

STATIONARY SOLUTIONS OF NEUTRAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH DELAYS IN THE HIGHEST-ORDER DERIVATIVES

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ABSTRACT. In this work, we shall consider the existence and uniqueness of stationary solutions to stochastic partial functional differential equations with additive noise in which a neutral type of delay is explicitly presented. We are especially concerned about those delays appearing in both spatial and temporal derivative terms in which the coefficient operator under spatial variables may take the same form as the infinitesimal generator of the equation. We establish the stationary property of the neutral system under investigation by focusing on distributed delays. In the end, an illustrative example is analyzed to explain the theory in this work.

1. Introduction. First of all, let us consider some simple stochastic systems to motivate our theory in this work. Let $w(t)$, $t \geq 0$, be a standard real Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the following stochastic partial differential equation

$$\begin{cases} dy(t, \xi) = \frac{\partial^2}{\partial \xi^2} y(t, \xi) dt + b(\xi) dw(t), & t \geq 0, \quad \xi \in (0, \pi), \\ y(t, 0) = y(t, \pi) = 0, & t \geq 0, \\ y(0, \cdot) = y_0(\cdot) \in L^2(0, \pi), \end{cases} \quad (1)$$

where $b(\cdot) \in L^2(0, \pi)$. It is well-known (see, e.g., Prévôt and Röckner [12]) that equation (1) has a unique stationary solution. That is, there exists a random initial $y_0 \in L^2(0, \pi)$ such that the corresponding (strong) solution $y(t, y_0)$, $t \geq 0$, is stationary, i.e., for any $t \geq 0$, $t_k \geq 0$ and Borel set $\Gamma_k \in \mathcal{B}(H)$, the Borel σ -field on H , $k = 1, \dots, n$,

$$\mathbb{P}\{y(t + t_k, y_0) \in \Gamma_k, k = 1, \dots, n\} = \mathbb{P}\{y(t_k, y_0) \in \Gamma_k, k = 1, \dots, n\}.$$

Moreover, this stationary solution is unique in the sense that any two stationary solutions of (1) with different initial data have the same finite dimensional distribution.

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Next, let $r > 0$ and consider a time delay version of (1) in the form

$$\begin{cases} dy(t, \xi) = \frac{\partial^2}{\partial \xi^2} y(t, \xi) dt + \int_{-r}^0 \beta(\theta) \frac{\partial^2}{\partial \xi^2} y(t + \theta, \xi) d\theta dt + b(\xi) dw(t), \\ t \geq 0, \quad \xi \in (0, \pi), \\ y(t, 0) = y(t, \pi) = 0, \quad t \geq 0, \\ y(0, \xi) = \phi_0(\xi), \quad y(\theta, \xi) = \phi_1(\theta, \xi), \quad \theta \in [-r, 0], \quad \xi \in (0, \pi), \end{cases} \quad (2)$$

where $\beta : [-r, 0] \rightarrow \mathbb{R}$ is some measurable function. The novelty in equation (2) is that a time delay appears in the highest-order, i.e., second order derivative term which usually leads to an unbounded operator from an advanced analysis viewpoint. On the other hand, due to the time delay in (2), it is essential to set up proper initial data, e.g., $\phi_0 \in L^2(0, \pi)$ and $\phi_1 \in L^2([-r, 0], H_0^1(0, \pi))$ where $H_0^1(0, \pi)$ is the classical Sobolev space, to secure a solution, and further a stationary solution, to equation (2). As a matter of fact, it has been shown in Liu [10] that if

$$\|\beta\|_{L^1([-r, 0], \mathbb{R})} < 1,$$

there would exist a unique stationary solution to (2).

In this work, we are interested in a neutral type of version of (2) in the form

$$\begin{cases} d\left(y(t, \xi) - \int_{-r}^0 \gamma(\theta) y(t + \theta, \xi) d\theta\right) = \frac{\partial^2}{\partial \xi^2} y(t, \xi) dt + \int_{-r}^0 \beta(\theta) \frac{\partial^2}{\partial \xi^2} y(t + \theta, \xi) d\theta dt \\ \quad + b(\xi) dw(t), \quad t \geq 0, \quad \xi \in (0, \pi), \\ y(t, 0) = y(t, \pi) = 0, \quad t \geq 0, \\ y(0, \xi) = \phi_0(\xi), \quad y(\theta, \xi) = \phi_1(\theta, \xi), \quad \theta \in [-r, 0], \quad \xi \in (0, \pi), \end{cases} \quad (3)$$

or equivalently, the form

$$\begin{cases} d\left(y(t, \xi) - \int_{-r}^0 \gamma(\theta) y(t + \theta, \xi) d\theta\right) = \frac{\partial^2}{\partial \xi^2} \left(y(t, \xi) - \int_{-r}^0 \gamma(\theta) y(t + \theta, \xi) d\theta\right) dt \\ \quad + \int_{-r}^0 (\beta(\theta) + \gamma(\theta)) \frac{\partial^2}{\partial \xi^2} y(t + \theta, \xi) d\theta dt + b(\xi) dw(t), \quad t \geq 0, \\ y(t, 0) = y(t, \pi) = 0, \quad t \geq 0, \\ y(0, \xi) = \phi_0(\xi), \quad y(\theta, \xi) = \phi_1(\theta, \xi), \quad \theta \in [-r, 0], \quad \xi \in (0, \pi), \end{cases} \quad (4)$$

where $\gamma : [-r, 0] \rightarrow \mathbb{R}$ is a measurable function. As a result of our theory, we may show later that if

$$\|\gamma\|_{L^1([-r, 0], \mathbb{R})} + \|\beta + \gamma\|_{L^1([-r, 0], \mathbb{R})} < 1,$$

there would exist a unique stationary solution to (3).

The organization of this work is as follows. In Section 2, we first develop a C_0 -semigroup theory so as to lift up the original time delay system into a non time delay one. To identify a stationary solution for our system, it is important to know when the associated “lift-up” solution semigroup is exponentially stable. To this end, we establish some stability results by means of a spectrum analysis method in Section 3. In contrast with point delay situation, it turns out in Section 4 that we can apply a norm continuity result of C_0 -semigroups in [11] to our case to locate a stationary solution for the system under consideration. Last, we shall apply the results established in this work to a concrete example to illustrate our theory.

2. Strongly continuous semigroup. For arbitrary Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, we always denote by $\mathcal{L}(X, Y)$ the space of all bounded, linear operators from X into Y . If $X = Y$, we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. Let V be a separable Hilbert space and $a : V \times V \rightarrow \mathbb{R}$ a bilinear form satisfying the so-called Gårding’s inequalities

$$|a(x, y)| \leq \beta \|x\|_V \|y\|_V, \quad a(x, x) \leq -\alpha \|x\|_V^2, \quad \forall x, y \in V, \tag{5}$$

for some constants $\beta > 0, \alpha > 0$. In association with the form $a(\cdot, \cdot)$, let A be a linear operator defined by

$$a(x, y) = \langle x, Ay \rangle_{V, V^*}, \quad x, y \in V,$$

where V^* is the dual space of V and $\langle \cdot, \cdot \rangle_{V, V^*}$ is the dual pairing between V and V^* . Then $A \in \mathcal{L}(V, V^*)$ and A generates an analytic semigroup $e^{tA}, t \geq 0$, on V^* . We also introduce the standard interpolation Hilbert space $H = (V, V^*)_{1/2, 2}$, which is described by

$$H = \left\{ x \in V^* : \int_0^\infty \|Ae^{tA}x\|_{V^*}^2 dt < \infty \right\}$$

with inner product

$$\langle x, y \rangle_H = \langle x, y \rangle_{V^*} + \int_0^\infty \langle Ae^{tA}x, Ae^{tA}y \rangle_{V^*} dt, \quad x, y \in V^*.$$

We identify the dual H^* of H with H , then it is easy to see that

$$V \hookrightarrow H = H^* \hookrightarrow V^* \tag{6}$$

where the imbedding \hookrightarrow is dense and continuous with $\|x\|_H^2 \leq \nu \|x\|_V^2, x \in V$, for some constant $\nu > 0$. Hence, $\langle x, Ay \rangle_H = \langle x, Ay \rangle_{V, V^*}$ for all $x \in V$ and $y \in V$ with $Ay \in H$. Moreover, for any $T \geq 0$ it is well known (see [8]) that

$$L^2([0, T], V) \cap W^{1,2}([0, T], V^*) \subset C([0, T], H)$$

where $W^{1,2}([0, T], V^*)$ is the Sobolev space consisting of all functions $y : [0, T] \rightarrow V^*$ such that y and its first order distributional derivative are in $L^2([0, T], V^*)$ and $C([0, T], H)$ is the space of all continuous functions from $[0, T]$ into H , respectively. It can be also shown (see, e.g., [13]) that the semigroup $e^{tA}, t \geq 0$, is bounded and analytic on both V^* and H such that $e^{tA} : V^* \rightarrow V$ for each $t > 0$ and for some constant $M > 0$,

$$\|e^{tA}\|_{\mathcal{L}(V^*)} \leq M, \quad \|e^{tA}\|_{\mathcal{L}(H)} \leq e^{-\alpha t} \quad \text{for all } t \geq 0.$$

Let $T \geq 0$ and $f \in L^2([0, T], V^*)$. Consider an abstract evolution equation in V^* as follows:

$$\begin{cases} dx(t) = Ax(t)dt + f(t)dt, & t \in [0, T], \\ x(0) = \phi_0. \end{cases} \tag{7}$$

The proofs of the following results are referred to Theorems 2.3 and 2.4 in [3].

Theorem 2.1. (i) *If function $f \in L^2([0, T], V^*)$, then*

$$(e^A * f)(t) := \int_0^t e^{(t-s)A} f(s) ds \in L^2([0, T], V) \cap W^{1,2}([0, T], V^*),$$

and

$$\|e^A * f\|_{L^2([0, T], V)} \vee \|e^A * f\|_{W^{1,2}([0, T], V^*)} \leq C_1 \|f\|_{L^2([0, T], V^*)}$$

where $C_1 = C_1(T) > 0$ and $a \vee b := \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

- (ii) If $\phi_0 \in H$, then function $t \rightarrow e^{tA}\phi_0$ belongs to $L^2([0, T], V) \cap W^{1,2}([0, T], V^*)$ and

$$\|e^{tA}\phi_0\|_{L^2([0,T],V)} \leq \|e^{tA}\phi_0\|_{W^{1,2}([0,T],V^*)} \leq C_2\|\phi_0\|_H$$

where $C_2 = \max\{M, \sqrt{T}, 1\}$.

- (iii) If $f \in L^2([0, T], V^*)$ and $\phi_0 \in H$, then equation (7) has a unique solution given by

$$x(t) = e^{tA}\phi_0 + \int_0^t e^{(t-s)A}f(s)ds.$$

In addition, there exists a constant $C_0 > 0$ such that $x \in C([0, T], H)$ and

$$C_0\|x\|_{C([0,T],H)} \leq \|x\|_{L^2([0,T],V)} \vee \|x\|_{W^{1,2}([0,T],V^*)} \leq C_1\|f\|_{L^2([0,T],V^*)} + C_2\|\phi_0\|_H.$$

- (iv) If $f \in W^{1,2}([0, T], V^*)$, $\phi_0 \in V$ and $A\phi_0 + f(0) \in H$, then the solution x of (7) satisfies

$$x \in W^{1,2}([0, T], V) \cap W^{2,2}([0, T], V^*) \subset C^1([0, T], H),$$

$$\|x'\|_{L^2([0,T],V)} \vee \|x'\|_{W^{1,2}([0,T],V^*)} \leq C_1\|f'\|_{L^2([0,T],V^*)} + C_2\|A\phi_0 + f(0)\|_H,$$

and

$$x'(t) = e^{tA}(A\phi_0 + f(0)) + \int_0^t e^{(t-s)A}f'(s)ds, \quad t \in [0, T].$$

- (v) If $f \in L^2([0, T], V)$, $\phi_0 \in V$ and $A\phi_0 \in H$, then the solution x of (7) satisfies

$$Ax \in L^2([0, T], V) \cap W^{1,2}([0, T], V^*) \subset C([0, T], H),$$

$$\|Ax\|_{L^2([0,T],V)} + \|Ax\|_{W^{1,2}([0,T],V^*)} \leq C_1\|Af\|_{L^2([0,T],V^*)} + C_2\|A\phi_0\|_H$$

and for any $t \geq 0$,

$$Ax(t) = e^{tA}A\phi_0 + \int_0^t e^{(t-s)A}Af(s)ds.$$

Let $r > 0$ and $T \geq 0$. For $x \in L^2([-r, T], V)$, we always write $x_t(\theta) := x(t + \theta)$ for any $t \geq 0$ and $\theta \in [-r, 0]$ in this work. Now suppose that $D_1 \in \mathcal{L}(V)$, $D_2 \in \mathcal{L}(L^2([-r, 0], V), V)$, $F_1 \in \mathcal{L}(V, V^*)$ and $F_2 \in \mathcal{L}(L^2([-r, 0], V), V^*)$. We introduce two linear mappings D and F on $C([-r, T], V)$, respectively, by

$$Dx_t = D_1x(t-r) + D_2x_t, \quad t \in [0, T], \quad \forall x(\cdot) \in C([-r, T], V),$$

and

$$Fx_t = F_1x(t-r) + F_2x_t, \quad t \in [0, T], \quad \forall x(\cdot) \in C([-r, T], V).$$

Lemma 2.1. *Both the mappings D and F have a bounded, linear extension to $L^2([-r, T], V)$ such that for any $x \in L^2([-r, T], V)$,*

$$\int_0^T \|Dx_t\|_V^2 dt \leq C_1 \int_{-r}^T \|x(t)\|_V^2 dt \tag{8}$$

and

$$\int_0^T \|Fx_t\|_{V^*}^2 dt \leq C_2 \int_{-r}^T \|x(t)\|_V^2 dt \tag{9}$$

with

$$C_1 = (\|D_1\|_{\mathcal{L}(V)} + \|D_2\|_{\mathcal{L}(L^2([-r,0],V),V)} \cdot r^{1/2})^2 > 0,$$

$$C_2 = (\|F_1\|_{\mathcal{L}(V,V^*)} + \|F_2\|_{\mathcal{L}(L^2([-r,0],V),V^*)} \cdot r^{1/2})^2 > 0.$$

Proof. We only prove (8) since the relation (9) can be obtained in an analogous manner. By using Hölder’s inequality and Fubini’s theorem, we can obtain that for any $x(\cdot) \in C([-r, T], V)$,

$$\begin{aligned} & \left(\int_0^T \|Dx_t\|_V^2 dt \right)^{1/2} \\ & \leq \left(\int_0^T \|D_1\|^2 \|x(t-r)\|_V^2 dt \right)^{1/2} + \|D_2\| \left(\int_0^T \int_{-r}^0 \|x(t+\theta)\|_V^2 d\theta dt \right)^{1/2} \\ & \leq \|D_1\| \left(\int_0^T \|x(t-r)\|_V^2 dt \right)^{1/2} + \|D_2\| \left(\int_{-r}^0 \int_{-r}^T \|x(t)\|_V^2 dt d\theta \right)^{1/2} \\ & \leq [\|D_1\| + \|D_2\|r^{1/2}] \left(\int_{-r}^T \|x(t)\|_V^2 dt \right)^{1/2}. \end{aligned}$$

Since $C([-r, T], V)$ is dense in $L^2([-r, T], V)$, D admits a bounded linear extension, still denote it by D , from $L^2([-r, T], V)$ to $L^2([0, T], V)$. The proof is thus complete. \square

Let $\mathcal{H} = H \times L^2([-r, 0], V)$ and consider the following deterministic functional differential equation of neutral type in V^* ,

$$\begin{cases} x(t) - Dx_t = e^{tA}\phi_0 + \int_0^t e^{(t-s)A}Fx_s ds, & t \geq 0, \\ x_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H}. \end{cases} \tag{10}$$

We say that x is a (strict) solution of (10) in $[0, T]$ if $x \in L^2([0, T], V) \cap W^{1,2}([0, T], V^*)$ and the equation (10) is satisfied almost everywhere in $[0, T]$, $T \geq 0$.

Theorem 2.2. *Given arbitrarily $\phi = (\phi_0, \phi_1) \in H \times L^2([-r, 0], V)$ and $T \geq 0$, there exists a function $x(t) \in V$, $t \in [-r, T]$, which is the unique solution of equation (10) with $x_0 = \phi_1$ such that*

$$x(\cdot) \in L^2([-r, T], V),$$

and

$$y(\cdot) := x(\cdot) - Dx \in L^2([0, T], V) \cap W^{1,2}([0, T], V^*) \subset C([0, T], H).$$

Moreover, we have the relations

$$\|x\|_{L^2([-r, T], V)} \leq M \left(\|\phi_0\|_H + \|\phi_1\|_{L^2([-r, 0], V)} \right) \tag{11}$$

and

$$\|y\|_{L^2([0, T], V)} + \|y\|_{W^{1,2}([0, T], V^*)} \leq M \left(\|\phi_0\|_H + \|\phi_1\|_{L^2([-r, 0], V)} \right) \tag{12}$$

for some positive number $M = M(T) > 0$.

Proof. We shall use a fixed point argument to the integral equation (10). Let $t_0 \in (0, r)$ and Σ be a closed subspace of $L^2([-r, t_0], V)$ such that $x_0 = \phi_1$ for $x \in \Sigma$. Define a mapping S on Σ as follows: for $x \in \Sigma$,

$$\begin{cases} (Sx)(t) = Dx_t + e^{tA}\phi_0 + \int_0^t e^{(t-s)A}Fx_s ds & \text{for } t \in [0, t_0], \\ (Sx)_0 = \phi_1, \quad (\phi_0, \phi_1) \in \mathcal{H}. \end{cases} \tag{13}$$

Note that we have

$$\begin{aligned} \|D_2x\|_{L^2([0,t_0],V)} &\leq \|D_2\| \left(\int_0^{t_0} \|x_t\|_{L^2([-r,0],V)}^2 dt \right)^{1/2} \\ &= \|D_2\| \left(\int_0^{t_0} \int_{-r}^0 \|x(t+\theta)\|_V^2 d\theta dt \right)^{1/2} \\ &\leq \|D_2\| \left(\int_0^{t_0} \int_{-r}^{t_0} \|x(s)\|_V^2 ds dt \right)^{1/2} = \sqrt{t_0} \|D_2\| \|x\|_{L^2([-r,t_0],V)}. \end{aligned} \tag{14}$$

Thus, by virtue of Theorem 2.1 (i) and (ii), it is immediate that $Sx(\cdot) \in L^2([0, t_0], V)$ and $Sx(\cdot) - Dx \in L^2([0, t_0], V) \cap W^{1,2}([0, t_0], V^*)$ for each $x \in \Sigma$.

To obtain a unique solution of (10), it suffices to show that S is a contraction from Σ into itself for sufficiently small $t_0 > 0$ and then implement a successive interval argument to extend the solution onto the whole interval $[0, T]$. Indeed, for any $x, \bar{x} \in \Sigma$ and $t \in [0, t_0]$, we have

$$\begin{aligned} (Sx - S\bar{x})(t) &= D_1(x(t-r) - \bar{x}(t-r)) + D_2(x_t - \bar{x}_t) \\ &\quad + \int_0^t e^{(t-s)A} [F_1(x(s-r) - \bar{x}(s-r)) + F_2(x_s - \bar{x}_s)] ds \\ &= D_2(x_t - \bar{x}_t) + \int_0^t e^{(t-s)A} F_2(x_s - \bar{x}_s) ds. \end{aligned} \tag{15}$$

which, in addition to (14) and (15), immediately yields that

$$\begin{aligned} \|Sx - S\bar{x}\|_{L^2([0,t_0],V)} &\leq \|D_2(x - \bar{x})\|_{L^2([0,t_0],V)} + \left\| \int_0^t e^{(\cdot-s)A} F_2(x_s - \bar{x}_s) ds \right\|_{L^2([0,t_0],V)} \\ &\leq \|D_2\| \sqrt{t_0} \|x(\cdot) - \bar{x}(\cdot)\|_{L^2([-r,t_0],V)} + M_0 t_0 \|F_2\| \|x(\cdot) - \bar{x}(\cdot)\|_{L^2([-r,t_0],V)} \\ &= \delta(t_0) \|x(\cdot) - \bar{x}(\cdot)\|_{L^2([-r,t_0],V)}, \end{aligned} \tag{16}$$

where $\delta(t_0) = \|D_2\| \sqrt{t_0} + M_0 \|F_2\| t_0 \rightarrow 0$ as $t_0 \rightarrow 0$. The map S is thus a contraction in Σ and the equation (10) has a unique solution x on $[-r, t_0]$.

To show the relation (11), we notice that for $t \in [0, t_0]$, $t_0 < r$,

$$x(t) = Dx_t + e^{tA} \phi_0 + \int_0^t e^{(t-s)A} Fx_s ds.$$

Then, from Theorem 2.1 (i) and (ii) and (14), we obtain

$$\begin{aligned} \|x\|_{L^2([-r,t_0],V)} &\leq \sqrt{t_0} \|D_2\| \|x\|_{L^2([-r,t_0],V)} + C_2(t_0) \|\phi_0\|_H \\ &\quad + C_1(t_0) \left(\int_0^{t_0} \|F_1 \phi_1(t-r)\|_{V^*}^2 dt \right)^{1/2} + C_1(t_0) \left(\int_0^{t_0} \|F_2 x_t\|_{V^*}^2 dt \right)^{1/2}, \end{aligned} \tag{17}$$

where $C_1(t_0), C_2(t_0) > 0$ are those numbers given in Theorem 2.1. On the other hand, it is immediate that

$$\left(\int_0^{t_0} \|F_1 \phi_1(t-r)\|_{V^*}^2 dt \right)^{1/2} \leq \|F_1\| \|\phi_1\|_{L^2([-r,0],V)}, \tag{18}$$

and similarly to (16), we have

$$\left(\int_0^{t_0} \|F_2 x_t\|_{V^*}^2 dt \right)^{1/2} \leq \|F_2\| t_0 \left(\int_0^{t_0} \|x(t)\|_V^2 dt \right)^{1/2}. \tag{19}$$

Hence, by letting t_0 be sufficiently small, we have from (17), (18) and (19) that

$$\|x\|_{L^2([-r, t_0], V)} \leq M(t_0) (\|\phi_0\|_H + \|\phi_1\|_{L^2([-r, 0], V)}), \tag{20}$$

where

$$M(t_0) := (1 - \sqrt{t_0} \|D_2\| - C_1(t_0) \|F_2\| t_0)^{-1} (C_1(t_0) \|F_1\| + C_2(t_0)) > 0.$$

In a similar manner, we can show the relation (12).

Last, by repeating the above argument on $[t_0, 2t_0]$, $[2t_0, 3t_0]$, \dots , we can finally show the existence and uniqueness of a solution x to (10) on $[0, T]$ satisfying the estimate (11) or (12) for each $T \geq 0$. The proof is thus complete. \square

The following results give conditions on the initial data in order to obtain a solution which is more regular with respect to time or space variables.

Theorem 2.3. *Suppose that $(\phi_0, \phi_1) \in V \times W^{1,2}([-r, 0], V)$ with*

$$\phi_0 = \phi_1(0) - D\phi_1 \in V \quad \text{and} \quad A\phi_0 + F\phi_1 \in H,$$

then the solution x of (10) satisfies

$$x(\cdot) \in W^{1,2}([-r, T], V), \tag{21}$$

and

$$x(\cdot) - Dx \in W^{1,2}([-r, T], V) \cap W^{2,2}([0, T], V^*) \subset C^1([0, T], H), \tag{22}$$

for each $T \geq 0$.

Proof. In correspondence with the time $t_0 \in [0, r]$ in the proof of Theorem 2.2, let us consider the following closed subspace Σ_0 of $W^{1,2}([-r, t_0], V)$ such that $x(0) = \phi_0 + D\phi_1$ and $x_0 = \phi_1$ for $x \in \Sigma_0$. Define the same mapping S as in (13) on Σ_0 by

$$Sx(t) = Dx_t + e^{tA}\phi_0 + \int_0^t e^{(t-s)A} Fx_s ds \quad \text{for any } t \in [0, t_0].$$

Once again, it is immediate from Theorem 2.1 (iv) and (v) that $Sx(\cdot) \in W^{1,2}([0, t_0], V)$ and $Sx(\cdot) - Dx \in W^{1,2}([0, t_0], V) \cap W^{2,2}([0, t_0], V^*)$ for each $x \in \Sigma_0$.

We shall show that S is a contraction from Σ_0 into itself for sufficiently small $t_0 > 0$. Indeed, first we note that $Fx_t \in W^{1,2}([0, T], V^*)$ for $x \in \Sigma_0$ and $A\phi_0 + F\phi_1 \in H$. Then for any $x, \bar{x} \in \Sigma_0$ and $t \in [0, t_0]$, we have

$$\begin{aligned} & \|Sx - S\bar{x}\|_{W^{1,2}([0, t_0], V)} \\ & \leq \|D_2(x - \bar{x})\|_{W^{1,2}([0, t_0], V)} + \left\| \int_0^t e^{(t-s)A} F_2(x_s - \bar{x}_s) ds \right\|_{W^{1,2}([0, t_0], V)} \\ & \leq \|D_2\| \sqrt{t_0} \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,2}([-r, t_0], V)} + M_0 t_0 \|F_2\| \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,2}([-r, t_0], V)} \\ & = \delta(t_0) \|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,2}([-r, t_0], V)}, \end{aligned}$$

where $\delta(t_0) = \|D_2\| \sqrt{t_0} + M_0 \|F_2\| t_0 \rightarrow 0$ as $t_0 \rightarrow 0$. Hence, map S is a contraction in Σ_0 and the equation (10) has a unique solution x such that

$$x(\cdot) \in W^{1,2}([-r, t_0], V),$$

and, similarly,

$$y(\cdot) = x(\cdot) - Dx \in W^{1,2}([-r, t_0], V) \cap W^{2,2}([0, t_0], V^*) \subset C^1([0, t_0], H).$$

Moreover, we can proceed as in the proof of Theorem 2.2 to get a solution x in $[-r, T]$ and the relation (21) or (22) for all $T \geq 0$. The proof is easily completed. \square

Let $x(t)$, $t \geq -r$ (and $y(t)$, $t \geq 0$) denote the unique solution of system (10) with $x_0 = \phi_1$, $\phi = (\phi_0, \phi_1) \in \mathcal{H}$. We define a family of operators $\mathcal{S}(t) : \mathcal{H} \rightarrow \mathcal{H}$, $t \geq 0$, by

$$\mathcal{S}(t)\phi = (x(t) - Dx_t, x_t) \quad \text{for any } \phi \in \mathcal{H}. \quad (23)$$

Theorem 2.4. *The family $t \rightarrow \mathcal{S}(t)$ is a strongly continuous semigroup on \mathcal{H} , i.e.,*

- (i) $\mathcal{S}(t) \in \mathcal{L}(\mathcal{H})$ for each $t \geq 0$;
- (ii) $\mathcal{S}(0) = I$, $\mathcal{S}(s+t) = \mathcal{S}(s)\mathcal{S}(t)$ for any $s, t \geq 0$;
- (iii) $\lim_{t \rightarrow 0^+} \mathcal{S}(t)\phi = \phi$ for each $\phi \in \mathcal{H}$.

Proof. Let $y(t, \phi) = x(t, \phi) - Dx_t(\phi)$ for each $t \geq 0$, $\phi \in \mathcal{H}$ and the solution $x(t, \phi)$ of (10). Then we have by virtue of Theorem 2.2 that for any $t \in [0, T]$,

$$\begin{aligned} \|\mathcal{S}(t)\phi\|_{\mathcal{H}}^2 &= \|x(t, \phi) - Dx_t(\phi)\|_H^2 + \int_{-r}^0 \|x(t+\theta, \phi)\|_V^2 d\theta \\ &\leq C_1 \left(\|x(\cdot) - Dx\|_{L^2([0, T], V)}^2 + \|x(\cdot) - Dx\|_{W^{1,2}([0, T], V^*)}^2 + \int_{-r}^T \|x(t, \phi)\|_V^2 dt \right) \\ &\leq C_2(T) (\|\phi_0\|_H^2 + \|\phi_1\|_{L^2([-r, 0], V)}^2), \end{aligned}$$

where $C_1, C_2(T) > 0$, which shows (i). To show (ii), it is easy to see from (10) that for any $t \geq s \geq 0$,

$$\begin{aligned} y(t-s, \mathcal{S}(s)\phi) &= e^{(t-s)A}(\mathcal{S}\phi)_0 + \int_0^{t-s} e^{(t-s-u)A} Fx_u(\mathcal{S}(s)\phi) du \\ &= e^{tA}\phi_0 + \int_0^s e^{(t-u)A} Fx_u(\phi) du + \int_s^t e^{(t-u)A} Fx_{u-s}(\mathcal{S}(s)\phi) du. \end{aligned}$$

On the other hand, for $t \geq s$,

$$y(t, \phi) = e^{tA}\phi_0 + \int_0^s e^{(t-u)A} Fx_u(\phi) du + \int_s^t e^{(t-u)A} Fx_u(\phi) du, \quad (24)$$

which immediately implies that for any $t \geq s \geq 0$,

$$\begin{aligned} x(t-s, \mathcal{S}(s)\phi) - Dx_{t-s}(\mathcal{S}(s)\phi) &= \int_s^t e^{(t-u)A} Fx_{u-s}(\mathcal{S}(s)\phi) du \\ &= x(t, \phi) - Dx_t(\phi) - \int_s^t e^{(t-u)A} Fx_u(\phi) du. \end{aligned} \quad (25)$$

Thus, by the uniqueness of solutions to (25), it follows that

$$x(t-s, \mathcal{S}(s)\phi) = x(t, \phi), \quad t \geq s, \quad (26)$$

and further

$$y(t-s, \mathcal{S}(s)\phi) = x(t-s, \mathcal{S}(s)\phi) - Dx_{t-s}(\mathcal{S}(s)\phi) = x(t, \phi) - Dx_t(\phi) = y(t, \phi)$$

for any $t \geq s \geq 0$. Hence, $[\mathcal{S}(t-s)\mathcal{S}(s)\phi]_0 = [\mathcal{S}(t)\phi]_0$ and similarly we can show that $[\mathcal{S}(t-s)\mathcal{S}(s)\phi]_1 = [\mathcal{S}(t)\phi]_1$ for any $t \geq s \geq 0$.

Finally, to show (iii) we notice that $y(t) = x(t) - Dx_t$, $t \in [0, T]$, and

$$\|\mathcal{S}(t)\phi - \phi\|_{\mathcal{H}}^2 = \|y(t) - y(0)\|_H^2 + \int_{-r}^0 \|x(t+\theta) - \phi_1(\theta)\|_V^2 d\theta \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

since $y \in C([0, T], H)$ and $x \in L^2([-r, T], V)$ for each $T \geq 0$. The proof is complete. \square

The following theorem whose finite dimensional version was established in Ito and Tarn [6] gives a complete description of the generator \mathcal{A} of semigroup $e^{t\mathcal{A}}$, $t \geq 0$.

Theorem 2.5. *The generator \mathcal{A} of the strongly continuous semigroup $e^{t\mathcal{A}}$, $t \geq 0$, is given by*

$$\begin{aligned} & \mathcal{D}(\mathcal{A}) \\ &= \left\{ (\phi_0, \phi_1) \in \mathcal{H} : \phi_1 \in W^{1,2}([-r, 0], V), \phi_0 = \phi_1(0) - D\phi_1 \in V, A\phi_0 + F\phi_1 \in H \right\} \\ & \text{and for each } \phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A}), \\ & \mathcal{A}\phi = (A\phi_0 + F\phi_1, \phi_1') \in \mathcal{H}. \end{aligned}$$

This theorem will result from the following several propositions according to Theorem 1.9 in Davies [2].

Proposition 2.1. *For each $t \geq 0$, we have $\mathcal{S}(t)\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$.*

Proof. If $(\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$ and $(x(\cdot) - Dx_\cdot, x_\cdot)$ is the unique solution of equation (10) with initial data (ϕ_0, ϕ_1) , then for each $T \geq 0$, we get from Theorem 2.3 that $x \in W^{1,2}([-r, T], V)$ and

$$x(\cdot) - Dx_\cdot \in W^{1,2}([0, T], V) \cap W^{2,2}([0, T], V^*) \subset C^1([0, T], H).$$

Therefore, for each $t \geq 0$, we have $x_t \in W^{1,2}([-r, 0], V)$ and

$$A(x(t) - Dx_t) + Fx_t = (x(t) - Dx_t)' \in H \quad \text{and} \quad x_t' \in L^2([-r, 0], V),$$

a fact which immediately implies that $(x(t) - Dx_t, x_t') \in \mathcal{D}(\mathcal{A})$ for each $t \geq 0$. The proof is thus complete. \square

Proposition 2.2. *The domain $\mathcal{D}(\mathcal{A})$ is dense in \mathcal{H} .*

Proof. Since $\mathcal{S}(t)$ is strongly continuous, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_0^\varepsilon \mathcal{S}(t)\phi dt = \phi \quad \text{for each } \phi \in \mathcal{H}.$$

Hence, it suffices to prove that for each $\varepsilon > 0$,

$$\int_0^\varepsilon \mathcal{S}(t)\phi dt = \left(\int_0^\varepsilon (x(t) - Dx_t) dt, \int_0^\varepsilon x_t(\cdot) dt \right) \in \mathcal{D}(\mathcal{A}),$$

where x is the unique solution of equation (10).

Since $x \in L^2([-r, T], V)$ for each $T > 0$, we have that for any $\theta \in [-r, 0]$,

$$\left(\int_0^\varepsilon x_s ds \right)(\theta) = \int_0^\varepsilon x(s + \theta) ds = \int_0^{\varepsilon + \theta} x(u) du - \int_0^\theta x(u) du,$$

which immediately implies

$$\left(\int_0^\varepsilon x_t dt \right)(\cdot) \in W^{1,2}([-r, 0], V) \quad \text{and} \quad \left(\int_0^\varepsilon x_t dt \right)(0) = \int_0^\varepsilon x(t) dt.$$

Hence, we have

$$\int_0^\varepsilon x(t) dt - \int_0^\varepsilon Dx_t dt = \left(\int_0^\varepsilon x_t dt \right)(0) - D \int_0^\varepsilon x_t dt.$$

Further, since $x \in L^2([0, \varepsilon], V) \cap W^{1,2}([0, \varepsilon], V^*) \subset C([0, \varepsilon], H)$, we obtain

$$\begin{aligned} A \int_0^\varepsilon (x(t) - Dx_t)dt + F_1 \int_0^\varepsilon x(t-r)dt + F_2 \int_0^\varepsilon x_t dt \\ = \int_0^\varepsilon \frac{d(x(t) - Dx_t)}{dt} dt = x(\varepsilon) - Dx_\varepsilon - \phi_0 \in H. \end{aligned}$$

The proof is thus complete. □

Proposition 2.3. *If $\phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$, then we have*

$$\lim_{t \downarrow 0} \frac{\mathcal{S}(t)\phi - \phi}{t} = \mathcal{A}\phi = (A\phi_0 + F\phi_1, \phi'_1).$$

Proof. Let $\phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$. Then it is known by Theorem 2.3 that the corresponding solution x of (10) with initial ϕ is in $C^1([0, T], H)$ for each $T > 0$. Hence, by the equality (23) we have

$$\lim_{t \downarrow 0} \left\| \frac{x(t) - Dx_t - \phi_0}{t} - x'(0) - Dx'_0 \right\|_H = \lim_{t \downarrow 0} \left\| \frac{x(t) - Dx_t - \phi_0}{t} - A\phi_0 - F\phi_1 \right\|_H = 0. \tag{27}$$

On the other hand, as $x \in W^{1,2}([-r, T], V)$ for each $T > 0$ according to Theorem 2.3, we can write for $\theta \in [-r, 0]$ and $t > 0$ that

$$\frac{x_t(\theta) - \phi_1(\theta)}{t} = \frac{1}{t} \int_\theta^{t+\theta} x'(s)ds. \tag{28}$$

Since $x' \in L^2([-r, T], V)$ for each $T > 0$, we have

$$\lim_{t \downarrow 0} \int_{-r}^0 \left\| \frac{1}{t} \int_\theta^{\theta+t} x'(s)ds - x'(\theta) \right\|_V^2 = 0.$$

Therefore, it is easy to get from (28) that

$$\lim_{t \downarrow 0} \left\| \frac{x_t - \phi_1}{t} - \phi'_1 \right\|_{L^2([-r, 0], V)} = 0. \tag{29}$$

The conclusion follows from (27) and (29) and the proof is thus complete. □

Proposition 2.4. *The map $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is a closed operator.*

Proof. Suppose that there exist a sequence $\{(\phi_{0,n}, \phi_{1,n})\}_{n \geq 1} \in \mathcal{D}(\mathcal{A})$ such that

$$(\phi_{0,n}, \phi_{1,n}) \rightarrow (\phi_0, \phi_1) \text{ as } n \rightarrow \infty \text{ in } \mathcal{H} \tag{30}$$

and

$$\mathcal{A}(\phi_{0,n}, \phi_{1,n}) \rightarrow (\psi_0, \psi_1) \text{ as } n \rightarrow \infty \text{ in } \mathcal{H}. \tag{31}$$

We need to show $(\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}(\phi_0, \phi_1) = (\psi_0, \psi_1)$.

Indeed, since $\phi_{1,n} \in W^{1,2}([-r, 0], V)$ for each $n \geq 1$, it follows from (30) and (31) that

$$\lim_{n \rightarrow \infty} \|\phi_{1,n} - \phi_1\|_{L^2([-r, 0], V)} = 0, \quad \lim_{n \rightarrow \infty} \|\phi'_{1,n} - \psi_1\|_{L^2([-r, 0], V)} = 0.$$

Hence, $\phi_1 \in W^{1,2}([-r, 0], V)$ and $\phi'_1 = \psi_1$. This implies

$$\phi_{1,n} \rightarrow \phi_1 \text{ in } W^{1,2}([-r, 0], V) \text{ as } n \rightarrow \infty, \tag{32}$$

and by Sobolev's imbedding theorem,

$$\phi_{1,n}(0) \rightarrow \phi_1(0) \quad \text{and} \quad \phi_{1,n}(-r) \rightarrow \phi_1(-r) \text{ in } V \text{ as } n \rightarrow \infty. \tag{33}$$

From (30), we thus have the following equalities

$$\begin{aligned} \phi_1(0) &= \lim_{n \rightarrow \infty} \phi_{1,n}(0) = \lim_{n \rightarrow \infty} (\phi_{0,n} - D_1\phi_{1,n}(-r) - D_2\phi_{1,n}) \\ &= \phi_0 - D_1\phi_1(-r) - D_2\phi_1 \\ &= \phi_0 - D\phi_1 \quad \text{in } H. \end{aligned} \tag{34}$$

Further, by virtue of (32) and (33) we have

$$A\phi_{0,n} + F_1\phi_{1,n}(-r) + F_2\phi_{1,n} \rightarrow A\phi_0 + F_1\phi_1(-r) + F_2\phi_1 \quad \text{in } V^* \quad \text{as } n \rightarrow \infty. \tag{35}$$

On the other hand, we have from (31) that

$$A\phi_{0,n} + F_1\phi_{1,n}(-r) + F_2\phi_{1,n} \rightarrow \psi_0 \quad \text{in } H \quad \text{as } n \rightarrow \infty. \tag{36}$$

Hence, from (35) and (36) we obtain

$$\psi_0 = A\phi_0 + F_1\phi_1(-r) + F_2\phi_1 \in H$$

as desired. Hence, the proof is complete now. □

3. Semigroup and resolvent. For each $\lambda \in \mathbb{C}$, we define a linear operator $D(e^\lambda) : V \rightarrow V$ by

$$D(e^\lambda)x = D(e^\lambda x) \quad \text{for any } x \in V.$$

Then it is easy to see that $D(e^\lambda) \in \mathcal{L}(V)$. Indeed, for any $x \in V$,

$$\begin{aligned} \|D(e^\lambda)x\|_V &= \|D(e^\lambda x)\|_V \\ &\leq \|D_1(e^{-\lambda r}x)\|_V + \|D_2\left(\int_{-r}^0 \|e^{\lambda\theta}x\|_V^2 d\theta\right)^{1/2} \\ &\leq \|D_1\|e^{|\lambda|r}\|x\|_V + \|D_2\|\sqrt{r}e^{|\lambda|r}\|x\|_V. \end{aligned}$$

In a similar way, one can show that $F(e^\lambda) \in \mathcal{L}(V, V^*)$. For each $\lambda \in \mathbb{C}$, we define a linear operator $\Delta(\lambda, A, D, F)$ (or $\Delta(\lambda)$) : $V \rightarrow V^*$ by

$$\Delta(\lambda, A, D, F) = (\lambda I - A)(I - D(e^\lambda)) - F(e^\lambda) \in \mathcal{L}(V, V^*). \tag{37}$$

The resolvent set $\rho(A, D, F)$ is defined as the family of all values λ in \mathbb{C} for which the operator $\Delta(\lambda, A, D, F)$ has a bounded inverse $\Delta(\lambda, A, D, F)^{-1}$ on V^* . The operator $\Delta(\lambda, A, D, F)^{-1}$ is called the *resolvent* of (A, D, F) .

The following proposition can be used to establish useful relations between the resolvent $\Delta(\lambda, A, D, F)^{-1}$ and resolvent $(\lambda I - \mathcal{A})^{-1}$ of \mathcal{A} .

Proposition 3.1. *Let $\lambda \in \mathbb{C}$ and $\psi = (\psi_0, \psi_1) \in \mathcal{H}$. If $\phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$ satisfies*

$$\lambda\phi - \mathcal{A}\phi = \psi, \tag{38}$$

then

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_{\theta}^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad -r \leq \theta \leq 0, \tag{39}$$

and

$$\Delta(\lambda)\phi_1(0) = (\lambda I - A)D\left(\int_{-r}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + F\left(\int_{-r}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + \psi_0. \tag{40}$$

Conversely, if $\phi_1(0) \in V$ satisfies the equation (40) and let $\phi_0 = \phi_1(0) - D\phi_1$ where

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_{\theta}^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad -r \leq \theta \leq 0, \tag{41}$$

then we have that $\phi_1 \in W^{1,2}([-r, 0], V)$, $\phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$ and ϕ satisfies (38).

Proof. The equation (38) can be equivalently written as

$$\lambda\phi_1(0) - \lambda D\phi_1 - A\phi_1(0) + AD\phi_1 - F\phi_1 = \psi_0, \quad (42)$$

and

$$\lambda\phi_1(\theta) - d\phi_1(\theta)/d\theta = \psi_1(\theta) \quad \text{for } \theta \in [-r, 0]. \quad (43)$$

It is easy to see that (43) is equivalent to (39). Hence, if (38) holds, we get $\phi_1(0) \in V$ and by virtue of (42) and (39), we have

$$\begin{aligned} & \Delta(\lambda, A, D, F)\phi_1(0) \\ &= \lambda\phi_1(0) - A\phi_1(0) + AD(e^{\lambda \cdot})\phi_1(0) - \lambda D(e^{\lambda \cdot})\phi_1(0) - F(e^{\lambda \cdot})\phi_1(0) \\ &= \lambda D\phi_1 - AD\phi_1 + F\phi_1 + \psi_0 + AD(e^{\lambda \cdot})\phi_1(0) - \lambda D(e^{\lambda \cdot})\phi_1(0) - F(e^{\lambda \cdot})\phi_1(0) \\ &= \lambda D(e^{\lambda \cdot})\phi_1(0) + \lambda D\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) - AD(e^{\lambda \cdot})\phi_1(0) \\ &\quad - AD\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + F(e^{\lambda \cdot})\phi_1(0) + F\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + \psi_0 \\ &\quad + AD(e^{\lambda \cdot})\phi_1(0) - \lambda D(e^{\lambda \cdot})\phi_1(0) - F(e^{\lambda \cdot})\phi_1(0) \\ &= (\lambda I - A)D\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + F\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + \psi_0 \end{aligned}$$

which is the equality (40).

Conversely, if $\phi_1(0) \in V$, then by a simple calculation it is easy to see that ϕ_1 , defined by (41), belongs to $W^{1,2}([-r, 0], V)$. In addition, let $\phi_0 = \phi_1(0) - D\phi_1 \in V$ and assume that (40) holds true. Then from (40) and (41), we get

$$\begin{aligned} \lambda\phi_0 - A\phi_0 &= \lambda\phi_1(0) - A\phi_1(0) - \lambda D\phi_1 + AD\phi_1 \\ &= (\lambda I - A)D\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + F\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + \psi_0 \\ &\quad + (\lambda I - A)D(e^{\lambda \cdot})\phi_1(0) + F(e^{\lambda \cdot})\phi_1(0) - \lambda D\phi_1 + AD\phi_1 \\ &= (\lambda I - A)D\phi_1 + F\phi_1 + \psi_0 - \lambda D\phi_1 + AD\phi_1 \\ &= F\phi_1 + \psi_0. \end{aligned} \quad (44)$$

Therefore, $A\phi_0 + F\phi_1 = \lambda\phi_0 - \psi_0 \in H$, which is the first coordinate relation of (38). The second coordinate equality of (38) is obvious. The proof is thus complete. \square

Proposition 3.2. *For each $\lambda \in \mathbb{C}$, the mapping $\Delta(\lambda)$ is injective if and only if $\lambda I - \mathcal{A}$ is injective.*

Proof. Let $\psi_0 = 0$, $\psi_1(\cdot) \equiv 0$ in Proposition 3.1. If $\Delta(\lambda)x = 0$ with $x \neq 0$, then we may take $\phi = (\phi_0, \phi_1)$ where

$$\phi_0 = x - D\phi_1, \quad \phi_1(\theta) = e^{\lambda\theta}x, \quad \theta \in [-r, 0].$$

Hence, $\phi = (\phi_0, \phi_1) \neq 0$ and

$$(\lambda I - \mathcal{A})\phi = 0.$$

Conversely, suppose that there exists a non-zero $\phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$ satisfying $(\lambda I - \mathcal{A})\phi = 0$. Then it can't happen that $\phi_1(\theta) = 0$ for all $\theta \in [-r, 0]$ since $\phi_0 = \phi_1(0) - D\phi_1 = 0$ otherwise. Hence, there exists a value $\theta \in [-r, 0]$ such that

$$\phi_1(0) = e^{-\lambda\theta}\phi_1(\theta) \neq 0.$$

Let $x = \phi_1(0) \neq 0$, then by virtue of (40) we have

$$\Delta(\lambda)x = 0.$$

The proof is thus complete. □

Proposition 3.3. *Suppose that $\Delta(\lambda)V = V^*$ for some $\lambda \in \mathbb{C}$, then*

$$(\lambda I - \mathcal{A})\mathcal{D}(\mathcal{A}) = \mathcal{H}.$$

Proof. For $\lambda \in \mathbb{C}$ and $\psi = (\psi_0, \psi_1) \in \mathcal{H}$, since $\Delta(\lambda)V = V^*$, there exists an element $\phi_1(0) \in V$ such that

$$\Delta(\lambda)\phi_1(0) = (\lambda I - A)D\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + F\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau\right) + \psi_0 \in V^*.$$

Let $\phi_0 = \phi_1(0) - D\phi_1 \in V$ where ϕ_1 is given by

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_{\theta}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau, \quad \theta \in [-r, 0].$$

Then by Proposition 3.1, we have that $\phi_1 \in W^{1,2}([-r, 0], V)$, $\phi = (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A})$ and ϕ satisfies $\lambda\phi - \mathcal{A}\phi = \psi \in \mathcal{H}$ as desired. The proof is complete now. □

Proposition 3.4. *Let $\lambda \in \mathbb{C}$. If there exists $C_1 > 0$ such that*

$$\|x\|_V \leq C_1\|\Delta(\lambda)x\|_{V^*} \quad \text{for each } x \in V, \tag{45}$$

then there exists a constant $C_2 > 0$ such that

$$\|\phi\|_{\mathcal{H}} \leq C_2\|(\lambda I - \mathcal{A})\phi\|_{\mathcal{H}} \quad \text{for each } \phi \in \mathcal{D}(\mathcal{A}).$$

Proof. First note that for any $\phi_1(0) \in V$ and $\psi_1 \in L^2([-r, 0], V)$, the function

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_{\theta}^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad \theta \in [-r, 0], \tag{46}$$

satisfies the relations

$$\begin{aligned} \|\phi_1\|_{L^2([-r, 0], V)} &\leq \|e^{\lambda\cdot}\phi_1(0)\|_{L^2([-r, 0], V)} + \left\| \int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\psi_1(\tau)d\tau \right\|_{L^2([-r, 0], V)} \\ &\leq c_1(\lambda)\|\phi_1(0)\|_V + c_2(\lambda)\|\psi_1\|_{L^2([-r, 0], V)}, \end{aligned} \tag{47}$$

where $c_1(\lambda), c_2(\lambda) > 0$.

For $\lambda \in \mathbb{C}$, we define a linear operator $\Sigma_{\lambda} : L^2([-r, 0], V) \rightarrow V^*$ by

$$\Sigma_{\lambda}\varphi = (\lambda I - A)D\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\varphi(\tau)d\tau\right) + F\left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)}\varphi(\tau)d\tau\right), \forall \varphi \in L^2([-r, 0], V). \tag{48}$$

Then it is easy to see that $\Sigma_\lambda \in \mathcal{L}(L^2([-r, 0], V), V^*)$. Indeed, for each $\varphi \in L^2([-r, 0], V)$ we have

$$\begin{aligned}
& \|\Sigma_\lambda \varphi\|_{V^*} \\
& \leq \left\| (\lambda I - A)D \left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)} \varphi(\tau) d\tau \right) \right\|_{V^*} + \left\| F \left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)} \varphi(\tau) d\tau \right) \right\|_{V^*} \\
& \leq \|\lambda I - A\|_{\mathcal{L}(V, V^*)} \|D\|_{\mathcal{L}(L^2([-r, 0], V), V)} \left\| \int_{\cdot}^0 e^{\lambda(\cdot-\tau)} \varphi(\tau) d\tau \right\|_{L^2([-r, 0], V)} \\
& \quad + \|F\|_{\mathcal{L}(L^2([-r, 0], V), V^*)} \left\| \int_{\cdot}^0 e^{\lambda(\cdot-\tau)} \varphi(\tau) d\tau \right\|_{L^2([-r, 0], V)} \\
& \leq c_2(\lambda) \left[\|\lambda I - A\|_{\mathcal{L}(V, V^*)} \|D\|_{\mathcal{L}(L^2([-r, 0], V), V)} + \|F\|_{\mathcal{L}(L^2([-r, 0], V), V^*)} \right] \|\varphi\|_{L^2([-r, 0], V)} \\
& =: c_3(\lambda) \|\varphi\|_{L^2([-r, 0], V)},
\end{aligned} \tag{49}$$

where $c_3(\lambda) > 0$.

For $\phi = (\phi_0, \phi_1) = (\phi_1(0) - D\phi_1, \phi_1) \in \mathcal{D}(\mathcal{A})$, we set $\psi = \lambda\phi - \mathcal{A}\phi$ as in (38). Then by virtue of (40), (41) and (49), the condition (45) implies that

$$\begin{aligned}
\|\phi_1(0)\|_V & \leq C_1 \|\Delta(\lambda)\phi_1(0)\|_{V^*} \\
& = C_1 \|\Sigma_\lambda \psi_1 + \psi_0\|_{V^*} \\
& \leq C_1 \cdot c_3(\lambda) \|\psi_1\|_{L^2([-r, 0], V)} + C_1 \|\psi_0\|_{V^*},
\end{aligned} \tag{50}$$

and, in addition to (46) and (50), that

$$\begin{aligned}
& \|D\phi_1\|_V \\
& \leq \|De^{\lambda\cdot}\phi_1(0)\|_V + \left\| D \left(\int_{\cdot}^0 e^{\lambda(\cdot-\tau)} \psi_1(\tau) d\tau \right) \right\|_V \\
& \leq \left(\|D_1\|e^{|\lambda|r} + \|D_2\|\sqrt{r}e^{|\lambda|r} \right) \|\phi_1(0)\|_V + c_2(\lambda) \|D\|_{\mathcal{L}(L^2([-r, 0], V), V)} \cdot \|\psi_1\|_{L^2([-r, 0], V)} \\
& \leq \left[C_1 c_3(\lambda) \left(\|D_1\|e^{|\lambda|r} + \|D_2\|\sqrt{r}e^{|\lambda|r} \right) + c_2(\lambda) \|D\|_{L^2([-r, 0], V), V} \right] \|\psi_1\|_{L^2([-r, 0], V)} \\
& \quad + C_1 \left(\|D_1\|e^{|\lambda|r} + \|D_2\|\sqrt{r}e^{|\lambda|r} \right) \|\psi_0\|_{V^*} \\
& =: c_4(\lambda) \|\psi_1\|_{L^2([-r, 0], V)} + c_5(\lambda) \|\psi_0\|_{V^*}.
\end{aligned} \tag{51}$$

Combining (50) and (51), we thus have

$$\begin{aligned}
\|\phi_0\|_V & \leq \|\phi_1(0)\|_V + \|D\phi_1\|_V \\
& \leq (C_1 \cdot c_3(\lambda) + c_4(\lambda)) \|\psi_1\|_{L^2([-r, 0], V)} + (C_1 + c_5(\lambda)) \|\psi_0\|_{V^*}.
\end{aligned} \tag{52}$$

Now from (47), (50), (52) and the fact that $\|\cdot\|_H \leq \nu \|\cdot\|_V$, $\nu > 0$, it follows for $\phi \in \mathcal{D}(\mathcal{A})$ that

$$\begin{aligned}
\|\phi\|_{\mathcal{H}} & \leq \sqrt{2} (\|\phi_0\|_H + \|\phi_1\|_{L^2([-r, 0], V)}) \\
& \leq \sqrt{2}\beta (C_1 \cdot c_3(\lambda) + c_4(\lambda)) \|\psi_1\|_{L^2([-r, 0], V)} + \sqrt{2}\nu (C_1 + c_5(\lambda)) \|\psi_0\|_{V^*} \\
& \quad + \sqrt{2}c_1(\lambda) \|\phi_1(0)\|_V + \sqrt{2}c_2(\lambda) \|\psi_1\|_{L^2([-r, 0], V)} \\
& \leq \left(\sqrt{2}\nu (C_1 \cdot c_3(\lambda) + c_4(\lambda)) + \sqrt{2}c_2(\lambda) + \sqrt{2}c_1(\lambda)C_1c_3(\lambda) \right) \|\psi_1\|_{L^2([-r, 0], V)} \\
& \quad + \left(\sqrt{2}c_1(\lambda)C_1^2c_3(\lambda) + \sqrt{2}\nu (C_1 + c_5(\lambda)) \right) \|\psi_0\|_{V^*} \\
& =: c_6(\lambda) \|\psi_1\|_{L^2([-r, 0], V)} + c_7(\lambda) \|\psi_0\|_{V^*}.
\end{aligned} \tag{53}$$

Since $\|\cdot\|_{V^*} \leq \nu \|\cdot\|_H$ for some $\nu > 0$, it further follows from (53) that

$$\begin{aligned} \|\phi\|_{\mathcal{H}}^2 &\leq 2c_6^2(\lambda)\|\psi_1\|_{L^2([-r,0],V)}^2 + 2c_7^2(\lambda)\nu\|\psi_0\|_H^2 \\ &= 2(c_6^2(\lambda) + c_7^2(\lambda))\|\psi\|_{\mathcal{H}}^2 = 2(c_6^2(\lambda) + c_7^2(\lambda))\|\lambda\phi - \mathcal{A}\phi\|_{\mathcal{H}}^2, \end{aligned}$$

with $c_6(\lambda) > 0$, $c_7(\lambda) > 0$. The proof is thus complete. □

Proposition 3.5. *Let $\lambda \in \mathbb{C}$. If $(\lambda - \mathcal{A})\mathcal{D}(\mathcal{A})$ is dense in \mathcal{H} , then $\Delta(\lambda)V$ is dense in V^* .*

Proof. For any $\psi_0 \in H$, let $\psi = (\psi_0, 0) \in \mathcal{H}$. Then by assumption for each $\varepsilon > 0$, there exist $\psi_\varepsilon = (\psi_{0,\varepsilon}, \psi_{1,\varepsilon}) \in \mathcal{H}$ and $\phi_\varepsilon = (\phi_{0,\varepsilon}, \phi_{1,\varepsilon}) \in \mathcal{D}(\mathcal{A})$ such that

$$\lambda\phi_\varepsilon - \mathcal{A}\phi_\varepsilon = \psi_\varepsilon \quad \text{and} \quad \|\psi_\varepsilon - \psi\|_{\mathcal{H}} < \varepsilon. \tag{54}$$

From (40), we have

$$\Delta(\lambda)\phi_{1,\varepsilon}(0) = \Sigma_\lambda\psi_{1,\varepsilon} + \psi_{0,\varepsilon},$$

where the operator Σ_λ is given in (48). Therefore,

$$\|\psi_0 - \Sigma_\lambda\psi_{1,\varepsilon} - \psi_{0,\varepsilon}\|_{V^*} \leq \|\psi_0 - \psi_{0,\varepsilon}\|_{V^*} + \|\Sigma_\lambda\psi_{1,\varepsilon}\|_{V^*},$$

and from (54) and the relation $\|\cdot\|_{V^*} \leq \nu \|\cdot\|_H$ for some $\nu > 0$, we have

$$\|\psi_0 - \psi_{0,\varepsilon}\|_{V^*} \leq \nu\|\psi_0 - \psi_{0,\varepsilon}\|_H < \varepsilon\nu,$$

and again from (54), it follows that

$$\|\psi_{1,\varepsilon}\|_{L^2([-r,0],V)} < \varepsilon.$$

Hence, we have the relations

$$\|\Sigma_\lambda\psi_{1,\varepsilon}\| \leq \|\Sigma_\lambda\| \cdot \|\psi_{1,\varepsilon}\|_{L^2([-r,0],V)} \leq \|\Sigma_\lambda\|\varepsilon.$$

In other words, we just show that for any $\psi_0 \in H$ and $\varepsilon > 0$, there exists $\Sigma_\lambda\psi_{1,\varepsilon} + \psi_{0,\varepsilon} \in \Delta(\lambda)V$ such that

$$\|\psi_0 - (\Sigma_\lambda\psi_{1,\varepsilon} + \psi_{0,\varepsilon})\|_{V^*} \leq \varepsilon(\nu + \|\Sigma_\lambda\|).$$

Since H is dense in V^* , the desired result is thus proved. □

4. Spectrum and stationary solution. First, let us consider the following deterministic functional differential equation of neutral type in V^* ,

$$\left\{ \begin{aligned} &d\left(y(t) - \alpha_1 y(t-r) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta\right) \\ &= A\left(y(t) - \alpha_1 y(t-r) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta\right)dt \\ &+ \alpha_2 Ay(t-r)dt + \int_{-r}^0 \beta(\theta)Ay(t+\theta)d\theta dt, \quad t \geq 0, \\ &y(0) = \phi_0, \quad y_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H}, \end{aligned} \right. \tag{55}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta, \gamma \in L^2([-r, 0]; \mathbb{R})$. By virtue of (23) and Theorem 2.5, the equation (55) can be equivalently lifted up into a deterministic equation without time delay

$$\left\{ \begin{aligned} &dY(t) = \mathcal{A}Y(t)dt, \quad t \geq 0, \\ &Y(0) = \phi \in \mathcal{H}, \end{aligned} \right. \tag{56}$$

where \mathcal{A} is the generator given in Theorem 2.5 and

$$Y(t) = \left(y(t) - \alpha_1 y(t-r) - \int_{-r}^0 \gamma(\theta) y(t+\theta) d\theta, y_t \right) \quad \text{for all } t \geq 0.$$

On the other hand, the characteristic operator $\Delta(\lambda)$ defined in (37) is given in this case by

$$\begin{aligned} \Delta(\lambda)x &= \left(1 - \alpha_1 e^{-\lambda r} - \int_{-r}^0 \gamma(\theta) e^{\lambda\theta} d\theta \right) (\lambda I - A)x - \alpha_2 e^{-\lambda r} Ax - \int_{-r}^0 \beta(\theta) e^{\lambda\theta} d\theta Ax \\ &=: m(\lambda)x - n(\lambda)Ax \quad \text{for each } \lambda \in \mathbb{C}, x \in V, \end{aligned}$$

where

$$m(\lambda) = \lambda \left(1 - \alpha_1 e^{-\lambda r} - \int_{-r}^0 \gamma(\theta) e^{\lambda\theta} d\theta \right), \quad \lambda \in \mathbb{C}, \tag{57}$$

and

$$n(\lambda) = 1 - \alpha_1 e^{-\lambda r} - \int_{-r}^0 \gamma(\theta) e^{\lambda\theta} d\theta + \alpha_2 e^{-\lambda r} + \int_{-r}^0 \beta(\theta) e^{\lambda\theta} d\theta, \quad \lambda \in \mathbb{C}. \tag{58}$$

Definition 4.1. The *point spectrum* $\sigma_p(\Delta)$ is defined to be the set

$$\sigma_p(\Delta) = \{ \lambda \in \mathbb{C} : \Delta(\lambda) \text{ is not injective} \},$$

the *continuous spectrum* $\sigma_c(\Delta)$ is defined by

$$\sigma_c(\Delta) = \{ \lambda \in \mathbb{C} : \Delta(\lambda) \text{ is injective } \mathcal{R}(\Delta(\lambda)) \neq V^* \text{ and } \overline{\mathcal{R}(\Delta(\lambda))} = V^* \},$$

where $\mathcal{R}(\Delta(\lambda))$ is the range of $\Delta(\lambda)$, and the *residual spectrum* $\sigma_r(\Delta)$ is defined by

$$\sigma_r(\Delta) = \{ \lambda \in \mathbb{C} : \Delta(\lambda) \text{ is injective and } \overline{\mathcal{R}(\Delta(\lambda))} \neq V^* \}.$$

Note that $\lambda \in \sigma_p(\Delta)$ if and only if there exists a nonzero $x \in V$ such that $\Delta(\lambda)x = 0$. The value $\lambda \in \sigma_p(\Delta)$ is often called a *characteristic value* of Δ .

Let $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ and $\sigma_p(\mathcal{A})$, $\sigma_c(\mathcal{A})$ and $\sigma_r(\mathcal{A})$ denote the point, continuous and residual spectrum sets of A and \mathcal{A} , respectively. In connection with (57) and (58), we further define

$$\begin{cases} \Gamma_c = \{ \lambda \in \mathbb{C} : n(\lambda) \neq 0, m(\lambda)n(\lambda)^{-1} \in \sigma_c(A) \}, \\ \Gamma_r = \{ \lambda \in \mathbb{C} : n(\lambda) \neq 0, m(\lambda)n(\lambda)^{-1} \in \sigma_r(A) \}, \\ \Gamma_p = \{ \lambda \in \mathbb{C} : n(\lambda) \neq 0, m(\lambda)n(\lambda)^{-1} \in \sigma_p(A) \}, \\ \Gamma_0 = \{ \lambda \in \mathbb{C} : \lambda \neq 0, n(\lambda) = 0 \}, \\ \Gamma_1 = \{ \lambda \in \mathbb{C} : n(\lambda) \neq 0, m(\lambda)n(\lambda)^{-1} \in \sigma(A) \}. \end{cases} \tag{59}$$

By using Propositions 3.1–3.5, one can obtain the following result whose proof is similar to that one of Theorem 3.9 in [4].

Proposition 4.1. *For the characteristic operator $\Delta(\lambda)$ and the associated generator \mathcal{A} of the equation (55), we have*

- (i) $\Gamma_0 \subset \sigma_c(\mathcal{A}) \subset \sigma_c(\Delta) = \Gamma_c \cup \Gamma_0$;
- (ii) $\sigma_r(\mathcal{A}) = \sigma_r(\Delta) = \Gamma_r$;
- (iii)

$$\sigma_p(\mathcal{A}) = \sigma_p(\Delta) = \begin{cases} \Gamma_p & \text{if } 1 - \alpha_1 + \alpha_2 + \int_{-r}^0 (\beta(\theta) - \gamma(\theta)) d\theta \neq 0, \\ \Gamma_p \cup \{0\} & \text{if } 1 - \alpha_1 + \alpha_2 + \int_{-r}^0 (\beta(\theta) - \gamma(\theta)) d\theta = 0. \end{cases}$$

If $\alpha_1 = 0, \gamma(\cdot) \equiv 0$ and $\beta(\cdot) \equiv 0$, then the equation (55) reduces to a simple form

$$\begin{cases} dy(t) = Ay(t)dt + \alpha_2 Ay(t-r)dt, & t \geq 0, \\ y(0) = \phi_0, \quad y_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H}. \end{cases} \tag{60}$$

Let us suppose at present that A is some linear operator, e.g., Laplace operator, in conjunction with the form $a(\cdot, \cdot)$ in (5) to generate a compact semigroup. It was shown, however, by Di Blasio, Kunisch and Sinestrari [4] that the associated C_0 -semigroup $e^{tA}, t \geq 0$, is generally not compact or even not eventually norm continuous, as shown by Jeong [7]. A direct consequence of this fact is that we cannot use the well-known spectral mapping theorem to establish stability, based on the spectrum knowledge of \mathcal{A} for system (60).

Bearing the above statement in mind, let us consider the following version of equation (55) with distributed delay by taking $\alpha_1 = \alpha_2 = 0$,

$$\begin{cases} d\left(y(t) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta\right) = A\left(y(t) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta\right)dt \\ \quad + \int_{-r}^0 \beta(\theta)Ay(t+\theta)d\theta, & t \geq 0, \\ y(0) = \phi_0, \quad y_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H}. \end{cases} \tag{61}$$

It was shown by Liu [11] that when the weight functions $\gamma(\cdot), \beta(\cdot)$ satisfy

$$\gamma(\cdot) \in L^2([-r, 0], \mathbb{C}), \quad \beta(\cdot) \in L^2([-r, 0], \mathbb{C}), \tag{62}$$

the associated solution semigroup $e^{tA}, t \geq 0$, in (55) is eventually norm continuous for $t > r$, i.e., $e^{tA} : [0, \infty) \rightarrow \mathcal{L}(\mathcal{H})$ is continuous on (r, ∞) , which implies further that the spectral mapping theorem is fulfilled,

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{A})\} = \inf\{\mu \in \mathbb{R} : \|e^{tA}\| \leq Me^{\mu t} \text{ for some } M > 0\}.$$

Now let us consider the following stochastic functional differential equation of neutral type with additive noise,

$$\begin{cases} d\left(y(t) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta\right) = A\left(y(t) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta\right)dt \\ \quad + \int_{-r}^0 \beta(\theta)Ay(t+\theta)d\theta + b dw(t), & t \geq 0, \\ y(0) = \phi_0, \quad y_0 = \phi_1, \quad \phi = (\phi_0, \phi_1) \in \mathcal{H}, \end{cases} \tag{63}$$

where $b \in H$ and $w(\cdot)$ is a standard real-valued Brownian motion. We can re-write (63) as a stochastic differential equation without time delay in \mathcal{H} ,

$$\begin{cases} dY(t) = AY(t)dt + \mathcal{B}dw(t), & t \geq 0, \\ Y(0) = \phi \in \mathcal{H}, \end{cases} \tag{64}$$

where \mathcal{A} is the generator given in Theorem 2.5, $\mathcal{B} = (b, 0) \in \mathcal{H}$ and

$$Y(t) = \left(y(t) - \int_{-r}^0 \gamma(\theta)y(t+\theta)d\theta, y_t\right) \quad \text{for all } t \geq 0.$$

For equation (64), if we can find conditions by showing

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{A})\} < 0, \tag{65}$$

to secure an exponentially stable semigroup $e^{tA}, t \geq 0$, then we will obtain a unique stationary solution to the equation (63) (cf, e.g., Prévôt and Röckner [12]).

Proposition 4.2. *Suppose that the spectrum set $\sigma(A) \subset (-\infty, -c_0]$ for some $c_0 > 0$ and the functions γ, β in (63) satisfy*

$$\|\gamma\|_{L^1([-r,0],\mathbb{R})} + \|\beta\|_{L^1([-r,0],\mathbb{R})} < 1. \quad (66)$$

Then there exists a unique stationary solution for the equation (64).

Proof. Note that from Proposition 4.1 we have $\sigma(\mathcal{A}) \subset \Gamma_0 \cup \Gamma_1$. We shall show that under the assumptions in this proposition, there is a constant $\mu > 0$ such that $\operatorname{Re} \lambda \leq -\mu$ for all $\lambda \in \Gamma_0 \cup \Gamma_1$ and hence for all $\lambda \in \sigma(\mathcal{A})$.

First, for elements in Γ_0 , if there exist a sequence $\{\lambda_n\} \subset \mathbb{C}$ such that $\operatorname{Re} \lambda_n \geq 0$ or $\operatorname{Re} \lambda_n \rightarrow 0$ as $n \rightarrow \infty$, then by using (57), (58) and Dominated Convergence Theorem, we have

$$1 \leq \limsup_{n \rightarrow \infty} \left| \int_{-r}^0 (\gamma(\theta) + \beta(\theta)) e^{\lambda_n \theta} d\theta \right| \leq \int_{-r}^0 (|\gamma(\theta)| + |\beta(\theta)|) d\theta < 1,$$

which is clearly a contradiction. Thus the desired result is obtained.

Now we consider the elements of $\sigma(\mathcal{A})$ in Γ_1 . If there exists a λ of $\sigma(\mathcal{A})$ in Γ_1 such that $\operatorname{Re} \lambda \geq 0$ (the case that $\operatorname{Re} \lambda \rightarrow 0$ can be similarly proved) with

$$\frac{m(\lambda)}{n(\lambda)} =: -\delta_\lambda \leq -c_0 < 0,$$

then we get by taking the real part of the equation into account that $\operatorname{Re} m(\lambda) \geq 0$. On the other hand, by assumption, we have

$$\begin{aligned} |1 - n(\lambda)| &= \left| \int_{-r}^0 \gamma(\theta) e^{\lambda \theta} d\theta - \int_{-r}^0 \beta(\theta) e^{\lambda \theta} d\theta \right| \\ &\leq \int_{-r}^0 (|\gamma(\theta)| + |\beta(\theta)|) d\theta < 1. \end{aligned}$$

Hence, by putting $m(\lambda) = a(\lambda) + ib(\lambda)$, we have the following relations

$$\begin{aligned} 1 \leq 1 + \frac{a(\lambda)}{\delta_\lambda} &= 1 - \frac{a(\lambda)}{m(\lambda)} \cdot n(\lambda) \\ &= \left| \frac{a(\lambda)(1 - n(\lambda)) + ib(\lambda)}{m(\lambda)} \right| \\ &= \frac{\sqrt{|1 - n(\lambda)|^2 a^2(\lambda) + b^2(\lambda)}}{\sqrt{a^2(\lambda) + b^2(\lambda)}} < \frac{\sqrt{a^2(\lambda) + b^2(\lambda)}}{\sqrt{a^2(\lambda) + b^2(\lambda)}} = 1, \end{aligned}$$

which, once again, yields a contradiction. Combining the above results, we thus obtain that

$$\operatorname{Re} \lambda \leq -\mu \quad \text{for some } \mu > 0 \quad \text{and all } \lambda \in \sigma(\mathcal{A}).$$

Therefore, the solution semigroup $e^{t\mathcal{A}}$, $t \geq 0$, is exponentially stable. This fact further implies the existence of a unique stationary solution to (61). The proof is complete. \square

Example 4.1. Consider the following initial-boundary value problem of Dirichlet type of the stochastic partial functional differential equation of neutral type,

$$\begin{cases} d\left(y(t, x) - \int_{-r}^0 \kappa e^{\mu\theta} y(t + \theta, x) d\theta\right) = \frac{\partial^2}{\partial x^2} \left(y(t, x) - \int_{-r}^0 \kappa e^{\mu\theta} y(t + \theta, x) d\theta\right) dt \\ \quad + \int_{-r}^0 \alpha \frac{\partial^2 y(t + \theta, x)}{\partial x^2} d\theta dt + b(x) dw(t), \quad t \geq 0, \quad x \in \mathcal{O}, \\ y(0, \cdot) = \phi_0(\cdot) \in L^2(\mathcal{O}; \mathbb{R}), \\ y(t, \cdot) = \phi_1(t, \cdot) \in H_0^1(\mathcal{O}; \mathbb{R}), \quad \text{a.e. } t \in [-r, 0). \end{cases} \tag{67}$$

Here \mathcal{O} is a bounded open subset of \mathbb{R}^n with regular boundary $\partial\mathcal{O}$, $\alpha, \kappa, \mu \in \mathbb{R}$, $r > 0$ and $b(\cdot) \in L^2(\mathcal{O}; \mathbb{R})$.

We can re-write (67) as a stochastic neutral initial boundary problem (61) in the Hilbert space $H = L^2(\mathcal{O}; \mathbb{R})$ by setting

$$\begin{cases} A = \frac{\partial^2}{\partial x^2}, \\ V = H_0^1(\mathcal{O}; \mathbb{R}), \\ \gamma(\theta) = \kappa e^{\mu\theta}, \quad \beta(\theta) \equiv \alpha, \quad \theta \in [-r, 0]. \end{cases}$$

Then for any random initial $(\phi_0, \phi_1) \in \mathcal{H}$, there exists a unique solution to (67) defined in $[0, \infty)$. Furthermore, by applying the results derived in this section to (67), one can obtain a unique stationary solution. In fact, note that $A = \partial^2/\partial x^2$ is a self-adjoint and negative operator in H and its spectrum satisfies $\sigma(A) = \sigma_p(A) \subset (-\infty, -c_0]$ for some $c_0 > 0$. Then by Proposition 4.2 and a direct computation, we obtain that if

$$|\alpha| < \frac{1}{r} \quad \text{and} \quad |\kappa| \leq \begin{cases} e^{r\mu}(1 - |\alpha|r)/r & \text{if } \mu \leq 0, \\ (1 - |\alpha|r)/r & \text{if } \mu > 0, \end{cases}$$

the associated solution semigroup of (67) is exponentially stable. Moreover, the lift-up system (64) of equation (67) in this case has a unique stationary solution.

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