Classification of Quasi-Realistic Heterotic String Vacua

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

by

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Abstract

The free fermionic formulation of heterotic strings has been able to provide some of the most phenomenologically viable string models to date. Within this formulation, classifications of string models with an $SO(10)$ GUT embedding and its subgroups have been shown to admit three chiral generations of matter, necessary Higgs representations and various other aspects of phenomenology associated with the Standard Model.

The classification method is extended to models where the $SO(10)$ symmetry is broken directly at the string scale to the Left-Right Symmetric subgroup. The method involves defining a set of basis vectors where fixed boundary conditions are assigned to the free fermions, before exploring the string vacua generated by varying the Generalised GSO (GGSO) projection coefficients. The method admits the derivation of algebraic expressions for each of the GGSO projections on a sector in order to generate the complete massless spectrum of the models. The derived algebraic expressions can be written in a computer code so as to facilitate a computerised analysis of the entire spectrum for each choice of GGSO projection coefficients. The classification procedure has been previously applied to models with the following subgroups of the observable $SO(10)$ gauge group: $SO(6) \times SO(4)$ (Pati-Salam), $SU(5) \times U(1)$ (Flipped SU(5)), $SU(4) \times SU(2) \times U(1)$ (SU(421)) and $SU(3) \times SU(2) \times U(1)^2$ (Standard-like Models).

A statistical sampling of models with the Left-Right Symmetric observable gauge group is performed and the results are presented. In contrast to the previous classification of Pati-Salam models, no three generation exophobic vacua were found. However, contrary to the SU(421) case which was found to be overconstrained and no complete generations of matter exist, complete generations of matter were found to exist for the Left-Right Symmetric case. The results of the classification performed demonstrate the existence of Left-Right Symmetric models with three chiral generations of matter, the necessary Higgs representations for spontaneous symmetry breaking and a leading top quark Yukawa coupling.

Ar Gyfer Fy Nheulu

For My Family

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Contents

1 Introduction

Unification of the fundamental forces of Nature has long been a goal of theoretical physics. One of the most remarkable achievements of theoretical physics in the 20^{th} century is the formulation of the Standard Model of particle physics. The Standard Model unifies three of the four known fundamental forces of Nature into a single framework of Quantum Field Theory (QFT). Specifically, quantum field theory is the synthesis of quantum mechanics and special relativity which is used as the framework of the Standard Model to give a description of electromagnetism and the strong and weak nuclear forces.

However, despite the overwhelming success of the Standard Model, it has always been known that it is not the complete description of the known Universe. The largest shortcoming of the Standard Model is its incompatibility with the fourth fundamental force, gravity, as described by the theory of general relativity.

This incompatibility arises as the Standard Model lacks a quantum description of gravity, leading to the breakdown of the Standard Model when quantum gravity effects become non-negligible. It is therefore reasonable to assume that the Standard Model should be embedded in a quantum theory of gravity. One approach of describing this embedding is string theory. String theory is a promising proposition as the proposed quantum particle of gravity (the graviton) must appear in the spectrum. This is in fact one of the consistency conditions of the theory. Furthermore, the consistency conditions of string theory allow for the existence of the matter and gauge structures which appear in the Standard Model.

However, string theory is not without issues. It is currently experimentally unproven due to the mass scale of strings, referred to as the string scale, being relatively high when compared with quantum field theories such as the Standard Model. In fact, in the work presented in this thesis, the string scale is assumed to be comparable to the Planck scale. While this may be a reasonable feature of a quantum theory of gravity, it means direct detection of strings is unachievable for any current or planned experiments. Indeed, without vast leaps in technology, the possibility of direct detection is out of the question. String theory also predicts more dimensions than the currently observed four dimensions of spacetime. Despite these issues, among others, string phenomenology as a field of theoretical physics has emerged from studying some of the implications of string theory. String phenomenology is the topic of this thesis.

This thesis focuses on the free fermionic formulation of heterotic string theory and the building of potentially relevant phenomenological models. The advantage of the free fermionic formulation is that the theory can be developed directly in four spacetime dimensions, in line with the current experimental observations.

The basics of the Standard Model are presented before discussing some of the problems which the Standard Model exhibits. The Left-Right Symmetric extension to the Standard Model is then briefly outlined before giving a brief discussion of how it solves some of the Standard Model's problems. Left-Right Symmetric heterotic string models will form the basis for the research presented in this thesis.

1.1 The Standard Model

The Standard Model is a description of all the known elementary particles and their interactions via the electromagnetic, strong and weak nuclear forces. It is a relativistic quantum gauge theory formulated in four spacetime dimensions.

The known fundamental particles are modelled as zero-dimensional objects, referred to as point particles. They are grouped into three generations of matter with distinctions between leptons and quarks. The matter content of the Standard Model can be found in table 1. There also exists the antimatter counterparts of all the particles displayed in the table. The representations of the matter particles under the Standard Model gauge group can be found in table 2.

The forces in the Standard Model are mediated by gauge bosons found in the adjoint representation of the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$

<table>
<tr><td></td><td colspan="3">Generation</td><td></td></tr>
<tr><td></td><td>1st</td><td>2nd</td><td>3rd</td><td>Q_{em}</td></tr>
<tr><td rowspan="2">Quarks</td><td>Up (u)</td><td>Charm (c)</td><td>Top (t)</td><td>$+\frac{2}{3}$</td></tr>
<tr><td>Down (d)</td><td>Strange (s)</td><td>Bottom (b)</td><td>$-\frac{1}{3}$</td></tr>
<tr><td rowspan="2">Leptons</td><td>Electron (e)</td><td>Muon (μ)</td><td>Tau (τ)</td><td>-1</td></tr>
<tr><td>Electron Neutrino (ν_e)</td><td>Muon Neutrino (ν_μ)</td><td>Tau Neutrino (ν_τ)</td><td>0</td></tr>
</table>

Table 1: *The matter content of the Standard Model.*

[1]. The $SU(3)_C$ symmetry of the gauge group provides the description for the strong nuclear force, responsible for the colour charge. The electromagnetic and the weak nuclear forces are unified into a single framework called the electroweak force at the energy level of the Standard Model (known as the electroweak scale). The gauge group of the electroweak symmetry is $SU(2)_L \times U(1)_Y$. However, this unification is not observed in the normal conditions of the current Universe and the electroweak symmetry in the Standard Model must be spontaneously broken. The method of spontaneous symmetry breaking is referred to as the Higgs mechanism and postulates a spin-zero scalar boson, known commonly as the Higgs boson. The Higgs boson was discovered using the LHC at CERN in 2012 [2], which can be considered as further proof of the accuracy of the Standard Model. Explicitly, the Higgs mechanism breaks the $SU(2)_L \times U(1)_Y$ electroweak gauge group to the observed electromagnetic $U(1)_{em}$ gauge group, which provides a description of the electromagnetic force. During the breaking of the gauge group the bosons $Z^0, W^{\pm}$ obtain a mass whereas the photon is left massless. Further information on the Higgs mechanism can be found in reference [3]. The gauge bosons of the Standard Model can be found in table 3.

Although the Standard Model provides an extremely accurate description of particle physics, there are shortcomings within the theory. Some of these include a lack of reasonable candidates for dark matter, hierarchy issues relating to the mass of the matter particles and the fact that the Standard Model itself offers no motivation for the values of the 19 free

	Gauge Group			
Notation	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	Particle Rep.
Q^i_L	$\mathbf{3}$	$\mathbf{2}$	$+\frac{1}{3}$	$\binom{u_L}{d_L}^i$
$(u^c_R)^i$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-\frac{4}{3}$	$(u^c_R)^i$
$(d^c_R)^i$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$+\frac{2}{3}$	$(d^c_R)^i$
L^i_L	$\mathbf{1}$	$\mathbf{2}$	-1	$\binom{\nu_L}{e_L}^i$
$(e^c_R)^i$	$\mathbf{1}$	$\mathbf{1}$	$+2$	$(e^c_R)^i$

Table 2: *The gauge representations of the Standard Model matter content where $i = 1, 2, 3$ accounts for the three generations of matter and c denotes a conjugate field.*

Force	Notation	Q_{em}
Electromagnetic	γ	0
Weak Nuclear	Z^0	0
	$W^{\pm}$	± 1
Strong Nuclear	g	0

Table 3: *The gauge bosons of the Standard Model.*

parameters it relies upon for it predictivity [4]. In fact, these need to be placed in the theory manually from their observed values measured in experiments, which is unappealing. Another issue is the gauge hierarchy problem, which refers to the fine tuning needed in order to obtain the scale of weak interactions and the need to stabilize it against radiative corrections [5]. Expressed in another way, why is the weak scale $\sim \mathcal{O}(10^3)$ GeV so many orders of magnitude lower than the Planck scale (or some other scale of grand unification), which is $\sim \mathcal{O}(10^{18})$ GeV? [6]

In light of these shortcomings, among others, it is natural to consider potential extensions to the Standard Model which may be able to solve some of these issues. One potential solution to the gauge hierarchy problem in particular involves the addition of supersymmetry to the Standard Model in order to create the Minimal Supersymmetric Standard Model

(MSSM) [7]. Another of the proposed extensions to the Standard Model is the embedding of the gauge group of the Standard Model into the so-called Left-Right Symmetric gauge group $SU(3) \times SU(2) \times SU(2) \times U(1)$. This proposed extension is the topic of the next section.

1.2 Left-Right Symmetric Models

Left-Right Symmetric (LRS) models extend the gauge symmetry of the Standard Model (SM) by

$$SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \, ,$$

where B is the baryon number and L is the lepton number. The anomaly free matter content of LRS models can be found in table 4.

This extension solves some of the undesirable features of the SM. Firstly, LRS models have a symmetry between left- and right-chiralities. At the level of the SM this symmetry is not present, meaning there is a distinction between left- and right-handed fermions [8]. This is commonly referred to as parity violation. However, the SM provides no motivation for why parity should be violated. However, LRS models motivate the violation of parity in the SM due to the spontaneous symmetry breaking of the $SU(2)_R$ gauge group [9].

Secondly the SM considers the left-handed neutrino as massless, contrary to experimental observation [10]. However, in LRS models a small mass for left-handed neutrinos can be introduced quite naturally by the introduction of a see-saw mechanism [11].

Finally, LRS models have the potential to provide a solution to the strong CP and supersymmetric CP problems. This will not be discussed further here but can be found in references [12, 13].

LRS models contain two instances of spontaneous symmetry breaking. Firstly, the $SU(2)_R$ symmetry must be broken before the $SU(2) \times U(1)_Y$ electroweak symmetry of the SM is broken. In this work, the breaking of the $SU(2)_R$ symmetry is achieved using a Higgs mechanism and therefore introduces a so-called Heavy Higgs (this indicates the energy scale of

	Gauge Group				
Notation	$SU(3)_C$	$SU(2)_L$	$SU(2)_R$	$U(1)_C$	Particle Rep.
Q_L^i	$\mathbf{3}$	$\mathbf{2}$	$\mathbf{1}$	$+\frac{1}{6}$	$\binom{u}{d}^i$
Q_R^i	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{6}$	$\binom{d^c}{u^c}^i$
L_L^i	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$-\frac{1}{2}$	$\binom{\nu}{e}^i$
L_R^i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$+\frac{1}{2}$	$\binom{e^c}{\nu^c}^i$
h	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	0	$\begin{pmatrix} h_+^u & h_0^d \\ h_0^u & h_-^d \end{pmatrix}$

Table 4: *The gauge representation of the Left-Right Symmetric matter content where $i = 1, 2, 3$ accounts for the three generations of matter and c denotes a conjugate field.*

Force	Notation	Q_{em}
Electromagnetic	γ	0
Weak Nuclear	Z^0	0
	$W^{\pm}$	± 1
Right Handed	Z'	0
Charged Currents	$W_R^{\pm}$	± 1
Strong Nuclear	g	0

Table 5: *The gauge bosons of Left-Right Symmetric models.*

the spontaneous symmetry breaking of the $SU(2)_R$ is higher than that of the SM electroweak symmetry breaking). The electroweak symmetry breaking is the same as for the Standard Model and therefore introduces a SM Higgs into the spectrum.

The spontaneous symmetry breaking of the $SU(2)_R$ symmetry results in the addition of three gauge bosons to the gauge boson content of the Standard Model. In the literature, these are commonly referred to as the $W_R^{\pm}$ and Z' bosons. These bosons obtain a mass during the spontaneous symmetry breaking and have a electromagnetic charge Q_{em} the same as their SM electroweak counterparts $W^{\pm}, Z^0$, which can be seen in table 5.

Although LRS models do not solve all of the shortcomings of the SM, their study warrants some attention. In addition to some of the solutions LRS models offer, the proposed existence of the Heavy Higgs and the gauge bosons $W_R^{\pm}, Z'$ give a testable prediction for either current or planned collider experiments.

1.3 Thesis Outline

The outline of the thesis is as follows. Chapter 2 introduces the basics of string theory. The fundamental concepts of the classical bosonic string are introduced before presenting a method of quantisation which is then applied to the bosonic string. The spectrum of the quantised bosonic string is then outlined. Considerations then move to the classical superstring, which utilises supersymmetry in order to introduce spacetime fermions to the spectrum of the theory. Quantisation of the superstring is considered before outlining the spectrum of the superstring.

In chapter 3 the Free Fermionic Formulation of the heterotic superstring is discussed. A brief introduction to the heterotic superstring is given before introducing the basic formulation of the free fermions and the important notation. Modular invariance is discussed before constructing the one-loop partition function of the theory. The derivations of the necessary rules for phenomenological model building are presented before outlining the complete Hilbert space of the theory.

In chapter 4 the rules on model building derived in the previous chapter are presented in a condensed manner, resulting in a self contained chapter which contains all the necessary rules for building consistent free fermionic string theories. A simple model consisting of two basis vectors is built in order to outline the use of the equations presented in the first sections of the chapter and the general method of deriving the spectrum of a free fermionic model.

Chapter 5 presents the classification of Left-Right Symmetric heterotic string vacua. The work presented is the subject of a publication by the author in collaboration with A. E. Faraggi and J. Rizos which can be found in [14].

Conclusions to the thesis and an outlook for the future of this research is given in chapter 6.

Appendix A contains a brief overview of previous instances where research groups have performed analysis on large sets of string vacua. Some details of the research methods and the results obtained are outlined while some of the similarities and differences to the work presented in Chapter 5 are highlighted.

2 The Basics of String Theory

This chapter concerns the fundamental concepts of string theory. The chapter begins with the construction of the classical bosonic string before applying a method of quantisation. The spectrum of the quantised bosonic string is then outlined. The chapter progresses by introducing supersymmetry in order to introduce spacetime fermions to the theory. This theory is referred to as superstring theory. The superstring is then quantised using a procedure analogous to the bosonic string and the spectrum is discussed. The chapter concludes with a brief discussing of the GSO projection.

2.1 The Bosonic String

String theory modifies the description of what constitutes the fundamental particles. Particles in the Standard Model are represented as zero-dimensional objects, referred to as point particles. String theory modifies this concept by representing the fundamental particles as one-dimensional objects, referred to as strings. This modification has some immediate effects. Where a point particle travelling through spacetime is said to trace out a one-dimensional worldline, a string traces out a two-dimensional worldsheet. This can be seen in figure 1.

The action of a point particle is proportional to the proper length of the particle's worldline. It is therefore reasonable to begin constructing string theory using an action proportional to the proper area of the string worldsheet. This results in the Nambu-Goto action defined as

$$\begin{aligned} S_{NG} &= -\frac{1}{2\pi\alpha'}\int_M dA \\ &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\,\sqrt{-\det\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}} \\ &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det\gamma} \end{aligned} \tag{2.1}$$

where α' is the Regge slope, M is the string worldsheet and $\eta^{\mu\nu}$ is the flat spacetime metric. The quantity $\sigma^\alpha = (\tau, \sigma)$ parameterises the string

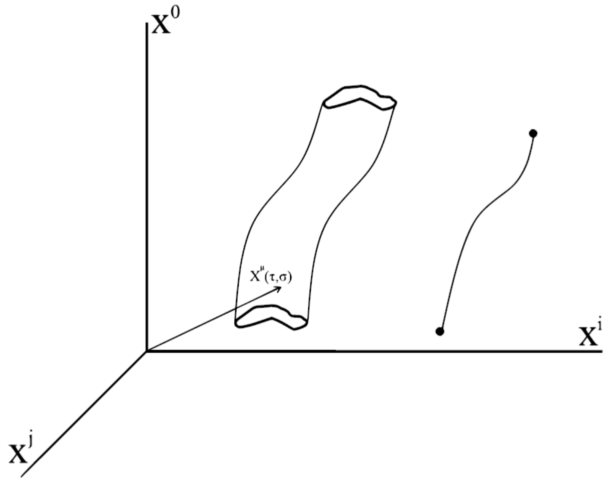

Figure 1: *When a point particle propagates through spacetime, a one dimensional worldline is traced out. String theory considers one dimensional strings which trace out a two dimensional worldsheet when propagating through spacetime.*

worldsheet where τ is a time-like coordinate defined as the proper time and σ is a space-like coordinate. The variable X^μ maps the string worldsheet to spacetime where μ runs over the number of spacetime dimensions. The prefactor to the integration is interpreted as the string tension T and is defined as the mass per unit length of the string. The quantity $\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}$ is often referred to as the induced metric on the string worldsheet.

The Nambu-Goto action contains a square root of the determinant of the induced metric γ. This can often become inconvenient to work with during calculations, specifically as it is relatively difficult to quantise using path integral techniques [15]. A solution to this is to use the Polyakov approach to string theory which is achieved by redefining the action as

$$\begin{aligned} S_P &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \\ &= -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\gamma_{\alpha\beta} \end{aligned} \tag{2.2}$$

where $h^{\alpha\beta}$ is defined as the two-dimensional metric of the string worldsheet and the notations $\partial_\alpha X^\mu = \frac{\partial X^\mu}{\partial \sigma^\alpha}$ and $h \equiv \det h_{\alpha\beta}$ have been introduced. An important quantity of any string theory is the energy momentum tensor $T_{\alpha\beta}$, defined as

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_P}{\delta h^{\alpha\beta}} \ . \tag{2.3}$$

The energy momentum tensor describes how the action is affected by infinitesimal variations of the metric.

The Polyakov action is invariant under the following

- **Poincaré Transformations**

  Poincaré transformations are a global symmetry of the worldsheet which take the form

  $$X^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + c^\mu \ ,$$

  where $\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$ and c^μ is some constant. $\Lambda^\mu{}_\nu$ and c^μ account for Lorentz transformations and translations of the fields X^μ respectively.

- **Reparameterisations of the string worldsheet**

  The string worldsheet possesses a gauge symmetry defined by

  $$(\tau,\sigma) \to (\tilde{\tau},\tilde{\sigma}) \ ,$$

  which reflects the fact that a reparameterisation of the worldsheet coordinates does not alter the underlying physics.

- **Weyl rescalings**

  There is another gauge symmetry of the string worldsheet described by

  $$h^{\alpha\beta}(\tau,\sigma) \to e^{2\omega(\tau,\sigma)} h^{\alpha\beta}(\tau,\sigma) \ ,$$

  known as Weyl rescaling. Using Weyl rescaling, the two-dimensional metric $h_{\alpha\beta}$ can be set equal to the two-dimensional flat metric $\eta_{\alpha\beta}$. Specifically, the flat metric is defined in this case as $\eta_{\alpha\beta} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This is known as the flat gauge and can be used to simplify the form of the action as described below.

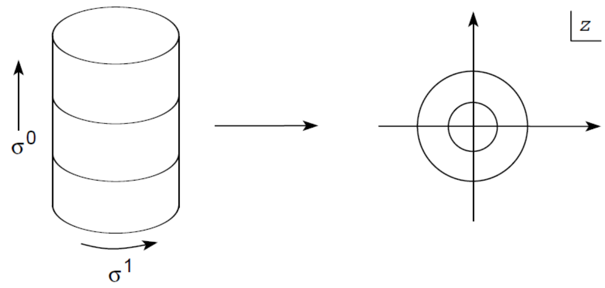

Figure 2: *The coordinates τ, σ can be mapped to the complex plane.*

With the flat gauge selected, the Polyakov action becomes

$$S_P = -\frac{T}{2} \int d\tau d\sigma \; \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \; ,$$

which is the form used for the subsequent analysis.

The equation of motion for the fields X^μ which results from this action is a two-dimensional wave equation, expressed as

$$\partial_\alpha \partial^\alpha X^\mu = 0 \; .$$

As an aside, we can introduce complex coordinates, which can be useful for writing some later expressions. By making the definitions

$$z = \tau + i\sigma \quad \text{and} \quad \bar{z} = \tau - i\sigma \tag{2.4}$$

the coordinates τ and σ can be mapped to the complex plane, as depicted in figure 2. Using the complex coordinates, the two-dimensional wave equation can be expressed as

$$\partial_z \partial_{\bar{z}} X^\mu = 0 \; .$$

As this thesis concerns the heterotic string, only the closed string solution to this equation is considered. The general solution for the closed string, which has the boundary condition

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \; ,$$

is then

$$X^\mu(\tau, \sigma) = X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma) \tag{2.5a}$$

$$X^\mu(z,\bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z}) \tag{2.5b}$$

where equations (2.5a) and (2.5b) give the general solution in both coordinate systems outlined above, but are entirely equivalent. This shows the closed string can be split arbitrarily into left- and right-moving solutions.

These solutions can be expanded in terms of their Fourier modes to become

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha^{'}}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha^{'}}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau+\sigma)} \ , \tag{2.6a}$$

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha^{'}}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha^{'}}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in(\tau-\sigma)} \ , \tag{2.6b}$$

where x^μ and p^μ are the position and momentum of the string's centre of mass respectively. The Fourier modes $\alpha^\mu_n, \tilde{\alpha}^\mu_n$ obey

$$\alpha^\mu_n = (\alpha^\mu_{-n})^* \ ,$$
$$\tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^* \ ,$$

in order to preserve the reality condition $X^\mu = (X^\mu)^*$. Classically, these Fourier modes are interpreted as amplitudes of the n^{th} order oscillations.

There is another equation of motion which arises from varying the action with respect to the worldsheet metric. This is simply the equation of motion which requires the vanishing of the energy momentum tensor, *i.e.* $T_{\alpha\beta} = 0$. This leads to the constraints

$$\left(\dot{X} \pm X'\right)^2 = 0 \tag{2.7}$$

and is a direct consequence of the vanishing of the energy momentum tensor. These constraints are commonly referred to as the Virasoro constraints. Further discussion of the vanishing of the energy momentum tensor is given in section 2.1.2.

This concludes the discussion of the classical bosonic string. It remains to perform the quantisation of the bosonic string and also to consider the addition of fermions to the theory. This is the subject of the following sections.

2.1.1 Quantising the Bosonic String

In quantising the bosonic string, the fields X^μ along with the quantities x^μ, p^μ, α_n^μ, $\tilde{\alpha}_n^\mu$ are promoted to operators. The equal time commutation relations must then be imposed, which are

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \ , \tag{2.8a}$$

$$[\tilde{\alpha}_m^\mu, \alpha_n^\nu] = 0 \ , \tag{2.8b}$$

$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = [\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \ . \tag{2.8c}$$

In addition to this, imposing the reality condition $X^\mu = (X^\mu)^*$ results in the conditions

$$(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu \quad \text{and} \quad (\tilde{\alpha}_m^\mu)^\dagger = \tilde{\alpha}_{-m}^\mu \ . \tag{2.9}$$

Redefining the oscillators according to

$$a_m^\mu \to \frac{\alpha_m^\mu}{\sqrt{m}} \quad \text{and} \quad (a_m^\mu)^\dagger \to \frac{\alpha_{-m}^\mu}{\sqrt{m}} \quad , \quad m > 0$$

then gives the familiar result for the commutation relation of the harmonic oscillator $[a_m^\mu, (a_n^\nu)^\dagger] = \delta_{m,n}\eta^{\mu\nu}$. In this form, it is clear that the modes a_{-m}^μ can be interpreted as creation operators and the modes a_m^μ as annihilation operators when $m > 0$.

Therefore, in momentum space defined by

$$\hat{p}^\mu \left|0; p\right\rangle = p^\mu \left|0; p\right\rangle \ ,$$

where p^μ is the eigenvalue of the momentum operator $\hat{p}^\mu$ and the ground state is defined as

$$a_m^\mu \left|0; p\right\rangle = \tilde{a}_m^\mu \left|0; p\right\rangle = 0 \ .$$

From this, a Fock space can be constructed by applying creation operators to the ground state. Explicitly, a state $\left|\Phi\right\rangle$ is constructed by

$$\left|\Phi\right\rangle = (a_{-1}^{\mu_1})^{n_{\mu_1}}(a_{-2}^{\mu_2})^{n_{\mu_2}}\ldots(\tilde{a}_{-1}^{\nu_1})^{n_{\nu_1}}(\tilde{a}_{-2}^{\nu_2})^{n_{\nu_2}}\ldots\left|0; p\right\rangle \ . \tag{2.10}$$

2.1.2 The Energy Momentum Tensor and the Virasoro Conditions

The energy momentum tensor is found as a result of varying the action with respect to the worldsheet metric. Explicitly, imposing the constraint

$$\frac{\delta S}{\delta h_{\alpha\beta}} = 0$$

gives the constraint

$$T_{\alpha\beta} = 0 \tag{2.11}$$

on the two-dimensional energy momentum tensor. It should be noted that the vanishing of the energy momentum tensor is equivalent to the theory having conformal symmetry [15]. When expressed in complex coordinates, this constraint on the energy momentum tensor is $T_{z\bar{z}} = 0$, *i.e.* the energy momentum tensor is traceless. This result follows directly from the definition of the energy momentum tensor given in equation (2.3). There are then two constraint operators [16, 17]

$$\begin{aligned} T(z) = T_{zz} &= -\frac{1}{2}\partial_z X^\mu \partial_z X_\mu \ , \\ T(\bar{z}) = T_{\bar{z}\bar{z}} &= -\frac{1}{2}\partial_{\bar{z}} X^\mu \partial_{\bar{z}} X_\mu \ . \end{aligned} \tag{2.12}$$

Classically, the energy momentum tensor can be expanded in terms of Fourier modes to give [15, 23]

$$T(z) = \sum_{n\in\mathbb{Z}} z^{(-n-2)} L_n \quad \text{and} \quad T(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{(-n-2)} \tilde{L}_n \tag{2.13}$$

where the Laurent modes L and $\tilde{L}$ have been introduced and are defined by

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n \quad \text{and} \quad \tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \ . \tag{2.14}$$

The classical modes L satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} \tag{2.15}$$

and similarly for the modes $\tilde{L}$. When defined in this manner, notable expressions are the Hamiltonian H and the canonical momentum P

$$H = L_0 + \tilde{L}_0 \quad \text{and} \quad P = L_0 - \tilde{L}_0 \tag{2.16}$$

which are the classical expressions of the Virasoro generators.

Upon quantisation, the Laurent modes $L, \tilde{L}$ are promoted to operator status and must therefore account for normal ordering. This is done by modifying the definitions of the classical modes given in equation (2.14) to become

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{m-n}\cdot\alpha_n : \quad \text{and} \quad \tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n : \tag{2.17}$$

where the colon notation denotes the normal ordered product of the creation and annihilation operators. Specifically, the annihilation operators always appear on the right hand side of the expressions of $L, \tilde{L}$.

The quantum operators now obey a modified Virasoro algebra

$$[L_n, L_m] = (n-m)L_{(n+m)} + \frac{c}{12}(n(n^2-1))\delta_{n+m} \tag{2.18}$$

which includes the variable c referred to as the central charge of the algebra and the conformal anomaly in string theories. Its appearance is a consequence of the presence of Weyl rescaling in the theory [15].

The addition of normal ordering introduces an ambiguity into the definition of L_0 and $\tilde{L}_0$. The result is that the general quantum versions of these operators differ from the normal ordered definitions by a constant. In practice, this means in the quantum expressions $L_0 \to L_0 + a$, where a is a constant. Similarly, $\tilde{L}_0$ has a redefinition of the same form. Although no further motivation for this is discussed here, proofs of this fact can be found in references [18, 21, 22, 23, 24, 26].

It is now instructive to consider the constraint that the energy momentum tensor must equal zero, as described by equation (2.11). In the quantised theory this leads to conditions on the physical states $|\Phi\rangle$ given by

$$(L_0 - \tilde{L}_0)\,|\Phi\rangle = 0 \tag{2.19a}$$

$$L_m\,|\Phi\rangle = 0 \quad \text{and} \quad \tilde{L}_m\,|\Phi\rangle = 0 \quad , \quad m > 0 \tag{2.19b}$$

$$(L_0 - a)\,|\Phi\rangle = 0 \quad \text{and} \quad (\tilde{L}_0 - a)\,|\Phi\rangle = 0 \tag{2.19c}$$

where a is currently left undefined. Noting the physical interpretations of these Virasoro generators, the generators L_{-1} and $\tilde{L}_{-1}$ generate translations in the conformal plane, whereas L_0 and $\tilde{L}_0$ generate scaling and rotations in the conformal plane [15].

The spectrum at this level contains states with a negative norm. An example of these states is

$$a_m^{0\dagger} |0\rangle$$

which has a norm

$$\langle 0| a_m^0 a_m^{0\dagger} |0\rangle = -1$$

where the ground state is normalised by $\langle 0|0\rangle = 1$ [23]. These states are referred to as ghosts and lead to an inconsistent theory as they violate unitarity and causality. They arise as the commutation relations are time-like and there is a residual gauge freedom [16]. However, these negative norm states can be removed from the spectrum by using the light-cone gauge, which is the topic of the next section.

2.1.3 Light-Cone Coordinates and Gauge

In order to define the light-cone gauge, the introduction of light-cone coordinates is necessary. The definition of light-cone coordinates on the string worldsheet is

$$\sigma^{\pm} = \tau \pm \sigma \tag{2.20}$$

and the introduction of spacetime light-cone coordinates is done accordingly

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}) \; . \tag{2.21}$$

The remaining $D-2$ spacetime coordinates are denoted by X^I and remain unchanged. These coordinates contain a residual gauge symmetry which can be seen by making the transformations

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad \text{and} \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \tag{2.22}$$

which has the effect of multiplying the flat metric by an overall factor, which can be negated by the use of a Weyl transformation. The solution to the equation of motion which was given in equation (2.5a) can now be written in light-cone coordinates as

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-)$$

which we can now gauge fix by utilising the reparameterisation invariance. The coordinates are chosen in the following manner

$$X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha^{'} p^+\sigma^+ \quad , \quad X_R^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha^{'} p^+\sigma^-$$

and subsequently it is found that

$$X^+(\tau,\sigma) = x^+ + \alpha^{'} p^+\tau \ , \tag{2.23}$$

which is the light-cone gauge. Now that the choice of X^+ has been made, it remains to calculate the corresponding form of X^- in this gauge. From the constraints given in equation (2.7) it can be shown that these constraints become

$$2\partial_+ X^-\partial_+ X^+ = \sum_{I=1}^{D-2} \partial_+ X^I \partial_+ X^I \ , \tag{2.24}$$

which can be rewritten using the light-cone gauge given in equation (2.23) as

$$\partial_+ X_L^- = \frac{1}{\alpha' p^+}\sum_{I=1}^{D-2} \partial_+ X^I \partial_+ X^I \tag{2.25}$$

for the left-movers, and

$$\partial_- X_R^- = \frac{1}{\alpha' p^+}\sum_{I=1}^{D-2} \partial_- X^I \partial_- X^I \tag{2.26}$$

for the right-movers. It can now be seen that X^- is completely determined by the transverse oscillations X^I. This can be seen by writing the mode expansions for X^- in a manner similar to equations (2.6a) and (2.6b), which are

$$\begin{aligned} X_L^-(\sigma^+) &= \frac{1}{2}x^- + \frac{1}{2}\alpha^{'} p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^+} \ , \\ X_R^-(\sigma^-) &= \frac{1}{2}x^- + \frac{1}{2}\alpha^{'} p^-\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^-} \ , \end{aligned} \tag{2.27}$$

then p^-, α_n^- and $\tilde{\alpha}_n^-$ are fixed by the constraints above and x^- is an integration constant. A more robust derivation of this can be seen in references [15, 26].

This concludes the discussion on the light-cone coordinates and gauge. The spectrum of the quantised bosonic string can now be considered.

2.1.4 Mass Squared and Spectrum of Quantised Bosonic String

Using the definition of the mass squared[1] $M^2 = -p_\mu p^\mu$ (where p_μ is the total momentum of the string) and results from the previous sections, the mass-shell condition can be shown to be

$$M^2 = \frac{4}{\alpha'}(N - a) \quad \text{and} \quad M^2 = \frac{4}{\alpha'}(\tilde{N} - a) \ , \tag{2.28}$$

where $N, \tilde{N}$ are the number operators defined as

$$N = \sum_{I=1}^{D-2} \sum_{n>0} \alpha_{-n}^I \alpha_n^I \quad \text{and} \quad \tilde{N} = \sum_{I=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I \ . \tag{2.29}$$

As the mass squared must be equal for the left- and right-movers, equating the two results in equation (2.28) shows

$$N = \tilde{N} \ , \tag{2.30}$$

which is known as the level matching condition. This result simply shows that the number of left- and right-moving oscillators must be equal. The mass-shell condition can therefore be redefined as

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2a) \ . \tag{2.31}$$

Using the result of the mass-shell condition, the spectrum can now be discussed. It is again noted that only the spectrum of the closed string is considered, as it is only the closed string discussed in later chapters.

[1]Specifically, the mass squared takes this form when using the metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ [15, 18, 23, 25].

- **Ground State** $|0; p^\mu\rangle$

  The mass-shell condition gives the result[2] $M^2 = -\frac{4a}{\alpha'}$. As discussed below, it is found that $a = 1$ and therefore the ground state has a negative mass squared. This state is therefore tachyonic and its removal from the spectrum by the addition of further constraints is discussed later.

- **First Excited States** $\alpha^I_{-1}\tilde{\alpha}^J_{-1}|0; p^\mu\rangle$

  These states are also referred to as level one and therefore $N = \tilde{N} = 1$. The mass-shell condition is then $M^2 = \frac{4}{\alpha'}(1 - a)$. These states fit into a representation of $SO(D-2)$ [15, 18]. In order for Lorentz invariance to be satisfied, all massless particles must fit into a representation of $SO(D-2)$ and massive states must fit into a representation of $SO(D-1)$ [23]. Therefore, as the states at this level fit into $SO(D-2)$ they must be massless. From the mass-shell condition stated in equation (2.31), it can be seen that massless states can only be created if $a = 1$.

  The form of a is explicitly stated here, but its derivation can be found in references [15, 18, 21, 22], and is

  $$a = \frac{D-2}{24} \ .$$

  Substituting in the result that $a = 1$ shows the critical spacetime dimension to be $D = 26$.

  The states of this level fit into the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$ which can be decomposed into three irreducible representations by

  $$\text{traceless symmetric} \oplus \text{anti-symmetric} \oplus \text{singlet } (\textit{i.e.} \text{ the trace})$$

  which correspond generally to the graviton, anti-symmetric tensor field and dilaton fields respectively.

[2] It should be noted this result is for closed strings only. The mass squared for open strings does not include the factor of four and therefore have a mass squared a quarter than that of the closed string case.

- **Higher Mass States**

  Considering the mass-shell condition presented in equation (2.31), when $N, \tilde{N} \geq 2$ it is found that $M^2 > 0$ and therefore all further excited states are massive.

The theory therefore has been shown to have 26 spacetime dimensions and a normal ordering constant of $a = 1$. It should be noted that this is only one method to obtain the value for a which utilised the requirement that the theory should be Lorentz invariant. There are other approaches which justify this value for a, but these will not be discussed here.

The spectrum contains tachyonic states and only bosonic fields. This leaves the task of formulating a string theory where the spectrum possesses no tachyons and contains fermions. This requires the use of supersymmetry in order to create the superstring and is the subject of the next section.

2.2 The Superstring

The string theory described up to this point contains only spacetime bosons and has a tachyon in the spectrum. This is undesirable as the theory needs to include spacetime fermions and tachyonic particles should be removed from the spectrum. The method described here to achieve this utilises supersymmetry, leading to the formulation of superstring theory. Firstly, fermionic degrees of freedom will be added to the string theory by invoking supersymmetry and secondly the superstring theory will be quantised.

2.2.1 The Ramond-Neveu-Schwarz Formalism

The Ramond-Neveu-Schwarz (RNS) formalism is an extension to the bosonic string considered in previous sections which introduces new dynamical fields. These dynamical fields are vectors with respect to spacetime but are spinors with respect to the worldsheet [27]. This has the result of introducing fermions to the spectrum of the string theory.

The RNS construction begins by demanding that for each field $X^\mu(\tau,\sigma) \to X^\mu(\tau,\sigma), \psi^\mu(\tau,\sigma)$. These dynamical fields are related to each other by the supersymmetric transformations

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \ ,$$
$$\delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \ ,$$

where $\epsilon, \bar{\epsilon}$ are constants and ρ^α are the two dimensional gamma matrices, defined as

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{2.32}$$

The matrices ρ are Grassmann variables and satisfy the anti-commutation relation $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$, commonly referred to as the Dirac algebra[3].

The field $\psi^\mu(\tau,\sigma)$ is a two component Majorana spinor which lives on the worldsheet and is defined by

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \ , \tag{2.33}$$

when written in light-cone coordinates (the definitions can be seen explicitly in equation (2.41)). The Dirac conjugate of this spinor $\overline{\psi}$ is defined as $\overline{\psi} = \psi^\dagger i\rho^0$. As the spinor is a two-dimensional Dirac spinor and is real by definition, *i.e.* $\psi^*_\pm = \psi_\pm$, the spinor is Majorana [18, 23]. The conjugate of the Majorana spinor therefore simplifies to $\psi^T\rho^0$. When written explicitly,

$$\overline{\psi}^\mu = \begin{pmatrix} \psi^\mu_+ \\ -\psi^\mu_- \end{pmatrix} \ . \tag{2.34}$$

Contrary to the bosonic fields X^μ, these spinors anti-commute according to the equation

$$\{\psi^\mu_A(\tau,\sigma), \psi^\nu_B(\tau,\sigma')\} = \pi\delta_{AB}\delta(\sigma-\sigma')\eta^{\mu\nu} \ , \tag{2.35}$$

where A, B denote the worldsheet spinor indices. Upon addition of these fields, the modified action is

$$S = -\frac{T}{2}\int d\tau d\sigma \ (\partial_\alpha X_\mu \partial^\alpha X^\mu + i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \tag{2.36}$$

[3]The Dirac algebra is also commonly called a Clifford algebra [23].

when written in the conformal gauge. The bosonic term in this action is the same as the bosonic string described in the previous section. Therefore, only the fermionic terms will be considered below. This will be achieved by considering only the fermionic term in light-cone coordinates, in which calculations are clearer. The fermionic term of this action in light-cone coordinates is then

$$S_f = iT \int d^2\sigma(\psi^\mu_- \partial_+ \psi_{-\mu} + \psi^\mu_+ \partial_- \psi_{+\mu}) \ . \tag{2.37}$$

There is a conserved supersymmetric current J^μ due to the worldsheet supersymmetric and translational invariances. Using the Noether method, it can be shown that the Virasoro constraints defined in equation (2.7) lead to the conclusions

$$J_+ = J_- = T_{++} = T_{--} = 0 \ . \tag{2.38}$$

Further information on these results can be found in reference [23].

2.2.2 Fermionic Boundary Conditions and Fourier Modes

Both the Fourier modes and the boundary conditions of the bosonic term are as described in previous sections. It then remains to find the Fourier mode expansions and boundary conditions of the newly introduced fermionic term of the action. The variation of the fermionic term of the action (2.37) is

$$\begin{aligned} \delta S_f = -T\Bigg[\int d\tau d\sigma(\delta\psi^\mu_- \partial_+ \psi_{-\mu} + \delta\psi^\mu_+ \partial_- \psi_{+\mu}) \\ + \frac{1}{2}\int d\tau[\psi^\mu_+ \delta\psi_{+\mu} - \psi^\mu_- \delta\psi_{-\mu}]^{\sigma=\pi}_{\sigma=0}\Bigg] \ , \end{aligned} \tag{2.39}$$

meaning the equations of motion are

$$\partial_+ \psi_- = 0 \quad \text{and} \quad \partial_- \psi_+ = 0 \ . \tag{2.40}$$

These are referred to as the Weyl conditions for spinors in two dimensions [23]. The fields $\psi_\pm$ are therefore Majorana-Weyl spinors. The equations of motion also imply

$$\psi^\mu_+ = \psi^\mu_+(\tau + \sigma) \quad \text{and} \quad \psi^\mu_- = \psi^\mu_-(\tau - \sigma) \ . \tag{2.41}$$

The boundary conditions can now be found by requiring that the last term of the action in equation (2.39) vanishes. Explicitly,

$$[\psi_+^\mu(\tau,\sigma)\delta\psi_{+\mu}(\tau,\sigma) - \psi_-^\mu(\tau,\sigma)\delta\psi_{-\mu}(\tau,\sigma)]_{\sigma=0}^{\sigma=\pi} = 0 \tag{2.42}$$

leads to the two boundary conditions below.

In the case of the closed superstring, the bosonic part corresponds to a tensor product of left- and right-moving modes (as shown in previous sections). The fermionic case is the same in this respect and the boundary conditions for the left- and right-moving modes are defined separately as

$$\begin{aligned} \psi_+^\mu(\tau,\sigma) &= \pm\psi_+^\mu(\tau,\sigma+\pi)\ , \\ \psi_-^\mu(\tau,\sigma) &= \pm\psi_-^\mu(\tau,\sigma+\pi)\ . \end{aligned} \tag{2.43}$$

The cases where the sign is positive are referred to as Ramond (R) boundary conditions, which are periodic. The cases where the sign is negative are referred to as Neveu-Schwarz (NS) boundary conditions and are anti-periodic.

In a procedure analogous to the bosonic case, the fermionic mode expansions of $\psi_\pm^\mu$ are found to be

$$\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}\ , \tag{2.44}$$

$$\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)}\ , \tag{2.45}$$

for the Ramond boundary conditions, and

$$\psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}\ , \tag{2.46}$$

$$\psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(\tau-\sigma)}\ , \tag{2.47}$$

for the NS boundary conditions. Upon quantisation, the operators d_n^μ and b_r^μ become raising operators for $n, r < 0$ and lowering operators for $n, r > 0$.

The open superstring has not been discussed here as the remainder of this document will utilise only closed strings in order to build phenomenological string models. However, discussion of open strings can be found in references [18, 19, 21, 22, 23, 24, 25, 26].

2.2.3 Quantisation of the Closed Superstring

Quantisation of the closed superstring can now be performed using the method of canonical quantisation. As before, the commutation relations given in equations (2.8a), (2.8b) and (2.8c) are imposed on the bosonic fields. The canonical anti-commutation relation given in equation (2.35) is applied to the fermionic fields which leads to the following anti-commutation relations for the fermionic Fourier coefficients

$$\{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{n+m,0} \tag{2.48a}$$

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{2.48b}$$

$$\{\tilde{d}_n^\mu, \tilde{d}_m^\nu\} = \eta^{\mu\nu}\delta_{n+m,0} \tag{2.48c}$$

$$\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{2.48d}$$

where $n, m \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$. All other anti-commutation relations vanish. As was the case for the bosonic string, there exist negative norm states. These appear from the time components of the fermionic modes as the spacetime metric appears on the right-hand side of the above equations. However, the negative norm states decouple due to the superconformal symmetry present in the RNS string, which is the required result [23].

The ground state $|0; p^\mu\rangle$, with momentum p^μ, of both the R and NS sectors is defined as

$$\alpha_m^\mu |0; p^\mu\rangle_{NS} = b_r^\mu |0; p^\mu\rangle_{NS} = 0 \qquad \forall m, r > 0 \ , \tag{2.49a}$$

$$\alpha_m^\mu |0; p^\mu\rangle_{R} = d_m^\mu |0; p^\mu\rangle_{R} = 0 \qquad \forall m > 0 \ , \tag{2.49b}$$

which is the state defined as being annihilated by the lowering operators. There are similar expressions for the ground state of the left-moving modes.

The excited states of the string are obtained by acting on these ground states with creation operators. In the same manner as for the bosonic string, the excited states form a Fock space. The Fock space consists of states which are formed using both fermionic and bosonic creation operators, such as

$$d^{\mu}_{-1}\tilde{d}^{\mu}_{-1}\left|0\right\rangle_{R} \quad , \quad \alpha^{\nu}_{-2}\left|0\right\rangle_{NS} \quad , \quad etc.$$

Acting on a state with a creation operator raises the energy of that state while acting with an annihilation operator lowers the energy of that state. In the NS sector, acting with the operator b^{μ}_{r} changes the energy of a state by a half integer unit. The result is that bosons have half integer energy spacings. In the R sector, acting with the operator d^{μ}_{n} changes the energy of a state by an integer unit. This results in fermions having integer energy spacings. This is in direct contrast with the requirements of unbroken supersymmetry and must be resolved [18, 23]. The method of resolution for this asymmetry between bosons and fermions is the GSO projection which is described in section 2.2.6.

The NS sector has a unique ground state, as opposed to the R sector where the ground state is degenerate. The NS ground state corresponds to states in spacetime with spin 0 and the excited states correspond to spacetime bosons. It should also be noted that the zero mode operator d^{μ}_{0} does not change the mass squared of a given state. This is due to the fact that the operator d^{μ}_{0} commutes with the number operator defined in equation (2.59). By considering the definition given in equation (2.48a) for the case where $n, m = 0$, it can be seen that the coefficients realise the algebra

$$\{d^{\mu}_{0}, d^{\nu}_{0}\} = \eta^{\mu\nu} \ .$$

Upon inspection, this result is a Clifford algebra (referred to as the Dirac algebra previously) which is missing a factor of two. Therefore, the zero mode oscillators d^{μ}_{0} can be identified as the gamma matrices Γ^{μ} by

$$d^{\mu}_{0} = \frac{1}{\sqrt{2}}\,\Gamma^{\mu} \ , \tag{2.50}$$

which satisfy the Dirac algebra

$$\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2\eta^{\mu\nu} \ . \tag{2.51}$$

The ground state is therefore degenerate. Furthermore, as all the states in the R sector can be found by applying the raising operators (which are spacetime vectors) on the degenerate R sector ground state, all the states from the R sector are spacetime fermions [23].

2.2.4 The Super Virasoro and Mass Shell Conditions

Analogously to defining the Virasoro generators for the bosonic case, the super Virasoro generators are defined as the Fourier modes of the energy momentum tensor $T_{\alpha\beta}$ and the supercurrent J^{μ}. Therefore,

$$L_m = L_m^{(b)} + L_m^{(f)} \,, \tag{2.52}$$

where the bosonic modes are defined as

$$L_m^{(b)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_{m+n}: \qquad m\in\mathbb{Z}$$

and the fermionic modes are defined by

$$\begin{aligned} L_m^{(f)} &= \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}\left(r+\frac{m}{2}\right) :b_{-r}\cdot b_{r+m}: \qquad \text{in the NS sector,} \\ L_m^{(f)} &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n+\frac{m}{2}\right) :d_{-n}\cdot d_{n+m}: \qquad \text{in the R sector,} \end{aligned}$$

where $m\in\mathbb{Z}$ in both cases.

The Fourier modes of the supercurrents in the NS sector (G_r) and R sector (F_m) are defined as

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{n+r} \qquad \text{for } r\in\mathbb{Z}+\frac{1}{2}\,,$$

$$F_m = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n} \qquad \text{for } m\in\mathbb{Z}\,,$$

where the left-moving modes have similarly defined expressions, but use the tilde notation $\tilde{G}_r$ and $\tilde{F}_m$ to distinguish them from the right-movers.

These expressions adhere to the super Virasoro algebra. For the right-movers in the R sector, the algebras are

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0} \ , \\
[L_m, F_n] &= \left(\frac{m}{2} - n\right)F_{m+n} \ , \\
\{F_m, F_n\} &= 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0} \ ,
\end{aligned} \tag{2.54}$$

and for the right-movers in the NS sector the algebras are

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n,0} \ , \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r} \ , \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \ .
\end{aligned} \tag{2.55}$$

where the left-movers realise similarly defined algebras. By quantising the theory in this way, analogously to the bosonic case, the following conditions on the physical states $|\Phi\rangle$ are found in the NS sector

$$L_m\,|\Phi\rangle = \tilde{L}_m\,|\Phi\rangle = 0 \qquad \text{for } m > 0 \ , \tag{2.56a}$$

$$G_r\,|\Phi\rangle = \tilde{G}_r\,|\Phi\rangle = 0 \qquad \text{for } r > 0 \ , \tag{2.56b}$$

$$(L_0 - a_{NS})\,|\Phi\rangle = (\tilde{L} - a_{NS})\,|\Phi\rangle = 0 \ , \tag{2.56c}$$

and the conditions on physical states in the R sector are

$$L_m\,|\Phi\rangle = \tilde{L}_m\,|\Phi\rangle = 0 \qquad \text{for } m > 0 \ , \tag{2.57a}$$

$$F_n\,|\Phi\rangle = \tilde{F}_n\,|\Phi\rangle = 0 \qquad \text{for } n > 0 \ , \tag{2.57b}$$

$$(L_0 - a_R)\,|\Phi\rangle = (\tilde{L} - a_R)\,|\Phi\rangle = 0 \ . \tag{2.57c}$$

The number operators can therefore be defined for the right-movers in the NS and R sectors respectively as

$$N_{NS} = \sum_{n=1}^{+\infty} \alpha^I_{-n}\alpha^I_n + \sum_{r=1/2}^{+\infty} r\ b^I_{-r}b^I_r \ , \tag{2.58}$$

$$N_R = \sum_{n=1}^{+\infty} \alpha^I_{-n}\alpha^I_n + \sum_{n=1/2}^{+\infty} n\, d^I_{-n} d^I_n \;, \tag{2.59}$$

where there are similarly defined expressions for the number operators of the left-movers and are denoted by $\tilde{N}_{NS/R}$. The mass shell condition of the superstring is therefore

$$M^2 = \frac{2}{\alpha'}\big(N_A + \tilde{N}_B - a_A - a_B\big) \tag{2.60}$$

where A, B refers to either the NS or R sector. The level matching condition for the superstring is therefore modified from the bosonic case defined in equation (2.30) to become

$$N_A - a_A = \tilde{N}_B - a_B \;. \tag{2.61}$$

Using the same argument given for the bosonic string, the normal ordering constants can be determined using Lorentz invariance. This results in finding

$$a_{NS} = \frac{1}{2} \quad \text{and} \quad a_R = 0 \tag{2.62}$$

which, in turn, can be used to calculate the critical dimension of the superstring. The solution to the calculation of the critical dimension is $D = 10$ [18, 23].

It is instructive at this point to briefly mention the conformal anomaly. The derivation of the equation which gives the conformal anomaly is more intuitive by considering the path integral quantisation of the superstring, as opposed to the covariant approach to quantisation given above. This method of quantisation will not be discussed here but can be found in the references [18, 19]. The total conformal anomaly is given by the equation

$$c_{total} = c_{bg} + c_{fg} + c_{X^\mu} \cdot D + c_{\psi^\mu} \cdot D \;, \tag{2.63}$$

where D is the number of spacetime dimensions and c_{bg} and c_{fg} is the contribution from bosonic and fermionic ghosts respectively. The term ghosts refers to Fadeev-Popov ghost fields which arise from quantisation using the path integral method.

2.2.5 The Massless Spectrum of the Superstring

Since the mass-shell condition of the superstring has been defined, the spectrum of the superstring can be considered. It should be noted that while the complete spectrum of the superstring contains both massive and massless states, only the massless states are of interest in string phenomenology. This is due to the massive states obtaining a mass comparable to the string scale, which is unobtainable by any current or planned experiments. Analysis for the remainder of the document is therefore restricted to the massless spectrum.

The spectrum generated by a single set of NS or R modes is outlined, which corresponds to one side of the closed string [33].

Neveu-Schwarz Sector

- **Ground state**
  The ground state $|0; p^\mu\rangle_{NS}$ has a mass squared value of $M^2 = -\frac{1}{2\alpha'}$. As this value is negative, the state is tachyonic. This should be removed from the spectrum as it is unstable. The method of removal is the GSO projection and is discussed in the next section.

- **First Excited State**
  The first excited state is $b^i_{-1/2}|0; p^\mu\rangle_{NS}$ which is the ground state acted upon by the lowest frequency oscillators. This state is a massless vector which has eight transverse components.

- **Higher Mass States**
  Considering the mass-shell condition presented in equation (2.60), it is found that for second and higher level mass states $M^2 > 0$. These states are therefore massive, meaning they are omitted from further analysis.

Ramond Sector

- **Ground state**
  The ground state is denoted by $|0; p^\mu\rangle_R$. This is a degenerate state as applying the oscillators d^i_0 to it do not change the mass squared. As the number of dimensions is $D = 10$, the spinors are restricted

by Majorana and Weyl conditions. Also the Dirac-Ramond equation must be satisfied [18, 23]. Therefore, the minimal Ramond ground state has eight physical degrees of freedom and corresponds to an irreducible spinor of $Spin(8)$. These states are all massless.

- **First Excited States**
  The first excited states are built as $\alpha^i_{-1} |0; p^\mu\rangle_R$ and $d^i_{-1} |0; p^\mu\rangle_R$. These are massive and are therefore not of interest to string phenomenology.

- **Higher Mass States**
  As was the case for the NS sector, any higher mass states have a positive mass squared and are therefore omitted from further analysis.

The full closed string spectrum is then found by considering the tensor products of the left- and right-moving sectors consisting of the states above. There is more than one way of achieving this, which leads to different string theories such as Type IIA or Type IIB. These will not be discussed here but further information can be found in the references [18, 19, 22, 23, 24, 25].

2.2.6 The GSO Projection

As found in the previous section, the ground state of the NS sector contains a tachyon. This can be removed from the spectrum by applying a method of projection on the spectrum. This is referred to as the Gliozzi-Scherk-Olive (GSO) projection and was originally introduced in the research article found in reference [34].

The GSO projection involves applying a projection operator on a physical state according to the equation

$$|\psi\rangle \rightarrow P_{GSO} |\psi\rangle \ . \tag{2.64}$$

In the NS sector the projection operator is given by

$$P_{GSO} = \frac{1}{2}\left[1 - (-1)^F\right] \tag{2.65}$$

where F is the fermion number operator defined by

$$F = \sum_{r=1/2}^{\infty} \eta_{\mu\nu} b^{\mu}_{-r} b^{\nu}_{r} \ . \tag{2.66}$$

Upon applying the GSO projection to the NS sector, the states with an even number of b oscillator excitations are removed which has the effect of removing the tachyon from the spectrum. In the R sector the projection operator takes the slightly modified form

$$P^{\pm}_{GSO} = \frac{1}{2}\left[1 \mp \Gamma^{11}(-1)^{F}\right] \tag{2.67}$$

where the operator $\Gamma^{11} = \Gamma^0\Gamma^1 \ldots , \Gamma^9$ is the $10D$ analogue of the chirality matrix γ^5 in $4D$. This has the effect of defining whether a spinor has a positive or negative chirality by

$$\Gamma^{11}\psi = \pm\psi \ .$$

The introduction of the GSO projection may seem like an ad-hoc procedure at first, but it is motivated by the need for modular invariance and the fact that spacetime supersymmetry requires there to be the same number of spacetime bosons and fermions at each energy level. Application of the GSO projection results in an equal number of degrees of freedom in the NS and R ground states, as well as ensuring spacetime supersymmetry between the NS sector bosons and R sector fermions at every mass level [35].

3 The Free Fermionic Formulation

This chapter concerns the construction of the free fermionic formulation of string theory. While the previous chapter concerned a general approach to the construction of closed superstrings, this chapter concerns only the heterotic construction of closed superstrings. This is due to the fact that the LRS models presented in later chapters are constructed exclusively from free fermionic heterotic superstrings.

The chapter begins with a brief introduction to the construction of heterotic strings before outlining how the free fermionic formulation (FFF) modifies this construction. The FFF is then elaborated upon, providing the necessary details and basis for string model building.

3.1 Construction of Heterotic Strings

The models which are ultimately obtained in this work have four spacetime dimensions, $\mathcal{N} = 1$ spacetime supersymmetry and are heterotic string theories, which are by definition, closed string theories. Therefore, the construction of heterotic string theories is outlined along with the modifications made in order to obtain 4D heterotic string theories in the free fermionic formulation.

In order to obtain supersymmetry on the heterotic superstring, the left- and right-moving modes are decoupled from each other. Supersymmetry is then imposed on the left-moving modes which leads to the left-moving currents carrying the supersymmetric charges. The right-moving modes are left as purely bosonic, meaning the right-moving worldsheet fields are described by the bosonic formulation of the string.

It can be shown that when the heterotic string is constructed in this manner, the equation which can be used to calculate the conformal anomaly is

$$c_{total} = c_{bg} + c_{fg} + c_{X^\mu} \cdot D + c_{\psi^\mu} \cdot D \ , \tag{3.1}$$

where D is the number of spacetime dimensions and c_{bg} and c_{fg} is the contribution from bosonic and fermionic ghosts respectively [4]. The left- and right-moving conformal anomalies of the heterotic string are therefore

$$\begin{aligned} c_L &= -26 + 11 + D + \frac{D}{2} \ , \\ c_R &= -26 + D \ . \end{aligned} \tag{3.2}$$

The conformal anomalies are then cancelled by requiring both equations be equal to zero and solving for D. The left-moving sector has the critical dimension $D = 10$, which leads to the consideration of the superstring fields X_+^μ and ψ_+^μ where $\mu = 0, \ldots, 9$. The right-moving sector has the critical dimension $D = 26$, which consists of ten bosonic fields X_-^μ, where $\mu = 0, \ldots, 9$, along with 32 Majorana-Weyl free fermions denoted by λ_-^i. For the right-movers, 32 Majorana-Weyl free fermions are necessary to cancel the conformal anomaly as they carry a conformal weight of $\frac{1}{2}$.

It should be noted that this theory still contains ten spacetime dimensions as the coordinates X^μ in both the left- and right-moving sectors have the spacetime index $\mu = 0, \ldots, 9$, whereas the internal fermions λ_-^i do not contain a spacetime index.

Summarising, this ten-dimensional theory contains the fields

$$\begin{aligned} &X_+^\mu \ , \ \psi_+^\mu \text{ in the left-moving sector,} \\ &X_-^\mu \ , \ \lambda_-^i \text{ in the right-moving sector,} \end{aligned} \tag{3.3}$$

where $\mu = 0, \ldots, 9$ and $i = 1, \ldots, 32$. The action for the heterotic string is therefore

$$S = \frac{1}{\pi} \int d^2\sigma \left(2\partial_- X_\mu \partial_+ X^\mu + i\psi^\mu \partial_- \psi_\mu + i \sum_{i=1}^{32} \lambda^i \partial_+ \lambda^i \right). \tag{3.4}$$

3.2 The Free Fermionic Formulation of the Heterotic String

In the free fermionic formulation, the extra degrees of freedom required in order to cancel the conformal anomalies are interpreted as free fermions

[4]The term ghosts refers to Fadeev-Popov ghost fields which arise from path integral quantisation of the string. This will not be discussed further in this work, but the method of path integral quantisation can be found in references [18, 19]

propagating on the string worldsheet. This leads to the modification of equation (3.2) in the following manner

$$\begin{aligned} c_L &= -26 + 11 + D + \frac{D}{2} + \frac{N_{f_L}}{2} \ , \\ c_R &= -26 + D + \frac{N_{f_R}}{2} \ , \end{aligned} \tag{3.5}$$

where N_{f_L} and N_{f_R} is the number of left- and right-moving free fermions respectively. In the same procedure as presented above, cancelling the conformal anomalies involves setting both equations equal to zero and solving. Formulating the theory directly in four spacetime dimensions ($i.e$ $D = 4$) and solving the equations gives the results

$$N_{f_L} = 18 \ \text{ and } \ N_{f_R} = 44 \ . \tag{3.6}$$

Therefore, 18 real left-moving and 44 real right-moving Majorana-Weyl fermions are necessary to cancel the conformal anomalies[5]. Therefore, the four dimensional theory contains the fields

$$\begin{aligned} &X^\mu_+ \ , \ \psi^\mu_+ \ , \ \lambda^i_+ \text{ in the left-moving sector,} \\ &X^\mu_- \ , \ \lambda^j_- \text{ in the right-moving sector,} \end{aligned} \tag{3.7}$$

where $\mu = 0, \ldots, 3$, $i = 1, \ldots, 18$ and $j = 1, \ldots, 44$.

It is now instructive to make a coordinate change in order to further construct the theory using complex coordinates. This coordinate change is performed by defining the complex coordinates

$$z = \tau + i\sigma \ \text{ and } \ \bar{z} = \tau - i\sigma \ , \tag{3.8}$$

which leads to the redefinition of the fields as such

$$\begin{aligned} X^\mu(z, \bar{z}), &\quad \mu = 1, 2 \ , \\ \psi^\mu(z), &\quad \mu = 1, 2 \ , \\ \lambda^i(z), &\quad i = 1, \ldots, 18 \ , \\ \bar{\lambda}^j(\bar{z}), &\quad j = 1, \ldots, 44 \ . \end{aligned} \tag{3.9}$$

[5]These are Majorana-Weyl fermions as they satisfy the same conditions as the fermionic fields which were introduced in section 2.2.2

The spacetime fermions and bosons now have only two degrees of freedom, which are the transverse coordinates.

The action for the heterotic string in this formulation is therefore

$$S = \frac{1}{\pi}\int d^2z\left(\partial_z X_\mu \partial_{\bar{z}} X^\mu - 2i\psi^\mu\partial_z\psi_\mu - 2i\sum_{i=1}^{18}\lambda^i\partial_z\lambda^i - 2i\sum_{j=1}^{44}\bar{\lambda}^j\partial_{\bar{z}}\bar{\lambda}^j\right) . \tag{3.10}$$

When the heterotic string is constructed in this way, it is referred to as the free fermionic formulation.

3.3 Free Fermionic Formulation and the Partition Function

Here the conventional notation is introduced, as is commonly used in the literature. This consists of the left-moving bosonic components X^μ and their supersymmetric partners ψ^μ, which carry a spacetime index $\mu = 0, \ldots, 3$. The left-moving free fermions λ^i are split into three groupings, as can be seen in table 6. The right-movers consist of the bosonic coordinates $\overline{X}^\mu$ which also carry a spacetime index $\mu = 0, \ldots, 3$, along with the free fermions $\overline{\lambda}^j$ which are split into 12 real fermions and 16 complex fermions. This can be be seen in table 6, where the descriptions motivate why certain free fermions are grouped in this manner. This notation convention will be used for the remainder of this document.

Following the worldsheet content as defined in table 6, it is noted that any two of the real fermions can form a single complex fermion according to the following prescription

$$\begin{aligned} \lambda_{ab} &= \frac{1}{\sqrt{2}}(\lambda_a + i\lambda_b) \ , \\ \lambda^*_{ab} &= \frac{1}{\sqrt{2}}(\lambda_a - i\lambda_b) \ . \end{aligned} \tag{3.11}$$

This will be a useful tool for model building in later chapters.

A mention should also be made with regards to the supercurrent T_F. In the case of the heterotic string, only the left-moving sector is supersymmetric. The left-moving sector consists of the spacetime coordinates,

	Notation	Description
Left-Movers (SUSY Sector)	X^{μ} ψ^{μ}	Bosonic coordinates, where $\mu = 0, \ldots, 3$ Majorana-Weyl superpartners of the bosonic coordinates, where $\mu = 0, \ldots, 3$
	$\chi^{1,\ldots,6}$ $y^{1,\ldots,6}$, $w^{1,\ldots,6}$	Real Majorana-Weyl superpartners to the six compactified dimensions in the bosonic formulation Real Majorana-Weyl fermions which describe the six compactified dimensions
Right-Movers (Non-SUSY Sector)	$\overline{X}^{\mu}$ $\overline{y}^{1,\ldots,6}, \overline{w}^{1,\ldots,6}$	Bosonic coordinates, where $\mu = 0, \ldots, 3$ Real Majorana-Weyl fermions which describe the six compactified dimensions
	$\overline{\psi}^{1,\ldots,5}, \overline{\eta}^{1,2,3}$ $\overline{\phi}^{1,\ldots,8}$	Complex fermions which describe the visible gauge sector Complex fermions which describe the hidden gauge sector

Table 6: *The worldsheet content is displayed and the standard notation from the literature is introduced.*

their supersymmetric partners and 18 real fermions which non-linearly realise supersymmetry [29]. When defined generally, the supercurrent is

$$T_F = \psi^{\mu}\partial X_{\mu} + f_{abc}\psi^a\psi^b\psi^c \ , \tag{3.12}$$

where f_{abc} are structure constants of a semi-simple Lie group G with 18 generators. It can be shown that all other forms of the supercurrent define string theories which only contain massive fermions and broken spacetime supersymmetry [31]. There are therefore only three admissible groups for the choice of G, these are $SU(2)^6$, $SU(4) \times SU(2)$ and $O(5) \times SU(3)$. The models built in later chapters will only use the case where the group G is $SU(2)^6$ in the adjoint representation. This means the fermions $\chi^i y^i w^i$ transforms with the same sign as $\psi^{\mu}\partial X_{\mu}$ and therefore ensures a well defined supercurrent. Specifically, the supercurrent used is

$$T_F = \psi^{\mu}\partial X_{\mu} + \sum_i \chi^i y^i w^i \ , \tag{3.13}$$

where $i = 1, \ldots, 6$. In this form, the supercurrent is unique (up to a sign) and therefore preserves worldsheet supersymmetry [16].

3.3.1 String Amplitude

When considering the Polyakov approach, string theory is formulated as a perturbative sum over the path integral of the string worldsheet. The string worldsheet therefore defines a genus-g Riemann surface. Using conformal invariance, the string states can be described as vertex operators on this genus-g Riemann surface. In this approach to string theory, the string amplitude can be calculated according to the following equation

$$A_n = \sum_{g=0}^{\infty} \int \mathcal{D}h \mathcal{D}X^{\mu} \int d^2 z_1 \ldots d^2 z_n V_1(z_1, \bar{z}_1) \ldots V_n(z_n, \bar{z}_n) \qquad (3.14)$$

where A_n is the string amplitude, g is the genus of the Riemann surface, h is the worldsheet metric and V_i are the vertex operators of the external string states. As there exist symmetries of this amplitude, the integration is performed over physically inequivalent paths to avoid the overcounting of identical physical states in the partition function.

Using conformal invariance, it is found at tree level (where $g = 0$) that the string amplitude is mapped to a Riemann surface which is topologically a sphere. For the one-loop amplitude (where $g = 1$) the string worldsheet maps to a Riemann surface which is topologically a torus. Multi-loop amplitudes (where $g \geq 2$) are isomorphic to a linear chain of g number of tori.

3.3.2 The Torus and Modular Invariance

Before detailing the partition function of the theory, the torus of the one-loop amplitude is considered before introducing the constraints necessary to preserve modular invariance.

To find these constraints, the one-loop string amplitude with no external states is considered. This amplitude is therefore isomorphic to the torus.

The torus can be mapped to the complex plane by cutting along the two non-contractible loops, which are displayed in figure 3. Unravelling

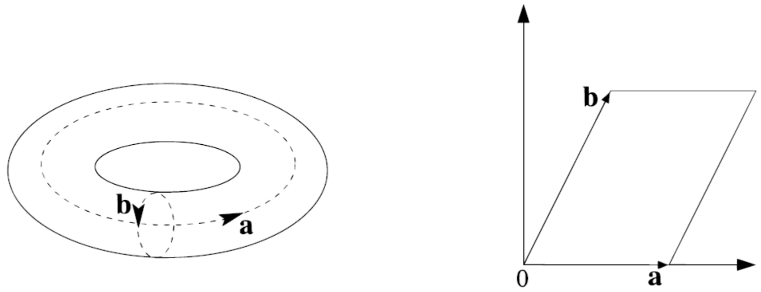

Figure 3: *The two non-contractible loops of the torus described by a and b can be cut along and form a parallelogram on the complex plane.*

the torus along these two directions leads to the torus being identified as a parallelogram in the complex plane. Two lengths can be associated with the two non-contractible loops of the torus, which are now the two non-parallel lengths of the parallelogram in the complex plane. These lengths are denoted by λ_1, λ_2 and are finite, non-zero and periodic. As the parallelogram in the complex plane is periodic in lengths λ_1, λ_2, for a point z in the complex plane the following identification can be made

$$z \sim z + \lambda_1 \ , \quad z \sim z + \lambda_2 \ . \tag{3.15}$$

Furthermore, a two-dimensional lattice $\Lambda_{(\lambda_1,\lambda_2)}$ can be defined in the complex plane by

$$\Lambda_{(\lambda_1,\lambda_2)} = \{m\lambda_1 + n\lambda_2;\ m, n \in \mathbb{Z}\} \ , \tag{3.16}$$

where it can be seen that the torus is modular with respect to the lattice defined in this way.

By applying the reparameterisation $z \to \frac{z}{\lambda_2}$ to equation (3.15), the lattice is modular with the periods

$$m\frac{\lambda_1}{\lambda_2} \quad \text{and} \quad n \ ,$$

the smallest unit of which is when $m, n = 1$, leading to the conclusion that the torus has the periods 1 and $\frac{\lambda_1}{\lambda_2}$. The definition $\tau = \frac{\lambda_1}{\lambda_2}$ can be made and τ is the commonly called the *modular parameter*[6]. The torus can

[6]The quantity τ can also be called the *complex structure* or *Teichmüller parameter [28].*

therefore be shown to be invariant under the following transformations

$$\begin{aligned} &T: \quad \tau \to \tau + 1 \quad &&\text{redefines the same torus,} \\ &S: \quad \tau \to -\tfrac{1}{\tau} \quad &&\text{swaps the coordinates and reorients the torus.} \end{aligned} \tag{3.17}$$

These two transformations generate the modular group which possesses the group algebra $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$. Explicitly,

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.18}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

The fundamental domain of the modular parameter is therefore

$$\mathcal{F} = \{\tau \in \mathbb{C} \ : \ |\tau| \geq 1, \ -\frac{1}{2} < \tau_1 \leq \frac{1}{2}, \ \tau_2 > 0\} \ . \tag{3.19}$$

Due to the presence of the T and S transformations, any torus outside of this fundamental domain can always be acted upon by a combination of the modular transformations to become equivalent to a torus inside the fundamental domain. However, tori within the fundamental domain are physically inequivalent and cannot be transformed into one another. All the physically inequivalent tori must therefore be accounted for by integrating over the whole fundamental domain. A graphical representation of τ can be seen in figure 4 and the integration must be performed over the shaded region in order to account for all physically inequivalent tori.

3.3.3 Boundary Conditions

To calculate the one-loop partition function, the boundary conditions of the fermions must first be defined. Each worldsheet fermion can be parallel transported around the two non-contractible loops which define the torus. Under this parallel transportation, each fermion can pick up a shift in phase which is defined in terms of two boundary conditions of that fermion. Specifically, when a fermion is parallel transported around each non-contractible loop, it picks up a phase

$$f \to -e^{i\pi\alpha(f)} f \ . \tag{3.20}$$

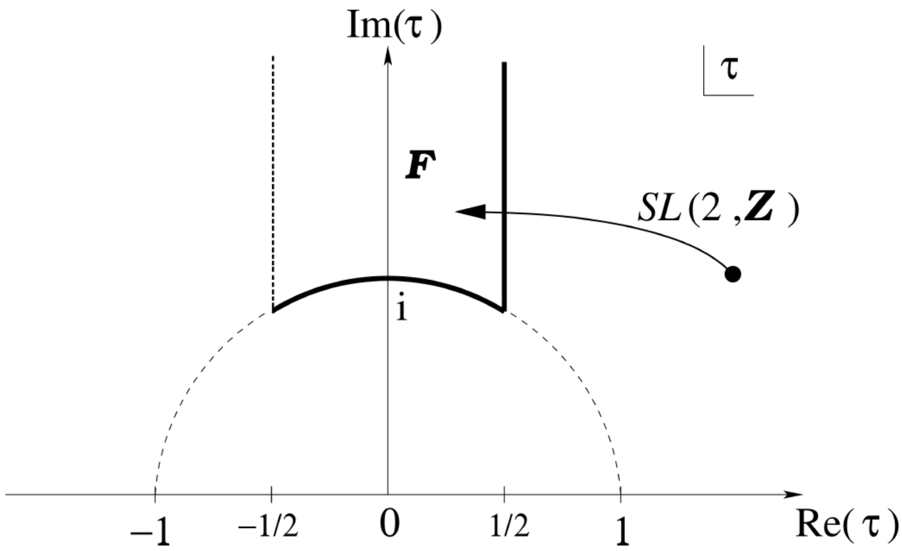

Figure 4: *The fundamental domain of the modular parameter τ of the torus is marked as* $\boldsymbol{F}$*. All tori in the regions outside the fundamental domain can be mapped to tori inside the fundamental domain using the* $PSL(2,\mathbb{Z})$ *transformations.*

Real fermions pick up a phase $\alpha(f) = 0, 1$ which signifies NS or R boundary conditions respectively, whereas complex fermions pick up a phase $\alpha(f) = (-1, 1]$. As there are two non-contractible loops of the torus, there are two phases for each fermion. Therefore, for each fermion these two phases can be written as

$$\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} .$$

If the phases for one non-contractible loop are defined for every fermion, this is called a *spin structure*. A spin structure is therefore a 64-dimensional vector and is written as

$$\alpha = \{\alpha(\psi_1^\mu), \ldots, \alpha(\bar{\phi}^8)\} .$$

As there are two non-contractible loops, the complete spin structure of all the fermions, around both loops, can be defined as two 64-dimensional vectors as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} .$$

3.3.4 The One-Loop Partition Function

The construction of the one-loop partition function can now be considered. As previously stated, the partition function of the one-loop amplitudes with no external states is isomorphic to the torus. In order to build the one-loop partition function completely, the path integral of a torus with a complex parameter $\tau = \tau_1 + i\tau_2$ will be considered. In this construction, τ_1 is associated as a spatial coordinate and τ_2 as a Euclidean time coordinate. When defined in this way, it is found that the generator of translations in time is the Hamiltonian $H = L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24}$ and the generator of translations in space is the momentum operator $P = L_0 - \overline{L}_0$ [24, 28, 37]. These are the zero modes of the energy momentum tensor and are the same expressions defined in equation (2.16).

Construction of the partition function begins by considering the trace over the Hilbert space of the states in the vacuum to vacuum amplitude, written as

$$\begin{aligned} Z(\tau_1, \tau_2) &= \sum_{s \in \mathcal{H}} \langle s| \, e^{2i\pi\tau_1 P} e^{-2i\pi\tau_2 H} \, |s\rangle \\ &= \operatorname*{Tr}_{\mathcal{H}} \, e^{2i\pi\tau_1 P} e^{-2i\pi\tau_2 H} \quad . \end{aligned} \tag{3.21}$$

By defining the quantities $q \equiv e^{2i\pi\tau}$ and $\bar{q} \equiv e^{-2i\pi\bar{\tau}}$ for the right- and left-movers respectively, along with the definition of the generators given above, this can be rewritten as [28, 37]

$$Z(\tau) = q^{\,-c/24} \bar{q}^{\,\bar{c}/24} \operatorname*{Tr}_{\mathcal{H}} \, q^{\,L_0} \bar{q}^{\,\bar{L}_0} \ . \tag{3.22}$$

As it is known how L_0 and $\bar{L}_0$ act on the Fock space (due to discussions in the previous chapter), this result for the partition function can be calculated. However, it remains to specify the boundary conditions for each worldsheet fermion. The total partition function will then be the product of the partition functions for each fermion and their respective boundary conditions. If the time boundary condition is fixed to be anti-periodic (NS) then the partition function is given by the trace of L_0 acting on either the NS or R Fock space, explicitly

$$Z_{\text{NS}}^{\text{NS}}(\tau) = \operatorname*{Tr}_{\text{NS}} q^{L_0 - 1/48} \quad \text{and} \quad Z_{\text{R}}^{\text{NS}}(\tau) = \operatorname*{Tr}_{\text{R}} q^{L_0 - 1/48} \,. \tag{3.23}$$

If the time boundary condition is periodic (Ramond) then the trace definitions are modified to become

$$Z_{\text{NS}}^{\text{R}}(\tau) = \operatorname*{Tr}_{\text{NS}} (-1)^F q^{L_0 - 1/48} \quad \text{and} \quad Z_{\text{R}}^{\text{R}}(\tau) = \operatorname*{Tr}_{\text{R}} (-1)^F q^{L_0 - 1/48} \,, \tag{3.24}$$

where F is the fermion number operator defined as

$$\begin{aligned} F(f) &= +1 \,, \quad \text{if } f \text{ is a fermionic oscillator} \,, \\ F(f^*) &= -1 \,, \quad \text{where } f^* \text{ is the complex conjugate of a fermionic oscillator} \,, \\ F\,|+\rangle_R &= 0 \,, \\ F\,|-\rangle_R &= -1 \,, \end{aligned} \tag{3.25}$$

where $|+\rangle_R = |0\rangle$ is a state of degenerate Ramond vacua with no oscillator and $|-\rangle_R = f_0^\dagger\,|0\rangle$ is a state of degenerate Ramond vacua with a single zero mode oscillator. In building the one-loop partition function, the $c, \bar{c} = \frac{1}{2}$ representations of the Virasoro algebra have been used [37]. These values have therefore been substituted accordingly into equations (3.23) and (3.24).

The complete partition function can now be constructed. The complete one-loop partition function is therefore

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\text{Im }\tau)^2} Z_B^2 \sum_{\substack{\text{spin} \\ \text{structure}}} C\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \prod_{f=1}^{64} Z_F \begin{bmatrix}\alpha(f) \\ \beta(f)\end{bmatrix} \tag{3.26}$$

where

- The integration is the path integral over the Fock space
- The integration measure $\frac{d\tau d\bar{\tau}}{(\text{Im }\tau)^2}$ is invariant under the modular transformations of the torus

- Z_B is the bosonic contribution to the partition function and is defined as

$$Z_B = \frac{1}{\sqrt{|\mathrm{Im}\tau|}\ |\eta(\tau)|^2} \ ,$$

where $\eta(\tau)$ is the Dedekind eta function defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

which uses the definition $q = e^{2i\pi\tau}$. The bosonic contribution is inherently modular invariant.

- $C\binom{\alpha}{\beta}$ are spin structure coefficients which are currently undefined, but will be discussed in section 3.4

- $Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}$ is the fermionic contribution to the partition function of each fermion f. It can be seen that Z_F depends on the boundary conditions α, β of the fermion f. The value of the contribution of each fermion can be calculated using equation (3.22) for each of the possible boundary conditions configurations. Explicitly, these four configurations are

$$Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \sqrt{\frac{\vartheta_3}{\eta}} \ , \tag{3.27a}$$

$$Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{\frac{\vartheta_4}{\eta}} \ , \tag{3.27b}$$

$$Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sqrt{\frac{\vartheta_2}{\eta}} \ , \tag{3.27c}$$

$$Z \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sqrt{\frac{\vartheta_1}{\eta}} \ , \tag{3.27d}$$

where

$$\vartheta_1 = \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad \vartheta_2 = \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad \vartheta_3 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \quad \vartheta_4 = \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{3.28}$$

Here a new modular function ϑ has been defined. The general definition of ϑ with the characteristics θ, ϕ is

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \eta \, e^{2\pi i \theta \phi} q^{\frac{1}{2}\theta^2 - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n+\theta-\frac{1}{2}} e^{2\pi i \phi})(1 + q^{n-\theta-\frac{1}{2}} e^{-2\pi i \phi}) \ , \tag{3.29}$$

which can be equally expressed as an infinite sum

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\theta)^2} e^{2\pi i (n+\theta)\phi} \tag{3.30}$$

due to arguments relating to bosonisation/fermionisation [36]. These equations describe the left-movers and the right-movers are defined similarly.

3.4 Derivation of the Rules of Model Building

Now the one-loop partition function has been specified and the modular invariance of the torus has been discussed, it remains to derive the conditions which arise due to the requirement of modular invariance of the one-loop partition function. The rules outlined below were originally derived by I. Antoniadis, C. Bachas and C. Kounnas in the publications given in reference [29][7].

3.4.1 Modular Invariance of the Partition Function

The one-loop partition function must be modular invariant in order to produce consistent string theories. As stated in the previous section, the integration measure and the bosonic contribution of the partition function are *a priori* modular invariant. It remains to impose modular invariance on the remaining terms in the partition function. By imposing modular invariance additional constraints are introduced which must be satisfied in order to build consistent free fermionic models.

[7]It is noted that these rules were derived by another group in a different formalism given in reference [30].

Under the T transformation $\tau \to \tau+1$, which leaves the torus invariant, the following transformations are true

$$\eta \to e^{i\pi/12}\,\eta\ , \tag{3.31a}$$

$$\vartheta_1 \to e^{i\pi/4}\,\vartheta_1\ , \tag{3.31b}$$

$$\vartheta_2 \to e^{i\pi/4}\,\vartheta_2\ , \tag{3.31c}$$

$$\vartheta_3 \longleftrightarrow \vartheta_4\ , \tag{3.31d}$$

and under the S transformation $\tau \to -\frac{1}{\tau}$ the following transformations are true

$$\eta \to (-i\tau)^{1/2}\,\eta\ , \tag{3.32a}$$

$$\frac{\vartheta_1}{\eta} \to e^{-i\pi/2}\,\frac{\vartheta_1}{\eta}\ , \tag{3.32b}$$

$$\frac{\vartheta_2}{\eta} \longleftrightarrow \frac{\vartheta_4}{\eta}\ , \tag{3.32c}$$

$$\frac{\vartheta_3}{\eta} \to \frac{\vartheta_3}{\eta}\ . \tag{3.32d}$$

The partition function is a product of the spin structures of 64 fermions. By performing the modular transformations described as S and T, one spin structure can transition to another *i.e.* one product of ϑ_i functions will transition to another product of ϑ_i functions.

Modular invariance requires that spin structures related by a transformation must contribute to the partition function with an equal weight. This is accounted for in the partition function by the spin structure coefficients $C\binom{\alpha}{\beta}$. In order to ensure modular invariance, the following constraints on the coefficients must be imposed

$$C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -e^{\frac{i\pi}{4}(\alpha\cdot\alpha+\mathbb{1}\cdot\mathbb{1})}C\begin{pmatrix}\alpha\\ \beta-\alpha+\mathbb{1}\end{pmatrix}\ , \tag{3.33}$$

$$C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = e^{\frac{i\pi}{2}\alpha\cdot\beta}C\begin{pmatrix}\beta\\ \alpha\end{pmatrix}^*\ , \tag{3.34}$$

where the vector $\mathbb{1}$ is a 64-dimensional vector in which every fermion has periodic boundary conditions and the Lorentzian product $\alpha\cdot\beta$ is defined as

$$\alpha \cdot \beta = \left\{ \sum_{\text{Complex Left}} + \frac{1}{2} \sum_{\text{Real Left}} - \left(\sum_{\text{Complex Right}} + \frac{1}{2} \sum_{\text{Real Right}} \right) \right\} \alpha(f)\beta(f) \, . \tag{3.35}$$

The next constraint considered regards higher order loops and is therefore not necessary for calculations regarding the one-loop partition function. However, it is instructive in deriving further constraints on the one-loop coefficients included in the one-loop partition function. As stated previously, a higher loop calculation can be considered as a linear chain of tori and the spin structure is found by specifying the spin structure of every fermion on each torus separately. This leads to the constraint in the two loop case being

$$C\binom{\alpha}{\beta} C\binom{\alpha'}{\beta'} = \delta_\alpha \delta_{\alpha'} \, e^{-\frac{i\pi}{2}\alpha\cdot\alpha'} C\binom{\alpha}{\alpha' + \beta} C\binom{\alpha'}{\alpha + \beta'} \tag{3.36}$$

where δ_α is the spacetime spin statistics index defined by

$$\delta_\alpha = e^{i\pi\alpha(\psi^\mu_{1,2})} = \begin{cases} -1 & \text{if } \alpha(\psi^\mu_{1,2}) = 1 \, , \\ +1 & \text{if } \alpha(\psi^\mu_{1,2}) = 0 \, . \end{cases} \tag{3.37}$$

By considering equation (3.36) and setting $\alpha' = \alpha$ and $\beta = \beta' = 0$, as well as using equation (3.34), the following result can be found

$$C\binom{\alpha}{0}^2 = \delta_\alpha \, C\binom{\alpha}{0} C\binom{0}{0} \, . \tag{3.38}$$

This result admits the two solutions

$$C\binom{\alpha}{0} = 0 \qquad \text{or} \qquad C\binom{\alpha}{0} = \delta_\alpha C\binom{0}{0} \, .$$

The normalisation $C\binom{0}{0} = 1$ is free to be made and the first solution will be discarded from further analysis. Using the second result, a set of vectors can be defined as

$$\Xi = \left\{ \alpha \middle| C\binom{\alpha}{0} = \delta_\alpha \right\} \, . \tag{3.39}$$

It can then be shown that the set of vectors Ξ form an Abelian additive group if the group action is defined as the standard addition of the boundary conditions of each fermion separately [37]. If Ξ is taken to be finite, *i.e.* the boundary conditions in the vector α are rational, then Ξ is found to be isomorphic to a direct sum of $\mathbb{Z}_N$ factors [38]

$$\Xi = \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_k} \ . \tag{3.40}$$

Therefore, there exists a basis of vectors $\{b_1, \ldots, b_k\}$ which generates the additive group Ξ such that [29]

$$\sum_{i=1}^{k} m_i b_i = 0 \qquad \text{if and only if } m_i = 0 \mod N_i \quad \forall i \ , \tag{3.41}$$

where N_i is the smallest positive integer for which $N_i b_i = 0$ is true. It should be noted that the condition $\mathbb{1} \in \Xi$ is true, *i.e.* the basis vector $\mathbb{1}$ must be included in the additive group. This is a direct consequence of equation (3.33) and can be derived by setting $\alpha = \beta = 0$ in the equation. For convenience, the choice $b_1 = \mathbb{1}$ will be used.

In fact, equation (3.36) can be written in a more general form by utilising the result found in equation (3.38). For the case where $\alpha, \beta, \gamma \in \Xi$

$$C\begin{pmatrix} \alpha \\ \beta + \gamma \end{pmatrix} = \delta_\alpha C\begin{pmatrix} \alpha \\ \beta \end{pmatrix} C\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \ . \tag{3.42}$$

Using this result and the fact that β generates a finite group of order N_β [37], the phase can be expressed as

$$\begin{aligned} C\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \delta_\alpha e^{\frac{2i\pi}{N_\beta} n} \ , \\ &= \delta_\beta e^{\frac{i\pi}{2}\alpha\cdot\beta} e^{\frac{2i\pi}{N_\alpha} m} \ . \end{aligned} \tag{3.43}$$

Another useful result used in model building can be derived by considering equation (3.33) and setting $\beta = \alpha$. This leads to the equation

$$C\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = -e^{\frac{i\pi}{4}\alpha\cdot\alpha} C\begin{pmatrix} \alpha \\ \mathbb{1} \end{pmatrix} \ , \tag{3.44}$$

where the result $\mathbb{1} \cdot \mathbb{1} = -12$ and therefore $e^{\frac{i\pi}{4} \mathbb{1}\cdot\mathbb{1}} = -1$ has been used.

Finally, it only remains to derive two more conditions on the modular nature of basis vectors. The first condition is found by making the assignments $\alpha = b_i$ and $\beta = b_j$ in equation (3.34), before raising both sides of the equation to the power of the least common multiple between the two basis vectors, denoted by N_{ij}. Using the known result which can be derived from equation (3.42) [29], the equation

$$e^{\frac{i\pi}{2}(N_{ij} b_i \cdot b_j)} = \left(\delta_{b_i} \delta_{b_j}\right)^{N_{ij}} \tag{3.45}$$

is found. As N_{ij} is always an even number, the right hand side of the equation is always positive. Therefore, the solution to this equation gives the constraint required in order to preserve modular invariance

$$N_{ij} b_i \cdot b_j = 0 \mod 4 \, . \tag{3.46}$$

This result applies to all the pairs of elements in the canonical basis. The final constraint is in the case where $i = j$. In the case where N_i is even, equation (3.46) becomes the stronger constraint

$$N_i b_i^2 = 0 \mod 8 \, . \tag{3.47}$$

This completes the derivations of the conditions which are necessary in order to build consistent and modular invariant free fermionic string models. The useful results found in this chapter which are most commonly used in model building are outlined in a more condensed manner in the next chapter.

3.4.2 Hilbert Space

Using the general form of equations (3.23) and (3.24) as well as the definition of the Hamiltonian, the fermionic contribution to the partition function can be expressed in the form

$$Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} = \mathrm{Tr}_\alpha [q^{H_\alpha} e^{i\pi\beta\cdot F_\alpha}] \, . \tag{3.48}$$

This result can be used to rewrite the partition function in equation (3.26) as

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\ \tau)^2} Z_B^2 \sum_{\substack{\text{spin} \\ \text{structure}}} C\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{Tr}_{\mathscr{H}_\alpha} [q^{H_\alpha} e^{i\pi\beta\cdot F_\alpha}] \tag{3.49}$$

where $\mathscr{H}_\alpha$ is the Hilbert space sector defined by the vector $\alpha = \sum_i n_i b_i$. H_α is the Hamiltonian and $\beta \cdot F_\alpha$ is the Lorentzian product of the vector β and the fermion number operator F_α defined similarly to equation (3.35). The notation here is changed in order to account for a model which consists of basis vectors b_i. Noting that the basis vectors b_i are the generators of a discrete group $\mathbb{Z}_{N_i}$ [37] and using equation (3.42), the partition function is found to be

$$\begin{aligned} Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\text{Im }\tau)^2} Z_B^2 \sum_{\alpha\in\Xi} \delta_\alpha \,\text{Tr}\Bigg\{ \prod_{b_i} \Bigg(\delta_\alpha C\binom{\alpha}{b_i} e^{i\pi b_i \cdot F_\alpha} + \dots \\ \dots + \left(\delta_\alpha C\binom{\alpha}{b_i} e^{i\pi b_i \cdot F_\alpha} \right)^{N_i - 1} + 1 \Bigg) q^{H_\alpha} \Bigg\} , \end{aligned} \tag{3.50}$$

where it can be seen that the sum is finite. The states which are included in the spectrum are those which satisfy the Generalised GSO (GGSO) projection equation

$$e^{i\pi b_i \cdot F_\alpha} \left|s\right\rangle_\alpha = \delta_\alpha C\binom{\alpha}{b_i}^* \left|s\right\rangle_\alpha \ . \tag{3.51}$$

Therefore, the complete Hilbert space of the model is

$$\mathscr{H} = \bigoplus_{\alpha\in\Xi} \prod_{i=1}^{k} \left\{ e^{i\pi b_i \cdot F_\alpha} = \delta_\alpha C\binom{\alpha}{b_i}^* \right\} \mathscr{H}_\alpha \ . \tag{3.52}$$

4 Rules on Model Construction

In this chapter the rules of model construction will be recalled and outlined in a condensed format. The chapter then concludes by detailing the construction of a simple model consisting of two basis vectors in order to outline the main process of model building.

A model requires the definition of two sets of quantities in order to be fully specified. Firstly, a set of basis vectors must be defined, where every basis vector consists of the boundary conditions of each of the free fermions. Secondly, the Generalised GSO (GGSO) phases $C\binom{v_i}{v_j}$ between these basis vectors must be specified.

When each fermion is propagated around the non-contractible loops of the torus associated with the one-loop partition function, each fermion obtains a non-trivial phase. These phases are commonly referred to as the boundary conditions of the free fermions. Basis vectors are defined as the set of boundary conditions for each free fermion. Therefore, each basis vector is a collection of 64 boundary conditions. Explicitly, basis vectors b_i are defined as

$$b_i = \{\alpha(\psi_1^\mu), \dots, \alpha(w^6) \mid \alpha(\overline{y}^1), \dots, \alpha(\overline{\phi}^8)\} \tag{4.1}$$

where $\alpha(f)$ is the boundary condition of the fermion f defined as

$$f \to -e^{i\pi\alpha(f)} f. \tag{4.2}$$

The boundary conditions $\alpha(f)$ can take the values $0, 1$ or $\pm\frac{1}{2}$, meaning the fermions are anti-periodic, periodic or complex respectively.

The basis vectors form an additive group Ξ which is defined as

$$\Xi = \sum_{i=1}^{n} m_i b_i \qquad \text{where } m_i = 0, \dots, N_i - 1, \tag{4.3}$$

where N_i is the smallest positive integer which satisfies $N_i b_i = 0$.

The scalar product used in this definition and those in the following rules is given by

$$b_i \cdot b_j = \left\{ \sum_{\text{Complex Left}} + \frac{1}{2} \sum_{\text{Real Left}} - \left(\sum_{\text{Complex Right}} + \frac{1}{2} \sum_{\text{Real Right}} \right) \right\} b_i(f) b_j(f) \tag{4.4}$$

where the sum over 'complex left' refers to the sum over all left-moving complex fermions, the sum over 'real left' refers to sum over all left-moving real fermions *etc.*

4.1 Rules on the Basis Vectors

Due to the constraints outlined in the previous chapter, all basis vectors must conform to the following rules in order to preserve modular invariance

$$1) \quad \sum_{i=1}^{n} m_i b_i = 0 \qquad \text{where } m_i = 0 \mod N_i \ , \ \forall i \tag{4.5a}$$

$$2) \quad \mathbb{1} \in \Xi \tag{4.5b}$$

$$3) \quad N_{ij} b_i \cdot b_j = 0 \mod 4 \tag{4.5c}$$

$$4) \quad N_i b_i^2 = 0 \mod 8 \qquad \text{if } N_i \text{ is even} \tag{4.5d}$$

$$5) \quad \text{There must be an even number of real fermions} \tag{4.5e}$$

The second rule states the basis vector which has periodic boundary conditions for all fermions, defined as $\mathbb{1}$, must be included in the additive group. N_i is the smallest positive integer for which $N_i b_i = 0$ and N_{ij} is the least common multiple of N_i and N_j.

4.2 Rules on the One-Loop Phase Coefficients

Now the basis vectors have been defined and the rules they must satisfy have been outlined, the rules which govern how they intersect must be given. These intersections between the basis vectors are described as the one-loop phases and are denoted by $C\binom{b_i}{b_j}$ in the common notation. The following rules apply to the one-loop phases:

$$1)\quad C\binom{b_i}{b_j} = \delta_{b_i}\, e^{\frac{2i\pi}{N_j}n} = \delta_{b_j} e^{\frac{i\pi}{2}b_i\cdot b_j}\, e^{\frac{2i\pi}{N_i}m} \tag{4.6a}$$

$$2)\quad C\binom{b_i}{b_i} = -e^{\frac{i\pi}{4}b_i\cdot b_i}\, C\binom{b_i}{\mathbb{1}} \tag{4.6b}$$

$$3)\quad C\binom{b_i}{b_j} = e^{\frac{i\pi}{2}b_i\cdot b_j}\, C\binom{b_j}{b_i}^* \tag{4.6c}$$

$$4)\quad C\binom{b_i}{b_j+b_k} = \delta_{b_i}\, C\binom{b_i}{b_j} C\binom{b_i}{b_k} \tag{4.6d}$$

where δ_{b_i} is the spacetime spin statistics index defined as

$$\delta_{b_i} = e^{i\pi b_i(\psi^\mu_{1,2})} = \begin{cases} -1 & b_i(\psi^\mu_{1,2}) = 1 \\ +1 & b_i(\psi^\mu_{1,2}) = 0 \end{cases}. \tag{4.7}$$

This quantity ensures the spacetime fermions and bosons have the correct space-time statistics.

4.3 The GGSO Projection

The next equation to be defined is the Generalised GSO (GGSO) projection. The term 'generalised' refers to an extension of the GSO projection presented in section 2.2.6. Specifically, the GSO projection acts on a single state whereas the Generalised GSO projection is simply the common nomenclature to denote that the projection operator is being applied to a collection of multiple states (*e.g.* a basis vector) at once.

In addition to the arguments presented in section 2.2.6, where it was stated that the GSO projection was necessary in order to preserve modular invariance, it is also necessary so as to avoid the overcounting of states within the spectrum. As the partition function is a sum over the spectrum at all masses, when the partition function is expanded for a sector it can be taken as a sum over the intersection with other sectors. This can result in cancellations of terms between the two sectors which is ultimately reflected in the spectrum. The GGSO projection accounts for those cancellations and therefore prevents the overcounting of states from occurring.

The GGSO projection equation is given by the equation

$$e^{i\pi\, b_i \cdot F_\xi}\, |S_\xi\rangle = \delta_\xi\, C\begin{pmatrix} \xi \\ b_i \end{pmatrix}^* |S_\xi\rangle \tag{4.8}$$

where F_ξ is the fermion number operator defined in equation (3.25) and $|S_\xi\rangle$ is the state in the sector $\xi \in \Xi$. If a state satisfies this equation, this state contributes to the one-loop partition function of the model. Therefore, the state is contained in the spectrum and is said to be 'kept in'. If the state does not satisfy the equation, the state is said to be 'projected out' and no longer appears in the spectrum as it does not contribute to the one-loop partition function.

4.4 The Massless Spectrum

The necessary components needed in order to define a model, as well as the rules these components need to satisfy, have been outlined above. Presuming a model is well defined, *i.e* it satisfies the above constraints, the spectrum of the model can begin to be calculated.

As the heterotic string is being considered, the left- and right-moving masses can be considered separately. However, the Virasoro condition must be satisfied, which states that the mass squared of the left- and right-moving modes must be equal in order to provide a physically viable string model. In effect, the condition $M_L^2 = M_R^2$ must hold. The left- and right-moving masses of a sector, defined as ξ_L and ξ_R respectively, are calculated using the following equations

$$M_L^2 = -\frac{1}{2} + \frac{\xi_L \cdot \xi_L}{8} + N_L, \tag{4.9a}$$

$$M_R^2 = -1 + \frac{\xi_R \cdot \xi_R}{8} + N_R, \tag{4.9b}$$

where N_L, N_R are the sum over the left- and right-moving oscillators respectively, given explicitly as

$$N_L = \sum \nu_L = \sum_{\substack{f \\ L-osc}} \nu_f + \sum_{\substack{f^* \\ L-osc}} \nu_{f^*}, \tag{4.10a}$$

$$N_R = \sum \nu_R = \sum_{\substack{f \\ R-osc}} \nu_f + \sum_{\substack{f^* \\ R-osc}} \nu_{f^*}. \tag{4.10b}$$

The frequencies ν_{f,f^*} of the fermionic oscillators for a given fermion f or the complex conjugate f^* are

$$\nu_f = \frac{1+\alpha(f)}{2} \quad , \quad \nu_{f^*} = \frac{1-\alpha(f)}{2} \, . \tag{4.11}$$

The sectors which are interesting phenomenologically are massless, and therefore $M_L^2 = M_R^2 = 0$. This is due to massive sectors obtaining a mass comparable to the Planck scale which is physically unobtainable for experiments. Therefore, in building quasi-realistic heterotic string vacua the restriction is made to only consider sectors which are massless in order to consider only the low energy field content.

4.5 $U(1)$ Charges

It only remains to comment on the $U(1)$ currents generated by the fermions. Every complex fermion f (or two real fermions which have been complexified according to equation (3.11)) generate a $U(1)$ current. The corresponding charge Q of each current can be calculated using the following equation

$$Q(f) = \frac{1}{2}\alpha(f) + F(f) \tag{4.12}$$

where $F(f)$ is the fermion number operator as defined in equation (3.25). It should be noted that the charge Q is the charge of the $U(1)$ current with respect to the unbroken Cartan generators of the four dimensional gauge group [44].

This result is simply stated here, but a more detailed derivation of this result can be found in reference [37].

4.6 Building a Simple Model

This section concerns how to build simple models utilising the rules given in the previous sections of this chapter. The simplest model consists of a single basis vector. Due to the rule described in equation (4.5b), this basis vector must contain periodic boundary conditions for all fermions, which was defined as $\mathbb{1}$.

In the case where the model only contains the basis vector $\mathbb{1}$, there exists only two sectors

$$\Xi = \{\mathbb{1}, NS\}$$

due to equation (4.5a). According to this rule, $m_i = 2$ and therefore $2 \cdot \mathbb{1} = 0$. In this sector all the fermions have anti-periodic boundary conditions. The sector is therefore the Neveu-Schwarz (NS) sector.

The rules on the basis vectors are all trivially satisfied for a model consisting of only the $\mathbb{1}$ basis vector.

We proceed by considering the sector $\mathbb{1}$. In this sector, the left- and right-moving masses are found to be

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{10}{8} + N_L \quad > 0, \\ M_R^2 &= -1 + \frac{44}{8} + N_R \quad > 0. \end{aligned} \tag{4.13}$$

Therefore, the sector $\mathbb{1}$ cannot give rise to massless states and is excluded in the following analysis.

The masses of the NS sector can be calculated using the Virasoro mass equation, which gives the result

$$M_L^2 = -\frac{1}{2} + \frac{0}{8} + N_L = -1 + \frac{0}{8} + N_R = M_R^2 \tag{4.14}$$

As all the fermions in the sector have anti-periodic boundary conditions, the following is found to be true for the oscillators of all fermions

$$\nu_{f,f^*} = \frac{1}{2} \,. \tag{4.15}$$

Massless states can therefore be created by requiring that the sector contains one left-moving fermionic oscillator and either one bosonic or two fermionic right-moving oscillators. The massless states admitted by the NS sector are found to be

- $$\psi^{\mu}_{12}\, \partial \bar{X}^{\nu} \left|0\right\rangle_{NS} \tag{4.16}$$

where ψ^{μ}_{12} is the left-moving fermion and $\partial\bar{X}^{\nu}$ is the bosonic creation operator, which acts as the right-moving boson. The bosonic states which arise from this sector are the graviton, the anti-symmetric tensor and the dilaton, as defined previously in section 2.1.4.

- $$\psi^{\mu}_{12}\ \bar{\Phi}^{a}\ \bar{\Phi}^{b}\,|0\rangle_{NS} \tag{4.17}$$

  where $a, b = 1, \ldots, 44$. This state consists of one left-moving and two right-moving fermionic oscillators. The two right-moving oscillators correspond to gauge bosons which generate an $SO(44)$ gauge group in the adjoint representation.

- $$\{\chi^{i}\ ,\ y^{i}\ ,\ w^{i}\}\ \partial\bar{X}^{\nu}\,|0\rangle_{NS} \tag{4.18}$$

  where $i = 1, \ldots, 6$. This state consists of one left-moving fermionic oscillator and one right-moving bosonic oscillator. The left-moving fermionic oscillators correspond to gauge bosons which generate an $SU(2)^{6}$ in the adjoint representation.

- $$\{\chi^{i}\ ,\ y^{i}\ ,\ w^{i}\}\ \bar{\Phi}^{a}\ \bar{\Phi}^{b}\,|0\rangle_{NS} \tag{4.19}$$

  where $i = 1, \ldots, 6$ and $a, b = 1, \ldots, 44$. This state consists of one left-moving and two right-moving fermionic oscillators. The left-movers correspond to gauge bosons which generate an $SU(2)^{6}$ gauge group in the adjoint representation and the right-movers correspond to gauge bosons which generate an $SO(44)$ gauge group in the adjoint representation.

There also exists tachyonic states in the spectrum which arise due to the following state

$$\overline{\Phi}^{a}\,|0\rangle_{NS}\,. \tag{4.20}$$

The state is tachyonic as $M_L^2 = -\frac{1}{2}$, meaning the ground state is unstable and is therefore an undesirable state to have present in the spectrum. The appearance of tachyonic states are a well known feature of non-supersymmetric string theories [18, 21][8].

In order to find which states are actually present in the spectrum, the GGSO projections of all these states must now be performed. The GGSO projection equation of the basis vector $\mathbb{1}$ on the NS sector is described by the equation

$$e^{i\pi\, \mathbb{1}\cdot F_{NS}}\, |S\rangle_{NS} = \delta_{NS}\, C\binom{NS}{\mathbb{1}}\, |S\rangle_{NS} \tag{4.21}$$

where the states $|S\rangle_{NS}$ are given in equations (4.16) - (4.20). Using the definition of the spacetime spin statistics index given in equation (4.7), it is found that $\delta_{NS} = +1$. Using equation (4.6a), it can be shown that

$$C\binom{NS}{b_j} = \delta_{b_j}. \tag{4.22}$$

Again, using the definition of the spacetime spin statistics index δ_{b_j}, the result

$$\begin{aligned} e^{i\pi\, \mathbb{1}\cdot F_{NS}}\, |S\rangle_{NS} &= \delta_{NS}\, \delta_{\mathbb{1}}\, \; |S\rangle_{NS} \\ &= -\, |S\rangle_{NS} \end{aligned} \tag{4.23}$$

is found. The left hand side of the GGSO projection equation is now considered. The values depend on each state being considered from the sector. As an example, the state described in equation (4.16) which contains the graviton, dilaton and anti-symmetric tensor will be considered. Using the definition of the scalar product given in equation (4.4)

$$\begin{aligned} \mathbb{1}\cdot F_{NS} &= \big(\mathbb{1}(\psi^{\mu}_{12})\cdot F_{NS}(\psi^{\mu}_{12}) \; + \; \mathbb{1}(\partial\bar{X}^{\nu})\cdot F_{NS}(\partial\bar{X}^{\nu})\big) \\ &= -1 \end{aligned} \tag{4.24}$$

The GGSO equation is therefore satisfied so this state remains in the spectrum. Applying the same procedure to all other states in the NS sector gives the same result *i.e* no states are projected out.

It should be noted that the graviton cannot be projected out by the GGSO projection. This is due to the fact that the identity given in equation (4.22) is always true for the NS sector. Therefore any free fermionic

[8]It should be noted there are examples of non-supersymmetric heterotic string models, which can be seen in references [32].

string model, *a priori*, always contains at least one graviton and therefore gravity is always present in any of the string theories built.

This simple model contains all the states listed in equations (4.16) - (4.19), but the spectrum still contains a tachyon. In order to remove the tachyon from the spectrum, the method of introducing supersymmetry to the model will be utilised.

4.7 Adding Supersymmetry to the Simple Model

Supersymmetry is included in the model by addition of a basis vector S. The vector S is defined as

$$S = \{\psi^{\mu}_{12}, \chi^{12}, \chi^{34}, \chi^{56}\} \ , \tag{4.25}$$

where the fermions have been complexified.

In the previous model which only contained the sectors $\{\mathbb{1}, NS\}$, it can be seen there were no gravitinos present in the spectrum. In this section we will now see that gravitinos are present in the spectrum and therefore spacetime supersymmetry is realised.

It can easily be shown that the addition of the S basis vector satisfies the rules on the basis vectors listed in equations (4.5a - 4.5e). Inclusion of this basis vector extends the additive group, meaning the following sectors are now present in the model

$$\Xi = \{\mathbb{1},\ NS,\ \mathbb{1} + S,\ S\} \ . \tag{4.26}$$

As in the previous section, the sector $\mathbb{1}$ gives no massless states. The states from the NS sector are the same as in the previous section, but now the GGSO projection of the basis vector S must be performed on the sector. Using equation (4.22) the GGSO projection is found to be

$$\begin{aligned} e^{i\pi S\cdot F_{NS}} \left|s\right\rangle_{NS} &= \delta_{NS}\, C\binom{NS}{S} \left|s\right\rangle_{NS} \\ &= \delta_{NS}\, \delta_{S} \left|s\right\rangle_{NS} \\ &= - \left|s\right\rangle_{NS} \end{aligned} \tag{4.27}$$

It now remains to calculate the left hand side of the GGSO projection equation for the states in the NS sector. For the state given in equation (4.16) the following result is found

$$\begin{aligned} S \cdot F_{NS} &= \big(S(\psi^{\mu}_{12}) \cdot F(\psi^{\mu}_{12}) + S(\partial\bar{X}^{\nu}) \cdot F(\partial\bar{X}^{\nu})\big) \\ &= 1 \end{aligned} \tag{4.28}$$

meaning the state remains in the spectrum. It should be noted that this result was expected as the graviton state cannot be projected out, as mentioned previously. The state given in equation (4.17) is also found to be kept in the spectrum, as

$$\begin{aligned} S \cdot F_{NS} &= \big(S(\psi^{\mu}_{12}) \cdot F(\psi^{\mu}_{12}) + S(\bar{\Phi}^{a}) \cdot F(\bar{\Phi}^{a}) + S(\bar{\Phi}^{b}) \cdot F(\bar{\Phi}^{b})\big) \\ &= 1 \end{aligned} \tag{4.29}$$

meaning the gauge group of $SO(44)$ in the adjoint representation is kept intact. The result now changes for the following states. Performing the GGSO projection on the states given in equation (4.18) gives the results

$$\chi^{i}\, \partial\bar{X}^{\nu}\, |0\rangle_{NS} : \quad \begin{aligned} S \cdot F_{NS} &= \big(S(\chi^{i}) \cdot F(\chi^{i}) + S(\partial\bar{X}^{\nu}) \cdot F(\partial\bar{X}^{\nu})\big) \\ &= 1 \end{aligned} \tag{4.30a}$$

$$y^{i}\, \partial\bar{X}^{\nu}\, |0\rangle_{NS} : \quad \begin{aligned} S \cdot F_{NS} &= \big(S(y^{i}) \cdot F(y^{i}) + S(\partial\bar{X}^{\nu}) \cdot F(\partial\bar{X}^{\nu})\big) \\ &= 0 \end{aligned} \tag{4.30b}$$

$$w^{i}\, \partial\bar{X}^{\nu}\, |0\rangle_{NS} : \quad \begin{aligned} S \cdot F_{NS} &= \big(S(w^{i}) \cdot F(w^{i}) + S(\partial\bar{X}^{\nu}) \cdot F(\partial\bar{X}^{\nu})\big) \\ &= 0 \end{aligned} \tag{4.30c}$$

where $i = 1,\ldots,6$. It can be seen from these results that only the states in equation (4.30a) survive, while the states in equations (4.30b - 4.30c) are projected out. The result is the same for the states given in equation (4.19), meaning the states which survive the projection are

$$\chi^{1,\ldots,6}\, \bar{\Phi}^{a}\, \bar{\Phi}^{b}\, |0\rangle_{NS} \ . \tag{4.31}$$

The final state to be considered is the tachyon. Calculating the relevant information from the left hand side of the GGSO projection gives

$$\begin{aligned} S \cdot F_{NS} &= \big(S(\bar{\Phi}^a) \cdot F_{NS}(\bar{\Phi}^a)\big) \\ &= 0 \end{aligned} \tag{4.32}$$

and therefore the tachyon is projected out. Addition of the S basis vector will always generate this result. As the tachyon is formed only by a right moving fermion, then $S(\bar{\Phi}^a) = 0$ is always true. The right hand side of the GGSO projection equation is fixed meaning the tachyon can never survive the addition of the supersymmetry basis vector S to the additive group. For the remainder of this document, removal of the tachyonic instability will be achieved by the addition of supersymmetry to the string models. However, it should again be noted that approaches to resolving instabilities in heterotic string theories which are non-supersymmetric have been considered by other groups in the references [32].

This concludes all the states which remain the NS sector after the addition of the supersymmetric S basis vector. It remains to consider if the sectors $\{\mathbb{1} + S, S\}$ admit massless states and to calculate which, if any, of these states remain in the spectrum.

4.7.1 $\mathbb{1} + S$ Sector

The sector $\mathbb{1} + S$ is found to only produce massive states. When the Virasoro condition is considered,

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{6}{8} + N_L \quad > 0 \, , \\ M_R^2 &= -1 + \frac{22}{8} + N_R \quad > 0 \, . \end{aligned} \tag{4.33}$$

This sector is therefore omitted from further analysis.

4.7.2 S Sector

The final sector to be considered is S and when the left- and right-mass squared is calculated as

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{4}{8} + N_L \ , \\ M_R^2 &= -1 + \frac{0}{8} + N_R \ , \end{aligned} \tag{4.34}$$

it can be seen that massless states can be formed using either one right-moving bosonic oscillator or two right-moving fermionic oscillators. The left-moving sector can be seen to be Ramond and requires no oscillators. This means the possible states which arise from the S sector are

- $$\left|S\right\rangle_L \ \partial\bar{X}^\mu \left|0\right\rangle_{NS} \tag{4.35}$$

  These states are the supersymmetric partners of the gravitons and are therefore referred to as gravitinos. The gravitinos possess spin $\frac{3}{2}$.

- $$\left|S\right\rangle_L \ \bar{\Phi}^a\bar{\Phi}^b \left|0\right\rangle_{NS} \tag{4.36}$$

  These states are the supersymmetric partners of the gauge bosons defined in equations (4.17) and (4.19). These are therefore referred to as gauginos and possess spin $\frac{1}{2}$.

In the above two states, L denotes the left-moving components of the sector and $a, b = 0, \ldots, 44$.

In order to represent the left-moving fermions in a clearer manner, a combinatorial notation will be introduced. As the left-moving component of the sector is

$$\left|S\right\rangle_L = \psi^\mu_{12}\, \chi^{12}\chi^{34}\chi^{56} \left|0\right\rangle_{NS} \tag{4.37}$$

and each complexified fermion is in the state $\left|\pm\right\rangle$ before the GGSO projection, the notation

$$\binom{\text{Total}}{i} \tag{4.38}$$

is introduced, where 'total' is the total number of fermions being considered and $i = (0, \ldots, \text{total})$ counts the number of 'negatives' ($i.e\ |-\rangle$) in the sector. For example,

$$\begin{aligned} \binom{4}{0} &= |+\rangle\,|+\rangle\,|+\rangle\,|+\rangle \ , \\ \binom{4}{3} &= |+\rangle\,|-\rangle\,|-\rangle\,|-\rangle \ , \end{aligned} \tag{4.39}$$

and any possible cyclic permutations in the position of the negative states.

Now the GGSO projection of these states can be considered. The GGSO projection equation for the projection with the basis vector $\mathbb{1}$ on the sector S is

$$\begin{aligned} e^{i\pi\ \mathbb{1}\cdot F_S}\,|s\rangle_S &= \delta_S\ C\binom{S}{\mathbb{1}}^{*}|s\rangle_S \\ &= -C\binom{S}{\mathbb{1}}^{*}|s\rangle_S \end{aligned} \tag{4.40}$$

where s denotes the state and S denotes the sector. In contrast to the phase $\binom{\mathbb{1}}{\mathbb{1}}$, which must equal -1 due to arguments presented in the previous section (specifically the result given in equation (4.22)), the phase $C\binom{S}{\mathbb{1}}$ can take the values ± 1. Therefore, a choice must be made. Without loss of generality, the choice for this calculation will be $C\binom{S}{\mathbb{1}} = -1$. This leaves the right hand side of the GGSO projection equation as

$$e^{i\pi\ \mathbb{1}\cdot F_S}\,|s\rangle_S = +\,|s\rangle_S \ . \tag{4.41}$$

Considering the state in equation (4.35), we find

$$\begin{aligned} \mathbb{1}\cdot F_S &= \big(\mathbb{1}(|s\rangle_L)\cdot F(|s\rangle_L) - (\mathbb{1}(\partial\bar{X}^\mu)\cdot F(\partial\bar{X}^\mu)\big) \\ &= \big(\mathbb{1}(|s\rangle_L)\cdot F(|s\rangle_L)\big) \end{aligned} \tag{4.42}$$

where $|s\rangle_L$ is given by equation (4.37). In order to satisfy the GGSO projection equation, the number of negative contributions from the fermions in state $|s\rangle_L$ is found to be even. Therefore, using the combinatorial notation, the states which survive are

$$\begin{aligned} |s\rangle_S &= \left[\binom{4}{even}\right]\partial\bar{X}^\mu\,|0\rangle_{NS} \\ &= \left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\partial\bar{X}^\mu\,|0\rangle_{NS} \ . \end{aligned} \tag{4.43}$$

Similarly, the gaugino states which remain in the spectrum after the GGSO projection with the $\mathbb{1}$ vector are

$$|s\rangle_S = \left[\binom{4}{even} \right] \bar{\Phi}^a \bar{\Phi}^b \, |0\rangle_{NS} \ . \tag{4.44}$$

It now remains to project with the S basis vector. The GGSO projection equation for this case is

$$\begin{aligned} e^{i\pi \ S \cdot F_S} \, |s\rangle_S &= \delta_S \ C\binom{S}{S} \, |s\rangle_S \\ &= -C\binom{S}{S} \, |s\rangle_S \end{aligned} \tag{4.45}$$

In fact, due to the rule given in equation (4.6b)

$$C\binom{S}{S} = C\binom{S}{\mathbb{1}} \ , \tag{4.46}$$

and therefore

$$e^{i\pi \ S \cdot F_S} \, |s\rangle_S = + \, |s\rangle_S \ , \tag{4.47}$$

which necessarily finds that there is no further projections performed by the S vector.

Finally, it is instructive to calculate the number of gravitinos present in the spectrum at this point. This can be found by considering the fermions ψ^μ_{12} separately from the $\chi^{1,\dots,6}$, like so

$$\binom{1}{0}\left[\binom{3}{0} + \binom{3}{2}\right] + \binom{1}{1}\left[\binom{3}{1} + \binom{3}{3}\right] . \tag{4.48}$$

Considering the fermions in this way is valid as the two components of ψ^μ_{12} (as given above) form the two components of a spacetime Weyl spinor [16]. It can now be seen that there are four gravitinos present due to the combinatorics of the states of $\chi^{1,\dots,6}$ *i.e* using the combinatorics equation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \ , \tag{4.49}$$

the fermions $\chi^{1,\dots,6}$ in the first square bracket of equation (4.48) give

$$\frac{3!}{0!(3-0)!} + \frac{3!}{2!(3-2)!} = 4 \ . \tag{4.50}$$

The second square bracket also gives this result. Due to the spectrum containing four gravitinos, the model possesses $\mathcal{N} = 4$ spacetime supersymmetry.

At this point of construction, the spectrum of this model has a gauge group of $SO(44)$, has $\mathcal{N} = 4$ supersymmetry and is free of tachyonic instabilities. However, the model contains no matter. Therefore, this model will form the initial configuration of the Left-Right Symmetric models constructed in the remainder of this document, but more basis vectors must be added in order to obtain quasi-realistic LRS string vacua. This will be the focus of the next chapter.

5 Classification of Left-Right Symmetric Heterotic String Vacua

This chapter concerns the classification of Left-Right Symmetric models in the free fermionic formulation of heterotic string theory. The contents of this chapter is the subject of a research paper published by the author and collaborators and can be found in reference [14].

The classification method of the free fermionic heterotic string vacua is extended to models where the $SO(10)$ GUT symmetry is broken directly at the string scale to the Left-Right Symmetric subgroup. The method involves using a fixed set of basis vectors which are defined by the boundary conditions assigned to the free fermions before enumerating the string vacua by varying the Generalised GSO (GGSO) projection coefficients. It allows the derivation of algebraic expressions for the GGSO projections for each sector that generates massless states in the models. This enables a computerised analysis of the entire massless spectrum of a given choice of GGSO projection coefficients. The total number of vacua in the class of models chosen is $2^{66} \approx 7.38 \times 10^{19}$. A statistical sampling is performed and a sample size of 10^{11} vacua with the Left-Right Symmetric gauge group is extracted. We present the results of the classification, noting that contrary to the previous classification of Pati-Salam models, no three generation exophobic models were found. The results obtained demonstrate the existence of three generation models with the necessary Higgs representations needed for viable spontaneous symmetry breaking and with a leading top quark Yukawa coupling.

5.1 Left-Right Symmetric Free Fermionic Models

This chapter concerns the extension of the free fermionic classification method, utilised in [39, 40, 48, 50, 52], to vacua which possess the Left-Right Symmetric (LRS) subgroup of $SO(10)$.

The free fermionic formulation provides a set of rules which enables extraction of the physical states in a string model and provides a straightforward approach to studying the phenomenological properties of the string vacua. The models are constructed by defining a set of basis vectors and the Generalised Gliozzi-Scherk-Olive (GGSO) projection coefficients of the one-loop partition function. The details are outlined in the following section.

The breaking of the $SO(10)$ GUT symmetry occurs directly at the string scale. All the models which are classified possess $\mathcal{N} = 1$ spacetime supersymmetry and preserve the $SO(10)$ embedding of the weak hypercharge. The unbroken subgroup of $SO(10)$ in the low energy effective field theory considered here is $SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R$. The matter states which give rise to the Standard Model fermionic representations are found in the spinorial **16** representation of $SO(10)$ decomposed under the unbroken $SO(10)$ subgroup. Similarly, the SM light Higgs states occur from the vectorial **10** representation of $SO(10)$.

5.1.1 The Free Fermionic Formulation

The notable features of the free fermionic formulation used in model building and classification will be briefly outlined. A more detailed discussion of these features can be found in reference [29, 30].

The free fermionic formulation of string theory is directly formulated in four spacetime dimensions, whereby the extra degrees of freedom found in string theories are interpreted as free fermions propagating on the two dimensional string worldsheet. The approach considered here utilises the four dimensional heterotic string in the light-cone gauge, meaning there are 20 left-moving and 44 right-moving free fermions introduced to account for all the extra degrees of freedom. In the standard notation the left-movers are represented by ψ^μ_{12} , $\chi^{1,\dots,6}$, $y^{1,\dots,6}$, $w^{1,\dots,6}$ and the right-movers by $\overline{y}^{1,\dots,6}$, $\overline{w}^{1,\dots,6}$, $\overline{\psi}^{1,\dots,5}$, $\overline{\eta}^{1,2,3}$, $\overline{\phi}^{1,\dots,8}$.

When these fermions are parallel transported around the two non-contractible loops of the one-loop partition function, they obtain a non-trivial phase[9]. These phases can be either periodic, anti-periodic or complex, denoted by 0,1 and $\pm\frac{1}{2}$ respectively. The boundary conditions of the fermions are specified in 64-dimensional vectors called 'basis vectors' which are given in the form

$$v_i = \{\alpha_i(f_1), \dots, \alpha_i(f_{20}) \mid \alpha_i(\overline{f}_1), \dots, \alpha_i(\overline{f}_{44})\},$$

where the boundary condition α is defined as the transformation property for a fermion f. Accordingly,

$$f_j \to -e^{i\pi\alpha_i(f_j)} f_j, \qquad j = 1, \dots, 64.$$

Each model is specified by a set of basis vectors $v_1, \dots, v_N$, which must satisfy modular invariance constraints. The basis vectors of the model span a space Ξ, which consists of 2^{N+1} sectors. Each sector is formed as a linear combination of the basis vectors and is given by

$$\xi = \sum_{i=1}^{N} m_j v_i \qquad m_j = 0, 1, \dots, N_j - 1, \tag{5.1}$$

where $N_j \cdot v_j = 0 \mod 2$. The string states in each sector, denoted by $|S_\xi\rangle$, must also conform to modular invariance constraints. This is imposed on the string states in the form of the one-loop GGSO projections via the equation,

$$e^{i\pi v_i \cdot F_\xi} \, |S_\xi\rangle = \delta_\xi \; C \binom{\xi}{v_i}^* |S_\xi\rangle \,, \tag{5.2}$$

where F_ξ is the fermion number operator, $\delta_\xi = \pm 1$ is the space-time spin statistics index and $C\binom{\xi}{v_i} = \pm 1$; $\pm\frac{1}{2}$ is the GGSO projection coefficient. By varying the choice of the GGSO coefficients, distinct vacua of the string model are obtained.

[9]In the common nomenclature, these phases are also referred to as 'boundary conditions' of the free fermions.

Summarising, a model is constructed by using a set of basis vectors v_i and by a set of distinct GGSO projection coefficients $C\binom{v_i}{v_j}$, with $i > j$, of which there are $2^{N(N-1)/2}$.

5.1.2 $SO(10)$ Models

In order to build the Left-Right Symmetric models that are studied in this chapter, a set of thirteen basis vectors are used. The first twelve basis vectors considered generate $SO(10)$ models and are common to the previous publications [40, 46, 50, 52]. These basis vectors are also included in the basis of the LRS models discussed here and are defined as:

$$
\begin{aligned}
v_1 = \mathbb{1} &= \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \\
&\qquad \overline{y}^{1,\dots,6}, \overline{\omega}^{1,\dots,6}, \overline{\eta}^{1,2,3}, \overline{\psi}^{1,\dots,5}, \overline{\phi}^{1,\dots,8}\}, \\
v_2 = S &= \{\psi^\mu, \chi^{1,\dots,6}\}, \\
v_{2+i} = e_i &= \{y^i, \omega^i \mid \overline{y}^i, \overline{\omega}^i\},\ i = 1, \dots, 6, \\
v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} \mid \overline{y}^{34}, \overline{y}^{56}, \overline{\eta}^1, \overline{\psi}^{1,\dots,5}\}, \\
v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} \mid \overline{y}^{12}, \overline{y}^{56}, \overline{\eta}^2, \overline{\psi}^{1,\dots,5}\}, \\
v_{11} = z_1 &= \{\overline{\phi}^{1,\dots,4}\}, \\
v_{12} = z_2 &= \{\overline{\phi}^{5,\dots,8}\},
\end{aligned}
\tag{5.3}
$$

where the fermions which appear in the basis vectors have periodic (Ramond) boundary conditions, whereas those not included have antiperiodic (Neveu-Schwarz) boundary conditions.

The basis vector $\mathbb{1}$ is required by the rules set out in the papers listed in reference [29, 30] and generates a model with an $SO(44)$ gauge group from the Neveu-Schwarz (NS) sector. Addition of the S basis vector generates $\mathcal{N} = 4$ space-time supersymmetry and leaves the gauge group intact. The e_i vectors break the gauge group to $SO(32) \times U(1)^6$ but preserve the $\mathcal{N} = 4$ supersymmetry. These vectors correspond to all the possible internal symmetric shifts of the six internal bosonic coordinates. Addition of the vectors b_1 and b_2 corresponds to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twists and breaks the spacetime supersymmetry firstly to $\mathcal{N}=2$ and subsequently to $\mathcal{N}=1$. They also break the $U(1)^6$ gauge symmetry, therefore reducing the rank of the gauge group, while simultaneously decomposing the $SO(32)$ to $SO(10)\times U(1)^3\times SO(16)$. Addition of the basis vectors z_1 and z_2 then break the hidden $SO(16)$ gauge group, generated by the fermions $\overline{\phi}^{1,\ldots,8}$, to $SO(8)\times SO(8)$. The untwisted vector bosons present due to this choice of basis vectors generate the gauge group $SO(10)\times U(1)^3\times SO(8)^2$ in the adjoint representation.

At this point, it is instructive to briefly mention orbifolds and how some of the underlying structure of the FFF can be interpreted in the orbifold construction.

In the formulation used throughout the following work, the six internal dimensions are compactified on a flat six-torus $\mathbf{T}^6$. This is factorisable, as the $\mathbf{T}^6$ structure can be split as $\mathbf{T}^6=\mathbf{T}^4\times\mathbf{T}^2=\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$ by applying the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold to the six-torus. The $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold action acts on a $\mathbf{T}^4$, therefore distinguishing between the $\mathbf{T}^4$ the orbifold is acting on and the $\mathbf{T}^2$ it is not acting on. This introduces a $\mathbb{Z}_2\times\mathbb{Z}_2$ 'twist' of the $\mathbf{T}^6$. There are three combinations of how $\mathbb{Z}_2\times\mathbb{Z}_2$ can act on $\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$, which generates three distinct twists and results in three orbifold planes. The three orbifold planes are defined as $B^{1,2,3}_{pqrs}$ and are commonly referred to as the twisted sectors. In contrast, sectors which give rise to states invariant under the orbifold action are designated untwisted sectors.

The orbifold action of $\mathbb{Z}_2$ on each of the target spaces ($\mathbf{T}^2$) has four fixed points. Therefore, when considering $\mathbb{Z}_2\times\mathbb{Z}_2$ twists on two of the two-tori, *i.e.* $\mathbf{T}^4=\mathbf{T}^2\times\mathbf{T}^2$, there are $4\times4=16$ fixed points. These sixteen fixed points are represented in the free fermionic formulation by the sixteen different configurations of each of $p,q,r,s=0,1$ which appear in the twisted sectors $B^{1,2,3}_{pqrs}$, presented in section 5.3.2 and the sections following.

Further information on the orbifold formulation can be found in references [28, 37, 53, 54].

5.1.3 Left-Right Symmetric Models

Previous constructions of free fermionic LRS models used two or more basis vectors to break the observable gauge group. Firstly, one basis vector with either the assignment $\overline{\psi}^{1,2,3} = 1$ (as in [44]), or equivalently $\overline{\psi}^{4,5} = 1$ (as in [40]), is used to obtain the $SO(6) \times SO(4)$ Pati-Salam gauge group and a second basis vector with the assignment $\overline{\psi}^{1,2,3} = \pm\frac{1}{2}$ breaks the Pati-Salam gauge group to the LRS.

However, the model under consideration uses only one additional basis vector, given by

$$\alpha = \{\overline{\psi}^{1,2,3} = \frac{1}{2}\ ,\ \overline{\eta}^{1,2,3} = \frac{1}{2}\ ,\ \overline{\phi}^{1,\ldots,6} = \frac{1}{2}\ ,\ \overline{\phi}^{7}\} \tag{5.4}$$

where the restriction that the phase on the complex right-moving fermions is positive is made, *i.e* $\overline{\psi}^{1,2,3} = +\frac{1}{2}$. The assignment of $\overline{\eta}^{1,2,3} = +\frac{1}{2}$ is made due to the constraint that $b_j \cdot \alpha = 0 \mod 1$, where $j = 1, 2, 3$, must be true in order to satisfy modular invariance.

It should be noted that while the assignments on the fermions $\overline{\psi}^{1,2,3}, \overline{\eta}^{1,2,3}$ must be as above, this choice of α is not unique due to possible variations of assignments for the fermions $\overline{\phi}^{1,\ldots,8}$. However, in this paper only models with the α defined above are considered.

With this choice of basis vectors, we note two sectors which are combinations of the basis vectors and facilitate the classification and presentation of the physical spectrum. The first is the composite vector defined as 'b_3' which is given by

$$\begin{aligned} b_3 &= \mathbb{1} + S + \sum_{i=1}^{6} e_i + b_1 + b_2 + z_1 + z_2 \\ &= \{\chi^{12}, \chi^{34}, y^{12}, y^{34},\ |\ \overline{y}^{12}, \overline{y}^{34}, \overline{\psi}^{1,\ldots,5}, \overline{\eta}^{3}\}\ . \end{aligned} \tag{5.5}$$

This combination of basis vectors corresponds to the third twisted plane of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, where the first two are related to b_1 and b_2, respectively. The second composite vector is given by the linear combination denoted by 'x', given by

$$
\begin{aligned}
x &= \mathbb{1} + S + \sum_{i=1}^{6} e_i + z_1 + z_2 \\
&= \{\overline{\psi}^{1,2,3,4,5}, \ \overline{\eta}^{1,2,3}\} \ .
\end{aligned}
\tag{5.6}
$$

This linear combination produces the spinorial 128 multiplet in the 248 adjoint representation of the observable E_8, generated by the subset $\{1, S, x, z_1 + z_2\}$ of the basis set (5.3). It generates the so–called x–map [56] that exchanges spinorial and vectorial representations from the twisted sectors B_j, to be defined below, and $B_j + x$, respectively. It should be noted that this linear combination is not generated in the LRS models of ref. [44] and therefore the models presented there do not admit the x–map. This is an important distinction between the models considered here and those of ref. [44]. We note that the x–map is crucial in our classification method as the sectors $B_j + x$ are those that give rise to the Standard Model electroweak doublets. Therefore, the basis of the models considered consists of the basis vectors $\{\mathbb{1}, S, e_1, e_2, e_3, e_4, e_5, e_6, b_1, b_2, z_1, z_2, \alpha\}$ with two notable linear combinations $\{b_3, x\}$.

5.1.4 GGSO Projections

Now that the basis has been specified, the next components of the model which need defining are the GGSO projection coefficients $C\binom{v_i}{v_j}$ which are necessary in order to completely describe the one-loop partition function.

The GGSO coefficients span a 13×13 matrix. The lower triangle of the matrix containing 78 coefficients is fixed by the corresponding 78 coefficients in the upper triangle by modular invariance constraints. In addition, the phases on the diagonal are also fixed by modular invariance.

Accordingly,

$$\begin{aligned}
&C\binom{e_i}{e_i} = -C\binom{e_i}{\mathbb{1}} \qquad i = 1, \ldots, 6 \\
&C\binom{b_k}{b_k} = C\binom{b_k}{\mathbb{1}} \qquad k = 1, 2 \\
&C\binom{z_k}{z_k} = C\binom{z_k}{\mathbb{1}} \qquad k = 1, 2 \\
&C\binom{\alpha}{\alpha} = C\binom{\alpha}{\mathbb{1}}
\end{aligned} \tag{5.7}$$

To ensure $\mathcal{N} = 1$ supersymmetry, without loss of generality, the following coefficients are fixed

$$C\binom{\mathbb{1}}{\mathbb{1}} = C\binom{S}{\mathbb{1}} = C\binom{S}{S} = C\binom{S}{e_i} = C\binom{S}{b_k} = C\binom{S}{z_k} = C\binom{S}{\alpha} = -1, \tag{5.8}$$

where $i = 1, \ldots, 6$ and $k = 1, 2$. We are therefore left with 66 independent coefficients, which generates $2^{66} \approx 7.38 \times 10^{19}$ distinct string vacua.

It should be noted that all the phases are real and take the discrete values ± 1 except for the phase $C\binom{\mathbb{1}}{\alpha}$ which takes values $\pm i$ due to the fact that $\mathbb{1} \cdot \alpha = -7$.

5.2 String Spectrum

Adapting the methodology of previous cases [40, 46, 50, 52], the sectors which can contribute massless states are enumerated and the corresponding algebraic conditions for the GGSO projections are derived for each sector.

Spacetime vector bosons that arise from the untwisted sector, generate the $SO(10)$ symmetry and its unbroken subgroups. There are further sectors in these models that can give rise to additional physical spacetime vector bosons, which enhance the untwisted gauge symmetry. Furthermore, if the additional spacetime vector bosons are charged with respect to the Cartan generators of the $SO(10)$ GUT symmetry, the unbroken $SO(10)$ subgroup is enhanced. Thus, a pivotal requirement in the construction is that the additional spacetime vector bosons are projected out.

The twisted sectors produce matter multiplets which possess $\mathcal{N} = 1$ supersymmetry and can be grouped depending on which $SO(10)$ subgroup they leave unbroken. Sectors which contain the α basis vector in the linear combination break the $SO(10)$ symmetry to the LRS and gives rise to exotic states. If the linear combination contains 2α then the $SO(10)$ gauge group is broken to the Pati-Salam $SO(6) \times SO(4)$ gauge group and also contains exotics. As α is the only $SO(10)$ breaking basis vector, all the remaining sectors which, *a priori*, do not include α in the linear combination do not break the $SO(10)$ symmetry.

The sectors in a model can be categorised according to the left- and right-moving vacuum. The physical states satisfy the Virasoro condition, defined as

$$M_L^2 = -\frac{1}{2} + \frac{\xi_L \cdot \xi_L}{8} + N_L = -1 + \frac{\xi_R \cdot \xi_R}{8} + N_R = M_R^2 \qquad (5.9)$$

where N_L and N_R are the sums over the left- and right-moving oscillators respectively. Sectors that have the products $\xi_L \cdot \xi_L = 0$ and $\xi_R \cdot \xi_R = 0, 4, 6, 8$ can produce spacetime vector bosons, which determine the gauge symmetry in a given vacuum. Sectors where the products are $\xi_L \cdot \xi_L = 4$ and $\xi_R \cdot \xi_R = 4, 6, 8$ produce matter states which are outlined in section 5.3. All the models considered preserve $\mathcal{N} = 1$ spacetime supersymmetry, which is generated by the basis vector S where the products are $(S_L \cdot S_L \ ; \ S_R \cdot S_R) = (4; 0)$.

5.2.1 The Gauge Symmetry

Vector bosons from the untwisted sector correspond to generators of the following observable and hidden gauge group symmetries

$$\begin{aligned} \text{Observable} \ &: \ SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R \times U(1)_1 \times U(1)_2 \times U(1)_3 \\ \text{Hidden} \ &: \ SU(4) \times U(1)_4 \times SU(2)_5 \times U(1)_5 \times U(1)_7 \times U(1)_8 \end{aligned}$$

and the weak hypercharge is given by[10]

$$U(1)_Y = \frac{1}{3}U(1)_C + \frac{1}{2}U(1)_L. \tag{5.10}$$

Depending on the choice of GGSO projection coefficients, additional spacetime vector bosons may arise from the following 26 sectors

$$\mathbf{G} = \left\{ \begin{array}{cccc} x & z_1 & z_2 & z_1+z_2 \\ & & & \\ z_1+2\alpha & z_1+z_2+2\alpha & 2\alpha+x & z_2+2\alpha+x \\ z_1+2\alpha+x & z_1+z_2+2\alpha+x & & \\ & & & \\ \alpha & 3\alpha & z_1+\alpha & z_1+3\alpha \\ z_2+\alpha & z_2+3\alpha & z_1+z_2+\alpha & z_1+z_2+3\alpha \\ \alpha+x & 3\alpha+x & z_1+\alpha+x & z_1+3\alpha+x \\ z_2+\alpha+x & z_2+3\alpha+x & z_1+z_2+\alpha+x & z_1+z_2+3\alpha+x \end{array} \right\}, \tag{5.11}$$

where x is defined in equation (5.6). The sectors in (5.11) have been organised such that the sectors which do not break the $SO(10)$ symmetry are on row 1; rows 2-3 break the $SO(10)$ symmetry to the Pati-Salam $SO(6) \times SO(4)$ gauge group and finally rows 4-7 break the $SO(10)$ symmetry to the LRS $SU(3) \times U(1) \times SU(2) \times SU(2)$ gauge group.

It should be noted that any projections on sectors containing 3α can be inferred from the projections made on the corresponding sector which contains only α. Therefore, in the following analysis these sectors will not be discussed in detail.

If any of the gauge bosons from the sectors in eq. (5.11) survive the projections, the untwisted gauge symmetry is enhanced. We restrict the classification analysis to vacua with no enhancements, meaning the gauge symmetry of all the vacua classified is identical. In the classification method the GGSO projection coefficients of the 26 sectors listed above were derived and expressed in an analytic form so that a computer code can easily detect if a particular vacua is enhanced. Of the vacua that

[10]It should be noted that $U(1)_C = \frac{3}{2}U(1)_{B-L}$ and $U(1)_L = 2U(1)_{T_{3_R}}$

were scanned in the classification, approximately 29.1% contained extra vector bosons and were therefore enhanced.

5.3 The Twisted Matter Spectrum

5.3.1 General Remarks

In the table below, the hypercharge and electromagnetic charge have been normalised according to the equations

$$Y = \frac{1}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{2}(Q_4 + Q_5) \tag{5.12a}$$

$$Q_{em} = Y + \frac{1}{2}(Q_4 - Q_5) \tag{5.12b}$$

In these equations, the $U(1)$ charges $Q_{1,\ldots,5}$ are the $U(1)$ charges generated by the fermions $\overline{\psi}^{1,\ldots,5}$ respectively and are calculated according to the equation

$$Q(f) = \frac{1}{2}\alpha(f) + F(f) \tag{5.13}$$

where $\alpha(f)$ is the boundary condition of the fermion in the sector and $F(f)$ is the fermion number given by

$$F(f) = \begin{cases} +1 & \text{for } f \\ -1 & \text{for } f^* \end{cases} \tag{5.14a}$$

for fermionic oscillators and their complex conjugates, and

$$\begin{aligned} F\,|+\rangle_R &= 0 \\ F\,|-\rangle_R &= -1 \end{aligned} \tag{5.14b}$$

for the degenerate Ramond vacua where $|+\rangle_R = |0\rangle$ is a degenerated vacuum with no oscillator and $|-\rangle_R = f_0^\dagger\,|0\rangle$ is the degenerated vacua with one zero mode oscillator.

The table below outlines the electromagnetic charges, and the charges under the electroweak $SU(2)\times U(1)$ Cartan generators, of the states which are contained in the observable LRS chiral matter representations:

Representation	$\overline{\psi}^{1,2,3}$	$\overline{\psi}^{4,5}$	Y	Q_{em}
$(\mathbf{3}, +^1\!/_2\,, \mathbf{2}, \mathbf{1})$	$(+,+,-)$	$(+,-)$	1/6	2/3 , -1/3
	$(+,+,-)$	$(+,+)$	2/3	2/3
$(\mathbf{3}, +^1\!/_2\,, \mathbf{1}, \mathbf{2})$	$(+,+,-)$	$(-,-)$	-1/3	-1/3
$(\overline{\mathbf{3}}, -^1\!/_2\,, \mathbf{2}, \mathbf{1})$	$(+,-,-)$	$(+,-)$	-1/6	1/3 , -2/3
	$(+,-,-)$	$(+,+)$	1/3	1/3
$(\overline{\mathbf{3}}, -^1\!/_2\,, \mathbf{1}, \mathbf{2})$	$(+,-,-)$	$(-,-)$	-2/3	-2/3
$(\mathbf{1}, +^3\!/_2\,, \mathbf{2}, \mathbf{1})$	$(+,+,+)$	$(+,-)$	1/2	1 , 0
	$(+,+,+)$	$(+,+)$	1	1
$(\mathbf{1}, +^3\!/_2\,, \mathbf{1}, \mathbf{2})$	$(+,+,+)$	$(-,-)$	0	0
$(\mathbf{1}, -^3\!/_2\,, \mathbf{2}, \mathbf{1})$	$(-,-,-)$	$(+,-)$	-1/2	0 , -1
	$(-,-,-)$	$(+,+)$	0	0
$(\mathbf{1}, -^3\!/_2\,, \mathbf{1}, \mathbf{2})$	$(-,-,-)$	$(-,-)$	-1	-1

where the representation is decomposed as $SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R$. The notation '+' above denotes a state of the degenerated Ramond vacua with no oscillator, *i.e* a state with a fermion number $F = 0$, whereas the notation '$-$' denotes a state of the degenerated Ramond vacua with a zero mode oscillator and therefore a state where $F = -1$. The values for Y and Q_{em} are calculated using equations (5.12a) and (5.12b) respectively.

It is when these representations are decomposed under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ that the particle states of the Standard Model are obtained. The leptons and quarks are realised by the following representations

$$Q^i_L = (\mathbf{3}, \mathbf{2}, \mathbf{1})_{\frac{1}{6}} = \begin{pmatrix} u \\ d \end{pmatrix}^i, \tag{5.15a}$$

$$Q^i_R = (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{\frac{1}{3},-\frac{2}{3}} = \begin{pmatrix} d^c \\ u^c \end{pmatrix}^i, \tag{5.15b}$$

$$L^i_L = (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-\frac{1}{2}} = \begin{pmatrix} \nu \\ e \end{pmatrix}^i, \tag{5.15c}$$

$$L^i_R = (\mathbf{1}, \mathbf{1}, \mathbf{2})_{1,0} = \begin{pmatrix} e^c \\ \nu^c \end{pmatrix}^i, \tag{5.15d}$$

$$h = (\mathbf{1}, \mathbf{2}, \mathbf{2})_0 = \begin{pmatrix} h^u_+ & h^d_0 \\ h^u_0 & h^d_- \end{pmatrix} \tag{5.15e}$$

where h^u and h^d are the low energy supersymmetric superfields associated with the Minimally Supersymmetric Standard Model (MSSM).

5.3.2 The Observable Matter Sectors

The chiral matter spectrum is obtained from the twisted sectors, which are as follows

$$\begin{aligned}
B^{(1)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1, \bar{\psi}^{1,\ldots,5}\} \qquad (5.16) \\
B^{(2)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6 \\
B^{(3)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4
\end{aligned}$$

where $p, q, r, s = 0, 1$ and $b_3 = b_1 + b_2 + x$. These 48 sectors contain the **16** and $\overline{\mathbf{16}}$ spinorial representations of the $SO(10)$ observable gauge group decomposed under $SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R$ as

$$\begin{aligned}
\mathbf{16} &= (\mathbf{3}, +\tfrac{1}{2}, \mathbf{2}, \mathbf{1}) + (\overline{\mathbf{3}}, -\tfrac{1}{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, -\tfrac{3}{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, +\tfrac{3}{2}, \mathbf{1}, \mathbf{2}), \\
\overline{\mathbf{16}} &= (\overline{\mathbf{3}}, -\tfrac{1}{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{3}, +\tfrac{1}{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, +\tfrac{3}{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, -\tfrac{3}{2}, \mathbf{1}, \mathbf{2}).
\end{aligned}$$

In this construction, each of the sectors $B^{(i)}_{pqrs}$ with $i = 1, 2, 3$, can contribute at most a single multiplet to the physical spectrum. The integers $\{pqrs\}$ essentially label the sixteen fixed points of the i^{th} twisted plane. For this reason the identification of the $\{pqrs\}$–sectors can be interchanged with states in the physical spectrum, *i.e.* the spectrum of states that survive the GGSO projections. The power of the formalism is that all the states producing sectors can be expressed in a similar fashion.

In addition to the twisted matter spectrum, there are vector-like states which contribute to the observable matter spectrum. These states arise from the sectors

$$
\begin{aligned}
B^{(4)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 + x \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \quad (5.17) \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{2,3}\} \\
B^{(5)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6 + x \\
B^{(6)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4 + x
\end{aligned}
$$

which have four periodic right-moving complex fermions. Massless states can be obtained by acting on the vacuum with a Neveu-Schwarz right-moving fermionic oscillator. If the oscillator is from either the fermions $\overline{\psi}^{1,\ldots,5}$ or their complex conjugates $\overline{\psi}^{*1,\ldots,5}$ then these sectors give rise to the vectorial **10** representation of $SO(10)$ decomposed under $SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R$ as

$$\mathbf{10} = (\mathbf{3}, -1, \mathbf{1}, \mathbf{1}) + (\overline{\mathbf{3}}, +1, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, 0, \mathbf{2}, \mathbf{2})$$

where the first and second representations are generated by the fermions $\{\overline{\psi}^{1,2,3}\}$ and $\{\overline{\psi}^{*1,2,3}\}$ respectively and the final representation is generated by the fermions $\{\overline{\psi}^{4,5}\}$ and $\{\overline{\psi}^{*4,5}\}$. It can be seen that the first two representations are colour triplets, usually referred to as leptoquarks in the literature, which mediate proton decay via dimension five operators. Therefore, these states must be either sufficiently heavy so as to agree with the current proton lifetime of $\geq 10^{33}$ years [57] or must be projected out of the string spectrum by the GGSO projections. This is a constraint which is considered when the classification is performed. The representation $(\mathbf{1}, 0, \mathbf{2}, \mathbf{2})$ gives rise to the light Standard Model Higgs.

The remaining right-moving complex fermions can give rise to states which are singlets under the observable gauge group but form the following representations

- $\{\overline{\eta}^i\}\,|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\eta}^{*i}\}\,|R\rangle^{(4,5,6)}_{pqrs}$, $i = 1, 2, 3$, where $|R\rangle^{(4,5,6)}_{pqrs}$ is the degenerated Ramond vacuum of the sectors $B^{(4,5,6)}_{pqrs}$ respectively. These states transform as vector-like representations of the $U(1)_i$'s.

- $\{\overline{\phi}^{1,\dots,4}\}\,|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*1,\dots,4}\}\,|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector-like representations of the $SU(4)\times U(1)_4$ gauge group.
- $\{\overline{\phi}^{5,6}\}\,|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*5,6}\}\,|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector-like representations of the $SU(2)_5\times U(1)_5$ gauge group.
- $\{\overline{\phi}^{7,8}\}\,|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*7,8}\}\,|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector-like representations of the $U(1)_7$ and $U(1)_8$ gauge groups respectively.

5.3.2.1 Chirality Operators

In order to calculate the number of families of a model, the number of chiral **16** and $\overline{\mathbf{16}}$ representations of $SO(10)$ decomposed under the LRS gauge group have to be counted. In these models families and anti-families are formed from the following representation

$$\begin{aligned}
\mathbf{16} &= (\mathbf{3},+\tfrac{1}{2},\mathbf{2},\mathbf{1})+(\overline{\mathbf{3}},-\tfrac{1}{2},\mathbf{1},\mathbf{2})+(\mathbf{1},-\tfrac{3}{2},\mathbf{2},\mathbf{1})+(\mathbf{1},+\tfrac{3}{2},\mathbf{1},\mathbf{2})\\
&= Q_L+Q_R+L_L+L_R\\
\overline{\mathbf{16}} &= (\overline{\mathbf{3}},-\tfrac{1}{2},\mathbf{2},\mathbf{1})+(\mathbf{3},+\tfrac{1}{2},\mathbf{1},\mathbf{2})+(\mathbf{1},+\tfrac{3}{2},\mathbf{2},\mathbf{1})+(\mathbf{1},-\tfrac{3}{2},\mathbf{1},\mathbf{2})\\
&= \overline{Q}_L+\overline{Q}_R+\overline{L}_L+\overline{L}_R
\end{aligned} \tag{5.18}$$

A model must then have three families in order to be phenomenologically viable *i.e*

$$N_{Q_L}-N_{\overline{Q}_L}=N_{Q_R}-N_{\overline{Q}_R}=N_{L_L}-N_{\overline{L}_L}=N_{L_R}-N_{\overline{L}_R}=3 \tag{5.19}$$

The number of these representations that occur in a model depends on the choice of the GGSO coefficients. Firstly, in order to distinguish between the **16** and $\overline{\mathbf{16}}$ an $SO(10)$ chirality operator is defined. These chirality operators for the sectors $B^{(1,2,3)}_{pqrs}$ are defined, respectively, as

$$\begin{aligned}
X^{(1)SO(10)}_{pqrs} &= C\begin{pmatrix} B^{(1)}_{pqrs}\\ (1-r)e_5+(1-s)e_6+b_2\end{pmatrix}\\
X^{(2)SO(10)}_{pqrs} &= C\begin{pmatrix} B^{(2)}_{pqrs}\\ (1-r)e_5+(1-s)e_6+b_1\end{pmatrix}\\
X^{(3)SO(10)}_{pqrs} &= C\begin{pmatrix} B^{(3)}_{pqrs}\\ (1-r)e_3+(1-s)e_4+b_1\end{pmatrix}
\end{aligned} \tag{5.20}$$

and can take the values $X^{(1,2,3)SO(10)}_{pqrs} = \pm 1$. Another chirality operator needs defining to determine whether the representations $((\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1}))$ of the $SU(2)_L \times SU(2)_R$ occur. These are defined for the sectors $B^{(1,2,3)}_{pqrs}$ respectively as

$$
\begin{aligned}
X^{(1)SU(2)_{L/R}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{2\alpha + x} \\
X^{(2)SU(2)_{L/R}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{2\alpha + x} \\
X^{(3)SU(2)_{L/R}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{2\alpha + x}
\end{aligned}
\tag{5.21}
$$

where x is the linear combination $x = \mathbb{1} + S + \sum_{i=1}^{6} e_i + z_1 + z_2$.

Furthermore, there is one final chirality operator which needs to be defined in order to determine the representations under the $SU(3)_C \times U(1)_C$ gauge group. These are

$$
\begin{aligned}
X^{(1)SU(3)\times U(1)}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{(1-p)e_3 + (1-q)e_4 + b_3 + x + 2\alpha} \\
X^{(2)SU(3)\times U(1)}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{(1-p)e_1 + (1-q)e_2 + b_3 + x + 2\alpha} \\
X^{(3)SU(3)\times U(1)}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{(1-p)e_1 + (1-q)e_2 + b_2 + x + 2\alpha} .
\end{aligned}
\tag{5.22}
$$

By performing the GGSO projections of these chirality operators the surviving states and therefore the number of families are calculated.

5.3.2.2 Projectors

The projectors are a set of equations which determine whether a sector is either projected out or kept in the string spectrum. These projectors consist of the relevant GGSO coefficients for the sector. For the observable chiral matter there are 48 projectors which are calculated to be

$$
\begin{aligned}
P^{(1)}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_2}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(1)}_{pqrs}}\right) \\
P^{(2)}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_3}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_4}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_1}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(2)}_{pqrs}}\right) \\
P^{(3)}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_5}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_6}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_1}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(3)}_{pqrs}}\right)
\end{aligned}
\tag{5.23}
$$

The analysis of the physical spectrum is formulated as algebraic equations. The projectors can be expressed as a system of linear equations where p, q, r, s take unknown values. The sectors which survive the GGSO projections are found by solving the systems of equations for p, q, r, s. Using this formalism allows for a computer analysis of the models as the systems of linear equations are easy to express in a computer code.

The following notation is used in the algebraic representation of the GGSO projections

$$C\binom{v_i}{v_j} = e^{i\pi(v_i|v_j)} \tag{5.24}$$

when the GGSO coefficients are expressed in this way the analytic expressions for the projectors $P^{(1,2,3)}_{pqrs}$ are given in matrix form $\Delta^i W^i = Y^i$ as

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_1|b_1) \\ (z_2|b_1) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_1|b_2) \\ (z_2|b_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3) \\ (e_6|b_3) \\ (z_1|b_3) \\ (z_2|b_3) \end{pmatrix}$$

respectively. Such algebraic matrix equations can be written for the entire physical spectrum. In the ensuing discussion we list all the sectors that can *a priori* produce physical states, but do not list explicitly all the algebraic matrix equations for the corresponding GGSO projections.

5.3.3 Exotic Sectors

Additional sectors exist in the string models that can give rise to states that carry fractional charges under the LRS gauge group. This leads to states with a fractional electric charge at the level of the Standard Model. The term 'exotic states' used here is reserved purely for the states with fractional electric charge which arise from the sectors containing the basis vector α. Exotic states arise from these sectors due to Wilson line breaking of the non-Abelian GUT symmetries. These exotics states are a generic feature of string compactifications [58, 59, 60] and experimental searches are being conducted in order to find them [61]. There are interesting phenomenological aspects to exotic states as charge conservation implies that the lightest of these states is necessarily stable. To date however, no such exotic states have been observed, leading to strong upper bounds on their abundance [61]. In addition, if these states are too plentiful in the early universe they can cause problems during the reheating phase as the lightest of these states is necessarily stable, meaning they continue to scatter and cannot decouple from the plasma in the early Universe due to their charge.

There are two solutions to the lack of experimental data for the existence of exotics. The first solution is by demanding that the exotics are confined to integrally charged states [41]. The second is to demand that the exotic states are sufficiently heavy and diluted in the cosmological evolution of the universe [60]. However, there are issues with the integrally charged state solution as these states affect the renormalisation group running of the weak-hypercharge and gauge group unification. This leads to the preferred solution of demanding that the exotic states are sufficiently massive and dilute. A sufficient mass for these states is above the GUT scale so that they are diluted during the inflationary period of the universe as during the reheating phase they will not be reproduced.

Previous classifications of heterotic-string models found examples of vacua in which massless exotics states were absent and only appeared in the massive spectrum. These models were dubbed 'exophobic heterotic string vacua'. In the case of the Pati-Salam models, three generation exophobic vacua were found [40] and in the FSU5 case exophobic vacua were found in models with an even number of generations [50]. A question of interest for the current research is therefore whether any exophobic LRS models can be found.

5.3.3.1 Spinorial Exotics

The term spinorial exotics refers to sectors which involve the basis vector α and have the products $\xi_L \cdot \xi_L = 4$ and $\xi_R \cdot \xi_R = 8$, therefore requires no oscillators to produce massless states.

The sectors below all give rise to states with the representations $(\mathbf{1}, -\frac{3}{4}, \mathbf{1}, \mathbf{2})$ and $(\mathbf{1}, -\frac{3}{4}, \mathbf{2}, \mathbf{1})$ under the $SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R$ observable gauge group. These states are defined in the analysis as $n_{L_L e}$ and $n_{L_R e}$ respectively. It can be seen that these are singlets under the $SU(3)_C$ gauge group but are still charged under $U(1)_C$. The corresponding sectors with 3α in the linear combination of basis vectors give states with the representations $(\mathbf{1}, +\frac{3}{4}, \mathbf{1}, \mathbf{2})$ and $(\mathbf{1}, +\frac{3}{4}, \mathbf{2}, \mathbf{1})$. It can be seen that the only change is the sign reversal of the charge under $U(1)_C$. The following are the sectors which give rise to these representations

$$
\begin{aligned}
B^{(7)}_{pqrs} &= B^{(1)}_{pqrs} + \alpha \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = -\tfrac{1}{2}, \qquad (5.25)\\
&\qquad \bar{\eta}^{2,3} = \tfrac{1}{2}, \bar{\psi}^{1,2,3} = -\tfrac{1}{2}, \bar{\psi}^{4,5}, \bar{\phi}^{1,\dots,6} = \tfrac{1}{2}, \bar{\phi}^7\} \\
B^{(8,9)}_{pqrs} &= B^{(2,3)}_{pqrs} + \alpha
\end{aligned}
$$

$$
\begin{aligned}
B^{(13)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + \alpha \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = -\tfrac{1}{2}, \qquad (5.26)\\
&\qquad \bar{\eta}^{2,3} = \tfrac{1}{2}, \bar{\psi}^{1,2,3} = -\tfrac{1}{2}, \bar{\psi}^{4,5}, \bar{\phi}^{1,\dots,4} = -\tfrac{1}{2}, \bar{\phi}^{5,6} = \tfrac{1}{2}, \bar{\phi}^7\} \\
B^{(14,15)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + \alpha
\end{aligned}
$$

$$
\begin{aligned}
B^{(22)}_{pqrs} &= B^{(1)}_{pqrs} + z_2 + \alpha \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = -\tfrac{1}{2}, \qquad (5.27) \\
&\qquad \bar{\eta}^{2,3} = \tfrac{1}{2}, \bar{\psi}^{1,2,3} = -\tfrac{1}{2}, \bar{\psi}^{4,5}, \bar{\phi}^{1,\ldots,4} = \tfrac{1}{2}, \bar{\phi}^{5,6} = -\tfrac{1}{2}, \bar{\phi}^8\} \\
B^{(23,24)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_2 + \alpha
\end{aligned}
$$

$$
\begin{aligned}
B^{(31)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + z_2 + \alpha \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = -\tfrac{1}{2}, \qquad (5.28) \\
&\qquad \bar{\eta}^{2,3} = \tfrac{1}{2}, \bar{\psi}^{1,2,3} = -\tfrac{1}{2}, \bar{\psi}^{4,5}, \bar{\phi}^{1,\ldots,4} = -\tfrac{1}{2}, \bar{\phi}^{5,6} = -\tfrac{1}{2}, \bar{\phi}^8\} \\
B^{(32,33)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + z_2 + \alpha
\end{aligned}
$$

5.3.3.2 Vectorial Exotics

The following are vectorial states, meaning they have the products $\xi_L \cdot \xi_L = 4$ and $\xi_R \cdot \xi_R = 6$, and therefore requiring one $\frac{1}{4}$ oscillator to produce massless states. Firstly, there are the sectors

$$
\begin{aligned}
B^{(46)}_{pqrs} &= B^{(1)}_{pqrs} + \alpha + x \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = \tfrac{1}{2}, \qquad (5.29) \\
&\qquad \bar{\eta}^{2,3} = -\tfrac{1}{2}, \bar{\psi}^{1,2,3} = \tfrac{1}{2}, \bar{\phi}^{1,\ldots,6} = \tfrac{1}{2}, \bar{\phi}^7\} \\
B^{(47,48)}_{pqrs} &= B^{(2,3)}_{pqrs} + \alpha + x
\end{aligned}
$$

Using $B^{(46)}_{pqrs}$ as an example to show the states that can be obtained from these sectors, the possible states are

- $\{\overline{\psi}^{*1,2,3}\}\,|R\rangle^{(46)}_{pqrs}$, where $|R\rangle^{(46)}_{pqrs}$ is the degenerate Ramond vacua of the $B^{(46)}_{pqrs}$ sector. These states transform as vector-like representations of the observable $SU(3)_C \times U(1)_C$.
- $\{\overline{\eta}^{*1}\}\,|R\rangle^{(46)}_{pqrs}$. These states transform as vector-like representations of $U(1)_1$.
- $\{\overline{\eta}^{2,3}\}\,|R\rangle^{(46)}_{pqrs}$. These states transform as vector-like representations of $U(1)_2$ and $U(1)_3$ respectively.

- $\{\overline{\phi}^{*1,\ldots,4}\}\,|R\rangle^{(46)}_{pqrs}$. These states transform as vector-like representations of the hidden $SU(4) \times U(1)_4$.
- $\{\overline{\phi}^{*5,6}\}\,|R\rangle^{(46)}_{pqrs}$. These states transform as vector-like representations of the hidden $SU(2)_5 \times U(1)_5$.

The states obtained from the sectors $B^{(47,48)}_{pqrs}$ transform in the same manner as those above.

Secondly, there are the following 48 sectors

$$
\begin{aligned}
B^{(52)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + \alpha + x \\
&= \{\psi^{\mu}, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = \tfrac{1}{2}, \qquad (5.30) \\
&\qquad \bar{\eta}^{2,3} = -\tfrac{1}{2}, \bar{\psi}^{1,2,3} = \tfrac{1}{2}, \bar{\phi}^{1,\ldots,4} = -\tfrac{1}{2}, \bar{\phi}^{5,6} = \tfrac{1}{2}, \bar{\phi}^7\} \\
B^{(53,54)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + \alpha + x
\end{aligned}
$$

The states found from these sectors only differ from $B^{(47,48,49)}_{pqrs}$ by a negative sign on the $\frac{1}{2}$ boundary conditions of the fermions $\overline{\phi}^{1,2,3,4}$. This has the effect of changing the sign of the $U(1)_4$ charges while leaving the other charges unaffected. The structure and charges generated by the other worldsheet fermions therefore remain identical.

Similarly, in the sectors

$$
\begin{aligned}
B^{(58)}_{pqrs} &= B^{(1)}_{pqrs} + z_2 + \alpha + x \\
&= \{\psi^{\mu}, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = \tfrac{1}{2}, \qquad (5.31) \\
&\qquad \bar{\eta}^{2,3} = -\tfrac{1}{2}, \bar{\psi}^{1,2,3} = \tfrac{1}{2}, \bar{\phi}^{1,\ldots,4} = \tfrac{1}{2}, \bar{\phi}^{5,6} = -\tfrac{1}{2}, \bar{\phi}^8\} \\
B^{(59,60)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_2 + \alpha + x,
\end{aligned}
$$

the observable states are identical to those in the sectors $B^{(47,48,49)}_{pqrs}$ and only the hidden charges differ by a slight change in the Ramond vacua and a sign difference of the boundary conditions of the fermions $\overline{\phi}^{5,6}$, which only affects the sign of the charges under $U(1)_5$.

The final 48 sectors are

$$\begin{aligned}
B^{(64)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + z_2 + \alpha + x \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^1 = \tfrac{1}{2}, \qquad (5.32) \\
&\qquad\qquad \bar{\eta}^{2,3} = -\tfrac{1}{2}, \bar{\psi}^{1,2,3} = \tfrac{1}{2}, \bar{\phi}^{1,\ldots,6} = -\tfrac{1}{2}, \bar{\phi}^8\} \\
B^{(65,66)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + z_2 + \alpha + x
\end{aligned}$$

These differ from sectors $B^{(58,59,60)}_{pqrs}$ by changing the sign on the $\frac{1}{2}$ boundary conditions of the fermions $\overline{\phi}^{1,2,3,4}$ and therefore, as above, there is a sign change on the charges under $U(1)_4$. All other states are unaffected and remain as in the sectors $B^{(58,59,60)}_{pqrs}$.

5.3.3.3 Pati-Salam Exotics

In the case of left-right symmetric models, there can be states which are exotic with respect to the Pati-Salam gauge group $SO(6) \times SO(4)$. The sectors from which these states can arise are those which contain the vector combination 2α. This is due to the fermions $\overline{\psi}^{1,2,3}$ or $\overline{\psi}^{4,5}$ having periodic boundary conditions in the sector (therefore generating the Pati-Salam gauge subgroup), while still having a fractional electric charge with respect to the Standard Model.

In the model being discussed, all of the Pati-Salam exotics are found in the following sectors:

$$\begin{aligned}
B^{(70)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + 2\alpha \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \qquad (5.33) \\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{2,3}, \bar{\psi}^{4,5}, \bar{\phi}^{5,6}\} \\
B^{(71,72)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + 2\alpha
\end{aligned}$$

These states transform in representations of the gauge group $SU(2)_L \times$

$SU(2)_R \times SU(2)_5 \times U(1)_5$.

$$\begin{aligned}
B^{(34)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + z_2 + 2\alpha \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \qquad (5.34)\\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{2,3}, \bar{\psi}^{4,5}, \bar{\phi}^{7,8}\} \\
B^{(35,36)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + z_2 + 2\alpha
\end{aligned}$$

These states transform as representations of the gauge group $SU(2)_L \times SU(2)_R \times U(1)_7 \times U(1)_8$. The states from the previous 96 sectors are defined in the analysis as $n_{L_Ls}, n_{L_Rs}, n_{\overline{L}_Ls}$ and $n_{\overline{L}_Rs}$.

$$\begin{aligned}
B^{(40)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + 2\alpha + x \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \qquad (5.35)\\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{1}, \bar{\psi}^{1,2,3}, \bar{\phi}^{5,6}\} \\
B^{(41,42)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + 2\alpha + x
\end{aligned}$$

These states transform as representations of the gauge group $SU(3)_C \times U(1)_C \times SU(2)_5 \times U(1)_5$

$$\begin{aligned}
B^{(43)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + z_2 + 2\alpha + x \\
&= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \qquad (5.36)\\
&\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{1}, \bar{\psi}^{1,2,3}, \bar{\phi}^{7,8}\} \\
B^{(44,45)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + z_2 + 2\alpha + x
\end{aligned}$$

These states transform as representations of the gauge group $SU(3)_C \times U(1)_C \times U(1)_7 \times U(1)_8$. The states from the previous 96 sectors are defined in the analysis as n_{3v} and $n_{\bar{3}v}$.

5.3.4 Hidden Matter Spectrum

The hidden matter spectrum refers to sectors which produce states that transform under the hidden gauge group but are singlets under the observable SO(10) GUT gauge group. This means that the states produced are not exotic with respect to the Standard Model gauge charges.

There are 48 sectors present from $B_{pqrs}^{(1,2,3)} + z_1 + x$ which are

$$\begin{aligned} B_{pqrs}^{(19)} &= B_{pqrs}^{(1)} + z_1 + x \\ &= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\ &\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{2,3}, \bar{\phi}^{1,2,3,4}\} \\ B_{pqrs}^{(20,21)} &= B_{pqrs}^{(2,3)} + z_1 + x \end{aligned} \tag{5.37}$$

These sectors contain states which transform under the hidden $SU(4) \times U(1)_4$ gauge group with the representations $(\mathbf{1}, +2)$, $(\mathbf{1}, -2)$, $(\mathbf{4}, +1)$, $(\overline{\mathbf{4}}, -1)$, $(\mathbf{6}, 0)$.

There exists another 48 sectors $B_{pqrs}^{(1,2,3)} + z_2 + x$ given by

$$\begin{aligned} B_{pqrs}^{(28)} &= B_{pqrs}^{(1)} + z_2 + x \\ &= \{\psi^\mu, \chi^{1,2}, (1-p)y^3\bar{y}^3, pw^3\bar{w}^3, (1-q)y^4\bar{y}^4, qw^4\bar{w}^4, \\ &\qquad (1-r)y^5\bar{y}^5, rw^5\bar{w}^5, (1-s)y^6\bar{y}^6, sw^6\bar{w}^6, \bar{\eta}^{2,3}, \bar{\phi}^{5,6,7,8}\} \\ B_{pqrs}^{(29,30)} &= B_{pqrs}^{(2,3)} + z_2 + x \end{aligned} \tag{5.38}$$

These sectors produce states which transform under the $SU(2)_5 \times U(1)_5 \times U(1)_7 \times U(1)_8$ gauge group with the representations: $(\mathbf{1}, +1, \pm\frac{1}{2}, \pm\frac{1}{2})$, $(\mathbf{2}, 0, \pm\frac{1}{2}, \pm\frac{1}{2})$, $(\mathbf{1}, -1, \pm\frac{1}{2}, \pm\frac{1}{2})$ where the charges of $U(1)_7$ and $U(1)_8$ can take all possible permutations of the values given, meaning there are 12 distinct representations in total.

5.4 Classification Results and Analysis

The classification process involves utilising the calculated algebraic conditions which were presented in the previous sections. By using the projectors and chirality operators for each sector the entire massless spectrum can be analysed for a specific choice of the one-loop GGSO projection coefficients. These algebraic conditions can be written in a computer program which enables a scan over the different choices of GGSO projection coefficients. As the total number of possible configurations, and therefore vacua, is $2^{66} \approx 7.38 \times 10^{19}$ a complete scan of the entire space of string vacua is not possible. Therefore, a random generation of the GGSO projection coefficients is used in order to provide a random sample of vacua[11] from which models with desirable phenomenological criteria can be found.

The algebraic conditions were programmed into a JAVA code in order to perform the classification and the accuracy of this program was checked against an independently written FORTRAN code. In the JAVA program, a random generator was used in order to provide the different GGSO configurations. This program initially produces a random GGSO configuration, before running these values through the algebraic conditions calculated for each sector in order to produce the full spectrum of each model. By repeating this process, the statistics associated with classification can be developed while also fishing for single models which are of phenomenological significance.

Previous papers which have utilised this technique can be seen in references [40, 46, 50, 52]. In the case of the classification of Pati-Salam models, this method was shown to produce three-generation models which contained no exotic massless states with fractional electric charge, and were therefore exophobic.

Therefore, an example question of phenomenological interest is whether exophobic LRS models can be found.

The observable sector of a heterotic string Left-Right Symmetric model is characterised by 27 integers which are defined in table 7. These contain the relevant quantities of phenomenological interest. Notable numbers defined in table 7 are n_g, n_h and n_H as these give the number of generations of a model and whether the model contains non-chiral light and heavy Higgs representations.

The numbers given in the first two columns of table 7 are as described above in section 5.3.2. The first four numbers form a complete **16** of $SO(10)$ and the last four form a complete $\overline{\mathbf{16}}$. The first four in the LRS Exotics column arise from the spinorial exotic sectors and the last two

[11] It is noted here that analysis of large sets of string vacua have been performed by other research groups. A discussion on these can be found in Appendix A.

Spinorial $SO(10)$ Observable	Vectorial $SO(10)$ Observable	LRS Exotic	Pati-Salam Exotic
$n_{L_L} = (\mathbf{1}, -^3/_2, \mathbf{2}, \mathbf{1})$	$n_h = (\mathbf{1}, 0, \mathbf{2}, \mathbf{2})$	$n_{L_L s} = (\mathbf{1}, +^3/_4, \mathbf{2}, \mathbf{1})$	$n_{L_L e} = (\mathbf{1}, 0, \mathbf{2}, \mathbf{1})$
$n_{L_R} = (\mathbf{1}, +^3/_2, \mathbf{1}, \mathbf{2})$	$n_3 = (\mathbf{3}, -1, \mathbf{1}, \mathbf{1})$	$n_{L_R s} = (\mathbf{1}, +^3/_4, \mathbf{1}, \mathbf{2})$	$n_{L_R e} = (\mathbf{1}, 0, \mathbf{1}, \mathbf{2})$
$n_{Q_L} = (\mathbf{3}, +^1/_2, \mathbf{2}, \mathbf{1})$	$n_{\overline{3}} = (\overline{\mathbf{3}}, +1, \mathbf{1}, \mathbf{1})$	$n_{\overline{L}_L s} = (\mathbf{1}, -^3/_4, \mathbf{2}, \mathbf{1})$	$n_{3e} = (\mathbf{3}, +^1/_2, \mathbf{1}, \mathbf{1})$
$n_{Q_R} = (\overline{\mathbf{3}}, -^1/_2, \mathbf{1}, \mathbf{2})$		$n_{\overline{L}_R s} = (\mathbf{1}, -^3/_4, \mathbf{1}, \mathbf{2})$	$n_{\overline{3}e} = (\overline{\mathbf{3}}, +^1/_2, \mathbf{1}, \mathbf{1})$
$n_{\overline{L}_L} = (\mathbf{1}, +^3/_2, \mathbf{2}, \mathbf{1})$		$n_{3v} = (\mathbf{3}, +^1/_4, \mathbf{1}, \mathbf{1})$	$n_{1e} = (\mathbf{1}, +^3/_2, \mathbf{1}, \mathbf{1})$
$n_{\overline{L}_R} = (\mathbf{1}, -^3/_2, \mathbf{1}, \mathbf{2})$		$n_{\overline{3}v} = (\overline{\mathbf{3}}, -^1/_4, \mathbf{1}, \mathbf{1})$	$n_{\overline{1}e} = (\mathbf{1}, -^3/_2, \mathbf{1}, \mathbf{1})$
$n_{\overline{Q}_L} = (\overline{\mathbf{3}}, -^1/_2, \mathbf{2}, \mathbf{1})$		$n_{1v} = (\mathbf{1}, +^3/_4, \mathbf{1}, \mathbf{1})$	
$n_{\overline{Q}_R} = (\mathbf{3}, +^1/_2, \mathbf{1}, \mathbf{2})$		$n_{\overline{1}v} = (\mathbf{1}, -^3/_4, \mathbf{1}, \mathbf{1})$	

$$n_g = n_{L_L} - n_{\overline{L}_L} = n_{L_R} - n_{\overline{L}_R} = n_{Q_L} - n_{\overline{Q}_L} = n_{Q_R} - n_{\overline{Q}_R}$$
$$n_H = n_{\overline{L}_R}$$

Table 7: *The 27 integers used to categorise the quantities of phenomenological interest. The first column contains states from the* ***16*** *and* $\overline{\mathbf{16}}$ *representations of* $SO(10)$*. The second contains the states from the* ***10*** *representation of* $SO(10)$*. The third and fourth list the states which are exotic with respect to the Left-Right Symmetric and Pati-Salam gauge groups respectively.*

arise from the vectorial exotic sectors.

To perform the classification, the analytic formulae for all the sectors which contribute to these numbers were derived so as to describe the complete spectrum of each model.

For a model to be phenomenologically viable, it must satisfy the following phenomenological criteria:

$n_g = 3$	Three light chiral generations
$n_H \geq 1$	At least one heavy Higgs pair to break the $SU(2)_R$ symmetry
$n_h \geq 1$	At least one light Higgs bi-doublet
$n_3 = n_{\bar{3}}$	Heavy mass can be generated for the colour triplets
$n_{3e} = n_{\bar{3}e}$	Heavy mass can be generated for the colour triplets
$n_{1e} = n_{\bar{1}e}$	Heavy mass can be generated for vector-like exotics
$n_{3v} = n_{\bar{3}v}$	Heavy mass can be generated for the colour triplets
$n_{1v} = n_{\bar{1}v}$	Heavy mass can be generated for the vector-like exotics
$n_{L_L s} = n_{\bar{L}_L s}$	Heavy mass can be generated for vector-like exotics
$n_{L_R s} = n_{\bar{L}_R s}$	Heavy mass can be generated for vector-like exotics

where the constraints which generate the heavy masses have been imposed in order to generate LRS models which contain no chiral exotics in the massless spectrum. By applying these constraints, all the exotic states (including those which are confined to being integrally charged states) are vector-like and can therefore obtain a superpotential term consisting of the vector-like state coupled to a SM singlet state. The singlet state then obtains a Vacuum Expectation Value (VEV) which generates a heavy mass and therefore decouples the exotic state from the massless spectrum.

An initial classification run of 10^9 distinct models was performed and the results are displayed in section 5.4.2. Due to a relative lack in abundance of three generation models a second run of 10^{11} distinct models was performed with the constraints on the vector-like chiral exotic states relaxed. Namely, the condition that $n_{1e} = n_{\bar{1}e}$ which arise from the Pati-

Salam exotic sectors were relaxed, along with the conditions $n_{L_L s} = n_{\overline{L}_L s}$, $n_{L_R s} = n_{\overline{L}_R s}$ and $n_{1v} = n_{\overline{1}v}$ which arise from the LRS exotic sectors. A remark should be made, that whereas a 10^9 run typically takes 2 days, a corresponding 10^{11} run can take 28 weeks, which becomes prohibitive. The results of these two runs is presented and commented on in section 5.4.2.

5.4.1 Top Quark Mass Coupling

For a model to be phenomenologically viable, it must reproduce the spectrum of the Standard Model while also reproducing the Standard Model interactions at the low energy limit. Therefore, our analysis extends to classifying the number of models which give the necessary conditions to include the top quark mass.

In order to ensure a model permits a top quark mass, the condition that there exists a top quark mass coupling at the tree level of the superpotential is imposed. As previously stated, all models considered in the classification possess $\mathcal{N} = 1$ supersymmetry. Therefore, the coupling of the top quark arises due to a superpotential interaction of the form

$$\lambda_t \int d^2\theta \; \Phi_S \Phi_S \Phi_V \ , \tag{5.39}$$

where λ_t is the coupling constant, Φ are superfields and S, V denote the spinorial or vectorial representation of the fields respectively. The value of the coupling constant λ_t can be calculated as it is proportional to the correlation function of the set of vertex operators of the massless string modes [62]

$$\lambda_t \sim \left\langle S^F S^F V^B \right\rangle \tag{5.40}$$

where S, V denote the spinorial and vectorial part of the vertex operator associated with the superfields $\Phi_{S/V}$ respectively. The superscripts F, B correspond to the fermionic or bosonic part of the vertex operator respectively.

Due to a result from conformal field theory, the total charge of the vertex operators in equation (5.40) under the three $U(1)$'s must equal zero in order for the non-vanishing of the correlator [62]. The derivation of all the types of couplings permitted is beyond the scope of the thesis, but can be found in reference [62]. It can then be shown that there are only three non-vanishing correlation functions [63]. These are

$$\big\langle (R)^1(R)^2(R)^3 \big\rangle \ , \quad \big\langle (R)^i(R)^i(NS) \big\rangle \ , \quad \big\langle (NS)(NS)(NS) \big\rangle \ , \tag{5.41}$$

where R represents a generic Ramond (twisted) field, NS represents a generic Neveu-Schwarz (untwisted) field and the superscript denotes the orbifold plane ($i = 1, 2, 3$).

An important feature of the models being considered in the classification is that the leptons and quarks only arise from the twisted sectors of the $SO(10)$ spinorials, whereas the light SM Higgs are only admitted by the vectorial representation of the twisted sectors. Therefore, only the correlation functions of the form

$$\big\langle (R)^1(R)^2(R)^3 \big\rangle \tag{5.42}$$

are viable options which admit a top quark mass coupling for the particular class of models under consideration.

In this class of models, the top quark mass coupling term in the superpotential is

$$\lambda_t Q^F u^{cF} h_u^B \ . \tag{5.43}$$

By comparing equations (5.42) and (5.43), it can be seen that suitable conditions to impose are: Q must arise from the first orbifold plane, u^c from the second and h_u from the third. Therefore, in the case of the models under consideration, the orbifold planes which can give rise to the necessary states are $B^{(1)}_{pqrs}$, $B^{(2)}_{pqrs}$ and $B^{(6)}_{pqrs} = B^{(3)}_{pqrs} + x$ which are the first, second and third orbifold planes respectively.

This leads to a straightforward general analytical method for these models. The general method details, without loss of generality, that if Q, u^c and h_u arise from the sectors $B^{(1)}_{pqrs}$, $B^{(2)}_{pqrs}$ and $B^{(6)}_{pqrs} = B^{(3)}_{pqrs} + x$ respectively, there exists a top quark mass coupling.

5.4.2 Results

We now explore the space of the Left-Right Symmetric free fermionic heterotic string vacua. The sample size used in the first classification was 10^9 vacua out of a possible total of 2^{66}. Some of the results are presented in Figures 5 - 7 and table 8.

In Figure 5 the number of generations is presented against the natural logarithm of the number of models found. The results show the greatest number of models have zero generations and the number of models decreases as the number of generations increases. The maximum number of generations found was $n_g = 5$. Figure 6 shows that only exophobic models with zero generations were found. Figure 7 displays the number of three generation models with no chiral exotic multiplets found with respect to the total number of exotic multiplets they contain. The results show minimally exotic models to have 22 exotic multiplets while maximally exotic models have 90 exotic multiplets. The greatest number of models contained 50 exotic multiplets and the results show an approximately normal distribution, skewed slightly to models containing more than 50 multiplets. It can be seen in table 8 that $\approx 62.2\%$ of the non-enhanced models with complete families had no chiral exotics. The inclusion of the constraint demanding that the model must have three generations then drastically drops the probability of finding a viable model. The probability of finding a model which satisfies all these criteria is 1.49×10^{-6}. Of these models, the probabilities that they contain no Higgs particles, only SM light Higgs particles or only heavy Higgs particles are 5.42×10^{-7}, 9.39×10^{-7} and 7.00×10^{-9} respectively. Table 8 shows that requiring the model to contain both a light SM Higgs and a heavy Higgs yielded one model. Although this suggests models with interesting phenomenology exist, this result is not statistically significant and therefore does not allow meaningful conclusions to be drawn. This result also does not allow for any analysis involving further constraints.

Due to the lack of models with suitable phenomenology found during the 10^9 sample, the sample size was increased to 10^{11} and some of the con-

	Constraints	Total models in sample	Inferred Frequency	Estimated number of models in class
	No Constraints	1000000000	1	7.38×10^{19}
(1)	+ No Enhancements	708830165	7.09×10^{-1}	5.23×10^{19}
(2)	+ Complete Families	70241057	7.02×10^{-2}	5.18×10^{18}
(3)	+ No Chiral Exotics	43660665	4.37×10^{-2}	3.30×10^{18}
(4)	+ Three Generations	1486	1.49×10^{-6}	1.10×10^{14}
(5)	+ SM Light Higgs + & Heavy Higgs	1	1.00×10^{-9}	7.38×10^{10}
(6)	+ Minimal Heavy Higgs & Minimal SM Light Higgs	0	0	N/A
(7)	+ Top Quark Mass Coupling	0	0	N/A

Table 8: *Statistics for the LRS models with respect to phenomenological constraints for* 10^9 *models.*

straints were relaxed. Specifically, the constraints concerning the chiral exotic triplets in the models were included (*i.e* $n_3 = n_{\bar{3}}$ and $n_{3v} = n_{\bar{3}v}$), whereas the constraints concerning the vector-like chiral colour–singlet exotics were omitted. We note that relaxing these constraints entails that in some of the scanned models $U(1)_C$ is anomalous. It should also be noted that vacua which have an anomalous $U(1)_C$ are not phenomenologically viable, as the weak hypercharge $U(1)_Y$ defined in equation (5.10) contains a description of the $U(1)_C$. Therefore if the $U(1)_C$ is anomalous, the resulting weak hypercharge is also anomalous, which is not in agreement with the Standard Model.

The sample size was then increased to perform a classification on 10^{11} vacua out of a possible total of 2^{66} and the program was run again. Some of the results are presented in Figures 8 - 10 and Tables 9 - 10.

In Figure 8 the number of models versus the number of full generations is displayed for the 10^{11} model run. The greatest number of models can be seen to have zero generations and the number of models decreases as the number of generations increases. This result is in accordance with the 10^9

run and the previous results of classifications [40, 50, 52]. It can be seen that once the number of generations is greater than six, there is an absence of models. This result indicates that for this choice of basis vectors, models with $n_g \geq 7$ are either completely forbidden or are extremely unlikely in the total space of model possibilities. As an aside, the upper limit on the number of generations is $n_g = 12$. This can be calculated by considering that there are 48 possible states which can arise from the sectors $B^{1,2,3}_{pqrs}$, which can potentially give rise to $n_{L_L}, n_{L_R}, n_{Q_L}, n_{Q_R} = 12$ states (where the 'n' values are defined in table 7), meaning the maximal number of generations is $n_g = 12$ for these LRS models. This decreases to an upper limit of $n_g = 11$ if the requirement of having a heavy Higgs is imposed, as this necessarily requires the existence of at least one $n_{\overline{L}_L}$ which reduces the number of states from which full generations can be made.

Figure 9 displays the number of exophobic models versus the number of generations. Analogously to the 10^9 classification run, the results show a relative abundance of zero generation exophobic models but an absence of any exophobic models with $n_g \geq 1$. This result leads to the conclusion that there are no three generation exophobic models with a statistical frequency larger than $1 : 10^{11}$. It should however be noted that the lack of exophobic models with $n_g \geq 1$ does not suggest that exophobic Left-Right Symmetric models are completely forbidden, only that for the choice of basis vectors used in this analysis none were found with a reasonable statistical likelihood.

This result is in contrast to the case of the results of both the Pati-Salam and Flipped $SU(5)$ classifications [40, 50]. In the Pati-Salam case, exophobic models with $n_g = 0, \ldots, 6$ were found and where $n_g \geq 7$ exophobic models with an even number of generations were found with a notable absence for $n_g = 14$. In the flipped $SU(5)$ case, exophobic models with an even number of generations were found. While this means no three generation exophobic models were found, the flipped $SU(5)$ case admits many more exophobic models than the current LRS case.

In Figure 10 the total number of three generation models with matched number of colour triplets is displayed against the number of exotic fractionally charged multiplets in a given three generation model. It can be seen that the minimal number of exotic multiplets was again found to be 22, while the maximally exotic models contained 98, an increase from the previous run. The results again show a roughly normal distribution with a central peak at 50 exotic multiplets with a slight skew toward models where the number of exotic multiplets greater than 50. This result is similar to what was found in the classification of Pati-Salam models [40], but in the case of the LRS the average number of exotic multiplets is much higher. In the Pati-Salam case, there was a central peak at 18 exotic multiplets with maximally exotic models having 54 multiplets. This result is, in general, to be expected as in the LRS models both Pati-Salam and LRS exotic sectors exist, therefore there is the potential for many more exotic states to enter the spectrum.

Table 9 shows the number of non-enhanced three generation models which have matched numbers of colour triplets with respect to the number of Pati-Salam, spinorial and vectorial exotic multiplets. It can be seen that of the total number of models, there were models found which contained no spinorial exotic multiplets. This is also true in the case of vectorial exotic multiplets. However, no models were found which were exophobic with respect to the Pati-Salam exotic multiplets, which is a leading reason for the lack of exophobic three generation models. This result is in contrast to the results of the classification performed in [40] as three generation exophobic Pati-Salam models were found.

Of the total models sampled, $\approx 61.1\%$ of the non-enhanced full generation models were found to have matched numbers of colour triplets. This is a slight increase from the 10^9 classification run due to the relaxing of some of the conditions as mentioned previously. This can be seen in table 10. It should be noted that the probability of finding non-enhanced three generation models is actually lower in the 10^{11} classification run than in the 10^9 case. This is expected to be a statistical fluctuation due to the random nature of the classification method. Further analysis on the effect of relaxing the requirement that all colour–singlet exotics are vector–like in three generation models may be an interesting area of research. However, this is beyond the scope of this analysis and is left for future work.

If the constraint of having a top quark mass coupling is included, then of the total number of non-enhanced full generation models only $\approx 0.015\%$ were found to be viable. While three generation models with a top quark mass coupling were found, it can be seen from table 10 that their appearance was not found to be frequent, as the probability for finding such a model was found to be 4.0×10^{-11}.

Of all the non-enhanced models with complete generations, $\approx 46.0\%$ contained at least one light Higgs. This is much higher than for the case of the heavy Higgs, where only $\approx 14.0\%$ of the total non–enhanced, generation complete models contained at least one heavy Higgs. When considering non–enhanced three generation models in which all exotic colour triplet are vector–like, the number which had at least one Standard Model Higgs is approximately 57.5% and the number which had at least one heavy Higgs is approximately 0.57%. Only 0.03% of the non–enhanced three generation models with vector–like exotics contain both light and heavy Higgs multiplets. Comparing with previous classifications, we note that in the three generation Pati-Salam models classified in [40], 7.9% had a heavy Higgs and 81.0% of these had a SM Higgs. Whereas, in the flipped $SU(5)$ case [50], the non–enhanced and anomaly free three generation models had $\approx 95.7\%$ which contained a SM Higgs and $\approx 6.3\%$ contained a heavy Higgs. In comparison, it can be seen that the number of three generation free fermionic LRS models, free of $U(1)_C$ anomalies and enhancements, which contain either Higgs is drastically lower. This outcome should nevertheless be compared with the case of the SU421 models in which no viable models can be constructed at all.

5.4.3 A Model of Notable Phenomenology

The random classification method can be used to trawl models with specified phenomenological properties. Using the notation convention

$$C\binom{v_i}{v_j} = e^{i\pi(v_i|v_j)} \tag{5.44}$$

the model defined by the GGSO projection coefficients in eq. (5.45) provides an example of a non–enhanced three generation model, with potentially viable phenomenology.

$$(v_i|v_j) = \begin{array}{c} \\ \mathbb{1} \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} \mathbb{1} & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha \end{array} \\ \left(\begin{array}{ccccccccccccc}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & -\frac{1}{2} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right) \end{array} \tag{5.45}$$

The observable matter sectors of this model produce three chiral generations, a minimal SM Higgs ($n_h = 1$) and a minimal heavy Higgs ($n_H = 1$). There exists colour triplets from the vectorial **10** representation of $SO(10)$ as $n_3 = 1$ and $n_{\overline{3}} = 1$, but as there are equal numbers of them heavy mass can be generated and there exists no anomaly in the LRS gauge group from these sectors. This model also contains no enhancements. The numbers defined in table 7 for the spinorial LRS exotic sectors of this model are as follows: $n_{L_Ls} = n_{\overline{L}_Ls} = 1$, $n_{L_Rs} = n_{\overline{L}_Rs} = 1$. The vectorial LRS exotics have the values $n_{3v} = n_{\overline{3}v} = 1$ and $n_{1v} = n_{\overline{1}v} = 5$.

The Pati-Salam exotic states have the values $n_{L_L e} = 4$, $n_{L_R e} = 10$ and $n_{3e} = n_{\bar{3}e} = n_{1e} = n_{\bar{1}e} = 0$. The model therefore has no anomaly under the LRS gauge group (*i.e.* all the exotic states are in vector–like representations) but is pseudo-anomalous under the $U(1)_2$ and $U(1)_3$ gauge groups as one linear combination of the charges can be defined which is anomaly-free, or alternatively another linear combination of the charges can be defined which is anomalous, but can then be cancelled by the Green-Schwarz mechanism [64, 65]. The model contains exotic multiplets and is therefore not exophobic. The model also admits a top quark mass coupling of order one.

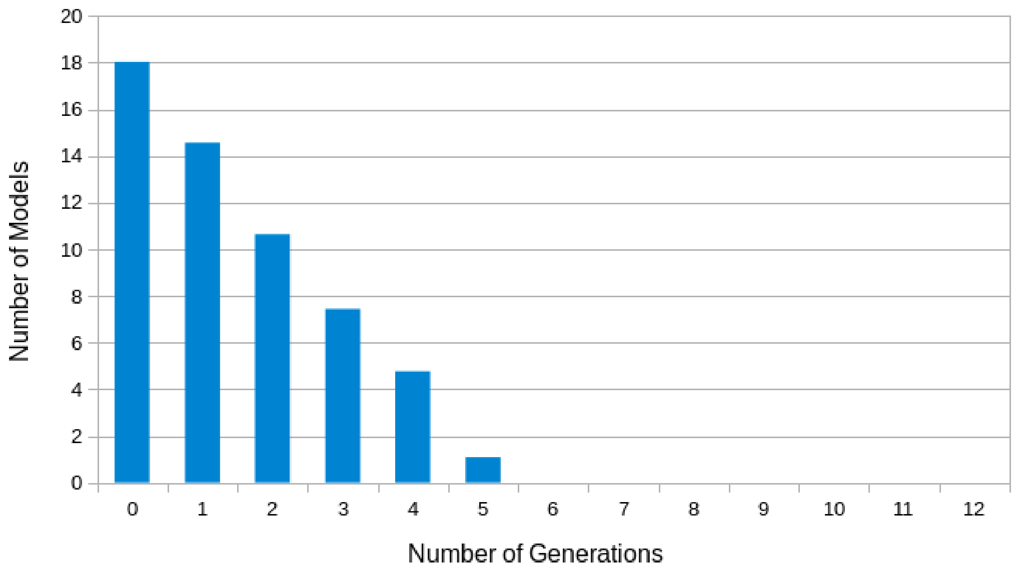

Figure 5: *Natural logarithm of the number of models against the number of generations (n_g) in a random sample of* 10^9 *GGSO configurations.*

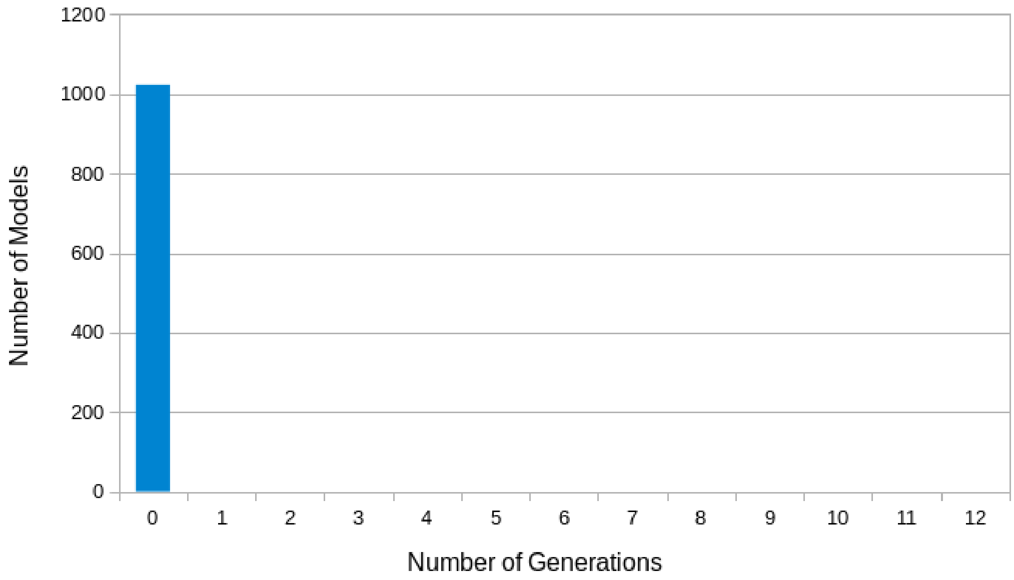

Figure 6: *Number of exophobic models against the number of generations in a random sample of* 10^9 *GGSO configurations.*

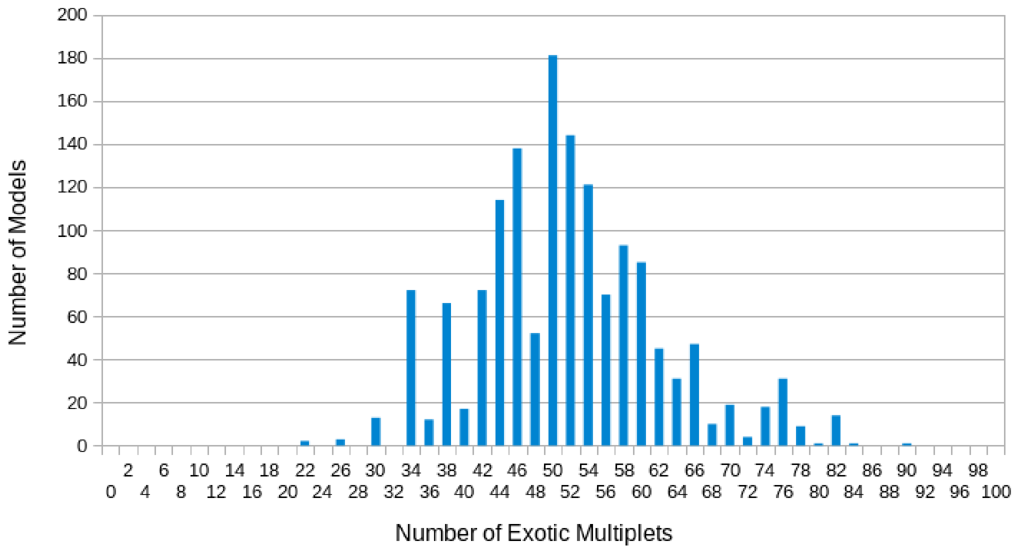

Figure 7: *The number of three generation models with no chiral exotic multiplets against the number of exotic multiplets in a random sample of* 10^9 *GGSO configurations.*

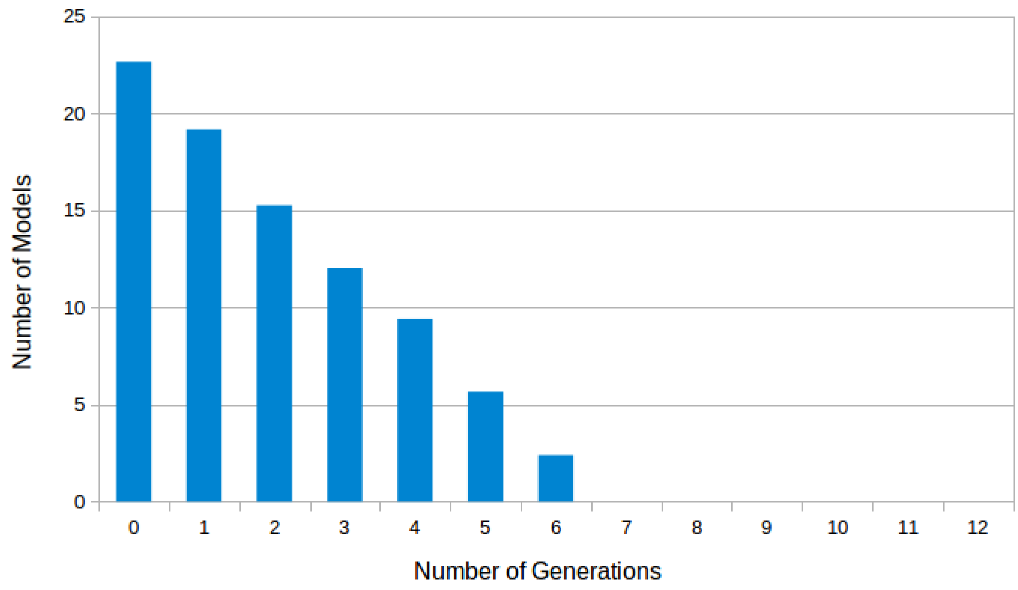

Figure 8: *Natural logarithm of the number of models against the number of generations (n_g) in a random sample of* 10^{11} *GGSO configurations.*

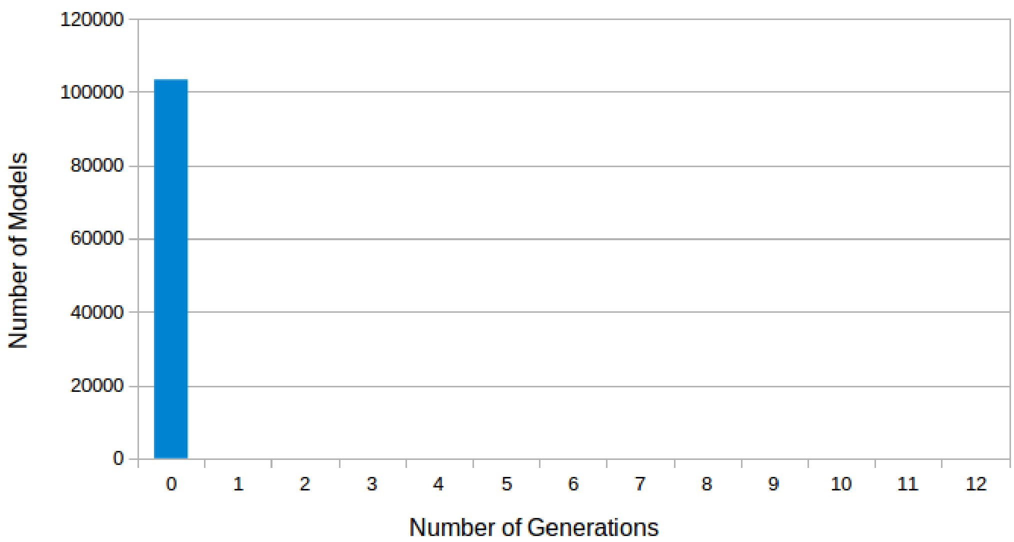

Figure 9: *Number of exophobic models against the number of generations in a random sample of* 10^{11} *GGSO configurations.*

# Exotic Multiplets	Pati-Salam	Spinorial	Vectorial
0	0	5536	1720
2	0	0	0
4	0	20854	3215
6	0	0	0
8	0	26727	19764
10	0	0	0
12	319	19102	4272
14	3030	0	0
16	894	10616	19750
18	15580	0	0
20	18598	1648	2157
22	13014	0	0
24	8703	3796	18673
26	15918	0	0
28	3528	739	1532
30	3386	0	0
32	1797	169	8093
34	2632	0	0
36	1169	0	952
38	398	0	0
40	25	73	7209
42	233	0	0
44	0	0	600
46	35	0	0
48	0	0	1212
50	1	0	0
52	0	0	9
54	0	0	0
56	0	0	46
58	0	0	0
60	0	0	40
62	0	0	0
64	0	0	16

Table 9: *The number of models is presented with respect to the number of Pati-Salam, Spinorial and Vectorial exotic multiplets.*

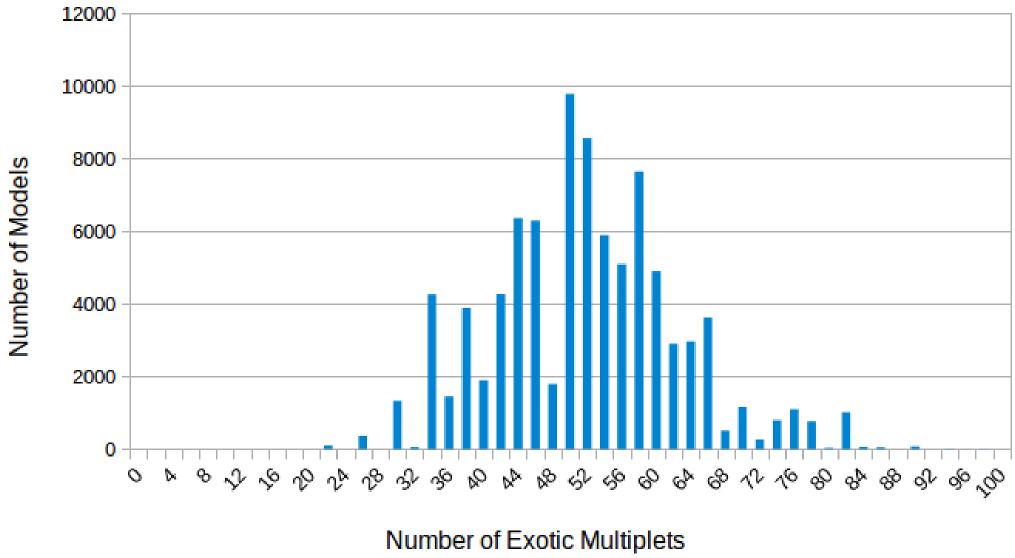

Figure 10: *The number of three generation models with no chiral exotic triplets against the number of exotic multiplets in a random sample of* 10^{11} *GGSO configurations.*

	Constraints	Total models in sample	Inferred Frequency	Estimated number of models in class
	No Constraints	100000000000	1	7.38×10^{19}
(1)	+ No Enhancements	70882805410	7.09×10^{-1}	5.23×10^{19}
(2)	+ Complete Families	7023975614	7.02×10^{-2}	5.18×10^{18}
(3)	+ No Chiral Exotic Triplets	4291254503	4.29×10^{-2}	3.17×10^{18}
(4)	+ Three Generations	89260	8.93×10^{-7}	6.59×10^{13}
(5)	+ SM Light Higgs + & Heavy Higgs	29	2.9×10^{-10}	2.14×10^{10}
(6)	+ Minimal Heavy Higgs & Minimal SM Light Higgs	22	2.2×10^{-10}	1.62×10^{10}
(7)	+ Top Quark Mass Coupling	4	4.0×10^{-11}	2.95×10^{9}

Table 10: *Statistics for the LRS models with respect to phenomenological constraints for* 10^{11} *models.*

6 Conclusions

The free fermionic formulation of the heterotic string offers one of the best formulations in order to study the field of string phenomenology. While it is true that the string models presented in this thesis are still referred to as quasi-realistic, the accuracy of the phenomenology of free fermionic models in general has steadily progressed since the models were first presented in the mid- to late-1980's [29, 30, 41, 42] and the early 2000's [43, 44]. The classification procedure has also been elaborated upon in order to gain more information from the models found during a classification scan [47]. For example, in the classification procedure presented in the previous chapter, the number of models which possess a top quark Yukawa coupling was investigated, which has not previously been achieved.

In this thesis, a new approach to the construction of Left-Right Symmetric models was presented and the classification procedure was applied to them. In previous constructions, LRS models utilised two separate basis vectors α, γ which resulted in breaking the observable $SO(10)$ symmetry to the $SO(6) \times SO(4)$ Pati-Salam gauge group before finally reaching the $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ LRS gauge group [44], whereas construction in the current work utilised the addition of only one basis vector α in order to break the $SO(10)$ directly to the LRS gauge group[12]. The LRS models in [44] also contained a 'NAHE'-basis [41, 66] whereas the LRS models presented here use a basis which has symmetric boundary conditions, as found in [46, 49, 50, 52]. The result of these differences between the LRS models here and in reference [44], is that the models here admit an x-map [56] whereas the previous LRS models do not. It is due to the presence of the x-map and of the vector 2α that these models

[12]It is noted that as a consequence of the complex boundary conditions contained in the basis vector α, the linear combination 2α also exists which breaks the observable $SO(10)$ gauge group to the observable Pati-Salam gauge group. However, the breaking of the $SO(10)$ to the observable LRS gauge group does not fundamentally require the intermediate Pati-Salam step, in contrast with the previous LRS models of reference [44].

appear more constrained than for the cases of the FSU(5), PS and SLM models. The inclusion of these two vectors also gives rise to many more exotic sectors, which can potentially give rise to exotic states. This is the reason why there are, in general, more exotic states for these LRS models as opposed to the FSU(5), PS and SLM models and is likely to be the reason why there were no exophobic models found with $n_g > 0$ for the case of the LRS models.

In the case of the LRS models presented, the addition of one basis vector α breaks the $SO(10)$ gauge group to the LRS observable gauge group. Consequently, due to the inclusion of complex boundary conditions in α, the linear combination 2α exists which also breaks the SO(10) gauge group to the Pati-Salam observable gauge group. In these models, there are therefore two vectors which break the observable $SO(10)$ gauge symmetry. This is the same as the SLM case where there are necessarily two basis vectors which break the observable $SO(10)$. The presence of two $SO(10)$ breaking vectors constrains these models to a higher degree than for the FSU(5) and PS cases where there is only one vector which breaks the $SO(10)$. This is the cause for the relative suppression in the number of three generation models in the LRS and SLM cases. Some new approaches to the classification procedure which create a better yield for the number of three generation LRS and SLM models are discussed in section 6.1.

In the case of free fermionic $SU(4) \times SU(2) \times U(1)$ (SU(421)) models and the LRS models, the vector 2α breaks the $SO(10)$ symmetry. This is not the case for FSU(5), PS and SLM models where the vector 2α leaves the $SO(10)$ symmetry unbroken. In the case of the SU(421) this leads to an absence of models which had complete generations of matter [45, 46]. This is not the case for LRS models which is proven by the classification presented. In the case of the SU(421) models, the 2α projection selects either the left- or right-handed Standard Model states to survive. However, due to a remaining freedom in the phase of the $U(1)_{1/2/3}$ symmetries in the LRS models, there exist both left- and right-handed Standard Model states after the 2α projection. This is the reason why LRS models admit complete families of chiral matter whereas SU(421) models do not.

6.1 Outlook

The future of this area of research is set to take an interesting direction. One line of research has been into the investigation of so-called 'fertile regions' of free fermionic heterotic string models. In short, certain conditions can be specified which leads to fertile regions which can then be explored in order to produce a greater yield of models with desirable phenomenology, such as three chiral generations of matter. The method has so far been applied to Standard-Like models with success as can be seen in the reference [52], along with a more detailed description of the method. In fact, fertile regions have been found for the class of LRS models presented in this thesis and a publication detailing a classification which utilises these fertile regions will appear in the future.

Another interesting direction of research is into the application of machine learning techniques to free fermionic heterotic string models. Use of these techniques has been shown to greatly improve the frequency with which phenomenologically interesting free fermionic models (such as models which have three generations of matter) are found when compared to the pseudo-random approach used in previous classifications [46, 50]. Such an example is the application of Genetic Algorithms (GA's) which can be found in reference [67].

The use of deep reinforcement learning[13] has also been applied to the class of LRS free fermionic models presented in the previous chapter by the author and collaborators. The aim of this research is to utilise deep reinforcement learning in order to increase the frequency with which free fermionic models of notable phenomenology are found. Another interesting application is the ability of the program to discover the conditions on the fertile regions of particular models. While the latter described ap-

[13]The term deep reinforcement learning applies to the use of reinforcement learning techniques where an artificial neural network is applied.

plication is theoretically possible, the concept is currently practically unproven. The application of deep reinforcement learning to free fermionic models is still a work in progress but will appear in a future publication.

In conclusion, the use of free fermionic heterotic string models provide a detailed framework in order to bring string theory closer in line with possible detection by experiment. In the hope that string theory proves somehow relevant to a description of the known Universe, the field of string phenomenology will already be relatively well defined. Even in the case that string theory is proven to not be a complete description of the Universe, the hope is that the methods developed in order to study string phenomenology, such as the creation of the classification procedure and the application of machine learning techniques, will be transferable to other areas of interest to research.

A An Overview of String Model Scans Performed by Other Research Groups

There have been various different approaches to performing computerised scans over different constructions of string vacua. For example, reference [68] performed a counting of supersymmetric solutions to the tadpole cancellation conditions in Type IIA intersecting D-brane models. The analysis then extended to finding the statistical distribution of various quantities in six-dimensional models on a $T^4/\mathbb{Z}_2$ orientifold, such as the probability of the appearance of an $SU(M)$ gauge factor, the rank of the gauge group and the number of generations. This was done by employing a 'semi-analytical saddle point' method (the details of which can be found in [68]) in order to make the problem tractable. This is in contrast to the brute force computer search method utilised by the classification method performed for the LRS string vacua presented in Chapter 5, as applying the brute force search method to the construction of string models in reference [68] would not have produced results in a reasonable time. The research presented in [68] then generalised this semi-analytical saddle point method to analyse the same distribution of intersecting branes on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold.

String models with the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold were also considered in reference [69] and therefore studied the same ensemble of intersecting brane models as was introduced in reference [68]. Reference [69] consists of the construction of tools for analysing the space of intersecting brane models before specifically applying them to Type IIB $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds. However, in the approach of reference [69] algorithms were produced which could enumerate all (in principle) of the configurations on this orientifold which satisfy the Diophantine equations arising from supersymmetry, which are finite in number. Another contrast between the two works is that reference [69] had a greater focus on the enumeration of models which have a specific visible gauge group and charged matter content, therefore allowing for the searching of SMs with three chiral generations of matter.

There have also been computerised scans involving the use of Gepner models. The paper [70] considered open string Gepner models which are supersymmetric, tadpole free, four-dimensional models with $\mathcal{N} = 1$ SUSY. The scan found $\sim 1.8 \times 10^5$ of these models which had only the SM as their chiral spectrum, but most also had non-chiral exotics and / or mirror pairs. Similar to the classification scan of the LRS vacua presented in this thesis, one aspect of the scan in [70] was to calculate the number of Higgs particles each model had. This allowed for the investigation into the statistical distribution of the number of Higgs particles of each model. Reference [70] reported a distribution peak of 3 Higgs pairs for the models analysed, with a maximum number of 56 Higgses.

Reference [71] considered an extension to the search of Gepner models approach in [70]. Specifically, the intention was to reduce the impact of some of the assumptions made in [70] and to investigate a large number of ways to realise the SM with D-branes. In total, approximately 1.9×10^4 models with a chirally distinct top-down SM spectrum before tadpole cancellation and 1.9×10^3 chirally distinct models which solved the tadpole conditions were found. Among various other results, the first examples of supersymmetric $SU(5)$ and $SU(5) \times U(1)$ orientifold vacua were found which had the correct chiral spectrum (*i.e.* no extra gauge groups and no states which were exotic with respect to the Chan-Paton gauge group).

The search method of reference [71] was further extended in reference [72], which built string models from orientifolds of Type IIB closed strings in a free fermionic construction. Specifically, it used an algebraic approach to constructing orientifold vacua, as opposed to a more geometric approach considered when building the LRS free fermionic models which are the topic of this thesis. The search method was employed to find, among various other phenomenological criteria, three generation standard model configurations with tadpole cancellation in the particular class of models constructed. Although no such models were found, one special case which utilised complex free fermions was found if the criteria of tadpole cancellation was relaxed.

Although there are some similarities between the search methods employed in the references [68] - [72] and the method used to classify the LRS models presented in this thesis, the comparison of the models constructed by these other groups and the free fermionic LRS models is limited due to the fact they use primarily different constructions of string theory. However, an interesting line of research could be to generalise some of the search methods developed by other groups to the case of free fermionic heterotic string models, such as those constructed in this thesis, to investigate if more phenomenologically interesting vacua could be obtained in a shorter computational period.

Scans over classes of heterotic string models have also been performed by other groups, which allow for a more direct comparison with the free fermionic heterotic string models constructed in Chapter 5. In particular, in 1989 D. Sénéchal used the free fermionic formulation of string theory in order to utilise a computerised method for the generation of four-dimensional heterotic string models [73]. A random generator was implemented in order to generate spin structures and therefore produce distinct string models of which the massless spectrum could be analysed. In contrast to the LRS string models found in this thesis (for which $\mathcal{N} = 1$), the search for models in reference [73] allowed a varying number of supersymmetries. In reference [73], 900 models with $\mathcal{N} = 4$ and over 3.2×10^4 models with $\mathcal{N} = 1$ were found.

Also in contrast to the LRS models presented in this thesis, models in reference [73] were found with varying observable gauge groups. These included a direct SM gauge group embedding, in addition to the SM embedding in a grand unified gauge structure, specifically the $SU(4) \times SU(2)_L \times SU(2)_R$ (Pati-Salam) and the $SU(5)$ gauge groups.

However, in the same manner as with the LRS classification, low energy requirements of favourable phenomenology were imposed in the search method of reference [73] and models were found which obeyed them. Specifically, reference [73] required a non-anomalous linear combination of $U(1)$ factors which fit the standard hypercharge assignment in addition to the requiring at least one complex Higgs isodoublet, among various other requirements. The difference between the two approaches is the LRS classification method built a statistical distribution of how many models satisfied the phenomenological criteria, while recording the massless spectrum of those which did, whereas in [73] no such statistical distribution was presented.

The computer program developed in reference [73] was then used by K. R. Dienes to generate non-supersymmetric four-dimensional heterotic string models [74]. The aim was to investigate the conjecture that these 4D non-SUSY models lead to vanishing one-loop cosmological constants by searching for models with specific criteria (as opposed to a statistical analysis of the total of the scanned models). Although no specific models were found, the scan produced $\sim 1.23 \times 10^5$ distinct models with 4303 different partition functions which satisfied various constraints imposed by the physical consistency of the underlying string models. In reference [75] the total sample generated by the scan in reference [74] was statistically analysed. The focus was on the statistical correlations which emerge between quantities of the models such as the gauge group and their one-loop cosmological constants.

Using an updated version of the code used to generate the string models found in references [73, 74, 75], reference [76] presented a larger data set of approximately 10^7 heterotic string models which were randomly generated. This data set of string models was then analysed in order to study the statistical correlations between the gauge symmetries and the degree of spacetime supersymmetry of the generated string models. This analysis found that string models with unbroken spacetime supersymmetry at the string scale tended to favour gauge group factors with a larger rank. The statistical analysis also showed that nearly 50% of the models were non-supersymmetric but were still tachyon-free at tree level, in addition to less than a 25% of the tree-level models exhibiting no supersymmetry at the string scale.

The classifications performed in this thesis involve both the searching method of finding single string models which are of phenomenological interest, such as in references [73] and [74], but also perform the statistical analysis approach of references [75, 76]. In contrast to the work of references [73, 74, 75, 76], models with only $\mathcal{N} = 1$ SUSY and one gauge group were analysed. This, combined with the increase of computing power available, allowed for the scan size of vacua to increase from $\mathcal{O}(10^5) - \mathcal{O}(10^7)$ distinct models to $\mathcal{O}(10^{11})$ vacua performed in the present classification. It also allows for a reduced computation time when searching for more imposed phenomenological criteria while producing the complete massless spectrum of each string model.

The classification method in this thesis utilises a random generation method which provides the data set of heterotic string models. There have been other approaches used in order to generate the data sets of string models, such as in reference [77]. In reference [77], a 'fertile region' of the heterotic landscape was found by considering the $E_8 \times E_8$ heterotic string compactified on a $\mathbb{Z}_6$-II orbifold [78, 79, 80] with $SO(10)$ and E_6 local GUT structures. The search strategy was based on the concept of local GUTs which inherit certain features of standard grand unification, such that while the local GUTs are specific to certain points in the compact space, the 4D gauge symmetry is that of the SM. Using this search method, it was shown that approximately 1% of models allowed the exact MSSM spectrum out of a total of 3×10^4 inequivalent models that were found.

While the execution of finding fertile regions for the LRS models constructed in this thesis is fundamentally different to that of reference [77] (due to the differences in the constructions of the string models), the concept of investigating regions in the string landscape which give rise to a higher statistical likelihood of phenomenologically interesting models remains an interesting one. This is the focus of a current research project by the author for the class of LRS models constructed in this thesis and the results will appear in a future publication.

There are other examples of research groups performing analysis of large sets of string models which are not mentioned here. For the interested reader, some of these can be found in reference [81] and references therein.

References

[1] S. P. Martin, arXiv:hep-ph/9709356

[2] The Atlas Collaboration, arXiv:1207.7214, *Phys. Lett.* **B716** (2012) 1.
The CMS Collaboration, arXiv:1207.7235, *Phys. Lett.* **B716** (2012) 30.

[3] M. E. Peskin, D. V. Schroeder *'An Introduction to Quantum Field Theory'*. Perseus Books. 1995.

[4] M. Harris, '*Left-Right Symmetric Model* '. Uppsala University.

[5] Y. Kawamura, arXiv:1311.2365

[6] C. Wetterich, *Phys. Lett.* **B718** (2012) 573

[7] C. Csaki, arXiv:hep-ph/9606414

[8] C. Quigg, arXiv:hep-ph/0404228

[9] G. Senjanovic, R. N. Mohapatra, *Phys. Rev.* **D12** (1975) 1502

[10] Super-Kamiokande Collaboration, *Phys. Rev. Lett.* **81** (1998) 1562

[11] M. Lindner, T. Ohlsson, G. Seidl, arXiv:hep-ph/0109264, *Phys. Rev.* **D65** (2002) 053014

[12] R. N. Mohaptra, arXiv:hep-ph/9912272 and references therein.

[13] G.B. Cleaver, A.E. Faraggi and C. Savage, *Phys. Rev.* **D63** (2001) 066001

[14] A.E. Faraggi, G. Harries, J. Rizos, arXiv:1806.04434, *Nucl. Phys.* **B936** (2018) 472

[15] D. Tong, arXiv:0908.0333.

[16] Alon Faraggi, *'Lectures on String Phenomenology'*, Ohio State University.

[17] L. J. Dixon, *'Introduction to Conformal Field Theory and String Theory'*. SLAC-PUB-5149 (1989)

[18] M. B. Green, J. H. Schwarz, E. Witten *'Superstring Theory. Vol 1: Introduction'*. Cambridge Monogr. Math. Phys 1988.

[19] M. B. Green, J. H. Schwarz, E. Witten *'Superstring Theory. Vol 2: Loop Amplitudes, Anomalies And Phenomenology'*. Cambridge Monogr. Math. Phys 1988.

[20] R. Blumenhagen, R. Plauschinn *'Introduction to Conformal Field Theory'*. Springer. 2009.

[21] R. Blumenhagen, D. Lüst, S. Theisen *'Basic Concepts of String Theory*. Springer. 2013.

[22] E. Kiritsis *'String Theory in a Nutshell'*. Princeton University Press. 2007.

[23] K. Becker, M. Becker, J. H. Schwarz *'String theory and M-theory: A modern introduction'*. Cambridge University Press. 2006.

[24] J. Polchinki *'String theory. Vol. 1: An introduction to the bosonic string.'*. Cambridge University Press. 2007.

[25] J. Polchinki *'String theory. Vol. 2: Superstring theory and beyond.'*. Cambridge University Press. 2007.

[26] B. Zwiebach *'A First Course in String Theory'*. Cambridge University Press. 2009.

[27] T. Mohaupt, arXiv:hep-th/0207249

[28] P. Athanasopoulos, *'Relations in the space of (2,0) heterotic string models'*, University of Liverpool. 2012.

[29] I. Antoniadis, C. Bachas, and C. Kounnas, *Nucl. Phys.* **B289** (1987) 87;
I. Antoniadis and C. Bachas, *Nucl. Phys.* **B298** (1988) 586.

[30] H. Kawai, D.C. Lewellen, and S.H.-H. Tye, *Nucl. Phys.* **B288** (1987) 1

[31] I. Antoniadis, C. Bachas, C. Kounnas and P. Windey, *Phys. Lett.* **B171** (1986) 51

[32] K.R. Dienes, *Phys. Rev. Lett.* **65** (1990) 1979; *Phys. Rev.* **D42** (1990) 2004;
S. Abel and K.R. Dienes, *Phys. Rev.* **D91** (2015) 126014;
J.M. Ashfaque, P. Athanasopoulos, A.E. Faraggi and H.Sonmez, *Eur. Phys. Jour.* **C76** (2016) 208;
M. Blaszczyk, S. Groot Nibbelink, O. Loukas and F. Ruehle, *JHEP* **1510** (2015) 166;
S. Groot Nibbelink and E. Parr, *Phys. Rev.* **D94** (2016) 041704;
I. Florakis and J. Rizos, *Nucl. Phys.* **B913** (2016) 495;
I. Florakis, arXiv:1611.10323;
B. Aaronson, S. Abel and E. Mavroudi, *Phys. Rev.* **D95** (2017) 106001;
S. Abel, K.R. Dienes and E. Mavroudi, arXiv:1712.06894;
T. Coudarchet and H. Partouche, arXiv:1804.00466.

[33] C. Núñez, 'Introduction to Superstring Theory'. https://inis.iaea.org/collection/NCLCollectionStore/_Public/40/084/40084716.pdf

[34] F. Gliozzi, J. Scherk and D. I. Olive, *Nucl. Phys.* **B122** (1977) 253.

[35] S. Abel, 'Cargese Lectures: A String Phenomenology Primer'. http://www.maths.dur.ac.uk/ dma0saa/cargese_001.pdf. 2007.

[36] For review and references see *e.g:* L.E Ibanez and A.M Uranga, *String theory and particle physics: an introduction to string phenomenology*, Cambrdige University Press, 2012.

[37] S. E. M. Nooij, arXiv:hep-th:0603035.

[38] S. Lang, *'Algebra'*. Addison-Wesley. 1965.

[39] A.E. Faraggi, C. Kounnas and J. Rizos, *Phys. Lett.* **B648** (2007) 84; *Nucl. Phys.* **B774** (2007) 208; *Nucl. Phys.* **B799** (2008) 19.

[40] B. Assel, C. Christodoulides, A.E. Faraggi, C. Kounnas and J. Rizos *Phys. Lett.* **B683** (2010) 306; *Nucl. Phys.* **B844** (2011) 365;
C. Christodoulides, A.E. Faraggi and J. Rizos, *Phys. Lett.* **B702** (2011) 81.

[41] I. Antoniadis, J. Ellis, J. Hagelin and D.V. Nanopoulos, *Phys. Lett.* **B231** (1989) 65

[42] I. Antoniadis. G.K. Leontaris and J. Rizos, *Phys. Lett.* **B245** (1990) 161;
G.K. Leontaris and J. Rizos, *Nucl. Phys.* **B554** (1999) 3.

[43] A.E. Faraggi, D.V. Nanopoulos and K. Yuan, *Nucl. Phys.* **B335** (1990) 347;
A.E. Faraggi, *Phys. Lett.* **B278** (1992) 131; *Nucl. Phys.* **B387** (1992) 239;
G.B. Cleaver, A.E. Faraggi and D.V. Nanopoulos, *Phys. Lett.* **B455** (1999) 135;
A.E. Faraggi, E. Manno and C.M. Timirgaziu, *Eur. Phys. Jour.* **C50** (2007) 701.

[44] G.B. Cleaver, A.E. Faraggi and C. Savage, *Phys. Rev.* **D63** (2001) 066001;
G.B. Cleaver, D.J Clements and A.E. Faraggi, *Phys. Rev.* **D65** (2002) 106003;

[45] G.B. Cleaver, A.E. Faraggi and S.E.M. Nooij, *Nucl. Phys.* **B672** (2003) 64.

[46] A.E. Faraggi and H. Sonmez, *Phys. Rev.* **D91** (066006) 2015

[47] A. Gregori, C. Kounnas and J. Rizos, *Nucl. Phys.* **B549** (1999) 16.

[48] A.E. Faraggi, C. Kounnas, S.E.M Nooij and J. Rizos, hep-th/0311058; *Nucl. Phys.* **B695** (2004) 41.

[49] L. Bernard *et al*, *Nucl. Phys.* **B868** (2013) 1.

[50] A.E. Faraggi, J. Rizos and H. Sonmez, *Nucl. Phys.* **B886** (2014) 202.

[51] H. Sonmez, *Phys. Rev.* **D93** (2016) 125002.

[52] A.E. Faraggi, J. Rizos and H. Sonmez, *Nucl. Phys.* **B927** (2018) 1.

[53] A.E. Faraggi, arXiv: hep-ph/9501288.
A.E. Faraggi, *Phys. Lett.* **B326** (1994) 62.

[54] L.E. Ibáñez and A.M. Uranga *'String Theory and Particle Physics - An Introduction to String Phenomenology'*. Cambridge University Press. 2012.

[55] A. Salam and J.C. Pati, *Phys. Rev.* **D10** (1975) 275:
R.N. Mohapatra and J.C. Pati, *Phys. Rev.* **D11** (1975) 566; *Phys. Rev.* **D11** (1975) 2558.

[56] A.E. Faraggi, *Nucl. Phys.* **B407** (1993) 57; *Eur. Phys. Jour.* **C49** (2007) 803;
G. Cleaver and A.E. Faraggi, *Int. J. Mod. Phys.* **A14** (1999) 2335.

[57] K. Abe, *et al.*, *Phys. Rev.* **D95** (2017) 012004

[58] X.G. Wen and E. Witten, *Nucl. Phys.* **B261** (1985) 651;
G. Athanasiu, J. Atick, M. Dine, and W. Fischler, *Phys. Lett.* **B214** (1988) 55.

[59] A.N. Schellekens, *Phys. Lett.* **B237** (1990) 363

[60] A.E. Faraggi, *Phys. Rev.* **D46** (1992) 3204;
S. Chang, C. Coriano and A.E. Faraggi, *Nucl. Phys.* **B477** (1996) 65;
C. Coriano, A.E. Faraggi and M. Plumacher, *Nucl. Phys.* **B614** (2001) 233.

[61] See e.g. V. Halyo *et al.*, *Phys. Rev. Lett.* **84** (2000) 2576.

[62] J. Rizos, K. Tamvakis, *Phys. Lett.* **B262** (1991) 227

[63] J. Rizos, *Eur. Phys. Jour.* **C74** (2014) 2905.

[64] M.B. Green and J.H. Schwarz, *Phys. Lett.* **B147** (1984) 117.

[65] M. Dine, N. Seiberg and E. Witten, *Nucl. Phys.* **B289** (1987) 589.

[66] A.E. Faraggi and D.V. Nanopoulos, *Phys. Rev.* **D48** (1993) 3288.

[67] S. Abel, J. Rizos, arXiv:1404.7359, *JHEP* **1408** (2014) 010

[68] R. Blumenhagen *et al*, *Nucl. Phys.* **B713** (2005) 83.

[69] M.R. Douglas and W. Taylor, *JHEP* **0701** (2007) 031.

[70] T.P.T. Dijkstra, L. Huiszoon and A.N. Schellekens, *Nucl. Phys.* **B710** (2005) 3.

[71] P. Anastasopoulos, T.P.T. Dijkstra, E. Kiritsis and A.N. Schellekens, *Nucl. Phys.* **B759** (2006) 83.

[72] E. Kiritsis, M. Lennek and A.N. Schellekens, *JHEP* **0902** (2009) 030.

[73] D. Senechal, *Phys. Rev.* **D39** (1989) 3717.

[74] K.R. Dienes, *Phys. Rev. Lett.* **65** (1990) 1979.

[75] K.R. Dienes, *Phys. Rev.* **D73** (2006) 106010.

[76] K.R. Dienes, M. Lennek, D. Senechal and V. Wasnik, *Phys. Rev.* **D75** (2007) 126005.

[77] O. Lebedev *et al*, *Phys. Lett.* **B645** (2007) 88.

[78] T. Kobayashi, S. Raby, and R. J. Zhang, *Phys. Lett.* **B593** (2004) 262.

[79] T. Kobayashi, S. Raby, and R. J. Zhang, *Nucl. Phys.* **B704** (2005) 3.

[80] W. Buchmuller, K. Hamaguchi, O. Lebedev and M. Ratz, arXiv:hep-th/0606187.

[81] M.R. Douglas, *JHEP* **0305** (2003) 046;
F. Denef and M.R. Douglas, *JHEP* **0405** (2004) 072;
B.S. Acharya, F. Denef and R. Valadro, *JHEP* **0506** (2005) 056;
L.B. Anderson, A. Constantin, J. Gray, A. Lukas and E. Palti, *JHEP* **1401** (2014) 047;
J. Halverson and P. Langacker, arXiv:1801.03503.