On the time to ruin for a dependent-delayed capital injection risk model

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Abstract

In this paper, we propose a generalisation to the Cramér-Lundberg risk model, by allowing for a delayed receipt of the required capital injections whenever the surplus of insurance firm is negative. Delayed capital injections often appear in practice due to the time taken for administrative and processing purposes of the funds from a third party or the shareholders of an insurance firm.

The delay time of the capital injection depends on a critical value of the deficit in the following way: If the deficit of the firm is less than the fixed critical value, then it can be covered by available funds and therefore the required capital injection is received instantaneously. On the other hand, if the deficit of the firm exceeds the fixed critical value, then the funds are provided by an alternative source and the required capital injection is received after some time delay. In this modified model, we derive a Fredholm integral equation of the second kind for the ultimate ruin probability and obtain an explicit expression in terms of ruin quantities for the Cramér-Lundberg risk model. In addition, we show that other risk quantities, namely the expected discounted accumulated capital injections and the expected discounted overall time in red, up to the time of ruin, satisfy a similar integral equation, which can also be solved explicitly. Finally, we extend the capital injection delayed risk model, such that the delay of the capital injections depends explicitly on the amount of the deficit. In this generalised risk model, we derive another Fredholm integral equation for the ultimate ruin probability, which is solved in terms of a Neumann series.

Keywords: Ruin Probability, Deficit Dependent Delayed Capital Injections, Fredholm Integral Equation, Neumann Series Solution.

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1 Introduction

Over the years, the fundamental Cramér-Lundberg risk model has experienced a large number of generalisations, in order to capture the reality of insurance business (whilst keeping its mathematical integrity). One such generalisation is the requirement of capital injections to restore the capital whenever the surplus drops into deficit. In the discussion of the seminal paper of Hans Gerber and Elias Shiu, Pafumi (1998) introduces the framework for capital injections when the company experiences a deficit below zero. In this model, the well known ruin time no longer exists and the process continues indefinitely. Since then, capital injections in the classical risk model have received a lot of attention with extensions to reinsurance and optimality under dividend strategies (see Kulenko and Schmidli (2009), Eisenberg and Schmidli (2009),(2011), Wu (2013) and Zhou and Yuen (2012), (2015)). Nie et al. (2011), (2015) and Dickson and Qazvini (2016) studied the infinite and finite-time ruin probabilities and the Gerber-Shiu function, respectively, in a risk model where capital injections are required if the surplus falls below some non-negative threshold $k \geqslant 0$, in order to regain this level. In this model it is assumed that the injections are funded by a reinsurer, with an instantaneous transaction time, in return for a single net premium paid at time zero.

An important assumption throughout the current literature on capital injections is their instantaneous receipt. However, in the real world markets, insurance firms are required to raise capital when their surplus falls below the Solvency Capital Requirements (SCR) (in the context of the modern regulatory directives such as Solvency II, etc.), by means of capital injections, which are not usually received instantaneously. Capital injections are one the most popular recapitalisation mechanisms in insurance business [see for example the report of ING insurance group (2010), or MOODY's report of April (2016)] and thus, to better reflect the reality, we have to consider that the transaction of capital injections need a certain amount of time to be carried out after the decision to inject capital is made. Time delays, for the receipt of capital injections, occur naturally in insurance business due to decision-making problems or regulatory delays (for example, preparatory and administrative work), and need to be taken into account when the companies make decisions due to the uncertainty of insolvency during these delays. Hence, empirical studies indicate that traditional surplus models with instantaneous capital injections do not capture the realistic process of capital raising transactions.

In order to model more accurately the reality of capital injection transactions, we have to consider that a certain amount of time is needed, after making the decision to inject capital and the receipt of the capital, to accommodate for the financial processing of the injection. The concept of delayed capital injections has been introduced in Jin and Yin (2014), for a pure diffusion risk model without jumps. In the aforementioned paper, the authors study the optimal dividends by means of a stochastic control problem, with mixed singular and delayed impulse controls, assuming that random injections occur at random stopping times throughout the time horizon.

In this paper, we are going to generalise the present models by incorporating a time delay for the receipt of capital injections that depends on the magnitude of the deficit below zero. That is, if the deficit below zero of an insurance firm is small enough (below some threshold), the shareholders are in a position to capital inject the required capital instantaneously. On the other hand, if the deficit of the insurance firm is large enough, then the shareholders need time to raise the required capital for a capital injection. Therefore, there exists a natural dependence between the amount of the required capital injection and the time delay of its receipt (the greater the deficit, the more time required to raise the necessary capital). Based on the above set up, we calculate closed form expressions for the ultimate ruin probability (and other risk quantities of interest) in three different scenarios: (a) discrete random and deterministic delay times, (b) continuous random delay times and (c) the delay time for the capital injection depends on the exact size of the deficit.

The rest of this paper is organised as follows. In Section 2, we introduce the proposed risk process with deficit dependent delayed capital injections. In Section 3, we obtain an integral equation for the ultimate survival probability of the delayed surplus process and derive explicit results for this quantity in terms of the well known ruin quantities of the Cramér-Lundberg risk model. In the same section, we construct a system of simultaneous equations to solve the case of discrete time delays and use these results to analyse the deterministic delay time setting, where we present some special cases. Moreover, we derive and solve a Fredholm integral equation of the second kind for the case of continuous random time delays and consider exponential claim sizes as an example. In Section 4, we generalise the previous model and consider multiple critical values of the deficit which provide a stronger dependence structure between the size of the deficit and the corresponding delay time for the required capital injection. In Section 5, we consider further quantities of interest, such as the expected accumulated capital injections up to time of ultimate ruin and the expected overall time in deficit and show that these quantities also satisfy the Fredholm integral equation of the previous sections. Finally in Section 6, we further generalise the dependence of the corresponding delay for the capital injections by considering the case where the delay time for the capital injections depends on the exact size of the deficit. An inhomogeneous Fredholm equation of the second kind is derived for the ultimate probability of ruin and solved in terms of Neumann series.

2 The model

The surplus process in the Cramér-Lundberg risk model is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geqslant 0, \tag{2.1}$$

where $u \geqslant 0$ is the insurer's initial capital, $c > 0$ is the continuously received premium rate, $\{N(t)\}_{t\geqslant 0}$ is a Poisson process with parameter $\lambda > 0$, which denotes the number of claims received up to time $t \geqslant 0$ and is characterised by the sequence of random variables $\{\sigma_i\}_{i\in\mathbb{N}^+}$, denoting the claim arrival epochs and $\tau_i = \sigma_i - \sigma_{i-1}$, the inter-arrival time between the $(i-1)$-th and i-th claim. The sequence of inter-arrival times, $\{\tau_i\}_{i\in\mathbb{N}^+}$, are independent and identically distributed (i.i.d.) random variables with common distribution function (d.f.) $F_\tau(t) = 1 - e^{-\lambda t}$ and density $f_\tau(t) = \lambda e^{-\lambda t}$, $t \geqslant 0$. The random variables $\{X_k\}_{k\in\mathbb{N}^+}$, form another sequence of i.i.d. random variables representing the amount of the k-th claim, having common d.f. $F_X(\cdot)$, and finite mean $\mu = \mathbb{E}(X) < \infty$. Within the Cramér-Lundberg risk model, it is assumed that the sequence of individual claim sizes, $\{X_k\}_{k\in\mathbb{N}^+}$, and the counting process, $\{N(t)\}_{t\geqslant 0}$, are mutually independent.

It is further assumed that the net profit condition holds, i.e. $c > \lambda\mu$, where the positive safety loading parameter, $\eta > 0$, is given by $\eta = \frac{c}{\lambda\mu} - 1$.

Let us denote the random time T to be the time of classic ruin, defined by

$$T = \inf\{t \geqslant 0 : U(t) < 0\}, \quad (\text{with } T = \infty \text{ if } U(t) \geqslant 0 \text{ for all } t \geqslant 0), \tag{2.2}$$

from which it follows that the probability of ruin, denoted $\psi(u)$, can be expressed as

$$\psi(u) = \mathbb{P}(T < \infty \big| U(0) = u), \quad u \geq 0,$$

with corresponding survival probability $\phi(u) = 1 - \psi(u), u \geq 0$. This quantity has received a great deal of attention over the years and there exists an extensive library of results.

Under the framework of capital injections it is assumed that if the random time T occurs, the company experiences a deficit of some random amount $|U(T)| > 0$, at which point they receive a capital injection, equal to this amount, instantaneously restoring the surplus back to the zero level and allowing the company to continue, see for example Pafumi (1998) and Eisenberg and Schmidli (2011). In order to extend the model, we introduce the delay time setting, with a dependency structure, in the following way.

Consider a deterministic value $k \geqslant 0$, which, in the following, will be referred to as the *critical value* for the magnitude of the deficit, indicating whether or not the receipt of a capital injection comes with some time delay. Note that throughout this paper, we assume that the critical value $k \geqslant 0$ is connected with the deficit below zero, i.e. when the surplus process becomes negative, however, for an environment with capital requirement regulations (such as SII), $k \geqslant 0$ may be associated with the deficit below the SCR of an insurance firm, without any loss of generality. Intuitively, the critical value $k \geqslant 0$ can be interpreted as the size of the deficit below which the injection is considered small enough to be covered by available funds and thus received instantaneously, whilst a deficit greater than the critical value requires time for the firm to raise the necessary funds and thus, a delay is required. That is, at the moment the surplus process, $\{U(t)\}_{t\geqslant 0}$, first becomes negative (which occurs at time T) we have two different possibilities:

(a) The deficit is at most $k \geqslant 0$, i.e. $|U(T)| \leqslant k$, which occurs with probability $G(u, k)$, where

$$G(u, y) = \mathbb{P}(T < \infty, |U(T)| \leqslant y \big| U(0) = u), \tag{2.3}$$

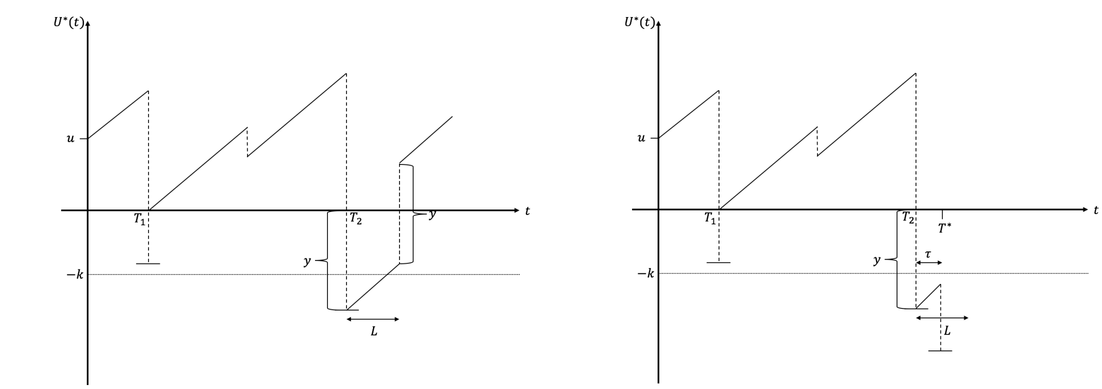
with density $g(u,y) = \frac{\partial}{\partial y}G(u,y)$ [the d.f. $G(\cdot,\cdot)$, of the well known deficit at ruin was first defined in Gerber et al. (1987) and has been extensively studied for the Cramér-Lundberg model]. Then, a capital injection of size $|U(T)| \leqslant k$ is required to restore the surplus back to the zero level which occurs instantaneously, since the amount of the capital injection is of a size that can be covered by readily available funds.

(b) The deficit is larger than the critical value $k \geqslant 0$, which occurs with probability

$$\overline{G}(u,k) = \int_k^\infty g(u,y)\,dy = \psi(u) - G(u,k). \tag{2.4}$$

The available funds are unable to cover the required capital injection and thus, the injection is received after some delay time, denoted by the random variable L, with d.f. $F_L(\cdot)$, to account for administration and processing time (see Fig: 1 for the two cases, respectively).

Based on the above set up, it is clear that the company is allowed to continue when in deficit and it is assumed they will receive premium income during this time. However, if a subsequent claim occurs before the capital injection is received, i.e. $\tau < L$, then the company is considered to be facing too much risk at any one time and is declared as 'ruined'. We call this time 'ultimate ruin' to distinguish from the classical ruin time defined in equation (2.2).



(a) Delayed capital injection arriving before subsequent claim in deficit.

(b) Subsequent claim arriving before delayed capital injection, resulting in ultimate ruin.

Figure 1: Possible cases when dropping into deficit.

We can now consider the amended surplus process under such a framework, denoted by $\{U^*(t)\}_{t\geqslant 0}$, which is defined by

$$U^*(t) = U(t) + \sum_{i=1}^{\infty} |U^*(T_i)| \mathbb{I}_{(\{|U^*(T_i)|\leqslant k\} \cup \{(|U^*(T_i)|>k) \cap (T_i+L_i\leqslant t)\})}, \tag{2.5}$$

where

$$T_i = \inf\{t > T_{i-1} : U^*(t) < 0, U^*(t-) \geqslant 0\},$$

is the i-th time the surplus falls below zero, due to a claim, with $T_0 = 0$ and L_i is the delay time corresponding to the i-th deficit, given that the deficit is larger than $k \geqslant 0$. Note that $T_1 = T$ is the classic ruin time defined in equation (2.2). We can now define the time of ultimate ruin by

$$T^* = \inf\left\{\sigma_i > 0 : U^*(\sigma_{i-1}) < -k, \sigma_i < \sigma_{i-1} + L_j\right\}, \tag{2.6}$$

for some $i = 1, 2, \ldots$, where $\{\sigma_i\}_{i\in\mathbb{N}^+}$ is the sequence of claim arrival epochs for the Poisson process, as defined previously, and some j corresponding to the j-th deficit larger than $k \geqslant 0$. Then, it follows that the ultimate ruin probability can be expressed as

$$\psi^*(u) = \mathbb{P}(T^* < \infty \big| U^*(0) = u), \qquad u \geqslant 0,$$

with the corresponding ultimate survival probability, given by

$$\phi^*(u) = 1 - \psi^*(u).$$

Note that a natural extension of this model is that ruin does not occur in the case that $\{T_j = \sigma_{i-1}, \sigma_i < T_j + L_j, U(\sigma_i) \geq 0\}$, for some i and j. However, in order to keep the mathematical tractability of our results (without altering the key findings of the paper), we avoid to extend to this case. Also, the following market practice, usually the value of $k \geqslant 0$ is sufficiently large, so the probability of such event is minimal.

3 Ultimate ruin probabilities for a single critical value

In this section, we consider three separate types of delay times, for which, by using a conditioning argument and the Markov property, we derive integral equations and obtain explicit expressions for the ultimate ruin probability, $\psi^*(u)$, for $u \geqslant 0$.

In the first case, where the delay time of the capital injections is represented by a discrete time random variable, we derive a system of simultaneous equations, which are solved by the use of general matrix algebra, to obtain a linear expression for the ultimate ruin probability. We then proceed to a second case by considering a deterministic delay time for the capital injections, which can be seen as a special case of the aforementioned discrete time model, with similar methods of solution. Finally, in the third case, we consider a continuous time delay for the capital injections and derive a inhomogeneous Fredholm integral equation of the second kind, which is solved to obtain an explicit expression in terms of the classic ruin quantities for the Cramér-Lundberg risk model.

3.1 Capital injections with discrete time random delays

Let us first consider the case where the capital injection delay time random variable, namely L, can take finitely many discrete values. That is, $L \in \{m_1, \ldots, m_N\}$ with probability $p_i = \mathbb{P}(L = m_i) > 0$, where $m_i \geqslant 0$ for all $i = 1, \ldots, N$ and $\sum_{i=1}^N p_i = 1$. Then, by conditioning on the amount of the first drop below zero ($y > 0$), the delay time random variable and the subsequent claim inter-arrival time, the law of total probability gives

$$\phi^*(u) = \phi(u) + G(u,k)\phi^*(0) + \int_k^\infty g(u,y) \int_0^\infty f_\tau(s) \sum_{i=1}^N p_i \phi^*(cm_i) \mathbb{I}_{\{m_i < s\}} \, ds \, dy, \tag{3.1}$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator function and $\phi(u)$ is the well known (classic) survival probability of the surplus process $\{U(t)\}_{t \geqslant 0}$, i.e. without the presence of capital injections for which numerous results and explicit expressions exist in the actuarial literature. Following from the definition of an indicator function, the above equation can be written as

$$\begin{aligned}
\phi^*(u) &= \phi(u) + G(u,k)\phi^*(0) + \int_k^\infty g(u,y) \sum_{i=1}^N p_i \int_{m_i}^\infty f_\tau(s) \phi^*(cm_i) \, ds \, dy \\
&= \phi(u) + G(u,k)\phi^*(0) + \overline{G}(u,k) \sum_{i=1}^N p_i \overline{F}_\tau(m_i) \phi^*(cm_i),
\end{aligned} \tag{3.2}$$

where $\overline{F}_\tau(t) = 1 - F_\tau(t) = e^{-\lambda t}$, $t \geqslant 0$, is the tail of the inter-arrival time distribution for the Poisson process. Thus, equation (3.2) reduces to

$$\phi^*(u) = \phi(u) + G(u,k)\phi^*(0) + \overline{G}(u,k) \sum_{i=1}^N p_i e^{-\lambda m_i} \phi^*(cm_i). \tag{3.3}$$

In order to complete the expression for $\phi^*(u)$, in equation (3.3), (since the risk quantities $\phi(u)$ and $G(u,y)$ are well known for the Cramér-Lundberg risk model for various classes of claim size distributions) we need to determine the boundary value $\phi^*(0)$ and individual values $\phi^*(cm_i)$, for $i = 1, \ldots, N$.

Setting $u = 0$, in the above equation, and solving with respect to $\phi^*(0)$, yields

$$\phi^*(0) = \frac{\phi(0) + \overline{G}(0,k) \sum_{i=1}^N p_i e^{-\lambda m_i} \phi^*(cm_i)}{1 - G(0,k)}, \tag{3.4}$$

which, after substituting this expression for $\phi^*(0)$ back into equation (3.3) and re-arranging, yields

$$\phi^*(u) = w(u,k) + v(u,k) \sum_{i=1}^N p_i e^{-\lambda m_i} \phi^*(cm_i), \tag{3.5}$$

where

$$w(u,k) = \phi(u) + \frac{G(u,k)\phi(0)}{1 - G(0,k)} > 0, \tag{3.6}$$

and

$$v(u,k) = \frac{G(u,k)\overline{G}(0,k)}{1 - G(0,k)} + \overline{G}(u,k) = \psi(u) - \frac{G(u,k)\phi(0)}{1 - G(0,k)} < 1, \tag{3.7}$$

such that $w(u,k) + v(u,k) = 1$, for all $u, k \geqslant 0$. The strict inequalities in equations (3.6) and (3.7), for the functions $w(u,k)$ and $v(u,k)$, follow from that fact that, under the net profit condition, the classical ruin function $\psi(u) < 1$, for all $u \geqslant 0$ [see Asmussen and Albrecher (2010)].

Remark 1. *The function $w(u,k) > 0$ (above) corresponds to the survival probability in the capital injection model without delays, as studied in Nie et al. (2011). Moreover, the function $v(u,k) = 1 - w(u,k) < 1$ is the corresponding ruin probability.*

Now, in order to uniquely determine $\phi^*(u)$ in equation (3.5), it remains to determine the values $\phi^*(cm_i)$, for $i = 1, \dots, N$.

To do this, we will construct and solve N linear simultaneous equations. Setting $u = cm_j$, for $j = 1, \dots, N$, in equation (3.5), results in the simultaneous equation system

$$\phi^*(cm_j) = w(cm_j,k) + v(cm_j,k) \sum_{i=1}^{N} p_i e^{-\lambda m_i} \phi^*(cm_i), \quad j = 1, \dots, N,$$

or equivalently

$$\left(1 - v(cm_j,k) p_j e^{-\lambda m_j}\right) \phi^*(cm_j) = w(cm_j,k) + v(cm_j,k) \sum_{i=1, i \neq j}^{N} p_i e^{-\lambda m_i} \phi^*(cm_i),$$

which can be written as the following first order matrix equation system

$$\mathbf{A}\vec{\phi}^* = \vec{w},$$

where

$$\mathbf{A} = \begin{pmatrix} \left(1 - v(cm_1,k)p_1 e^{-\lambda m_1}\right) & -v(cm_1,k)p_2 e^{-\lambda m_2} & \cdots & -v(cm_1,k)p_N e^{-\lambda m_N} \\ -v(cm_2,k)p_1 e^{-\lambda m_1} & \left(1 - v(cm_2,k)p_2 e^{-\lambda m_2}\right) & \cdots & -v(cm_2,k)p_N e^{-\lambda m_N} \\ \vdots & \vdots & \ddots & \vdots \\ -v(cm_N,k)p_1 e^{-\lambda m_1} & -v(cm_N,k)p_2 e^{-\lambda m_2} & \cdots & \left(1 - v(cm_N,k)p_M e^{-\lambda m_N}\right) \end{pmatrix},$$

is an N-dimensional square matrix, with $v(\cdot,\cdot)$ given by equation (3.7), $\vec{\phi}^* = (\phi^*(cm_1), \dots, \phi^*(cm_N))^\top$ and $\vec{w} = (w(cm_1,k), \dots, w(cm_N,k))^\top$ are both N-dimensional column vectors, where $(\cdot)^\top$ denotes the transpose of a vector/matrix. In order to evaluate the vector of unknowns, $\vec{\phi}^*$, we will show in the following Lemma that that the matrix $\mathbf{A}$ is non-singular and thus invertible.

Lemma 1. *For $u \geqslant 0$, $0 < p_i \leqslant 1$, $i = 1, \ldots, N$ and $\sum_{j=1}^{N} p_j = 1$, the matrix $\mathbf{A}$ is non-singular.*

Proof. To show that $\mathbf{A}$ is a non-singular matrix, by the Lévy-Desplanques Theorem [see Horn and Johnson (1990)], it suffices to prove that $\mathbf{A}$ is a strictly diagonally dominant matrix, i.e.

$$|1 - v(cm_i, k)p_i e^{-\lambda m_i}| > \sum_{j \neq i} |-v(cm_i, k)p_j e^{-\lambda m_j}|,$$

for all $i = 1, \ldots, N$, or equivalently

$$1 - v(cm_i, k)p_i e^{-\lambda m_i} > v(cm_i, k) \sum_{j \neq i} p_j e^{-\lambda m_j},$$

since, from equation (3.7), we have $0 \leqslant v(u, k) < 1$, for all $u \geqslant 0$, which guarantees that $v(u, k)p_j e^{-\lambda m_j} \geqslant 0$ and $v(u, k)p_j e^{-\lambda m_j} < p_j e^{-\lambda m_j} < 1$, for every $i, j = 1, \ldots, N$.

Employing the fact that $v(u, k) < 1$, for all $u \geqslant 0$ (under the net profit condition), from equation (3.7), we have that

$$1 > v(cm_i, k) = v(cm_i, k) \sum_{j=1}^{N} p_j \geqslant v(cm_i, k) \sum_{j=1}^{N} p_j e^{-\lambda m_j}, \qquad i = 1, \ldots, N,$$

from which it follows that $\mathbf{A}$ is strictly diagonally dominant and thus, the result follows. □

Now, since the matrix $\mathbf{A}$ is non-singular, and thus invertible, the forms of $\phi^*(cm_i)$, $i = 1, \ldots, N$, can be determined by

$$\vec{\phi^*} = \mathbf{A}^{-1}\vec{w},$$

where $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$. Finally, the ultimate survival probability, for capital injections with a discrete random time delay, is given by the linear expression

$$\phi^*(u) = w(u, k) + v(u, k) \sum_{i=1}^{N} p_i e^{-\lambda m_i} \left[\mathbf{A}^{-1}\vec{w}\right]_i$$

where $\left[\mathbf{A}^{-1}\vec{w}\right]_i$ is the i-th element of the vector $\mathbf{A}^{-1}\vec{w}$.

Theorem 1. *For $u \geqslant 0$, the ultimate ruin probability under capital injections with discrete time random delays, namely $\psi^*(u)$, is given by*

$$\psi^*(u) = v(u, k) \left(1 - \sum_{i=1}^{N} p_i e^{-\lambda m_i} \left[\mathbf{A}^{-1}\vec{w}\right]_i\right), \tag{3.8}$$

where

$$v(u, k) = \psi(u) - \frac{\eta G(u, k)}{1 + \eta - F_e(k)}$$

and $F_e(x) = \frac{1}{\mu} \int_0^x \overline{F}_X(y)\, dy$ is the integrated tail distribution of the claim sizes.

Remark 2. *For $N = 0$, the ultimate ruin probability, $\psi^*(u) = v(u,k)$, reduces to the ruin probability in a risk model with instantaneous capital injections when below the critical value and ultimate ruin when larger than the critical value, as studied in Nie et al. (2011). Thus, it should be clear that, for $N > 0$, the term in the brackets of equation* (3.8) *is the contribution to $\psi^*(u)$ due to the possible delays.*

3.2 Capital injections with deterministic delay times

In practice, market studies indicate that the delay times for the capital injections may not be random, but instead a fixed amount of time, i.e. number of days needed to gather required capital injection or number of days needed for financial or regulatory purposes. Thus, a natural consideration is to consider the case of deterministic delay times. Let the delay time $L = \rho \geqslant 0$. Note that this is equivalent to the discrete time case with $N = 1$ and random time delay $m_1 = \rho$, with $p_1 = 1$. Thus, equation (3.5) reduces to

$$\phi^*(u) = w(u,k) + v(u,k)e^{-\lambda\rho}\phi^*(c\rho). \tag{3.9}$$

and from Theorem 1, we have the following Corollary.

Corollary 1. *For $u \geqslant 0$, the ultimate ruin probability under capital injections with deterministic time delay $L = \rho \geqslant 0$, namely $\psi^*(u)$, is given by*

$$\psi^*(u) = v(u,k)\left(\frac{1 - e^{-\lambda\rho}}{1 - v(c\rho,k)e^{-\lambda\rho}}\right), \tag{3.10}$$

where

$$v(u,k) = \psi(u) - \frac{\eta G(u,k)}{1 + \eta - F_e(k)}.$$

Remark 3 ($\rho \to \infty$)**.** *As $\rho \to \infty$, since* $\lim_{\rho\to\infty} e^{-\lambda\rho} = 0$, *equation* (3.10) *reduces to*

$$\psi^*(u) = v(u,k) = \psi(u) - G(u,k)\frac{\phi(0)}{1 - G(0,k)},$$

which is equivalent to the results given in Nie et al. (2011).

3.3 Capital injections with continuous time random delays

In this section, we will consider the case where the delay time random variable, L, is a continuous time random variable having probability density function $f_L(\cdot)$ and finite mean $\mathbb{E}(L) < \infty$. If we apply a similar conditioning argument as in the discrete time case, i.e. conditioning on the amount of the first drop below zero, the delay time and the subsequent claim inter-arrival time, we obtain the continuous time form of equation (3.1), given by

$$\begin{aligned}\phi^*(u) &= \phi(u) + G(u,k)\phi^*(0) + \int_k^\infty g(u,y)\int_0^\infty f_L(t)\int_0^\infty f_\tau(s)\phi^*(ct)\mathbb{I}_{\{t<s\}}\,ds\,dt\,dy \\ &= \phi(u) + G(u,k)\phi^*(0) + \overline{G}(u,k)\int_0^\infty f_L(t)\overline{F}_\tau(t)\phi^*(ct)\,dt, \qquad (3.11)\end{aligned}$$

or equivalently

$$\phi^*(u) = \phi(u) + G(u,k)\phi^*(0) + \overline{G}(u,k)\int_0^\infty f_L(t)e^{-\lambda t}\phi^*(ct)\,dt. \qquad (3.12)$$

Now, as in the discrete case (since $G(\cdot,\cdot)$ and $\phi(\cdot)$ are well known for the Cramér-Lundberg model), in order to complete the expression for $\phi^*(u)$ in equation (3.12), we first need to determine the boundary value $\phi^*(0)$.

Setting $u = 0$, in equation (3.12), and solving with respect to $\phi^*(0)$, we have that

$$\phi^*(0) = \frac{\phi(0) + \overline{G}(0,k)\int_0^\infty f_L(t)\,e^{-\lambda t}\phi^*(ct)\,dt}{1 - G(0,k)},$$

which is simply the continuous analogue of the expression given in equation (3.4). Substituting this form of the boundary value $\phi^*(0)$ into equation (3.12), yields

$$\phi^*(u) = w(u,k) + v(u,k)\int_0^\infty f_L(t)e^{-\lambda t}\phi^*(ct)\,dt, \qquad (3.13)$$

where $w(u,k)$ and $v(u,k)$ are defined as in equations (3.6) and (3.7), respectively.

Now, using a change of variables, the above equation can be written as

$$\phi^*(u) = w(u,k) + \frac{1}{c}v(u,k)\int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}\phi^*(t)\,dt, \qquad (3.14)$$

which is the form of an inhomogeneous Fredholm integral equation of the second kind over a semi-infinite interval, with degenerate kernel [see Polyanin and Manzhirov (2008)]

$$K(u,t) = v(u,k)f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}. \qquad (3.15)$$

Following the general general theory of integral equations to derive a closed form expression for the inhomogeneous Fredholm equation with degenerate kernel [see Polyanin and Manzhirov (2008)], we point out that the integral in equation (3.14) evaluates to a constant, say C_1 (the existence of this constant is shown in Proposition 1, below).

Proposition 1. *The constant* $C_1 = \int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}\phi^*(t)\,dt$ *is finite and bounded by the premium rate* $c > 0$.

Proof. The function $\phi^*(x)$ is a probability measure, hence $e^{-\frac{\lambda t}{c}}\phi^*(t) \leqslant 1$, for all $t \geqslant 0$. Therefore, it follows that

$$C_1 = \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}\phi^*(t)\, dt \leqslant \int_0^\infty f_L(\frac{t}{c})\, dt = c,$$

since $f_L(\cdot)$ is a proper density function. □

Then, the general solution to equation (3.14) is given by the linear combination

$$\phi^*(u) = w(u,k) + \frac{C_1}{c}v(u,k), \tag{3.16}$$

where C_1 is some constant [see Proposition 1], that needs to be determined.

To complete the solution for $\phi^*(u)$, in equation (3.16), it remains to calculate explicitly the constant C_1. In order to do this, let us: replace the variable u, in equation (3.14), by t; multiply through by $f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}$ and integrate from 0 to ∞, to obtain the expression

$$\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}\phi^*(t)\, dt = \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt + \frac{C_1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt.$$

Note that the left hand side of the above equality is simply the constant C_1. Further, since we have that $w(u,k) \leqslant 1$ and $v(u,k) < 1$, from equations (3.6) and (3.7), we can use a similar argument as in the proof of Proposition 1 to show that both $\int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}} w(t,k)\, dt$ and $\int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}} v(t,k)\, dt$ exist and are bounded by $c > 0$.

Now, solving this equation with respect to C_1, we find that

$$C_1 = \frac{\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt}{1 - \frac{1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt},$$

as long as $\frac{1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}} v(t,k)\, dt \neq 1$, which can be verified since $v(u,k) < 1$, for all $u \geqslant 0$.

Substituting this form of C_1 back into equation (3.14), we obtain the explicit expression for the survival probability given by

$$\phi^*(u) = w(u,k) + \frac{\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt}{c - \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt} v(u,k). \tag{3.17}$$

Finally, defining the Laplace-Stieltjes transform of the delay time distribution by $\widehat{f}_L(s) = \int_0^\infty e^{-sx}\, dF_L(x)$ and recalling that $w(u,k) = 1 - v(u,k)$, we have the following Theorem.

Theorem 2. *For all $u \geqslant 0$, the ultimate ruin probability under capital injections with continuous time delays, namely $\psi^*(u)$, is given by*

$$\psi^*(u) = v(u,k)\left(\frac{1-\widehat{f}_L(\lambda)}{1-\int_0^\infty f_L(t)v(ct,k)e^{-\lambda t}\,dt}\right), \tag{3.18}$$

where $\widehat{f}_L(s)$ is the Laplace-Stieltjes transform of the delay time distribution and

$$v(u,k) = \psi(u) - \frac{\eta G(u,k)}{1+\eta-F_e(k)}. \tag{3.19}$$

Remark 4. *Note that the two integral terms appearing in the expression for C_1 are both finite. This can be proved using a similar argument as the proof of Proposition 1.*

In order to illustrate the applicability of Theorem 2, in the next proposition we give an explicit expression for the ultimate ruin probability, namely $\psi^*(u)$, in the case where both the delay time of the capital injections and the individual claim sizes follow an exponential distribution with different parameters.

Proposition 2. *Assume that the delay time, L, follows an exponential distribution with parameter $\alpha > 0$. Further, assume that the claim sizes also follow an exponential distribution with parameter $\beta > 0$. Then, the probability of ultimate ruin under delayed capital injections is given by*

$$\psi^*(u) = Ke^{-\frac{\lambda\eta}{c}u}, \qquad u \geqslant 0, \tag{3.20}$$

where K is a constant of the form

$$K = \frac{\lambda(\alpha+\beta c)}{(\alpha+\lambda)\left(\beta c+(\alpha+\beta c)\eta e^{\beta k}\right)}$$

Proof. For a delay time, L, which is exponentially distributed with parameter $\alpha > 0$, we have that $F_L(x) = 1-e^{-\alpha x}$, with corresponding density $f_L(x) = \alpha e^{-\alpha x}$ and Laplace transform $\widehat{f}_L(s) = \frac{\alpha}{\alpha+s}$. In addition, the forms of the quantities $G(u,y)$ and $\overline{G}(u,y)$, for the classical Cramér-Lundberg risk model, are known explicitly for the case of exponentially distributed claim sizes, i.e. when $F_X(x) = 1-e^{-\beta x}$, $\beta > 0$, and are given by $G(u,y) = \psi(u)\left(1-e^{-\beta k}\right)$ and $\overline{G}(u,y) = \psi(u)e^{-\beta k}$, where $\psi(u) = \frac{1}{1+\eta}e^{-\frac{\lambda\eta}{c}u}$, for $u \geqslant 0$. Thus, from equation (3.19), it follows that

$$v(u,k) = e^{-\frac{\lambda\eta}{c}u}\left(\frac{1}{1+\eta e^{\beta k}}\right),$$

and

$$\int_0^\infty f_L\left(t\right)v(ct,k)e^{-\lambda t}\,dt = \frac{\alpha}{(1+\eta e^{\beta k})(\alpha+\beta c)}.$$

Employing equation (3.21) of Theorem 2, the result follows. □

Remark 5. *In this section, we have discussed three different methods of obtaining an explicit expression for the ruin probability, corresponding to the different structures of the delay time random variable. It is noted here that the method employed in the final subsection for a continuous time delay (Fredholm integral equations) can be generalised to incorporate all the previous results in one step. This is seen by considering a general distribution function $F_L(\cdot)$, resulting in the generalised constant*

$$C_1 = \frac{c\int_0^\infty e^{-\lambda s} w(cs,k)\,dF_L(s)}{1-\int_0^\infty e^{-\lambda s} v(cs,k)\,dF_L(s)},$$

from which, using equation (3.16)*, we obtain the following Theorem.*

Theorem 3. *Let $F_L(\cdot)$ be a general distribution function for the delay time random variable L. Then, for all $u \geqslant 0$, the ultimate ruin probability under delayed capital injections, namely $\psi^*(u)$, is given by*

$$\psi^*(u) = v(u,k)\left(1 - \frac{\int_0^\infty e^{-\lambda s} w(cs,k)\,dF_L(s)}{1-\int_0^\infty e^{-\lambda s} v(cs,k)\,dF_L(s)}\right). \tag{3.21}$$

In the remainder of this paper, we consider the case of a continuous delay time random variable as it makes the methodologies clearer to follow. However, as in Remark 5, we point out that the results can be generalised to incorporate a general delay time distribution function.

4 Extension to a model with N critical values

In this section, we generalise the previous model for a continuous time delay, L, to allow for N independent deficit critical values, introducing a dependence between the size of the deficit and the corresponding delay time.

Let k_i, $i = 0, 1, \dots, (N+1)$ be ordered, positive constants denoting the magnitude of the critical values, between which the deficit lies (deficit thresholds) such that $0 = k_0 < k_1 < \dots < k_N < k_{N+1} = \infty$. Similarly to Section 2, we define the joint probability functions $G_i(u) = \mathbb{P}(T < \infty, k_i < |U(T)| \leqslant k_{i+1} \big| U(0) = u)$ which can be expressed in terms of the deficit at ruin functions $G(u, y)$ since

$$G_i(u) = \int_{k_i}^{k_{i+1}} g(u,y)\,dy = G(u,k_{i+1}) - G(u,k_i),$$

with $G_0(u) = G(u, k_1)$ and $G_N(u) = \overline{G}(u, k_N) = \mathbb{P}(T < \infty, |U(T)| > k_N \big| U(0) = u)$ being the probability that ruin occurs with a deficit larger than the greatest deficit critical value, namely k_N.

Similarly to the previous section, we assume that if ruin occurs with a deficit less than the smallest barrier k_1, i.e. $|U(T)| \leqslant k_1$, then the required capital injection can be covered by available funds and is received instantaneously. On the other hand, if ruin occurs and the deficit has magnitude $|U(T)| = y \in (k_i, k_{i+1}]$, $i = 1, 2, \ldots, N$, then the capital injection, of size y, is received after some random time delay, L_i, having d.f. $F_{L_i}(\cdot)$ and density $f_{L_i}(\cdot)$. Finally, it is assumed that the time delay time random variable L_i is 'less than' the time delay random variable L_{i+1}, in the sense of stochastic ordering, i.e. $L_i \leqslant_{st} L_{i+1}$, such that there exists a positive correlation between the size of the required injection and the corresponding delay time.

Using the same conditioning argument as in Section 2, we obtain an equation for the ultimate survival probability, under N deficit threshold barriers and continuous delay times, given by

$$\begin{aligned}\phi^*(u) &= \phi(u) + G(u,k_1)\phi^*(0) + \sum_{i=1}^{N} \int_{k_i}^{k_{i+1}} g(u,y) \int_0^\infty f_{L_i}(t) \int_0^\infty f_\tau(s)\phi^*(ct)\mathbb{I}_{\{t<s\}}\, ds\, dt\, dy \\ &= \phi(u) + G(u,k_1)\phi^*(0) + \sum_{i=1}^{N} G_i(u) \int_0^\infty f_{L_i}(t)\overline{F}_\tau(t)\phi^*(ct)\, dt,\end{aligned}$$

or equivalently

$$\phi^*(u) = \phi(u) + G(u,k_1)\phi^*(0) + \sum_{i=1}^{N} G_i(u) \int_0^\infty f_{L_i}(t)e^{-\lambda t}\phi^*(ct)\, dt. \tag{4.1}$$

To complete the the solution for $\phi^*(u)$ in equation (4.1), as in the previous sections, we need to determine the boundary value $\phi^*(0)$. Setting $u = 0$, in the above equation, and solving with respect to $\phi^*(0)$, yields

$$\phi^*(0) = \frac{\phi(0) + \sum_{i=1}^{N} G_i(0) \int_0^\infty f_{L_i}(t)e^{-\lambda t}\phi^*(ct)\, dt}{1 - G(0,k_1)},$$

which, after substitution back into equation (4.1), gives

$$\phi^*(u) = w(u,k_1) + \sum_{i=1}^{N} v_i(u) \int_0^\infty f_{L_i}(t)e^{-\lambda t}\phi^*(ct)\, dt, \tag{4.2}$$

where $w(u,k)$ is defined as in equation (3.6) and $v_i(u)$, for $i = 1, 2, \ldots, N$, is defined by

$$v_i(u) = \frac{G(u,k_1)G_i(0)}{1 - G(0,k_1)} + G_i(u), \tag{4.3}$$

with $\sum_{i=1}^{N} v_i(u) = 1 - w(u,k_1)$.

Now, using a change of variables, equation (4.2) takes the form of an inhomogeneous Fredholm equation of the second kind, given by

$$\phi^*(u) = w(u,k_1) + \frac{1}{c}\sum_{i=1}^{N} v_i(u) \int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi^*(t)\, dt, \tag{4.4}$$

with degenerate kernel of the form

$$K(u,t) = \sum_{i=1}^{N} v_i(u) f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}.$$

Following similar arguments as in Section 3.3 and Proposition 1, we note that the integral terms on the right hand side of the Fredholm integral equation, given in equation (4.4), evaluate to constants, say $C_i = \int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi^*(t)\, dt < \infty$. Thus, the general solution to equation (4.4) is given by the linear combination

$$\phi^*(u) = w(u,k_1) + \frac{1}{c}\sum_{i=1}^{N} C_i v_i(u). \tag{4.5}$$

It remains to calculate explicitly the constants $C_i, i = 1,2,\ldots,N$. Following similar arguments to Section 3.3, we first replace the variable u, in equation (4.5), by t, multiply through by $f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}$, for $j = 1,2,\ldots,N$, and integrate from 0 to ∞, to obtain the expression

$$\int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi^*(t)\, dt = \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k_1)\, dt + \frac{1}{c}\sum_{i=1}^{N} C_i \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\, dt,$$

which, after recalling the definition of the constants C_i, $i = 1,2,\ldots,N$, reduces to the form

$$C_j = \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k_1)\, dt + \frac{1}{c}\sum_{i=1}^{N} C_i \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\, dt,$$

or equivalently, leads to the system of N simultaneous equations, of the form

$$\begin{aligned}\int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k_1)\, dt \;&=\; \left(1 - \frac{1}{c}\int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\, dt\right) C_j \\ &\quad -\frac{1}{c}\sum_{i\neq j}^{N} C_i \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\, dt, \quad j = 1,2,\ldots,N.\end{aligned}$$

In a more concise matrix form, the above linear system of equation for C_i, $i = 1,\ldots,N$, can be expressed by

$$\mathbf{M}\vec{C} = \vec{w},$$

where $\mathbf{M}$ is an N dimensional square matrix given by

$$\mathbf{M} = \begin{pmatrix} 1 - \frac{1}{c}\int_0^\infty f_{L_1}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_1(t)\,dt & \cdots & -\frac{1}{c}\int_0^\infty f_{L_1}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_N(t)\,dt \\ \vdots & \ddots & \vdots \\ -\frac{1}{c}\int_0^\infty f_{L_N}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_1(t)\,dt & \cdots & 1 - \frac{1}{c}\int_0^\infty f_{L_N}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_N(t)\,dt \end{pmatrix},$$

$\vec{C} = (C_1, \ldots, C_N)^\top$ and $\vec{w} = \left(\int_0^\infty f_{L_1}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k_1)\,dt, \ldots, \int_0^\infty f_{L_N}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k_1)\,dt\right)^\top$ are both N-dimensional column vectors. In order to evaluate the vector of unknowns, $\vec{C}$, we will show in the following Lemma that the matrix $\mathbf{M}$ is non-singular and thus invertible.

Lemma 2. *The N-dimensional square matrix $\mathbf{M}$ is non-singular.*

Proof. As in the proof of Lemma 1, in order to prove the matrix $\mathbf{M}$ is non-singular, it suffices to prove that it is a strictly diagonally dominant matrix. That is, the i-th diagonal element of $\mathbf{M}$, for all $i = 1, \ldots, N$, satisfies

$$\left|1 - \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\,dt\right| > \sum_{j \neq i} \left|-\frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\,dt\right|,$$

or equivalently

$$1 - \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\,dt > \sum_{j \neq i} \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\,dt,$$

since (similarly to the proof of Lemma 1) $v_i(u) < 1$, for $u \geqslant 0$, which guarantees that $0 \leqslant \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\,dt < 1$, for all $i = 1, \ldots, N$.

Now, since $\sum_{i=1}^N v_i(u) = 1 - w(u,k_1) < 1$, for all $u \geqslant 0$, we have that

$$\begin{aligned} 1 = \int_0^\infty f_{L_i}(t)\,dt > \int_0^\infty f_{L_i}(t)(1 - w(ct,k_1))\,dt \geqslant \int_0^\infty f_{L_i}(t) e^{-\lambda t} \sum_{j=1}^N v_j(ct)\,dt \\ = \sum_{j=1}^N \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\,dt, \end{aligned}$$

which completes the proof. □

Using the results of Lemma 2, the constants C_i can be evaluated by

$$\vec{C} = \mathbf{M}^{-1}\vec{w},$$

where $\mathbf{M}^{-1}$ is the inverse of the matrix $\mathbf{M}$. Now, since the constants C_i, for $i = 1, \ldots, N$, are uniquely determined, we can employ the form of general solution to the Fredholm integral equation, given by equation (4.5), to obtain the following Theorem for the corresponding probability of ruin.

Theorem 4. *For $u \geqslant 0$, the ultimate ruin probability under capital injections with continuous time random delays and N critical values, namely $\psi^*(u)$, is given by*

$$\psi^*(u) = \frac{1}{c} \sum_{i=1}^{N} \left(c - \left[\mathbf{M}^{-1} \vec{w} \right]_i \right) v_i(u), \tag{4.6}$$

where $\left[\mathbf{M}^{-1} \vec{w} \right]_i$ is the i-th element of the vector $\mathbf{M}^{-1}\vec{w}$.

Remark 6. *It is worth pointing out that the methodologies used in subsections 3.1 and 3.2, for the discrete time random delays and the deterministic time delays for the capital injections, can also be extended to the model with N critical values.*

5 Further quantities with continuous delay times

In this section, we consider two further quantities that will be of interest to an insurance company when it comes to risk management and mitigation. The first is the expected discounted accumulated capital injections up to the time of ultimate ruin, which gives an indication of the (discounted) amount of funds needed to keep the company solvent during its lifetime. This particular quantity can be used to determine the net single premium of a reinsurance contract, which may provide the necessary capital injections, as seen in Pafumi (1998) and Nie et al. (2011), or to determine the present value of dividends to be paid to the companies shareholders, who may contribute to such injections when needed.

The second, closely related, quantity of interest is the discounted expected overall time in red (deficit), up to the time of ultimate ruin. This is a natural consideration, since knowledge of the expected time in deficit (or below the SCR) provides valuable information to an insurance firm. For example, if we assume the firm is subject to a continuous constant penalty during the time in which it is in a deficit, the discounted expected overall time in red, up to the time of ultimate ruin, provides the present value of this penalised time in red, allowing the company to more accurately calculate its capital requirements.

For simplicity of calculations, we revert back to the simplest model of a single critical value, given by $k \geqslant 0$ as in Section 2, but point out that the following results hold for the N barrier setting by employing a similar method to that discussed in Section 4.

5.1 The expected discounted accumulated capital injections up to the time of ultimate ruin

Let $\{Z_u^*(t)\}_{t \geqslant 0}$ be a pure jump process denoting the accumulated capital injections in a continuous time delayed setting, up to time $t \geqslant 0$, for the risk process $U^*(t)$, defined in equation (2.5), with initial capital $u \geqslant 0$. We are interested in the expected discounted accumulated capital injections, up to the time of ultimate ruin, i.e. $z_\delta^*(u) = \mathbb{E}\left(e^{-\delta T^*} Z_u^*(T^*)\right)$, where $\delta \geqslant 0$ is a constant discount rate and T^* is the time of ultimate ruin, defined in equation (2.6).

Further, let us first define

$$W(u,y,t)=\mathbb{P}\left(T\leqslant t,|U(T)|\leqslant y\middle|U(0)=u\right),$$

to be the joint probability of classic ruin time (before time $t\geqslant 0$) and the deficit at ruin for the Cramér-Lundberg risk process $U(t)$, defined in equation (2.1), and let

$$w(u,y,t)=\frac{\partial^2}{\partial t\partial y}W(u,y,t),$$

denote the (defective) joint density of T and $|U(T)|$. Note that $\lim_{t\to\infty}W(u,y,t)=G(u,y)$, where $G(u,y)$ is defined in equation (2.3). The risk quantity $W(u,y,t)$ has been studied in Dickson and Drekic (2006), Landriault and Willmot (2009) and Nie et al. (2011), (2015), for the capital injection model without delays, and explicit expressions exist for certain claim size distributions. Finally, we denote by

$$g_\delta(u,y)=\int_0^\infty e^{-\delta t}w(u,y,t)\,dt,\quad\text{and}\quad G_\delta(u,y)=\int_0^y g_\delta(u,x)\,dx,$$

the (defective) discounted density function and d.f., respectively, of the deficit at ruin, with initial surplus $u\geqslant 0$ and force of interest $\delta\geqslant 0$.

Conditioning on the time and amount of the first fall into deficit and the subsequent delay and claim inter-arrival times, we obtain that

$$\begin{aligned}z^*_\delta(u) &= \int_0^\infty\int_0^k e^{-\delta t}w(u,y,t)[y+z^*_\delta(0)]\,dy\,dt\\ &\quad+\int_0^\infty\int_k^\infty e^{-\delta t}w(u,y,t)\int_0^\infty e^{-\delta s}f_L(s)\int_0^\infty f_\tau(v)[y+z^*_\delta(cs)]\mathbb{I}_{\{s<v\}}\,dv\,ds\,dy\,dt.\end{aligned}\tag{5.1}$$

Then, by recalling that in the Cramér-Lundberg model, the inter-arrival times are exponentially distributed with parameter $\lambda>0$, equation (5.1) can be re-written as

$$\begin{aligned}z^*_\delta(u) &= \int_0^k yg_\delta(u,y)\,dy+G_\delta(u,k)z^*_\delta(0)+\int_k^\infty g_\delta(u,y)\int_0^\infty e^{-s(\delta+\lambda)}f_L(s)[y+z^*_\delta(cs)]\,ds\,dy\\ &= \int_0^k yg_\delta(u,y)\,dy+G_\delta(u,k)z^*_\delta(0)+\int_k^\infty yg_\delta(u,y)\int_0^\infty e^{-s(\delta+\lambda)}f_L(s)\,ds\,dy\\ &\quad+\overline{G}_\delta(u,k)\int_0^\infty e^{-s(\delta+\lambda)}f_L(s)z^*_\delta(cs)\,ds.\end{aligned}\tag{5.2}$$

To complete the solution for $z^*_\delta(u)$, in equation (5.2), we need to determine an explicit expression for the boundary value $z^*_\delta(0)$. Setting $u=0$, in equation (5.2), and solving with respect to $z_\delta^*(0)$, yields

$$z_\delta^*(0) = \frac{1}{1-G_\delta(0,k)}\left(\int_0^k y g_\delta(0,y)\,dy + \int_k^\infty y g_\delta(0,y)\int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\,ds\,dy \right.$$
$$\left. + \overline{G}_\delta(0,k)\int_0^\infty e^{-s(\delta+\lambda)} f_L(s) z_\delta^*(cs)\,ds\right),$$

and thus, equation (5.1), can be written in the form

$$z_\delta^*(u) = h_\delta(u,k) + v_\delta(u,k)\int_0^\infty e^{-(\delta+\lambda)t} f_L(t)\, z_\delta^*(ct)\,dt, \tag{5.3}$$

where

$$\begin{aligned} h_\delta(u,k) \;=\; & \int_0^k y g_\delta(u,y)\,dy + \int_k^\infty y g_\delta(u,y)\int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\,ds\,dy \\ & + \frac{G_\delta(u,k)}{1-G_\delta(0,k)}\left(\int_0^k y g_\delta(0,y)\,dy + \int_k^\infty y g_\delta(0,y)\int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\,ds\,dy\right), \end{aligned} \tag{5.4}$$

and

$$v_\delta(u,k) = \frac{G_\delta(u,k)\overline{G}_\delta(0,k)}{1-G_\delta(0,k)} + \overline{G}_\delta(u,k) < 1, \tag{5.5}$$

such that, when $\delta = 0$, we have $v_0(u,k) = v(u,k)$ given by equation (3.7).

Note that, equation (5.3) is of a similar form to equation (3.13). Thus, by a change of variable in the integral term, we have that

$$z_\delta^*(u) = h_\delta(u,k) + \frac{1}{c} v_\delta(u,k)\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) z_\delta^*(t)\,dt, \tag{5.6}$$

which is an inhomogeneous Fredholm equation of the second kind and of similar form to equation (3.14). Hence, provided that both $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) z_\delta^*(t)\,dt < \infty$ and $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) h_\delta(t,k)\,dt < \infty$, the general solution of equation (3.14), given by equation (3.17), can be employed to solve equation (5.6).

Proposition 3. *Let $g(x)$ be a continuous function defined on the positive half line $[0,\infty)$, which is bounded by its finite maximum $M = \max_{x\in[0,\infty)}\{g(x)\} < \infty$. Then, $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) g(t)\,dt$ is finite and we have $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) g(t)\,dt < cM$.*

Proof. Firstly, by dividing $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) g(t)\,dt$ through by M, we obtain the normalised integral $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) \omega(t)\,dt$, where $\omega(t) = \frac{g(t)}{M} \leqslant 1$ for all $t \geqslant 0$. Now, applying similar arguments as the proof of Proposition 1, we have

$$\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) \omega(t)\,dt < c.$$

The result follows by multiplying the above inequality through by the maximum value $M < \infty$. $\square$

From Proposition 3 and the assumption that the expected deficit at ruin is finite, i.e. $\int_0^\infty y g_0(u,y)\,dy < \infty$, such that $h_\delta(u,k)$ and consequently $z_\delta^*(u)$ are finite, for all $u \geqslant 0$, we have the following Theorem.

Theorem 5. *Let $z_\delta^*(u)$ denote the expected discounted accumulated capital injections, in the continuous time delayed capital injection setting, up to the time of ultimate ruin with initial capital $U^*(0) = u$. Then, if $\int_0^\infty y g_0(u,y)\,dy < \infty$, the solution to the Fredholm integral equation* (5.6) *is given by*

$$z_\delta^*(u) = h_\delta(u,k) + \frac{\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{(\delta+\lambda)t}{c}} h_\delta(t,k)\,dt}{c - \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{t(\delta+\lambda)}{c}} v_\delta(t,k)\,dt} v_\delta(u,k), \tag{5.7}$$

where $h_\delta(u,k)$ and $v_\delta(u,k)$ are given by equation (5.4) *and* (5.5)*, respectively.*

5.2 Expected overall time in red up to the time of ultimate ruin

We will now turn our attention to another quantity, namely the expected discounted time in red, which reflects the expected discounted duration in deficit or below the SCR, up to the time of ruin. That is, let $\{V_u^*(t)\}_{t\geqslant 0}$ be a stochastic process denoting the the overall time in red up to time $t \geqslant 0$, from initial capital $u \geqslant 0$, defined by

$$V_u^*(t) = \int_0^\infty \mathbb{I}_{\{U^*(s)<0\}}\,ds, \quad \text{with} \quad U^*(0) = u.$$

We are interested in the expected discounted overall time in red up to the time of ultimate ruin, i.e. $\nu_\delta^*(u) = \mathbb{E}\left(e^{-\delta T^*} V_u^*(T^*)\right)$. Using a similar conditioning argument to the previous subsection, that is conditioning on the time and amount of the first fall into deficit, the subsequent delay and claim inter-arrival time, and recalling that the capital injection is received instantaneously if the deficit is less than $k \geqslant 0$, we have

$$\begin{aligned}\nu_\delta^*(u) &= \int_0^\infty \int_0^k e^{-\delta t} w(u,y,t) \nu_\delta^*(0)\,dydt + \int_0^\infty \int_k^\infty e^{-\delta t} w(u,y,t) \int_0^\infty f_L(s) \int_0^\infty f_\tau(w) \\ &\qquad \times \left[e^{-\delta w} w \mathbb{I}_{\{w<s\}} + e^{-\delta s}(s + \nu_\delta^*(cs))\mathbb{I}_{\{s<w\}}\right] dw\,ds\,dy\,dt \\ &= G_\delta(u,k)\nu_\delta^*(0) + \overline{G}_\delta(u,k) \left(\int_0^\infty s\left[\lambda \overline{F}_L(s) + f_L(s)\right] e^{-(\delta+\lambda)s}\,ds \right. \\ &\qquad \left. + \int_0^\infty e^{-\delta s} f_L(s) \overline{F}_\tau(s) \nu_\delta^*(cs)\,ds \right).\end{aligned} \tag{5.8}$$

To complete the solution for $\nu_\delta^*(u)$, in equation (5.8), we need to determine an explicit expression for the boundary value $\nu_\delta^*(0)$. Setting $u=0$, in the above equation, and solving with respect to $\nu_\delta^*(0)$, yields

$$\nu_\delta^*(0) = \frac{\overline{G}_\delta(0,k)}{1-G_\delta(0,k)}\left(\int_0^\infty s\left[\lambda\overline{F}_L(s)+f_L(s)\right]e^{-(\delta+\lambda)s}\,ds + \int_0^\infty e^{-\delta s} f_L(s)\overline{F}_\tau(s) v_\delta^*(cs)\,ds\right),$$

and thus, equation (5.8), can be written in the form

$$\nu_\delta^*(u) = b_\delta(u,k) + v_\delta(u,k)\int_0^\infty e^{-(\delta+\lambda)t} f_L(t)\nu_\delta^*(ct)\,dt, \tag{5.9}$$

where

$$b_\delta(u,k) = v_\delta(u,k)\int_0^\infty s\left[\lambda\overline{F}_L(s)+f_L(s)\right]e^{-(\delta+\lambda)s}\,ds, \tag{5.10}$$

and $v_\delta(u,k)$ is defined in equation (5.5).

Now, equation (5.9) is again of a similar form to equation (3.13) and thus the general solution of equation (3.13) can be employed to solve the Fredholm integral equation in equation (5.9), provided both $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right)\nu_\delta^*(t)\,dt<\infty$ and $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) b_\delta(t,k)\,dt<\infty$.

In order to show that these conditions are satisfied, let us consider the behaviour of the function $b_\delta(u,k)$, given by equation (5.10) and recall that the function $v_\delta(u,k)<1$, for all $u\geqslant 0$. Then, we have

$$\begin{aligned} b_\delta(u,k) = v_\delta(u,k)\int_0^\infty s\left[\lambda\overline{F}_L(s)+f_L(s)\right]e^{-(\delta+\lambda)s}\,ds &< \int_0^\infty s\left[\lambda\overline{F}_L(s)+f_L(s)\right]e^{-(\delta+\lambda)s}\,ds \\ \leqslant \lambda\int_0^\infty se^{-\lambda s}\,ds + \int_0^\infty sf_L(s)\,ds = 1+\mathbb{E}(L)<\infty, \end{aligned}$$

since it is assumed that the delay time distribution has finite mean $\mathbb{E}(L)<\infty$ [see Section 3.3]. Using this result, the fact that the function $\nu_\delta^*(u)$ is bounded and applying the result of Proposition 3 to show the two integrals $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right)\nu_\delta^*(t)\,dt$ and $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) b_\delta(t,k)\,dt$ are finite, we have the following Theorem.

Theorem 6. *Let $\nu_\delta^*(u)$ denote the expected discounted time in red, in the continuous time delayed capital injection setting, up to the time of ultimate ruin with initial capital $U^*(0)=u$. Then, the solution to the Fredholm integral equation (5.9) is given by*

$$\nu_\delta^*(u) = b_\delta(u,k) + \frac{\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) b_\delta(t,k)\,dt}{c-\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) v_\delta(t,k)\,dt} v_\delta(u,k), \tag{5.11}$$

where $b_\delta(u,k)$ is given by equation (5.10).

Remark 7. *We point out that the second moments (and thus the variance) can be calculated for the above two quantities using similar arguments, however, due to these calculations being somewhat cumbersome, we omit them from this paper.*

6 Capital injections with explicit delay time dependence

In the previous sections we have considered a dependency structure based on a deficit falling between certain threshold barriers. In this section, we generalise the dependence between the deficit and the delay of the capital injections such that, when the deficit is greater than the critical value $k \geqslant 0$ (there exists a delay), the random delay time depends explicitly on the size of the deficit ($y > 0$), in the following way:

Let the delay time be denoted by a continuous random variable, L, (the argument holds true for the discrete and deterministic settings as well) which depends on the size of the deficit via the its conditional distribution $F_{L|Y=y}(\cdot) =: F_{L|Y}(\cdot; y)$ and corresponding density $f_{L|Y}(\cdot; y)$, where $Y = |U(T)|$ is a random variable denoting the size of the deficit. Intuitively, if the insurance company experiences a deficit of $Y = y > k$, then the delay time, L, increases as Y increases (the more capital the firm requires through a capital injection, the more time that will be needed to gather and process the funds), hence it is assumed that the conditional distribution, $F_{L|Y}(\cdot; y)$, is a decreasing function of $y > 0$.

Then, conditioning on the size of the deficit, the subsequent delay time and claim inter-arrival time, we have

$$\begin{aligned}\phi^*(u) &= \phi(u) + G(u,k)\phi^*(0) + \int_k^\infty g(u,y) \int_0^\infty \int_0^\infty f_{L|Y}(t;y) f_\tau(s) \phi^*(ct) \mathbb{I}_{\{t<s\}}\, ds\, dt\, dy \\ &= \phi(u) + G(u,k)\phi^*(0) + \int_k^\infty g(u,y) \int_0^\infty e^{-\lambda t} f_{L|Y}(t;y) \phi^*(ct)\, dt\, dy. \end{aligned} \tag{6.1}$$

In order to determine the boundary value, $\phi^*(0)$, we set $u = 0$, in equation (6.1), and solve for $\psi^*(0)$, to obtain

$$\phi^*(0) = \frac{\phi(0) + \int_k^\infty g(0,y) \int_0^\infty e^{-\lambda t} f_{L|Y}(t;y)\phi^*(ct)\, dt\, dy}{1 - G(0,k)}.$$

Substituting this form of $\phi^*(0)$, into equation (6.1), and changing the order of integration in the resulting integral, yields

$$\phi^*(u) = w(u,k) + \int_0^\infty e^{-\lambda t} \left(\int_k^\infty z(u,k,y) f_{L|Y}(t;y)\, dy \right) \phi^*(ct)\, dt, \tag{6.2}$$

where $w(u,k)$ is given by equation (3.6) and

$$z(u,k,y) = \frac{G(u,k)g(0,y)}{1 - G(0,k)} + g(u,y). \tag{6.3}$$

We note that, since $\int_k^\infty z(u,k,y)\, dy = v(u,k)$, defined in equation (3.7), it is not difficult to show that the right hand side of equation (6.2) is less than equal to 1 and thus, the integral equation is well defined.

Now, using a change of variables, equation (6.2) can be transformed to

$$\phi^*(u) = w(u,k) + \frac{1}{c}\int_0^\infty e^{-\frac{\lambda t}{c}} \left(\int_k^\infty z(u,k,y) f_{L|Y}\left(\frac{t}{c};y\right)\,dy\right)\phi^*(t)\,dt, \tag{6.4}$$

which is an inhomogeneous Fredholm integral equation of the second kind with kernel

$$K(u,t) = e^{-\frac{\lambda t}{c}} \left(\int_k^\infty z(u,k,y) f_{L|Y}\left(\frac{t}{c};y\right)\,dy\right). \tag{6.5}$$

Remark 8. *The kernel $K(u,t)$, given above, is non-degenerate and an explicit solution is no longer obtainable, however, it is possible to derive a solution in terms of the Neumann series. For details of the following method of solution see Zemyan (2012).*

To derive the Neumann series solution, let us first rewrite equation (6.4) in the following form

$$\phi^*(u) = w(u,k) + \alpha\int_0^\infty K(u,t)\phi^*(t)\,dt, \tag{6.6}$$

where $\alpha = c^{-1} > 0$ and $K(u,t)$ is given in equation (6.5). Then, by the method of successive substitution (see Chapter 2 of Zemyan (2012)), i.e. substituting the form of $\phi^*(u)$, given in equation (6.6), back into the integral itself, we have

$$\begin{aligned}\phi^*(u) &= w(u,k) + \alpha\int_0^\infty K(u,t)\left[w(t,k) + \alpha\int_0^\infty K(t,s)\phi^*(s)\,ds\right]\,dt \\ &= w(u,k) + \alpha\int_0^\infty K(u,t)w(t,k)\,dt + \alpha^2\int_0^\infty\int_0^\infty K(u,t)K(t,s)\phi^*(s)\,ds\,dt,\end{aligned}$$

which, after changing the order of integration in the last term, yields

$$\phi^*(u) = w(u,k) + \alpha\int_0^\infty K(u,t)w(t,k)\,dt + \alpha^2\int_0^\infty K_2(u,t)\phi^*(t)\,dt,$$

where

$$K_2(u,t) = \int_0^\infty K(u,s)K(s,t)\,ds.$$

Repeating the above iterative process, n times, we get that

$$\phi^*(u) = w(u,k) + \sum_{m=1}^{n}\alpha^m\int_0^\infty K_m(u,t)w(t,k)\,dt + \alpha^{n+1}\int_0^\infty K_{n+1}(u,t)\phi^*(t)\,dt,$$

where $K_1(u,t) = K(u,t)$ and

$$K_m(u,t) = \int_0^\infty K_{m-1}(u,s)K(s,t)\,ds,$$

or equivalently

$$\phi^*(u) = w(u,k) + \alpha\sigma_n(u) + \rho_n(u), \tag{6.7}$$

where

$$\sigma_n(x) = \sum_{m=1}^{n} \alpha^{m-1} \left(\int_0^\infty K_m(u,t) w(t,k)\, dt \right)$$

and

$$\rho_n(u) = \alpha^{n+1} \int_0^\infty K_{n+1}(u,t)\phi^*(t)\, dt.$$

Following the methodology of Fredholm integral equations of the second kind with general kernels (sometimes called iterated kernels), equation (6.7) has a unique solution as long as the sequence $\{\sigma_n(u)\}_{n\in\mathbb{N}^+}$ of continuous functions converges uniformly to a continuous limit function on the interval $[0,\infty)$, and the sequence $\rho_n(u) \to 0$, as $n \to \infty$ (see Zemyan (2012) for more details).

Theorem 7. *Assume that the conditional density $f_{L|Y}(\cdot;y)$ is bounded for all $y \geqslant k$ and let $M = \max\{f_{L|Y}(x;y) : x \in [0,\infty), y \in [k,\infty)\}$ be its maximum value. Then, the ruin probability under an explicit delay dependence, namely $\psi^*(u)$, is given by*

$$\psi^*(u) = v(u,k) - \sum_{m=1}^{\infty} c^{-m} \left(\int_0^\infty K_m(u,t) w(t,k)\, dt \right), \tag{6.8}$$

provided

$$\lambda > M,$$

where $w(u,k)$ and $v(u,k)$ are given by equations (3.6) and (3.7), respectively, and $K_n(u,t)$ is the n-th iterated kernel of $K(u,t)$, given in equation (6.5).

Proof. Let $M = \max\{f_{L|Y}(x;y) : x \in [0,\infty), y \in [k,\infty)\}$ be the maximum value of all delay time density functions, for $y \geqslant k$. Then, it follows that

$$\begin{aligned} |K(u,t)| &= e^{-\frac{\lambda t}{c}} \int_k^\infty z(u,k,y) f_L\left(\frac{t}{c};y\right) dy \leqslant M e^{-\frac{\lambda t}{c}} \int_k^\infty z(u,k,y)\, dy, \qquad \forall t \geqslant 0, \\ &= M e^{-\frac{\lambda t}{c}} v(u,k) < M e^{-\frac{\lambda t}{c}}, \qquad \forall u \geqslant 0, \end{aligned}$$

since $v(u,k) < 1$. Now, using the bound for $K(u,t) = K_1(u,t)$, we can determine an upper bound for $|K_2(u,t)|$, since

$$|K_2(u,t)| = \int_0^\infty K(u,s)K(s,t)\, ds < M^2 e^{-\frac{\lambda t}{c}} \int_0^\infty e^{-\frac{\lambda s}{c}}\, ds = \frac{cM^2}{\lambda} e^{-\frac{\lambda t}{c}}.$$

By repeating this argument it is not hard to show that

$$|K_m(u,t)| < \left(\frac{cM}{\lambda}\right)^{m-1} Me^{-\frac{\lambda t}{c}},$$

for all $m \in \mathbb{N}$. Now, using the bound for $|K_m(u,t)|$, we can show that $\{\sigma_n(u)\}_{n\geqslant 1}$ uniformly converges and that $\rho_n \to 0$, as $n \to \infty$. For the former, first note that each summand of the summation in $\sigma_n(u)$, satisfies the inequality

$$\begin{aligned}\left|\alpha^{m-1}\left(\int_0^\infty K_m(u,t)w(t,k)\,dt\right)\right| &< \left(\frac{\alpha cM}{\lambda}\right)^{m-1} M\int_0^\infty e^{-\frac{\lambda t}{c}} w(t,k)\,dt\\ &\leqslant \left(\frac{\alpha cM}{\lambda}\right)^{m-1}\frac{cM}{\lambda} = c\left(\frac{M}{\lambda}\right)^m,\end{aligned}$$

since $\alpha = c^{-1}$. Then, provided $\lambda > M$, the sequence, $\{\sigma_n(u)\}_{n\in\mathbb{N}^+}$, of partial sums is a Cauchy sequence, i.e. for some arbitrary $\epsilon > 0$, we have that

$$|\sigma_n(x) - \sigma_p(x)| < c\sum_{m=p+1}^{n}\left(\frac{M}{\lambda}\right)^m < \frac{c(M/\lambda)^p}{1-(M/\lambda)} < \epsilon,$$

for large enough p. Thus, the sequence $\{\sigma_n(u)\}_{n\in\mathbb{N}^+}$ converges uniformly to the continuous limit function given by

$$\sum_{m=1}^{\infty}\alpha^{m-1}\left(\int_0^\infty K_m(u,t)w(t,k)\,dt\right).$$

Finally, we have that $|\rho_n(u)| < (M/\lambda)^{n+1} \to 0$ as $n \to \infty$, since $\lambda > M$, which after using the fact that $\psi^*(u) = 1 - \phi^*(u)$, in equation (6.7), completes the proof. □

Example 1 (Exponential delay time and exponential claim sizes)**.** *Assume that the conditional distribution of the delay time random variable, given a deficit size $|U(T)| = y$, follows an exponential distribution, with parameter y^{-1}, i.e. $f_{L|Y}(x;y) = y^{-1}e^{-\frac{x}{y}}, y \geqslant k$. Then, since a delay occurs only when the deficit is larger than $k \geqslant 0$, we have that*

$$\begin{aligned}M &= \max\{y^{-1}e^{-\frac{x}{y}} : x \in [0,\infty), y \in [k,\infty)\}\\ &= k^{-1}.\end{aligned}$$

Then, by Theorem 7, the ruin probability is given by

$$\psi^*(u) = v(u,k) - \sum_{m=1}^{\infty} c^{-m}\left(\int_0^\infty K_m(u,t)w(t,k)\,dt\right), \tag{6.9}$$

as long as $\lambda k > 1$.

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