over k with coefficients in $\mathbb{Q}$, see [23]. Since $\mathscr{M}$ is a tensor additive category, one can construct its Gothendieck ring $K_0(\mathscr{M})$ in a standard way, i.e. taking direct sums as sums and tensors products as products in K_0. In [9] Gillet and Soulé constructed a motivic measure

$$\mu_{GS} : K_0(Var) \longrightarrow K_0(\mathscr{M})$$

sending the class $[X]$ of a smooth projective variety X to the class $[M(X)]$ of its Chow motive $M(X)$. For any given Chow motive M let

$$\zeta_M(t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n M] t^n$$

be the corresponding zeta function with coefficients $K_0(\mathscr{M})$. By the result of Del Baño and Navarro Aznar, [4], $\mu_{GS}[\mathrm{Sym}^n X] = [\mathrm{Sym}^n M(X)]$, whence

$$\zeta_{M(X)} = \zeta_{X,\mu_{GS}}$$

for any X. If M is a Chow motive which is finite-dimensional in the sense of Kimura, then $\zeta_M(t)$ is rational, see [1] and [2].

A board generalization can be done as follows. Let $\mathscr{A}$ be any pseudoabeian[1] symmetric monoidal $\mathbb{Q}$-linear category with a monoidal product $\otimes$. Then we have wedge and symmetric powers of objects X in $\mathscr{A}$ as images of the corresponding idempotents in the group algebra of the symmetric group Σ_n acting on $X^{\otimes n}$. Let $\mathbb{Z}[\mathscr{A}]$ be the free abelian group generated by isomorphisms classes of objects in $\mathscr{A}$, and let $K_0(\mathscr{A})$ be the Grothendieck group of the category $\mathscr{A}$, i.e. the quotient of $\mathbb{Z}[\mathscr{A}]$ by the minimal subgroup generated by expressions of type $[X \oplus Y] - [X] - [Y]$. Clearly, it has a ring structure induced by the monoidal product in $\mathscr{A}$.

Recall that a lambda-structure on a commutative ring A with unit 1 is just a chain of maps $\lambda^i : A \to A$, $i = 0, 1, 2, \ldots$, such that $\lambda^0(a) = 1$, $\lambda^1(a) = a$ and $\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$ for all a and b in A. It can be also defined as a group homomorphism

$$\lambda_t : A \longrightarrow 1 + A[[t]]^+$$
$$a \mapsto \lambda_t(a) = \sum_i \lambda^i(a) t^i$$

from the additive group of A to the multiplicative group $1 + A[[t]]^+$ of formal power series of type $1 + a_1 t + a_2 t^2 + \ldots$. Considering this multiplicative structure as additive we can think of $1 + A[[t]]^+$ as an abelian group the addition in which is just the multiplication of series. This group has its own multiplication and, moreover, is a lambda-ring. Indeed, let $\xi_1, \xi_2, \ldots$ and $\eta_1, \eta_2, \ldots$ be infinite collections of indeterminantes over $\mathbb{Z}$.

[1]an additive category is pseudoabelian if it contains kernels of its idempotents

Thinking of the coefficients of a serious $1+a_1t+a_2t^2+\dots$ as algebraically independent symmetric functions one can write it as an infinite product $\prod(1+\xi_i t)$. Then the multiplication in $1+A[[t]]^+$ can be defined by the rule:

$$\prod(1+\xi_i t)\circ\prod(1+\eta_j t)=\prod(1+\xi_i\eta_j t)\ ,$$

and the lambda-structure by the rule:

$$\Lambda^n\left(\prod(1+\xi_i t)\right)=\prod(1+\xi_{i_1}\xi_{i_2}\dots\xi_{i_r}t)\ ,$$

see [3] and [17]. The lambda-structure on A is called special if λ_t is a ring homomorphism commuting with Λ^i, i.e.

$$\Lambda^i\lambda_t=\lambda_t\lambda^i$$

for any i. Finally, given two lambda structures λ and σ on the same ring, they are called opposite if

$$1+\sum_{i=1}^{+\infty}\lambda^i(a)t^i=\left(1+\sum_{i=1}^{+\infty}\sigma^i(a)(-t)^i\right)^{-1}$$

for all a in A.

Turning back to the category $\mathscr{A}$, wedge and symmetric powers in it give rise to special λ-structures in the ring $K_0(\mathscr{A})$, opposite each other, [12, 4.1]. We will denote these λ-structures by λ_+ for wedge and λ_- for symmetric powers respectively. For example, if $X\in \mathrm{Ob}(\mathscr{A})$ then $\lambda_+^n[X]=[\wedge^n X]$ and $\lambda_-^n[X]=[\mathrm{Sym}^n X]$. Let also

$$\lambda_t^{\pm}: K_0(\mathscr{A})\longrightarrow 1+K_0(\mathscr{A})[[t]]^+$$

be the group homomorphism corresponding to the λ-structure $\lambda_{\pm}$. Then for any object X in $\mathscr{A}$ we can define its zeta function $\zeta_X(t)$ by the formula:

$$\zeta_X(t):=\lambda_t^-([X])=[\mathbb{1}]+[X]t+[\mathrm{Sym}^2X]t^2+\dots\ ,$$

If $\mathscr{A}$ is the category of Chow motives $\mathscr{M}$ then we arrive to the above motivic zeta function with respect to the measure constructed by Gillet and Soulé.

Below we are mainly interested in motivic measures μ which can be factored through μ_{GS}, i.e. $\mu=\tau\circ\mu_{GS}$ for some homomorphism τ from $K_0(\mathscr{M})$ to A. In that case, if we know rationality of $\zeta_{M(X)}$, then we also know rationality of $\zeta_{X,\mu}$.

Any reasonable motivic measure which can be defined in terms of appropriate cohomology groups can be factored through $K_0(\mathscr{M})$. For example, given any quasi-projective variety X over $\mathbb{C}$ we define its Hodge numbers $h^{p,q}$ as dimensions of the corresponding bigraded pieces of the mixed Hodge structure on the cohomology with compact support $H_c^{p+q}(X,\mathbb{Q})$ of X. Then the motivic measure sending X to its Hodge polynomial

$\sum h^{p,q}u^p v^q$ can be defined in terms of mixed Hodge realizations. Therefore it factors through $K_0(\mathscr{M})$. Another interesting example of a motivic measure factoring through μ_{GS} can be provided by conductors of l-adic representations over a number field, see [6].

3. *A result*

Let $\mathscr{T}$ be a small triangulated category with shift functor $X \mapsto \Sigma X$, and let $\mathbb{Z}[\mathscr{T}]$ be the free abelian group generated by isomorphism classes of objects in $\mathscr{T}$. Let, furthermore, $S(\mathscr{T})$ be the minimal subgroup in $\mathbb{Z}[\mathscr{T}]$ generated by elements $[Y]-[X]-[Z]$ whenever $Y = X \oplus Z$, and let $T(\mathscr{T})$ be the minimal subgroup generated by the same expressions whenever $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a distinguished triangle in $\mathscr{T}$. The quotient $\mathbb{Z}[\mathscr{T}]/S(\mathscr{T})$ is the above Grothendieck group of $\mathscr{T}$ as an additive category, since now denoted by $K_0^s(\mathscr{T})$ to stress that it is built up by splittings of type $Y = X \oplus Z$. While $K_0(\mathscr{T})$ will be reserved for the quotient $\mathbb{Z}[\mathscr{T}]/T(\mathscr{T})$, which is the Grothendieck group of the triangulated category $\mathscr{T}$. Evidently, $K_0(\mathscr{T})$ is a quotient of $K_0^s(\mathscr{T})$ by $T(\mathscr{T})/S(\mathscr{T})$. If $\mathscr{T}$ is a derived category of a nice abelian category $\mathscr{A}$, then $K_0(\mathscr{T})$ is isomorphic to $K_0(\mathscr{A})$, so that the triangulated K_0 does make sense, [21].

Now suppose that $\mathscr{T}$ is a symmetric monoidal and triangulated category. If $\Sigma : \mathscr{T} \to \mathscr{T}$ is the shift functor and $\otimes$ is the product in $\mathscr{T}$, they are compatible in the following sense. For any two objects X and Y in $\mathscr{T}$ there are natural isomorphisms

$$X \otimes \Sigma Y \cong \Sigma(X \otimes Y) \cong \Sigma X \otimes Y \ ,$$

and for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

and any object V in $\mathscr{T}$ the induced triangle

$$X \otimes V \to Y \otimes V \to Z \otimes V \to \Sigma X \otimes V$$

is distinguished, [13]. In that case K_0 of $\mathscr{T}$ is a commutative ring. Suppose, furthermore, that $\mathscr{T}$ is pseudo-abelian and $\mathbb{Q}$-linear. Then wedge and symmetric powers are defined in $\mathscr{T}$. Since $\mathscr{T}$ is additive $K_0^s(\mathscr{T})$ has two canonical lambda-structures $\lambda_\pm$ by Lemma 4.1 from [12] (see below). The question is then whether or not these lambda-structures induce lambda structures on the triangulated group $K_0(\mathscr{T})$. The positive answer to that question would have quite interesting corollaries when applying to zeta functions in the triangulated category of motives over a field. Below we will show an existence of two opposite special lambda-structures in $K_0(\mathscr{T}')$ induced by wedge and symmetric powers in $\mathscr{T}$, where $\mathscr{T}'$ is a thick symmetric monoidal subcategory generated by compact objects in $\mathscr{T}$, and $\mathscr{T}$ is the homotopy category of a simplicial model monoidal category.

To be more precise, let $\mathscr{C}$ be a pointed simplicial model monoidal category, [13], and let

$$\mathscr{T} = Ho(\mathscr{C})$$

be the homotopy category of $\mathscr{C}$. Then $\mathscr{T}$ is a triangulated category whose shift functor is induced by the simplicial suspension

$$\Sigma X = X \wedge S^1$$

in $\mathscr{C}$, loc.cit. Assume furthermore that $\mathscr{C}$ is symmetric monoidal with monoidal product $\wedge$, model and monoidal structures are compatible in the sense of [13], 4.2. Such a triangulated category has a symmetric monoidal product $\otimes$ induced by the product $\wedge$ in the category $\mathscr{C}$, and it can be viewed as an abstract prototype for all reasonable stable homotopy categories appearing in algebraic topology and algebraic geometry, [13]. As we are interested in the study of wedge and symmetric powers in $\mathscr{T}$ we will assume that $\mathscr{T}$ is $\mathbb{Q}$-localized, i.e. all Hom-groups are vector spaces over $\mathbb{Q}$.

There are several examples of such triangulated categories arising in algebraic topology and motivic algebraic geometry. The homotopy category of $\mathbb{Q}$-local topological symmetric spectra over a point is just the category of graded $\mathbb{Q}$-vector spaces. However, the rational stable homotopy theory of S^1-equivariant spectra is still interesting, see [10]. But the main example for our purposes is the homotopy category $\mathscr{SH}(X)$ of $\mathbb{Q}$-local motivic symmetric spectra over a Noetherian base scheme X, see [14] and [24]. If $X = \mathrm{Spec}(k)$ we will write $\mathscr{SH}$ instead of $\mathscr{SH}(\mathrm{Spec}(X))$. As it was announced by Morel, [20], if -1 is a sum of squares in the ground field k, the category $\mathscr{SH}$ is equivalent to the big category $\mathscr{DM}$ of triangulated motives over k. Using this equivalence we can apply results obtained in $\mathscr{T} = Ho(\mathscr{C})$ to the category $\mathscr{DM}$.

Since $\mathscr{T}$ is a homotopy category it has direct sums. Then $K_0(\mathscr{T}) = 0$ by so-called "Eilenberg swindle". Indeed, let X be an object in $\mathscr{T}$ and let $[X]$ be its class in K_0. Let $Y = X \oplus X \oplus \dots$ be a direct sum of a countable number of copies of X in $\mathscr{T}$. Then $X \oplus Y = Y$, whence $[X] = 0$ in K_0. Therefore, dealing with $\mathscr{T} = Ho(\mathscr{C})$ it is reasonable to work with a thick triangulated subcategory

$$\mathscr{T}' := \mathscr{T}^{\aleph_0}$$

of compact objects in $\mathscr{T}$, see [21]. For example, if $\mathscr{T} = \mathscr{DM}$ over a field then $\mathscr{T}' = \mathscr{DM}'$ is nothing but the triangulated category of geometrical motives DM_{gm} over k, see [25].

Theorem 1. *Let $\mathscr{T}'$ be a thick triangulated monoidal subcategory of compact objects in the homotopy category $\mathscr{T} = Ho(\mathscr{C})$ of a simplicial model symmetric monoidal category $\mathscr{C}$, and assume that all* Hom*-groups are vector spaces over $\mathbb{Q}$. Then wedge and symmetric powers in $\mathscr{T}'$ induce* *two special lambda-structures in the ring $K_0(\mathscr{T}')$, which are opposite each other.*

Actually that theorem is a consequence of the existence of a special Postnikoff tower connecting wedge (symmetric) powers of the vertex Y in a given distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

with wedge (symmetric) powers of two another two vertices X and Z, constructed in [11]. To be precise, for any $i = 0, \dots, n$ there exist two distinguished triangles

$$I_+^i \longrightarrow I_+^{i+1} \longrightarrow J_+^{i+1} \longrightarrow \Sigma I_+^i$$

and

$$I_-^i \longrightarrow I_-^{i+1} \longrightarrow J_-^{i+1} \longrightarrow \Sigma I_-^i$$

– one for the alternated case, and the second one for the symmetric case. When $i = 0$, then

$$I_+^0 = \wedge^n X \ , \quad I_+^n = \wedge^n Y \ ,$$

and, similarly,

$$I_-^0 = \mathrm{Sym}^n X \ , \quad I_-^n = \mathrm{Sym}^n Y \ .$$

The key point is that the above cones $J_\pm^i$ can be computed by the Künneth rule:

$$J_+^i = \wedge^i Z \otimes \wedge^{n-i} X$$

and

$$J_-^i = \mathrm{Sym}^i Z \otimes \mathrm{Sym}^{n-i} X \ .$$

The precise construction of these Postnikoff towers is given in [11].

Let now M be an abelian monoid and let M^+ be the group completion of M with a universal morphism $M \to M^+$, $m \mapsto [m]$. Given an additive equivalence relation R on M, we can construct a quotient additive monoid M/R. Let R^+ be a subgroup in M generated by elements $[a] - [a']$, such that $(a, a') \in R$. Then the abelian group M^+/R^+ is canonically isomorphic to the abelian group $(M/R)^+$.

Let S be a set and let $\mathbb{N}[S]$ be a free abelian monoid generated by S. Let ρ be a reflexive and symmetric relation on $\mathbb{N}[S]$. If $(a, b) \in \rho$ then we will say that a is elementary ρ-equivalent to b. Build an additive equivalence relation $\langle\rho\rangle$ on $\mathbb{N}[S]$ generated by ρ as follows: two linear combinations a and a' from $\mathbb{N}[S]$ are called to be equivalent if there exist a chain of linear combinations $a_0, a_1, \dots a_n$, such that $a_0 = a$, $a_n = a'$ and for each i the element a_{i+1} can be obtained from a_i by a replacement of a summand in a_i by an elementary ρ-equivalent summand. For short, let $\mathbb{N}[S]/\rho$ be the quotient of $\mathbb{N}[S]$ by $\langle\rho\rangle$ and $\rho^+ = \langle\rho\rangle^+$. Certainly, $(\mathbb{N}[S])^+ = \mathbb{Z}[S]$. From the previous observation we have that $(\mathbb{N}[S]/\rho)^+ = \mathbb{Z}[S]/\rho^+$.

For example, let $\mathscr{A}$ be an additive category, $\mathbb{N}[\mathscr{A}] := \mathbb{N}[\mathrm{Ob}(\mathscr{A})]$ and let σ be the set of pairs $([X \oplus Y], [X] + [Y])$ and their transposes. We see that $\mathbb{Z}[\mathscr{A}]/\sigma^+$ is the additive Grothendieck group $K_0^s(\mathscr{A})$ of $\mathscr{A}$. The above isomorphism shows then that $K_0^s(\mathscr{A})$ can be also described as a completion of the monoid $\mathbb{N}[\mathscr{A}]/\sigma$.

In the triangulated situation we have the following. Let $\mathscr{T}$ be a small triangulated category, $\mathbb{N}[\mathscr{T}] := \mathbb{N}[\mathrm{Ob}(\mathscr{T})]$ and let Δ be the set of pairs $([Y], [X] + [Z])$, where XYZ is a distinguished triangle, and their transposes. Again we see that $\mathbb{Z}[\mathscr{T}]/\Delta^+$ is the triangulated Grothendieck group $K_0(\mathscr{T})$ and the above isomorphism shows then that it can be also described as a completion of the monoid $\mathbb{N}[\mathscr{T}]/\Delta$.

Let now $\mathscr{T} = Ho(\mathscr{C})$ be the homotopy category from the assumptions of Theorem 1, and let $\mathscr{T}'$ be the subcategory of compact objects in it. To prove Theorem 1 we will consider the case of wedge powers only. The case of symmetric powers is analogous.

Consider a collection of maps

$$\lambda^n : \mathbb{Z}[\mathscr{T}] \longrightarrow \mathbb{Z}[\mathscr{T}] \; , \quad n = 0, 1, 2, \dots \; ,$$

where

$$\lambda^n[X] = [\wedge^n X] \; ,$$

for any object X in $\mathscr{T}$ and the value of λ^n on $[X] + [Y]$ is defined by Künneth's rule. This collection of maps gives a λ-structure

$$\lambda^n : K_0^s(\mathscr{T}) \longrightarrow K_0^s(\mathscr{T}) \; , \quad n = 0, 1, 2, \dots \; ,$$

in the group $K_0^s(\mathscr{T})$, see. [12], Lemma 4.1. Monoidal product of a finite number of compact objects is compact and a direct summand of a compact object is compact. Hence, the above maps λ^n induce maps $\lambda^n : \mathbb{Z}[\mathscr{T}'] \to \mathbb{Z}[\mathscr{T}']$ in the groups generated by compact objects in $\mathscr{T}$. Therefore, to define lambda-operations by the same rule in the group $K_0(\mathscr{T}')$ we need only to show that the maps $\lambda^n : \mathbb{Z}[\mathscr{T}'] \to \mathbb{Z}[\mathscr{T}']$ preserve the subgroup $T(\mathscr{T}')$. Since $K_0(\mathscr{T}') = \mathbb{Z}[\mathscr{T}']/T(\mathscr{T}') = (\mathbb{N}[\mathscr{T}']/\Delta)^+$, we need actually to show that, if two linear combinations a and b in $\mathbb{N}[\mathscr{T}']$ are elementary Δ-equivalent, the element $\lambda^n a$ is Δ-equivalent to the element $\lambda^n b$.

Without loss of generality we can assume that $a = [Y]$, $b = [X] + [Z]$ and we have a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma Z$$

in $\mathscr{T}'$. For each n let

$$I_+^i \longrightarrow I_+^{i+1} \longrightarrow J_+^{i+1} \longrightarrow \Sigma I_+^i \; ,$$

$$i = 0, 1, \dots, n-1 \; ,$$

be the Postnikoff system as above. Let

$$a_i = [I_+^i] \; , \quad c_i = [J_+^i] \; , \quad x_i = [\wedge^i X] \; , \quad y_i = [\wedge^i Y] \quad \text{and} \quad z_i = [\wedge^i Z]$$

for all non-negative integers i. The above Postnikoff tower can be rewritten in terms of the ring $K_0(\mathscr{T}')$ as follows:

$$a_0 = x_n \ , \quad a_n = y_n \ ,$$

$$c_i = z_i x_{n-i} \ , \quad i = 0, 1, \ldots, n \ ,$$

and

$$a_{i+1} = a_i + c_{i+1} \ , \quad i = 0, 1, \ldots, n-1 \ ,$$

whence we get:

$$y_n = \sum_{i=0}^{n} z_i x_{n-i}$$

in the ring $K_0(\mathscr{T}')$, where $y_n = \lambda^n[Y]$, $x_0 = 1$ and $z_0 = 1$. It means that the element $\lambda^n([X] + [Z])$ is Δ-equivalent to the element $\lambda^n[Y]$, as desired.

Now we have to show that both lambda-structures in $K_0(\mathscr{T}')$ are special and opposite each other. Consider a commutative square

$$\begin{tikzcd}
K_0^s(\mathscr{T}') \arrow[r] \arrow[d, "\lambda_t"] & K_0(\mathscr{T}') \arrow[d, "\lambda_t"] \\
1 + K_0^s(\mathscr{T}')[[t]]^+ \arrow[r] & 1 + K_0(\mathscr{T}')[[t]]^+
\end{tikzcd}$$

The left vertical map respects the multiplication as the lambda-structure in $K_0^s(\mathscr{T}')$ is special. Since the diagram is commutative and the homomorphism $K_0^s(\mathscr{T}') \to K_0(\mathscr{T}')$ is surjective, the vertical homomorphism from the right hand side respects the multiplication too.

In order to show that the constructed lambda-operations in $K_0(\mathscr{T}')$ commute with the operations Λ^i we consider a 3-dimensional diagram

$$\begin{tikzcd}
K_0^s(\mathscr{T}') \arrow[rr, "\lambda^i"] \arrow[dr] \arrow[dd, "\lambda_t"] & & K_0^s(\mathscr{T}') \arrow[dr] \arrow[dd, "\lambda_t"] & \\
& K_0(\mathscr{T}') \arrow[rr, "\lambda^i"] \arrow[dd, "\lambda_t"] & & K_0(\mathscr{T}') \arrow[dd, "\lambda_t"] \\
1 + K_0^s(\mathscr{T}')[[t]]^+ \arrow[rr, "\Lambda^i"] \arrow[dr] & & 1 + K_0^s(\mathscr{T}')[[t]]^+ \arrow[dr] & \\
& 1 + K_0(\mathscr{T}')[[t]]^+ \arrow[rr, "\Lambda^i"] & & 1 + K_0(\mathscr{T}')[[t]]^+
\end{tikzcd}$$

in which all the faces, except for the front one, are commutative squares. Precompositions of $\Lambda^i\lambda_t$ and $\lambda_t\lambda^i$ with the surjective homomorphism $K_0^s(\mathscr{T}') \longrightarrow K_0(\mathscr{T}')$ coincide because the lambda-structure in K_0^s is special. Therefore, the front facet in the diagram is commutative as well, which completes the proof of speciality of the lambda-structures in $K_0(\mathscr{T}')$.

Let now θ be the composition of a map sending t into $-t$ and of a map taking the inverse to any element in $1 + A(\mathscr{T}')[[t]]^+$, where A is either $K_0^s(\mathscr{T})$ or $K_0(\mathscr{T}')$. Consider a diagram

$$\begin{tikzcd}
K_0^s(\mathscr{T}') \arrow[rr, "\lambda_t"] \arrow[d] \arrow[ddr, "\sigma_t"'] & & 1 + K_0^s(\mathscr{T}')[[t]]^+ \arrow[d] \\
K_0(\mathscr{T}') \arrow[rr, "\lambda_t"] \arrow[ddr, "\sigma_t"'] & & 1 + K_0(\mathscr{T}')[[t]]^+ \\
& 1 + K_0^s(\mathscr{T}')[[t]]^+ \arrow[uur, "\theta"'] \arrow[d] & \\
& 1 + K_0(\mathscr{T}')[[t]]^+ \arrow[uur, "\theta"'] &
\end{tikzcd}$$

in which λ_t and σ_t are induced by the lambda-operations constructed by wedge and symmetric powers in $K_0^s(\mathscr{T}')$ and in $K_0(\mathscr{T}')$. The fact that λ and σ are opposite in $K_0^s(\mathscr{T}')$ can be expressed by the formula $\theta \circ \sigma_t = \lambda_t$. In order to show their opposition in $K_0(\mathscr{T}')$ we observe that the precomposition of $\theta \circ \sigma_t$ and λ_t with the map $K_0^s(\mathscr{T}') \to K_0(\mathscr{T})$ coincide because they do coincide on $K_0^s(\mathscr{T}')$-level. Since the last map is surjective, $\theta \circ \sigma_t = \lambda_t$ on $K_0(\mathscr{T}')$-level too, as required.

4. *Some applications*

A formal power series $\xi(t)$ in variable t with coefficients in a commutative ring A is called rational, if there exists two polynomials $a(t)$ and $b(t)$ in $A[t]$, such that $a\xi = b$ in $A[[t]]$.

For any element $a \in K_0(\mathscr{T}')$ let $\zeta_a(t) = \lambda_t^-(a)$ be the corresponding zeta function of a, and let $\zeta_X = \zeta_{[X]}$ if $X \in \mathrm{Ob}\,\mathscr{T}'$.

Corollary 2.

$$\zeta_{\Sigma X} = (\zeta_X)^{-1}$$

for any object X in $\mathscr{T}'$. In particular, the suspension does not change rationality of zeta-function.

Proof. The suspension $\Sigma : \mathscr{T} \to \mathscr{T}$ induces an involution $a \mapsto -a$ in $K_0(\mathscr{T}')$, and Theorem 1 gives the formula $\zeta_{-a} = \zeta_a^{-1}$ for any a in K_0. □

Corollary 3. *Let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle in $\mathscr{T}$. If two out of three zetas ζ_X, ζ_Y and ζ_Z are rational functions, then the third zeta function is also rational.*

Proof. Since XYZ is a distinguished triangle, $[Y] = [X] + [Z]$ in $K_0(\mathscr{T})$. Since we work with zeta-functions induced by λ-structures in K_0,

$$\zeta_Y = \zeta_X \cdot \zeta_Z \ ,$$

whence the proof. □

Assume now that either $\mathrm{char}(k) = 0$ or -1 is a sum of squares in k. In that case, as we have mentioned above, the category $\mathscr{DM}$ is equivalent to the category $\mathscr{SH}$ (announced in [20]). Therefore, Corollary 3 can be applied to the category of geometrical motives $\mathscr{DM}' = DM_{gm} \otimes \mathbb{Q}$.

For any motive $M \in \mathscr{DM}'$ let $\zeta_M(t)$ be the corresponding motivic zeta function of M. The benefit of Theorem 1 is that it extends the range of varieties whose motivic zeta function is rational. Indeed, let $\mathscr{DM}^{\otimes}_{\leq 1}$ be a thick symmetric monoidal subcategory in $\mathscr{DM}$ generated by motives of quasi-projective curves over k.

Corollary 4. *For any motive M in $\mathscr{DM}^{\otimes}_{\leq 1}$ its zeta function $\zeta_M(t)$ is rational.*

Proof. The motive $M(X)$ of a quasi-projective curve X/k is finite-dimensional, see [11] or [19]. Therefore, $\zeta_{M(X)}$ is rational by André's result, [2, 13.3.3.1]. If M is in $\mathscr{DM}^{\otimes}_{\leq 1}$, then its class $[M]$ belongs to a subring in $K_0(\mathscr{DM}')$ generated by classes of motives of quasi-projective curves. Since the lambda-structure on $K_0(\mathscr{DM}')$ is special, rationality of zeta functions is closed under sums and products of elements in $K_0(\mathscr{DM}')$, whence the result. □

Corollary 3 says that motivic zeta function has 2-out-of-3 property in distinguished triangles in $\mathscr{DM}'$. This correlates with Lemma 3.1 in [18] via Gyzin distinguished triangles in the category $\mathscr{DM}'$. In general, there are several canonical distinguished triangles in $\mathscr{DM}$ each of which gives new examples of varieties whose motivic zeta function is rational.

Example 5. Recall that a Weil hypersurface in $\mathbb{P}^{n+1}$ is defined by an equation of type

$$W(d) \ : \ W_0(x_0, x_{n+1}) + W_{\geq 1}(x_1, \dots, x_n) = 0 \ ,$$

where

$$W_{\geq 1} = \sum_{i=1}^{s} W_i(x_{2i-1}, x_{2i}) \quad \text{if} \quad n = 2s \ ,$$

$$W_{\geq 1} = \sum_{i=1}^{s} W_i(x_{2i-1}, x_{2i}) + x_{2s+1}^d \quad \text{if} \quad n = 2s+1 \ ,$$

and W_i is a form of degree d. As it was shown in [22] the motive $M(W(d))$ is generated by motives of curves. In our terminology it means that such a motive is always an object of the category $\mathscr{DM}^{\otimes}_{\leq 1}$. Therefore, the zeta-function $\zeta_{W(d)}$ is rational by Corollary 4.

In particular, zeta-function $\zeta_{W(4)}$ of a smooth quartic $W(4)$ in $\mathbb{P}^3$ is rational. Such a quartic is a $K3$-surface. As it was shown by O'Sullivan, [19], there exists a Zariski open subset U in $X = W(4)$, such that the motive $M(U)$ is not finite-dimensional. However, its motivic zeta-function ζ_U is rational. Indeed, the complement $Z = X - U$ is a union of curves. The motive of a quasi-projective curve is finite-dimensional, whence we get rationality of ζ_Z. As ζ_X is rational, we apply Corollary 3 to the Gyzin distinguished triangle in $\mathscr{DM}$ associated with the pair (X, U) getting rationality of ζ_U.

Example 6. Let $X = X_1 \cup \dots \cup X_n$ be a union of quasi-projective surfaces whose zeta functions ζ_{X_i} are rational. Then ζ_X is rational. Indeed, if $n = 2$, we apply Corollary 3 and rationality of zeta functions of quasiprojective curves to the corresponding Mayer-Vietoris distinguished triangle in $\mathscr{DM}$. If $n > 2$, we just write down an appropriate Postnikoff tower and apply Corollary 3 several additional times.

Example 7. Let X be a scheme of finite type over k, let Z be a regularly imbedded closed subscheme in it, $f : \tilde{X} \to X$ be a blow up of X along Z and $E = f^{-1}(Z)$ be the exceptional divisor. As the closed imbedding $Z \hookrightarrow X$ is regular $E \to Z$ is a projective bundle on Z, see [8, B.6.2]. Assume now that the motive $M(Z)$ is finite-dimensional. The motive $M(E)$ is finite-dimensional by the projective bundle formula in Voevodsky's category, [25]. Then the functions $\zeta_{M(Z)}$ and $\zeta_{M(E)}$ are rational. If the function $\zeta_{M(\tilde{X})}$ is rational the function associated with the direct sum $M(\tilde{X}) \oplus M(Z)$ is rational by Corollary 3 applied to the corresponding split triangle in $\mathscr{DM}$. Consider the canonical distinguished triangle

$$M(E) \to M(\tilde{X}) \oplus M(Z) \to M(X) \to M(E)[1]$$

of the blowing up $\tilde{X} \to X$, see [25, 2.2]. Applying Corollary 3 once again we see that the function $\zeta_{M(X)}$ is rational. Vice versa, if $\zeta_{M(X)}$ is rational then the function of the sum $M(\tilde{X}) \oplus M(Z)$ is so. Since the function $\zeta_{M(Z)}$ is rational, the function $\zeta_{M(\tilde{X})}$ is rational too. Here we applied Corollary 3 a few times again.

Remark 8. By recent result of Bondarko, [5], $K_0(\mathscr{M})$ is isomorphic to the triangulated $K_0(\mathscr{DM}')$ if we consider both categories over a field of characteristic zero. Most probably this isomorphism also gives the two above canonical lambda-structures in Voevodsky's category $\mathscr{DM}'$. However, our approach is intrinsic and Theorem 1 allows to work with lambda-structures in the 2-functor $\mathscr{SH}(-)$ on schemes in general. On the one hand, we can use $\mathscr{SH}(-)$ in order to build new varieties whose motivic zeta functions are rational, on the other hand, we can use Bondarko's isomorphism in order to factor interesting motivic measures through $K_0(\mathscr{DM}')$.

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