



A SIMPLICIAL APPROACH TO STRATIFIED HOMOTOPY
THEORY

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Abstract

This thesis provides a framework to study the homotopy theory of stratified spaces, in a way that is compatible with previous approaches. In particular our approach will be closely related to the work of Frank Quinn on homotopically stratified sets. We introduce a stratified analogue of the geometric realisation-singular simplicial set adjunction, allowing us to relate simplicial sets to stratified spaces. This allows us to cofibrantly transfer the Joyal model structure from simplicial sets to the category of fibrant stratified spaces. We have chosen to use the Joyal model structure on simplicial sets over the Quillen model structure. This choice allows a partial (ordered) composition of simplices, which under the stratified adjunction corresponds to concatenation of stratified paths. One of the biggest advantages of working in a simplicially enriched model structure is the ability to exploit the combinatorial nature of simplicial sets, which helps us to prove results about stratified spaces. By studying the cofibrations and fibrations that we transfer to the category of stratified spaces, we see that the cofibrant stratified spaces satisfy one condition that Quinn imposed for homotopically stratified sets, and that the fibrant stratified spaces satisfy the other condition. Consequently, the cofibrant-fibrant stratified spaces in our model structure are closely related to homotopically stratified sets.

To use our framework to study homotopy theory, we need a notion of basepoint for a stratified space. We define the basing of a stratified space to be a factoring of the counit on the underlying poset through a choice of continuous map to the underlying topological space. The requirement of a stratified space to be based provides a restriction on the stratified spaces, and as such there are examples of cofibrant-fibrant stratified spaces which cannot be based. To justify this approach we are able to construct an adjunction between stratified suspension and loop space functors. In addition, we are able to construct an \mathbb{N} -indexed family of categories for a based fibrant stratified space, which we call the homotopy categories of a stratified space. Importantly, in the case of a trivially stratified connected space, the homotopy categories coincide with the homotopy groups of the underlying topological space. The homotopy categories of a based fibrant stratified space behave analogously to homotopy groups. For example, we are able to extract a long exact sequence of homotopy categories from a stratified fibration. Furthermore, we are able to provide partial results towards construction of a Postnikov Tower of a based fibrant stratified space. Further research is required to complete this construction, which would hopefully lead to a stratified analogue of Eilenberg-Mac Lane spaces.

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Chapter 1

Introduction

In this PhD thesis, we study the homotopy theory of stratified spaces by introducing a model structure on an appropriate category of stratified spaces. Once we have settled on a notion of basing of a stratified space, this approach allows us to construct homotopy invariants of fibrant stratified spaces, which behave analogously to homotopy groups of topological spaces. The main motivation of this thesis is to construct a framework in which we can prove results about stratified spaces that closely parallel their topological counterparts. By way of an illustration of the power of the categorical framework, we can prove an analogue of Whitehead’s Theorem for stratified spaces. Another motivation behind this project was to unify the language and definitions used in the study of stratified spaces. Throughout the literature there are various different definitions of stratified spaces, and in particular of fibrations between stratified spaces. The hope is that by introducing a model structure that captures the homotopy theory of stratified spaces that we desire, the natural definitions should be apparent.

Stratified spaces naturally arise in many areas of mathematics; classical examples include algebraic geometry, differential geometry, singularity theory, and more recently in theoretical physics (specifically within string theory), computer science, and even in neuroscience. As such, there have been many definitions provided of stratified spaces, each designed to suit the author’s intended purpose. Originally, Hassler Whitney studied stratified spaces from the perspective of differential geometry. As such, Whitney stratified spaces are built as collections of strata (submanifolds) which fit together smoothly to give a stratified space. Whitney stratified spaces arise across mathematics; some illustrative examples are given by CW-complexes (stratified by skeleta), triangulations (stratified by simplices), configuration spaces (stratified by “collisions”), or complex analytic varieties (as proven in [Whi64]). Work of Thom and Mather extended Whitney’s definition to Thom-Mather stratified spaces, and such spaces satisfy Thom’s Isotopy Lemmas. In particular this means that Thom-Mather stratified spaces satisfy a local triviality condition on the stratifications, local triviality along a stratum, and topological stability of smooth mappings. For more details see [Tho69] and [Mat71], and a

historical guide to their works can be found in [Gor12].

In particular our approach to stratified spaces will be closely related to the study of homotopically stratified sets, introduced by Frank Quinn in [Qui88]. Quinn was interested in quotients by group actions on manifolds, and the stratifications that arise in this context (the orbit type stratification) are not Thom-Mather stratified. Motivated to study the topological phenomena of stratified spaces, Quinn importantly defined homotopically stratified sets by topological conditions (a path fibration condition) rather than imposing geometric conditions (a bundle structure on neighbourhoods of strata). Specifically, they are required to satisfy both a stratum neighbourhood condition, and a holink path space fibration condition. The holink captures the homotopy type of the stratified path space between strata. As an illustration, Thom-Mather stratified spaces provide examples of homotopically stratified sets. Homotopically stratified sets were studied further by Bruce Hughes in [Hug96] and [Hug99], and recently by David Miller in [Mil09] and [Mil13].

The notion of a poset stratified space is used in our work, and was first written down explicitly by Jacob Lurie [Lur14, §A.5], although the idea of a stratified space indexed by a poset has been inherent in the literature dating back to Hassler Whitney [Whi64]. The idea is to abstractly define a category of spaces for which the strata form a poset. For the category of poset stratified spaces to provide a suitable setting for studying homotopy theory, we require it to satisfy certain properties. Chapter 6 is devoted to a proof of Theorem A (a culmination of Proposition 6.1.4.1 and Proposition 6.2.0.7).

Theorem A. *The category of surjective poset stratified spaces is bi-complete and cartesian closed.*

On the category of surjectively poset stratified spaces, a suitable choice of model structure will give rise to conditions picking out classes of homotopically well behaved stratified spaces. We wish to mimic classical homotopy theory, in which the model structure on topological spaces is transferred from the Quillen model structure on simplicial sets; in that context, retracts of CW complexes are the homotopically well behaved spaces. To do this, we construct a stratified analogue of the geometric realisation-singular simplicial set adjunction between topological spaces and simplicial sets.

To construct a framework in which to study the homotopy theory of stratified spaces, we need to select the right model structure on simplicial sets to transfer. We choose not to use the Quillen model structure on simplicial sets, but instead the Joyal model structure introduced in [Joy]. This choice is because the Joyal model structure allows a partial (ordered) composition of arrows, corresponding (under the stratified adjunction) to concatenation of stratified paths. Under the stratified adjunction, higher dimensional horns are interpreted as families of stratified paths, and fillers for such horns correspond to higher associativity relations of composition of stratified paths. We construct the following model structure in Theorem 8.2.3.2.

Theorem B. *There is a cofibrantly transferred model structure on the category of fibrant stratified spaces, from the Joyal model structure on simplicial sets.*

A model structure on a category is defined by relaxing the closure under limits and colimits axiom of a model category. Currently we are unable to prove that this transfer gives us a model category on stratified spaces, but we suspect that this situation can at least be improved to give a cofibrantly transferred J -semi model category. The main obstruction to our progress is the difficulty in understanding stratified weak equivalences between arbitrary stratified spaces.

In the unstratified case, the adjunction between topological spaces and simplicial sets provides an equivalence between the homotopy theory of topological spaces and of simplicial sets. This result is stated mathematically as the existence of a Quillen equivalence between model structures on the categories of topological spaces and simplicial sets. A Quillen equivalence is a stronger requirement than an equivalence between homotopy categories, and is the correct notion of equivalence between model categories. We do not expect a Quillen equivalence between stratified spaces and simplicial sets, because the simplicial sets arising from stratified spaces will satisfy the property that any endomorphism is an isomorphism.

By studying the transferred cofibrations and fibrations in the model structure, we discover that the cofibrant stratified spaces satisfy one condition that Quinn imposed for homotopically stratified sets, and that the fibrant objects satisfy the other condition. As a consequence of the choice of the Joyal model structure, the cofibrant-fibrant stratified spaces in our model structure satisfy properties closely related to homotopically stratified sets. This is strong evidence that our transferred model structure is a good setting to study the homotopy theory of stratified spaces, because it is coherent with previous approaches.

Theorem C. *Cofibrant stratified spaces satisfy the stratum neighbourhood condition required of a homotopically stratified set, and fibrant stratified spaces satisfy the stratified path fibration condition.*

The details of this result can be found in Corollary 9.1.0.7 and Corollary 9.2.1.15. One advantage of our work, over Frank Quinn’s original work on homotopically stratified spaces, is that we do not need to impose local finiteness of strata or metric topology restrictions to prove results. Hence, the categorical framework in which we work provides cleaner statements and more direct analogies with topology. An example of this is that the model structure directly gives an analogue of Whitehead’s Theorem; a formal result which justifies the use of stratum preserving homotopy when studying stratified spaces. The transferred model structure on fibrant stratified spaces is enriched over the Joyal model structure on simplicial sets; this is a powerful statement because it allows us to prove topological results, exploiting the combinatorial nature of simplicial sets.

When studying the stratified spaces in the transferred model structure, it becomes clear that invariants of stratified spaces must contain more information than homotopy groups of the underlying topological space. This is best illustrated through considering the choice of model structure on simplicial sets, and the homotopy theory presented by each model structure. The Quillen model structure on simplicial sets has Kan complexes as fibrant objects, which provides a model for $(\infty, 0)$ -categories (or alternatively ∞ -groupoids). The Joyal model structure on the other hand has quasi-categories as fibrant objects, which are a model for

$(\infty, 1)$ -categories. This suggests that weak equivalences between stratified spaces should be detected by more than homotopy groups of the underlying topological space, because there is an extra layer of information required.

Theorem D. *A stratified morphism between fibrant stratified spaces which is a bijection on posets, is a weak equivalence if and only if it induces weak homotopy equivalences between strata and holinks.*

This result is Theorem 9.4.0.5 of this thesis. To construct homotopy invariants for stratified spaces, one first needs to define what it means for a stratified space to be based. The question is more subtle than in the unstratified case, and there are several possible approaches. The notion that we introduce, whilst natural, imposes a genuine constraint on the stratified space. Somewhat counter-intuitively, there are even cofibrant-fibrant stratified spaces which cannot be based. In particular, a basing of a stratified space means that poset relations are detected by stratified paths. Using this notion of basing for a stratified space, we construct the stratified loop space, and by introducing an even stronger notion of well-based spaces, we are also able to construct the stratified reduced suspension. These functors generalise to stratified spaces their topological counterparts.

Theorem E. *Restricting to well-based fibrant stratified spaces, stratified reduced suspension is left adjoint to the stratified loop space functor.*

This result follows from Theorem 10.2.0.25 of this thesis. In other works, poset stratified spaces have been studied for a variety of reasons. Jacob Lurie extended a classical correspondence of Robert MacPherson to the ∞ -categorical level. MacPherson noticed that the category of constructible sheaves of sets on a stratified space is equivalent to presheaves of sets on the exit path category of the space (this result is explained and extended to an equivalence between constructible stacks, and the 2-category of \mathbf{Cat} -valued 2-functors in [Tre09]). Lurie [Lur14, §A] extended this result to give an equivalence between the category of constructible sheaves of spaces on a poset-stratified space and the category of space valued functors out of the exit ∞ -path category (referred to as an exodromy equivalence). This result has been further generalised by Clark Barwick, Saul Glasman, and Peter Haine [BGH18], in which the notion of a stratified ∞ -topos is defined, and an associated exodromy equivalence is proved.

Works on exit path categories and their higher dimensional generalisations motivate the information that homotopy invariants of a stratified space must capture, and inspire the construction that we give. Our invariants are built by looking at homotopy classes of stratified paths, and therefore capture information about strata of a space as well as holinks. There are additional reasons to believe this is the correct approach; when working with stratified spaces, stratified paths between different strata will not have inverses, suggesting that homotopy invariants must also depend on the homotopy type of holinks. Further evidence is provided by Theorem D, which suggests that the homotopy invariants for fibrant stratified spaces should depend on both strata and holinks.

For a based fibrant stratified space X , we can construct an $\mathbb{N}_{\geq 1}$ -indexed family of categories which we call the homotopy categories of X . These homotopy invariants are built from the homotopy groups of strata and holinks of X . In the case of a trivially stratified path-connected space, the homotopy categories coincide with the homotopy groups of the underlying topological space. Moreover, the homotopy categories can detect weak equivalences between based fibrant stratified spaces. The homotopy categories of a based fibrant stratified space behave analogously to homotopy groups.

Theorem F. *For a stratified fibration between based fibrant stratified spaces, there is an associated long exact sequence of homotopy categories.*

This result is contained as Theorem 11.1.0.5 of this thesis. We conjecture (and give an outline of a proof) that we can construct a Postnikov Tower of a based fibrant stratified space.

There are a number of other works closely related to the research contained in this thesis. The sequence of papers ([AFT17], [AFT16], and [AFR15]) develop a number of tools for working with conically smooth stratified spaces, and prove that the exit path ∞ -category construction is a fully faithful functor from the homotopy theory of conically smooth stratified spaces to ∞ -categories. Examples of the tools developed in these papers are the tangent classifier, open handlebody decompositions, a tubular neighbourhood theorem, and an isotopy extension theorem. Using these they are able to extend the definition of factorisation homology to conically smooth stratified spaces. This sequence of papers build up to a statement and proof of the stratified homotopy hypothesis.

Towards developing a homotopy theory of stratified spaces, Sylvain Douteau [Dou18] introduces a category of filtered simplicial sets and builds an associated model category. Using this model category, Douteau is able to construct invariants of filtered simplicial sets (filtered homotopy groups), and is able to translate some results to the study of stratified spaces. The invariants Douteau constructs are closely related to, but distinct from the invariants that we introduce in this thesis. Recently (in unpublished work), Douteau has been working on different model structures on the category of stratified spaces, which provide an analogue of the Quillen model structure on stratified spaces (meaning that every stratified space is fibrant).

Peter Haine [Hai18] introduces the Joyal-Kan model structure on simplicial sets, whose underlying ∞ -category should be the ∞ -category of stratified spaces. The Joyal-Kan model structure is motivated by the desire for a model structure that imitates the behaviour of stratified spaces, meaning that the Joyal-Kan model structure captures the fact that paths in a stratum of a stratified space have inverses, whereas paths in a stratum of a stratified simplicial set may not. Haine (in unpublished work) claims that the restriction of the Joyal-Kan model structure to the layered ∞ -categories should give a Quillen equivalent model structure to the model structure on stratified spaces. Layered ∞ -categories are those for which any endomorphism is an isomorphism (which reflects the nature of stratified paths).

This thesis takes a different approach to those of Douteau and Haine by focusing on the topology of stratified spaces, rather than working simplicially, and provides an understanding

of the restrictions that define homotopically well behaved spaces within the transferred model structure. Understanding the cofibrant-fibrant stratified spaces in this model structure and the key features to detect weak equivalences between such spaces, indicates the vital information that invariants of fibrant stratified spaces must encode. We use this intuition to build our homotopy invariants, which as far as we are aware, are new.

1.1 Thesis Outline

In *Part I* of this thesis, we summarise the necessary background material from category theory, abstract homotopy theory and the study of stratified spaces.

Chapter 2 gives a brief overview of the category theory used in this thesis. The main motivation for this section is to introduce left Kan extensions, which are used to construct the stratified geometric realisation functor. *Chapter 3* outlines some of the homotopical frameworks that can be used to formally understand the homotopy category associated to a category with weak equivalences. The most relevant for this thesis is the concept of a model structure. *Chapter 4* introduces simplicial sets, and the geometric realisation - singular simplicial sets adjunction between them and topological spaces. We introduce the Joyal model structure, and discuss how this is related to the Quillen model structure. *Chapter 5* gives an overview of the classical approach to studying the homotopy theory of stratified spaces. We introduce the notion of homotopically stratified sets, mention some of the different notions of stratified fibrations used, and elaborate on a theorem of David Miller, which gives criteria for detecting stratified homotopy equivalences between homotopically stratified sets.

In *Part 2* of this thesis we discuss the category of poset-stratified spaces, and introduce a model structure in which we can study the homotopy theory of fibrant stratified spaces.

Chapter 6 explores the category of stratified spaces. In this chapter, we discuss topological aspects of poset stratified spaces, and prove that the category has small limits and colimits. We also introduce a notion of a stratified mapping space when restricting to surjective stratified spaces, giving a cartesian closed category (proving Theorem A). *Chapter 7* builds the stratified analogue of the geometric realisation - singular simplicial sets adjunction. The stratified adjunction follows from a general categorical argument that allows the construction of adjunctions on presheaf categories. This chapter also contains the result that stratified geometric realisation preserves finite products, and shows that the category of stratified spaces has a structure of a simplicial category. *Chapter 8* constructs a model structure on the category of fibrant stratified spaces. This is built by initially showing that we have a homotopical category, and then further a category of fibrant objects, using definitions transferred from the Joyal model structure along the stratified adjunction. We are able to build this structure into a model structure on fibrant stratified spaces by introducing the transferred cofibrations, and are able to show that it is enriched over the Joyal model structure (stated as Theorem B). We end this section by discussing the difficulties that we faced in constructing a model category,

and suggest a possible approach to showing that there is a transferred semi model structure. *Chapter 9* analyses the transferred model structure on stratified spaces. By this, we are able to show the cofibrant stratified spaces satisfy the tameness assumption that Frank Quinn places on homotopically stratified sets, while the fibrant stratified spaces satisfy the pairwise holink fibration condition (this is Theorem C). We then consider the weak equivalences between stratified spaces and are able to characterise them as maps which induce weak homotopy equivalences between strata and holinks, providing a theorem similar to David Miller's criterion for detecting stratified homotopy equivalences (proving Theorem D).

Part 3 of this thesis concerns the homotopy theory of stratified spaces. In it, we begin to explore the consequences of having a model structure on fibrant stratified spaces.

In *Chapter 10* we introduce the notion of a based stratified space, and of a well-based stratified space. The existence of a basing is a non-trivial condition; there are even some cofibrant-fibrant stratified spaces that cannot be based. We construct stratified suspension and loops functors, for well-based stratified spaces, prove that they are adjoint, and moreover that there are isomorphisms between stratified mapping spaces, and between the based homotopy classes (stated as Theorem E). Using the intuition built from understanding weak equivalences of fibrant stratified spaces, we define new homotopy invariants of based fibrant stratified spaces. Our invariants are categories built from homotopy groups of links and strata of stratified spaces, hence we refer to them as the homotopy categories of a based fibrant stratified space. We show that a stratified map of fibrant stratified spaces is a weak equivalence if and only if it induces an isomorphism on homotopy categories. *Chapter 11* uses our notion of a based stratified space to construct the long exact sequence associated to a stratified fibration between based fibrant stratified spaces. To do this, we need to make sense of the fiber of a stratified fibration, and therefore we need to replace the original stratified fibration by one which is an isomorphism on posets. The notion of a stratified fiber gives rise to an associated long exact sequence of homotopy categories, for which we need to define a long exact sequence between categories (this is Theorem F). We end this thesis with a conjecture that we can construct Postnikov Towers of based fibrant stratified spaces. The issue that prevents us from completing this construction is the same issue that arises when we try to construct a model category on the category of stratified spaces. This provides an open question for further research, and may lead to a notion of stratified Eilenberg-Mac Lane spaces which would be interesting to study.

Part I

Background

Chapter 2

Category Theory

The first chapter of this thesis will review the background material necessary to comprehend the tools from categorical homotopy theory that we apply to the category of stratified spaces.

2.1 Basic Category Theory

This section will provide a brief overview of category theory, to introduce the concepts from category theory that will be used in this thesis. The main point of this section is to introduce a scenario in which we are guaranteed that left Kan extensions exist, and to provide an explicit formula to calculate a Kan extension. We will use this to relate the study of stratified spaces to the study of simplicial sets.

There are many good introductory books on category theory which we will cite for further details. Examples are Tom Leinster's introductory book [Lei14], Emily Riehl's slightly more advanced book with plenty of examples [Rie17], or the classical reference is Categories for the Working Mathematician by Saunders Mac Lane [Lan98].

Definition 2.1.0.1. A category \mathcal{C} consists of:

1. a collection of objects $\text{Ob}(\mathcal{C})$;
2. for every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a collection $\text{Mor}_{\mathcal{C}}(X, Y)$ of arrows from X to Y , also called morphisms or maps;
3. a composition law on arrows, meaning that for each triple $X, Y, Z \in \mathcal{C}$ there is a map $\circ: \text{Mor}_{\mathcal{C}}(X, Y) \times \text{Mor}_{\mathcal{C}}(Y, Z) \rightarrow \text{Mor}_{\mathcal{C}}(X, Z)$;
4. an identity map on X , $\mathbb{1}_X \in \text{Mor}_{\mathcal{C}}(X, X)$;

such that the following conditions hold:

1. composition of arrows is associative, meaning that $H \circ (G \circ F) = (H \circ G) \circ F$;
2. the identity arrows act as left and right units, so that for a morphism $F: X \rightarrow Y$ in $\text{Mor}_{\mathcal{C}}(X, Y)$, we have $F \circ \mathbb{1}_X = F$ and $\mathbb{1}_Y \circ F = F$.

We say a category \mathcal{C} is *locally small* if for every pair X, Y then the morphisms $\text{Mor}_{\mathcal{C}}(X, Y)$ is a set, and that \mathcal{C} is *small* if \mathcal{C} is locally small and $\text{Ob}(\mathcal{C})$ is a set.

Notation. The set of morphisms $\text{Mor}_{\mathcal{C}}(X, Y)$ is frequently denoted by $\mathcal{C}(X, Y)$ or $\text{Hom}_{\mathcal{C}}(X, Y)$, and we will write $X \in \mathcal{C}$ to mean $X \in \text{Ob}(\mathcal{C})$. For simplicity, when we say category we mean a locally small category.

Example 2.1.0.2. For any category \mathcal{C} , the opposite category \mathcal{C}^{op} is defined as having the same objects as \mathcal{C} and where the arrows are reversed; explicitly this says that for each morphism $F: X \rightarrow Y$ in \mathcal{C} there is a morphism $F^{\text{op}}: Y \rightarrow X$ in \mathcal{C}^{op} . This is an important example throughout category theory, which provides a duality to statements proven. This follows because if a result is true of any category \mathcal{C} then the dual statement is true of the opposite category \mathcal{C}^{op} .

Examples of categories are ubiquitous throughout mathematics. We provide a number of examples of categories that we will use throughout this thesis:

1. the category **Set** has sets as objects and functions (or set maps) as morphisms;
2. the category Δ has finite ordered sets as objects and monotonic (or order preserving) maps as morphisms;
3. the category **Top** has topological spaces as objects and continuous maps as morphisms;
4. the category **Top**_{*} has pointed topological spaces as objects and pointed maps as morphisms;
5. the category **Grp** has as objects groups and group homomorphisms as morphisms;
6. the category **Ab** has abelian groups as objects and group homomorphisms as morphisms.
7. the category **Ring** has rings as objects and ring homomorphisms as morphisms.

Definition 2.1.0.3. An object $\emptyset \in \mathcal{C}$ is an *initial object* if for any $X \in \mathcal{C}$, there is a unique morphism $\emptyset \rightarrow X$. Dually, an object $*$ $\in \mathcal{C}$ is a *terminal object* in \mathcal{C} if for any $X \in \mathcal{C}$, there is a unique morphism $X \rightarrow *$.

Definition 2.1.0.4. A *functor* $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ between categories \mathcal{C} and \mathcal{D} defines for each object $X \in \mathcal{C}$ an object $\mathcal{F}(X) \in \mathcal{D}$, and functorially defines for each arrow $F: X \rightarrow Y$ in \mathcal{C} an arrow $\mathcal{F}(F): \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ in \mathcal{D} . Functoriality means that both $\mathcal{F}(G \circ F) = \mathcal{F}(G) \circ \mathcal{F}(F)$, and that $\mathcal{F}(\mathbb{1}_X) = \mathbb{1}_{\mathcal{F}(X)}$. A functor is *faithful* if it is injective on Hom-sets, meaning that for all objects $X, Y \in \mathcal{C}$, the map of sets $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is injective. A functor is *full* if it is surjective on Hom-sets, so for all $X, Y \in \mathcal{C}$, the map $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is surjective. A functor which is both full and faithful is said to be an *embedding*.

A *natural transformation* $\alpha: \mathcal{F} \Rightarrow \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G}: \mathcal{C} \rightarrow \mathcal{D}$ assigns to each $X \in \mathcal{C}$ a morphism $\alpha_X: \mathcal{F}(X) \rightarrow \mathcal{G}(X)$ which is natural. Naturality means that for any arrow $F: X \rightarrow Y$ in \mathcal{C} , we have $\mathcal{G}(F) \circ \alpha_X = \alpha_Y \circ \mathcal{F}(F)$; morally it says that α_X and α_Y are compatible with $\mathcal{F}(F)$ and $\mathcal{G}(F)$. Naturality is best understood as commutativity of Figure 2.1 for any choice of morphism $F \in \mathcal{C}$.

$$\begin{array}{ccc}
 \mathcal{F}(X) & \xrightarrow{\mathcal{F}(F)} & \mathcal{F}(Y) \\
 \downarrow \alpha_X & & \downarrow \alpha_Y \\
 \mathcal{G}(X) & \xrightarrow{\mathcal{G}(F)} & \mathcal{G}(Y)
 \end{array}$$

 Figure 2.1: Naturality of a natural transformation α .

Example 2.1.0.5. Examples of functors include homotopy groups of a pointed space which define a functor $\pi_i: \mathbf{Top}_* \rightarrow \mathbf{Grp}$ and integral homology defines a functor $H_i(-; \mathbb{Z}): \mathbf{Top} \rightarrow \mathbf{Ab}$.

Remark 2.1.0.6. When considering what it means to be an equivalence of categories, the notion of isomorphism between categories (meaning inverse functors) is in general too strong. Instead, an alternative notion of equivalence is used between categories.

Definition 2.1.0.7. A pair of functors $\mathcal{F}: \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ defines an *equivalence of categories* if there are natural isomorphisms $\mathcal{G} \circ \mathcal{F} \cong \mathbb{1}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathbb{1}_{\mathcal{D}}$.

Proposition 2.1.0.8. Equivalently, a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ defines an equivalence of categories if \mathcal{F} is:

1. an essentially surjective functor (meaning that for all $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that $\mathcal{F}(c) \cong d$);
2. and fully faithful.

Proof. For details see [Lei14, Proposition 1.3.18]. ■

Definition 2.1.0.9. For locally small categories \mathcal{C} and \mathcal{D} , we can form the *functor category*, denoted by $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$. This is defined as the category with functors from \mathcal{C} to \mathcal{D} as objects and with natural transformations between functors as morphisms.

Definition 2.1.0.10. Given categories \mathcal{C}, \mathcal{D} and \mathcal{E} , with functors $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{E}$ and $\mathcal{G}: \mathcal{D} \rightarrow \mathcal{E}$, we can form the *comma category* $\mathcal{F} \downarrow \mathcal{G}$. This category has as objects triples (C, D, F) , where objects $C \in \mathcal{C}, D \in \mathcal{D}$, and $F: \mathcal{F}(C) \rightarrow \mathcal{G}(D)$ a morphism in \mathcal{E} . A morphism between triples (C, D, F) and (C', D', F') is a pair of morphisms (H, H') where $H: C \rightarrow C'$ in \mathcal{C} and $H': D \rightarrow D'$ in \mathcal{D} , such that the diagram in \mathcal{E} , depicted in Figure 2.2, commutes.

$$\begin{array}{ccc}
 \mathcal{F}(C) & \xrightarrow{\mathcal{F}(H)} & \mathcal{F}(C') \\
 \downarrow F & & \downarrow F' \\
 \mathcal{G}(D) & \xrightarrow{\mathcal{G}(H')} & \mathcal{G}(D')
 \end{array}$$

 Figure 2.2: A morphism (H, H') in a comma category.

The construction of a slice category is a specific example of the general construction of a comma category. For a category \mathcal{C} and an object $X \in \mathcal{C}$, we can form the *slice (or over) category*

of \mathcal{C} over X , denoted \mathcal{C}/X . This is the category which has as objects morphisms in \mathcal{C} of the form $Y \rightarrow X$, and morphisms between $Y \rightarrow X$ and $Z \rightarrow X$ is a morphism in \mathcal{C} of the form $Y \rightarrow Z$ which commutes over X . In particular, the slice category of \mathcal{C} over X is obtained by taking $\mathcal{F} = \mathbb{1}_{\mathcal{C}}$ and \mathcal{G} to be the functor from the one object category with only the identity morphism, denoted $*$, to the object X . As expected, there is a dual notion of a *co-slice* or an *under category*.

Definition 2.1.0.11. For a category \mathcal{C} , a *subcategory* \mathcal{D} is a sub-collection of the objects and morphisms of \mathcal{C} such that:

1. for any object in $X \in \mathcal{D}$ the identity on X is contained in \mathcal{D} ;
2. if a morphism of \mathcal{C} is contained in \mathcal{D} then so is its source and its target;
3. composable morphisms have their composite contained in \mathcal{D} .

A subcategory of \mathcal{C} containing all the morphisms of \mathcal{C} between some of the objects is a *full subcategory*, and a subcategory containing all the objects of \mathcal{C} is a *wide (or lluf) subcategory*.

Definition 2.1.0.12. For an index category \mathbb{I} , a category \mathcal{C} and a functor $\mathcal{F}: \mathbb{I} \rightarrow \mathcal{C}$, a *cone at $X \in \mathcal{C}$ over \mathcal{F}* is a natural transformation $\alpha: \mathcal{G} \Rightarrow \mathcal{F}$, where \mathcal{G} denotes the constant functor $\mathcal{G}: * \rightarrow \mathcal{C}$ taking value X . There is a dual notion of a *cocone at $X \in \mathcal{C}$ over \mathcal{F}* , where the direction of the natural transformation is reversed, so we instead have $\alpha: \mathcal{F} \Rightarrow \mathcal{G}$. Alternatively, a cocone is a cone in the opposite category.

Definition 2.1.0.13. A *diagram \mathcal{F} in a category \mathcal{C}* is a functor $\mathcal{F}: \mathbb{I} \rightarrow \mathcal{C}$ from an index category \mathbb{I} . The *limit of the diagram \mathcal{F}* is a cone over $\mathcal{F}(\mathbb{I})$ in \mathcal{C} , which is universal in the sense that it is terminal over all cones over $\mathcal{F}(\mathbb{I})$. Dually, the *colimit of a diagram \mathcal{F} in \mathcal{C}* is the initial cocone over \mathcal{F} in \mathcal{C} . A category with small limits (where the indexing category is small) is *complete*, with all small colimits is *cocomplete*, and *bicomplete* if it is both complete and cocomplete.

The terminal object in a category is the limit over the empty diagram. Products are the limit over a discrete diagram, fibered / amalgamated products are pullbacks; i.e. limits over a diagram of the shape $\bullet \rightarrow \bullet \leftarrow \bullet$. Another example of a limit is an equalizer; this is the limit over a *fork diagram*. A fork diagram is a diagram of the shape $\bullet \rightrightarrows \bullet$. Dually, the initial object is a colimit over the empty diagram, and coproducts are colimits over a discrete diagram. Fibered coproducts are pushouts, which colimits over a diagram of the shape $\bullet \leftarrow \bullet \rightarrow \bullet$. Another example is of a coequalizer; this is the colimit over a fork diagram.

Definition 2.1.0.14. Consider categories \mathcal{C} and \mathcal{D} with functors $\mathcal{F}: \mathcal{C} \rightleftarrows \mathcal{D}: \mathcal{G}$. We say that \mathcal{F} is a *left adjoint* to \mathcal{G} , or that \mathcal{G} is a *right adjoint* to \mathcal{F} if there is a bijection between Hom-sets $\mathcal{C}(X, \mathcal{G}(Y)) \cong \mathcal{D}(\mathcal{F}(X), Y)$, which is natural in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and is denoted by $\mathcal{F} \dashv \mathcal{G}$.

Remark 2.1.0.15. From an adjunction we extract the *unit natural transformation* $\eta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{G}\mathcal{F}(\mathcal{C})$, defined by sending C to $\mathcal{G}\mathcal{F}(C)$ along the map $\eta_{\mathcal{C}}: C \rightarrow \mathcal{G}\mathcal{F}(C)$ which is the adjoint morphism of $\mathbb{1}_{\mathcal{F}(C)}$. To show that this comprises a natural transformation, for a map $F: C \rightarrow D$, we need to show that $\mathcal{G}\mathcal{F}(F) \circ \eta_{\mathcal{C}} = \eta_{\mathcal{D}} \circ F$. Allowing $\varphi(F)$ to denote the adjoint morphism of F , we

see that naturality of an adjunction gives:

$$\eta_D \circ F = \varphi(\mathbb{1}_{\mathcal{F}(D)}) \circ F = \varphi(\circ \mathbb{1}_{\mathcal{F}(D)} \circ \mathcal{F}(F)) = \varphi(\mathbb{1}_{\mathcal{F}(F) \circ \mathbb{1}_{\mathcal{F}(C)}}) = \mathcal{G}\mathcal{F}(F) \circ \varphi(\mathbb{1}_{\mathcal{F}(C)}) = \mathcal{G}\mathcal{F}(F) \circ \eta_C.$$

Dually, there is a counit natural transformation $\epsilon_C: \mathcal{F}\mathcal{G}(C) \rightarrow C$.

Adjunctions appear all over mathematics, and roughly speaking, allow us to translate work and ideas carried out in one category to another. An important consequence of an adjunction is the following proposition, which relates limits and colimits to adjoint functors.

Proposition 2.1.0.16. *Right adjoints preserve limits, and by duality left adjoints preserve colimits.*

Proof. There are many versions of this proof; for example see [Awo06, Proposition 9.14], [Lei14, Theorem 6.3.1] or [Lan98, §V.5 Theorem 1]. ■

Definition 2.1.0.17. We say that an object $A \in \mathcal{C}$ in a category \mathcal{C} with finite products, is *exponentiable* if the functor $- \times A: \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint, called *the internal hom functor* of \mathcal{C} and denoted by $\underline{\mathcal{C}}(A, -): \mathcal{C} \rightarrow \mathcal{C}$. A category \mathcal{C} is *cartesian closed* if it has finite products, and every object of \mathcal{C} is exponentiable.

Example 2.1.0.18. In the category of sets, for $A, B \in \mathbf{Set}$, the internal hom of two sets is defined as the function set $\mathbf{set}(A, B) \in \mathbf{Set}$ of functions from A to B .

Notation. For a general category, we will drop the capitalisation when we denote the internal hom; for example the internal hom in \mathbf{Set} is denoted by \mathbf{set} .

Definition 2.1.0.19. Consider a complete category \mathcal{C} with a functor $\underline{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$. Then for $F: X \rightarrow Y$ and $G: A \rightarrow B$ in \mathcal{C} , define *the pullback power of F and G* to be the morphism in \mathcal{C} of the form:

$$\langle F, G \rangle: \underline{\mathcal{C}}(B, Y) \rightarrow \underline{\mathcal{C}}(A, X) \times_{\underline{\mathcal{C}}(A, Y)} \underline{\mathcal{C}}(B, Y),$$

which is the unique morphism to the limit on the right hand side.

2.2 Kan Extensions

Definition 2.2.0.1 (Yoneda Functor). For a category \mathcal{C} , there is a functor $\mathcal{Y}: \mathcal{C} \rightarrow \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ called the *Yoneda embedding*, defined on objects $X \in \mathcal{C}$ by $\mathcal{C}(-, X)$, and on morphisms by pre-composition.

Lemma 2.2.0.2 (Yoneda Lemma [Yon60]). *For a locally small category \mathcal{C} we can form the category of presheaves on \mathcal{C} ; this is the functor category $\mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$. Then, for any presheaf $Z \in \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ and any $X \in \mathcal{C}$, there is a natural isomorphism $\text{Mor}_{\mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})}(\mathcal{Y}(X), Z) \cong Z(X)$.*

It follows from the Yoneda Lemma that the Yoneda embedding is full and faithful, hence it is an embedding. When we come to study stratified spaces, we will use the concept of a left

Kan extension to construct a stratified analogue of the geometric realisation functor, giving a functor from simplicial sets to stratified spaces.

Definition 2.2.0.3 (Kan Extensions). Given three categories \mathcal{C}, \mathcal{D} and \mathcal{E} and functors $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{E}$ and $\mathcal{K}: \mathcal{C} \rightarrow \mathcal{D}$, the *left Kan extension of \mathcal{F} along \mathcal{K}* is a functor $\text{Lan}_{\mathcal{K}}\mathcal{F}: \mathcal{D} \rightarrow \mathcal{E}$ with a natural transformation $\eta: \mathcal{F} \Rightarrow \text{Lan}_{\mathcal{K}}\mathcal{F} \circ \mathcal{K}$ so that η satisfies an appropriate universal property.

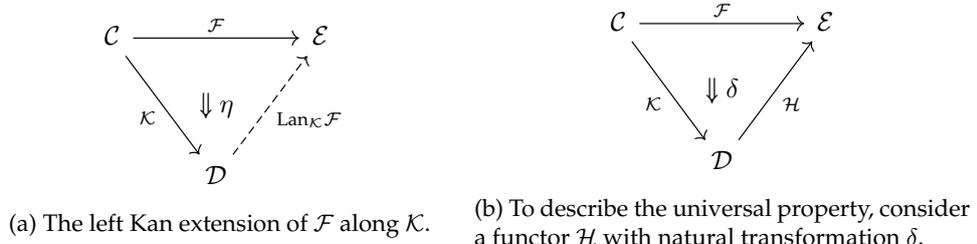


Figure 2.3: Describing the left Kan extension of \mathcal{F} along \mathcal{K} .

To explain the universal property, consider another functor $\mathcal{H}: \mathcal{D} \rightarrow \mathcal{E}$ with a natural transformation $\delta: \mathcal{F} \Rightarrow \mathcal{H} \circ \mathcal{K}$, then there exists a unique natural transformation $\gamma: \text{Lan}_{\mathcal{K}}\mathcal{F} \Rightarrow \mathcal{H}$ which fits into the commutative diagram in Figure 2.4a. This diagram naturally lives in a 2-categorical world, hence it may be easier to visualise γ as in Figure 2.4b, remembering that we also have the natural transformation δ not pictured.

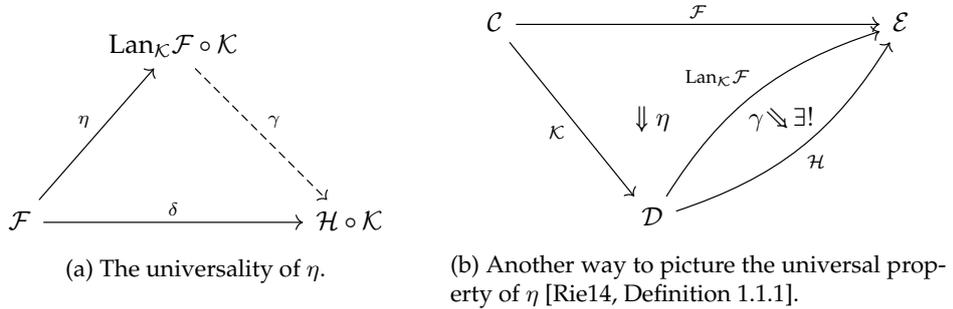


Figure 2.4: Describing the universal property of the functor η .

Remark 2.2.0.4. Dually we can define a right Kan extension; this is a functor $\text{Ran}_{\mathcal{K}}\mathcal{F}: \mathcal{D} \rightarrow \mathcal{E}$ with a universal 2-morphism $\epsilon: \text{Ran}_{\mathcal{K}}\mathcal{F} \circ \mathcal{K} \Rightarrow \mathcal{F}$. In effect, it is only the direction of the universal morphism that changes direction.

Saunders Mac Lane famously asserted that “The notion of Kan extensions subsumes all other fundamental concepts of category theory” [Lan98, §X.7]. We choose not to elaborate a great deal on this, but note that it illustrates the power of Kan extensions. For example, colimits can be expressed using left Kan extensions of the diagram functor along the functor to the terminal object. Dually, limits can be expressed as a right Kan extension of the diagram functor along

the functor to the terminal object. In these cases, the universality of the Kan extension is identified with the universality of the (co)limit. Similarly, note that if we have an adjunction then the left Kan extension of the identity map along the left adjoint is the right adjoint with universal natural transformation given by the unit. Dually, the right Kan extension of the identity map along the right adjoint is the left adjoint and in this case the universal natural transformation is the counit.

To give an explicit construction of a left Kan extension, we need to know when a left Kan extension exists. To describe a criterion for this, we use the language of coends, which are specific examples of colimits.

Definition 2.2.0.5. Let $\mathcal{F}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{E}$ be a functor between categories \mathcal{C} and \mathcal{E} . Define the *coend* $\int^{X \in \mathcal{C}} \mathcal{F}(X, X)$ to be an object of \mathcal{E} , which comes with arrows $\gamma_X: \mathcal{F}(X, X) \rightarrow \int^{X \in \mathcal{C}} \mathcal{F}(X, X)$ for all $X \in \mathcal{C}$ fitting into the commutative diagram in Figure 2.5 for any $F: X \rightarrow Y$ in \mathcal{C} , and which are universal in the appropriate sense.

$$\begin{array}{ccc} \mathcal{F}(Y, X) & \xrightarrow{F_*} & \mathcal{F}(Y, Y) \\ \downarrow F^* & & \downarrow \gamma_Y \\ \mathcal{F}(X, X) & \xrightarrow{\gamma_X} & \int^{X \in \mathcal{C}} \mathcal{F}(X, X) \end{array}$$

Figure 2.5: The commutative diagram for arrows F and morphisms γ_X and $\gamma_{X'}$.

In Figure 2.5, $F_* = F \circ -$ is the pushforward of F , so post-composes an element of $\mathcal{F}(Y, X)$ with F to give an element of $\mathcal{F}(Y, Y)$. Similarly, $F^* = - \circ F$ pre-composes an element of $\mathcal{F}(Y, X)$ with F .

If the category \mathcal{E} is cocomplete then the coend has a natural description, as the coequalizer of the diagram below (this follows from Figure 2.5):

$$\coprod_{F \in \text{Mor}(\mathcal{C})} \mathcal{F}(\text{Cod}(F), \text{Dom}(F)) \begin{array}{c} \xrightarrow{F_*} \\ \xleftarrow{F^*} \end{array} \coprod_{X \in \text{Ob}(\mathcal{C})} \mathcal{F}(X, X) \rightarrow \int^{X \in \mathcal{C}} \mathcal{F}(X, X),$$

where Cod is the codomain of an arrow F and Dom is the domain of F .

Remark 2.2.0.6. If the functor \mathcal{F} is constant in the first term, then the coend reduces to the usual colimit of \mathcal{F} .

Remark 2.2.0.7. The use of an integral sign may be confusing to the reader. This notation was introduced by Yoneda in [Yon60]. This is for many reasons, but one particularly useful one is because they satisfy an analogue of Fubini's Theorem; this says that for a functor $\mathcal{F}: \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{D}^{\text{op}} \times \mathcal{D}$, such that the coends $\int^{X \in \mathcal{C}} \mathcal{F}(X, X)$ and $\int^{Y \in \mathcal{C}} \mathcal{F}(Y, Y)$ exist, then $\int^{X \in \mathcal{C}} \int^{Y \in \mathcal{C}} \mathcal{F}(X, X, Y, Y) \cong \int^{Y \in \mathcal{C}} \int^{X \in \mathcal{C}} \mathcal{F}(X, X, Y, Y)$ [Lan98, IX.8 Corollary]. Therefore we are justified as writing a double coend as simply $\int^{X, Y \in \mathcal{C}} \mathcal{F}(X, X, Y, Y)$.

The following theorem gives us a strong enough criterion for the existence of left Kan extensions, for our purposes. The following theorem can be found in [Rie14, Theorem 1.2.1], proof of which uses [Lan98, X.4 Theorems 1 and 2].

Theorem 2.2.0.8. *Consider small category \mathcal{C} , a locally small category \mathcal{D} , a cocomplete category \mathcal{E} , and functors $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{E}$ and $\mathcal{K}: \mathcal{C} \rightarrow \mathcal{D}$. Then, the left Kan extension of \mathcal{F} along \mathcal{K} exists and is computed pointwise at $Y \in \mathcal{D}$ by $\text{Lan}_{\mathcal{K}}\mathcal{F}(Y) = \int^{X \in \mathcal{C}} \mathcal{D}(\mathcal{K}(X), Y) \odot \mathcal{F}(X)$.*

Remark 2.2.0.9. For a locally small category \mathcal{E} , object $X \in \mathcal{E}$ and a set S , the tensor $S \odot X$ of S by X is $\coprod_S X$, a coproduct of X indexed by S . This coproduct exists when \mathcal{E} is cocomplete, when it defines a bifunctor $\mathbf{Set} \times \mathcal{E} \rightarrow \mathcal{E}$ (a bifunctor is a functor of the form $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ for categories \mathcal{C}, \mathcal{D} and \mathcal{E}).

Chapter 3

Homotopical Frameworks

We provide a brief introduction to differing approaches to studying the homotopy category obtained from a category. The idea is to construct a categorical framework in which we can define weak equivalences and depending on the additional structure we have, obtain alternative ways to study their homotopy categories. The weakest framework that we will discuss is a category with weak equivalences, where we have a class of morphisms that we would like to invert (there is a weaker notion called a relative category, however this is not of direct relevance to this thesis). Whilst the construction of a category with weak equivalences is the easiest to describe, it is the most difficult to work with because it can be difficult to understand the maps in the localised (at the weak equivalences) category. At the other end of the scale we have the axiomatic approach of working with model categories. This framework mirrors the role of cofibrations, fibrations and weak homotopy equivalences in topology, to give a framework that naturally allows the study of homotopical algebra. In some sense this is analogous to abelian categories being a setting for homological algebra. This chapter by no means contains a complete collection of homotopical frameworks, however it illustrates some of the settings that have been studied.

3.1 Categories with Weak Equivalences

We start by describing a category with weak equivalences; the intuitive idea here is that we have a class of morphisms in a category we wish to invert (or localise), for example weak homotopy equivalences between topological spaces. The difficulty here is that the morphisms of the localised category are very tough to understand, hence to work with such categories it can be helpful to require extra structure.

Definition 3.1.0.1. In a category \mathcal{C} , a sub-class of morphisms $\mathcal{W} \subseteq \text{Mor}(\mathcal{C})$ satisfies the 2-out-of-3 property if for any pair of composable morphisms F and G such that any two of F , G and $G \circ F$ are in \mathcal{W} , then so is the third morphism.

Definition 3.1.0.2. Consider a category \mathcal{C} and a sub-class $\mathscr{W} \subseteq \text{Mor}(\mathcal{C})$ of the morphisms of \mathcal{C} . We say that \mathcal{C} is a *category with weak equivalences* (or a *we-category* in the terminology of [DHKS04]), if the sub-class \mathscr{W} contains all isomorphisms and satisfies the 2-out-of-3 property.

We introduce the concept of localising a category at a subcategory, before applying this technique to a category with weak equivalences.

Definition 3.1.0.3. For a category \mathcal{C} and a subcategory \mathcal{D} , there is a localisation of \mathcal{C} with respect to \mathcal{D} . This is a category $\mathcal{C}[\mathcal{D}^{-1}]$ with a universal localisation functor $\gamma: \mathcal{C} \rightarrow \mathcal{C}[\mathcal{D}^{-1}]$ which is a bijection on objects and sends each morphism of \mathcal{D} to an isomorphism in $\mathcal{C}[\mathcal{D}^{-1}]$. The localisation has the universal property that for any category \mathcal{E} and functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{E}$ which carries each morphism of \mathcal{D} to an isomorphism in \mathcal{E} , then there is a unique functor $\mathcal{F}_\gamma: \mathcal{C}[\mathcal{D}^{-1}] \rightarrow \mathcal{E}$ such that $\mathcal{F}_\gamma \circ \gamma = \mathcal{F}$.

We would like to construct the homotopy category generally for a category with weak equivalences. The idea is that we would like to invert the weak equivalences, and note that in Definition 3.1.0.3, we need the weak equivalences to form a subcategory of \mathcal{C} , which follows automatically from the 2-out-of-3 property.

Definition 3.1.0.4. For a category \mathcal{C} with weak equivalences \mathscr{W} , define the *homotopy category* $\mathbf{Ho}(\mathcal{C})$ as the localisation of \mathcal{C} at the subcategory \mathscr{W} .

Following [DHKS04] (although originally from [GZ67], we are able to explicitly describe the morphisms of the category $\mathbf{Ho}(\mathcal{C})$. For objects $X, Y \in \mathcal{C}$, the morphism set $\mathbf{Ho}(\mathcal{C})(X, Y)$ consists of equivalence classes of (finite but of arbitrary length) zig-zag arrows in \mathcal{C} where all arrows in the reverse direction are weak equivalences. The equivalence relation is the smallest relation defined by saying that two arrows are equivalent if one can be obtained from the other by either omitting an identity arrow, by replacing composable arrows by a composite arrow, or by omitting arrows which are the same but go in opposing directions (the picture here is $\cdot \xrightarrow{F} \cdot \xleftarrow{F} \cdot$).

3.2 Homotopical Categories

When working in a category with weak equivalences, we would hope that the isomorphisms in the homotopy category were related to the weak equivalences of the original category. The process of localisation does not necessarily ensure that only weak equivalences are inverted, and in order to have the required universal property, more morphisms may end up as isomorphisms in the homotopy category.

Definition 3.2.0.1. A category with weak equivalences \mathcal{C} is *saturated* if a morphism in \mathcal{C} is a weak equivalence if and only if its image under the localisation functor is an isomorphism in $\mathbf{Ho}(\mathcal{C})$.

Definition 3.2.0.2. In a category \mathcal{C} , a sub-class of morphisms $\mathscr{W} \subseteq \text{Mor}(\mathcal{C})$ satisfies the 2-out-of-6 property if for any composable triple of morphisms F, G and H such that $G \circ F$ and $H \circ G$ are in \mathscr{W} , then so are F, G, H and $H \circ G \circ F$.

Remark 3.2.0.3. Notice that the 2-out-of-6 property implies the 2-out-of-3 property for a sub-class of morphisms. Furthermore, the saturation property for a category with weak equivalences implies the 2-out-of-6 property.

Definition 3.2.0.4. A homotopical category \mathcal{C} is a category with weak equivalences such that the weak equivalences satisfy the 2-out-of-6 property. Alternatively, a homotopical category \mathcal{C} is a category with a wide-subcategory $\mathscr{W} \subseteq \mathcal{C}$ satisfying the 2-out-of-6 property.

Remark 3.2.0.5. In defining a homotopical category, strengthening the weak equivalences to satisfy the 2-out-of-6 property implies that we can require that the weak equivalences contain only the identity morphism on each object rather than requiring they contain all isomorphisms.

We now can relate homotopical categories to saturation; this is done via the notion of 3-arrow calculus. This is a requirement on the weak equivalences and morphisms of a category, which ensures that arrows in the homotopy category from X to Y can be described as equivalence classes of zig-zags in \mathcal{C} of the form:

$$X \overset{\sim}{\leftarrow} \cdot \rightarrow \cdot \overset{\sim}{\leftarrow} Y,$$

where the two backwards arrows are weak equivalences.

Proposition 3.2.0.6 (Proposition 36.4 of [DHKS04]). *A homotopical category which admits a 3-arrow calculus is saturated.*

Remark 3.2.0.7. In the proof of Proposition 3.2.0.6, it is important to use the full strength of the 2-out-of-6 property rather than simply the 2-out-of-3 property.

The theory of homotopical categories can be extended to constructions such as homotopy limit and colimit functors; for further details, see [DHKS04].

3.3 Categories of Fibrant Objects

The concept of a category of fibrant objects was first introduced by Kenneth Brown [Bro73]. The motivation behind introducing this concept is that categories of fibrant objects have additional structure, allowing more control over the maps in the homotopy category.

Definition 3.3.0.1. Let \mathcal{C} be a category and consider an object $X \in \mathcal{C}$. A *path object* of X is an object $\text{Path}(X) \in \mathcal{C}$ with a factorisation of the diagonal map $\Delta_X = (\mathbb{1}_X, \mathbb{1}_X): X \rightarrow X \times X$ as a weak equivalence R followed by a fibration (D_0, D_1) , as depicted in Figure 3.1.

$$\begin{array}{ccc}
 X & \xrightarrow[\substack{\Delta_X \\ (\mathbb{1}_X, \mathbb{1}_X)}]{} & X \times X \\
 & \searrow \substack{\sim \\ R} & \nearrow \substack{(D_0, D_1) \in \mathcal{F}} \\
 & \text{Path}(X) &
 \end{array}$$

 Figure 3.1: A path object of X .

Definition 3.3.0.2. A category with weak equivalences \mathcal{C} is a *category of fibrant objects* (for homotopy theory) if \mathcal{C} has finite products and an additional distinguished class of morphisms called fibrations satisfying four axioms. Calling a map that is a fibration and a weak equivalence an acyclic fibration, the four axioms are:

1. fibrations are closed under composition, and any isomorphism is a fibration;
2. pullbacks of fibrations exist, and furthermore the classes of fibrations and acyclic fibrations are stable under pullback;
3. there are path objects for every object $X \in \mathcal{C}$;
4. for every $X \in \mathcal{C}$, the unique map $X \rightarrow *$ is a fibration.

The final condition of Definition 3.3.0.2 says that every object of \mathcal{C} is *fibrant*.

Remark 3.3.0.3. Conditions 2 and 4 of a category of fibrant objects imply that for any X_1, X_2 , the two path object projection maps $D_i: \text{Path}(X) \rightarrow X$ are fibrations. In addition, the two product projection maps $\text{pr}_1, \text{pr}_2: X_1 \times X_2 \rightarrow X_i$ are fibrations, because they are constructed in the pullback of the cospan $X_1 \rightarrow * \leftarrow X_2$, where they are pullbacks of fibrations. Moreover, requiring that \mathcal{C} is a category with weak equivalences implies that the maps D_0 and D_1 are acyclic fibrations.

Lemma 3.3.0.4. Any map in a category of fibrant objects can be factorised as a weak equivalence followed by a fibration.

Proof. (Factorisation Lemma of [Bro73]) Let $F: X \rightarrow Y$ be a morphism in \mathcal{C} , and form the pullback $X \times_Y \text{Path}(Y)$. We claim that the factorisation:

$$X \xrightarrow{I} X \times_Y \text{Path}(Y) \xrightarrow{P} Y$$

is the desired factorisation, where the map $I = (\mathbb{1}_X, R \circ F)$ is defined using the map R from the path object of Y as in Figure 3.1, and the map $P = \text{pr}_2 \circ D_1$ where pr_2 is projection out of the fibered product onto the second factor.

Initially we wish to show that I is a weak equivalence; to do this, we note that I is a section of the map pr_1 . The map pr_1 is constructed as a pullback of D_0 (in the defining diagram of $X \times_Y \text{Path}(Y)$). The map D_0 is an acyclic fibration by Remark 3.3.0.3, hence pr_1 is also an

acyclic fibration. The 2-out-of-3 property for weak equivalences therefore implies that I is a weak equivalence.

To show that $P = \text{pr}_2 \circ (\mathbb{1}_X, D_1)$ is a fibration, note that $\text{pr}_2: X \times_Y \text{Path}(Y) \rightarrow X \times Y$ is a fibration, because it is obtained as a pullback of the fibration $(D_0, D_1): \text{Path}(Y) \rightarrow Y \times Y$ along the map $(F, \mathbb{1}_Y): X \times Y \rightarrow Y \times Y$. The map $\text{pr}_2: X \times Y \rightarrow Y$ is a fibration by Remark 3.3.0.3. Stability under composition implies that P is a fibration. ■

For a category of fibrant objects \mathcal{C} we can form an associated homotopy category $\mathbf{Ho}(\mathcal{C})$ by inverting the class of weak equivalences, as in the underlying category with weak equivalences. The goal now is to explain how the extra structure in a category of fibrant objects allows us to describe a notion of homotopy. The notion of homotopy that the fibrations give us allows us to simplify the description of the homotopy category of \mathcal{C} .

Definition 3.3.0.5. Consider morphisms $F, G: X \rightarrow Y$ in a category of fibrant objects \mathcal{C} . A *right homotopy* from F to G is the existence of a morphism H_r for some path object of Y , such that there is a factorisation of the map $(F, G): X \rightarrow Y \times Y$ as $(D_0, D_1) \circ H_r: X \rightarrow \text{Path}(Y) \rightarrow Y \times Y$, depicted in Figure 3.2.

$$\begin{array}{ccc}
 X & \xrightarrow{(F,G)} & Y \times Y \\
 \searrow^{H_r} & & \nearrow_{(D_0, D_1)} \\
 & \text{Path}(Y) &
 \end{array}$$

Figure 3.2: A right homotopy $F \overset{r}{\sim} G$.

If morphisms F and G are right homotopic, this is denoted by $F \overset{r}{\sim} G$. Commutativity of the appropriate diagram of cones over the product ensures that $D_0 \circ H_r = F$ and $D_1 \circ H_r = G$.

Using the notion of right homotopy, which is an equivalence relation if the source is fibrant, we are able to provide an alternative description of the homotopy category of a category of fibrant objects. This description can be found in [Bro73, Theorem 1].

Definition 3.3.0.6. Consider a category of fibrant objects \mathcal{C} . The category $\pi\mathcal{C}$ is defined as the category with the same objects as \mathcal{C} where hom sets $\pi\mathcal{C}(X, Y)$ for $X, Y \in \mathcal{C}$ are defined as the quotient of $\mathcal{C}(X, Y)$ by $F \sim G$ if and only if there is a weak equivalence $H: X' \rightarrow X$ such that $F \circ H \overset{r}{\sim} G \circ H$:

$$\pi\mathcal{C}(X, Y) = \mathcal{C}(X, Y) / \sim.$$

We want to use the category $\pi\mathcal{C}$ to give an explicit description of the homotopy category of \mathcal{C} . To do this, we construct a filtered colimit in $\pi\mathcal{C}$ which will play the role of morphism spaces in the homotopy category.

Definition 3.3.0.7. Consider objects $X, Y \in \mathcal{C}$ in a category of fibrant objects \mathcal{C} , and define a diagram \mathcal{F} in $\pi\mathcal{C}$ with objects the image in $\pi\mathcal{C}$ of any weak equivalence in \mathcal{C} which has target X . A map in the diagram is an arrow in the slice category $\pi\mathcal{C}_{/X}$. For objects $X, Y \in \mathcal{C}$ in a category of fibrant objects, define:

$$[X, Y] = \operatorname{colim}_{X' \xrightarrow{\sim} X} \pi\mathcal{C}(X', Y)$$

where the colimit formed is filtered colimit is taken over \mathcal{F} .

Theorem 3.3.0.8. For objects $X, Y \in \mathcal{C}$ in a category of fibrant objects \mathcal{C} , the homotopy category of \mathcal{C} is equivalent to the category $\mathbf{Ho}(\mathcal{C})$ defined to have the same objects as \mathcal{C} and with hom-sets defined by $[X, Y]$ as in Definition 3.3.0.7.

Proof. See [Bro73, Theorem 1]. ■

3.4 Model Categories

The goal of this subsection is to introduce the definition of a model category; this is more structure than a category of fibrant objects, and allows us to give a simpler description of the homotopy category associated to a model category. We start with some necessary definitions.

Definition 3.4.0.1. We say that a morphism $F: X \rightarrow Y$ is a *retract* of a morphism $G: X' \rightarrow Y'$ if there is a commutative diagram as depicted in Figure 3.3.

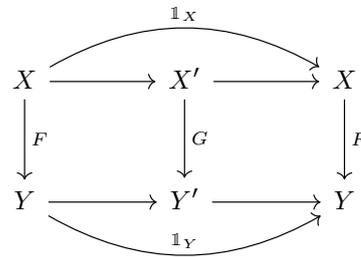


Figure 3.3: The morphism f is a retract of g .

Definition 3.4.0.2. Consider morphisms F and G in a category \mathcal{C} , as depicted in Figure 3.4.

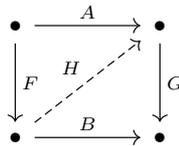


Figure 3.4: Lifting properties of f and g .

If a lift H exists for any choice of pair (A, B) in Figure 3.4, then we say that F has the *left lifting property with respect to G* , or equivalently that G has the *right lifting property with respect to F* . The lifting relationship between F and G is denoted by $F \boxdot G$.

Notation. Consider subclasses $\mathcal{R}, \mathcal{L} \subset \text{Mor}(\mathcal{C})$. We will use $\boxdot\mathcal{R}$ to denote the class of morphisms in \mathcal{C} which have the left lifting property with respect to the class \mathcal{R} ; i.e. the class of morphisms

$$\boxdot\mathcal{R} = \{F \mid F \in \text{Mor}(\mathcal{C}) \text{ and } \{F\} \boxdot \mathcal{R}\}.$$

Dually, we will use \mathcal{L}^\boxdot to denote the class of morphisms with the right lifting property with respect to \mathcal{L} . Using the notation introduced in Definition 3.4.0.2, we see that $F \boxdot G$ implies that $F \in \boxdot\{G\}$ and $G \in \{F\}^\boxdot$.

Definition 3.4.0.3. In a category \mathcal{C} , a *weak factorisation system* is two classes of morphisms \mathcal{L} and \mathcal{R} such that the following conditions hold:

1. every morphism in \mathcal{C} can be functorially factored as a morphism of one in \mathcal{L} followed by one in \mathcal{R} ;
2. we have the equalities $\mathcal{L} = \boxdot\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^\boxdot$.

We will use the following definition of a model structure and of a model category. When Quillen originally proposed the concept of a model category he differentiated between a closed model category and simply a model category, however the difference hasn't turned out to be important, and closed model categories have become the standard context to work in. The changes in the accepted definition of a model category are discussed in [DHKS04, §1 2.1]. The definition we use comes from [GJ09].

Definition 3.4.0.4. A *model structure* on a category \mathcal{C} is a choice of three classes of maps; the fibrations \mathcal{F} , cofibrations \mathcal{C} , and weak equivalences \mathcal{W} , satisfying four axioms. We call a cofibration that is also a weak equivalence an *acyclic cofibration*, and a fibration which is also a weak equivalence an *acyclic fibration*. These classes of morphisms must satisfy the following axioms:

- MC1 (2-out-of-3 property) for composable weak equivalences F and G in \mathcal{C} , if any two of F , G or $G \circ F$ are weak equivalences, then so is the third;
- MC2 (closure under retracts) all three of the classes are closed under taking retracts; i.e. if F is a retract of G , and G is either a weak equivalence (or cofibration or a fibration), then so respectively is F ;
- MC3 (lifting) suppose we are trying to lift a cofibration I on the left against a fibration P . Then, a lift exists if either I or P is also a weak equivalence;
- MC4 (factorisation) any morphism of \mathcal{C} can be functorially factored as a cofibration followed by an acyclic fibration, and as an acyclic cofibration followed by a fibration.

A bicomplete category equipped with a model structure is a *model category*.

Notation. In general, we will use \emptyset to denote the initial object of a category, and $*$ to denote the terminal object. These exist in any model category due to the requirement that \mathcal{C} is closed under finite limits and finite colimits, because the initial object is the colimit over the empty diagram, and the terminal object is the limit over the empty diagram.

Before giving some examples of model categories, we relate the definition of a model structure to the notion of a category with weak equivalences, and to the notion of a category of fibrant objects.

Definition 3.4.0.5 (Succinct Definition of a Model Structure). A model structure (resp. model category) is a (bicomplete) category \mathcal{C} with weak equivalences \mathcal{W} with specified classes of cofibrations \mathcal{C} and fibrations \mathcal{F} such that both $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ define weak factorisations systems on \mathcal{C} .

Remark 3.4.0.6. Notice that restricting a model category to its sub-category of fibrant objects defines a category of fibrant objects (for homotopy theory), in the sense of Definition 3.3.0.2.

Our first example of a model structure is to construct a model structure on the opposite category of a given model structure.

Example 3.4.0.7. A model structure on \mathcal{C} gives rise to a model structure on the category \mathcal{C}^{op} by setting $F \in \mathcal{C}^{\text{op}}$ to be a cofibration if the map in \mathcal{C} is a fibration, a morphism $F \in \mathcal{C}^{\text{op}}$ to be a fibration if the map in \mathcal{C} is a cofibration, and a morphism $F \in \mathcal{C}^{\text{op}}$ is a weak equivalence if $F \in \mathcal{C}$ is a weak equivalence. Factorisations in the opposite category become inverted.

This gives dual statements for results in a model structure, because we know that there will always exist a naturally arising model structure on the opposite category \mathcal{C}^{op} if there exists a model structure on \mathcal{C} .

Example 3.4.0.8. For model categories \mathcal{C}, \mathcal{D} , there is a natural model category structure on the product category $\mathcal{C} \times \mathcal{D}$ given by defining (F, G) as a cofibration (or fibration or weak equivalence) if and only if both F and G are cofibrations (resp. fibrations or weak equivalences). More generally, for a product $\prod_{i \in I} \mathcal{C}_i$ indexed by a set I , we define a morphism $\prod_{i \in I} F_i$ to be a cofibration (or fibration or weak equivalence) if and only if projection onto each component F_i individually gives a cofibration (resp. fibration or weak equivalence).

Example 3.4.0.9. A category \mathcal{C} which is closed under small limits and colimits has a unique minimal model structure; this has as weak equivalences all isomorphisms, and all morphisms are both cofibrations and fibrations. This is minimal in the sense that all weak equivalences are already inverted in \mathcal{C} .

At the other end of the scale, any bicomplete category \mathcal{C} has at least two maximal model structures, which means that all morphisms of \mathcal{C} are weak equivalences. We can always define such a maximal model structure by setting all isomorphisms to be fibrations and all morphisms to be cofibrations, or dually we can set all isomorphisms as cofibrations and all morphisms as

fibrations. An example where there are more maximal model structures is the category of sets; in this case we could let the cofibrations be the monomorphisms and the epimorphisms as fibrations, or the dual [DHKS04, 9.7 Example (v)].

Definition 3.4.0.10. In a model category, we say that an object X is *cofibrant* if the unique map from the initial object to X is a cofibration. A *cofibrant replacement* of X is a cofibrant object QX with an acyclic fibration $QX \rightarrow X$. Dually, we can define a *fibrant* object X to be one for which the unique morphism from X to the terminal object is a fibration, and a *fibrant replacement* of X is a fibrant object RX with an acyclic cofibration $X \rightarrow RX$.

The functoriality of factorisations in a model category implies that there is a *cofibrant replacement functor*, defined at any object X by factoring the map $\emptyset \rightarrow X$ as a cofibration followed by an acyclic fibration, giving a weakly equivalent object QX to X . For a model category \mathcal{C} , this defines a functor $\mathcal{C} \rightarrow \mathcal{C}_{\mathcal{C}}$ to the category of cofibrant objects in \mathcal{C} . Dually, there is a *fibrant replacement functor* using the dual functorial factorisation which gives a functorial fibrant replacement $X \rightarrow RX$, and a functor $\mathcal{C} \rightarrow \mathcal{C}_{\mathcal{F}}$.

Remark 3.4.0.11. If we are already working with a fibrant object X , then we can let the identity map $X \rightarrow X$ be the fibrant replacement. Dual logic holds for a cofibrant object.

Lemma 3.4.0.12. *In a model structure, a map F is a cofibration if and only if it has the left lifting property with respect to all acyclic fibrations, and an acyclic cofibration if and only if it has the left lifting property with respect to all fibrations. Similarly, a map F is a fibration if and only if it has the right lifting property with respect to all acyclic cofibrations, and an acyclic fibration if and only if it has the right lifting property with respect to all cofibrations.*

Remark 3.4.0.13. This lemma says that any two classes of morphisms determine the third in a model structure, in the sense that the closure under lifting gives relations $\mathcal{C} = {}^{\square}(\mathcal{F} \cap \mathcal{W})$, $\mathcal{C} \cap \mathcal{W} = {}^{\square}\mathcal{F}$, $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\square}$ and $\mathcal{F} \cap \mathcal{W} = \mathcal{C}^{\square}$, which allow us to determine the third class.

Proof of Lemma 3.4.0.12. We will prove the first statement, noting that proof of the other statements are similar. A cofibration lifts on the left against any acyclic fibration by MC4. In reverse, consider a map $F: X \rightarrow Z$ with the left lifting property with respect to all acyclic fibrations; we wish to prove that F is a cofibration. We can factor the map F as a cofibration $I: X \rightarrow Y$ followed by an acyclic fibration $J: Y \rightarrow Z$. This gives us Figure 3.5, in which we know a lift exists because F lifts on the left against any acyclic fibration.

$$\begin{array}{ccc}
 X & \xrightarrow{I} & Y \\
 \downarrow F & \nearrow H & \downarrow J \\
 Z & \xrightarrow{\mathbb{1}_Z} & Z
 \end{array}$$

Figure 3.5: Lifting F on the left against J .

Expanding the commutative square of Figure 3.5 gives us Figure 3.6, which shows that F is a retract of the cofibration I . Hence the closure under retracts axiom MC2 shows us that F is also a cofibration.

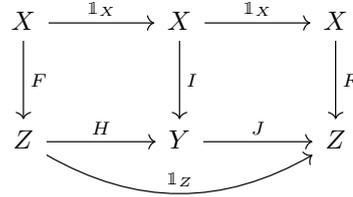


Figure 3.6: Exhibiting F as a retract of I .

■

Remark 3.4.0.14. In proving this lemma, we have proven the Retract Argument of Hovey [Hov98a, Lemma 1.1.9]. This states that if we have a factorisation $F = I \circ J$ in a category \mathcal{C} , and F has the left lifting property with respect to I , then F is a retract of J . By duality if F has the right lifting property with respect to J , then F is a retract of I .

Remark 3.4.0.15. Lemma 3.4.0.12 is frequently applied to show a model structure exists once we have factorisations of morphisms in a category, showing that axiom MC4 (lifting) holds.

Lemma 3.4.0.12 gives us the following corollary, as in [GJ09].

Corollary 3.4.0.16. *In a model category, the classes of cofibrations and acyclic cofibrations are closed under composition, pushouts and coproducts. Furthermore, any isomorphism is a cofibration. Similarly for fibrations and acyclic fibrations, these classes are closed under composition, pullbacks and products, with any isomorphism also being a fibration.*

Proof. To show that these classes are closed under composition, consider two composable cofibrations F and G such that their composite is $G \circ F$. By Lemma 3.4.0.12, the classes of morphisms are determined by their lifting properties. Let P be an acyclic fibration, and consider the commutative diagram in Figure 3.7.

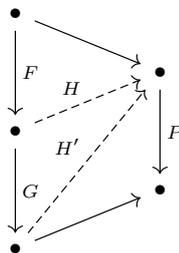


Figure 3.7: Constructing the lift H' of $G \circ F$ on the left against P .

The lift H constructed in this figure is of F against P , using the composite of G and the arrow along the bottom to give a commutative square. We can now use the fact that G is a cofibration, so also lifts against P (this time using H to give the commutative diagram), to give us a lift H' which is also a lift of $G \circ F$ by the commutativity of the top triangle.

To show cofibrations (resp. fibrations) are closed under pushouts (resp. pullbacks), construct the appropriate pushout (resp. pullback) diagram, which we wish to show lifts against the acyclic fibration (resp. acyclic cofibration). We know that our original map lifts against this, so we can use the universality of the pushout (resp. pullback) to find the lift. In Figure 3.8, we have illustrated the construction for a cofibration F with pushout denoted P .

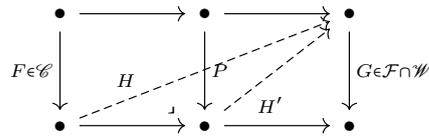


Figure 3.8: The pushout of a cofibration is also a cofibration.

Initially, construct the relative lift H of F against G . The morphism H' , the lift of P against G , exists because of the universal property of a pushout, and we use H to create a cone over the pushout to the source of G . The same argument holds for an acyclic cofibration F and fibration G .

That the class of (co)fibrations is closed under (co)products follows directly from the universal property of the (co)product. To illustrate the use of the universal property, we show that for two cofibrations F and G , the coproduct $F \amalg G$ is also a cofibration. In the Figure 3.9, H is any acyclic fibration.

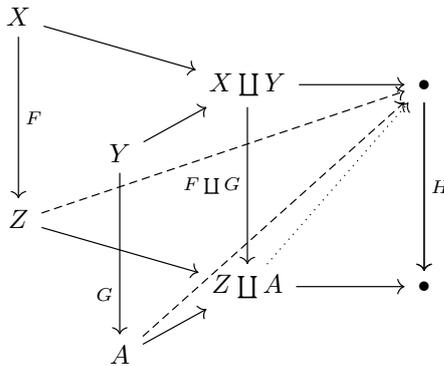


Figure 3.9: The coproduct of two cofibrations is a cofibration.

The two dashed lines exist because F and G are cofibrations so lift against any acyclic fibration such as H , which then form a cocone over the discrete diagram of Z and A , so the dotted line exists by the universal properties of the coproduct.

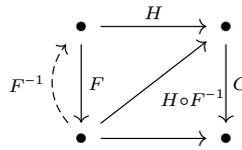


Figure 3.10: Any isomorphism is a cofibration.

For an isomorphism, we can always construct the lift in a commuting diagram, by using the inverse morphism. A diagram is provided in Figure 3.10 to clarify, where the map F is an isomorphism and F^{-1} is its inverse, and the lift $H \circ F^{-1}$ shows that F is a cofibration. ■

Corollary 3.4.0.17. *In a model category \mathcal{C} , any isomorphism is a weak equivalence.*

Proof. As in Figure 3.10, we see that an isomorphism has the left and right lifting properties with respect to any other morphism. Fixing an isomorphism $F: X \rightarrow Y$, we choose to factor F as a cofibration followed by an acyclic fibration I . Because F lifts on the left against I , the Retract Argument of Hovey (see Remark 3.4.0.14) implies that F is a retract of the acyclic fibration, and in particular must be a weak equivalence. ■

Proposition 3.4.0.18. *In a model structure, a morphism is a weak equivalence if and only if it can be factored as an acyclic cofibration followed by an acyclic fibration.*

Proof. Applying the factorisation axiom MC5, any weak equivalence F can be factored as an acyclic cofibration followed by a fibration. Using the 2-out-of-3 property satisfied by weak equivalences, we see that the fibration must also be a weak equivalence. To show the converse, if a morphism F can be expressed as an acyclic cofibration followed by an acyclic fibration, then the composite must be a weak equivalence by the 2-out-of-3 property satisfied by weak equivalences. ■

Proposition 3.4.0.19. *The cofibrations and fibrant objects in a model structure determine the model structure, if one exists. Dually, the fibrations and cofibrant objects determine a model structure, if one exists.*

Proof. See [Joy08a, 51.10]. ■

Definition 3.4.0.20. A model structure is *left proper* if weak equivalences are preserved by pushout along cofibrations, *right proper* if weak equivalences are preserved by pullback along fibrations, and *proper* if it is both left and right proper.

3.4.1 Quillen Model Category on the Category of Topological Spaces

We now present the standard model structure on the category **Top** of topological spaces and continuous maps. This model structure was originally constructed by Quillen in [Qui67, II.3].

Notation. We will denote the standard geometric n -simplex by $|\Delta^n|$. In \mathbb{R}^{n+1} , the standard geometric n -simplex consists of all points (x_0, \dots, x_n) such that $\sum_i x_i = 1$. We will use $|\partial\Delta^n|$ to denote the points on the boundary of $|\Delta^n|$ (all points in $|\Delta^n|$ such that in barycentric coordinates there is an i such that $x_i = 0$). Similarly, $|\Delta_k^n|$ will denote the topological space obtained from $|\partial\Delta^n|$ by removing the face opposite the k -vertex (removing the points of $|\partial\Delta^n|$ for which $x_k = 0$ and $x_i \neq 0$ for all $i \neq k$). This choice of notation will become apparent after introducing the geometric realisation functor in §4.2.

Theorem 3.4.1.1. *The Quillen model category on the category of topological spaces is given by the following three classes of maps:*

1. *the weak equivalences are the weak homotopy equivalences;*
2. *the fibrations are the Serre fibrations (continuous maps with the right lifting property with respect to all inclusions $|\Delta^n| \times \{0\} \hookrightarrow |\Delta^n| \times [0, 1]$);*
3. *the cofibrations are those with the left lifting property with respect to acyclic fibrations, precisely the retracts of relative cell complexes (see Definition 3.5.2.6).*

Remark 3.4.1.2. We can equivalently define a Serre fibration as a morphism with the right lifting property with respect to all inclusions $A \times \{0\} \hookrightarrow A \times [0, 1]$ for all CW complexes A .

Remark 3.4.1.3. In this model structure, the cofibrant objects are retracts of CW-complexes, and every object is fibrant. It is easy to see that if $\emptyset \rightarrow X$ is a cofibration then the space X must be a CW-complex. In terms of the fibrant objects, consider a commutative diagram as in Figure 3.11, for any CW-complex A .

$$\begin{array}{ccc}
 A \times \{0\} & \xrightarrow{F} & X \\
 \downarrow & \nearrow H & \downarrow \\
 A \times [0, 1] & \longrightarrow & *
 \end{array}$$

Figure 3.11: Lifting $A \times \{0\} \hookrightarrow A \times [0, 1]$ into X .

We can define a lift H by $H(a, t) = F(a)$, which is well defined, makes both diagrams commute and always exists. Hence any topological space X is fibrant in this model structure.

This means that one choice for a cofibrant replacement functor in the Quillen model category could be taken from the CW approximation theorem; this states that for any topological space X there exists a CW complex Z and a weak homotopy equivalence $Z \rightarrow X$ [Hat10, §4.1]. It also means that the homotopy theory of **Top** can be well represented by the full subcategory of CW complexes.

The first proof of Theorem 3.4.1.1 was given in [Qui67, II.3], and consists of checking of the model category axioms. Other ways of constructing the model structure exist, and many use the adjunction between **Top** and **sSet** (which we explain in §4.2) to transfer simplicial methods to a topological setting. The key to the proof is Quillen's Small Object Argument; this

is used to construct the desired factorisations in the category **Top**, in addition to showing that the fibrations given by lifting properties are precisely the set that lifts against the morphisms constructed as retracts of relative CW complexes.

3.5 Quillen's Small Object Argument

The aim of this subsection is to introduce Quillen's Small Object Argument; this is an essential ingredient in proving that a choice of morphisms satisfies the factorisation axioms for a model category. When defining or constructing a model category on a category \mathcal{C} , one may be in the situation in which two classes of morphisms are defined, and the third class would like to be defined via lifting properties. In this scenario, supposing that a required smallness property is satisfied, Quillen's small object argument gives us the required factorisations of morphisms in \mathcal{C} . Before we can discuss this, we need to recap the definitions of ordinals and cardinals.

3.5.1 Ordinals and Cardinals

We begin by introducing the notions of ordinals and cardinals, which will play an important role when discussing the smallness of objects in a category.

Definition 3.5.1.1. A *total order* is a set S with a binary relation \leq which is reflexive, antisymmetric, transitive and satisfies the totality condition; that for any $a, b \in S$ either $a \leq b$ or $a \geq b$.

A *well ordering* is a total order on S such that every non-empty subset of S has a least element with respect to \leq .

Definition 3.5.1.2. An *order isomorphism* is a map $F: S \rightarrow T$ such that for all $a, b \in S$, then $a \leq b$ if and only if $F(a) \leq F(b)$ in T .

Definition 3.5.1.3. An *ordinal* γ is an ordered isomorphism class of well ordered sets.

Every ordinal has an order isomorphism to the well ordered set of all preceding ordinals, and has a successor ordinal; the smallest ordinal larger than S . Every well ordered set is uniquely isomorphic to a unique ordinal, called the *order type* of the well ordered set.

Definition 3.5.1.4. We say that an ordinal is a *cardinal* if its cardinality is larger than any of the preceding ordinals. A cardinal κ is called *regular* if for every set of sets $\{X_j\}_{j \in J}$ indexed by a set J of cardinality less than κ and such that the cardinality of each X_j is also less than κ , then the cardinality of $\cup_j X_j$ is also less than κ .

Definition 3.5.1.5. For a cardinal γ , we say that an ordinal S is γ -*filtered* if S is a limit ordinal, and when $A \subseteq S$ and $|A| \leq \gamma$, then $\sup A < S$.

3.5.2 Quillen's Small Object Argument

Definition 3.5.2.1. Within a cocomplete category \mathcal{C} , consider a set $\mathcal{K} = \{B_\alpha \xrightarrow{I_\alpha} E_\alpha\}$ of morphisms in \mathcal{C} . A \mathcal{K} -cell attachment is a morphism $F: K \rightarrow L$ obtained as a pushout indicated in Figure 3.12. The left hand side vertical arrow is representing the coproduct of morphisms in \mathcal{K} .

$$\begin{array}{ccc} \coprod_{\alpha \in \mathcal{K}} B_\alpha & \longrightarrow & K \\ \downarrow \coprod_{\alpha \in \mathcal{K}} I_\alpha & & \downarrow F \\ \coprod_{\alpha \in \mathcal{K}} E_\alpha & \dashrightarrow & L \end{array}$$

Figure 3.12: The \mathcal{K} -cell attachment is the arrow F above.

Definition 3.5.2.2. For an ordinal Λ and a cocomplete category \mathcal{C} , define a Λ -sequence to be a colimit-preserving functor $\mathcal{F}: \Lambda \rightarrow \mathcal{C}$.

By assumption \mathcal{F} preserves colimits, so it follows that for all limit ordinals $\gamma < \Lambda$, the induced map $\text{colim}_{\beta < \gamma} \mathcal{F}_\beta \rightarrow \mathcal{F}_\gamma$ is an isomorphism.

Definition 3.5.2.3. The map $\mathcal{F}(\emptyset) \rightarrow \text{colim}_\Lambda X$ is called the *transfinite composite* of the maps of \mathcal{F} . We say that a subcategory $\mathcal{C}_1 \subset \mathcal{C}$ is *closed under transfinite composition* if for every ordinal Λ and every Λ -sequence $\mathcal{F}: \Lambda \rightarrow \mathcal{C}$ such that $\mathcal{F}(\alpha) \rightarrow \mathcal{F}(\alpha + 1)$ is in \mathcal{C}_1 for every ordinal $\alpha < \Lambda$, the induced map $\mathcal{F}(\emptyset) \rightarrow \text{colim}_\Lambda X$ is also in \mathcal{C}_1 .

Remark 3.5.2.4. When we think of an ordinal Λ , we are really referring to an ordered category which has a successor $\Lambda + 1$, and where there is a unique map from $\alpha \rightarrow \beta$ if and only if $\alpha \leq \beta$; i.e. the poset category associated to a well ordered set.

Definition 3.5.2.5. Let \mathcal{C} be a category with a set of morphisms $\mathcal{K} \subseteq \text{Mor}(\mathcal{C})$. A morphism $F: K \rightarrow L$ is a *relative \mathcal{K} -cell attachment* if it is the composite of a Λ -sequence:

$$K = L_0 \rightarrow \dots \rightarrow L_\nu \xrightarrow{F_\nu} L_{\nu+1} \rightarrow \dots \rightarrow \text{colim}_{\nu < \Lambda} L_\nu = L$$

for some ordinal Λ , such that each morphism F_ν is a \mathcal{K} -cell attachment (meaning obtained as a pushout of a coproduct of morphisms in \mathcal{K}).

Definition 3.5.2.6. A *relative cell complex* is a morphism $F: K \rightarrow L$ obtained as a relative \mathcal{K} -cell attachment where \mathcal{K} is a set of morphisms. An *absolute cell complex* is an object $L \in \mathcal{C}$ which can be obtained from a relative cell complex of the form $\emptyset \rightarrow L$.

Definition 3.5.2.7. An object X in a locally small cocomplete category \mathcal{C} is said to be *compact* if for all filtered categories \mathcal{D} and any functor $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{C}$, the canonical morphism:

$$\text{colim}_{D \in \mathcal{D}} \mathcal{C}(X, \mathcal{F}(D)) \rightarrow \mathcal{C}(X, \text{colim}_{D \in \mathcal{D}} \mathcal{F}(D))$$

is an isomorphism.

This definition says that X is compact if $\mathcal{C}(X, -)$ commutes with all filtered colimits.

Remark 3.5.2.8. There is a subtlety hidden in here; a compact object in the category **Top** isn't the same as a topological space being compact. It is true that a compact object in the category of topological spaces is a compact topological space. For a topological space X , being compact in the categorical sense is equivalent to X being finite and discrete; this is contained in [AR94, 1.2(10)].

Definition 3.5.2.9. Consider a cocomplete category \mathcal{C} , and a subcategory $\mathcal{C}_1 \subset \mathcal{C}$, which is closed under transfinite composition. If κ is a regular cardinal, an object $X \in \mathcal{C}$ is called κ -small relative to \mathcal{C}_1 if for every regular cardinal $\Lambda \geq \kappa$ and Λ -sequence \mathcal{F} in \mathcal{C} such that each map is in \mathcal{C}_1 , the map of sets:

$$\operatorname{colim}_{\Lambda} \operatorname{Hom}_{\mathcal{C}}(X, \mathcal{F}) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, \operatorname{colim}_{\Lambda} \mathcal{F})$$

is an isomorphism. We say that an object $X \in \mathcal{C}$ is *small relative to \mathcal{C}_1* , if there exists a regular cardinal κ , such that X is κ -small relative to \mathcal{C}_1 . If an object X is small relative to \mathcal{C} , then X is a *small object*. The object X is *finite relative to \mathcal{C}_1* if there is a finite cardinal κ such that X is κ -small relative to \mathcal{C}_1 . If an object X is finite relative to \mathcal{C} , then X is *finite*.

Remark 3.5.2.10. It is clear that being a compact object implies that our object is small, however the converse is not true because it is not necessarily true that any κ -colimit can be built from filtered diagrams (smallness says that $\operatorname{Hom}_{\mathcal{C}}(X, -)$ commutes with all cardinal-indexed colimits). If $\kappa = \aleph_0$, the notions of compactness and smallness coincide.

Example 3.5.2.11. Any set A is $|A|$ -small; for a proof of this result see [Hov98a, Example 2.1.5]. Similarly, for a ring R , then every R -module is small; this is also contained in [Hov98a, Example 2.1.6]. The situation is more complicated when working with topological spaces; in [Hov98a, Proposition 2.4.2], Mark Hovey is able to show that compact topological spaces are finite relative to the closed T_1 inclusions. In Hovey's context, this result is enough to construct the Quillen model structure on topological spaces.

Definition 3.5.2.12. Consider a set $I \subseteq \operatorname{Mor}(\mathcal{C})$. We define $\operatorname{cell}(I)$, *the relative I -cell complexes*, as the morphisms in \mathcal{C} obtained as a (possibly transfinite) composition of pushouts of elements of I (see Definition 3.5.2.5). We define $\operatorname{cof}(I)$ to be the class of retracts of the relative I -cell complexes in \mathcal{C} . We say that the set I *generates the cofibrations of \mathcal{C}* , if $\operatorname{cof}(I)$ is precisely the cofibrations of \mathcal{C} .

Remark 3.5.2.13. By construction, $\operatorname{cof}(I) \subseteq \square(I^{\square})$. This follows because every element of I lies in $\square(I^{\square})$, and the proof of Corollary 3.4.0.16 shows us that any map in $\operatorname{cof}(I)$ inherits left lifting properties against maps that lift on the right against I , so must also lie in $\square(I^{\square})$.

We are now able to state Quillen's Small Object argument; the original formulation can be found in [Qui67, II.3.2, Lemma 3], however we use the transfinite analogue as stated by [SS00, Lemma 2.1].

Lemma 3.5.2.14 (Quillen's Small Object Argument). *Let \mathcal{C} be a cocomplete locally small category with a set of maps I in \mathcal{C} , whose domains are small relative to the class of morphisms $\text{cell}(I)$. Then the following statements are true:*

1. *there is a functorial factorisation of any map F in \mathcal{C} as $F = J \circ I$, where $I \in {}^{\square}(I^{\square})$ and $J \in I^{\square}$;*
2. *every morphism in ${}^{\square}(I^{\square})$ is a retract of a morphism which can be obtained as a composition (possibly transfinite) of pushouts of maps in I .*

Remark 3.5.2.15. It is important that I is actually a set, and not a proper class.

A statement 2 of Lemma 3.5.2.14 is that ${}^{\square}(I^{\square}) \subseteq \text{cof}(I)$, and by Remark 3.5.2.13 above, $\text{cof}(I) = {}^{\square}(I^{\square})$.

Proof. The original proof dates back to [Qui67, II.3.3, Lemma 3], and has been re-written, and generalised in many places such as [Hov98a, Theorem 2.1.14], [SS00, Lemma 2.1] or [Gar09, Theorem 4.4]. Here, we outline the key ideas of the proof.

Consider a morphism $F: X \rightarrow Y$ in \mathcal{C} . To prove statement 1, we perform I -cell attachments to X relative to Y . Denote by $G_j: A_i \rightarrow B_i$ an element of I , and let S denote the set of all maps $H_i: A_i \rightarrow X$ such that there is a map $H'_i: B_i \rightarrow Y$ making Figure 3.13 commute.

$$\begin{array}{ccc} A_i & \xrightarrow{H_i} & X \\ \downarrow G_j & & \downarrow F \\ B_i & \dashrightarrow^{H'_i} & Y \end{array}$$

Figure 3.13: Extending a map $H_i: A_i \rightarrow X$ to a map $H'_i: B_i \rightarrow Y$.

For each diagram as in Figure 3.13, we wish to attach a copy of B_i to X along $H(A_i)$. This is obtained via the pushout shown in Figure 3.14.

$$\begin{array}{ccc} \coprod_S A_i & \xrightarrow{\coprod_S H_i} & X \\ \downarrow \coprod_S G_j & & \downarrow I_1 \\ \coprod_S B_i & \dashrightarrow^J & Z_1 \end{array} \quad \begin{array}{c} \xrightarrow{F} \\ \downarrow \\ \xrightarrow{J_1} \\ \downarrow \\ \xrightarrow{\coprod_S H'_i} \\ \downarrow \\ Y \end{array}$$

Figure 3.14: Cell attachment to X relative to Y .

Note that the pushout Z_1 comes with a unique map $J_1: Z_1 \rightarrow Y$ which identifies each filler added to X in X_1 , with its image in Y under H' .

Let κ be a regular cardinal such that the domain of each morphism of I is κ -small relative to $\text{cell}(I)$, and let λ be a κ -filtered ordinal. The idea of the proof is to set up a transfinite induction

using this construction. To explain this, suppose that for a limit ordinal γ we have Z_α for all $\alpha < \gamma$. Define Z_γ as $\text{colim}_{\alpha < \gamma} Z_\alpha$, and the map J_γ as the map out of the colimit induced by the maps J_α . The technique discussed above allows us to construct $Z_{\gamma+1}$ and the map $J_{\gamma+1}$. Let $Z = \text{colim}_{\alpha < \lambda} Z_\alpha$, and let J be the map induced out of Z .

The map $I: X \rightarrow Z$ is the colimit of the maps I_i and by construction is an element of $\text{cof}(I)$. Remark 3.5.2.13 shows that I lives in ${}^{\square}(I^{\square})$. To show that $J \in I^{\square}$, we need to construct a lift in Figure 3.15 for any $G_i \in I$.

$$\begin{array}{ccc} A_i & \xrightarrow{H} & Z \\ \downarrow G_i & \nearrow & \downarrow J \\ B_i & \longrightarrow & Y \end{array}$$

Figure 3.15: We need to construct a lift in this commutative diagram.

By assumption, the domains of each map of I are κ -small relative to $\text{cell}(I)$, hence we can find a $\beta < \lambda$ such that H factors through Z_β . In the construction, we attach a copy of B_i to Z_β along $H(A_i)$ at the stage of the construction $Z_{\beta+1}$. This defines a lift in Figure 3.15, which makes both triangles commute.

We now discuss the proof of statement 2; to do this, consider a map $F: X \rightarrow Y$ such that $F \in {}^{\square}(I^{\square})$, and use the factorisation constructed in part 1 to factor F as $J \circ I: X \rightarrow Z \rightarrow Y$. Consider Figure 3.16, in which there is a lift by the assumed properties of F and the map J .

$$\begin{array}{ccc} X & \xrightarrow{I} & Z \\ \downarrow F & \nearrow K & \downarrow J \\ Y & \xrightarrow{\mathbb{1}} & Y \end{array}$$

Figure 3.16: A lift K exists because $F \in {}^{\square}(I^{\square})$ and $J \in I^{\square}$.

To complete the proof, we use the lift K of Figure 3.16 to exhibit F as a retract of I in Figure 3.17; this shows that F lies in $\text{cof}(I)$.

$$\begin{array}{ccccc} X & \xrightarrow{\mathbb{1}_X} & X & \xrightarrow{\mathbb{1}_X} & X \\ \downarrow I & & \downarrow F & & \downarrow I \\ Y & \xrightarrow{K} & Z & \xrightarrow{J} & Y \\ & \searrow & \text{---} & \nearrow & \\ & & \mathbb{1}_Y & & \end{array}$$

Figure 3.17: Exhibiting F as a retract of I .

The proof is completed by noting that the left hand square commutes by the construction of

I and J , and the right hand square commutes by the construction of the lift K . The composite along the bottom row gives the identity on Y due to the lower triangle of Figure 3.16. ■

3.5.3 Cofibrantly Generated Model Categories

In this section, we introduce the concept of a cofibrantly generated model category; the idea is that there is a set of generating cofibrations and acyclic cofibrations, such that all (acyclic) cofibrations are generated from the generating set. Requiring smallness properties for the domains of these maps is enough to ensure that the maps generated from the generating set have the required lifting properties of an (acyclic) cofibration.

Definition 3.5.3.1. A model category \mathcal{C} is *cofibrantly generated* if it is cocomplete, and there exists a set of *generating cofibrations* I and a set of *generating acyclic cofibrations* J such that the following conditions hold:

1. the fibrations are precisely the maps J^\square ;
2. the acyclic fibrations are precisely the maps I^\square ;
3. the domain of each map in I or J is small with respect to maps obtained as a composition (possibly transfinite) of pushouts of maps in I or J .

In a cofibrantly generated model structure, the cofibrations are the maps of ${}^\square(I^\square)$, and the acyclic cofibrations are the maps of ${}^\square(J^\square)$.

Example 3.5.3.2. The Quillen model category on topological spaces is cofibrantly generated, with a set of generating cofibrations $I = \{|\partial\Delta^n| \hookrightarrow |\Delta^n|\}_{n \in \mathbb{N}}$, and a set of generating acyclic cofibrations $J = \{|\Lambda_k^n| \hookrightarrow |\Delta^n|\}_{n \in \mathbb{N}}$. Another choice of generating set for acyclic cofibrations is $J = \{D^n \times \{0\} \hookrightarrow D^n \times [0, 1]\}_{n \in \mathbb{N}}$.

Remark 3.5.3.3. One can also make sense of a cofibrantly generated model structure, under the assumption that pushouts of maps in I and J exist.

Remark 3.5.3.4. If we are able to apply the small object argument to our set of morphisms I in \mathcal{C} , then it follows that $\text{cof}(I) = {}^\square(I^\square)$. It is precisely this set I that we say is the *generating set* for the morphisms $\text{cof}(I)$. Equivalently, any morphism of $\text{cof}(I)$ can be obtained as a transfinite composite of pushouts of coproducts of morphisms in I .

When constructing model categories, it is possible to transport one along an adjunction. The following result gives us a construction of a cofibrantly transferred model category. The technical details and proof of this result can found in [Hir09, Theorem 11.3.2], and can be traced back to [Qui67, II.4].

Theorem 3.5.3.5. Consider a cofibrantly generated model category \mathcal{C} , with a generating set of cofibrations I and of acyclic cofibrations J . Suppose there are adjoint functors $\mathcal{F}: \mathcal{C} \rightleftarrows \mathcal{D}: \mathcal{G}$, with $\mathcal{F} \dashv \mathcal{G}$, and such that \mathcal{D} is a category with all small limits and colimits. Suppose that in \mathcal{D} :

1. the sets $\mathcal{F}(I)$ and $\mathcal{F}(J)$ permit the small object argument;

2. the functor \mathcal{G} carries relative $\mathcal{F}(J)$ -cell complexes to weak equivalences of \mathcal{C} .

Then, there is a cofibrantly generated model category on \mathcal{D} , where $\mathcal{F}(I)$ (resp. $\mathcal{F}(J)$) is a generating set of (acyclic) cofibrations, and weak equivalences (resp. fibrations) in \mathcal{D} are the morphisms which are sent to a weak equivalence (fibration) of \mathcal{C} under \mathcal{G} . Furthermore, the adjunction $\mathcal{F} \dashv \mathcal{G}$ defines a Quillen adjunction between the model category on \mathcal{C} and the transferred model category on \mathcal{D} .

Remark 3.5.3.6. A useful result is [Hir09, Proposition 12.4.12], which shows that if an object is small relative to $\mathcal{F}(I)$, then it is small relative to $\text{cell}(\mathcal{F}(I))$. Condition 1 of Theorem 3.5.3.5 says that the domains of each morphism in $\mathcal{F}(I)$ (resp. $\mathcal{F}(J)$) are small relative to $\text{cell}(\mathcal{F}(I))$ (resp. $\text{cell}(\mathcal{F}(J))$).

In practice, condition 2 of Theorem 3.5.3.5 can be problematic to check, however in certain circumstances an argument of Quillen gives us a simpler set of conditions to check. This argument, which is contained to [Qui67, II.4.9] and has been re-written in [Ste16, Lemma A.4], and [SS00, Remark 2.4], gives criteria for when condition 2 holds.

Proposition 3.5.3.7 (Quillen’s Path Object Argument). *Condition 2 of Theorem 3.5.3.5 is satisfied if the category \mathcal{D} has functorial path objects for fibrant objects, and a fibrant replacement functor.*

3.6 Constructions in a Model Category

This section will describe how the structure of a model category on \mathcal{C} can be used to study the homotopy category of \mathcal{C} , and explain how a model category models the homotopy category.

3.6.1 Homotopy In A Model Category

Any model category admits a notion of homotopy between maps, and in this section we explain in this section how this is constructed.

Remark 3.6.1.1. It is important that we are working in a model category rather than a model structure, because we will use coproducts and products. Many of these results will also hold in a model structure with coproducts and products.

The presence of the factorisation of any map as an acyclic cofibration followed by a fibration in a model structure implies there is always at least one path object for any object in a model structure, which are defined as in a category of fibrant objects (see Definition 3.3.0.1). Path objects give rise to a notion of homotopy, which is called right homotopy. To move towards a notion of homotopy between maps, we need to define the dual of path objects; cylinder objects, which exist by the factorisations as a cofibration followed by an acyclic fibration in any model category \mathcal{C} . Cylinder objects give rise to a dual notion of homotopy, which is called left homotopy.

Definition 3.6.1.2. Let $X \in \mathcal{C}$ be an object in a closed model category \mathcal{C} . We define a *cylinder object* of X to be a choice of object $\text{Cyl}(X) \in \mathcal{C}$, with a factorisation of the *canonical fold map*

$\nabla_X = (\mathbb{1}_X, \mathbb{1}_X): X \amalg X \rightarrow X$ as $X \amalg X \xrightarrow{(\partial_0, \partial_1)} \text{Cyl}(X) \xrightarrow[\sim]{S} X$. In the factorisation, the map (∂_0, ∂_1) is required to be a cofibration, and S a weak equivalence.

$$\begin{array}{ccc}
 & \text{Cyl}(X) & \\
 \mathcal{C} \ni (\partial_0, \partial_1) \nearrow & & \searrow S \\
 X \amalg X & \xrightarrow[\quad (\mathbb{1}_X, \mathbb{1}_X) \quad]{\nabla_X} & X
 \end{array}$$

Figure 3.18: A cylinder object of X .

A chosen cylinder object is said to be *very good* if in addition to S being a weak equivalence, it is also a fibration.

The dual, a *path object* is defined as in a category of fibrant objects (in Definition 3.3.0.1). If the weak equivalence R is also a cofibration then the path object is said to be *very good*.

Remark 3.6.1.3. The factorisation of a morphism as a cofibration followed by an acyclic fibration imply that there is at least one choice of very good cylinder object in a model category. Dually, factorisation of a morphism as an acyclic cofibration followed by a fibration imply that there is at least one choice of very good path object in a model category.

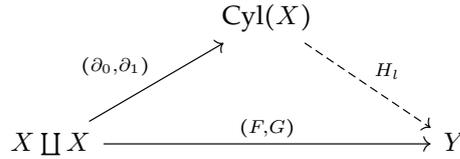
Intuition. Our intuition in **Top** shows that we can think of the map ∂_0 as embedding one copy of X as the bottom end of the cylinder, and ∂_1 as the top of the cylinder. It is important to note that in the Quillen model category, the product $X \times I$ will only be a cylinder object for a cofibrant space X ; this is necessary to ensure that the morphism $X \amalg X \rightarrow X \times I$ is a cofibration. In the Hurewicz (or Strøm) model structure (for more details see [Str72]), then $X \times I$ is a cylinder object for any topological space X .

Similarly, when thinking about path objects, we can think of R as taking a space X to the space of constant paths at each point of X , and the maps D_0 and D_1 as being respectively the start point and end point evaluation morphisms on a path in X .

Progressing as we did in a category of fibrant objects, we can make sense of the notion of right homotopy in a model category, which uses path objects (see Definition 3.3.0.5). The dual of this construction allows us to make sense of the notion of left homotopy between maps, which uses cylinder objects.

Definition 3.6.1.4. Let $F, G: X \rightarrow Y$ be two maps in a model category \mathcal{C} . A *left homotopy* from F to G , is a map $H_l: \text{Cyl}(X) \rightarrow Y$ for some choice of cylinder object $\text{Cyl}(X)$ for X , providing a factorisation of the map $(F, G): X \amalg X \rightarrow Y$ as $X \amalg X \xrightarrow{(\partial_0, \partial_1)} \text{Cyl}(X) \xrightarrow{H_l} Y$. If a left homotopy from F to G exists for some cylinder object for X , then we say that F is *left homotopic* to G and denoted the left homotopy by $F \stackrel{l}{\sim} G$.

A left homotopy from F to G is depicted in Figure 3.19. Commutativity of the appropriate diagram involving cocones over the coproduct shows that $H_l \circ \partial_0 = F: X \rightarrow Y$, as well as $H_l \circ \partial_1 = G: X \rightarrow Y$.


 Figure 3.19: A left homotopy from F to G .

Definition 3.6.1.5. A right homotopy between morphisms is defined as in Definition 3.3.0.5, for a choice of path object for the target. If both a left and right homotopy exist from F to G , then we say that F and G are *homotopic*, denoted by $F \sim G$. We say that a morphism $F: X \rightarrow Y$ is a *homotopy equivalence* between X and Y , if there is a map $G: Y \rightarrow X$ such that $G \circ F \sim \mathbb{1}_X$ and $F \circ G \sim \mathbb{1}_Y$.

There are a number of standard results that explain how the notions of left and right homotopy interact, and explain why we are justified in calling them homotopies. Here we present a few results relevant to this thesis. The proofs of these results can be found in [GJ09, §1]

Proposition 3.6.1.6. Assume we have a left homotopy between two morphisms $F \stackrel{l}{\sim} G: X \rightarrow Y$ in a model category \mathcal{C} . If either F or G is a weak equivalence, then so is the other. By duality, if $F \stackrel{r}{\sim} G$ and either F or G is a weak equivalence, then so is the other.

Proposition 3.6.1.7. For a left homotopy $F \stackrel{l}{\sim} G: X \rightarrow Y$ and a morphism $H: Y \rightarrow Z$, it follows that $H \circ F \stackrel{l}{\sim} H \circ G$. By duality, if we consider $F \stackrel{r}{\sim} G: X \rightarrow Y$ and a morphism $J: W \rightarrow X$, then $F \circ J \stackrel{r}{\sim} G \circ J$.

We can now consider how left and right homotopies behave in the presence of cofibrant and fibrant objects.

Lemma 3.6.1.8. If X is a cofibrant object in a model category \mathcal{C} , then left homotopy defines an equivalence relation on $\mathcal{C}(X, Y)$. By duality, if we have a fibrant object Y in a model category \mathcal{C} , then right homotopy defines an equivalence relation on maps $\mathcal{C}(X, Y)$.

Corollary 3.6.1.9. Consider a fibrant object A and cofibrant object B in a model category \mathcal{C} , and morphisms $F, G: A \rightarrow B$. Then the following statements are equivalent:

1. F is left homotopic to G ;
2. F and G are right homotopic with respect to a fixed choice of path object;
3. F is right homotopic to G ;
4. F and G are left homotopic with respect to a fixed choice of cylinder object.

From here, we can prove the model category analogue of Whitehead's Theorem.

Theorem 3.6.1.10 (Whitehead's Theorem). In a model category, a weak equivalence between cofibrant-fibrant objects is a homotopy equivalence.

Proof. This proof is taken from [GS07, Theorem 1.10].

To start, assume that a map $F: A \rightarrow B$ is a map between cofibrant-fibrant objects and that F is an acyclic fibration. Consider a cylinder object for A which is a factorisation of the form $\nabla_A: A \amalg A \xrightarrow{(\partial_0, \partial_1)} \text{Cyl}(A) \xrightarrow{\sim} A$. Then, we can form a commutative diagram as in Figure 3.20. In this figure, a lift θ exists by MC4, because B is cofibrant (hence the map $\emptyset \rightarrow B$ is a cofibration) and by assumption F is an acyclic fibration.

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & \nearrow \exists \theta & \downarrow F \\ B & \xrightarrow{\mathbb{1}_B} & B \end{array}$$

Figure 3.20: Constructing a section of the acyclic fibration f .

Reading off the commutivity of Figure 3.20, we see that $F \circ \theta = \mathbb{1}_B$. We are left to prove that $\theta \circ F \sim \mathbb{1}_A$. Using a similar method, we construct the commutative diagram in Figure 3.21.

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(\theta \circ F, \mathbb{1}_A)} & A \\ (\partial_0, \partial_1) \downarrow & \nearrow \exists \theta' & \downarrow F \\ \text{Cyl}(A) & \xrightarrow{F \circ S} & B \end{array}$$

Figure 3.21: Constructing a homotopy from $\theta \circ F$ to $\mathbb{1}_A$.

The diagram of Figure 3.21 is commutative by construction. This is because passing along the lower two arrows gives the composite $F \circ S \circ (\partial_0, \partial_1) = F \circ (\mathbb{1}_A, \mathbb{1}_A) = (F, F)$. Passing the other way gives us the composite $F \circ (\theta \circ F, \mathbb{1}_A) = (F \circ \theta \circ F, F \circ \mathbb{1}_A) = (\mathbb{1}_B \circ F, F \circ \mathbb{1}_A) = (F, F)$. By definition of a cylinder object for A , the morphism $(\partial_0, \partial_1): A \amalg A \rightarrow \text{Cyl}(A)$ is a cofibration. Therefore we can find a lift θ' in Figure 3.21.

The top triangle of Figure 3.21 provides the left homotopy that we need between $\theta \circ F$ and $\mathbb{1}_A$ with $H_t = \theta'$, proving that F is a homotopy equivalence if it is an acyclic fibration. By duality, we see that F is a homotopy equivalence if F is an acyclic cofibration between cofibrant-fibrant objects.

Now, consider any weak equivalence $F: A \rightarrow B$ of cofibrant-fibrant objects. We can factor F as a composite $F: A \xrightarrow{I} C \xrightarrow{J} B$ of acyclic cofibration I followed by an acyclic fibration P (by applying the 2-out-of-3 property for weak equivalences). The object C must also be cofibrant-fibrant, hence the morphisms I and J are homotopy equivalences. Thus, F is a homotopy equivalence. \blacksquare

3.6.2 Homotopy Category Associated To A Model Category

The proofs of results contained in this section can be found in [GS07, §1]. The main result is Theorem 3.6.2.9, characterising the homotopy category of a model category.

Definition 3.6.2.1. Let $\mathcal{C}_{\mathcal{C}, \mathcal{F}}$ denote the full subcategory of a model category \mathcal{C} , consisting of the cofibrant-fibrant objects of \mathcal{C} . The subcategories $\mathcal{C}_{\mathcal{C}}$ and $\mathcal{C}_{\mathcal{F}}$ are defined similarly.

Definition 3.6.2.2. Consider cofibrant-fibrant objects A, B in a model category \mathcal{C} . We define $\pi(A, B)$ to be the set of equivalence classes of morphisms $\mathcal{C}(A, B)$ under the homotopy equivalence relation, which is well-defined by Corollary 3.6.1.9.

Definition 3.6.2.3. To a model category \mathcal{C} , we can associate a category $\pi\mathcal{C}_{\mathcal{C}, \mathcal{F}}$. Objects are the cofibrant-fibrant objects of \mathcal{C} , and morphisms from A to B are elements of the set $\pi(A, B)$; i.e. a morphism in $\pi\mathcal{C}_{\mathcal{C}, \mathcal{F}}$ is a homotopy class of maps from A to B in \mathcal{C} . It should be noted that $\pi\mathcal{C}_{\mathcal{C}, \mathcal{F}}$ is also denoted as $\mathcal{C}_{\mathcal{C}, \mathcal{F}} / \sim$.

For any object X in a model category \mathcal{C} , we can assign a cofibrant-fibrant object RQX to X . Taking a cofibrant replacement does not give a canonical assignment to maps, however is well defined up to left homotopy. Similarly, when taking a fibrant replacement, it is well-defined on maps up to right homotopy. It follows that the assignment of a cofibrant-fibrant object to any object X is well-defined on morphisms up to homotopy.

Definition 3.6.2.4. We define the *homotopy category* $\mathbf{Ho}(\mathcal{C})$ of a model category \mathcal{C} to have the same objects as \mathcal{C} , and with morphisms between objects given by $\mathbf{Ho}(\mathcal{C})(X, Y) = \pi(RQX, RQY)$.

Remark 3.6.2.5. To define homotopy groups, $\pi_n X$ in the $\mathbf{Top}_{\text{Quillen}}$ model structure, for an object X , we simply define $\pi_n X$ to be the homotopy classes of maps from S^n to RQX . This is valid because S^n is a cofibrant-fibrant in this model structure, so we can let RQS^n be S^n .

Proposition 3.6.2.6. Consider a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ between model categories. If \mathcal{F} carries weak equivalences of \mathcal{C} to isomorphisms in \mathcal{D} , then any morphisms $F \stackrel{l}{\sim} G$ in \mathcal{C} are identified, meaning that $\mathcal{F}(F) = \mathcal{F}(G)$. By duality, any morphisms $F \stackrel{r}{\sim} G$ are identified under \mathcal{F} .

Theorem 3.6.2.7. The functor $\gamma: \mathcal{C} \rightarrow \mathbf{Ho}(\mathcal{C})$ is universal, in the sense that for any functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ which carries the weak equivalences of \mathcal{C} to isomorphisms in \mathcal{D} , then we can find a unique functor $\mathcal{F}': \mathbf{Ho}(\mathcal{C}) \rightarrow \mathcal{D}$ such that $\mathcal{F} = \mathcal{F}' \circ \gamma$.

Remark 3.6.2.8. Alternatively the homotopy category of a category \mathcal{C} with weak equivalences, is defined as the category which carries weak equivalences of \mathcal{C} to isomorphisms, and is universal in the sense of Theorem 3.6.2.7. Therefore the definition of the homotopy category presented, for a model category, coincides with the original definition provided for a category with weak equivalences.

Theorem 3.6.2.9. The homotopy category $\mathbf{Ho}(\mathcal{C})$ is equivalent, as a category, to the category of cofibrant-fibrant objects in \mathcal{C} under the appropriate homotopy equivalence relation; i.e. $\mathcal{C}_{\mathcal{C}, \mathcal{F}} / \sim$.

Remark 3.6.2.10. The theorem explains how a model category models the homotopy category. It is a generalisation of a number of well known results. An example is that the category of homotopy classes of maps between CW-complexes is equivalent to the full homotopy category of topological spaces. Another example is that the homotopy category of simplicial sets is equivalent to the homotopy category of Kan complexes.

3.6.3 Quillen Adjunctions

Before delving into Quillen adjunctions we introduce Ken Brown's Lemma, which introduces us to the idea of considering maps between model categories.

Lemma 3.6.3.1 (Ken Brown's Lemma). *Consider a model category \mathcal{C} , and a category with weak equivalences \mathcal{D} . If a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ carries acyclic cofibrations between cofibrant objects of \mathcal{C} to weak equivalences, then \mathcal{F} carries weak equivalences between cofibrant objects to weak equivalences. Dually, if \mathcal{F} carries acyclic fibrations between fibrant objects of \mathcal{C} to weak equivalences, then \mathcal{F} carries weak equivalences between fibrant objects of \mathcal{C} to weak equivalences.*

The proof that we explain is taken from [Hov98a, Lemma 1.1.12].

Proof. Consider a weak equivalence of cofibrant objects $F: A \rightarrow B$. To start, we consider the pushout diagram shown in Figure 3.22. By construction, both of the morphisms $A \rightarrow A \amalg B$ and $B \rightarrow A \amalg B$ are cofibrations (because these inclusions are constructed as pushouts of $\emptyset \rightarrow A$ and $\emptyset \rightarrow B$ which are both cofibrations by assumption).

$$\begin{array}{ccc} \emptyset & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \amalg B \end{array}$$

Figure 3.22: The coproduct of A and B .

We can factor the morphism $(F, \mathbb{1}_B): A \amalg B \rightarrow B$ as a cofibration $A \amalg B \rightarrow C$ followed by an acyclic fibration $C \rightarrow B$. Apply the two-out-of-three property for weak equivalences to see that the composite $A \rightarrow A \amalg B \rightarrow C$ is an acyclic cofibration (when we post-compose with the weak equivalence $C \rightarrow B$ the composite is F), and similarly for $B \rightarrow A \amalg B \rightarrow C$ (when we post-compose again with the weak equivalence $C \rightarrow B$ the composite is $\mathbb{1}_B$). By assumption, the functor \mathcal{F} carries both composites to weak equivalences in \mathcal{D} .

The composite $\mathbb{1}_B: B \rightarrow A \amalg B \rightarrow C \rightarrow B$ is also sent to a weak equivalence under \mathcal{F} . Therefore by the two-out-of-three property for weak equivalences, $\mathcal{F}(C \rightarrow B)$ must be a weak equivalence. Consider the composite $F: A \rightarrow A \amalg B \rightarrow C \rightarrow B$; this is sent to a weak equivalence under \mathcal{F} , because $A \rightarrow A \amalg B \rightarrow C$ is an acyclic cofibration between cofibrant objects and is therefore sent to a weak equivalence, as is $C \rightarrow B$. Therefore $\mathcal{F}(F)$ is a weak equivalence as claimed. ■

Proposition 3.6.3.2. *Let $\mathcal{F}:\mathcal{C} \rightleftarrows \mathcal{D}:\mathcal{G}$ be adjoint functors between model categories \mathcal{C} and \mathcal{D} . Then the following conditions are equivalent:*

1. \mathcal{F} preserves cofibrations and acyclic cofibrations;
2. \mathcal{G} preserves fibrations and acyclic fibrations;
3. \mathcal{F} preserves cofibrations and \mathcal{G} preserves fibrations;
4. \mathcal{F} preserves acyclic cofibrations and \mathcal{G} preserves acyclic fibrations.

Proof. Consider a lifting diagram as in the left hand side of Figure 3.23. To this diagram we apply the adjunction which gives the right hand side of Figure 3.23.

$$\begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{F} & C \\
 \mathcal{F}(I) \downarrow & \nearrow H & \downarrow J \\
 \mathcal{F}(B) & \xrightarrow{G} & D
 \end{array}
 \quad \Leftrightarrow \quad
 \begin{array}{ccc}
 A & \xrightarrow{F} & \mathcal{G}(C) \\
 I \downarrow & \nearrow H & \downarrow \mathcal{G}(J) \\
 B & \xrightarrow{G} & \mathcal{G}(D)
 \end{array}$$

Figure 3.23: A lifting diagram in \mathcal{C} on the left and \mathcal{D} on the right.

Naturality of the adjunction shows us that a solution on either side is equivalent to a solution on the other (for further explanation of this argument see [Rie14, Lemma 11.1.5]). It is then straightforward to see that conditions (i)-(iv) are equivalent, using the characterisation of morphisms by their lifting properties (see Lemma 3.4.0.12). ■

Definition 3.6.3.3. Let \mathcal{C} and \mathcal{D} denote two model categories, with adjoint functors $\mathcal{F}:\mathcal{C} \rightleftarrows \mathcal{D}:\mathcal{G}$ such that the four equivalent conditions of Proposition 3.6.3.2 hold. Then, we say that the functor \mathcal{F} is *left Quillen adjoint* to \mathcal{G} , or that \mathcal{G} is *right Quillen adjoint* to \mathcal{F} , and say the *Quillen adjunction* is the pair $(\mathcal{F}, \mathcal{G})$.

Remark 3.6.3.4. More generally, a functor between model categories which preserves cofibrations and acyclic cofibrations is called a *left Quillen functor*, and one which preserves fibrations and acyclic fibrations is called a *right Quillen functor*.

Corollary 3.6.3.5. *Consider a Quillen adjunction $\mathcal{F}:\mathcal{C} \rightleftarrows \mathcal{D}:\mathcal{G}$ between model categories. Then \mathcal{F} preserves weak equivalences between cofibrant objects, and \mathcal{G} preserves weak equivalences between fibrant objects.*

Proof. Apply Ken Brown's Lemma (Lemma 3.6.3.1) to the Quillen adjunction. ■

A Quillen adjunction will induce an adjunction between the homotopy categories (we defined homotopy categories in Definition 3.6.2.4).

3.6.4 Quillen Equivalence Between Model Categories

We now briefly introduce the concept of a Quillen equivalence, which should be the correct notion of equivalence between model categories. In particular, this means that it should be

stronger than asking for a Quillen adjunction and a weaker notion than asking for equivalence of the underlying categories.

Definition 3.6.4.1. For a left Quillen functor between model categories $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$, the *left derived functor* of \mathcal{F} , denoted by $\mathbb{L}\mathcal{F}: \mathbf{Ho}(\mathcal{C}) \rightarrow \mathbf{Ho}(\mathcal{C}_{\mathcal{C}}) \rightarrow \mathbf{Ho}(\mathcal{D})$, is obtained by applying the cofibrant replacement functor and then \mathcal{F} . Dually, there is a notion of a *right derived functor* obtained by applying the functor after applying the fibrant replacement functor.

Definition 3.6.4.2. A Quillen adjunction between model categories $\mathcal{F}: \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ is a *Quillen equivalence* if the left derived functor is an equivalence of homotopy categories or, equivalently, if the right derived functor is an equivalence of homotopy categories.

Example 3.6.4.3. For a model category \mathcal{C} , the identity maps $\mathbb{1}_{\mathcal{C}}: \mathcal{C} \rightleftarrows \mathcal{C} : \mathbb{1}_{\mathcal{C}}$ provide a Quillen equivalence between \mathcal{C} and \mathcal{C} .

In a more practical setting, if one wants to check whether a Quillen adjunction is a Quillen equivalence, the following proposition provides conditions which are often easier to check.

Proposition 3.6.4.4. *A Quillen adjunction is a Quillen equivalence if and only if for all cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{D}$, a map $F: \mathcal{F}(X) \rightarrow Y$ in \mathcal{D} is a weak equivalence if and only if its adjoint $F: X \rightarrow \mathcal{G}(Y)$ is a weak equivalence in \mathcal{C} .*

Proof. See [Hov98a, Proposition 1.3.13] ■

Remark 3.6.4.5. It is not true that any equivalence of homotopy categories lifts to a Quillen equivalence between model categories; a counter example to this claim is provided in [DS09].

3.7 J -Semi Model Categories

In this section, we introduce the notion of a J -semi model category. The concept was first introduced by Mark Hovey in [Hov98b, Theorem 3.3]. A J -semi model category structure is a slight weakening of the requirements of a model category; in particular it is a weakening of the axioms on the (acyclic cofibration, fibration) weak factorisation system.

Definition 3.7.0.1. A *J -semi model category* (or a *left semi model category*) is a bicomplete category \mathcal{C} with specified classes of cofibrations, fibrations and weak equivalences such that the initial object is cofibrant and such that the three classes satisfy two alterations to the axioms MC1-MC4 of a model structure (introduced in Definition 3.4.0.4). The first alteration is that we require factorisations as an acyclic cofibration followed by a fibration, only for those morphisms whose domain is cofibrant in \mathcal{C} . The second alteration is that we require liftings of acyclic cofibrations against fibrations, only when the domain of the acyclic cofibration is cofibrant.

To understand why this is a practical definition, we present results that show that working in a left semi model category gives a number of parallel results to a model category. These results come from Markus Spitzweck [Spi01]. Moreover, we can describe the homotopy category associated to a left semi model category, by way of the cofibrant replacement functor.

Proposition 3.7.0.2. *In a J -semi model category, a map is a cofibration if and only if it has the left lifting property against all acyclic fibrations, and dually acyclic fibrations are characterised by lifting against cofibrations.*

A map whose domain is cofibrant is a acyclic cofibration if and only if it lifts on the left against all fibrations. Similarly, a map whose domain is cofibrant is a fibration if and only if it lifts on the right against all acyclic cofibrations with cofibrant domain.

Cofibrations are stable under pushout, as are acyclic cofibrations with cofibrant domain along maps with a cofibrant codomain. There are corresponding dual statements, however these are more technical to state (for details see [Spi01, p.10]). ■

To define homotopy in a semi model category, note that we can define path and cylinder objects as in a model category. Cylinder objects exist for any object, however path objects are only guaranteed to exist for cofibrant objects.

Proposition 3.7.0.3. *For left homotopic maps $F \overset{l}{\sim} G: X \rightarrow Y$ and $H: Y \rightarrow Z$, we have $H \circ F \overset{l}{\sim} H \circ G$. Dually, for right homotopic maps $F \overset{r}{\sim} G: X \rightarrow Y$ and $J: W \rightarrow X$, we have $F \circ J \overset{r}{\sim} G \circ J$.* ■

Proposition 3.7.0.4. *For X, Y in a J -semi model category \mathcal{C} with X cofibrant, left homotopy defines an equivalence relation on $\mathcal{C}(X, Y)$. There is a dual statement which is technical to state; for details see [Spi01, Proposition 2.4]* ■

Proposition 3.7.0.5. *If two maps between cofibrant objects are left homotopic, then they are right homotopic. Dually, if the domain is cofibrant and the codomain is fibrant, then when two maps are right homotopic they are left homotopic. Consequently, when considering maps from a cofibrant object to a cofibrant-fibrant object, left and right homotopy coincide and homotopy defines an equivalence relation.*

Results about the homotopy category, analogous to §3.6.2 follow from the same proofs as the results for model categories. The theory of Quillen adjunctions and derived functors can be extended to the semi model category setting; for further details see [Bar07]. The main difference when working in a J -semi model category is the necessity to apply the cofibrant replacement functor before moving to the homotopy category.

Chapter 4

Simplicial Sets

The study of simplicial sets is of great interest to us because it allows a combinatorial study of topological spaces. We will use this machinery to study stratified spaces, and show how the tools translated in terms of stratified spaces coincide with a number of phenomena that have been studied previously in that context.

4.1 The Category of Simplicial Sets

We wish to introduce the notion of a simplicial set, and to do this we need to initially describe the simplicial category.

Definition 4.1.0.1. The *simplicial category* Δ is defined to be the category with finite ordered sets as objects and order preserving **Set** maps between them as morphisms. Up to isomorphism, the elements of Δ are uniquely represented by $[n] = \{0 < 1 < \dots < n\}$ for $n \in \mathbb{N}$, and morphisms in Δ are given by increasing maps, which are set maps $F: [n] \rightarrow [m]$ such that when $i \leq j$ then $F(i) \leq F(j)$.

Remark 4.1.0.2. In particular, Δ is a small category.

In the category Δ we have two important classes of maps, from which every other map can be built. The first of these will be referred to as *coface maps* $d^i: [n-1] \rightarrow [n]$, which are defined element-wise as:

$$d^i(k) = \begin{cases} k & k < i; \\ k+1 & k \geq i, \end{cases}$$

which is an injective **Set** map. Similarly, we also have *codegeneracy maps* $s^i: [n+1] \rightarrow [n]$, which are defined element-wise as:

$$s^i(k) = \begin{cases} k & k \leq i; \\ k-1 & k > i, \end{cases}$$

which is a surjective **Set** map. From these definitions, it is straightforward to see that we can express any morphism in Δ as a combination of these.

Proposition 4.1.0.3. *Any morphism $F: [n] \rightarrow [m]$ in Δ can be expressed uniquely in terms of coface and codegeneracy maps as below, with $k_1 < \dots < k_i$ and $j_1 < \dots < j_h$:*

$$F = d^{k_i} \circ \dots \circ d^{k_1} \circ s^{j_h} \circ \dots \circ s^{j_1}. \quad (4.1)$$

Using the notion of the simplicial category, we are now able to define a simplicial set.

Definition 4.1.0.4. A *simplicial set* is a functor $X: \Delta^{\text{op}} \rightarrow \mathbf{Set}$.

Notation. We let $X([n]) = X_n$ and as we will later justify, we will call elements of X_n the n -simplices of X .

By definition, X is a functor and so we have maps in **Set** induced by the coface and codegeneracy maps of Δ . There are two distinct types of map here; the first of these we call the *face* maps, written as $d_i = X d^i: X_n \rightarrow X_{n-1}$. The map d_i simply takes an n -simplex in X , and assigns its i^{th} face. We also have the *degeneracy* maps, denoted similarly by $s_i = X s^i: X_n \rightarrow X_{n+1}$. The degeneracy map s_i assigns to an n -simplex the $n + 1$ -simplex with its i^{th} vertex repeated.

Definition 4.1.0.5. Simplicial sets constitute a category, which we denote by **sSet**. The objects of **sSet** are simplicial sets, i.e. functors $X: \Delta^{\text{op}} \rightarrow \mathbf{Set}$, and the morphisms are natural transformations between simplicial sets.

A morphism $F: X \rightarrow Y$ in the category **sSet** is comprised of maps $F_n: X_n \rightarrow Y_n$ for all $n \in \mathbb{N}$ which commute with the face and degeneracy maps.

Definition 4.1.0.6. A *degenerate* simplex is a simplex $x \in X_n$, which can be written as $x = s_i(y)$ for some $y \in X_{n-1}$ and some i . We say that a simplex $x \in X_n$ is *non-degenerate* if it cannot be written in the form $x = s_i(y)$ for some y and some i .

Remark 4.1.0.7. The Eilenberg-Zilber Lemma [GZ67, p.26] states that a degenerate simplex $x \in X_n$ can be written uniquely in the form $(X \circ s)(y)$ for a pair (s, y) where s is a surjective map $s: [n] \rightarrow [m]$ in Δ , and $y \in X_m$ is a non-degenerate m -simplex of X where $m < n$. Because s is a surjective, it can be written as a unique composition of co-degeneracy maps.

Remark 4.1.0.8. It is enough to define a simplicial morphism F by its action on the non-degenerate simplices of a simplicial set X . To see why this is the case, consider a degenerate n -simplex in X . This can be expressed as $s_i(x)$ for some i and where x is a possibly non-degenerate n -simplex. Because the simplicial morphism F commutes with face and degeneracy maps, we see that $F(s_i(x)) = s_i(F(x))$. Similarly, a simplicial morphism F on a face of an n -simplex in X is determined entirely by the action of F on the n -simplex.

Remark 4.1.0.9. Like any presheaf category on a small category (in this case Δ), the category **sSet** is bicomplete, and limits and colimits are computed level-wise. The initial simplicial set

is the empty simplicial set \emptyset , and the terminal simplicial set $*$, is the simplicial set with one non-degenerate 0-simplex and one degenerate n -simplex for every $n \in \mathbb{N}_{\geq 1}$.

Consider the Yoneda embedding $\mathcal{Y}: \Delta \rightarrow \mathbf{Set}^{\Delta^{\text{op}}}$, defined by sending $[n]$ to the *simplicial n -simplex* $\Delta^n := \mathcal{Y}[n] = \Delta(-, [n])$. The k -simplices of the simplicial n -simplex are therefore $\Delta_k^n = \Delta([k], [n])$, given by maps $[k] \rightarrow [n]$ in Δ . The Yoneda Lemma says that the Yoneda embedding is full and faithful, hence simplicial morphisms $F: \Delta^n \rightarrow \Delta^m$ correspond bijectively to maps $F: [n] \rightarrow [m]$ in Δ , with the maps $\Delta_k^n \rightarrow \Delta_k^m$ defined by post composition with F .

In more generality, for any simplicial set X , the Yoneda Lemma tells us that there is a natural bijection between the n -simplices of X , and simplicial morphisms $\Delta^n \rightarrow X$. So, we can think of an n -simplex $x \in X_n$ as a map $x: \Delta^n \rightarrow X$, which sends the unique non-degenerate n -simplex in Δ^n to x [Rie11, Lemma 3.1].

Remark 4.1.0.10. A terminal simplicial set $*$ is isomorphic to Δ^0 .

Remark 4.1.0.11. As we discuss in Remark 4.2.0.3, the simplicial set Δ^n corresponds to a combinatorial model for the standard geometric n -simplex $|\Delta^n|$, via the geometric realisation functor.

For simplicial sets $X, Y \in \mathbf{sSet}$, the cartesian product of X and Y is defined as the simplicial set which has n -simplices given by $(X \times Y)_n = X_n \times Y_n$. This follows because limits in a presheaf category are computed level-wise (mentioned in Remark 4.1.0.9). Face and degeneracy maps are defined for $x \in X_n$ and $y \in Y_n$ by $d_i(x, y) = (d_i x, d_i y)$ and $s_i(x, y) = (s_i x, s_i y)$.

4.2 Adjunction Between Simplicial Sets And Topological Spaces

We introduce adjoint functors between \mathbf{sSet} and \mathbf{Top} . This adjunction allows one to exploit the combinatorial nature of simplicial sets to understand the category of topological spaces.

Notation. Let the *geometric n -simplex* denote the standard topological n -simplex, which we will denote by $|\Delta^n|$. Formally, for any affine linear collection of $n + 1$ points $e_0, \dots, e_n \in \mathbb{R}^{n+1}$, we define the geometric n -simplex to be $\mathbb{R}_{\geq 0}$ -linear combinations of the vertices e_0, \dots, e_n :

$$|\Delta^n| = \{x_0 e_0 + \dots + x_n e_n \mid x_i \in \mathbb{R}_{\geq 0} \text{ for all } i, \text{ and } \sum_i x_i = 1\},$$

where the coordinates x_0, \dots, x_n are the barycentric coordinates of a point in $|\Delta^n|$.

Definition 4.2.0.1. Define a covariant functor $|-|: \Delta \rightarrow \mathbf{Top}$ by sending $[n]$ to the geometric n -simplex $|\Delta^n|$, with actions on coface and codegeneracy maps given in the obvious manner.

Definition 4.2.0.2. Let X be a simplicial set, and endow each set X_n with the discrete topology. Define the *geometric realisation* of X , denoted by $|X|$, to be the quotient $|X| = \bigcup_n X_n \times |\Delta^n| / \sim$, where the equivalence relation \sim is defined by $(x, |s^i|(p)) \sim (s_i(x), p)$ for $x \in X_{n+1}$ and $p \in |\Delta^n|$, and $(x, |d^i|(p)) \sim (d_i(x), p)$ for $x \in X_{n-1}$ and $p \in |\Delta^n|$ [Fri12].

Geometric realisation extends the cosimplicial operator $|-|$ of Definition 4.2.0.1 from the category Δ to the category of simplicial sets.

Remark 4.2.0.3. The geometric realisation of a simplicial n -simplex Δ^n is the geometric n -simplex $|\Delta^n|$, justifying the notation we introduced.

This definition gives us a geometric n -simplex for each element of X_n and for all n . We will explain why the equivalence relation defined above is natural, and gives us the properties we desire from a geometric realisation. Consider $(x, |d^i|(p)) \sim (d_i(x), p)$; this takes $d_i(x)$ in $X_n \times |\Delta^n|$, and glues this as the i^{th} face of the $n+1$ -simplex assigned to x in $X_{n+1} \times |\Delta^{n+1}|$. This is also done for any simplex y and j so that $d_j(y) = d_i(x)$. Now consider $(x, |s^i|(p)) \sim (s_i(x), p)$; this suppresses the degenerate simplices in X , since the geometric data of these is contained in the non-degenerate simplices, which they are degeneracies of. The equivalence relation takes the $|\Delta^n|$ for the degenerate n -simplex, and collapses this down to the $n-1$ -simplex, of which it is a degeneracy. These considerations show that we can succinctly express geometric realisation in coend notation as:

$$|X| = \int^{[n] \in \Delta} X_n \odot |\Delta^n|.$$

Remark 4.2.0.4. For a simplicial set X , the geometric realisation $|X|$ is a CW-complex which has one n -cell for each non-degenerate simplex $x \in X_n$.

It is easy to show that the geometric realisation of a simplicial set defines a functor. To do this, we need to define how it acts on a simplicial morphism $F: X \rightarrow Y$. Using the construction given of the geometric realisation of a simplicial set, it is easy to see that F carries simplices of $|X|$ to simplices in $|Y|$. Continuity of this map follows, because F is a simplicial map.

Definition 4.2.0.5. Define the *singular simplicial set* of a topological space X as the simplicial set with n -simplices defined as $\text{Sing}(X)_n = \mathbf{Top}(|\Delta^n|, X)$, the set of continuous maps from the geometric n -simplex to X . To define the face and degeneracy maps, we identify the set $[n] \in \Delta$ with the ordered geometric n -simplex, which is written as $|\Delta^n|$. Therefore, an order-preserving function $F: [m] \rightarrow [n]$ induces a map by pre-composition, as $|\Delta^m| \rightarrow |\Delta^n| \rightarrow X$, and hence a map $F^*: \text{Sing}(X)_n \rightarrow \text{Sing}(X)_m$. In particular, there are induced morphisms $d_i: \text{Sing}(X)_n \rightarrow \text{Sing}(X)_{n-1}$ and $s_i: \text{Sing}(X)_n \rightarrow \text{Sing}(X)_{n+1}$.

In terms of the topological information contained, we can consider $\text{Sing}(X)$ and X to be essentially the same. This is true because the counit $\epsilon_X: |\text{Sing}(X)| \rightarrow X$ is a weak homotopy equivalence.

The geometric realisation functor $|-|$ turns out to be left adjoint to the singular simplicial set functor. This relation is expressed in this important theorem, proof of which follows because Theorem 2.2.0.8 gives conditions under which we can understand the left Kan extension.

Theorem 4.2.0.6. *The geometric realisation functor $|-|$ is left adjoint to the singular simplicial set functor. Explicitly this says that for a simplicial set X and a topological space Y , there is a family of bijections $\mathbf{Top}(|X|, Y) \cong \mathbf{sSet}(X, \text{Sing}(Y))$, which is natural in X and Y .*

We will return to this theorem later, and use it to explain further correspondences between topological spaces and simplicial sets. It is a specific example of a categorical nerve-realisation pair of adjoint functors.

If we let X be a topological space, then $|\text{Sing}(X)|$ will be large, often with uncountably many simplices in each dimension. However [Mil57, Theorem 4] shows that the counit morphism $\epsilon_X: |\text{Sing}(X)| \rightarrow X$ is a weak homotopy equivalence (it induces an isomorphism on homotopy groups). In particular, the counit constructs a CW-complex which is weakly homotopy equivalent to X . As a corollary of this result, we have the following.

Corollary 4.2.0.7. *If we let X be a connected CW-complex, then $|\text{Sing}(X)|$ and X are homotopy equivalent.*

Proof. This is simply proven by applying Whitehead's Theorem, which states that if a continuous map $F: X \rightarrow Y$ induces isomorphisms on homotopy groups, then F is a homotopy equivalence, provided both X and Y are connected CW-complexes. ■

The beauty of these results are that we can consider either the simplicial set, or the geometric realisation of this simplicial set, dependent on our purposes. The following theorem was first proven in Milnor's paper on realisations of simplicial sets [Mil57, Theorem 2].

Theorem 4.2.0.8. *The continuous bijection $|K \times K'| \rightarrow |K| \times |K'|$ is a homeomorphism precisely when either one of $|K|$ or $|K'|$ is locally finite, or both of K and K' are countable.*

4.3 Kan Complexes and ∞ -Categories

Before introducing Kan complexes, we need to introduce the notion of a simplicial subset.

Definition 4.3.0.1. A *simplicial subset* of a simplicial set X , is a simplicial set A so that $A_n \subseteq X_n$ for all n , and the face and degeneracy maps of A agree with those of X .

Definition 4.3.0.2. The simplicial subset of Δ^n generated by the faces $d_i(\Delta^n)$ for all $0 \leq i \leq n$ is the *boundary of the n -simplex* $\partial\Delta^n$. The k^{th} *simplicial horn of the n -simplex* Λ_k^n is the simplicial subset of Δ^n generated by the faces $\{d_0(\Delta^n), \dots, d_{k-1}(\Delta^n), d_{k+1}(\Delta^n), \dots, d_n(\Delta^n)\}$, by which we mean the subset of all simplices obtained by applying face and degeneracy maps to these.

Geometrically speaking, the k^{th} simplicial horn of the n -simplex is the union of all the faces of $|\Delta^n|$ except the k^{th} face. Equivalently, the horn Λ_k^n can be described as the union of the $(n-1)$ -dimensional faces of Δ^n which contain the k^{th} vertex.

Definition 4.3.0.3. We define a *horn* in a simplicial set X to be a simplicial morphism $\Lambda_k^n \rightarrow X$ for any $0 \leq k \leq n$. An *inner horn* in a simplicial set X is a simplicial morphism $\Lambda_k^n \rightarrow X$ for $0 < k < n$. An *outer horn* in a simplicial set X is a simplicial morphism $\Lambda_k^n \rightarrow X$ for $k = 0$ or $k = n$.

Definition 4.3.0.4. We say a simplicial morphism $P: E \rightarrow B$ is a *right fibration*, if P lifts on the right against horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ for $0 < k \leq n$, depicted in Figure 4.1.

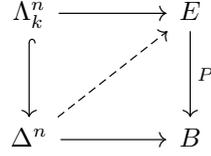


Figure 4.1: A right fibration P has a lift for any $0 < k \leq n$.

We also have the notion of a *left (resp. inner) fibration*, if we can find a lift $\Delta^n \rightarrow E$ in Figure 4.1, for any $0 \leq k < n$ (resp. $0 < k < n$).

We are now able to define what it means to say that simplicial set is a Kan complex.

Definition 4.3.0.5. A *Kan complex* is a simplicial set X such that each horn in X has a filler. By a filler, we mean that for every horn in X there exists an extension along the inclusion $\Lambda_k^n \hookrightarrow \Delta^n$, as indicated by the dotted arrow in Figure 4.2.

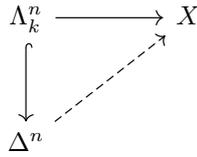


Figure 4.2: A filler of the horn $\Lambda_k^n \rightarrow X$ is indicated by the dotted arrow.

Example 4.3.0.6. It turns out that the singular simplicial set of any topological space is a Kan complex. To see why this is true, consider a simplicial morphism $F: \Lambda_k^n \rightarrow \text{Sing}(Y)$; this defines for us a singular $(n - 1)$ -simplex in Y , for each face $d_i(|\Delta^n|)$ with $i \neq k$, which is a continuous map $\sigma_i: d_i(|\Delta^n|) \rightarrow Y$. Any other simplex of Λ_k^n is either a degeneracy or a face of these non-degenerate simplices, and as we saw in Remark 4.1.0.8 it is enough to define a simplicial morphism by its action on the non-degenerate simplices of a simplicial set.

Consider a continuous retract of the form $\pi: |\Delta^n| \rightarrow |\Lambda_k^n|$; there are many maps of this kind, but we only need to choose one. We know that such a map exists, because there is a homeomorphism $(|\Delta^n|, |\Lambda_k^n|) \cong (I^{n-1} \times I, I^{n-1} \times \{0\})$, and for the second pair there is clearly a retract. We set $F \circ \pi: |\Delta^n| \rightarrow Y$, which is a singular n -simplex in Y and possible extension of F .

Remark 4.3.0.7. Returning to another example we introduced earlier, one can also consider the nerve of a category \mathcal{C} , which isn't always a Kan complex. Let $F: x_1 \rightarrow x_2$ be a morphism in \mathcal{C} which does not have a right inverse. We are therefore able to construct an outer horn Λ_2^2 in $\mathcal{N}\mathcal{C}$, which does not have a filler. Labelling the vertices of Δ^2 as $0, 1, 2$, define a horn by sending 0 to x_2 , 1 to x_1 and 2 to x_2 , the $\langle 0, 2 \rangle$ edge to $\mathbb{1}_{x_2}$ and $\langle 1, 2 \rangle$ to F , as illustrated in Figure 4.3.

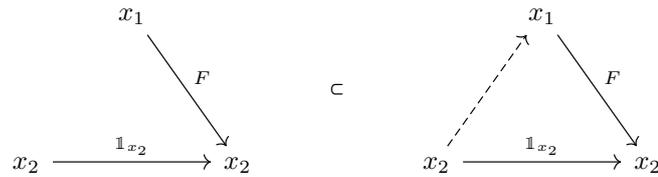


Figure 4.3: An outer horn in NC .

Asking for fillers of every horn in NC implies that each morphism of \mathcal{C} has left and right inverses, which will not hold in general. If the category \mathcal{C} is a groupoid, then every morphism is invertible, and therefore the nerve of a groupoid is a Kan complex.

Remark 4.3.0.8. The nerve of a category \mathcal{C} satisfies a property much akin to that of a Kan Complex. It can be shown that each inner horn in NC has a unique filler. Moreover, a simplicial set X is isomorphic to the nerve of a groupoid if and only if each horn in X has a unique filler [Joy08b, 1.3].

We now proceed to introduce the more general notion of an ∞ -category, which is best thought of as the common generalisation of Kan complexes and of the nerve of a category. In particular, any Kan complex and any nerve of a category will be examples of ∞ -categories.

Definition 4.3.0.9. An ∞ -category, commonly known as a *quasi-category*, is defined to be a simplicial set X , such that every inner horn in X has a filler.

Remark 4.3.0.10. An important remark is that the singular simplicial set of any topological space will be an ∞ -category.

We need to be a little careful with the name used here; there is already a notion of an ∞ -category, and in general quasi-category avoids the confusion that this might draw. Recently however, Jacob Lurie has developed the theory of ∞ -categories in the sense of Definition 4.3.0.9, and has referred to such objects as ∞ -categories [Lur77]. The original name *weak Kan complex* also appears in the literature, however much less frequently now.

4.4 Quillen Model Category on Simplicial Sets

We now present a model category on simplicial sets.

Theorem 4.4.0.1. *There is a model category on simplicial sets in which:*

1. *the cofibrations are the monomorphisms;*
2. *the weak equivalences are the weak homotopy equivalences (i.e. the morphisms whose geometric realization is a weak homotopy equivalence);*
3. *the fibrations are the Kan fibrations, the morphisms with the right lifting property with respect to all horn inclusions.*

This is referred to as the Quillen model category on simplicial sets, and is denoted by $s\mathbf{Set}_{\text{Quillen}}$.

Remark 4.4.0.2. In this model category, all simplicial sets are cofibrant and the fibrant objects are precisely the Kan complexes. It is both left and right proper; in particular, left properness follows because all objects are cofibrant. This model category is cofibrantly generated; a set of generating cofibrations is $I = \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \in \mathbb{N}}$, and a set of generating acyclic cofibrations is $J = \{\Lambda_k^n \hookrightarrow \Delta^n\}_{n \in \mathbb{N}}$. For further details about the Quillen model category on simplicial sets, see [Qui67] or [Hov98a, §3.2].

Proof of Theorem 4.4.0.1. The original proof can be found in [Qui67, II.3] or for an alternative proof see [JT99, Theorem 1.3.1]. ■

The following theorem relates the study of homotopy theories between simplicial sets and topological spaces, via the geometric realisation - singular simplicial set adjunction.

Theorem 4.4.0.3. *The Quillen adjunction $|-|: \mathbf{sSet} \rightleftarrows \mathbf{Top} : \text{Sing}(-)$ defines a Quillen equivalence between model categories on the category of simplicial sets and the category of topological spaces.*

Proof. The original proof can be found in [Qui67, I.4.8], and has been rewritten many times, for example in [GJ09, §I Theorem 11.4] or [Hov98a, Theorem 3.6.7] ■

In particular, Theorem 4.4.0.3 shows us that simplicial sets provide a combinatorial approach to studying the homotopy theory of topological spaces.

4.5 Simplicial Model Categories

We introduce additional structure to that of a model category; namely that it is simplicially enriched, so there is a simplicial set of maps between any two objects, in a way that makes homotopy-theoretic sense. One advantage of working in a simplicial model category is that lifts required from the model structure axioms are unique up to homotopy, a sentiment that we will explain towards the end of this section. The first structure we introduce is that of a simplicial category, choosing to use the definition from [GJ09, §II Definition 2.1].

Definition 4.5.0.1. *A simplicial category \mathcal{C} is a category \mathcal{C} which is endowed with a mapping space functor $\text{Hom}_{\mathcal{C}}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{sSet}$, so that for any objects $A, B \in \mathcal{C}$, the following three conditions hold:*

1. the simplicial set $\text{Hom}_{\mathcal{C}}(A, B)$ satisfies $\text{Hom}_{\mathcal{C}}(A, B)_0 = \mathcal{C}(A, B)$;
2. the functor $\text{Hom}_{\mathcal{C}}(A, -): \mathcal{C} \rightarrow \mathbf{sSet}$ has a left adjoint $A \otimes -: \mathbf{sSet} \rightarrow \mathcal{C}$, which is associative;
3. the functor $\text{Hom}_{\mathcal{C}}(-, B): \mathcal{C}^{\text{op}} \rightarrow \mathbf{sSet}$ has a left adjoint, denoted $\text{hom}_{\mathcal{C}}(-, B): \mathbf{sSet} \rightarrow \mathcal{C}^{\text{op}}$.

Remark 4.5.0.2. Associativity of the left adjoint \otimes in (2) means that there is an isomorphism $A \otimes (K \times L) \cong (A \otimes K) \otimes L$, which is natural in $A \in \mathcal{C}$, and $K, L \in \mathbf{sSet}$. For $A, B \in \mathcal{C}$ and $K \in \mathbf{sSet}$, the adjunction of (2) asks for a natural bijection of the form:

$$\mathcal{C}(A \otimes K, B) \cong \mathbf{sSet}(K, \text{Hom}_{\mathcal{C}}(A, B)),$$

and the adjunction of (3) asks for a natural bijection of the form:

$$\mathcal{C}^{\text{op}}(\text{hom}_{\mathcal{C}}(K, B), A) = \mathcal{C}(A, \text{hom}_{\mathcal{C}}(K, B)) \cong \mathbf{sSet}(K, \text{Hom}_{\mathcal{C}}(A, B)).$$

Remark 4.5.0.3. The use of tensor product notation (the \otimes symbol) should not be confused with the tensor product in algebra; this choice of notation goes back to Quillen's original monograph [Qui67].

Remark 4.5.0.4. We have called such a category a simplicial category, where we really mean that it is a category enriched over the category of simplicial sets, which is powered and copowered over the category of simplicial sets (the functor $\text{hom}_{\mathcal{C}}(-, B)$ is the powering and the functor $A \otimes -$ is the copowering over \mathbf{sSet}).

Example 4.5.0.5. The category of simplicial sets is a simplicial category where we can let the cartesian product be the copowering operation \otimes , and we let $\text{hom}_{\mathbf{sSet}} = \text{Hom}_{\mathbf{sSet}}$. Similarly, the category of pointed simplicial sets forms a simplicial category. Generalising further, the category of simplicial objects in any bicomplete category is naturally a simplicial category (details of this can be found in [GJ09, §II Theorem 2.5]).

To move towards the definition of a simplicial model category, consider a category \mathcal{C} which is both a simplicial category and a (closed) model category. We would like the model structure on \mathcal{C} to be enriched over the standard (Quillen) model structure on simplicial sets, which amounts to requiring the following compatibility condition.

Definition 4.5.0.6. Suppose a category \mathcal{C} is a simplicial category and a (closed) model category. We say that \mathcal{C} is a *simplicial model category* if for a cofibration $J: A \rightarrow B$ in simplicial sets and a fibration $F: X \rightarrow Y$ in \mathcal{C} , then the pullback power of J and F :

$$\langle J, F \rangle: \text{hom}_{\mathcal{C}}(B, X) \rightarrow \text{hom}_{\mathcal{C}}(A, X) \times_{\text{hom}_{\mathcal{C}}(A, Y)} \text{hom}_{\mathcal{C}}(B, Y)$$

is a fibration which is acyclic if either J or F are.

Remark 4.5.0.7. Notice that Definition 4.5.0.6 depends on the choice of model category for the category of simplicial sets, by which we mean the specific choice of fibrations and weak equivalences between simplicial sets. A simplicial model category in the sense of Definition 4.5.0.6 is also called a model category enriched over the Quillen model structure on simplicial sets.

In a simplicial model category, one can show that the notion of homotopy is controlled by the requirements on $\langle J, F \rangle$. For example, for a cofibrant $A \in \mathcal{C}$ it can be shown that $A \otimes \Delta^1$ is a cylinder object, and for a fibrant $X \in \mathcal{C}$, then $\text{hom}_{\mathcal{C}}(\Delta^1, X)$ is a path object for X .

One way of viewing this axiom is that it is a strengthening of the lifting axiom MC3 of a model category; that follows because any lifting diagram corresponds to a 0-simplex in the simplicial set $\text{Hom}_{\mathcal{C}}(A, X) \times_{\text{Hom}_{\mathcal{C}}(A, Y)} \text{Hom}_{\mathcal{C}}(B, Y)$, and a lift is a choice of 0-simplex in

$\text{Hom}_{\mathcal{C}}(B, X)$ which exists by the surjectivity of acyclic Kan fibrations. To elaborate on how we can view Definition 4.5.0.6 as a strengthening of the lifting axiom of a model category MC3, we have the following proposition.

Proposition 4.5.0.8. *For a simplicial model category \mathcal{C} and cofibrant $A \in \mathcal{C}$, consider a cofibration I and fibration P as in Figure 4.4, such that either I or P is acyclic.*

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & X \\
 \downarrow I & \nearrow H & \downarrow P \\
 B & \xrightarrow{\quad} & Y \\
 & \nwarrow H' &
 \end{array}$$

Figure 4.4: Lifting I on the left against P in a simplicial model category.

In this situation, any two lifts H and H' are homotopic below A and over Y .

Proof. See [GJ09, §II Proposition 3.8]. ■

Remark 4.5.0.9. In the case of Proposition 4.5.0.8 if A is not cofibrant, then it still follows that any two lifts are homotopic, but the homotopy cannot be required to live below A and over Y . A proof of this can be found in [Hir09, Proposition 9.6.1].

In a simplicial category, one is able to make sense of simplicially enriched limits. The concept of enriched limits is borrowed from enriched category theory, in the specific case that the category we are interested in is enriched over the category of simplicial sets.

Definition 4.5.0.10. In a simplicial category \mathcal{C} , a limit is a *simplicially enriched limit* if applying the covariant functor $\text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow \mathbf{sSet}$ to the limiting cone $(\lim_{i \in \mathbb{I}} A_i \rightarrow A_i)_{i \in \mathbb{I}}$ of the limit gives an isomorphism of simplicial sets for all $X \in \mathcal{C}$:

$$\text{Hom}_{\mathcal{C}}(X, \lim_{i \in \mathbb{I}} A_i) \rightarrow \lim_{i \in \mathbb{I}} \text{Hom}_{\mathcal{C}}(X, A_i).$$

The requirement that a limit is a simplicially enriched limit is a strengthening of the universal property of a limit object, and if a simplicially enriched limit exists then it arises from the usual limit of the underlying category.

4.6 The Joyal Model Category on Simplicial Sets

In this subsection, we introduce the Joyal model category on simplicial sets, also known as the model category on quasi-categories, because the cofibrant-fibrant objects in this model structure are precisely the quasi-categories. We need two important results from the theory of simplicial sets and of quasi-categories to describe the model category.

Proposition 4.6.0.1. *The category \mathbf{sSet} is cartesian closed.*

Proof. We make \mathbf{sSet} into a cartesian closed category by defining the internal hom functor as $\underline{\mathbf{sset}}(X, Y)_n = \mathbf{sSet}(X \times \Delta^n, Y)$, with face morphisms $d_i: \underline{\mathbf{sset}}(X, Y)_n \rightarrow \underline{\mathbf{sset}}(X, Y)_{n-1}$, defined for $F: X \times \Delta^n \rightarrow Y$ as pre-composition with $\mathbb{1} \times d^i$, giving $X \times \Delta^{n-1} \xrightarrow{\mathbb{1} \times d^i} X \times \Delta^n \xrightarrow{F} Y$. Degeneracy morphisms $s_i: \underline{\mathbf{sset}}(X, Y)_n \rightarrow \underline{\mathbf{sset}}(X, Y)_{n+1}$ are defined for $F: X \times \Delta^n \rightarrow Y$ as precomposition by $\mathbb{1} \times s^i$, so we have $X \times \Delta^{n+1} \xrightarrow{\mathbb{1} \times s^i} X \times \Delta^n \xrightarrow{F} Y$ [Rie11].

To complete the proof that this defines a cartesian closed category, we need to show that $\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, \underline{\mathbf{sset}}(Y, Z))$. The next step in this proof is taken from [Jar14, Lecture 4]. To do this, we introduce two pieces of notation; to start, the *classifying n -simplex* is the identity map $\mathbb{1}_{[n]}: [n] \rightarrow [n]$ in $\Delta(-, [n]) = \Delta^n$ the n -simplex. We define an *evaluation simplicial morphism*, $\text{ev}: \underline{\mathbf{sset}}(X, Y) \otimes X \rightarrow Y$ which sends $(F: X \times \Delta^n \rightarrow Y, x \in X_n)$, to the n -simplex in Y defined by $F_n(x, \mathbb{1}_{[n]})$.

Define a simplicial morphism $\mathbf{sSet}(X, \underline{\mathbf{sset}}(Y, Z)) \rightarrow \mathbf{sSet}(X \times Y, Z)$ as a morphism which carries $G: X \rightarrow \underline{\mathbf{sset}}(Y, Z)$ to the composite $X \times Y \xrightarrow{G \times \mathbb{1}} \underline{\mathbf{sset}}(Y, Z) \times Y \xrightarrow{\text{ev}} Z$, which is a bijection by construction. An inverse to G is defined by taking a morphism $H: X \times Y \rightarrow Z$ to the morphism $H_*: X \rightarrow \underline{\mathbf{sset}}(Y, Z)$, defined for a simplex $x \in X$ as $H_*(x): Y \times \Delta^n \xrightarrow{\mathbb{1} \otimes x} Y \otimes X \xrightarrow{H} Z$. ■

Remark 4.6.0.2. The classifying n -simplex is given its name thanks to the Yoneda Lemma, which says that there is a natural bijection $\mathbf{sSet}(\Delta^n, Y) \cong Y_n = Y([n])$, which sends the simplicial morphism $\sigma: \Delta^n \rightarrow Y$ to $\sigma(\mathbb{1}_{[n]}) \in Y_n$.

Remark 4.6.0.3. In more generality, the category of presheaves on a small category is cartesian closed. This fact is proven in [LM92, §I.6, Theorem 1].

Proposition 4.6.0.4. *For a quasi-category $X \in \mathbf{sSet}$ and any simplicial set A , the simplicial set internal hom functor $\underline{\mathbf{sset}}(A, X)$ is also a quasi-category.*

Proof. See [Rie11, p.9], or for a more in depth discussion [Rie14, Corollary 15.2.3]. ■

Using these as a starting point, we can discuss the necessary functors for the construction of weak equivalences in the Joyal model category. The nerve functor, $N: \mathbf{Cat} \rightarrow \mathbf{sSet}$ is defined as $N(\mathcal{C}) = \{NC_n\}_{n \in \mathbb{N}}$ where NC_n is the set of n -composable morphisms of \mathcal{C} . There is a left adjoint to the nerve functor, referred to as the *fundamental category functor* $\tau_1: \mathbf{sSet} \rightarrow \mathbf{Cat}$, defined by setting the objects of $\tau_1 X$ to be the 0-simplices of $X \in \mathbf{sSet}$, the elements of X_0 , and the morphisms are generated by the elements of X_1 (face maps define the source and target of a morphism), and where composition of arrows is freely generated by elements of X_1 , subject to the relations given by elements of X_2 . By this, we mean that if $x \in X_2$, we require that $d_1 x = d_0 x \circ d_2 x$. In particular, for an object $y \in \tau_1 X$, the identity map on y is given by $s_0 y$.

To elaborate on the construction, consider the class of objects X_0 and the free graph on this class of objects generated by elements of X_1 , and impose the condition that $H = G \circ F$ if there is $x \in X_2$ so that $H = d_1 x$, $F = d_2 x$ and $G = d_0 x$. Associativity of composition is taken care of by requiring the morphisms to be freely generated by the elements of X_1 . This functor is very

destructive, because it only relies on the low dimensional simplices of the simplicial set and ignores any higher dimensional simplices.

If we restrict ourselves to considering the fundamental category of a quasi-category, then the composite given by a 2-simplex can be interpreted by setting $d_0x = G$ and $d_2x = F$. The edge d_1x can be interpreted as a composite $G \circ F$ defined up to homotopy. In this situation, higher dimensional horns give homotopies between homotopies and so forth. For example 3-simplices give homotopies between the composites defined in a 2-simplex. The difference between the situation for a quasi-category as opposed to a simplicial set, is that such composites need not exist in a simplicial set, however they do for a quasi-category, by definition.

Remark 4.6.0.5. For any 1-category \mathcal{C} , it is interesting to note that $N(\mathcal{C})$ contains all the information of \mathcal{C} . In particular it is true that $\tau_1(N(\mathcal{C})) = \mathcal{C}$. This is true because the 1-composable morphisms of \mathcal{C} (comprising NC_1) includes all composite morphisms.

We define a functor $\tau_0: \mathbf{sSet} \rightarrow \mathbf{Set}$ by sending $A \in \mathbf{sSet}$ to the isomorphism classes of objects of $\tau_1(A)$. Using this, define a new category \mathbf{sSet}^{τ_0} as the cartesian closed category with the same objects as \mathbf{sSet} , and morphisms $A \rightarrow B$ are given by $\tau_0(B^A)$. The same construction works with any cartesian closed category in place of \mathbf{sSet} ; for more details see [Rie14, p.14].

Definition 4.6.0.6. We say that an ∞ -category $A \in \mathbf{sSet}$ is an EI ∞ -category if and only if $\tau_1 A$ is an EI-category, in the sense that any endomorphism is an isomorphism.

We now wish to describe Joyal's model category for simplicial sets. The weak equivalences are the weak categorical equivalences, which we define.

Definition 4.6.0.7. A map of simplicial sets is a *categorical equivalence* if its image in \mathbf{sSet}^{τ_0} is a bijection. A map $u: A \rightarrow B$ is a *weak categorical equivalence* if the contravariantly induced map of sets $\mathbf{sSet}^{\tau_0}(u, X): \mathbf{sSet}^{\tau_0}(B, X) \rightarrow \mathbf{sSet}^{\tau_0}(A, X)$ is a bijection for all quasi-categories X .

A weak categorical equivalence between quasi-categories is a categorical equivalence; this can be proven using the Yoneda Lemma applied to $\mathbf{QCat}^{\tau_0} \hookrightarrow \mathbf{sSet}^{\tau_0}$ (see [Rie08]). By definition, a categorical equivalence is necessarily a weak categorical equivalence. The following propositions provide us with examples of weak categorical equivalences, as well as a further understanding of their behaviour.

Proposition 4.6.0.8. *An acyclic Kan fibration of simplicial sets is a categorical equivalence, and therefore a weak categorical equivalence.*

Proof. See [Rie08, Proposition 3.5]. ■

Proposition 4.6.0.9. *The cartesian product of two weak categorical equivalences is a weak categorical equivalence.*

Proof. See [Joy, Proposition 2.28]. ■

Definition 4.6.0.10. A simplicial morphism is *inner anodyne* if it lies in the class of morphisms generated by the set of inner horn inclusions $I_m := \{\Lambda_k^n \hookrightarrow \Delta^n \mid n > 1 \text{ and } 0 < k < n\}$.

Proposition 4.6.0.11. *Every inner anodyne map is a weak categorical equivalence.*

Proof. See [Joy, Proposition 2.29]. ■

To further understand the weak categorical equivalences, we have the following result, sometimes informally known as the Fundamental Theorem of Quasi-Categories. To give this theorem, we need to introduce the simplicial mapping space within a simplicial set.

Definition 4.6.0.12. Consider a simplicial set A and vertices $a, b \in A_0$. Then the *simplicial mapping space from a to b in A* , denoted by $A(a, b)$, is obtained via the pullback in simplicial sets shown in Figure 4.5.

$$\begin{array}{ccc}
 A(a, b) & \overset{\text{-----}}{\longrightarrow} & \mathbf{sset}(\Delta^1, A) \\
 \downarrow & \lrcorner & \downarrow (E_0, E_1) \\
 * & \xrightarrow{(a, b)} & \mathbf{sset}(\{0, 1\}, X) \cong A \times A
 \end{array}$$

Figure 4.5: The simplicial mapping space from a to b in A .

The mapping space between vertices in a simplicial set allows us to state and understand the Fundamental Theorem of Quasi-Categories, which is stated in Theorem 4.6.0.13.

Theorem 4.6.0.13 (Fundamental Theorem of Quasi-Categories). *Let A and B be quasi-categories. Then a map $F: A \rightarrow B$ is a weak categorical equivalence if and only if:*

1. *the map f is fully faithful, by which we mean there is weak homotopy equivalence of Kan complexes $A(a, b) \rightarrow B(F(a), F(b))$ for all $a, b \in A$;*
2. *the map f is essentially surjective, meaning that $\tau_1(F): \tau_1(A) \rightarrow \tau_1(B)$ is an essentially surjective functor of categories.*

Proof. In [Joy, p.162] it is shown that a simplicial morphism between quasi-categories is a weak categorical equivalence if and only if it is a categorical equivalence. In [Joy, p.159], it is shown that a map of quasi categories is a categorical equivalence if and only if 1 and 2 above hold. ■

The following model category on the category of simplicial sets is due to Joyal. A proof that it defines a model structure can be found in [Joy, Theorem 6.12].

Theorem 4.6.0.14. *The Joyal model category on the category of simplicial sets, denoted by $\mathbf{sSet}_{\text{Joyal}}$ has monomorphisms as cofibrations (the same as in the Quillen model structure), weak equivalences are the weak categorical equivalences, and the fibrations in this model structure are determined by their lifting property, and are referred to as the quasi-fibrations. The Joyal model category is a cofibrantly generated.*

In the Joyal model category, every simplicial set is cofibrant, and quasi-categories are the fibrant objects. By definition, cofibrant objects are those for which the map $\emptyset \rightarrow X$ is a cofibration. This is clearly a monomorphism for any $X \in \mathbf{sSet}$, so any simplicial set is cofibrant. For the proof that the fibrant objects are the quasi-categories see [Rie08, Theorem 5.4].

In Joyal's construction of the model category for quasi-categories [Joy], it is proven that the model structure is cartesian; this means that the cartesian product functor is a left Quillen functor and that the terminal object is cofibrant. The Joyal model category is a Cisinski model structure, hence is cartesian because the product of two weak categorical equivalences is a weak categorical equivalence; see [Joy, p.161]. This implies that the model category is left proper, hence weak equivalences are preserved along pushout along cofibrations. It is not right proper; for further details see [Rie08, p.11].

Definition 4.6.0.15. Define the n -spine of the n -simplex $I^n \subseteq \Delta^n$ to be the simplicial set generated by the 1-simplices from 0 to 1, from 1 to 2, ..., from $n - 1$ to n .

As a particular example of a weak categorical equivalence, the inclusion $I^n \hookrightarrow \Delta^n$ is an acyclic cofibration, because it is inner anodyne (see [Joy, Proposition 2.13]).

Definition 4.6.0.16. Define the simplicial set \mathcal{J} to be the nerve of the groupoid consisting of two objects with a unique arrow in each Hom-set, also known as the free walking isomorphism. The groupoid can be pictured as $0 \rightleftarrows 1$.

Definition 4.6.0.17. The set of inner horn inclusions is $I_m := \{\Lambda_k^n \hookrightarrow \Delta^n \mid n > 1 \text{ and } 0 < k < n\}$. Define the *mid-fibrations* to be the class F_m of simplicial morphisms which have the right lifting property with respect to the class I_m . Equivalently we can write $F_m = (I_m)^\square$.

We refer to the fibrations in the Joyal model category as quasi-fibrations, and between quasi-categories, we can explicitly describe them. A map F of quasi-categories is a quasi-fibration if F is a mid-fibration, and F has the right lifting property with respect to the inclusion $\{*\} \hookrightarrow \mathcal{J}$.

In the Joyal model category, the simplicial set \mathcal{J} plays the role of the interval. We explain what this means in the following proposition.

Proposition 4.6.0.18. *By this we mean that for a simplicial set X , the simplicial set $X \times \mathcal{J}$ is a cylinder object for all X , and the simplicial set $\underline{\mathbf{sset}}(\mathcal{J}, X)$ is a path object for any quasi-category X .*

Proof. It is straightforward to show that $X \times \mathcal{J}$ is a cylinder object for any simplicial set X . To start, note that \mathcal{J} is a Kan complex, so in particular is a quasi-category. The map $\mathcal{J} \rightarrow *$ is an acyclic fibration because it has the right lifting property with respect to all monomorphisms. We can take the product of this acyclic fibration with $\mathbb{1}_X$, which also an acyclic fibration, to arrive at the simplicial morphism $\mathcal{J} \times X \rightarrow X$ which is an acyclic fibration because the class of acyclic fibrations is closed under taking products. It is evident that the simplicial morphism $X \amalg X \rightarrow X \times \mathcal{J}$ is a cofibration, therefore $X \times \mathcal{J}$ is a cylinder object for any simplicial set X .

It follows that $\mathbf{sset}(\mathcal{J}, X)$ is a path object for a quasi-category X , by noticing that the Joyal model category is enriched over itself as a model structure, because it is a closed monoidal model category (for further details see [Rie08, Theorem 5.4]). For an alternative proof that $\mathbf{sset}(\mathcal{J}, X)$ defines a path object when X is a quasi-category, see [Joy08b, Proposition 6.20]. ■

Remark 4.6.0.19. This model category is cofibrantly generated, where the cofibrations of $\mathbf{sSet}_{\text{Joyal}}$ are generated by the boundary simplex inclusions $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \in \mathbb{N}}$, but the acyclic cofibrations which are the monic categorical equivalences have no known explicit generating set. Recently a paper of Danny Stevenson has been published on arXiv, which claims to be able to describe the generating set, however we have not had time to study this paper. If correct, this would also allow arbitrary quasi-fibrations to be detected by their lifting properties against the set J . For details, see [Ste18]. However, as Joyal proved, there is a generating set of morphisms for the acyclic cofibrations and this alone is enough for our purposes.

Definition 4.6.0.20. [Sim11, Definition 8.5.1] Consider a locally κ -presentable category \mathcal{C} and a subclass \mathcal{A} of the arrows of \mathcal{C} . We say that the class of arrows \mathcal{A} is a κ -accessible class if any element $f: X \rightarrow Y$ of \mathcal{A} can be expressed as a κ -filtered colimit of morphisms $f_i: X_i \rightarrow Y_i$ where X_i and Y_i are compact objects in \mathcal{C} .

Remark 4.6.0.21. Whilst we have not defined what it means to be a locally κ -presentable category, for our purposes the category of simplicial sets satisfies the conditions to be locally κ -presentable. This follows because ([AR94, Example 1.10 (1)] shows that \mathbf{Set} is locally finitely presentable, allowing us to apply [AR94, Corollary 1.54]).

Lemma 4.6.0.22. *The class of acyclic cofibrations in Joyal's model structure on simplicial sets has a generating set of morphisms.*

Proof. In [Joy, Theorem 6.11] Joyal proves that the class of acyclic cofibrations is an accessible class, so we have a generating class $F_i: X_i \rightarrow Y_i$ for these morphisms, where X_i and Y_i are compact objects in \mathbf{sSet} . Using [Lur77, Lemma A.2.6.7], we are guaranteed that there exists a generating set for the acyclic cofibrations of $\mathbf{sSet}_{\text{Joyal}}$. ■

Definition 4.6.0.23. Consider a model category \mathcal{C} . A *left Bousfield localisation* of \mathcal{C} , denoted \mathcal{C}_{loc} , is another model category structure on the same underlying category \mathcal{C} , defined so that the cofibrations remain the same ($\mathcal{C}_{loc} = \mathcal{C}$), but with more weak equivalences (so $\mathcal{W}_{\mathcal{C}_{loc}} \supset \mathcal{W}_{\mathcal{C}}$).

Corollary 4.6.0.24. *The class of fibrations of \mathcal{C}_{loc} is contained in the class of fibrations in \mathcal{C} , but the class of acyclic fibrations remain the same. As a consequence, we get a Quillen adjunction between \mathcal{C} and \mathcal{C}_{loc} .*

Proof. The fibrations of \mathcal{C}_{loc} are determined by their lifting property against acyclic cofibrations, therefore it follows that the fibrations of \mathcal{C}_{loc} are contained in those of \mathcal{C} because:

$$\mathcal{F}_{\mathcal{C}_{loc}} = (\mathcal{C}_{loc} \cap \mathcal{W}_{\mathcal{C}_{loc}})^{\square} \subseteq (\mathcal{C}_{loc} \cap \mathcal{W}_{\mathcal{C}})^{\square} = (\mathcal{C} \cap \mathcal{W}_{\mathcal{C}})^{\square} = \mathcal{F}_{\mathcal{C}},$$

and in the case of acyclic fibrations we have:

$$\mathcal{F}_{\mathcal{C}_{loc}} \cap \mathcal{W}_{\mathcal{C}_{loc}} = (\mathcal{C}_{\mathcal{C}_{loc}})^{\square} = (\mathcal{C}_{\mathcal{C}})^{\square} = \mathcal{F}_{\mathcal{C}} \cap \mathcal{W}_{\mathcal{C}}.$$

We see that there is a Quillen adjunction $\mathbb{1}:\mathcal{C} \rightleftarrows \mathcal{C}_{loc}:\mathbb{1}$, where the left adjoint $\mathbb{1}:\mathcal{C} \rightarrow \mathcal{C}_{loc}$ preserves cofibrations and acyclic cofibrations, and the right adjoint $\mathbb{1}:\mathcal{C}_{loc} \rightarrow \mathcal{C}$ preserves fibrations and acyclic fibrations. ■

Remark 4.6.0.25. The Quillen model category for simplicial sets is a *Bousfield localisation* of the Joyal model category on simplicial sets. This result is proven in [Joy, Proposition 6.15]. From this observation, it follows that every weak categorical equivalence is a weak homotopy equivalence, and that every Kan fibration is a quasi-fibration. Furthermore, Ken Brown's Lemma (see Lemma 3.6.3.1) applied to the identity Quillen adjunction on simplicial sets implies that a weak homotopy equivalence between Kan complexes is a weak categorical equivalence. Explicitly, the right Quillen adjoint $\mathbb{1}:\mathbf{sSet}_{\text{Quillen}} \rightarrow \mathbf{sSet}_{\text{Joyal}}$ preserves acyclic fibrations, hence Ken Brown's Lemma states that it carries weak homotopy equivalences between Kan complexes to weak categorical equivalences (see Corollary 3.6.3.5).

Chapter 5

Classical Homotopy Theory of Stratified Spaces

5.1 Homotopically Stratified Sets

The goal of this section is to introduce homotopically stratified sets, and indicate their importance when studying the homotopy theory of stratified spaces. We will outline the key definitions and results in the study of the homotopy theory of this class of stratified spaces, which were first studied by Frank Quinn. Before the work of Frank Quinn, Hasler Whitney first studied stratified spaces by imposing geometric conditions on how strata are related. In Quinn's work, the goal was to study stratified spaces by imposing topological conditions on how the strata interact. Whitney stratified spaces provide examples of homotopically stratified sets, as do topologically stratified spaces (and hence so do Thom-Mather and Whitney stratified spaces). The study of homotopically stratified sets has provided a successful framework to answer topological questions of stratified spaces. We will be comparing our cofibrant-fibrant objects to the homotopically stratified sets introduced by Frank Quinn in [Qui88], and the comparison will be used to justify the choice of model structure that we transfer to the category of stratified spaces.

Definition 5.1.0.1. For a topological space X , a *filtration of X* by closed subsets, is a chain of inclusions $X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$, such that X^i is closed in X^{i+1} . Note that we may let n to be countably infinite, in which case we take colimit as the final stage of the filtration. A *filtered map $F: X \rightarrow Y$* between filtered spaces is a map of underlying topological spaces which respects filtration; explicitly this means that $F(X^i) \subseteq Y^i$ for all i . This defines a *category of filtered spaces*.

Example 5.1.0.2. A subspace $Y \subseteq X$ of a filtered space X has a natural filtration induced by the filtration on X .

Example 5.1.0.3. There is a natural filtration on any n -simplex; this is defined in barycentric coordinates by filtering the simplex by the last non-zero coordinate. For example, the 1-simplex is naturally filtered by the 0-vertex including into the whole simplex $\{0\} \subseteq |\Delta^1|$. Maps from the naturally filtered 1-simplex to a filtered space X are called *filtered paths*.

For some studies of stratified spaces, it is enough to work with filtered spaces; an example of this is the work in progress of Ryan Wissett and Jon Woolf (see [WW]), in which filtered spaces are shown to have a natural notion of filtered homology, which can be interpreted as a functorial approach to studying intersection homology. Importantly, the natural filtration on an n -simplex used in this work differs from that introduced in Example 5.1.0.3.

Definition 5.1.0.4. For a filtered space X , define the i -skeleton of X as $X_i = X^i \setminus X^{i-1}$. An i -stratum of X is a connected component of X_i .

Remark 5.1.0.5. Underlying a filtered space X is a partial order on X , which arises from the strata of X .

There are many different definitions of stratified space which have been classically studied; these arise as a filtered space, where we require the strata to be sufficiently nice (such as manifolds), and such that each inclusion $X^i \hookrightarrow X^{i+1}$ is suitably nice. An example of the inclusions being suitably nice is given by Whitney Stratified Spaces (see [Whi64]); intuitively the conditions on these filtered spaces ensure that the tangent space of a lower dimensional stratum is contained in the limiting tangent space of any sequence of points approaching the lower dimensional stratum. This notion of stratified space is accepted as the correct notion of a stratified space when working in the smooth setting. Stratified spaces are also often required to satisfy the frontier condition; this condition says that if $X_p \cap \overline{X_q} \neq \emptyset$ then $X_p \subseteq \overline{X_q}$.

When working with filtered or stratified spaces, regardless of the definition, the notion of homotopy equivalence that is used is that of stratum preserving homotopy equivalence.

Definition 5.1.0.6. We say that a map of filtered (or stratified) spaces $F: X \rightarrow Y$ is a *stratum preserving homotopy equivalence* if there is a homotopy inverse $G: Y \rightarrow X$ to F , such that each point remains in the same stratum throughout both homotopies.

Remark 5.1.0.7. Under a stratum preserving homotopy equivalence, the track (path traced out) of each point under the required homotopies remains in the same stratum.

Rather than digress on the differing definitions of stratified spaces, we will focus on Quinn's notion of homotopically stratified sets. These spaces are defined by homotopical requirements on how strata interact, making them suitable for studying homotopy theory. A more detailed overview of the different definitions of stratified spaces can be found in [HW00]. We introduce the following definitions, allowing us to move towards defining homotopically stratified sets.

Definition 5.1.0.8. Consider a filtered space X with filtered subspace $Y \subseteq X$. Then, we say that a deformation retract $\mathbb{1}_Y: Y \xrightarrow{i} X \xrightarrow{r} Y$ of the underlying topological space X onto Y , with

homotopy of underlying topological spaces $R: X \times |\Delta^1| \rightarrow X$ from $\mathbb{1}_X$ (at $1 \in |\Delta^1|$) to $r \circ i$ (at $0 \in |\Delta^1|$), is an *almost stratum preserving deformation retract* if for any $x \in X$ and $y \in Y$:

1. $R(x): |\Delta^1| \rightarrow X$ (is a filtered path);
2. $R(x, 0) = r(x) \in Y$;
3. $R(x, 1) = x$;
4. $R(y, t) = y$ for all $t \in |\Delta^1|$;
5. for $x \in X \setminus Y$ we have $R(x, t) \in X \setminus Y$ for all $t > 0$;
6. for $x \in X_j$, then $R(x, t) \in X_j$ for all $t > 0$.

Remark 5.1.0.9. Notice that if $x, x' \in X_q$ then $R(x)(0) = r(x)$ and $R(x')(0) = r(x')$ may lie in different strata of Y .

Intuitively, this definition says that an almost stratum preserving deformation retract is a deformation retract such that under the retract, the points of X remain in the same stratum throughout the retraction until the final moment, when they must move into Y .

Remark 5.1.0.10. In the literature, almost stratum preserving deformation retracts are also known as nearly stratum preserving deformation retracts.

Definition 5.1.0.11. Let X be a filtered space, with a filtered subspace $Y \subseteq X$. Then we say that Y is *stratified forward tame* in X , or that the inclusion of Y into X is *tame* if there is a neighbourhood U of Y in X , and an almost stratum preserving deformation retract of the neighbourhood U onto Y .

Notation. We say that a filtered space has *tame strata* if for any two strata X_p and X_q such that $X_p \subseteq \overline{X_q}$, then X_p is stratified forward tame in $X_p \cup X_q$.

We now introduce the notion of a homotopy link; the idea here is that they (topologically) behave analogously to regular neighbourhoods in the study of piecewise linear (PL) manifolds (this was Quinn's original motivation for their introduction). The idea behind the name is that in a homotopically stratified set, the link of a stratum will be well-defined up to homotopy. An alternative approach to the study of a stratified space is to consider Siebenmann's locally cone-like spaces [Sie72], however the link of a stratum in a locally cone-like space is not well-defined up to homotopy.

Definition 5.1.0.12. For a filtered space X with filtered subspace $Y \subseteq X$, define the *homotopy link* $holink(X, Y)$ to be the space of filtered paths γ in X , such that $\gamma(0) \in Y$ and $\gamma(0, 1] \in X \setminus Y$. For each path γ , the natural filtration on $|\Delta^1|$ implies that if we let the stratum of $\gamma(0)$ in X be denoted by X_p , then the stratum of $\gamma(t)$ for $t > 0$ must be X_q where q is any stratum such that $X_p \subseteq \text{PCI}(X_q)$, where PCI denotes the path closure of the stratum X_q .

We can now introduce homotopically stratified sets, as defined by Quinn. In his original definition, the spaces involved were required to be metric (meaning that the topology on the underlying topological space arises from a choice of metric) and with a finite filtration.

Definition 5.1.0.13. A filtered space X is said to be a *homotopically stratified set* if for any p and q such that $X_p \subseteq X_q$, the following two conditions hold:

1. the inclusion $X_p \hookrightarrow X_p \cup X_q$ is tame;
2. and the start point evaluation map $E_0: \text{Holink}(X_p \cup X_q, X_p) \rightarrow X_p$ is a Serre fibration.

Remark 5.1.0.14. There is the following chain of inclusions, where WSS denotes Whitney stratified spaces (see [Whi64]), $TMSS$ denotes Thom-Mather stratified spaces (see [Tho69], [Mat70] and [Mat71]), TSS denotes topologically stratified spaces (see Remark 6.1.3.4) and HSS denotes homotopically stratified sets:

$$WSS \subseteq TMSS \subseteq TSS \subseteq HSS.$$

The result that any Whitney stratified spaces is a Thom-Mather stratified spaces can be found in [Mat70]. The definition of a topologically stratified space is a generalisation of the definition of a Thom-Mather stratified space. Any topologically stratified space is a homotopically stratified set; this was shown by Quinn in [Qui88].

In Quinn's work on homotopically stratified sets, the filtered spaces used are assumed to be metric. This allows conditions 1 and 2 of a homotopically stratified set to be extended to pure subsets of the filtered space.

Definition 5.1.0.15. Consider a filtered space X and a subspace $K \subseteq X$. We say that K is a *pure subset* of X if K is closed within X and is a union of the strata of X .

Example 5.1.0.16. For a fixed q -stratum of a filtered space X , the union of strata $\bigcup_{p \leq q} X_p$ is closed in X , and hence is a pure subset of X .

Definition 5.1.0.17. A map $P: E \rightarrow B$ of homotopically stratified sets is a *stratified system of fibrations* with respect to a filtration of B , if P is a fibration over each stratum (so for each i , the map $P: P^{-1}(B_i) \rightarrow B_i$ is a fibration), and we can find an almost stratum preserving deformation retract of a neighbourhood of each B_i in B which is covered by an almost stratum preserving deformation retract of a neighbourhood in E of $P^{-1}(B_i)$.

Proposition 5.1.0.18. ([Qui88, Proposition 3.2]) *Let X be a homotopically stratified set with a pure subset $K \subseteq X$. Then there is an almost stratum-preserving deformation retract of a neighbourhood U in X onto K . Furthermore, the evaluation map $E_0: \text{Holink}(X, K) \rightarrow K$ is a stratified system of fibrations.*

Bruce Hughes defines stratified fibrations in [Hug99], in a different way to Frank Quinn. With this definition, Hughes shows that a stratified fibration between homotopically stratified metric spaces with a finite number of strata satisfy a similar path fibration condition to Quinn's.

Definition 5.1.0.19. A *stratified fibration* is a filtered map $F: X \rightarrow Y$ of filtered spaces such that for any space Z given the pre-image filtration under $G: Z \rightarrow X$, and continuous maps G and H as in Figure 5.1, a lift \tilde{F} can be found.

$$\begin{array}{ccc}
 Z \times \{0\} & \xrightarrow{G} & X \\
 \downarrow & \nearrow \tilde{F} & \downarrow F \\
 Z \times [0, 1] & \xrightarrow{H} & Y
 \end{array}$$

Figure 5.1: Hughes's definition of a stratified fibration.

Note that $[0, 1]$ denotes the trivially stratified interval (which is terminal amongst all stratifications of the interval), so that H and \tilde{F} are stratum preserving homotopies.

Definition 5.1.0.20. For a homotopically stratified set X with a filtered subspace $Y \subseteq X$, define the space of *almost stratum preserving paths in X starting in Y* , denoted $P_{\text{nsp}}(X, Y)$, to be the space of filtered paths in X with start point in Y .

Remark 5.1.0.21. Notice that $\text{Holink}(X, Y) \subseteq P_{\text{nsp}}(X, Y)$. This inclusion arises because the paths contained in $P_{\text{nsp}}(X, Y)$ start in Y akin to $\text{Holink}(X, Y)$, however paths may either not leave Y immediately, or that leave Y immediately and later return.

Theorem 5.1.0.22 (Theorem 6.1 of [Hug99]). *Suppose that X is a homotopically stratified metric space with a finite number of strata, and a filtered subspace $Y \subseteq X$ which is a closed union of strata. Then, the start point evaluation map:*

$$E_0: P_{\text{nsp}}(X, Y) \rightarrow Y$$

is a stratified fibration.

When studying homotopically stratified sets, we would like to introduce a lemma which gives an understanding of the space of stratified paths between two strata.

Definition 5.1.0.23. A *po-path* in a homotopically stratified set X is a path γ in the underlying topological space of X , such that for any $s \leq t$ in $|\Delta^1|$ where $\gamma(s) \in X_i$ and $\gamma(t) \in X_j$, then $i \leq j$ in the filtration (or the associated partial order) of X .

The following lemma shows us that po-paths between two strata, in a homotopically stratified set, are unique up to homotopy though po-paths in the holink.

Lemma 5.1.0.24. [Woo09, Lemma 3.2] *Consider a homotopically stratified set X , and a po-path γ in X which goes from a point $x_i \in X_i$ to $x_j \in X_j$. Then there is an end point preserving homotopy of γ to a path $\tilde{\gamma} \in \text{holink}(X_i \cup X_j, X_i)$. Moreover, $\tilde{\gamma}$ is unique up to homotopy through stratified paths in the homotopy link.*

David Miller extended this homotopy, which works on a path level, to give a homotopy equivalence at the level of path spaces.

Lemma 5.1.0.25 (Lemma 3.5 of [Mil09]). *For a homotopically stratified set with metric topology, the inclusion of the space of holink paths between two strata into the space of po-paths between the strata is a homotopy equivalence.*

We now wish to introduce David Miller’s theorem for detecting stratified homotopy equivalences between homotopically stratified sets. This provides some motivation as to why the understanding of holinks and po-paths is important in the study of stratified spaces. To do this, Miller uses a slightly stronger notion of morphism between homotopically stratified sets.

Remark 5.1.0.26. For a filtered space X , there is an associated partially ordered set which indexes the strata of X . This is induced by the ordering on successive differences $X_i = X^i \setminus X^{i-1}$, where we split elements apart corresponding to strata (the connected components of successive differences).

Definition 5.1.0.27. A map $F: X \rightarrow Y$ of homotopically stratified sets is a *strongly stratified map* if the induced map on partial orders is an isomorphism. This requirement means that X and Y have the same number of strata, that the pre-image of a stratum of Y is a stratum of X .

Restricting to strongly stratified maps, David Miller characterised stratified homotopy equivalences between homotopically stratified sets in terms of their induced map on homotopy groups of strata and holinks.

Theorem 5.1.0.28. ([Mil13, Theorem 6.3]) *A strongly stratified map $F: X \rightarrow Y$ of homotopically stratified sets is a stratified homotopy equivalence if and only if the induced map on strata and on holinks of pairs of strata is a weak homotopy equivalence.*

David Miller proves this result using an inductive argument proven using previous results in the paper, where the induction is carried out over the number of strata of X and Y (because the induced map on partial orders is an isomorphism).

In his initial paper on homotopically stratified sets [Qui88], Frank Quinn proves a range of (topologically) interesting properties for such spaces; such as analogues of a h -cobordism theorem, extensions of isotopies and obstructions to the existence of regular neighbourhoods and geometric stratifications. We do not discuss these results further, because they are not of direct relevance to this thesis.

5.2 Exit Path Category of a Stratified Space

In this section, we introduce the concept of an exit path category associated to a topologically stratified space. While the concept of an exit path category will not directly be used in our study of poset-stratified spaces, the ideas used in the study of exit path categories of topologically stratified spaces have influenced and shaped the direction of our work.

The motivation for such a category stems from an unpublished observation of Robert MacPherson, which relates constructible sheaves to presheaves on the exit path category. For

a stratified space, we can think of constructible sheaves as the sheaves which are locally constant with respect to each stratum. Bob MacPherson’s observation was that the category of constructible sheaves with respect to a fixed stratification of a topological space is equivalent to the functor category of **Set**-valued presheaves on the exit path category.

Definition 5.2.0.1. For a stratified space X , an *elementary exit path* is a stratified map from the naturally stratified 1-simplex $|\Delta^1|$ to X . A path in X is an *exit path* if whenever $s \leq t \in |\Delta^1|$, $\gamma(s) \in X_i$ and $\gamma(t) \in X_j$, we have that $i \leq j$.

Definition 5.2.0.2. For a topologically stratified space X , the *exit path category* of X , denoted $EP_{\leq 1}(X)$, is the category with objects the points of X with arrows from a to b defined as the stratum preserving homotopy classes of elementary exit paths in X from a to b .

Remark 5.2.0.3. This definition is analogous to the fundamental groupoid of a topological space, however we are using elementary exit paths instead of arbitrary paths and the notion of stratum preserving homotopy rather than homotopy.

Remark 5.2.0.4. We have only discussed 1-categories so far; these are a set of objects with 1-morphisms between them. There are associated notions of higher categories; the intuitive idea is that a 2-category is a 1-category which also has 2-morphisms between 1-morphisms, a 3-category is a 2-category which also has 3-morphisms between 2-morphisms, and so on. As expected, notions of functor can be extended to give a notion of a 2-functor, a notion of 3-functor, and so on for higher categories. As expected, this also allows one to make sense of an equivalence between 2-categories (with a notion of natural isomorphism between 2-functors), 3-categories and so on. This is a simplification of the situation, and in the study of 2-categories there are two common models used; either strict or weak 2-categories, and for higher categories there are a plethora of different models. We do not digress further on the topic of higher categories, because it is not of direct relevance to this thesis. For more information, and details on the technicalities that we have avoided here, see for example [CL19].

The observation of Robert MacPherson is captured by the following theorem which can be found in David Treumann’s paper [Tre09, Theorem 1.2].

Theorem 5.2.0.5. *For a topologically stratified space X , there is an equivalence between the category of constructible sheaves of sets on X and the functor category $\mathbf{Fun}(EP_{\leq 1}(X), \mathbf{Set})$.*

In Treumann’s paper, this result is generalised to give an equivalence between constructible stacks (for an explanation of constructible stacks on a stratified space see [Tre09, §2 or Appendix A]) and the appropriate 2-category of 2-functors out of the higher dimensional exit path category.

Definition 5.2.0.6. For a topologically stratified space X , the *2-exit path category*, denoted $EP_{\leq 2}(X)$ is the 2-category where objects are given by the points of X , morphisms are given by elementary exit paths between objects, and 2-morphisms are given by the stratum preserving homotopy classes of homotopies of exit paths (with an extra tameness condition imposed).

Theorem 5.2.0.7. *For a topologically stratified space X , there is an equivalence of 2-categories between the category of locally constant stacks on X and the 2-category of 2-functors $2\mathbf{Fun}(EP_{\leq 2}(X), \mathbf{Cat})$.*

This result has generalisations to higher dimensions. One such generalisation is by Jacob Lurie; instead of considering a finite truncation of the exit path category, Jacob introduces the ∞ -exit path category and proves that the ∞ -category of constructible objects of sheaves on a stratified space is equivalent to the ∞ -category of ∞ -functors out of the ∞ -exit path category to the ∞ -category of spaces. Details of this equivalence and its explicit construction can be found in [Lur14, §A.9]. A different generalisation is due to Clark Barwick, Saul Glasman and Peter Haine; one aspect of their paper [BGH18] is to introduce definitions of stratified ∞ -topoi and an associated notion of constructible sheaves. Using these definitions, they are able to prove a so called “exodromy equivalence for stratified ∞ -topoi” between the ∞ -category of functors valued in π -finite spaces and the appropriate category of constructible sheaves (further details can be found in [BGH18, Theorem 11.7]).

Part II

A Simplicial Approach to Stratified Homotopy Theory

Chapter 6

The Category of Stratified Spaces

6.1 The Category of Stratified Spaces

An understanding of partially ordered sets and the cartesian closed category of compactly generated topological spaces is important to the study of stratified spaces.

6.1.1 Posets and k -Spaces

Definition 6.1.1.1. A *poset* (short for a partially ordered set), denoted (P, \leq) , is a set P with a binary relation \leq defined on the elements of P which is anti-symmetric (if $p \leq q$ and $q \leq p$, then $p = q$), reflexive (for $p \in P$ we have $p \leq p$) and transitive (if $p \leq q$ and $q \leq r$, then $p \leq r$). A morphism of posets is a map of the underlying sets $f: P \rightarrow Q$ such that for $p \leq q$ in P , then $f(p) \leq f(q)$. Posets and maps of posets form a cartesian closed category denoted by **POSet**.

Definition 6.1.1.2. We can define a functor $\mathcal{I}: \mathbf{POSet} \rightarrow \mathbf{Top}$ by giving any poset the upwards closed topology, so a subset $U \subseteq P$ is open if when $p \in U$ then for every $p \leq q$ in P , we also have $q \in U$. This topology is also referred to as the Alexandrov topology arising from the partial order on P .

Remark 6.1.1.3. It is easy to see that a map between posets embedded into **Top** is continuous if and only if it arises as \mathcal{I} applied to a morphism of posets. Hence, the functor \mathcal{I} is fully faithful, a fact which can be exploited to show that the category of posets is isomorphic to the category of T_0 -Alexandrov spaces.

Notation. For a poset P and element $p \in P$, denote by U_p the upwards closure of p in P . Explicitly, U_p is the set $\{q \mid q \in P \text{ and } q \geq p\}$.

Proposition 6.1.1.4. A basis for the Alexandrov topology on a poset P is given by $\{U_p \mid p \in P\}$, where U_p is the upwards closure of p in P .

Proof. A general open set V of a poset P can be expressed as $V = \bigcup_{p \in V} U_p$. ■

Definition 6.1.1.5. A topological space X is *locally compact* if for each point $x \in X$ has an open neighbourhood U of x such that there is a compact neighbourhood $K \subseteq X$ with $U \subset K$.

Lemma 6.1.1.6. Any poset with the Alexandrov topology is locally compact.

Proof. A poset P is locally compact because the upwards closure of a poset element is compact. Explicitly, for any $p \in P$, the open neighbourhood U_p is compact. This is because any open cover $\bigcup_{i \in I} U_i = U_p$ contains an open set U_i such that $p \in U_i$, and hence $U_p \subseteq U_i$. ■

For the category of stratified spaces to be a convenient category to work with, we do not want to consider stratifications of arbitrary topological spaces. Instead we work with the category of k -spaces and emphasise the properties satisfied by this category. Further details on the category of k -spaces can be found in [Lew78, Appendix A] or alternatively [Rez17b], and an expository of different convenient categories of topological spaces can be found in [Str09].

Definition 6.1.1.7. For a topological space X , define a subset $A \subseteq X$ to be *compactly open* if any continuous morphism $F: K \rightarrow X$ from any compact Hausdorff space K , has open pre-image $F^{-1}(A)$ in K . A space X is said to be a k -space if every compactly open subset is open.

Remark 6.1.1.8. If a topological space X is (weak) Hausdorff, then X is a k -space if any $A \subseteq X$ is open if and only if $A \cap K$ is open in K for any compact $K \subseteq X$.

Denote the full subcategory of \mathbf{Top} spanned by the k -spaces as $k\mathbf{Top}$.

Definition 6.1.1.9. For a topological space X , define the k -ification of X , to be the topological space $k(X)$ which is a refinement of the topology on X by adding in the compactly open subsets of X as open subsets. The construction is functorial, and hence k -ification defines a functor $k(-): \mathbf{Top} \rightarrow k\mathbf{Top}$, which is a right adjoint to the inclusion $j: k\mathbf{Top} \rightarrow \mathbf{Top}$.

Note that for a topological space X , the identity map on underlying $k(X) \rightarrow X$ is continuous; in particular X is a k -space if and only if $k(X) \rightarrow X$ is a homeomorphism. Colimits in $k\mathbf{Top}$ coincide with colimits in \mathbf{Top} and limits in $k\mathbf{Top}$ are obtained by k -ifying to the limit in \mathbf{Top} . In addition, a map $X \rightarrow Y$ from a k -space X to a topological space Y is continuous if and only if $X \rightarrow k(Y)$ is.

Example 6.1.1.10. Any first countable or locally compact topological space is a k -space (for a proof see [Rez17a, Proposition 7.1]). In particular, Lemma 6.1.1.6 shows that any poset is a k -space.

Definition 6.1.1.11. Let X and Y be k -spaces, Between two k -spaces X and Y , define the k -compact-open topology as the topology on the set of maps $\mathbf{Top}(X, Y)$ generated by the sub-basis elements:

$$N(t, U) = \{f: X \rightarrow Y \mid f(t(K)) \subseteq U\},$$

for an open subset $U \subseteq Y$ and any continuous $t: K \rightarrow X$ from a compact-Hausdorff space K . Denote $\mathbf{Top}(X, Y)$ equipped with the k -compact-open topology by $\underline{\mathbf{top}}_0(X, Y)$.

If X has compact subspaces which are not Hausdorff, the k -compact-open topology will be coarser than the compact-open topology, however if X is Hausdorff then the two notions coincide. Note however that $\underline{\mathbf{top}}_0(X, Y)$ may not be a k -space.

Proposition 6.1.1.12. *The category $k\mathbf{Top}$ is cartesian closed, with internal hom defined by k -ification of the k -compact-open topology $\underline{k\mathbf{top}}(X, Y) = k(\underline{\mathbf{top}}_0(X, Y))$. Moreover, the natural bijection of sets defines a homeomorphism of mapping spaces $\underline{k\mathbf{top}}(X \times Y, Z) \cong \underline{k\mathbf{top}}(X, \underline{k\mathbf{top}}(Y, Z))$.*

Proof. See for example [Rez17b, Proposition 4.7]. ■

6.1.2 Category of Stratified Spaces

Definition 6.1.2.1. Define a *stratified space* to be a k -space X with a continuous, surjective map $S_X: X \rightarrow \mathcal{I}(P)$ to a poset P . The map S_X is called the *stratification* of X . For $p \in P$, define the p -*stratum* of X to be $X_p := S_X^{-1}(p)$.

Notation. When there is no danger of confusion, we denote $\mathcal{I}(P)$ by P , and denote the stratified space $S_X: X \rightarrow P$ simply by $S_X: X \rightarrow P$ or X . We may also refer to the topological space X as the *underlying space* of the stratified space $S_X: X \rightarrow P$.

Remark 6.1.2.2. Note that any stratified space can be interpreted as a filtered space, in the sense of Definition 5.1.0.1. The Szpilrajn extension theorem [Szp30] gives a way of taking a partial order to a total order, and choice of a poset element allows us to pick out an initial stratum of the stratified space to act as the initial 0-stratum in the filtration.

Definition 6.1.2.3. The objects of the category \mathbf{Strat} are stratified spaces and *stratified morphisms* are commutative squares in \mathbf{Top} (equivalently $k\mathbf{Top}$) between stratified spaces. We illustrate a stratified morphism between the stratified spaces $S_X: X \rightarrow P$ and $S_Y: Y \rightarrow Q$ in Figure 6.1. In this situation, we say that the continuous map F lives above the poset map f .

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow S_X & & \downarrow S_Y \\ P & \xrightarrow{f} & Q \end{array}$$

Figure 6.1: A stratified morphism from X to Y .

Recalling that the functor $k(-)$ is the k -ification of a topological space and \mathcal{I} is the inclusion of \mathbf{POSet} into $k\mathbf{Top}$. The category \mathbf{Strat} can be described as a restriction of the comma category $k(-) \downarrow \mathcal{I}$ to those arrows which are surjective, because $k\mathbf{Top}$ is a full subcategory of \mathbf{Top} , and the functor \mathcal{I} is fully faithful. In particular, it follows that \mathbf{Strat} is locally small.

Remark 6.1.2.4. The requirement that our stratifications are surjective provides an adjunction with the category of all (not necessarily surjective) stratified spaces. Denote the category of

all stratified spaces by $\mathbf{Strat}_{\text{All}}$. The inclusion $\mathbf{Strat} \hookrightarrow \mathbf{Strat}_{\text{All}}$ has a right adjoint, defined by throwing away poset elements for which the corresponding stratum is empty. Alternatively, the right adjoint sends a stratified space $S_X: X \rightarrow P$ to the stratified space $S_X: X \rightarrow S_X(X)$, where partial order on $S_X(X)$ is given by restriction of the partial order on P .

In particular, requiring stratification maps to be surjective will imply that if a continuous map F between the underlying spaces of stratified spaces X and Y defines a stratified morphism, then the poset map will be uniquely determined. This will be important when defining and studying stratified morphism spaces. The converse of this is not true; a choice of poset map may live below many continuous maps of the underlying spaces. When the stratification map is not surjective, a continuous map $F: X \rightarrow Y$ which determines a stratified morphism is only enough to uniquely determine a map between $S_X(X)$ and $S_Y(Y)$.

Notation. We will generally denote a stratified morphism by $F: X \rightarrow Y$ between stratified spaces $X \rightarrow P$ and $Y \rightarrow Q$, where the stratifications of X and Y are understood. We denote the underlying poset map of F by $f: P \rightarrow Q$.

Remark 6.1.2.5. The initial object in \mathbf{Strat} is $\emptyset \rightarrow \emptyset$ where \emptyset is the empty set and as a poset is given the empty partial order. The terminal object is $* \rightarrow *$.

Example 6.1.2.6. Any topological space X can be *trivially stratified* using the terminal object of \mathbf{POSet} (a one point set with no partial ordering), embedded into \mathbf{Top} (the one point space). This gives a unique continuous map $S: X \rightarrow *$. Any continuous map $Y \rightarrow X$ for a trivially stratified space X is a stratified morphism $Y \rightarrow X$, because there is a unique map from Y to $*$.

Remark 6.1.2.7. We did not require that the strata of our stratified spaces be path connected; therefore each stratum may have many path-connected components. Let \mathbf{Strat}^C denote the category of stratified spaces with path-connected strata. The canonical inclusion $\mathbf{Strat}^C \hookrightarrow \mathbf{Strat}$ has a right adjoint $C: \mathbf{Strat} \rightarrow \mathbf{Strat}^C$ given by splitting apart poset elements corresponding to the path-connected components of strata. We do not use this adjunction in this thesis, so do not make this precise.

Example 6.1.2.8. Any subspace of a stratified space inherits a stratification. For example, consider a stratified space $S_X: X \rightarrow P_X$, and a subspace $Y \subseteq X$ with inclusion map $I: Y \hookrightarrow X$. Let $P_Y = S_X(I(Y))$ with partial order restricted from P_X , and let $S_Y = S_X \circ I: Y \rightarrow P_Y$ be the unique map making Figure 6.2 commute.

$$\begin{array}{ccc}
 Y & \xleftarrow{I} & X \\
 \downarrow S_Y & & \downarrow S_X \\
 P_Y & \xleftarrow{\quad} & P_X
 \end{array}$$

Figure 6.2: The subspace $Y \subseteq X$ inherits a stratification from X .

More generally, for a stratified space $S_X: X \rightarrow P_X$ and a continuous map $F: Y \rightarrow X$, the space Y inherits a stratification from X . Let $P_Y = S_X(F(Y))$ with partial order restricted from P_X , and let $S_Y = S_X \circ F: Y \rightarrow P_Y$ be the unique map making Figure 6.3 commute.

$$\begin{array}{ccc} Y & \xrightarrow{F} & X \\ \downarrow S_Y & & \downarrow S_X \\ P_Y & \xrightarrow{\quad} & P_X \end{array}$$

Figure 6.3: Stratifying Y by its image in X .

6.1.3 Topology of Stratified Spaces

Proposition 6.1.3.1. *For a stratified space $X \rightarrow P$ and a poset element $p \in P$, the closure of the stratum X_p is contained in the union of strata $X_{\leq p}$.*

Proof. The closure of a stratum is the smallest closed subset of X which contains X_p . Consider the downwards closure of p in P , and its pre-image along S_X , which is $X_{\leq p}$. This is closed in X and contains X_p , hence the closure of X_p must be contained in $X_{\leq p}$. ■

Lemma 6.1.3.2. *Consider a stratified morphism as pictured in Figure 6.1, and suppose that the underlying map of topological spaces F and the induced map on posets f are both homeomorphisms. Then we have an isomorphism of stratified spaces. Conversely, an isomorphism of stratified spaces is a homeomorphism between underlying topological spaces, such that the underlying map of posets is an isomorphism.*

Remark 6.1.3.3. The poset map f being a homeomorphism is equivalent to f being an isomorphism of posets, because fully faithful functors reflect isomorphisms.

Proof. Because F and f are homeomorphisms, we have inverses F^{-1} and f^{-1} . The proof of the lemma is clear, once we have shown that the following square, in Figure 6.4 commutes.

$$\begin{array}{ccc} Y & \xrightarrow{F^{-1}} & X \\ \downarrow S' & & \downarrow S \\ Q & \xrightarrow{f^{-1}} & P \end{array}$$

Figure 6.4

We know that $f \circ S_Y = S_X \circ F$ by the stratified morphism we are given. Pre-composing with f^{-1} and post-compose with F^{-1} gives $S_X \circ F^{-1} = f^{-1} \circ S_Y$, which is exactly the condition that Figure 6.4 commutes.

To complete the proof that F and f define an isomorphism in the category **Strat**, note that both pre-composing or post-composing Figure 6.1 with the commutative square in Figure 6.4 gives the identity map on $Y \rightarrow Q$ (or $X \rightarrow P$). The converse statement is clear. ■

Remark 6.1.3.4. It is now appropriate to see how this definition of a stratified space fits in with a more classical definition of a stratified space, as given by René Thom in [Tho69]. This definition says that an n -dimensional topological stratification of the topological space X is given by a filtration of the form $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = X$, such that each X_i is closed for $-1 \leq i \leq n$, and for each point $x \in X_m \setminus X_{m-1}$ for $0 \leq m \leq n$, there is a neighbourhood $U_x \subseteq X$ and a compact $n - m - 1$ -dimensional stratified space Z and a filtration-preserving homeomorphism, such that $U_x \cong \mathbb{R}^m \times C(Z)$. Here, $C(Z) = Z \times [0, 1) / Z \times 0$ denotes the open cone on Z with the product stratification on $Z \times (0, 1)$ and $Z \times 0$ as a one point, deepest stratum. The filtration of X by $n + 1$ closed subsets determines a continuous map $X \rightarrow \mathbb{N}_{\leq n}$ made into a poset with it's natural order, with the i^{th} stratum $X^i = X_i \setminus X_{i-1}$ sent to $i \in \mathbb{N}$.

Notation. When we draw stratified spaces, we will draw the strata as being disconnected to make the stratification clear to the reader.

Example 6.1.3.5. The fundamental stratified space for this thesis is the stratification of the standard geometric n -simplex $|\Delta^n|$ over the poset $[n]$. For a point in the standard geometric n -simplex expressed in barycentric coordinates, define the stratification $S_{|\Delta^n|}: |\Delta^n| \rightarrow [n]$ by:

$$S_{|\Delta^n|}(x_0, \dots, x_n) = \max_{x_i \neq 0} i.$$

We will call this the *natural stratification of the geometric n -simplex* and denote it $\|\Delta^n\|$. For $n = 2$ the natural stratification of $|\Delta^2|$ is shown in Figure 6.5.

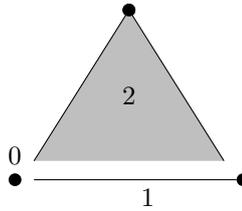


Figure 6.5: The natural stratification of $\|\Delta^2\|$, where numbers indicate the image in $[2]$.

Lemma 6.1.3.6. *Let $X \rightarrow P$ denote a stratified space. If there are $p, q \in P$ such that $X_p \cap \overline{X_q} \neq \emptyset$, then $p \leq q$ in P .*

Proof. Assume that $X_p \cap \overline{X_q} \neq \emptyset$ for some $p, q \in P$. We recall that the closure of any subset satisfies the property that for any $x \in \overline{X_q}$, then any neighbourhood U_x of x in X satisfies $U_x \cap X_q \neq \emptyset$. Let $x \in X_p \cap \overline{X_q}$ denote a point. In particular, the open neighbourhood $U_x = \bigcup_{p \leq r} X_r$ of x contains the point x , hence intersects X_q . Therefore we must have $p \leq q$. ■

If we considered **Strat** within the functor category $[\mathbb{2}, k\mathbf{Top}]$, then we would like to take $\lim (\mathcal{I}(P_i))$ as limiting poset. Limits in **Strat** can be constructed, but we initially show that they cannot be constructed in the naïve way, by considering **Strat** as the comma category restricted to surjective stratification maps.

Proposition 6.1.3.7. *The functor $\mathcal{I}: \mathbf{POSet} \hookrightarrow \mathbf{Top}$ does not preserve infinite limits.*

Proof. Consider an \mathbb{N} -indexed collection of posets $\{P_i\}_{i \in \mathbb{N}}$ where $P_i \cong [1]$ for all $i \in \mathbb{N}$. The product of these posets taken in **POSet** is the set $\prod_{i \in \mathbb{N}} P_i$, equipped with product partial order defined by $(p_i)_{i \in \mathbb{N}} \leq (p'_i)_{i \in \mathbb{N}}$ if and only if $p_i \leq p'_i$ for all $i \in \mathbb{N}$. When embedded into $k\mathbf{Top}$, the product poset is given the upwards-closed topology, which coincides with the box topology. Explicitly, the box topology is the topology with basis given by $\{\prod_{i \in \mathbb{N}} U_i \mid U_i \subseteq \mathcal{I}(P_i) \text{ is open}\}$. On the other hand, the product in $k\mathbf{Top}$, which is $\prod_{i \in \mathbb{N}} \mathcal{I}(P_i)$ has the product topology. In our example, the element $(1, 1, \dots)$ is open in the box topology but is not open in the product topology. ■

Remark 6.1.3.8. When we have a finite product, the box topology and product topology coincide so there is no issue. In general however, the box topology is finer than the product topology and so there will be a unique induced continuous map:

$$\mathcal{I}\left(\prod_{i \in \mathbb{N}} P_i\right) \rightarrow \prod_{i \in \mathbb{N}} (\mathcal{I}(P_i)).$$

For a general limit, there is no guarantee that $\lim_{i \in I} (\mathcal{I}(P_i))$ will correspond to a poset given the poset topology. This fact will be important in the next section §6.1.4, when we define limits in **Strat**.

6.1.4 Limits and Colimits in Strat

Proposition 6.1.4.1. *The category **Strat** is closed under small limits and colimits.*

Proof. Consider a functor $\mathcal{F}: I \rightarrow \mathbf{Strat}$ from a small category I to the category of stratified spaces. In **Strat**, the functor \mathcal{F} picks out stratified spaces $\mathcal{F}(i) = S_{X_i}: X_i \rightarrow \mathcal{I}(P_i)$ for all $i \in I$, with stratified morphisms corresponding to the morphisms of I .

The colimit of \mathcal{F} is a stratification of the colimit of the spaces X_i taken in $k\mathbf{Top}$:

$$S: \operatorname{colim}_{i \in I} X_i \rightarrow \mathcal{I}(\operatorname{colim}_{i \in I} P_i),$$

which is stratified over the poset constructed as the colimit constructed in **POSet** embedded into $k\mathbf{Top}$. The stratification map S is defined as the unique morphism in $k\mathbf{Top}$ out of $\operatorname{colim}_{i \in I} X$ to $\mathcal{I}(\operatorname{colim}_I P_i)$. This exists because colimits are initial cocones, and we can define a cocone at $\mathcal{I}(\operatorname{colim}_{i \in I} P_i)$ over the spaces X_i using each stratification morphism S_i and the maps to the colimit of the form $\mathcal{I}(\pi_i): \mathcal{I}(P_i) \rightarrow \mathcal{I}(\operatorname{colim}_{i \in I} P_i)$.

For this to define a stratified space, we need the map S to be surjective; to show this, consider a poset element $p \in \operatorname{colim}_{i \in I} P_i$. By construction of the colimit of posets, there must exist some $i \in I$ and an element $p' \in P_i$ such that $\mathcal{I}(\pi_i)(p') = p$, and by assumption the stratification of the space X_i over the poset $\mathcal{I}(P_i)$ is surjective, so there is a point $x_i \in X_i$ such that $S_{X_i}(x_i) = p'$. Finally, the construction of the map S implies that $S(x_i) = p$.

To show that this stratified space exhibits the universal property of a colimit, consider a stratified space $Y \rightarrow \mathcal{I}(Q)$ equipped with a cocone over F . We have induced unique maps in $k\mathbf{Top}$ of the form $\operatorname{colim}_{i \in I} X_i \rightarrow Y$ over the k -spaces X_i , and a unique map of posets $\mathcal{I}(\operatorname{colim}_{i \in I} P_i) \rightarrow \mathcal{I}(Q)$ over the posets $\mathcal{I}(P_i)$. This is depicted in Figure 6.6, with the curved arrows representing the cocone at $Y \rightarrow \mathcal{I}(Q)$ over F .

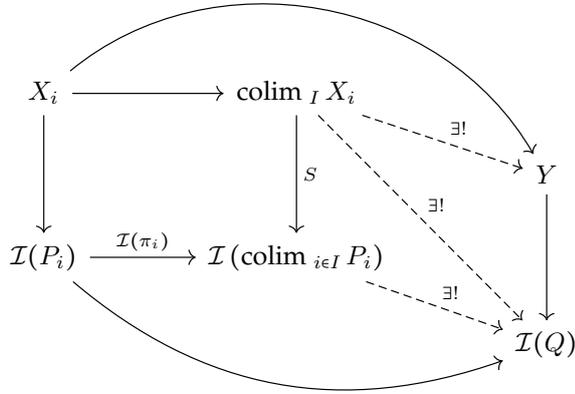


Figure 6.6: The universal property of a colimit in **Strat**.

We need to show that there is a unique map of stratified spaces out of the colimit. This follows because there is a unique continuous map $\operatorname{colim}_{i \in I} X_i \rightarrow \mathcal{I}(Q)$ which is equal to the two composites depicted in Figure 6.6.

To prove that **Strat** has all limits, we initially describe the poset that a limiting stratified space will live over. As in the colimit case, consider a functor $\mathcal{F}: I \rightarrow \mathbf{Strat}$ for which we want to construct the limit; the limit poset of \mathcal{F} will be the limit of the posets P_i constructed in **POSet** and then embedded into **Top**, denoted by $\mathcal{I}(\lim_{i \in I} P_i)$. From the construction, there are projection maps $\mathcal{I}(\operatorname{pr}_i): \mathcal{I}(\lim_{i \in I} P_i) \rightarrow \mathcal{I}(P_i)$ out of the limit in **POSet**. To define the limiting stratified space, take the limit over the entire diagram consisting of the spaces $\mathcal{F}(i)$ and the limiting poset, in $k\mathbf{Top}$. Denote the limit taken by PB and the map to $\mathcal{I}(\lim_{i \in I} P_i)$ by S . The diagram is shown on the right hand commutative square shown in Figure 6.7, where for simplicity we only show one specific $i \in I$.

By applying the forgetful functor from **Top** to **Set** (which preserves limits), it follows that PB has the same underlying set as $\lim_{i \in I} X_i$. The same idea shows that $S: PB \rightarrow \mathcal{I}(\lim_{i \in I} P_i)$ is a surjective map, hence defines a stratification of PB . As a topological space, PB is equipped

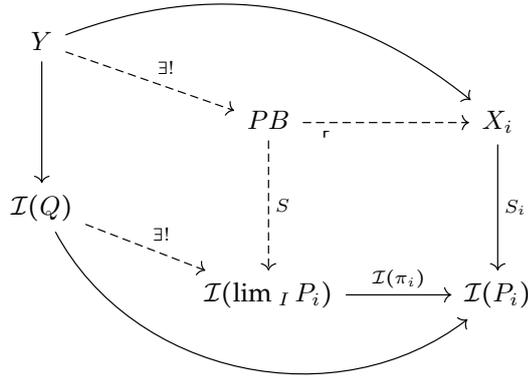


Figure 6.7: Constructing limits in **Strat**.

with the appropriate topology to make the entire diagram over $\mathcal{F}(I)$ and $\mathcal{I}(\lim_I P_i)$ commutative (i.e. the right hand square of Figure 6.7 commutes). In general this will be a refinement of the topology on the limit of X_i calculated in $k\mathbf{Top}$. However, for example if I is a finite discrete category, then the topology on PB coincides with the product topology.

We need to show that the stratified space $PB \rightarrow \mathcal{I}(\lim_{i \in I} P_i)$ is actually the limit in **Strat**. To do this, consider a stratified space $Y \rightarrow \mathcal{I}(Q)$ with compatible maps to $X_i \rightarrow \mathcal{I}(P_i)$ (indicated by the curved arrows in Figure 6.7). The poset Q has a unique map in **POSet** to $\lim_{i \in I} P_i$, which gives the lower unique dashed arrow when embedded into $k\mathbf{Top}$. Composition with this arrow defines a cone at Y in $k\mathbf{Top}$, over the diagram of which PB is defined to be the limit. Therefore we have a unique continuous map $Y \rightarrow PB$ making the required squares commute, showing there is a unique stratified morphism from $Y \rightarrow \mathcal{I}(Q)$ to $PB \rightarrow \mathcal{I}(\lim_{i \in I} P_i)$. This proves that $PB \rightarrow \mathcal{I}(\lim_{i \in I} P_i)$ is the limit of \mathcal{F} in **Strat**. ■

Remark 6.1.4.2. For a simple example where the topology on PB and $\lim_I X_i$ differ, consider an infinite product in **Strat**. The topology on PB will be the box topology with respect to open sets arising from strata, rather than the product topology.

6.2 Stratified Morphism Spaces

In this section we will construct stratified spaces from the space of stratified morphisms between two stratified spaces. It is important to note that we have restricted ourselves to stratifications of k -spaces; this restriction will give a cartesian closed category of stratified spaces. It should be noted however that this would work with any other choice of cartesian category of stratified spaces (for example numerically generated spaces). Initially, we discuss a natural way to turn a non-continuous stratification map into a continuous one.

Definition 6.2.0.1. Consider a stratification map $S_X: X \rightarrow P$ which is not continuous. By refining the topology on X by adding in the open sets $S_X^{-1}(U_p)$ for every $p \in P$, we arrive at the stratified space $R(X)$ which has a canonical stratification $S_{R(X)}: R(X) \rightarrow X \xrightarrow{S_X} P$.

Lemma 6.2.0.2. Suppose we have a commutative diagram in **Top**, as depicted in the lower square of Figure 6.8, where the maps F and f are continuous however the stratifications S_X and S_Y are not necessarily continuous (but are surjective). Then, the induced map $R(F): R(X) \rightarrow R(Y)$ is automatically continuous.

$$\begin{array}{ccc}
 R(X) & \xrightarrow{R(F)} & R(Y) \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{F} & Y \\
 \downarrow S_X & & \downarrow S_Y \\
 P & \xrightarrow{f} & Q
 \end{array}$$

$S_{R(X)}$ (left dashed arrow) and $S_{R(Y)}$ (right dashed arrow)

Figure 6.8: Refining the topology on X and Y , to achieve continuous stratification maps.

Proof. The topology on $R(Y)$ has a basis consisting of the open subsets of Y and the pre-images $S_Y^{-1}(U_q)$ for every $q \in Q$. Since $F^{-1}(S_Y^{-1}(U_q)) = S_X^{-1}(f^{-1}(U_q))$, and $f^{-1}(U_q)$ is open in P , we see that $R(F)$ is continuous. ■

To define the stratified space of stratified morphisms between any stratified spaces $X \rightarrow P$ and $Y \rightarrow Q$, it is vital that the stratification maps are surjective.

Definition 6.2.0.3. Let $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X, Y)$ denote the restriction of $\underline{k\mathbf{top}}(X, Y)$ to continuous maps from X to Y and which fit into some stratified morphism, equipped with the k -ification of the k -compact-open topology. Similarly, let $\underline{\mathbf{poset}}_{\mathbf{Strat}}(P, Q)$ denote the poset morphisms from P to Q which fit into some stratified morphism, with partial order defined by setting $f \leq g$ if and only if for all $p \in P$ we have $f(p) \leq g(p)$ in Q . This partial order describes the internal hom in **POSet**, making **POSet** into a cartesian closed category. We would like to define $\underline{\mathbf{strat}}(X, Y)$ as $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X, Y) \rightarrow \underline{\mathbf{poset}}_{\mathbf{Strat}}(P, Q)$ however this stratification map may not be continuous. As in Definition 6.2.0.1, we refine the topology on the underlying topological space. Therefore, we define the stratified mapping space $\underline{\mathbf{strat}}(X, Y)$ as the canonical map:

$$S: R\left(\underline{k\mathbf{top}}_{\mathbf{Strat}}(X, Y)\right) \rightarrow \underline{\mathbf{poset}}_{\mathbf{Strat}}(P, Q),$$

where surjectivity of the stratifications of X and Y ensure that the map S is well-defined.

Example 6.2.0.4. As a stratified space, the mapping space $\underline{\mathbf{strat}}(\|\Delta^0\|, X)$ is isomorphic to X .

Example 6.2.0.5. Consider the stratified space $\mathbb{1}: \mathbb{R} \rightarrow \mathbb{R}$, where the topological space \mathbb{R} is given the Euclidean topology and the partial order on \mathbb{R} is given by the standard ordering of the real numbers. Notice that the stratification map in this case is not continuous; to fix this we refine the topology on \mathbb{R} by adding the required open sets, giving a stratified space that we will denote by $\tilde{\mathbb{R}}$. The topology on $\tilde{\mathbb{R}}$ is generated by the subsets $[a, b)$ for $a \leq b$ in \mathbb{R} .

If we now consider the stratified mapping space $\mathbf{strat}(\tilde{\mathbb{R}}, \mathbb{R} \rightarrow \{*\})$ we see that this trivially stratified space consists of all right continuous functions on \mathbb{R} . If we reversed the partial order on \mathbb{R} , this would instead give all left continuous functions on \mathbb{R} . If we instead consider the stratified mapping space $\mathbf{strat}(\tilde{\mathbb{R}}, \tilde{\mathbb{R}})$, then we arrive at the set of increasing right continuous functions on \mathbb{R} .

Remark 6.2.0.6. If P and Q are finite, then the topology on $\mathbf{poset}(P, Q)$ embedded into \mathbf{Top} coincides with the k -ification of the k -compact-open topology on the space of continuous maps from P to Q .

Allowing arbitrary posets P and arbitrary topological spaces X implies that the topology on $R(k\mathbf{top}_{\mathbf{Strat}}(X, Y))$ may need to be much larger than the k -ification of the k -compact-open topology to ensure that the stratification map S is continuous.

Proposition 6.2.0.7. *The category \mathbf{Strat} is cartesian closed. Moreover, for any $X, Y, Z \in \mathbf{Strat}$, the bijection of sets $\mathbf{strat}(X \times Y, Z) \cong \mathbf{strat}(X, \mathbf{strat}(Y, Z))$ is an isomorphism of stratified spaces.*

Proof. Let our stratified spaces be $X \rightarrow P, Y \rightarrow Q$ and $Z \rightarrow R$. The key step in our proof is showing that the natural homeomorphism of Proposition 6.1.1.12:

$$k\mathbf{top}(X \times Y, Z) \xrightarrow{\cong} k\mathbf{top}(X, k\mathbf{top}(Y, Z))$$

restricts to a natural homeomorphism:

$$k\mathbf{top}_{\mathbf{Strat}}(X \times Y, Z) \xrightarrow{\cong} k\mathbf{top}_{\mathbf{Strat}}(X, k\mathbf{top}_{\mathbf{Strat}}(Y, Z)), \quad (6.1)$$

where $k\mathbf{Top}_{\mathbf{Strat}}(-, -)$ is defined as in Definition 6.2.0.3.

Consider $F \in k\mathbf{Top}_{\mathbf{Strat}}(X \times Y, Z)$ which lives above some uniquely defined poset morphism f . Using the cartesian closure of $k\mathbf{Top}$, there is a continuous map $G: X \rightarrow k\mathbf{top}(Y, Z)$ defined by $G(x)(y) = F(x, y)$. We need to show that G is a stratified morphism over a poset morphism g from P to $\mathbf{poset}_{\mathbf{Strat}}(Q, R)$. Fixing an element $p \in P$ determines a map $g(p)(-) = f(p, -): Q \rightarrow R$, which is a poset map because f is. Considering two elements $p \leq p'$ in P , we need to show that $g(p) \leq g(p')$ where $g(p), g(p') \in \mathbf{poset}_{\mathbf{Strat}}(Q, R)$. Expanding out the pointwise ordering given by the internal hom in \mathbf{POSet} , we see $g(p)(q) = f(p, q) \leq f(p', q) = g(p')(q)$ for all $q \in Q$, because f is a poset map.

To show that the converse also holds, consider a morphism $G \in k\mathbf{Top}_{\mathbf{Strat}}(X, k\mathbf{top}_{\mathbf{Strat}}(Y, Z))$ with an associated poset morphism $g: P \rightarrow \mathbf{poset}_{\mathbf{Strat}}(Q, R)$ living below G . The fact that $k\mathbf{Top}$ is cartesian closed allows us to define a continuous map $F: X \times Y \rightarrow Z$ by $F(x, y) = G(x)(y)$,

leaving us to show that $F \in k\mathbf{Top}_{\mathbf{Strat}}(X \times Y, Z)$. Define a map between posets under F by $f(p, q) = g(p)(q)$, and we need to show that f is a poset map. Taking $(p, q) \leq (p', q')$ in $P \times Q$, we see that $f(p, q) = g(p)(q) \leq g(p')(q') = f(p', q')$, which follows because both g and $g(-)$ are poset maps. Therefore, we have the required natural homeomorphism.

To complete the proof, Lemma 6.2.0.2 and the fact that X has a continuous stratification map already, gives a bijection of sets:

$$k\mathbf{Top}_{\mathbf{Strat}}(X, \underline{k\mathbf{top}}_{\mathbf{Strat}}(Y, Z)) \cong k\mathbf{Top}_{\mathbf{Strat}}(X, R(\underline{k\mathbf{top}}_{\mathbf{Strat}}(Y, Z))).$$

Hence, restriction of the internal-hom homeomorphism in $k\mathbf{Top}$ gives the natural homeomorphism of (6.1). Finally, notice that when we refine the topology on $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X \times Y, Z)$, the open sets that we add exactly correspond to open sets that are added to refine the topology of $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X, R(\underline{k\mathbf{top}}_{\mathbf{Strat}}(Y, Z)))$, because the category \mathbf{POSet} is cartesian closed. \blacksquare

Remark 6.2.0.8. In general the upwards-closed topology on $\mathbf{POSet}(P, Q)$ will not coincide with the internal hom of $k\mathbf{Top}$ between posets; this is only the case if both P and Q are finite (for details see [May, Corollary 2.2.11]). Furthermore, if both P and Q are finite, then the refinement on the topology of $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X, Y)$ to define $\mathbf{strat}(X, Y)$ (in Definition 6.2.0.3) adds in no open sets.

Using Proposition 6.2.0.7, we can show that when considering maps into these spaces (referring to the internal hom in $k\mathbf{Top}$ and the upwards closed topology placed on the internal hom in \mathbf{POSet}), it does not matter which we consider. This is illustrated by the following chain of Hom-set bijections for any posets P, Q, R :

$$\begin{aligned} k\mathbf{Top}(P, \underline{k\mathbf{top}}(Q, R)) &\cong k\mathbf{Top}(k(P \times Q), R) \\ &\cong \mathbf{Top}(P \times Q, R) \\ &\cong \mathbf{POSet}(P \times Q, R) \\ &\cong \mathbf{POSet}(P, \underline{\mathbf{poset}}(Q, R)) \\ &\cong k\mathbf{Top}(P, \underline{\mathbf{poset}}(Q, R)). \end{aligned}$$

The second isomorphism follows because $k(P \times Q) \cong P \times Q$, as any poset is locally compact.

Lemma 6.2.0.9. *For a stratified morphism $G: X \rightarrow X'$ of stratified spaces, there is an induced stratified contravariant morphism $G^*: \mathbf{strat}(X', Y) \rightarrow \mathbf{strat}(X, Y)$. Similarly, for a stratified morphism $G: Y \rightarrow Y'$ for any stratified spaces Y and Y' , we get an induced stratified covariant morphism $G_*: \mathbf{strat}(X, Y) \rightarrow \mathbf{strat}(X, Y')$.*

Proof. We prove the contravariant case, and the proof for the covariant case is similar. Consider the stratified morphism $G: X \rightarrow X'$ as shown in the left hand square of Figure 6.9, where we are considering stratified spaces $S_X: X \rightarrow P$, $S_{X'}: X' \rightarrow Q$ and $S_Y: Y \rightarrow R$.

$$\begin{array}{ccccc}
 X & \xrightarrow{G} & X' & \xrightarrow{F} & Y \\
 \downarrow S_X & & \downarrow S_{X'} & & \downarrow S_Y \\
 P & \xrightarrow{g} & Q & \xrightarrow{f} & R
 \end{array}$$

Figure 6.9: Composing stratified morphisms to give a morphism of mapping spaces.

The proof that G^* is a stratified morphism translates into to showing Figure 6.10 of topological spaces commutes, and that the arrows are continuous.

$$\begin{array}{ccc}
 R\left(\underline{k\mathbf{top}}_{\mathbf{Strat}}(X', Y)\right) & \xrightarrow{-\circ G} & R\left(\underline{k\mathbf{top}}_{\mathbf{Strat}}(X, Y)\right) \\
 \downarrow S & & \downarrow S' \\
 \underline{\mathbf{poset}}_{\mathbf{Strat}}(Q, R) & \xrightarrow{-\circ g} & \underline{\mathbf{poset}}_{\mathbf{Strat}}(P, R)
 \end{array}$$

 Figure 6.10: Demonstrating the morphism G^* .

By construction of the refinement R , the stratifications S and S' are continuous stratification maps. To proceed, we will use Lemma 6.2.0.2, so where necessary can ignore the refinements $R(-)$.

The proof that $-\circ g$ is continuous is equivalent to showing that it defines a poset morphism, by Remark 6.1.1.3. So, for $i \leq j$ in $\mathbf{POSet}_{\mathbf{Strat}}(Q, R)$ we wish to show that $i \circ g \leq j \circ g$ in $\mathbf{POSet}_{\mathbf{Strat}}(P, R)$. To show this, let $p \in P$ be a fixed poset element and notice that because i and j are poset maps such that $i \leq j$ pointwise, $i \circ g(p) = i(g(p)) \leq j(g(p)) = j \circ g(p)$. Therefore $-\circ g$ is a poset map, and hence defines a continuous map of topological spaces.

To show that $-\circ G$ is continuous, it is enough to prove continuity for sub-basis elements. Consider a continuous map $t: K \rightarrow X$ for any compact Hausdorff space K , and an open subset $U \subseteq Y$, to give the sub-basis element $N(t, U) = \{F: X \rightarrow Y \mid F(t(K)) \subseteq U\}$. The pre-image under $-\circ G$ in $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X', Y)$, is a set $\{F': X' \rightarrow Y \mid F'(G(t(K))) \subseteq U\}$, which is a sub-basis element for the topology on $\underline{k\mathbf{top}}_{\mathbf{Strat}}(X', Y)$.

To show the diagram in Figure 6.10 commutes, we use Remark 6.1.2.4, which says that because we have restricted ourselves to surjective stratification maps, the poset map is uniquely determined by the map on the underlying topological spaces. Consider a continuous map $F \in \underline{k\mathbf{Top}}_{\mathbf{Strat}}(X', Y)$, which determines a unique morphism $f \in \mathbf{POSet}_{\mathbf{Strat}}(Q, R)$. Commutativity of Figure 6.10 means that $f \circ g = S(F) \circ g = S'(F \circ G)$. The map $F \circ G$ lives above a unique poset morphism $S'(F \circ G): P \rightarrow Q$, and Figure 6.9 shows that the morphism $f \circ g = S(F) \circ g$ is also a poset morphism which is determined by $F \circ G$, implying that we must have $S(F) \circ g = S'(F \circ G)$. ■

Chapter 7

Linking Stratified Spaces and Simplicial Sets

7.1 Stratified Adjunction

Through construction of an adjunction between **Strat** and **sSet**, analogous to the unstratified case, we develop a framework for studying the homotopy theory of stratified spaces. In this section, our goal is to develop and understand a stratified analogue of the standard adjunction between **Top** and **sSet**, given by the geometric realisation functor which is left adjoint to the singular simplicial set functor. By way of recap, the (unstratified) geometric realisation of a simplicial set defines a functor $|-|: \mathbf{sSet} \rightarrow \mathbf{Top}$, and for a simplicial set A is defined to be the colimit (which is succinctly expressed as a coend):

$$|A| = \operatorname{colim}_{\Delta^n \rightarrow A} |\Delta^n| = \int^{[n] \in \Delta} A_n \odot |\Delta^n|.$$

This colimit can be thought of as taking a geometric n -simplex for each n -simplex of A and gluing these simplices according to face and degeneracy maps in the simplicial set A . Geometric realisation has a right adjoint, the singular simplicial set functor which is defined for a topological space X as the simplicial set with n -simplices defined by:

$$\operatorname{Sing}(X)_n = \mathbf{Top}(|\Delta^n|, X),$$

and where face and degeneracy maps are defined according to collapse and degeneracy maps of $|\Delta^n|$, in the obvious manner.

We now move on to describing the stratified analogue of these functors, which will be constructed to give an adjunction between **Strat** and **sSet**, and to give us a framework to study the homotopy theory of stratified spaces. The stratified analogues of the adjoint functors make

use of the naturally stratified n -simplex, introduced in Example 6.1.3.5.

Definition 7.1.0.1. The *stratified geometric realisation* for any $A \in \mathbf{sSet}$ is constructed as a colimit of naturally stratified n -simplices:

$$\|A\| = \operatorname{colim}_{\Delta^n \rightarrow A} \|\Delta^n\|,$$

which is analogous to the unstratified case. To remove any worries, notice that the stratification of each n -simplex gives a k -space, and that the category of k -spaces is closed under colimits. As in the unstratified case, the stratified realisation can be expressed using coends:

$$\|A\| = \int^{[n] \in \Delta} A_n \odot \|\Delta^n\|,$$

where \odot denotes the canonical copowering of a locally small category with all small coproducts, over \mathbf{Set} .

Remark 7.1.0.2. The underlying topological space of the stratified realisation of any simplicial set A is the same topological space as the unstratified geometric realisation of A . In particular this follows because colimits in $k\mathbf{Top}$ are constructed as colimits in \mathbf{Top} . The stratification on a path component of $|A|$ arises from the ordering of vertices, which is implicit in a simplex of any simplicial set A , and the gluing of simplices.

Notation. We will use $\|\Delta^1\|$ to refer to the naturally stratified 1-simplex, whereas $[0, 1]$ will refer to the trivially stratified geometric 1-simplex (for example this is a vital distinction in Definition 8.1.1.4).

Definition 7.1.0.3. The *stratified singular simplicial set of a stratified space X* , denoted $SS(X)$, has n -simplices defined as the set $SS(X)_n = \mathbf{Strat}(\|\Delta^n\|, X)$. The face and degeneracy maps are defined as pre-composition with face and degeneracy maps of the stratified n -simplex. We will refer to elements of $SS(X)_n$ as *stratified n -simplices of X* .

Example 7.1.0.4. Define a stratification of $[n]$ over itself via the identity map. In this case, we have $SS([n]) \cong \Delta^n$, because there are very few non-degenerate simplices to understand.

Notation. Elements of $SS(X)_1$ are also often referred to as *elementary exit paths*; these are the image of the naturally stratified 1-simplex in X ; paths which begin in some stratum and may stay in the same stratum, or immediately leave to enter a less deep stratum.

Proposition 7.1.0.5. *The stratified geometric realisation functor is left adjoint to the stratified singular simplicial set functor, providing an adjunction $\|- \| : \mathbf{sSet} \rightleftarrows \mathbf{Strat} : SS(-)$ between simplicial sets and stratified spaces.*

Proof. To prove this result, we follow the construction of [Rie14, Construction 1.5.1]. We have seen that \mathbf{Strat} is cocomplete in Proposition 6.1.4.1, and is locally small because it can be interpreted as a full subcategory of a comma category in \mathbf{Top} .

Define a covariant functor $\Delta^\bullet: \Delta \rightarrow \mathbf{Strat}$, the *standard cosimplicial stratified space*, by sending $[n]$ to the naturally stratified geometric n -simplex $\|\Delta^n\|$. We need to show that simplicial morphisms in Δ induce stratified morphisms, which we do by defining induced stratified morphisms from the co-face and co-degeneracy morphisms in the simplicial category Δ . Consider a co-face map $d^i: [n-1] \rightarrow [n]$; we can give each ordered set the Alexandrov topology to interpret them as stratified spaces over themselves, which allows us to apply $SS(-)$ and then $\|- \|$. Using Example 7.1.0.4, for d^i the end result is a map $\|SS(d^i)\|: \|\Delta^{n-1}\| \rightarrow \|\Delta^n\|$, which embeds the $(n-1)$ -simplex as the i -th face of the geometric n -simplex. Both geometric simplices come equipped with natural stratifications, and the morphism between posets is the continuous map d^i , because the map between posets is an increasing map.

$$\begin{array}{ccc} |\Delta^{n-1}| & \xrightarrow{\|SS(d^i)\|} & |\Delta^n| \\ \downarrow S_{|\Delta^{n-1}|} & & \downarrow S_{|\Delta^n|} \\ [n-1] & \xrightarrow{d^i} & [n] \end{array}$$

Figure 7.1: Explaining the stratified morphism $\|SS(d^i)\|$.

The same technique can be applied to the co-degeneracy maps s^i , to define a covariant functor Δ^\bullet .

The construction of Emily Riehl uses a left Kan extension to extend our cosimplicial stratified space functor Δ^\bullet to a geometric realisation functor from simplicial sets into stratified spaces, which turns out to be precisely the stratified geometric realisation functor that we introduced in Definition 7.1.0.1. Geometrically, the cosimplicial functor Δ^\bullet fits with the Yoneda embedding \mathcal{Y} to give Figure 7.2.

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^\bullet} & \mathbf{Strat} \\ \downarrow \mathcal{Y} & \nearrow \text{Lan}_{\mathcal{Y}} \Delta^\bullet & \\ \mathbf{sSet} & & \end{array}$$

Figure 7.2: Constructing the stratified geometric realisation functor.

We are now able to apply Theorem 2.2.0.8, which states that the left Kan extension $\text{Lan}_{\mathcal{Y}} \Delta^\bullet$ exists and is computed pointwise at $Y \in \mathbf{sSet}$ by:

$$\text{Lan}_{\mathcal{Y}} \Delta^\bullet(Y) = \int^{[n] \in \Delta} \mathbf{sSet}(\mathcal{Y}([n]), Y) \odot \|\Delta^n\| = \int^{[n] \in \Delta} Y_n \odot \|\Delta^n\| = \|Y\|.$$

The stratified geometric realisation functor is constructed as a colimit and we know that colimits commute with each other, hence the stratified geometric realisation functor is cocontinuous. The category \mathbf{sSet} is a symmetric monoidal category, hence we use [Kel82, Theorem

4.51], which guarantees the existence of a right adjoint. The right adjoint to stratified geometric realisation is precisely the stratified singular simplicial set functor of Definition 7.1.0.3; to see why, denote the right adjoint by R , and let X be a stratified space. Then, we have the following isomorphism:

$$(R(X))_n = \mathbf{sSet}(\Delta^n, RX) \cong \mathbf{Strat}(\|\Delta^n\|, X)$$

provided by expanding the n -simplices of $R(X)$ and applying the stratified adjunction. \blacksquare

The stratified adjunction constructed is actually a factorisation of the unstratified adjunction between \mathbf{Top} and \mathbf{sSet} , demonstrated in Figure 7.3.

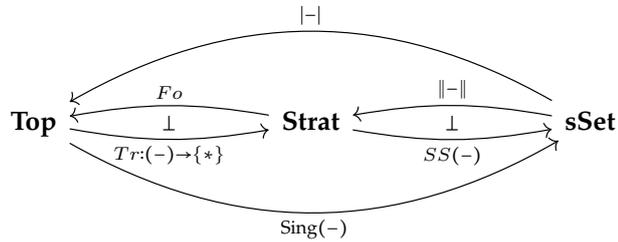


Figure 7.3: The relationship between the stratified and unstratified adjunction.

The functor Fo is the forgetful functor which forgets the stratification of the topological space X , and its adjoint gives any topological space the trivial stratification. It is easy to see that the functor $\text{Sing}(-)$ is equivalent to giving a topological space the trivial stratification, and then taking the stratified singular simplicial set, because all maps from the naturally stratified geometric n -simplex into a trivially stratified space are stratified maps. On the other hand, the geometric realisation of a simplicial set is equivalent to taking the stratified geometric realisation, and forgetting the stratification.

In general, the stratified singular simplicial set of a stratified space X will be large, however we can understand some aspects of it through the following example.

Example 7.1.0.6. For a stratified space X , the fundamental category $\tau_1(SS(X))$ has objects given by the points of X , and morphisms are the equivalence classes of finite compositions of elementary exit paths under a relation similar to Lemma 5.1.0.24 [Woo09, Lemma 3.2]. By this, we mean that we can replace the composite (by concatenation) of two elementary exit paths by a homotopic elementary exit path, where the homotopy given is through elementary exit paths for all $t > 0$. Note that in τ_1 , the 1-simplices are freely generated from $SS(X)$ so homotopic elementary exit path composites may be added. The structure for such a homotopic elementary exit path, from the composite exit path, is given by elements of $SS(X)_2$. In this case, $\tau_0(SS(X))$ is the set of path-components of strata of X .

If we let X be a homotopically stratified metric set with finitely many strata (or later a fibrant stratified space), then $\tau_1(SS(X))$ is the fundamental category $\Pi_1^{\text{po}}(X)$ of X thought of

as a po-space (as defined in [Woo09]). To see that this is the case, note that the fundamental category $\Pi_1^{\text{po}}(X)$ has as objects the points of X , and morphisms given as the path-connected components of the po-path space between any two points of X .

Remark 7.1.0.7. Consider a quasi-category which arises as the stratified singular simplicial set of a stratified space X . An endomorphism in $\tau_1(SS(X))$ will be a morphism in τ_1 from an element $x:|\Delta^0| \rightarrow X$ to itself. This is an elementary exit path γ in X , starting and ending at some $|\Delta^0| \rightarrow X$. Because the path γ is contained in one stratum, we can construct an elementary exit path $\tilde{\gamma}(t) = \gamma(1-t):|\Delta^1| \rightarrow X$, so that $\gamma \circ \tilde{\gamma}$ and $\tilde{\gamma} \circ \gamma$ are both stratum preserving homotopic to the identity morphism on x , hence define stratified 2-simplices in X where the composite edge is the identity. When we pass back to considering $\tau_1(SS(X))$, this forces every endomorphism to be an isomorphism, so $SS(X)$ is an EI ∞ -category (defined in Definition 4.6.0.6).

7.2 Stratified Realisation Preserves Finite Products

Proposition 7.2.0.1. *The stratified geometric realisation functor preserves products. Explicitly, for simplicial sets A, B there is an isomorphism of stratified spaces of the form $\|A \times B\| \cong \|A\| \times \|B\|$.*

Proof. We prove Proposition 7.2.0.1 by showing that for $n, m \in \mathbb{N}$, there is a stratified isomorphism $\|\Delta^n \times \Delta^m\| \xrightarrow{\cong} \|\Delta^n\| \times \|\Delta^m\|$. Because **Strat** is a cartesian closed category, it follows that products preserve colimits in both variables and hence the proof follows by formal arguments. We will elaborate on the formal arguments at the end of the proof.

Denote by P the colimit poset that $\|\Delta^n \times \Delta^m\|$ is stratified over. Initially, we wish to show that there is an isomorphism $P \cong [n] \times [m]$. To show this, note that a non-degenerate i -simplex in $\Delta^n \times \Delta^m$ is the product of a (possibly degenerate) i -simplex in Δ^n and a (possibly degenerate) i -simplex in Δ^m . In the obvious manner define a stratification $|\Delta^n \times \Delta^m| \rightarrow [n] \times [m]$, and by the universality of the colimit P , there is a unique induced morphism $f: P \rightarrow [n] \times [m]$. We claim that this map is injective, surjective and that $p \leq q$ in P if and only if $f(p) \leq f(q)$. Surjectivity follows by construction, and to show injectivity, consider distinct elements $p, q \in P$ which correspond to distinct 0-simplices of $\Delta^n \times \Delta^m$, hence must correspond to distinct elements in $[n] \times [m]$. Finally, consider two elements $p \leq q$ in P ; because $\|\Delta^n \times \Delta^m\|$ is constructed as a colimit of stratified n -simplices, this means that in $\Delta^n \times \Delta^m$ there is a 1-simplex $\gamma \in (\Delta^n \times \Delta^m)_1 = (\Delta^n)_1 \times (\Delta^m)_1$ so that $\|d_1\gamma\|$ is contained in the p -stratum and $\|d_0\gamma\|$ is contained in the q -stratum. We can also consider γ separately as a pair of (possibly degenerate) 1-simplices in Δ^n and Δ^m , which come with induced face maps, implying that $f(p) \leq f(q)$. The same argument in reverse shows that $f(p) \leq f(q)$ implies $p \leq q$.

To complete the proof, note that the poset isomorphism $P \cong [n] \times [m]$ fits into a stratified morphism under the identity map on $|\Delta^n \times \Delta^m|$. This is composed with the stratified morphism constructed by taking Milnor's homeomorphism on the underlying topological spaces (the homeomorphism $|\Delta^n \times \Delta^m| \xrightarrow{\cong} |\Delta^n| \times |\Delta^m|$, for further details see [Mil57, Theorem 2]) which

lives over the identity map on $[n] \times [m]$, to give the desired stratified isomorphism.

We can now elaborate on the formal arguments that allow us to complete the proof. The following proof can be found in [Lor15, p.23] or [Sch]. Express the simplicial sets A and B using the CoYoneda Lemma as $A = \int^{[n] \in \Delta} A([n]) \odot \Delta^n$ and $B = \int^{[m] \in \Delta} B([m]) \odot \Delta^m$. In **Strat**, we have the following chain of isomorphisms:

$$\begin{aligned}
 \|A \times B\| &= \left\| \left(\int^{[n] \in \Delta} A([n]) \odot \Delta^n \right) \times \left(\int^{[m] \in \Delta} B([m]) \odot \Delta^m \right) \right\| \\
 &\cong \left\| \int^{[n], [m] \in \Delta} A([n]) \odot B([m]) \odot (\Delta^n \times \Delta^m) \right\| \\
 &\cong \int^{[n], [m] \in \Delta} A([n]) \odot B([m]) \odot (\|\Delta^n \times \Delta^m\|) \\
 &\cong \int^{[n], [m] \in \Delta} A([n]) \odot B([m]) \odot (\|\Delta^n\| \times \|\Delta^m\|) \\
 &\cong \int^{[n] \in \Delta} A([n]) \odot \left(\|\Delta^n\| \times \left(\int^{[m] \in \Delta} B([m]) \odot \|\Delta^m\| \right) \right) \\
 &= \int^{[n] \in \Delta} A([n]) \odot (\|\Delta^n\| \times \|B\|) \\
 &\cong \int^{[n] \in \Delta} (A([n]) \odot \|\Delta^n\|) \times \|B\| \\
 &= \|A\| \times \|B\|.
 \end{aligned}$$

The first isomorphism follows because **sSet** is cartesian closed, hence every simplicial set is exponentiable, the fact that colimits commute with colimits, and that functors preserve isomorphisms. The second isomorphism follows because left adjoints preserve colimits, and the stratified isomorphism $\|\Delta^n \times \Delta^m\| \cong \|\Delta^n\| \times \|\Delta^m\|$, which also gives the third isomorphism. The fourth and fifth isomorphisms follows because **Strat** is cartesian closed, and in particular taking a product with $\|B\|$ is a left adjoint and therefore preserves colimits. ■

7.3 Simplicial Category of Stratified Spaces

In this section, we show that **Strat** can be considered as a simplicial category, using the same notation as in Definition 4.5.0.1. It will be important in this section that **Strat** is cartesian closed. Initially, we wish to relate the stratified singular simplicial set of a stratified mapping space to the simplicial mapping space. The following result will be used throughout this thesis, to exploit the theory of quasi-categories to understand stratified spaces.

Lemma 7.3.0.1. *For a stratified space X and a simplicial set A , there is an isomorphism of simplicial sets:*

$$SS(\mathbf{strat}(\|A\|, X)) \cong \mathbf{sset}(A, SS(X)).$$

Proof. Expanding the simplicial sets level-wise, we have the following chain of isomorphisms:

$$\begin{aligned}
 SS(\mathbf{strat}(\|A\|, X))_i &= \mathbf{Strat}(\|\Delta^i\|, \mathbf{strat}(\|A\|, X)) \\
 &\cong \mathbf{Strat}(\|A\| \times \|\Delta^i\|, X) \\
 &\cong \mathbf{Strat}(\|A \times \Delta^i\|, X) \\
 &\cong \mathbf{sSet}(A \times \Delta^i, SS(X)) = \mathbf{sset}(A, SS(X))_i,
 \end{aligned}$$

where the first isomorphism follows because \mathbf{Strat} is cartesian closed, second by Proposition 7.2.0.1 and third by the stratified adjunction. This gives an isomorphism of simplicial sets, because it is evidently compatible with face and degeneracy maps. \blacksquare

Proposition 7.3.0.2. *The category of stratified spaces has the structure of a simplicial category defined for $X, Y \in \mathbf{Strat}$ and $A, B \in \mathbf{sSet}$ by: $\mathrm{Hom}_{\mathbf{Strat}}(X, Y)_n = \mathbf{Strat}(\|\Delta^n\| \times X, Y)$, $X \otimes A = X \times \|A\|$, and $\mathrm{hom}_{\mathbf{Strat}}(A, X) = \mathbf{strat}(\|A\|, X)$.*

Remark 7.3.0.3. Since $\|\Delta^n\|$ is exponentiable, we have $\mathrm{Hom}_{\mathbf{Strat}}(X, Y) = SS(\mathbf{strat}(X, Y))$.

Proof. We need to check that the three conditions of Definition 4.5.0.1 are satisfied. It is straightforward to check axiom (1) holds:

$$\mathrm{Hom}_{\mathbf{Strat}}(X, Y)_0 = \mathbf{Strat}(\|\Delta^0\| \times X, Y) \cong \mathbf{Strat}(X, Y).$$

To show axiom (2), we exhibit a chain of isomorphisms:

$$\mathbf{Strat}(X \otimes A, Y) = \mathbf{Strat}(X \times \|A\|, Y) \cong \mathbf{Strat}(\|A\|, \mathbf{strat}(X, Y)) \cong \mathbf{sSet}(A, SS(\mathbf{strat}(X, Y))),$$

where the first isomorphism follows because $\|A\| \in \mathbf{Strat}$ is exponentiable, and the second isomorphism follows by the stratified adjunction. Associativity is equivalent to the following isomorphism $X \otimes (A \times B) = X \times \|A \times B\| \cong X \times \|A\| \times \|B\| = (X \otimes A) \otimes B$, which hold by Proposition 7.2.0.1. Axiom (3) holds because \mathbf{Strat} is cartesian closed and the stratified adjunction:

$$\begin{aligned}
 \mathbf{Strat}^{\mathrm{op}}(\mathrm{hom}_{\mathbf{Strat}}(A, X), Y) &= \mathbf{Strat}^{\mathrm{op}}(\mathbf{strat}(\|A\|, X), Y) \\
 &\cong \mathbf{Strat}(Y, \mathbf{strat}(\|A\|, X)) \\
 &\cong \mathbf{Strat}(Y \times \|A\|, X) \\
 &\cong \mathbf{Strat}(\|A\| \times Y, X) \\
 &\cong \mathbf{Strat}(\|A\|, \mathbf{strat}(Y, X)) \\
 &\cong \mathbf{sSet}(A, SS(\mathbf{strat}(Y, X))) \\
 &= \mathbf{sSet}(A, \mathrm{Hom}_{\mathbf{Strat}}(Y, X)).
 \end{aligned}$$

\blacksquare

Chapter 8

Homotopical Frameworks for Stratified Spaces

8.1 Basic Frameworks

Initially, we describe the most basic homotopical frameworks that we can use to study stratified spaces.

8.1.1 Homotopical Category

In this section, we introduce our transferred notion of weak equivalence between stratified spaces, and show that it is closely related to the notion of stratum preserving homotopy, which has long been studied between stratified spaces.

Definition 8.1.1.1. Define a *stratified weak equivalence* to be a stratified morphism $F: X \rightarrow Y$ such that $SS(F): SS(X) \rightarrow SS(Y)$ is a weak categorical equivalence.

Unpacking the definition of a weak categorical equivalence shows that a stratified morphism F is a stratified weak equivalence if for all quasi-categories Q , the induced map on sets $\tau_0(\underline{\mathbf{sset}}(SS(F), Q)): \tau_0(\underline{\mathbf{sset}}(SS(Y), Q)) \rightarrow \tau_0(\underline{\mathbf{sset}}(SS(X), Q))$ is a bijection.

Proposition 8.1.1.2. *The category \mathbf{Strat} is a homotopical category with stratified weak equivalences.*

Proof. This follows from functoriality of $SS(-)$, the definition of stratified weak equivalences and that weak categorical equivalences are the weak equivalences in the Joyal model structure on simplicial sets, hence satisfy the 2-out-of-6 property. ■

An important consequence of Definition 8.1.1.1 and Proposition 8.1.1.2 is that we have a homotopy category associated to \mathbf{Strat} , formed by inverting the stratified weak equivalences (as discussed in §3.2).

Remark 8.1.1.3. The class of stratified weak equivalences is closed retracts. This is true because weak categorical equivalences are closed under retracts in the category of simplicial sets, and functoriality of the stratified singular simplicial set functor.

We now wish to relate the notion of weak equivalence for stratified spaces to the notion of stratum preserving homotopy; the classically studied notion of homotopy equivalence between stratified spaces. Initially, we define a stratum preserving homotopy equivalence between stratified spaces (in Definition 5.1.0.6 it was defined only for filtered spaces with filtered maps, although the intuition remains the same).

Definition 8.1.1.4. Stratified morphisms $F, G: X \rightarrow Y$ are *stratum preserving homotopic* if there is a stratum preserving homotopy between them; explicitly this means that we can find a stratified morphism $H: X \times [0, 1] \rightarrow Y$ so that $H(X, 0) = F$ and $H(X, 1) = G$. A *stratum preserving homotopy equivalence* is a stratified map $F: X \rightarrow Y$ such that there exists a stratum preserving homotopy inverse $G: Y \rightarrow X$, such that $G \circ F$ is stratum preserving homotopic to the identity on X , and that $F \circ G$ is stratum preserving homotopic to the identity on Y .

Remark 8.1.1.5. It is important that in the definition of a stratum preserving homotopy the interval $[0, 1]$ is trivially stratified. This ensures that the track of any point $x \in X$ remains in the same stratum of X through the homotopy H .

Proposition 8.1.1.6. *A stratum preserving homotopy equivalence is a stratified weak equivalence.*

Proof. Consider a stratum preserving homotopy equivalence $F: X \rightarrow Y$, with stratum preserving homotopy inverse $G: Y \rightarrow X$. We claim that the simplicial map $SS(F): SS(X) \rightarrow SS(Y)$ is a categorical equivalence with inverse up to isomorphism in \mathbf{sSet}^{τ_0} given by $SS(G)$. To prove this, we show the composite $SS(G) \circ SS(F)$ is isomorphic to the identity map in the simplicial mapping space $\mathbf{sset}(SS(X), SS(X))$, and that $SS(F) \circ SS(G)$ is isomorphic to the identity map in the simplicial mapping space $\mathbf{sset}(SS(Y), SS(Y))$.

To provide a \mathcal{J} -homotopy from $SS(G) \circ SS(F)$ to $\mathbb{1}_{SS(X)}$, we denote the stratum preserving homotopy from $G \circ F$ to $\mathbb{1}_X$ by $H: X \times [0, 1] \rightarrow X$. Using this, we will construct a simplicial morphism $SS(X) \times \mathcal{J} \rightarrow SS(X)$ as shown in Figure 8.1.

$$\begin{array}{ccc}
 SS(X) \times SS([0, 1]) \cong SS(X \times [0, 1]) & \xrightarrow{SS(H)} & SS(X) \\
 \uparrow \mathbb{1}_{SS(X) \times SS(\rho)} & \nearrow & \uparrow \\
 SS(X) \times SS(\|\mathcal{J}\|) & & \\
 \uparrow \mathbb{1}_{SS(X) \times \eta_{\mathcal{J}}} & \nearrow & \\
 SS(X) \times \mathcal{J} & &
 \end{array}$$

Figure 8.1: Constructing an isomorphism in $\mathbf{sset}(SS(X), SS(X))$.

We need to describe the map $\rho: \|\mathcal{J}\| \rightarrow [0, 1]$, but first need an understanding of the stratified realisation $\|\mathcal{J}\|$. We have postponed a more detailed analysis until §8.2.1, but require that the stratified geometric realisation of $\|\mathcal{J}\|$ can be pictured as $S^\infty \rightarrow *$, with cell structure on S^∞ given by two 0-simplices and two n -cells attached in opposing directions, for $n \geq 1$. The map ρ is defined by sending the 0-simplices $\{0\}, \{1\} \in \|\mathcal{J}\|$ to the end points $\{0\}, \{1\} \in [0, 1]$ respectively, and then the 1-simplices are mapped to the path between $\{0\}$ and $\{1\}$ in $[0, 1]$. The 2-simplices are collapsed down onto the 1-simplex in $[0, 1]$ via their boundaries, and the same logic extends this construction to all higher dimensional simplices. This construction defines a map ρ such that restriction of the map $SS(X) \times SS(\|\mathcal{J}\|) \rightarrow SS(X)$ to 0 of \mathcal{J} give the map $SS(G) \circ SS(F)$ and restriction to 1 gives the identity map on $SS(X)$.

Pre-composing $\mathbb{1}_{SS(X)} \times SS(\rho)$ with the identity times the unit of the stratified adjunction at \mathcal{J} (the map $\eta_{\mathcal{J}}: \mathcal{J} \rightarrow SS(\|\mathcal{J}\|)$) defines the map $SS(X) \times \mathcal{J} \rightarrow SS(X)$, such that exponentiability of $SS(X)$ gives a non-trivial isomorphism $\mathcal{J} \rightarrow \mathbf{sset}(SS(X), SS(X))$ in the mapping space of $SS(X)$, between $SS(G) \circ SS(F)$ and the identity map on $SS(X)$. Importantly these maps become isomorphic in $\tau_1(\mathbf{sset}(SS(X), SS(X)))$. The same construction applied to the stratum preserving homotopy from $F \circ G$ to the identity map on Y proves that $SS(F) \circ SS(G)$ is isomorphic to the identity map in $\tau_1(\mathbf{sset}(SS(Y), SS(Y)))$. Therefore $SS(G)$ is an inverse for $SS(F)$ in \mathbf{sSet}^{τ_0} , which shows that any stratum preserving homotopy equivalence is a stratified weak equivalence. ■

On the topic of stratified weak equivalences, in general it is tough to prove that a map of stratified spaces is a weak equivalence. We are however able to prove that the following stratified morphisms are weak equivalences.

Theorem 8.1.1.7. *The spine inclusion $\|I\|: \|I^n\| \hookrightarrow \|\Delta^n\|$ is a stratified weak equivalence.*

Proof of Theorem 8.1.1.7 relies on the two following propositions.

Proposition 8.1.1.8. *The unit inclusion at the n -spine, $\eta_{I^n}: I^n \hookrightarrow SS(\|I^n\|)$, is a weak categorical equivalence.*

Proof. Define a homotopy inverse $SS(\|I^n\|) \rightarrow I^n$ by carrying an n -simplex $\|\Delta^n\| \rightarrow \|I^n\|$ to its underlying poset map, defining an n -simplex in I^n . The composite $I^n \hookrightarrow SS(\|I^n\|) \rightarrow I^n$ is the identity map on I^n . To complete the proof, we need to show that the other composite $SS(\|I^n\|) \rightarrow I^n \hookrightarrow SS(\|I^n\|)$ is homotopic (in the Joyal model structure) to the identity map. Note that any simplex in $\|I^n\|$ is uniquely homotopic, through the linear homotopy, to its image under the composite; this allows us to define a J -homotopy precisely through the linear homotopy on any simplex and its inverse. ■

Proposition 8.1.1.9. *The unit inclusion at the n -simplex $\eta_{\Delta^n}: \Delta^n \hookrightarrow SS(\|\Delta^n\|)$ is a weak categorical equivalence.*

Proof. This result is directly proven using the Joyal criteria for weak equivalences between quasi-categories (stated as Theorem 4.6.0.13). The stratified space $\|\Delta^n\|$ is conically smooth, and work of Jacob Lurie shows that $SS(\|\Delta^n\|)$ is a quasi-category; for further details see p.94. Notice that the map η_{Δ^n} is essentially surjective because each $x \in SS(\|\Delta^n\|)$ has a 1-simplex which provides a stratum preserving path γ_x to a vertex of $\|\Delta_0^n\|$. In barycentric coordinates at $x = (x_0, \dots, x_k, 0, \dots, 0)$ with final non-zero coordinate in the k th position, the stratum preserving path γ_x is defined by $\gamma_x(t) = (1-t)x + t(0, \dots, 0, 1, 0, \dots, 0)$ where the 1 appears in the k th coordinate. To complete the proof, note that for any $a, b \in \Delta^n$, the simplicial mapping space $\Delta^n(a, b)$ is contractible, as is $SS(\|\Delta^n\|)(\eta_{\Delta^n}(a), \eta_{\Delta^n}(b))$. Therefore the induced inclusion $\Delta^n(a, b) \hookrightarrow SS(\|\Delta^n\|)(\eta_{\Delta^n}(a), \eta_{\Delta^n}(b))$ is a weak homotopy equivalence for all $a, b \in \Delta^n$. ■

Proof of Theorem 8.1.1.7. We prove that $\|I\|: \|I^n\| \hookrightarrow \|\Delta^n\|$ is a stratified weak equivalence by applying the 2-out-of-3 property applied to Figure 8.2; the double arrow indicates the implication with respect to weak categorical equivalences.

$$\begin{array}{ccc}
 I^n & \xrightarrow[\sim]{I} & \Delta^n \\
 \downarrow \eta_{I^n} & \Downarrow & \downarrow \eta_{\Delta^n} \\
 SS(\|I^n\|) & \xrightarrow[\sim]{SS(\|I\|)} & SS(\|\Delta^n\|)
 \end{array}$$

Figure 8.2: Proving that $SS(\|I\|)$ is a weak categorical equivalence.

The vertical maps are weak categorical equivalences by Propositions 8.1.1.8 and 8.1.1.9. The simplicial map I is an inner anodyne map of simplicial sets (for details see [Joy, Proposition 2.13]) and hence is a weak categorical equivalence (this result was stated as Proposition 4.6.0.11). ■

8.1.2 Category of Fibrant Objects

Definition 8.1.2.1. Define a *stratified fibration* to be a stratified morphism $F: X \rightarrow Y$ such that $SS(F): SS(X) \rightarrow SS(Y)$ is a fibration in the Joyal model structure on simplicial sets. A stratified space X is *fibrant* if the unique map $X \rightarrow *$ to the terminal stratified space (a trivially stratified point) is a stratified fibration.

Explicitly, this says that X is fibrant if and only if the stratified singular simplicial set $SS(X)$ is a quasi-category. It immediately follows from the definition that the class of stratified fibrations is closed under retracts.

Remark 8.1.2.2. We may wish to compare this notion of stratified fibration to classical definitions. The easiest definition to compare with is Bruce Hughes' definition, which we introduced in Definition 5.1.0.19. The fibrations in our model structure are different to those studied by

Hughes. Definition 5.1.0.19 requires lifting of stratified fibrations against $Z \hookrightarrow Z \times [0, 1]$ for any space Z where as in the case of stratified fibrations we only require this lifting when Z is cofibrant. Conversely, we also require liftings against acyclic cofibrations which are not stratum preserving homotopies, for example the stratified inclusion $\|I^n\| \hookrightarrow \|\Delta^n\|$.

To compare to Quinn's notion of a stratified system of fibrations (Definition 5.1.0.17), we notice that stability under pullbacks implies that over each stratum of a stratified space, a stratified fibration is a Serre fibration. However, we do not have the requirement Quinn places relating to compatibility of almost stratum preserving neighbourhoods.

Proposition 8.1.2.3. *A stratified space X is fibrant if and only if for all simplicial sets A , the stratified mapping space $\underline{\mathbf{strat}}(\|A\|, X)$ is fibrant.*

Proof. Suppose that $\underline{\mathbf{strat}}(\|A\|, X)$ is fibrant for all $A \in \mathbf{sSet}$. Then, by taking $A = *$, it follows that $\underline{\mathbf{strat}}(\|*\|, X) \cong X$ is fibrant. Conversely, suppose that X is fibrant. Lemma 7.3.0.1 provides an isomorphism $SS(\|A\|, X) \cong \mathbf{sset}(A, SS(X))$ for any $A \in \mathbf{sSet}$. The latter is a quasi-category because $SS(X)$ is fibrant, allowing us to apply Lemma 4.6.0.4. ■

Corollary 8.1.2.4. *A stratified space X is fibrant if and only if for any stratified space B obtained as a retract of a stratified realisation, the stratified mapping space $\underline{\mathbf{strat}}(B, X)$ is fibrant in \mathbf{Strat} .*

Proof. If the stratified space B is a retract of the realisation $\|A\|$, then $\underline{\mathbf{strat}}(B, X)$ is a retract of $\underline{\mathbf{strat}}(\|A\|, X)$, and from Proposition 8.1.2.3 and the closure properties of stratified fibrations under retracts, it follows that $\underline{\mathbf{strat}}(B, X)$ is fibrant when X is. The other direction is clear. ■

Proposition 8.1.2.5. *There is a category of fibrant objects (for homotopy theory) given by the full subcategory on the fibrant stratified spaces.*

Proof. By restricting to fibrant stratified spaces, we have a category that has (small) products because right adjoints preserve limits, and the fact that fibrations in a model category are stable under products (Corollary 3.4.0.16). By definition, it also follows that fibrations are stable under composition, and that isomorphisms are fibrations. The category \mathbf{Strat} has all limits hence pullbacks of fibrations exist and because right adjoints preserve limits it follows that (acyclic) fibrations are stable under pullbacks.

We need to construct path objects for any fibrant stratified space; for a fibrant stratified space X , we claim that the stratified mapping space $\underline{\mathbf{strat}}([0, 1], X)$ is a path object. For this to be a valid construction, we initially need to show that $\underline{\mathbf{strat}}([0, 1], X)$ is fibrant. To show this, let $\mathcal{J}_{\leq 1}$ denote the 1-skeleton of \mathcal{J} and note that $[0, 1]$ can be obtained as a retract of $\|\mathcal{J}_{\leq 1}\|$ (for more information see Example 9.2.0.3). The stratified space $\underline{\mathbf{strat}}(\|\mathcal{J}_{\leq 1}\|, X)$ is fibrant by Proposition 8.1.2.3, and because fibrations are closed under retracts, it follows that $\underline{\mathbf{strat}}([0, 1], X)$ is also fibrant. The map $c: X \rightarrow \underline{\mathbf{strat}}([0, 1], X)$ is given by the constant path at each point in X , and the map $(d_0, d_1): \underline{\mathbf{strat}}([0, 1], X) \rightarrow X$ is start and end evaluation of each path. It is clear that the composite $(d_0, d_1) \circ c = \Delta_X$ provides a factorisation of the diagonal map on X .

The map c is a stratum preserving homotopy equivalence; to see this we define a stratum preserving homotopy inverse to c by start point evaluation, which is the map d_0 . The composite $d_0 \circ c = \mathbb{1}_X$, so we need to show that $c \circ d_0$ is stratum preserving homotopic to $\mathbb{1}_{\mathbf{strat}([0,1],X)}$. For a path $\gamma: [0,1] \rightarrow X$, parameterised by $t \in [0,1]$, we see that $c \circ d_0(\gamma)$ is the constant path at $\gamma(0)$. A stratum preserving homotopy from γ to the constant path at $\gamma(0)$ is given by $H: \mathbf{strat}([0,1], X) \times [0,1] \rightarrow \mathbf{strat}([0,1], X)$:

$$H(\gamma, h) = \begin{cases} \gamma(0) & \text{for } t \leq h, \\ \gamma(t-h) & \text{for } h \leq t. \end{cases}$$

If $h = 0$ we have the path γ and if $h = 1$ we have the constant path at $\gamma(0)$. Therefore c is a stratum preserving homotopy equivalence, and Proposition 8.1.1.6 shows this is a stratified weak equivalence.

Rather than manually proving that the map (d_0, d_1) is a stratified fibration, we note that it is a simple consequence of Proposition 8.2.3.5 applied to the inclusion $* \coprod * \hookrightarrow [0,1]$. ■

Following on from Remark 8.1.2.2, where we noticed the restrictive nature of stratified fibrations as opposed to classical definitions, it is important to consider whether the definition of a stratified fibration is too restrictive to be of interest. In a partial attempt to answer this question, we initially will show that there are many interesting examples of fibrant stratified spaces. The first example of fibrant stratified spaces are the conically stratified spaces, which was proved by Jacob Lurie (see [Lur14, §A.5]); these are stratified spaces $X \rightarrow P$, such that for each point $x \in X_p$, there is a stratified space $Y \rightarrow P_{>p}$ and a topological space Z with an open embedding $Z \times C(Y) \hookrightarrow X$ whose image contains x . Here, $C(Y)$ is the open cone on Y , formed by quotienting $Y \times \{0\}$ to a point within the product of Y with $\mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ is stratified over $[1]$ by $\{0\}$ sent to 0 and $\mathbb{R}_{\geq 0} \setminus \{0\}$ is sent to 1. As a particular example, Jacob Lurie constructs a stratification of the geometric realisation of an abstract simplicial complex, corresponding to the poset generated by face inclusions in the underlying simplicial complex. Using this stratification, Lurie shows that the stratified realisation of any abstract simplicial complex is a conically stratified space (see [Lur14, Corollary A.6.9]). As an example the realisation of Δ^2 stratified via the face inclusion poset is pictured in Figure 8.3.

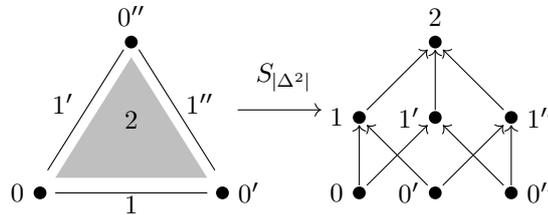


Figure 8.3: The stratification on $|\Delta^2|$ over the face inclusion poset, depicted on the right.

The following result proves that another large class of stratified spaces are fibrant with respect to the introduced notion of a stratified fibration.

Proposition 8.1.2.6. *A homotopically stratified metric space with finite stratification is fibrant.*

Proof. We use [Mil09, Theorem 4.9]; this states that for a homotopically stratified metric space $X \rightarrow P$ with finite stratification, then for any $p \leq q$ and $p \neq q$ in P , the path space inclusion $i: \text{Holink}(X_p \cup X_q, X_p) \hookrightarrow \text{pop}(X_p, X_q)$ is a special path inclusion. The space $\text{pop}(X_p, X_q)$ is the space of order preserving paths (endowed with the compact-open topology) in X with start point in the stratum X_p and end point in the stratum X_q . We also need to explain the what it means for i to be a special path inclusion; it means that we can find a homotopy (of topological spaces) $H: \text{pop}(X_p, X_q) \times I \rightarrow \text{pop}(X_p, X_q)$, so that the start points and end points of each path γ are fixed, that $H(\gamma, 0) = \gamma$, and for $s \in (0, 1]$ then $H(\gamma, s) \in \text{Holink}(X_p \cup X_q, X_p)$.

Any inner horn $\|\Lambda_k^n\|$ in X can be thought of as a family of $\|I^2\|$ -paths from the $\{0\}$ -vertex to the $\{n\}$ -vertex, passing through intermediate strata, allowing us to think of the inner horn in X as a continuous image from $\|\Lambda_{k-1}^{n-1}\|$ to $\text{pop}(X_p, X_q)$. To expand on this, we show how an inner horn can be considered as a family of $\|I^2\|$ -paths. The paths will be parameterised by the inner horn $\|\Lambda_{k-1}^{n-1}\| \subseteq \|d_0(\Delta^n)\| \subseteq \|\Delta^n\|$. We can then define a $\|\Lambda_{k-1}^{n-1}\|$ -family of $\|I^2\|$ -paths in $\|\Lambda_k^n\|$ by letting $x \in \|\Lambda_{k-1}^{n-1}\|$ and define the associated $\|I^2\|$ -path by:

$$\gamma(x, t) = \begin{cases} (1-2t)\{0\} + (2t)x & \text{for } 0 \leq t \leq \frac{1}{2} \\ (2-2t)x + (2t-1)\{n\} & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

To help the reader understand the paths γ , Figure 8.4 depicts the situation for $\|\Lambda_1^3\|$, where the $\|\Lambda_0^2\|$ is the outer horn given by the union of the $\langle 3, 1 \rangle$ and $\langle 1, 2 \rangle$ edges, and the dashed red lines indicate the stratified map $\|\Lambda_0^2\| \times \|I^2\| \rightarrow \|\Lambda_1^3\|$. The restriction to $\{0\}$ in $\|I^2\|$ is the constant map at the $\{0\}$ vertex, and restriction to $\{1\}$ is the constant map at the $\{3\}$ vertex.

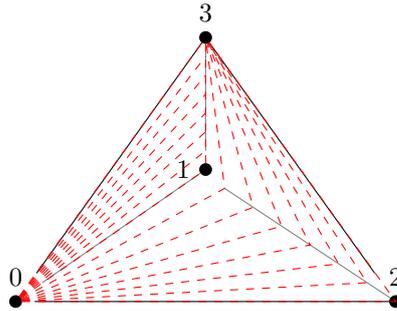


Figure 8.4: Interpreting $\|\Lambda_1^3\|$ as a $\|\Lambda_0^2\|$ -family of $\|I^2\|$ -paths.

Let $i: \text{Holink}(X_0 \cup X_n, X_0) \hookrightarrow \text{pop}(X_0, X_n)$ denote the special path inclusion, where we let 0 and n denote the strata in X corresponding to the image of these vertices. The inclusion i being a special path inclusion says there is a homotopy $H: \|\Lambda_{k-1}^{n-1}\| \times \|I^2\| \times \|\Delta^1\| \rightarrow X$, where

the restriction to $\{0\} \in \|\Delta^1\|$ is the map of the inner horn, and so that for any $t > 0$ in $\|\Delta^1\|$ our $\|I^2\|$ -paths are actually stratified paths; i.e. images of $\|\Delta^1\| \rightarrow X$. These conditions on the homotopy H and on the map mean that it is possible to pick a map (there are many possible choices of map):

$$\|\Delta^n\| \rightarrow \|\Lambda_{k-1}^{n-1}\| \times \|I^2\| \times \|\Delta^1\| \xrightarrow{H} X,$$

whose restriction to $\|\Lambda_k^n\| \subseteq \|\Delta^n\|$ is precisely the inner horn in X (which is the restriction $\|\Lambda_{k-1}^{n-1}\| \times \|I^2\| \times \{0\} \rightarrow X$), and hence defines a filler of this horn. \blacksquare

Example 8.1.2.7. We can easily describe another source of stratified spaces which are fibrant; any poset given the Alexandrov (upwards closed) topology which is stratified over itself via the identity map is automatically fibrant. This follows because for a poset P , the simplicial set $SS(P)$ coincides with $N(P)$, the nerve of P thought of as a category (and in particular the nerve of any category is a quasi-category; see Remark 4.3.0.8).

8.2 Model Structure on the Fibrant Stratified Spaces

8.2.1 Relationship with the Interval Object \mathcal{J}

In this short section, we will explore the stratified geometric realisation of the interval object \mathcal{J} in the Joyal model structure, and will explain why the path object for a fibrant stratified space X should really be thought of as $\mathbf{strat}(\|\mathcal{J}\|, X)$. Earlier we chose to use $[0, 1]$ instead, since this makes the relationship with stratum preserving homotopy evident.

Consider the unstratified realisation of the interval object \mathcal{J} in $\mathbf{sSet}_{\text{Joyal}}$; as a simplicial set this is the nerve of the groupoid $0 \begin{smallmatrix} \xrightarrow{F} \\ \xleftarrow{G} \end{smallmatrix} 1$ consists of two objects. The stratified geometric realisation $|\mathcal{J}|$ will have two 0-simplices (one corresponding to 0 and the other to 1), and two non-degenerate n -simplices for every $n \geq 1$. The n -simplices are attached corresponding to either composite $\dots \circ F \circ G \circ F$ or $\dots \circ G \circ F \circ G$. From this perspective, identity maps correspond to degenerate lower dimensional non-degenerate simplices.

Proposition 8.2.1.1. *The stratified space $\|\mathcal{J}\|$ is the stratified space $S^\infty \rightarrow *$.*

Proof. The geometric realisation $\|\mathcal{J}\|$ has two n -simplices for every n . A simplicial set can be built as a colimit of its skeleta, and the geometric realisation functor commutes with colimits being a left adjoint, hence we consider $\|\mathcal{J}\|$ inductively built through its skeleton. The geometric realisation of the 0-skeleton will have two 0-simplices x and y , stratified over the poset consisting of two elements with no relation between them. In the realisation of the 1-skeleton $\|\mathcal{J}_{\leq 1}\|$, we attach two stratified 1-simplices denoted F from x to y and G from y to x , in opposite directions between these 0-simplices; this process trivialises the stratification, by identifying the two elements in the poset corresponding to the 0-simplices. The realisation

of the 2-skeleton $\|\mathcal{J}_{\leq 2}\|$ is obtained from the realisation of the 1-skeleton by attaching two 2-simplices glued via their boundaries to the 1-skeleton. By this, we mean that one 2-simplex is glued via its boundary to the 1-skeleton by attaching the $\langle 0, 1 \rangle$ edge to F and the $\langle 1, 2 \rangle$ edge to G , and the $\langle 0, 2 \rangle$ edge to the degenerate 1-simplex at x . The other 2-simplex is glued via its boundary to the 1-skeleton by attaching the $\langle 0, 1 \rangle$ edge to G and the $\langle 1, 2 \rangle$ edge to F , and the $\langle 0, 2 \rangle$ edge to the degenerate 1-simplex at y . Because the stratification along the boundary of the 2-simplices glued on is trivial, the stratification on the interior of the 2-simplices is trivial. This argument can be extended to the higher dimensional simplices in the higher dimensional skeleta, showing that $\|\mathcal{J}\|$ is a trivially stratified space.

Now consider the cell structure that we described on $|\mathcal{J}|$ as being built as the colimit of the geometric realisation of the inductive skeleton of \mathcal{J} . It is clear from the above description that $|\mathcal{J}_0| = S^0$, $|\mathcal{J}_{\leq 1}| = S^1$ and $|\mathcal{J}_{\leq 2}| = S^2$. Continuing the inductive description, we see that $|\mathcal{J}_{\leq n}| = S^n$, and that $|\mathcal{J}| = \text{colim}(S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \dots)$ in the usual manner. From the colimit description, it is evident that this construction describes a cell structure of S^∞ (this has also been noticed for example in [DS11b, §2.3] or simplicially by Joyal in [Joy02, p.210]). Therefore, we can think of the stratified geometric realisation $\|\mathcal{J}\|$ as the stratified space $S^\infty \rightarrow *$. ■

8.2.2 Towards a Model Structure on Fibrant Stratified Spaces

To construct a model structure, we need to understand smallness properties of stratified spaces. We will prove a result analogous to [Hov98a, Proposition 2.4.2] in the category of stratified spaces. Hovey proves that compact topological spaces are finite relative to the closed T_1 inclusions, in order to construct the model structure on **Top**.

Definition 8.2.2.1. A closed T_1 inclusion of stratified spaces is a closed T_1 inclusion of the underlying topological spaces. A closed T_1 inclusion of topological spaces $F: Y \rightarrow Z$ is a closed inclusion such that each point in $Z \setminus F(Y)$ is closed.

Proposition 8.2.2.2. Compact k -spaces which are stratified over finite posets are finite relative to closed T_1 inclusions.

Proof. A compact topological space X being finite relative to the closed T_1 inclusions means that there is a finite cardinal κ for X , such that for every regular cardinal $\Lambda \geq \kappa$ and Λ -sequence \mathcal{F} in **Top** such that each map is a closed T_1 inclusion, the map of sets:

$$\text{colim}_\Lambda \text{Hom}_{\mathbf{Top}}(X, \mathcal{F}) \rightarrow \text{Hom}_{\mathbf{Top}}(X, \text{colim}_\Lambda \mathcal{F})$$

is an isomorphism.

Let $A \rightarrow P$ be a compact topological space stratified over a finite poset P . Consider a limit ordinal λ , and a λ -sequence $X: \lambda \rightarrow \mathbf{Strat}$ of closed T_1 inclusions. Let X_α the stratified space $X(\alpha) = X_\alpha \rightarrow Q_\alpha$. Considering a stratified morphism $A \rightarrow \text{colim}_{\alpha < \lambda} X_\alpha$, it follows that the

underlying continuous map of topological spaces factors through X_α for some α . It is not true that the factorisation $A \rightarrow X_\alpha$ is necessarily a stratified morphism.

To fix this, note that a colimit in **POSet** is constructed as the colimit in **Set** with the smallest partial order such that each map of sets in the defining cocone is a poset map and quotienting as necessary. There are two obstructions to the map $P \rightarrow Q_\alpha$ being a poset map. Suppose that there is a relation between two poset elements of P , but no relation in Q_α . The relation in P implies there is a relation in the image of P in $\text{colim}_{\alpha < \lambda} Q_\alpha$, and by construction of the colimit of posets there exists some $\beta > \alpha$ such that the image of P in Q_β has the required relation. The second obstruction is that one stratum, A_p , may be mapped to multiple strata of X_α . By construction of $\text{colim}_{\alpha < \lambda} X_\alpha$, the stratum A_p is entirely contained in one stratum, so there must exist some $\beta > \alpha$ such that A_p is contained within one stratum of X_β . Let δ denote the supremum of the β ; because we have assumed that P is finite, it follows that $\delta < \lambda$, and hence $A \rightarrow \text{colim}_{\alpha < \lambda} X_\alpha$ factors through a stratified morphism $A \rightarrow X_\delta$. ■

Proposition 8.2.2.3. *For a κ -small object $A \in \mathbf{sSet}$, the stratified geometric realisation $\|A\|$ is κ -small relative to closed T_1 inclusions.*

Proof. Consider a κ -small object A in **sSet**, and let $\Lambda \geq \kappa$ be a regular cardinal, with a Λ -sequence $X: \Lambda \rightarrow \mathbf{Strat}$ of closed T_1 -inclusions of the underlying topological spaces. Then, we have the following chain of isomorphisms of sets:

$$\begin{aligned}
 \text{colim}_{\alpha \in \Lambda} \mathbf{Strat}(\|A\|, X(\alpha))_i &\cong \text{colim}_{\alpha \in \Lambda} \mathbf{sSet}(A, SS(X(\alpha)))_i \\
 &\cong \mathbf{sSet}(A, \text{colim}_{\alpha \in \Lambda} (SS(X(\alpha)))_i) \\
 &= \mathbf{sSet}(A, \text{colim}_{\alpha \in \Lambda} (\mathbf{Strat}(\|\Delta^i\|, X(\alpha)))) \\
 &\cong \mathbf{sSet}(A, \mathbf{Strat}(\|\Delta^i\|, \text{colim}_{\alpha \in \Lambda} X(\alpha))) \\
 &= \mathbf{sSet}(A, SS(\text{colim}_{\alpha \in \Lambda} X(\alpha))_i) \\
 &\cong \mathbf{Strat}(\|A\|, \text{colim}_{\alpha \in \Lambda} X(\alpha))_i.
 \end{aligned}$$

The first and final isomorphisms are provided by the stratified adjunction. The second follows because A is κ -small in **sSet**, and the third isomorphism follows from Proposition 8.2.2.2 because $\|\Delta^i\|$ is a compact topological space stratified over a finite poset. ■

8.2.3 Model Structure on Fibrant Stratified Spaces

In this section, by restricting again to fibrant stratified spaces we show that we have a cofibrantly transferred model structure from the Joyal model structure on simplicial sets. To construct a model structure rather than merely a category of fibrant objects, we introduce a notion of cofibration between stratified spaces via their lifting properties, and proceed to show that these maps satisfy the required axioms of a model structure (see Definition 3.4.0.4).

Definition 8.2.3.1. We would like to define a model structure on stratified spaces using the given definitions of stratified fibrations and stratified weak equivalences. Therefore, the *stratified cofibrations* in the model structure must be defined as the morphisms which lift on the left against acyclic fibrations in **Strat**.

We now state the main result of this section.

Theorem 8.2.3.2. *The classes of stratified cofibrations, fibrations and weak equivalences define a model structure on the category of fibrant stratified spaces.*

The first step towards a proof of this theorem is to understand further the set of cofibrations and acyclic cofibrations given by Definition 8.2.3.1. Let $I = \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \in \mathbb{N}}$ and J be the set of generating acyclic cofibrations in $\mathbf{sSet}_{\text{Joyal}}$. Write $\|I\|$ and $\|J\|$ for the sets consisting of their stratified geometric realisations.

Proposition 8.2.3.3. *The stratified cofibrations are generated by the set of stratified boundary simplex inclusions $\|I\| = \{\|\partial\Delta^n\| \hookrightarrow \|\Delta^n\|\}_{n \in \mathbb{N}}$. Furthermore, the stratified morphisms generated by the stratified realisation of the generating set of acyclic cofibrations J in $\mathbf{sSet}_{\text{Joyal}}$, are the morphisms which lift on the left against the stratified fibrations, and satisfy $\text{cof}(\|J\|) = {}^{\square}(\|J\|^{\square})$.*

Proof. The stratified adjunction implies that $\|I\|^{\square}$ contains the set of stratified acyclic fibrations. To show that $\|I\|^{\square}$ is precisely the set of stratified acyclic fibrations, assume that there is a stratified morphism $F \in \|I\|^{\square}$ such that F is not a stratified acyclic fibration. By assumption, F lifts on the right against any morphism of $\|I\|$ hence under the adjunction $SS(F)$ will lift against any generating cofibration in $\mathbf{sSet}_{\text{Joyal}}$. This implies that F is a stratified acyclic fibration, contradicting the assumption that F is not. Hence the stratified morphisms in $\|I\|^{\square}$ are precisely the stratified fibrations.

To complete the proof for I , note that by definition $\text{cof}(\|I\|) \subseteq {}^{\square}(\|I\|^{\square})$. Equality follows by noting that for any map in ${}^{\square}(\|I\|^{\square})$, we can apply Quillen's Small Object Argument to the set $\|I\|$, by Proposition 8.2.2.2, completing the proof.

In both paragraphs, the set I can be replaced by J , and instead of applying Proposition 8.2.2.2, we need to apply Proposition 8.2.2.3 where κ is taken in turn to be the cardinality of the domain of each generating map in J . This allows us to apply Quillen's Small Object Argument, which shows that $\text{cof}(\|J\|) = {}^{\square}(\|J\|^{\square})$. ■

Remark 8.2.3.4. A personal grievance is that we are (currently) unable to show that $\|J\|$ is the generating set of acyclic cofibrations in **Strat**. This is because we are unable to show that arbitrary relative $\|J\|$ -cell complexes, whilst they have the correct lifting properties, are stratified weak equivalences.

Before we can prove Theorem 8.2.3.2, we need to understand a case in which the relative $\|J\|$ -cell complexes are acyclic cofibrations, which in particular happens when we restrict to the case of a relative $\|J\|$ -cell complex between fibrant stratified spaces. To do this, we need the following result.

Proposition 8.2.3.5. *Let $F: A \rightarrow B$ be an (acyclic) cofibration in $\mathbf{sSet}_{\text{Joyal}}$, and a fibrant stratified space X . Then the contravariantly induced morphism of stratified mapping spaces:*

$$- \circ \|F\|: \mathbf{strat}(\|B\|, X) \rightarrow \mathbf{strat}(\|A\|, X)$$

is a stratified (acyclic) fibration.

Similarly, consider a simplicial set A and a stratified (acyclic) fibration between fibrant stratified spaces $F: X \rightarrow Y$. Then the covariantly induced morphism on stratified mapping spaces:

$$F \circ -: \mathbf{strat}(\|A\|, X) \rightarrow \mathbf{strat}(\|A\|, Y)$$

is a stratified (acyclic) fibration.

Notation. In this proof, we shall use $\varphi(-)$ to denote the adjoint of a morphism, so for example if $F: A \rightarrow SS(X)$ then the adjoint of F is the morphism $\varphi(F): \|A\| \rightarrow X$.

Proof. We prove the first claim; the second follows by an almost identical proof. Initially, notice that Proposition 8.1.2.3 implies that $- \circ \|F\|$ is a stratified morphism between fibrant stratified spaces. By definition, the stratified morphism $- \circ \|F\|$ is an (acyclic) fibration if and only if $SS(- \circ \|F\|)$ is an (acyclic) fibration in $\mathbf{sSet}_{\text{Joyal}}$. We would like to apply the isomorphism of Lemma 7.3.0.1 to $SS(- \circ \|F\|)$ but need to know which map $SS(- \circ \|F\|)$ is sent to under this isomorphism. Naturality of the adjunction ensures that $SS(- \circ \|F\|)$ is sent to $\varphi(-) \circ F$. Acyclic fibrations are closed under retracts, implying that the proof of the proposition is completed if we can show that when F is an (acyclic) cofibration then:

$$\varphi(-) \circ F: \mathbf{sset}(B, SS(X)) \rightarrow \mathbf{sset}(A, SS(X))$$

is an (acyclic) fibration of quasi-categories. This is true because the Joyal model structure is enriched (as a model structure) over itself (this fact was mentioned in the sketch proof of Proposition 4.6.0.18). ■

Remark 8.2.3.6. To elaborate on the use of the naturality assumption, consider a stratified morphism $\alpha: \|B\| \rightarrow X$ so that under the induced functor we arrive at $\alpha \circ \|F\|: \|A\| \rightarrow \|B\| \rightarrow X$. Applying the stratified adjunction gives us two morphisms of the form $B \rightarrow A \rightarrow SS(X)$; these are $\varphi(\alpha \circ \|F\|)$ and $\varphi(\alpha) \circ F$. The naturality requirement on an adjunction states that these two morphisms are equal (as explained in [Lei14, p.42]). The dual naturality requirement is used in the dual proof (where we instead consider the morphism $F \circ -$).

Remark 8.2.3.7. Proposition 8.2.3.5 holds in slightly greater generality; we do not need to require that the stratified spaces X and Y are fibrant. However if X and Y are fibrant, then the induced morphism on mapping spaces is between fibrant stratified spaces.

Lemma 8.2.3.8. *A relative $\|J\|$ -cell complex between fibrant stratified spaces is a stratified weak equivalence.*

Proof. Let $F: X \rightarrow Y$ denote a relative $\|J\|$ -cell complex between fibrant stratified spaces X and Y . Define $E_0^{-1}(X) \subseteq \mathbf{strat}([0, 1], Y)$ to be the subspace of stratum preserving paths in Y , which start in $F(X)$. As a stratified space, $E_0^{-1}(X)$ can alternatively be described as the pullback shown in Figure 8.5.

$$\begin{array}{ccc}
 E_0^{-1}(X) & \overset{\text{r}}{\dashrightarrow} & \mathbf{strat}([0, 1], Y) \\
 \downarrow & & \downarrow (E_0, E_1) \\
 E_1 \left(\begin{array}{ccc} X \times Y & \xrightarrow{(F, 1_Y)} & Y \times Y \\ \downarrow \text{pr}_2 & & \downarrow \\ Y & & Y \end{array} \right. & &
 \end{array}$$

Figure 8.5: Constructing the end point evaluation morphism E_1 .

Proposition 8.2.3.5 applied to the cofibration $\{0\} \coprod \{1\} \hookrightarrow \mathcal{J}_{\leq 1}$ (where $\mathcal{J}_{\leq 1}$ is the 1-skeleton of \mathcal{J}) shows us that the map $(E_0, E_1): \mathbf{strat}(\mathcal{J}_{\leq 1}, Y) \rightarrow Y \times Y$ is a stratified fibration. Noting that $[0, 1]$ can be obtained as a retract of $\|\mathcal{J}_{\leq 1}\|$, shows that (E_0, E_1) is a stratified fibration. It follows that the pullback of (E_0, E_1) is also a stratified fibration, because right adjoints preserve pullbacks. The stratified projection morphism $\text{pr}_2: X \times Y \rightarrow Y$ is a stratified fibration because X is a fibrant stratified space, and fibrations are closed under cartesian product. Therefore the composite morphism $E_1: E_0^{-1}(X) \rightarrow X \times Y \rightarrow Y$ is also a stratified fibration.

We now construct the commutative diagram shown in Figure 8.6, where the morphism c is defined by sending each point $x \in X$ to the constant path at $F(x) \in Y$. By assumption F is a relative $\|J\|$ -cell complex, hence Proposition 8.2.3.3 shows that F lifts on the left against the stratified fibrations.

$$\begin{array}{ccc}
 X & \xrightarrow{c} & E_0^{-1}(X) \\
 \downarrow F & \nearrow H & \downarrow E_1 \\
 Y & \xrightarrow{1_Y} & Y
 \end{array}$$

Figure 8.6: Showing that X is a stratum preserving deformation retract of Y .

The existence of a lift H in Figure 8.6 proves that X is a stratum preserving deformation retract of Y , which in particular is a stratum preserving homotopy equivalence, and Proposition 8.1.1.6 shows that F is a stratified weak equivalence. \blacksquare

We are now in a position to prove Theorem 8.2.3.2

Proof of Theorem 8.2.3.2. By definition the stratified weak equivalences satisfy the 2-out-of-3 property (MC1). The stratified fibrations and weak equivalences are closed under retracts as

a simple consequence of their definitions, and because the class of stratified cofibrations are defined as a left lifting class against the acyclic fibrations, it also follows that these are closed under retracts (MC2). By definition the stratified cofibrations are those morphisms that lift on the left against acyclic fibrations. In Lemma 8.2.3.8, we show that a relative $\|J\|$ -cell complex between fibrant stratified spaces is a stratified weak equivalence, which combined with Proposition 8.2.3.3 shows that acyclic cofibrations lift on the left against fibrations (MC3). Proposition 8.2.2.2 allows us to applying Quillen's Small Object argument to the set $\|I\|$ in **Strat**, to any map between fibrant stratified spaces. This gives a factorisation (as in Proposition 8.2.3.3) of any map into a cofibration followed by an acyclic fibration, and because the latter map is a fibration, the factorisation is between fibrant stratified spaces. Lemma 8.2.3.8 proves that any relative $\|J\|$ -cell complex between fibrant stratified spaces is a weak equivalence, hence applying Quillen's Small Object Argument to the set $\|J\|$ in **Strat**, implies we also have factorisations of any map as an acyclic cofibration followed by a fibration (MC4). In this case, application of Quillen's Small Object is valid because any simplicial set is κ -small, allowing us to apply Proposition 8.2.2.3. ■

Remark 8.2.3.9. It is important to note that we cannot naïvely apply Quillen's Small Object Argument in the full subcategory of fibrant stratified spaces, because this subcategory lacks small colimits.

8.3 Simplicial Enrichment

We can show that limits in **Strat** are actually simplicially enriched limits (see Definition 4.5.0.10).

Proposition 8.3.0.1. *Limits in the category of stratified spaces are simplicially enriched.*

Proof. The category **Strat** is cartesian closed by Proposition 6.2.0.7, which in particular implies that $\mathbf{Strat}(Y, -)$ is a right adjoint and therefore preserves limits. We arrive at the following isomorphism of n -simplices:

$$\mathbf{Strat}(X \times \|\Delta^n\|, \lim_{i \in \mathbb{I}} A_i) \cong \lim_{i \in \mathbb{I}} (\mathbf{Strat}(X \times \|\Delta^n\|, A_i)),$$

which is compatible with face and degeneracies of $\|\Delta^n\|$, hence defines the necessary isomorphism of simplicial sets. ■

Remark 8.3.0.2. This is true in further generality; in any cartesian closed simplicial category, limits are automatically simplicial limits. For more details, see [RV18, Lemma A.5.1]

We are able to show that the model structure on fibrant stratified spaces it is naturally enriched over the Joyal model structure on simplicial sets.

Proposition 8.3.0.3. *The model structure on fibrant stratified spaces is enriched over the Joyal model structure on simplicial sets.*

Remark 8.3.0.4. By enrichment of the model structure over the Joyal model structure on simplicial sets, we mean that the definitions of fibrations and weak equivalences in the definition of a simplicial model category (see Definition 4.5.0.6), are replaced with the notions from the Joyal model structure.

Proof. For a cofibration $J: A \rightarrow B$ in \mathbf{sSet} and stratified fibration $F: X \rightarrow Y$, define $\langle \|J\|, F \rangle$ by substituting the internal hom in \mathbf{Strat} in to the pullback power $\langle \|J\|, F \rangle$ as in Definition 2.1.0.19. Proposition 8.3.0.3 is equivalent to proof that the pullback power:

$$\langle \|J\|, F \rangle: \mathbf{strat}(\|B\|, X) \rightarrow \mathbf{strat}(\|A\|, X) \times_{\mathbf{strat}(\|A\|, Y)} \mathbf{strat}(\|B\|, Y)$$

is a stratified fibration, which is acyclic if either J or F are. This is done by showing that as simplicial maps we have an isomorphism $SS(\langle \|J\|, F \rangle) \cong \langle J, SS(F) \rangle$, which follows from Lemma 7.3.0.1 and naturality of an adjunction.

Expanding as in the proof of Lemma 7.3.0.1, we would like to detect whether $\langle \|J\|, F \rangle$ is a stratified (acyclic) fibration. To check this, we apply $SS(-)$ to $\langle \|J\|, F \rangle$ and arrive at the following simplicial morphism:

$$SS(\langle \|J\|, F \rangle): SS(\mathbf{strat}(\|B\|, X)) \rightarrow SS\left(\mathbf{strat}(\|A\|, X) \times_{\mathbf{strat}(\|A\|, Y)} \mathbf{strat}(\|B\|, Y)\right).$$

Expand each simplicial set at its n -simplices. Using the fact that right adjoints preserve limits, and that stratified geometric realisation preserves finite products, we arrive at an expression for the n -simplices of $SS(\langle \|J\|, F \rangle)$ as:

$$\langle \mathbb{1}_{\Delta^n} \times J, F \rangle: \mathbf{Strat}(\|\Delta^n \times B\|, X) \rightarrow \mathbf{Strat}(\|\Delta^n \times A\|, X) \times_{\mathbf{Strat}(\|\Delta^n \times A\|, Y)} \mathbf{Strat}(\|\Delta^n \times B\|, Y).$$

From here we apply the adjunction and exponentiability of A and B in \mathbf{sSet} , and un-package the expansion to consider the n -simplices, to arrive at the simplicial morphism:

$$\langle J, SS(F) \rangle: \mathbf{sset}(B, SS(X)) \rightarrow \mathbf{sset}(A, SS(X)) \times_{\mathbf{sset}(A, SS(Y))} \mathbf{sset}(B, SS(Y)).$$

We see that $\langle J, SS(F) \rangle$ is a fibration which is acyclic if either J or F is, because the Joyal model structure is enriched over itself (this was mentioned in the proof of Proposition 4.6.0.18). Naturality of the adjunction implies that $\langle \|J\|, F \rangle$ is a fibration which is acyclic if either $\|J\|$ or F are. ■

8.4 Issues in Constructing a Model Category

We wish to briefly discuss the issues that we have had with constructing a model category on the category of stratified spaces. Because of the nature of this section, sketch proofs given will

not be provided, and hence they will not be as rigorous as other proofs given in other sections of this thesis. The main issue that we have faced in the construction, and that we have been unable to resolve, is to prove that a relative $\|J\|$ -cell complex (i.e. a morphism in **Strat** which is generated by the set $\|J\|$) is a stratified weak equivalence. In particular, this is required to prove statement (2) of a cofibrantly transferred model structure (detailed in Theorem 3.5.3.5). We do know that relative $\|J\|$ -cell complexes are cofibrations and Proposition 8.2.3.3 shows that they have the correct lifting properties for acyclic cofibrations. One of the difficulties is in showing that a map between stratified singular simplicial sets of arbitrary stratified spaces (not necessarily fibrant) is a weak categorical equivalence.

Initially we attempted to construct a fibrant replacement $X \rightarrow RX$ using the stratified inner horn inclusion factorisation system applied to the map $X \rightarrow *$, for any $X \in \mathbf{Strat}$. To show this gave a fibrant replacement, meaning that $X \rightarrow RX$ was a stratified weak equivalence, we need to prove that the map:

$$SS(X) \coprod_{SS(\|\Lambda_k^n\|)} SS(\|\Delta^n\|) \hookrightarrow SS\left(X \coprod_{\|\Lambda_k^n\|} \|\Delta^n\|\right)$$

is a weak categorical equivalence for the pushout of an inner horn in X along an inclusion $\|\Lambda_k^n\| \rightarrow \|\Delta^n\|$. In the unstratified case, this can be shown by a choice of retract of $\|\Delta^n\|$ onto $\|\Lambda_k^n\|$. However in the stratified case, an appropriate choice of retract will not give a stratum preserving homotopy but instead only a weak almost stratum preserving deformation retract. We are currently unable to use this to show that the inclusion is a weak categorical equivalence. We have also attempted to explicitly understand and manipulate the stratified simplices of:

$$SS(X) \coprod_{SS(\|\Lambda_k^n\|)} SS(\|\Delta^n\|) \text{ and } SS\left(X \coprod_{\|\Lambda_k^n\|} \|\Delta^n\|\right),$$

but have been unable to prove that the inclusion induces the required bijection of sets to define a weak categorical equivalence.

We are able to relate relative $\|J\|$ -cell complexes to $\tau_0(SS(-))$ bijections. However this result is not enough to prove that a relative $\|J\|$ -cell complex is a stratified weak equivalence.

Proposition 8.4.0.1. *A relative $\|J\|$ -cell complex $F: X \rightarrow Y$ induces a bijection of sets:*

$$\tau_0(SS(\mathbf{strat}(F, -))) : \tau_0(SS(\mathbf{strat}(Y, Q))) \rightarrow \tau_0(SS(\mathbf{strat}(X, Q))),$$

for any fibrant $Q \in \mathbf{Strat}$.

Proof. Proposition 8.2.3.3 implies that a relative $\|J\|$ -cell complex has the left lifting property against the maps $Q \rightarrow *$ and $\mathbf{strat}([0, 1], Q) \rightarrow Q \times Q$ for any fibrant $Q \in \mathbf{Strat}$. Lifting against $Q \rightarrow *$ shows that the map $\tau_0(SS(\mathbf{strat}(F, -)))$ is surjective. Lifting against the fibration $\mathbf{strat}([0, 1], Q) \rightarrow Q \times Q$ shows that $\tau_0(SS(\mathbf{strat}(F, -)))$ is injective. \blacksquare

Remark 8.4.0.2. For a relative $\|J\|$ -cell complex F to be a stratified weak equivalence, the map $\tau_0(\underline{\mathbf{sset}}(SS(F), Q))$ must be a bijection of sets for all quasi-categories Q .

Remark 8.4.0.3. It is tempting to attempt to believe that the map:

$$\tau_0(SS(\underline{\mathbf{strat}}(Y, \|Q\|))) \rightarrow \tau_0(\underline{\mathbf{sset}}(SS(Y), Q))$$

is a bijection for any quasi-category Q , however we can provide an example where this is not the case. Consider $Y = [1]$; the underlying topological space of Y is the Sierpinski space which is stratified over the poset $[1]$ via the identity map. If we let $Q = \Delta^1$, then the set $\tau_0(SS(\underline{\mathbf{strat}}(Y, \|Q\|)))$ has two objects corresponding to constant maps from Y into either stratum of $\|Q\|$. The set $\tau_0(\underline{\mathbf{sset}}(SS(Y), Q))$ however has three objects arising from the isomorphism classes of the two constant maps into Δ^1 , but also an isomorphism class of the non-constant map $SS(Y) \rightarrow \Delta^1$.

8.4.1 An Approach to Constructing a J -Semi Model Structure

To construct a J -semi model structure, we need to construct a functorial fibrant replacement for cofibrant stratified spaces. This would allow us to apply Quillen's Path Object (Proposition 3.5.3.7) argument to relative $\|J\|$ -cell complexes whose domain is cofibrant, to give the necessary factorisations. In this discussion, we will use the Joyal-Kan model structure on the category of simplicial sets. This model structure is a localisation of the Joyal model structure on simplicial sets, introduced & explored by Peter Haine in [Hai18].

Consider a stratified realisation $\|A\|$ of a simplicial set A , for which we would like to construct a fibrant replacement. We conjecture that:

$$\|R(-) \circ \eta_A\|: \|A\| \rightarrow \|R(SS(\|A\|))\|$$

constructs a fibrant replacement for $\|A\|$ in **Strat**. A proof that this constitutes a fibrant replacement can be broken down into two conjectures.

Conjecture 1. *The unit $\eta_A: A \rightarrow SS(\|A\|)$ is a Joyal-Kan weak equivalence.*

One approach to this conjecture is to attempt to use [Hai18, Proposition 3.12], which says that we can apply the functor VEx_P^∞ to η_A to detect the Joyal-Kan weak equivalence, by checking if the map $VEx_P^\infty(\eta_A)$ is a weak categorical equivalence. The idea of the functor VEx_P^∞ is to replace each stratum with its Ex^∞ fibrant replacement, which is a Kan complex. Because the functor VEx_P^∞ is not a fibrant replacement functor, we are unsure of the best way to approach this problem. Our intuition is that the path spaces of A and $SS(\|A\|)$ should be homotopic.

Therefore, one possible way to approach the issue would be to factor the map $VEx_P^\infty(\eta_A)$

as an acyclic cofibration I followed by a fibration P (in the Joyal model structure), giving us:

$$VEx_P^\infty(\eta_A): VEx_P^\infty(A) \xrightarrow[\sim]{I} B(VEx_P^\infty(A)) \xrightarrow{P} VEx_P^\infty(SS(\|A\|)),$$

and showing that the fibration P is in fact an acyclic fibration. One might establish this by applying [Joy, Exercise 2.4 (Descent property of fibrations)]. The idea would be to pull the map $B(VEx_P^\infty(A)) \rightarrow VEx_P^\infty(SS(\|A\|))$ back along the surjective map $\coprod \Delta^n \rightarrow VEx_P^\infty(SS(\|A\|))$ where the coproduct is taken over all possible simplices of all dimensions in $VEx_P^\infty(SS(\|A\|))$. If we are able to show that the pullback is a trivial Kan fibration (by showing that the pullback over any particular simplex $\Delta^n \rightarrow VEx_P^\infty(SS(\|A\|))$ is a trivial Kan fibration), then the descent property of fibrations implies that $B(VEx_P^\infty(A)) \rightarrow VEx_P^\infty(SS(\|A\|))$ is also a trivial Kan fibration. Fix a simplex $\Delta^n \rightarrow VEx_P^\infty(SS(\|A\|))$, and consider the pullback of $B(VEx_P^\infty(A)) \rightarrow VEx_P^\infty(SS(\|A\|))$. We would need to show that the pullback has contractible strata and contractible holinks (noting that the pullback is a quasi-fibration between quasi-categories, hence could apply the Fundamental Theorem for Quasi-Categories to check for a weak categorical equivalence).

Remark 8.4.1.1. We initially hoped that the unit η_A was a weak categorical equivalence for any $A \in \mathbf{sSet}$, but for $A = \mathcal{J}_{\leq 1}$, the map $\eta_A: A \rightarrow SS(\|A\|)$ is not a weak categorical equivalence. In this case $SS(\|A\|)$ is a Kan complex, but A is not; this map is not a weak categorical equivalence because $\tau_1(A)$ is $\mathcal{J}_{\leq 1}$ thought of as a category, however $\tau_1(SS(\|A\|))$ is a groupoid and hence is not equivalent to $\tau_1(A)$. In this case however, if we apply VEx_P^∞ to A , then we see that $\tau_1(VEx_P^\infty(A))$ is a groupoid, and is equivalent to $\tau_1(SS(\|A\|))$.

Corollary 8.4.1.2. *From Conjecture 1, it follows that the stratified geometric realisation of a Joyal-Kan weak equivalence is a stratified weak equivalence.*

Proof. To see this, consider a Joyal-Kan weak equivalence $F: A \rightarrow B$, and let $R(-)$ denote the fibrant replacement in the Joyal model structure on simplicial sets. We will use \sim to denote Joyal-Kan weak equivalences in Figure 8.7.

$$\begin{array}{ccc} A & \xrightarrow[\sim]{R(-) \circ \eta_A} & R(SS(\|A\|)) \\ \downarrow \sim F & & \downarrow R(SS(\|F\|)) \\ B & \xrightarrow[\sim]{R(-) \circ \eta_B} & R(SS(\|B\|)) \end{array}$$

Figure 8.7: Showing that $\|-$ carries Joyal-Kan equivalences to stratified weak equivalences.

The 2-out-of-3 property satisfied by Joyal-Kan weak equivalences implies that $R(SS(\|F\|))$ is also a Joyal-Kan weak equivalence. Because $R(SS(\|A\|))$ and $R(SS(\|B\|))$ are quasi-categories whose the strata are Kan complexes, it follows that both are fibrant in the Joyal-Kan model structure (see [Hai18, Proposition 1.9]). Therefore, the Joyal-Kan weak equivalence $R(SS(\|F\|))$

is a weak categorical equivalence because right Quillen functors preserve weak equivalences between fibrant objects (alternatively see [Hai18, Proposition 3.12]). To show $\|F\|$ is a stratified weak equivalence, consider Figure 8.8, in which \sim denotes a weak categorical equivalence.

$$\begin{array}{ccc}
 SS(\|A\|) & \xrightarrow[\sim]{R(-)} & R(SS(\|A\|)) \\
 \downarrow SS(\|F\|) & & \downarrow R(SS(\|F\|)) \\
 SS(\|B\|) & \xrightarrow[\sim]{R(-)} & R(SS(\|B\|))
 \end{array}$$

Figure 8.8: Showing that $SS(\|F\|)$ is a weak categorical equivalence.

The 2-out-of-3 property satisfied by weak categorical equivalences implies that $SS(\|F\|)$ is also a weak categorical equivalence. It follows from the definition that $\|F\|$ is a stratified weak equivalence. \blacksquare

In particular, Corollary 8.4.1.2 implies that for any simplicial set A , the stratified morphism $\|R(-) \circ \eta_A\|: \|A\| \rightarrow \|R(SS(\|A\|))\|$ is a stratified weak equivalence. For this result to provide functorial fibrant replacement for stratified geometric realisations, which we would like to extend to cofibrant stratified spaces, then we need to know that the realisation of a quasi-category is a fibrant stratified space.

Conjecture 2. *The stratified geometric realisation of a quasi-category is a fibrant stratified space.*

We have a general strategy for proving Conjecture 2, by breaking it into three parts. The first part is to show that for a cofibrant stratified space X , we can model the holink between strata X_p and X_q by an almost stratum preserving deformation neighbourhood of X_p in X_q ; for details see Subsection 9.2.3. Denote the almost stratum preserving neighbourhood of X_p in X_q by U . As a consequence, if there is a fibration:

$$E_0: \text{Holink}(X_p \cup U, X_p) \rightarrow X_p,$$

then there is a fibration:

$$E_0: \text{Holink}(X_p \cup X_q, X_p) \rightarrow X_p.$$

Next, we hope that for a quasi-category Q , the stratified realisation $\|Q\|$ satisfies the E_0 pairwise holink fibration condition. By the E_0 pairwise holink fibration condition, we mean that for every pair $p \leq q$ in the poset of $\|Q\|$, there is a stratified fibration:

$$E_0: \text{Holink}(\|Q\|_p \cup \|Q\|_q, \|Q\|_p) \rightarrow \|Q\|_p.$$

The third part is to show that for a simplicial set A , then the stratified realisation $\|A\|$ is fibrant if and only if the E_0 pairwise holink fibration condition is satisfied for $\|A\|$. If $\|A\|$ is fibrant, then we prove in Corollary 9.1.0.7 that the pairwise holink fibration condition is

satisfied. Conversely, suppose that $\|A\|$ satisfies the E_0 pairwise holink fibration condition. Consider a stratified inner horn $F: \|\Lambda_k^n\| \rightarrow \|A\|$, of which we would like to construct a filler. Notice that by construction $\|A\|$ is metric and the image of the horn is contained in a finite union of strata. In particular, this will allow us to use Hughes' result [Hug99, Corollary 6.2 (2)] which states that for a closed union of strata $Y \subseteq X$ there is a stratified fibration (in Hughes' sense, Definition 5.1.0.19) $E_0: \text{Holink}(X, Y) \rightarrow Y$. Consider the horn in $\|A\|$ as paths from the base to the n -vertex of the n -simplex, and restrict $\|A\|$ to the strata $\|A\|_{f(0)} \cup \dots \cup \|A\|_{f(n)}$. Temporarily ignore the stratification in the base $\|\Lambda_k^{n-1}\|$ and choose a map:

$$\|\Lambda_k^{n-1}\| \times [0, 1] \rightarrow \|d_n(\Delta^n)\| \rightarrow \|A\|_{f(0)} \cup \dots \cup \|A\|_{f(n-1)},$$

which is compatible with F restricted to the base of the horn. The idea is to use the stratified fibration of Hughes to extend the horn to give a full simplex in the holink; because we have ignored the stratification in the base, the paths constructed will be compatible extensions of the paths at $\|\Lambda_k^{n-1}\|$ to give stratified exit paths at the complete $\|\Delta^{n-1}\|$.

Remark 8.4.1.3. As we suggested when we discussed Lurie's work on conically smooth fibrant stratified spaces, by choice of the face partial order stratification for realisation on the n -simplices, it follows that the stratified realisation of an abstract simplicial complex is always fibrant. While we do not see a direct approach to use this result, it appears to prove that for any simplicial set A , then the stratified geometric realisation (in our sense) of the subdivision of A is fibrant. A different approach could be to show that for a quasi-category Q , any simplex in $\|Q\|$ is stratum preserving homotopic to a simplex in $\|\text{sd}(Q)\|$. This would then allow one to homotope a horn in $\|Q\|$ to one in $\|\text{sd}(Q)\|$, of which we know a filler exists.

8.4.2 A Possible Approach to Detecting Stratified Weak Equivalences

As an alternative approach to the issue of detecting stratified weak equivalences, we attempted to prove that stratified weak equivalences could be detected by considering the ∞ -exit path spaces of stratified spaces. Results of Dugger and Spivak in [DS11a] and [DS11b] show that in a quasi-category, the simplicial mapping spaces can be understood as the necklace space between vertices; we are proposing a stratified analogue of their work. Intuitively, the idea was to let X be a stratified space, and by considering the simplicial morphisms $I^n \hookrightarrow I^{n-1}$ defined by sending the final 1-simplex to a degenerate 0-simplex, we have contravariant induced morphisms of stratified spine mapping spaces:

$$\mathbf{strat}(\|I^{n-1}\|, X)(a, b) \hookrightarrow \mathbf{strat}(\|I^n\|, X)(a, b).$$

This allows us to define the ∞ -exit path space of a stratified space X by the directed limit $X^\infty(a, b) = \text{colim}_{n \in \mathbb{N}} \mathbf{strat}(\|I^n\|, X)(a, b)$. We had hoped that we could use ∞ -exit path spaces to detect stratified weak equivalences. The idea is that for a fibrant stratified space X , we can

show that the underlying topological space $X^\infty(a, b)$ is weak homotopy equivalent to $X(a, b)$. However, when we construct what we would hope to be the fibrant replacement for a stratified space (factorise the map $X \rightarrow *$ by applying the small object argument to the set of the inner horn inclusions), it is not clear that this map induces a weak homotopy equivalence of ∞ -exit path spaces. If we were able to show the required weak homotopy equivalence of ∞ -exit path spaces, then consider a map $F: X \rightarrow Y$ between stratified spaces which is an isomorphism on posets. Then F induces a weak homotopy equivalence on ∞ -exit path spaces if and only if F is a stratified weak equivalence.

Chapter 9

Topology of Stratified Spaces

9.1 Fibrant Stratified Spaces

Recall that a fibrant stratified space X is one for which $SS(X)$ is a quasi-category. Furthermore, we can detect fibrations between fibrant stratified spaces as the maps $F: X \rightarrow Y$ which lift against the stratified inner horn inclusions, and the acyclic cofibration $\|\ast\| \hookrightarrow \|\mathcal{J}\|$.

Proposition 9.1.0.1. *A stratified space $S_X: X \rightarrow P$ is fibrant if and only if the stratification morphism S_X is a stratified fibration.*

Proof. Any poset thought of as a stratified space is fibrant, therefore if S_X is a stratified fibration it is clear that X is a fibrant stratified space. To prove the converse, we need to show that if X is fibrant, then S_X is a stratified fibration. To prove this, we first need to show that for any $0 < k < n$ we can find a lift in Figure 9.1. We also need to show that there is a lift of S_X against the inclusion $\|\ast\| \hookrightarrow \|\mathcal{J}\|$.

$$\begin{array}{ccc}
 \|\Lambda_k^n\| & \longrightarrow & X \\
 \downarrow & \nearrow & \downarrow S_X \\
 \|\Delta^n\| & \longrightarrow & P
 \end{array}$$

Figure 9.1: Proving that S_X is a stratified fibration when X is fibrant.

By assumption X is fibrant, hence we can construct a lift in the upper half of the diagram. Because S_X is the stratification map of X living over the identity map on P , and the poset map on $[n]$ induced by $\|\Lambda_k^n\| \hookrightarrow \|\Delta^n\|$ is the identity map, it follows that the bottom triangle of the Figure 9.1 commutes with this choice of lift. The same logic applies to the inclusion $\|\ast\| \hookrightarrow \|\mathcal{J}\|$, where the lift is constructed by sending $\|\mathcal{J}\|$ to the constant map at the image of $\|\ast\|$ in X , completing the proof of Proposition 9.1.0.1. ■

A union of strata within a fibrant stratified space is also fibrant, when given the sub-space stratification.

Proposition 9.1.0.2. *Let X be a fibrant stratified space and consider a union of strata $Y \subseteq X$. When given the subspace stratification, the stratified space Y is fibrant.*

Proof. Consider a stratified inner horn in Y as a stratified inner horn in X . The stratified space X is fibrant, so we can fill this horn in X . It follows that this filler is contained in Y because Y is a union of strata of X , and the strata that the filler simplex is contained in are exactly the strata which contain the inner horn. ■

Theorem 9.1.0.3. *For a stratified space X and any $x \in X$, let $E_1^{-1}(x)$ denote the pullback of the endpoint evaluation map $E_1: \mathbf{strat}(\|\Delta^1\|, X) \rightarrow X$ along $x: * \rightarrow X$, and similarly for $E_0^{-1}(x)$. Then, the following conditions are equivalent:*

1. *the stratified singular simplicial set $SS(X)$ of X is an ∞ -category, i.e. X is fibrant;*
2. *for any $x \in X$, the evaluation $E_0: E_1^{-1}(x) \rightarrow X$ becomes a right Kan fibration under $SS(-)$;*
3. *for any $x \in X$, the evaluation $E_1: E_0^{-1}(x) \rightarrow X$ becomes a left Kan fibration under $SS(-)$.*

Remark 9.1.0.4. In the unstratified case, the evaluation morphisms E_0 and E_1 are Serre fibrations, and $\text{Sing}(X)$ is a Kan complex for any topological space X .

Proof. We begin by showing that statement 1 holds if and only if statement 2 holds. Note that for a stratified simplex $\sigma: \|\Delta^n\| \rightarrow X$ if we consider any point $x \in d_n(\|\Delta^n\|)$ there is a path $\gamma_x: \|\Delta^1\| \rightarrow \|\Delta^n\|$ defined by $\gamma_x(t) = (1-t)x + t(0, \dots, 0, 1)$, expressed in barycentric coordinates. All paths of this form will be stratified morphisms as the final coordinate of any point along the paths will be 0 for $t = 0$, so lie in a stratum corresponding to some $m \in [n]$ for $m < n$. This coordinate will be strictly greater than 0 for $t \in (0, 1]$, hence lying in the stratum corresponding to $n \in [n]$. The image of these paths allow us to consider the stratified simplex σ as a stratified morphism $\tilde{\sigma}: \|\Delta^{n-1}\| \rightarrow E_1^{-1}(\sigma(\{n\}))$ by identifying in the obvious manner $d_n(\|\Delta^n\|)$ with $\|\Delta^{n-1}\|$. This shows that the lifting problems in Figure 9.2 are therefore equivalent. By application of the stratified adjunction, the left hand lifting problem for $0 < k < n$ is equivalent to $SS(X)$ being an ∞ -category, and the right hand lifting problem for $0 < k \leq n - 1$ is equivalent to $SS(E_0)$ being a right Kan fibration.

$$\begin{array}{ccc}
 \|\Delta_k^n\| & \xrightarrow{F} & X \\
 \downarrow & \nearrow F' & \\
 \|\Delta^n\| & &
 \end{array}
 \quad \Leftrightarrow \quad
 \begin{array}{ccc}
 \|\Delta_k^{n-1}\| & \xrightarrow{f} & E_1^{-1}(y) \\
 \downarrow & \nearrow h & \downarrow E_0 \\
 \|\Delta^{n-1}\| & \xrightarrow{g} & X
 \end{array}$$

Figure 9.2: These lifting problems, where we identify $\|\Delta^{n-1}\|$ with $d_n(\|\Delta^n\|)$, are equivalent.

In order to prove that conditions 1 and 3 are equivalent, a very similar method can be used, however in this case we end up with a family of paths parameterised by the face $d_0(\Delta^n)$. The details are identical to the case proven here. ■

Remark 9.1.0.5. Adapting this proof, we are able to replace $E_1^{-1}(x)$ with $\mathbf{strat}(\|\Delta^1\|, X)$, upon assuming that paths for every point in the geometric horn $\|\Lambda_k^{n-1}\|$ end in the same path component of a stratum. The idea is that this condition mimics the start point fibration condition of Frank Quinn, however in further generality; our conditions allow the image of paths at each point of the horn in X to begin in different strata, corresponding to the stratification of $\|\Lambda_k^n\|$, so long as all paths end in the same path component of a stratum. Under this assumption, we can fill the horn $\|\Lambda_k^n\|$ by extending the ends of each path to a common path end point. The easiest way to do this is to use the canonical linear path to the end of the stratified path at the k vertex of $\|\Lambda_k^n\|$; this translates our situation into condition 2 of Theorem 9.1.0.3.

The same holds for condition 3 by replacing $E_0^{-1}(x)$ with $\mathbf{strat}(\|\Delta^1\|, X)$, assuming that every path in the geometric horn starts in the same path component of a stratum; in this situation, $SS(X)$ being a quasi-category allows us to extend the path at each point, to a path which begins at the chosen point x .

Corollary 9.1.0.6. *Suppose we are given a commutative diagram such as either (a) or (b) of Figure 9.3, for some stratified space X and $x \in X$.*

$$\begin{array}{ccc}
 \{n\} & \xrightarrow{F} & E_1^{-1}(x) \\
 \downarrow & \nearrow G & \downarrow E_0 \\
 \|\Delta^n\| & \longrightarrow & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \{0\} & \xrightarrow{F} & E_0^{-1}(x) \\
 \downarrow & \nearrow G & \downarrow E_1 \\
 \|\Delta^n\| & \longrightarrow & X
 \end{array}$$

(a) A lift over $\{n\} \in \|\Delta^n\|$. (b) A lift over $\{0\} \in \|\Delta^n\|$.

Figure 9.3: Constructing lifts when X is fibrant.

If X is fibrant we can find a lift G in the two commutative diagrams of Figure 9.3.

Proof. The simplest way to prove Corollary 9.1.0.6 is to note that the inclusion $\{n\} \hookrightarrow \Delta^n$ is right anodyne (for details see [Joy, Proposition 2.12]). In particular, stratified geometric realisation commutes with colimits hence the inclusion $\{n\} \hookrightarrow \|\Delta^n\|$ can be built using right stratified horn inclusions (inclusions $\|\Lambda_k^n\| \hookrightarrow \|\Delta^n\|$ for $0 < k \leq n$). We can now apply the right Kan fibration $SS(E_0)$ of Theorem 9.1.0.3 to each right horn inclusion, to build the lift G of Figure 9.3(a).

Dually, the inclusion $\{0\} \hookrightarrow \Delta^n$ is left anodyne, hence the inclusion $\{0\} \hookrightarrow \|\Delta^n\|$ can be built using left stratified horn inclusions. Applying the left Kan fibration $SS(E_1)$ of Theorem 9.1.0.3 to each left horn inclusion builds the lift G of Figure 9.3(b). ■

Corollary 9.1.0.7. *Consider a fibrant stratified space $X \rightarrow P$. For any $p \leq q \in P$, there is a Kan fibration $SS(E_0): SS(\text{Holink}(X_p \cup X_q, X_p)) \rightarrow SS(X_p)$, where E_0 is start point evaluation.*

Proof. It is important to note that $SS(X_p) = \text{Sing}(X_p)$ is a Kan complex because it is stratified over the terminal poset. Recalling [Lur77, Lemma 2.1.3.3], that a left/right Kan fibration over a Kan complex is a Kan fibration, we see that $SS(E_0)$ is a Kan fibration by Theorem 9.1.0.3 and Remark 9.1.0.5. ■

We are able to characterise stratified fibrations between fibrant stratified spaces.

Proposition 9.1.0.8. *Let F be a stratified morphism $F: X \rightarrow Y$ between fibrant stratified spaces X and Y . Then F is a fibration if and only if $\text{strat}(B, F): \text{strat}(B, X) \rightarrow \text{strat}(B, Y)$ is a fibration for any cofibrant stratified space B .*

Proof. A stratified morphism $F: X \rightarrow Y$ is a stratified fibration between fibrant stratified spaces if and only if $SS(F): SS(X) \rightarrow SS(Y)$ is a quasi-fibration between quasi-categories, which is true if and only if $\mathbf{sset}(A, SS(F)): \mathbf{sset}(A, SS(X)) \rightarrow \mathbf{sset}(A, SS(Y))$ is a quasi-fibration of quasi-categories for all simplicial sets A . By applying the isomorphism of Lemma 7.3.0.1, this holds if and only if $SS(\text{strat}(\|A\|, F)): SS(\text{strat}(\|A\|, X)) \rightarrow SS(\text{strat}(\|A\|, Y))$ is a quasi-fibration of quasi-categories. By the definition of fibration in **Strat**, this is true if and only if the induced stratified map $\text{strat}(\|A\|, F)$ is a fibration between fibrant stratified spaces. Since any cofibrant stratified space B is a retract of $\|A\|$ for some $A \in \mathbf{sSet}$, the last condition is equivalent to $\text{strat}(B, F)$ being a fibration for all cofibrant stratified spaces B . ■

9.1.1 Topology of Fibrant Stratified Spaces

Definition 9.1.1.1. Consider a path-connected stratum X_p in a stratified space X . Define the *path-closure* $\text{PCI}(X_p)$ to be the smallest enlargement of X_p within X , such that for any stratified path $\gamma: \|\Delta^1\| \rightarrow X$ where $\gamma((0, 1]) \subseteq X_p$, then $\gamma([0, 1]) \subseteq \text{PCI}(X_p)$.

Definition 9.1.1.2. We say that the stratified space $X \rightarrow P$ satisfies the *homotopy-theoretic frontier condition* if for any pair $p, q \in P$, if $X_p \cap \text{PCI}(X_q) \neq \emptyset$ then $X_p \subseteq \text{PCI}(X_q)$.

We can rephrase the property of a stratified space X satisfying the homotopy-theoretic frontier condition; if there is a stratified map $\gamma: \|\Delta^1\| \rightarrow X$ with $\gamma(\{0\}) \in X_p$ and $\gamma((0, 1]) \subseteq X_q$, then for any $x \in X_p$ there is a stratified path $\delta: \|\Delta^1\| \rightarrow X$ so that $\delta(\{0\}) = x$ and $\delta((0, 1]) \subseteq X_q$.

Proposition 9.1.1.3. *Fibrant stratified spaces with path-connected strata satisfy the homotopy-theoretic frontier condition.*

Proof. Let $X \rightarrow P$ denote a fibrant stratified space with path-connected strata and consider two elements $p, q \in P$. If we assume that $X_p \cap \text{PCI}(X_q) \neq \emptyset$, then we are guaranteed that there exists some $x_1 \in X_p$ and a path $\gamma: \|\Delta^1\| \rightarrow X$ such that $\gamma(\{0\}) = x_1 \in X_p$ and $\gamma((0, 1]) \subseteq X_q$. We wish to show that if this happens, then X_p lies in the path-closure of X_q .

Define an inner horn $\|\Lambda_1^2\| \rightarrow X$ as indicated in black in Figure 9.4; in this diagram, the horizontal axis along the bottom represents the path-connected stratum X_p , with the edge $\langle 0, 1 \rangle \subseteq \|\Lambda_1^2\|$ being sent to a path from any point $x_0 \in X_p$ to x_1 in the stratum X_p , and the edge $\langle 1, 2 \rangle \subseteq \|\Lambda_1^2\|$ being sent to the stratified path γ . Fibrancy of X and the stratified adjunction allow us to find the filler of the inner horn constructed, depicted in red in Figure 9.4, and ultimately the filler gives us a stratified 1-simplex $\tilde{\gamma}$ from x_0 to x_2 .

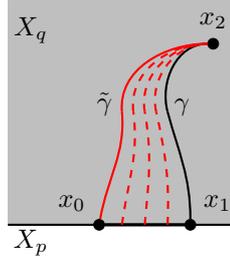


Figure 9.4: Homotopy-theoretic frontier condition.

The filler 1-simplex $\langle 0, 2 \rangle$ can be interpreted as a path $\tilde{\gamma}: \|\Delta^1\| \rightarrow X$, so that $\tilde{\gamma}(\{0\}) = x_0 \in X_p$ and $\tilde{\gamma}((0, 1]) \subseteq X_q$, which shows that $X_p \subseteq \text{PCI}(X_q)$. Therefore, in a fibrant stratified space X with path-connected strata, if $X_p \cap \text{PCI}(X_q) \neq \emptyset$, then $X_p \subseteq \text{PCI}(X_q)$. ■

Remark 9.1.1.4. We might hope that the converse is true; that an appropriate stratified space satisfying the frontier condition is fibrant. However as we will illustrate with a simple example, this is not true. Our example shows that we cannot even expect a cofibrant stratified space which satisfies the homotopy-theoretic frontier condition to be fibrant. A counter example is the cofibrant stratified space X pictured in Figure 9.5.

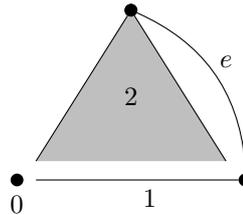


Figure 9.5: A cofibrant stratified space which satisfies the homotopy-theoretic frontier condition, but is not fibrant.

The stratified space pictured in Figure 9.5 is the naturally stratified of the 2-simplex, with a stratified 1-simplex e attached to the vertices corresponding to $\{1\}$ and $\{2\}$ of $\|\Delta^2\|$. This space is cofibrant and satisfies the homotopy-theoretic frontier condition. However it is not fibrant; for example we can map $\|\Lambda_1^2\|$ into X sending the 1-simplex spanned by $\{0\}$ and

$\{1\}$ to the corresponding base 1-simplex in Figure 9.5, which is $\|\Delta^1\| = \|d_2(\Delta^2)\| \subseteq X$, and then the remaining 1-simplex spanned by 1 and 2 to the attached simplex e . This defines an inner horn $\|\Lambda_1^2\|$ in Figure 9.5, which has no possible filler.

Remark 9.1.1.5. Path objects for fibrant stratified spaces and the notion of right homotopy (arising from choice of a path object) were discussed in §8.1.2.

9.2 Cofibrant Stratified Spaces

In this section, we will consider general cofibrations ignoring the restriction of fibrancy for the time being. If the model structure on fibrant stratified spaces can be improved to a model category or even a J -semi model category, then this understanding of cofibrant stratified spaces will still hold.

Corollary 9.2.0.1 (Stratified Homotopy Extension Property). *Cofibrations have a stratified homotopy extension property; by this we mean that any cofibration lifts on the left against the start point evaluation $E_0: \mathbf{strat}([0, 1], X) \rightarrow X$ for any fibrant stratified space X .*

Proof. This follows directly from Proposition 8.2.3.5 applied to the stratum preserving homotopy equivalence $* \hookrightarrow [0, 1]$, which is also a cofibration. ■

We now study and understand some properties of the cofibrant stratified spaces, using the transferred definition of cofibrations.

Proposition 9.2.0.2. *Cofibrant stratified spaces are retracts of absolute cell complexes.*

Proof. A cofibrant object $X \in \mathbf{Strat}$ is one for which the unique stratified morphism $F: \emptyset \rightarrow X$ satisfies $F \in \text{cof}(I)$, where we are using \emptyset to denote the initial object of \mathbf{Strat} (see Corollary 6.1.2.5). If F is an absolute cell complex (meaning that the source of F is \emptyset , and that $F \in \text{cell}(I)$ which says that F is a relative I -cell complex), then F is obtained by gluing simplices via their boundaries beginning with the initial simplicial set. Equivalently this follows because the stratified geometric realisation functor preserves cofibrations (see Remark 9.2.0.5).

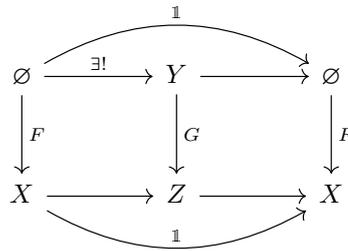


Figure 9.6: The morphism F is a retract of G .

However, if the morphism F is obtained as a retract of an element of $\text{cell}(I)$, we want to show that X must still be a retract of an absolute cell complex. Assume that X is not a retract

of an absolute cell complex. Consider a stratified morphism $G: Y \rightarrow Z$ where $G \in \text{cell}(I)$ is the stratified morphism of which F is a retract. The stratified morphisms F and G must fit into a commutative diagram as shown in Figure 9.6. Commutativity of the right hand square of Figure 9.6 forces $Y = \emptyset$. The top row of the commutative diagram now gives us no information, and we see that the space X is a retract of Z . Because $G \in \text{cell}(I)$, the space Z is an absolute cell complex. Therefore cofibrant stratified spaces are retracts of absolute cell complexes. ■

Example 9.2.0.3. Consider the interval $[0, 1]$ given the trivial stratification. In **Top**, this space is an absolute cell complex, however in **Strat** it is not. We have previously used that the interval $[0, 1]$ is cofibrant in disguise, where we briefly stated that it can be obtained as a retract of the simplicial set $\mathcal{J}_{\leq 1}$ which is the 1-skeleton of \mathcal{J} (in both Proposition 8.1.2.5 and Lemma 8.2.3.8). We will go into explicit detail here. The stratified space $\|\mathcal{J}_{\leq 1}\|$ is cofibrant and is trivially stratified by Proposition 8.2.1.1. The trivially stratified interval $[0, 1]$ is obtained as a retract via maps:

$$[0, 1] \xrightarrow{i} \|\mathcal{J}_{\leq 1}\| \xrightarrow{r} [0, 1],$$

where the map i is defined by mapping $[0, 1]$ to the stratified realisation of either 1-simplex of $\mathcal{J}_{\leq 1}$, and the map r is defined by the collapse of $\|\mathcal{J}_{\leq 1}\|$ onto that 1-simplex. It is clear that $r \circ i$ is the identity on $[0, 1]$.

Remark 9.2.0.4. In Example 9.2.0.3 we constructed a map $\|\mathcal{J}_{\leq 1}\| \rightarrow [0, 1]$, however just as easily we could have used the map $\rho: \|\mathcal{J}\| \rightarrow [0, 1]$ constructed in the proof of Proposition 8.1.1.6; this would instead show that $[0, 1]$ can be obtained as a retract of $\|\mathcal{J}\|$.

Remark 9.2.0.5. Notice that the stratified geometric realisation functor preserves cofibrations; this follows because the left adjoints preserve colimits (iterated pushouts of elements of the generating set), and noticing that set of generating cofibrations in (either model structure on) simplicial sets maps to the generating set for stratified cofibrations under stratified geometric realisation. Furthermore, the stratified singular simplicial set functor preserves cofibrations.

Definition 9.2.0.6. A stratified space $X \rightarrow P$ admits a compatible triangulation if there is a triangulation T of X , such that any open face of a simplex in the triangulation T is contained in a single stratum of X .

Corollary 9.2.0.7. A stratified space with path-connected strata and which admits a compatible triangulation is a cofibrant stratified space.

Proof. Initially, we show that given a compatible triangulation, we can find a subdivision of this triangulation which consists of stratified simplices in X . To do this, consider a simplex in the triangulation of X ; i.e. a continuous map of the form $\sigma: \|\Delta^n\| \rightarrow X$. We have assumed our triangulation is compatible, hence the pre-image of the closure of any stratum of X is a union of closed faces. Barycentrically subdivide the simplex $\|\Delta^n\|$; the restriction of σ to each simplex in the subdivision defines a stratified simplex in X .

We apply this to construct a map $\varphi: X \rightarrow \|SS(X)\|$, such that the post-composition with the counit $\epsilon_X: \|SS(X)\| \rightarrow X$ gives the identity on X . Assume our triangulation of X is given by stratified simplices $\sigma: \|\Delta^n\| \rightarrow X$. Construct the map $\varphi(x) = [(\sigma, \sigma^{-1}(x))]$, where we think of a point in $\|SS(X)\|$ as an equivalence class of an n -simplex given by a stratified simplex in X , and a point within the simplex. Note that the map φ gives a stratified morphism, precisely because we have assumed that the strata of X are path-connected. The map φ is well defined because each point $x \in X$ lies in the interior of a unique simplex σ in any triangulation of X , and it is evident that $\epsilon \circ \varphi = \mathbb{1}_X$. Therefore we have exhibited X as a retract of the stratified geometric realisation $\|SS(X)\|$, hence X is cofibrant. ■

Remark 9.2.0.8. In particular, Corollary 9.2.0.7 proves that any stratified PL pseudomanifold is a cofibrant stratified space (for more details see [FHLM14, pp.180-182] or [DR04, pp.397-398]).

9.2.1 Topology of Cofibrant Stratified Spaces

The next step in understanding the transferred model structure is to understand the class of cofibrations, and the properties of the cofibrant stratified spaces.

Lemma 9.2.1.1. *Any cofibrant stratified space has path-connected strata.*

Proof. This follows from the fact that the naturally stratified simplices have path connected strata. This implies that absolute cell complexes also have path-connected strata because the poset of an absolute cell complex is constructed as a colimit of the posets associated to its simplices. It is easy to see that the property of path-connected strata is preserved under taking retracts, hence the proof is completed by Proposition 9.2.0.2. ■

Lemma 9.2.1.2. *In a cofibrant stratified space $X \rightarrow P$, if $p \leq q$ in P , then we can find a finite chain of elements in P of the form $p = p_0 \leq p_1 \leq \dots \leq p_n = q$, such that there is stratified spine $Sp: \|I^n\| \rightarrow X$ where $Sp(\{i\})$ is contained in the stratum X_{p_i} for $i \in \{0 \leq i \leq n-1\}$.*

Proof. We prove the claim for any absolute cell complex, and therefore also holds for any retract. Consider the poset P of an absolute cell complex A . For simplicity we will denote the poset by $P = \text{colim } [n_h]$, where the colimit is indexed over the simplices of A . A colimit in **POSet** is constructed by taking the colimit of the underlying sets, and then we place the smallest pre order on the set (and collapsing as necessary), so that the maps $[n_h] \rightarrow \text{colim } [n_h]$ are order-preserving maps. We then quotient out by the equivalence relation $x \sim y$ if $x \leq y$ and $y \leq x$. In particular, taking the smallest pre order implies that the colimit partial order will only have the transitive property over a finite set of relations. Explicitly, this means that two elements p, q in the colimit poset have a relation $p \leq q$ if and only if we can find a finite chain $p = p_0 \leq \dots \leq p_n = q$ so that $p_i \leq p_{i+1}$ in some $[n_h]$. Therefore there is an elementary exit path in some $\|\Delta^n\| \subseteq A$ from A_{p_i} to $A_{p_{i+1}}$, and concatenating over the finite chain, using the homotopy theoretic frontier condition to concatenate paths, gives the desired stratified spine in A . ■

Remark 9.2.1.3. In this proof, we considered colimits constructed in **POSet**, and saw that transitivity only holds over finite sets. In particular, the construction implies that the inclusion $i: \mathbf{POSet} \rightarrow k\mathbf{Top}$ does not preserve colimits.

We wish to relate cofibrant stratified spaces to the homotopically stratified sets of Quinn, which is achieved by showing they satisfy the tameness condition (as in Definition 5.1.0.13).

Remark 9.2.1.4. In Quinn's work on homotopically stratified sets [Qui88], the spaces are introduced by imposing pairwise conditions on the strata of the space. Quinn then proceeds to show that in the presence of both the tameness neighbourhood for pairs of strata, and the pairwise holink path fibration, these conditions extend to a closed union of strata. For a cofibrant stratified space in our setting, it is easier to show that the closed union of strata in a cofibrant space is stratified forward tame, which implies the tameness condition for pairs of strata.

Definition 9.2.1.5. For a simplicial set A , consider the stratified geometric realisation $\|A\|$. We define the *open star of a stratum* $\|A\|_p$, denoted by $(\|A\|_p)^*$, is the union of all the open simplices of $\|A\|$, whose closures have non-empty intersection with the stratum $\|A\|_p$. The *open star of a union of strata* is the union of the open stars of each stratum. This definition extends to absolute cell complexes in the obvious manner.

Example 9.2.1.6. Suppose that $\|A\| = \|\Delta^n\|$. If we consider the open star of the q stratum, $(\|\Delta^n\|_q)^*$, this is the open subset of $\|\Delta^n\|$ consisting of all the points $(t_0, \dots, t_n) \in \|\Delta^n\|$ such that $t_q \neq 0$. Similarly, if we consider $(\|\Delta^n\|_{\leq q})^*$, then this consists of all the points $(t_0, \dots, t_n) \in \|\Delta^n\|$ such that $t_p \neq 0$ for some $p \leq q$ in $[n]$.

Remark 9.2.1.7. Consider a cofibrant stratified space $X \rightarrow P$. If $p \leq q$ in P , then X_p is contained in any closed union of strata containing X_q . This follows because any closed union of strata containing X_q must contain $X_{\leq q}$, by the construction of the poset attached to a cofibrant stratified space. This can be strengthened in a cofibrant-fibrant space, as in Proposition 9.3.0.6.

Definition 9.2.1.8. Let X denote a stratified space with stratified subspace $Y \subseteq X$. Then, we say that a deformation retract $\mathbb{1}_Y: Y \xrightarrow{i} X \xrightarrow{r} Y$ of the stratified space X onto Y , with homotopy of underlying topological spaces $R: X \times \|\Delta^1\| \rightarrow X$ from $\mathbb{1}_X$ (at $1 \in \|\Delta^1\|$) to $r \circ i$ (at $0 \in \|\Delta^1\|$), is a *strong almost stratum preserving deformation retract* if R provides a lift in **Strat** for Figure 9.7, where c is the constant path at $i(y)$ for all $y \in Y$.

$$\begin{array}{ccc}
 Y & \xleftarrow{c} & \mathbf{strat}(\|\Delta^1\|, X) \\
 \downarrow i & \nearrow R & \downarrow (E_0, E_1) \\
 X & \xrightarrow{(i \circ r, \mathbb{1}_X)} & X \times X
 \end{array}$$

Figure 9.7: The lifting condition on the homotopy R , to define a strong almost stratum preserving deformation retract.

Remark 9.2.1.9. Explicitly, a lift R in Figure 9.7 says that for each $x \in X$ and $y \in Y$:

1. $R(x): \|\Delta^1\| \rightarrow X$;
2. $R(x)(0) = r(x) \in Y$;
3. $R(x)(1) = x$;
4. $R(y)(t) = y$ for all $t \in \|\Delta^1\|$;
5. $R(x)(t) \in X_{S_X(x)}$ for all $t \in (0, 1]$;
6. if $x, x' \in X_p$ then $R(y)(0), R(y')(0) \in Y_q$ for some q in the poset of Y .

Remark 9.2.1.10. Note that we are missing the requirement of an almost stratum preserving deformation that for $x \in X \setminus Y$, then $R(x, t) \in X \setminus Y$ for all $t > 0$. If Y is a closed union of strata of X , then this condition is implied by the lift R in Figure 9.7.

Remark 9.2.1.11. There is a weak version, in which we do not require r or R to be stratified. In the case that Y is a closed union of strata, the weak almost stratum preserving deformation retract coincides with the definition of an almost stratum preserving of Definition 5.1.0.8.

Lemma 9.2.1.12. *Consider an absolute cell complex A and a closed union of strata $Y \subseteq A$. Then, Y is stratified forward tame in A . In particular this means that A has tame strata.*

Proof. Suppose that A is stratified over the poset P , and let $Y \subseteq A$ be a closed union of strata. Consider $Y^* \subseteq A$; this is a union of the open simplices in A whose closures have non-empty intersection with Y . Each such open simplex comes equipped with a linear strong almost stratum preserving deformation retract onto Y . To define the strong almost stratum preserving deformation retract of Y^* onto Y , consider a stratified simplex $\|\Delta^n\| \subseteq A$ allowing us to consider $Y \cap \|\Delta^n\| = A_{\leq q} \cap \|\Delta^n\| \neq \emptyset$, where q is the greatest element of P whose stratum is contained in Y and such that the q -stratum intersects $\|\Delta^n\|$. By assumption Y is a closed union of strata, meaning that the strata arising from downwards closure of q are also contained in Y .

Working in barycentric coordinates of $\|\Delta^n\|$, write:

$$Y \cap \|\Delta^n\| = \left\{ (x_0, \dots, x_r, 0, \dots, 0) \mid \sum_i x_i = 1 \right\},$$

noting that when the final coordinate of x , for example $x_j \neq 0$, then x lies in A_j . This expression is valid by Remark 9.2.1.7, explaining the relationship between the poset and corresponding closed union of strata in a cofibrant space. This allows us to define the strong almost stratum preserving deformation retract R from any point $x = (x_0, \dots, x_n) \in Y^* \cap \|\Delta^n\|$ (at $t = 1$) onto a point in Y (at $t = 0$):

$$R(x, T) = \frac{(x_0, \dots, x_r, Tx_{r+1}, \dots, Tx_n)}{x_0 + \dots + x_r + Tx_{r+1} + \dots + Tx_n}.$$

This is a well-defined deformation retract because any $x \in Y^* \cap \|\Delta^n\|$ has some $p \leq r$ such that $t_p \neq 0$ (because the closure of each open simplex in Y^* intersects Y), and any point $x \in Y^*$ is contained in some open simplex of A . We have defined R in barycentric coordinates, hence

it is clear that R defines a strong almost stratum preserving deformation retract. Note that R is continuous because across any join of simplices; if an open simplex of the join is contained in Y^* , then the corresponding open stars in both simplices must be. By construction of the stratification on a colimit of naturally stratified simplices, the linear retracts are compatible.

The strong almost stratum preserving deformation retraction of Y^* onto Y is defined by carrying out the strong almost stratum preserving deformation retracts R in each open simplex of Y^* simultaneously. Therefore a closed union of strata Y is stratified forward tame in A . ■

Remark 9.2.1.13. To provide an illustrative example as to why we require a closed union of strata, and not simply a union of strata, consider $Y = \|\Delta^2\|_0 \cup \|\Delta^2\|_2 \subseteq \|\Delta^2\|$. There is an evident almost stratum preserving deformation retract of $\|\Delta^2\|_{\leq 1}$ onto $\|\Delta^2\|_0$. The problem arises when we attempt to the deformation retract to an almost stratum preserving deformation retract of the entire 2-simplex onto Y , which we are unable to perform continuously.

Lemma 9.2.1.14. *For a cofibrant space X and a closed union of strata $Y \subseteq X$, the sub stratified space Y is stratified forward tame in X .*

Proof. Suppose that the cofibrant space X is obtained as a retract of an absolute cell complex A . Lemma 9.2.1.12 shows that a closed union of strata $Y \subseteq A$ is stratified forward tame in A . We extend this to show that a closed union of strata $Y \subseteq X$ is stratified forward tame in X .

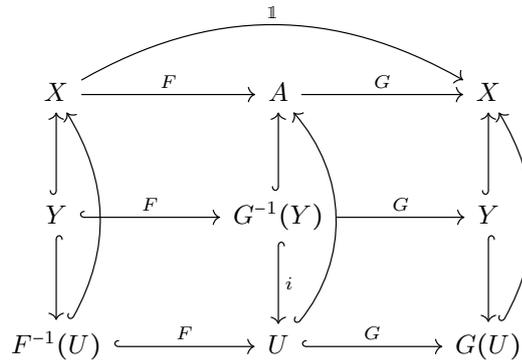


Figure 9.8: Exhibiting X as a retract of A , with almost stratum preserving deformation neighbourhood U of $G^{-1}(Y)$.

Let $Y \subseteq X$ be a closed union of strata, and exhibit X as a retract of the absolute cell complex A as in Figure 9.8. The pre-image of Y along G is a closed subset of A , and a union of strata because G is a stratified morphism.

The closed union of strata $G^{-1}(Y)$ allows us to apply Lemma 9.2.1.12. By this, we mean that can find a neighbourhood of tameness U with inclusion $i: G^{-1}(Y) \hookrightarrow U$ in A , and strong almost stratum preserving deformation retract given by the retract $r: U \rightarrow G^{-1}(Y)$, and a lift R as in Figure 9.9.

$$\begin{array}{ccc}
 G^{-1}(Y) & \xrightarrow{c} & \mathbf{strat}(\|\Delta^1\|, U) \\
 \downarrow i & \nearrow R & \downarrow (E_0, E_1) \\
 U & \xrightarrow{(r \circ i, \mathbb{1}_U)} & U \times U
 \end{array}$$

 Figure 9.9: The strong almost stratum preserving deformation retract of U onto $G^{-1}(Y)$.

We know that $F(Y) \subseteq G^{-1}(Y)$, hence the pre-image $F^{-1}(U)$ provides an open neighbourhood of Y in X . By assumption $G \circ F = \mathbb{1}_X$ and so it follows that $F^{-1}(U) = G(U)$. This allows us to denote the inclusion by $G(i): Y \hookrightarrow F^{-1}(U)$ and to define a retract by $G(r): F^{-1}(U) \rightarrow Y$. To complete the proof, we need to show that we have a lift in Figure 9.10.

$$\begin{array}{ccc}
 Y & \xrightarrow{c} & \mathbf{strat}(\|\Delta^1\|, F^{-1}(U)) \\
 \downarrow G(i) & \nearrow G(R) & \downarrow (E_0, E_1) \\
 F^{-1}(U) & \xrightarrow{(G(i) \circ G(r), \mathbb{1}_{F^{-1}(U)})} & F^{-1}(U) \times F^{-1}(U)
 \end{array}$$

 Figure 9.10: The strong almost stratum preserving deformation retract of $F^{-1}(U)$ onto Y .

Such a lift is constructed by $G(R)$, which can be seen by applying G to Figure 9.9. ■

Corollary 9.2.1.15. *Any cofibrant stratified space has tame strata.*

Proof. Consider an absolute cell complex $A \rightarrow P$ and $p \leq q$ in the poset P . Initially, we show that A has tame strata, meaning that A_p is tame in $A_p \cup A_q$. To do this, we follow Lemma 9.2.1.12. Initially, we construct a neighbourhood of tameness U of A_p in $A_p \cup A_q$ as $U = (A_p)^* \cap (A_p \cup A_q)$. Consider the strong almost stratum preserving deformation retract R constructed in Lemma 9.2.1.12; we need to check that this defines a strong almost stratum preserving deformation retract of U onto A_p . Working in barycentric coordinates inside a simplex $\|\Delta^n\|$ which intersects U , we define R at $x = (x_0, \dots, x_n) \in \|\Delta^n\|$ as follows:

$$R(x, T) = \frac{(x_0, \dots, x_k, Tx_{k+1}, \dots, Tx_n)}{x_0 + \dots + x_k + Tx_{k+1} + \dots + Tx_n}.$$

When $T = 1$, then $R(x, 1)$ is the identity map. If there is some $j \in [k+1, n]$ such that $t_j \neq 0$, then for any $T > 0$ the almost stratum preserving retract $R(x, T) \in U \setminus A_p$. If there is no such j , then $R(x, T)$ is the identity on the point x for all T . When $T = 0$, we are given a point in A_p because any point in $U \subseteq (A_p)^*$ has some $i \in [0, k]$ such that $t_i \neq 0$. Therefore we have shown that A_p is tame in $A_p \cup A_q$.

Corollary 9.2.1.15 follows by the above and Lemma 9.2.1.14, which shows that the property of a stratified space having tame strata is closed under retracts. ■

9.2.2 Cylinder Objects and Left Homotopy

Proposition 9.2.2.1. *For a cofibrant stratified space X , a cylinder object for X is given by $X \times [0, 1]$.*

Proof. To prove that for any cofibrant $X \in \mathbf{Strat}$, the stratified space $X \times [0, 1]$ is a cylinder object, we need to show that there is factorisation of the canonical fold map on X as follows:

$$\nabla_X = (\mathbb{1}_X, \mathbb{1}_X): X \amalg X \xrightarrow{(\partial_0, \partial_1)} \text{Cyl}(X) = X \times [0, 1] \xrightarrow{\sim} X,$$

where the map (∂_0, ∂_1) is a stratified cofibration and s is a stratified weak equivalence.

Define $s: X \times [0, 1] \rightarrow X$ by $s(x, t) = x$, and it is easy to see that this is a stratum preserving homotopy equivalence, thus a stratified weak equivalence by Proposition 8.1.1.6. For example, if we define $i: X \rightarrow X \times [0, 1]$ as $i(x) = (x, 0)$ then $s \circ i = \mathbb{1}_X$, and we can define a stratum preserving homotopy from $i \circ s$ to the identity on $(x, t) \in X \times [0, 1]$ by $H(x, T) = (x, Tt)$ for $T \in [0, 1]$. The homotopy H is stratum preserving because $[0, 1]$ is given a trivial stratification. Define the morphism ∂_0 to send $x \in X$ to $(x, 0) \in X \times [0, 1]$, and similarly ∂_1 sends $x \in X$ to $(x, 1) \in X \times [0, 1]$. The composite $s \circ (\partial_0, \partial_1)$ is the identity on each factor of X , hence provides a factorisation of the canonical fold map.

It remains to show that the map (∂_0, ∂_1) is a cofibration. The first step is to show that when $X = A$, for A an absolute cell complex, the map (∂_0, ∂_1) is a cofibration. To show this, recall that $[0, 1]$ can be obtained as a retract of $\mathcal{J}_{\leq 1}$ (as explained in Example 9.2.0.3). To show that the stratified morphism $A \amalg A \rightarrow A \times \mathcal{J}_{\leq 1}$ is a stratified cofibration, we use the dual statement of the pushout power axiom of Proposition 8.3.0.3. The dual statement, for our purposes, says that for a cofibrant stratified space A and cofibration $j: K \hookrightarrow L$ in simplicial sets, the induced map $\mathbb{1} \otimes j: A \times \|K\| \cong A \otimes K \rightarrow A \otimes L \cong A \times \|L\|$ is a stratified cofibration (for a proof of this statement see [GJ09, II Proposition 3.4]). The idea is to let $j: K = \mathcal{J}_0 = * \amalg * \hookrightarrow \mathcal{J}_{\leq 1} = L$, in which case the map $\mathbb{1} \otimes j$ is precisely the stratified morphism $A \amalg A \rightarrow A \times \|\mathcal{J}_{\leq 1}\|$. The stratified morphism $A \amalg A \rightarrow A \times [0, 1]$ is obtained as a retract of the map $A \amalg A \rightarrow A \times \|\mathcal{J}_{\leq 1}\|$, hence closure of cofibrations under retracts shows that it is also a stratified cofibration.

$$\begin{array}{ccc}
 X \amalg X & \xrightarrow{(\partial_0, \partial_1)} & X \times [0, 1] \\
 \downarrow (i, i) & & \downarrow (i, \mathbb{1}) \exists! \\
 A \amalg A & \xrightarrow{\epsilon_{\mathcal{C}}} & A \times [0, 1] \\
 \downarrow (r, r) & & \downarrow (r, \mathbb{1}) \exists! \\
 X \amalg X & \xrightarrow{(\partial_0, \partial_1)} & X \times [0, 1]
 \end{array}$$

(Large curved arrows labeled $\mathbb{1}$ indicate the retract relationship between the top and bottom rows.)

Figure 9.11: Obtaining (∂_0, ∂_1) as a retract of a stratified cofibration.

To complete the proof, consider a cofibrant stratified space X which is obtained as a retract

of an absolute cell complex A . The retract $X \xrightarrow{i} A \xrightarrow{r} X$ is used to exhibit $X \amalg X \rightarrow X \times [0, 1]$ as a retract of the cofibration $A \amalg A \hookrightarrow A \times [0, 1]$, demonstrated in Figure 9.11.

Therefore the inclusion $X \amalg X \hookrightarrow X \times [0, 1]$ is a stratified cofibration for any cofibrant stratified space X , because classes defined by a left lifting property are closed under retracts. ■

Remark 9.2.2.2. Notice that if X is fibrant, then so is $\text{Cyl}(X) = X \times [0, 1]$.

9.2.3 Modelling the Holink in a Cofibrant Stratified Space

In this subsection, we show that in a cofibrant stratified space, we can model the holink between strata by an almost stratum preserving neighbourhood of a stratum.

Definition 9.2.3.1. Let X be a stratified space and consider poset elements $p \leq q \in P$. We say that there is an *almost stratum preserving neighbourhood* of X_p in $X_p \cup X_q$ if there is an open neighbourhood U of X_p in X_q and an almost stratum preserving deformation retract of U onto X_p .

Remark 9.2.3.2. Notice that in a cofibrant stratified space, for poset elements $p \leq q \in P$, there is always an almost stratum preserving neighbourhood of X_p in $X_p \cup X_q$.

Proposition 9.2.3.3. Consider a stratified space X and suppose that for poset elements $p, q \in P$, there is an almost stratum preserving neighbourhood U of X_p in $X_p \cup X_q$, with almost stratum preserving deformation retract R . Then, the inclusion (induced point-wise) $R: U \hookrightarrow \underline{\text{strat}}(\|\Delta^1\|, X)_{p,q}$ is a weak homotopy equivalence.

Remark 9.2.3.4. The stratified space $\underline{\text{strat}}(\|\Delta^1\|, X)_{p,q} \subseteq \underline{\text{strat}}(\|\Delta^1\|, X)$ is the subspace of stratified paths in X which start in X_p and end in X_q . As a stratified space, this can be obtained as the pullback depicted in Figure 9.12.

$$\begin{array}{ccc}
 \underline{\text{strat}}(\|\Delta^1\|, X)_{p,q} & \dashrightarrow & \underline{\text{strat}}(\|\Delta^1\|, X) \\
 \downarrow & & \downarrow (E_0, E_1) \\
 X_p \times X_q & \hookrightarrow & X \times X
 \end{array}$$

Figure 9.12: The (trivially) stratified space of stratified paths in X from X_p to X_q .

Proof. The retraction R induces maps of homotopy groups $\pi_n(U) \rightarrow \pi_n(\underline{\text{strat}}(\|\Delta^1\|, X)_{p,q})$. To define an inverse on homotopy groups, consider a homotopy class $[F] \in \pi_n(\underline{\text{strat}}(\|\Delta^1\|, X)_{p,q})$, for which we let $F: S^n \times \|\Delta^1\| \rightarrow X$ be a representative (thinking of S^n as trivially stratified). By assumption, we have $F(S^n, \{0\}) \subseteq X_p$ and $F(S^n, \{t\}) \subseteq X_q$ for any $t \in (0, 1]$. In particular, there exists some $\epsilon > 0$ such that for any $0 \leq t \leq \epsilon$ we have $F(S^n, t) \subseteq U$. Thus $F(-, \epsilon): S^n \rightarrow U$ defines a homotopy class in $\pi_n(U)$. This map is well defined on homotopy classes, because

a homotopy between maps in $\mathbf{strat}(\|\Delta^1\|, X)_{p,q}$ will restrict to define a homotopy between representatives of the homotopy classes in U .

Consider the induced map composed with the inverse we constructed:

$$\pi_n(U) \rightarrow \pi_n(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}) \rightarrow \pi_n(U).$$

The composite of these arrows gives the identity on $\pi_n(U)$, because choosing a representative G of a homotopy class of $\pi_n(U)$ is sent to the homotopy class of $R(G(S^n), -)$. In this case, we consider a representative of the homotopy class, which without loss of generality can assume to be $R(G(S^n), -)$. In this case we can let $\epsilon = 1$, giving the same homotopy class in U that we started with. This shows that $\pi_n(U)$ is a retract of $\pi_n(\mathbf{strat}(\|\Delta^1\|, X)_{p,q})$.

To complete the proof, consider the other composite:

$$\pi_n(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}) \rightarrow \pi_n(U) \rightarrow \pi_n(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}).$$

This composite is equal to the identity map. To see this, let $[F] \in \pi_n(\mathbf{strat}(\|\Delta^1\|, X)_{p,q})$, and apply the retract R to a representative $F(-, \epsilon): S^n \rightarrow U$. This gives a family of stratified paths which are stratum preserving homotopic to the restriction of F to the $\langle 0, \epsilon \rangle$ segment of stratified paths. This family is stratum preserving homotopic to $F: S^n \times \|\Delta^1\| \rightarrow X$ via the stratum preserving homotopy $H((x, t), T) = F(x, (\frac{T+\epsilon}{1+T\cdot\epsilon})t)$. ■

Remark 9.2.3.5. Proposition 9.2.3.3 holds in greater generality; rather than restricting to a single stratum that our stratified paths must begin in, we are able to consider an arbitrary closed union. In this case, we must extend the definition of an almost stratum preserving neighbourhood to closed unions of strata, which we do in the obvious manner using (weak) almost stratum preserving deformation retracts. In this case, the map R may not be a stratified morphism but will still provide the necessary weak homotopy equivalence.

One particular example of this would be if we replaced X_p by the closed union of strata $X_{\leq p}$ in a cofibrant stratified space X . In this case, we know that such an almost stratum preserving deformation retract exists (by Lemma 9.2.1.14), and the almost stratum preserving neighbourhood can be taken as the open star of this closed union of strata in X_q for some $p \leq q$.

9.3 Cofibrant-Fibrant Stratified Spaces

Motivated by Quinn's notion of homotopically stratified sets, we make the following definition, which translates Definition 5.1.0.13 from filtered spaces to stratified spaces.

Definition 9.3.0.1. A stratified space $X \rightarrow P$ is said to be a *homotopically stratified space*, if for any $p \leq q$ in P the following two conditions hold:

1. the inclusion $X_p \hookrightarrow X_p \cup X_q$ is tame;
2. and the start point evaluation map $E_0: \text{Holink}(X_p \cup X_q, X_p) \rightarrow X_p$ is a Serre fibration.

Theorem 9.3.0.2. *Any cofibrant-fibrant object in \mathbf{Strat} is a homotopically stratified space.*

Proof. Recall that Corollary 9.2.1.15 shows that for a cofibrant stratified space X , the inclusion of $X_p \hookrightarrow X_p \cup X_q$ is tame. Corollary 9.1.0.7, shows that for a fibrant stratified space X there is a fibration $E_0: \text{Holink}(X_p \cup X_q, X_q) \rightarrow X_p$. ■

Example 9.3.0.3. Note that the converse is not true; there are homotopically stratified sets which are not cofibrant-fibrant objects in our model structure. An example is the Warsaw circle given the trivial stratification; this is a homotopically stratified space, however it is not a retract of a CW-complex and therefore is not cofibrant in \mathbf{Strat} .

Proposition 9.3.0.4. *The closure of a stratum in a cofibrant stratified space is equal to its path-closure.*

Proof. Consider a cofibrant stratified space $X \rightarrow P$ with poset elements $p, q \in P$. By definition, we see that $\text{PCI}(X_q) \subseteq \overline{X_q}$. To prove the converse, suppose there is a point x in the closure of X_q ; we will show that x must lie in $\text{PCI}(X_q)$. Within any stratified space, the closure of a stratum X_q is contained in $X_{\leq q}$, so we can assume $x \in X_p$ for some $p \leq q$ in P . The cofibrant stratified space X is a retract of some absolute cell complex A , as shown in Figure 9.13.

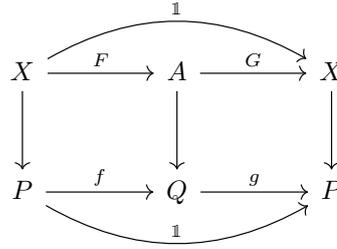


Figure 9.13: Exhibiting $X \rightarrow P$ as a retract of the absolute cell complex $A \rightarrow Q$.

Now we have $x \in \overline{X_q}$, and by continuity of F on the underlying topological spaces, we can write $F(x) \in F(\overline{X_q}) \subseteq \overline{F(X_q)} \subseteq \overline{A_{f(q)}}$ (the first inclusion follows because F is continuous [Wil70, Theorem 7.2]). It is straightforward to see that we have $\overline{A_q} = \text{PCI}(\|A\|_q)$. This means that $F(x) \in \text{PCI}(A_{f(q)})$, and so there is a path $\gamma: \|\Delta^1\| \rightarrow A$ satisfying $\gamma(0) = F(x)$ and $\gamma((0, 1]) \subseteq A_{f(q)}$. By post-composing γ with G we have a path $G \circ \gamma: \|\Delta^1\| \rightarrow X$ so that $G \circ \gamma(0) = G \circ F(x) = x$ and $G \circ \gamma((0, 1]) \subseteq G(A_{f(q)}) = X_q$ (equality follows because G and g are surjective as set maps). Therefore $x \in \text{PCI}(X_q)$ and $\text{PCI}(X_q) = \overline{X_q}$. ■

Corollary 9.3.0.5. *Any cofibrant-fibrant stratified space satisfies the frontier condition.*

Proof. This follows from Proposition 9.1.1.3, Lemma 9.2.1.1, and Proposition 9.3.0.4. ■

The following proposition shows that if a cofibrant stratified space is also fibrant, then a relation in the poset directly relates to the topology of the underlying topological space, i.e. is detectable by a stratified path. This is a strengthening of Remark 9.2.1.7.

Proposition 9.3.0.6. *For a cofibrant-fibrant $X \rightarrow P$ in **Strat**, there is a relation $p \leq q$ in P if and only if $X_p \subseteq \overline{X_q}$.*

Proof. To begin the proof, let $p \leq q$ in P . Lemma 9.2.1.2 says that we can find a finite chain $p = p_0 \leq p_1 \leq \dots \leq p_n = q$, so that for each i there is an elementary exit path from X_{p_i} to $X_{p_{i+1}}$. For each successive triple, starting with p_0, p_1 and p_2 , we let these paths define an inner horn $\|\Lambda_1^2\| \rightarrow X$. Using the fibrancy of X , we can fill this horn, which in particular gives us an elementary exit path from X_{p_0} to X_{p_2} which is homotopic through exit paths to the inner horn in X . Proceeding inductively (importantly using the finite chain of poset elements), we arrive at an elementary exit path from X_p to X_q . This shows that $X_p \cap \overline{X_q} \neq \emptyset$. The frontier condition satisfied by a cofibrant-fibrant stratified space (Corollary 9.3.0.5) implies that $X_p \subseteq \overline{X_q}$.

To show the converse, suppose that for p, q in P , we have $X_p \subseteq \overline{X_q}$. By Proposition 9.3.0.4 there is a stratified path $\gamma \|\Delta^1\| \rightarrow X$ such that $\gamma(\{0\}) \in X_p$ and $\gamma(\{1\}) \in X_q$. In particular there is a poset map $[1] \rightarrow P$ such that $\{0\} \rightarrow p$ and $\{1\} \rightarrow q$, hence $p \leq q$ in P . ■

Theorem 9.3.0.7. *A weak equivalence between cofibrant-fibrant stratified spaces is a stratum preserving homotopy equivalence.*

Proof. We first show that a homotopy equivalence in **Strat** is a stratum preserving homotopy equivalence, using our choice of path (Proposition 8.1.2.5) and cylinder (Proposition 9.2.2.1) objects. A choice of cylinder object naturally gives a notion of left homotopy between maps $F, G: X \rightarrow Y$, which is an equivalence relation for a cofibrant object X . If a left homotopy exists, it is given as the map H_l in the commutative diagram of Figure 9.14. The existence of a left homotopy is equivalent to the existence of a stratum preserving homotopy from F to G .

$$\begin{array}{ccc}
 & X \times [0, 1] & \\
 (\partial_0, \partial_1) \nearrow & & \searrow H_l \\
 X \amalg X & \xrightarrow{(F, G)} & Y
 \end{array}$$

Figure 9.14: A left homotopy in **Strat** from F to G .

We have the dual notion of right homotopy, which is an equivalence relation for a fibrant stratified space Y . This is expressed as the existence of the map H_r in Figure 9.15.

$$\begin{array}{ccc}
 & \mathbf{strat}([0, 1], Y) & \\
 H_r \nearrow & & \searrow (D_0, D_1) \\
 X & \xrightarrow{(F, G)} & Y \times Y
 \end{array}$$

Figure 9.15: A right homotopy in **Strat** from F to G .

The existence of such a right homotopy H_r is again equivalent to F and G being stratum preserving homotopic, because H_r is adjoint to a stratified map $X \times [0, 1] \rightarrow Y$. Between cofibrant-fibrant stratified spaces, left homotopy and right homotopy coincide (we can apply Corollary 3.6.1.9 because for a fibrant space both $\text{Cyl}(X)$ and $\text{Path}(X)$ are fibrant), giving a notion of homotopy equivalence which is stratum preserving homotopy equivalence. The proof follows as a formal consequence of Whitehead's Theorem in a model category (Theorem 3.6.1.10). We do not have a model category, however can use Whitehead's Theorem because for a fibrant stratified space X , then $X \coprod X$, $\text{Cyl}(X) = X \times [0, 1]$, $X \times X$ and $\text{Path}(X) = \mathbf{strat}([0, 1], X)$ are fibrant. In particular, this allows us to construct the required lifts (and in the final stage of the proof factorisation) for the proof of Whitehead's Theorem, using the lifting axiom MC3 (and in the final stage, the factorisation axiom MC4) from our model structure on fibrant stratified spaces. ■

9.4 Weak Equivalences of Stratified Spaces

Remark 9.4.0.1. To gain an understanding of weak equivalences between stratified spaces, we begin by translating the Fundamental Theorem for Quasi-Categories to the language of stratified spaces. This says that a map $F: X \rightarrow Y$ of fibrant stratified spaces is a stratified weak equivalence if F is a bijection on posets so that under F each path-connected component of a stratum of Y is mapped onto, and the induced map $SS(F): SS(X)(x, y) \rightarrow SS(Y)(F(x), F(y))$ is a weak homotopy equivalence for all $x, y \in X$. We will return to this result after Proposition 9.4.0.2, to see that we do not need to consider simplicial mapping spaces. We also have Whitehead's Theorem, the use of which is justified in the proof of Theorem 9.3.0.7, which says that any weak equivalence between cofibrant-fibrant stratified spaces is a stratum preserving homotopy equivalence.

When working with simplicial sets, there is no natural composition of simplicial mapping spaces within a simplicial set. This can be dealt with in various ways; an example is the homotopy coherent nerve (as described by Lurie in [Lur14, §1.1.5]) which takes a simplicial set to a simplicial category, which admits function complexes with natural composition maps. When working with stratified spaces, we do not have this issue.

Proposition 9.4.0.2. *For any stratified space X and any $x, y \in X$, we have an isomorphism of simplicial sets $SS(X)(x, y) \cong \text{Sing}(E_0^{-1}(x) \cap E_1^{-1}(y))$. In particular, the simplicial mapping space $SS(X)(x, y)$ is always a Kan complex.*

Proof. We note that $E_0^{-1}(x) \cap E_1^{-1}(y)$ is a one stratum stratified space, which implies that $SS(E_0^{-1}(x) \cap E_1^{-1}(y)) = \text{Sing}(E_0^{-1}(x) \cap E_1^{-1}(y))$. By comparison with defining diagram for $SS(X)(x, x')$ (similar to Figure 4.5), we will use the fact that pullbacks are defined up to isomorphism to show that $SS(X)(x, y) \cong \text{Sing}(E_0^{-1}(x) \cap E_1^{-1}(y))$. By definition, there is a pullback diagram in \mathbf{Strat} as shown in Figure 9.16.

$$\begin{array}{ccc}
 E_0^{-1}(x) \cap E_1^{-1}(y) = X(x, y) & \dashrightarrow & \mathbf{strat}(\|\Delta^1\|, X) \\
 \downarrow & \lrcorner & \downarrow (E_0, E_1) \\
 * & \xrightarrow{(x, y)} & X \times X
 \end{array}$$

 Figure 9.16: Defining the stratified mapping space in X from x to y .

Applying $SS(-)$ which preserves limits and the isomorphism of Lemma 7.3.0.1, which states that $SS(\mathbf{strat}(\|A\|, X)) \cong \mathbf{sset}(A, SS(X))$, it follows that $SS(E_0^{-1}(x) \cap E_1^{-1}(y))$ and $SS(X)(x, y)$ fit into the same pullback diagram. Hence they are isomorphic. ■

Remark 9.4.0.3. Continuing from Remark 9.4.0.1, we use the isomorphism of Proposition 9.4.0.2 $SS(X(x, y)) = \text{Sing}(E_0^{-1}(x) \cap E_1^{-1}(y))$. Therefore, we can detect whether a map $F: X \rightarrow Y$ of fibrant stratified spaces is a weak equivalence by checking if F is a bijection on posets so that under F each path-connected component of a stratum of Y is mapped onto, and if the map $X(x, y) \rightarrow Y(F(x), F(y))$ is a weak homotopy equivalence for all $x, y \in X$.

Proposition 9.4.0.4. *Consider a fibrant stratified space $X \rightarrow P$. Then for any $p \leq q$ in P , the stratified morphisms E_0 and E_1 depicted in Figure 9.17 are stratified fibrations, and hence Serre fibrations.*

$$X_p \xleftarrow{E_0} \mathbf{strat}(\|\Delta^1\|, X)_{p,q} \xrightarrow{E_1} X_q$$

 Figure 9.17: Proposition 9.4.0.4 claims that the maps E_0 and E_1 are Serre fibrations.

Recall that the stratified space $\mathbf{strat}(\|\Delta^1\|, X)_{p,q}$ is a single stratum subspace of $\mathbf{strat}(\|\Delta^1\|, X)$ consisting of stratified paths $\gamma: \|\Delta^1\| \rightarrow X$ such that $\gamma(\{0\}) \in X_p$ and $\gamma(\{1\}) \in X_q$.

Proof. The cleanest proof of this result is to apply Proposition 8.2.3.5 which shows that the map $(E_0, E_1): \mathbf{strat}(\|\Delta^1\|, X) \rightarrow \mathbf{strat}(* \amalg *, X) \cong X \times X$ is a stratified fibration because it is induced by the cofibration $\{0\} \amalg \{1\} \hookrightarrow \|\Delta^1\|$. Pullback the fibration (E_0, E_1) along the inclusion map $X_p \times X_q \hookrightarrow X \times X$, to give the stratified fibration $(E_0, E_1): \mathbf{strat}(\|\Delta^1\|, X)_{p,q} \rightarrow X_p \times X_q$ between Kan complexes. The projections $\text{pr}_1: X_p \times X_q \rightarrow X_p$ and $\text{pr}_2: X_p \times X_q \rightarrow X_q$ are fibrations because X_p and X_q are fibrant, completing the proof of Proposition 9.4.0.4. ■

We can now consider stratified weak equivalences between fibrant stratified spaces, and are able to characterise such maps if the underlying poset map is an isomorphism. Vital in the proof of this result is the fundamental theorem of quasi-categories (Theorem 4.6.0.13). It should be noted that the motivation for Theorem 9.4.0.5 was to provide an analogue of David Miller's Theorem (see [Mil13, Theorem 6.3], stated as Theorem 5.1.0.28) for detecting stratum preserving homotopy equivalences between homotopically stratified sets.

Theorem 9.4.0.5. *Consider a stratified morphism $F: X \rightarrow Y$ of fibrant stratified spaces such that the underlying poset map $f: P \rightarrow Q$ is a bijection. Then, F is a weak equivalence in **Strat** if and only if F induces weak homotopy equivalences between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and the strata of $\mathbf{strat}(\|\Delta^1\|, Y)$.*

Remark 9.4.0.6. For this theorem, whenever we refer to a stratum we will assume that it is path-connected. This is for simplicity when working with homotopy groups; the result does hold when strata are not path-connected.

Proof. Initially, we will prove that if F induces weak homotopy equivalences between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and of $\mathbf{strat}(\|\Delta^1\|, Y)$ then F is a weak equivalence in **Strat**. To do this, consider a stratified morphism F which induces a bijection on posets, and induces weak homotopy equivalences between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and the strata of $\mathbf{strat}(\|\Delta^1\|, Y)$. We can prove the following claim under these assumptions, which shows that our hypothesis is equivalent to requiring F to induce weak equivalences of the strata and holinks between X and Y .

Claim 9.4.0.7. The weak homotopy equivalences between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and of $\mathbf{strat}(\|\Delta^1\|, Y)$ induced by F imply that F induces weak homotopy equivalences between the strata of X and the strata of Y .

Proof. The mapping spaces $\mathbf{strat}(\|\Delta^1\|, X_p)$ and $\mathbf{strat}(\|\Delta^1\|, Y_{f(p)})$ are single stratum subspaces of $\mathbf{strat}(\|\Delta^1\|, X)$ and $\mathbf{strat}(\|\Delta^1\|, Y)$ respectively. Applying Lemma 7.3.0.1 allows us to consider $\mathbf{sset}(\Delta^1, \text{Sing}(X_p))$. The inclusion $\{0\}: * \hookrightarrow \Delta^1$ is a weak homotopy equivalence, and the Quillen model structure on simplicial sets is enriched over itself, hence the start point evaluation maps $E_0: \mathbf{sset}(\Delta^1, \text{Sing}(X_p)) \rightarrow \text{Sing}(X_p)$ and $E_0: \mathbf{sset}(\Delta^1, \text{Sing}(Y_{f(p)})) \rightarrow \text{Sing}(Y_{f(p)})$ are weak homotopy equivalences. The 2-out-of-3 property satisfied by weak homotopy equivalences implies that $\text{Sing}(F): \text{Sing}(X_p) \rightarrow \text{Sing}(Y_{f(p)})$ is a weak homotopy equivalence. By [Hov98a, Corollary 1.3.16], the Quillen equivalence between topological spaces and simplicial sets implies that $\text{Sing}(-)$ reflects weak equivalences between Kan complexes. Therefore $F: X_p \rightarrow Y_{f(p)}$ is a weak homotopy equivalence. ■

Let $x \in X_p$ and $x' \in X_q$ be basepoints. By assumption X and Y are fibrant, hence we have Serre fibrations E_0, E_1 as in Figure 9.17. There is a long exact sequence of homotopy groups which naturally arises from any Serre fibration (see for example [Hat10, pp.375-384] or [McC01, Corollary 4.31]); in particular we will use E_0 (a fibration by Proposition 9.4.0.4) first to give the long exact sequence of homotopy groups shown in Figure 9.18. Choice of basepoint $x \in X_p$ gives a corresponding topological space $E_0^{-1}(x) \subset \mathbf{strat}(\|\Delta^1\|, X)_{p,q}$ for any $p \leq q$ in P ; the subspace of stratified paths in X which start at $x \in X_p$ and end in the stratum X_q . We use the functoriality properties of homotopy groups to covariantly induce the group homomorphisms F_* , as pictured in Figure 9.18. Claim 9.4.0.7 and the fact that F induces weak

homotopy equivalences between strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and the strata of $\mathbf{strat}(\|\Delta^1\|, Y)$ gives us the isomorphisms pictured in the long exact sequence.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \pi_i(E_0^{-1}(x)) & \longrightarrow & \pi_i(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}) & \xrightarrow{(E_0)_*} & \pi_i(X_p) \longrightarrow \dots \\
 & & \downarrow F_* & & \cong \downarrow F_* & & \cong \downarrow F_* \\
 \dots & \longrightarrow & \pi_i(E_0^{-1}(F(x))) & \longrightarrow & \pi_i(\mathbf{strat}(\|\Delta^1\|, Y)_{f(p),f(q)}) & \xrightarrow{(E_0)_*} & \pi_i(Y_{f(p)}) \longrightarrow \dots
 \end{array}$$

Figure 9.18: The long exact sequence of homotopy groups arising from the Serre fibration E_0 .

An application of the Five Lemma to this long exact sequence shows that as well as the weak homotopy equivalences indicated, the morphism F also induces weak homotopy equivalences of the form $F_*: E_0^{-1}(x) \rightarrow E_0^{-1}(F(x))$ for all $x \in X_p$.

Now we can show that $E_1: E_0^{-1}(x) \rightarrow X_q$ is also a Serre fibration. This follows by a similar argument to the proof of Proposition 9.4.0.4; initially Theorem 9.1.0.3 shows us that $SS(E_1)$ is a right Kan fibration, which has base a Kan complex, therefore is a Kan fibration. Use of the adjunctions shows that E_1 is a Serre fibration.

Now consider the long exact sequence of homotopy groups arising from this Serre fibration; the situation we have is depicted in Figure 9.19.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \pi_i(E_0^{-1}(x) \cap E_1^{-1}(x')) & \longrightarrow & \pi_i(E_0^{-1}(x)) & \xrightarrow{(E_1)_*} & \pi_i(X_q) \longrightarrow \dots \\
 & & \downarrow F_* & & \cong \downarrow F_* & & \cong \downarrow F_* \\
 \dots & \longrightarrow & \pi_i(E_0^{-1}(F(x)) \cap E_1^{-1}(F(x'))) & \longrightarrow & \pi_i(E_0^{-1}(F(x))) & \xrightarrow{(E_1)_*} & \pi_i(Y_{f(q)}) \longrightarrow \dots
 \end{array}$$

Figure 9.19: The long exact sequence of homotopy groups arising from the Serre fibration E_1 .

In this case, the fiber over a point $x' \in X_q$ will be the space $E_0^{-1}(x) \cap E_1^{-1}(x')$, of stratified paths in X from x to x' , and the Five Lemma applied to Figure 9.19 shows that we have a weak homotopy equivalence:

$$\mathbf{strat}(\|\Delta^1\|, X)_{p,q} \supseteq E_0^{-1}(x) \cap E_1^{-1}(x') \rightarrow E_0^{-1}(F(x)) \cap E_1^{-1}(F(x')) \subseteq \mathbf{strat}(\|\Delta^1\|, Y)_{f(p),f(q)},$$

for all $x \in X_p, x' \in X_q$ and $p \leq q$ in P .

In the Quillen model structure on **Top**, every topological space is fibrant; therefore the weak equivalence:

$$E_0^{-1}(x) \cap E_1^{-1}(x') \rightarrow E_0^{-1}(F(x)) \cap E_1^{-1}(F(x'))$$

is a weak equivalence between fibrant objects. This allows us to use Ken Brown's Lemma, which in the presence of a Quillen adjunction says that the right Quillen adjoint preserves

weak equivalences between fibrant objects. Therefore we have a weak homotopy equivalence:

$$\text{Sing}(E_0^{-1}(x) \cap E_1^{-1}(x')) \rightarrow \text{Sing}(E_0^{-1}(F(x)) \cap E_1^{-1}(F(x')))$$

in \mathbf{sSet} . Applying Proposition 9.4.0.2 shows that we have a weak homotopy equivalence of Kan complexes:

$$SS(X)(x, x') \rightarrow SS(Y)(F(x), F(x'))$$

for all $x \in X_p$ and $x' \in X_q$.

Notice that this is condition (1) of the Fundamental Theorem of Quasi-Categories (detecting weak categorical equivalences between quasi-categories; see Theorem 4.6.0.13). To show that $SS(F)$ is a weak categorical equivalence we also need to show that it is essentially surjective. To show this, recall that by assumption the underlying map of F is a bijection on posets. Consider $\tau_1(SS(Y))$, and an object $y \in \tau_1(SS(Y))$. We need to show that y is isomorphic to an object in the image of $\tau_1(SS(F))$. We can think of y as a point in the stratum Y_q for some $q \in Q$. Because the underlying map of F on posets is a bijection and we have assumed that all strata are path-connected, there will be a stratified path from y to some point in $F(X)$ which remain in the Y_q stratum. This path will have an inverse path because it is stratum preserving. This shows that y is isomorphic to an object in the image of $\tau_1(SS(F))$, showing that $SS(F)$ is essentially surjective. Therefore Theorem 4.6.0.13 shows that $SS(F)$ is a weak categorical equivalence, and hence F is a stratified weak equivalence.

We now prove the converse. Suppose that F is a stratified weak equivalence between fibrant stratified spaces. Then by definition $SS(F)$ is a weak categorical equivalence between quasi-categories. In particular, Theorem 4.6.0.13 implies that $SS(F)$ induces a weak homotopy equivalence of Kan complexes $SS(X)(x, x') \rightarrow SS(Y)(F(x), F(x'))$ for all pairs of points $x, x' \in X$. Note that Proposition 9.4.0.2 shows us that we have an isomorphism $SS(X)(x, x') \cong \text{Sing}(E_0^{-1}(x) \cap E_1^{-1}(x'))$. Now recall that $|-| \dashv \text{Sing}(-)$ is actually a Quillen equivalence; in particular this implies that the right adjoint $\text{Sing}(-)$ reflects weak equivalences between fibrant objects [Hov98a, Corollary 1.3.16]. Therefore we have the following weak homotopy equivalence:

$$\mathbf{strat}(\|\Delta^1\|, X) \supseteq E_0^{-1}(x) \cap E_1^{-1}(x') \rightarrow E_0^{-1}(F(x)) \cap E_1^{-1}(F(x')) \subseteq \mathbf{strat}(\|\Delta^1\|, Y).$$

Note that we can consider the particular case in which $x = x'$ in X_p ; in this case we see that the trivially stratified path space $E_0^{-1}(x) \cap E_1^{-1}(x')$ is canonically isomorphic to ΩX_p . We also recall the well known isomorphism of homotopy groups $\pi_i(\Omega X) \cong \pi_{i+1}(X)$, and therefore it follows that F induces an isomorphism between the homotopy groups $\pi_i(\Omega X_p)$ and $\pi_i(\Omega Y_{f(p)})$ for $i \geq 1$. We have an isomorphism in the case that $i = 0$, which follows because the strata of X and Y are both non-empty and path-connected, and by assumption the underlying map of F on posets is a bijection.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \pi_i(E_0^{-1}(x) \cap E_1^{-1}(x')) & \longrightarrow & \pi_i(E_0^{-1}(x)) & \xrightarrow{(E_1)_*} & \pi_i(X_q) \longrightarrow \dots \\
 & & \cong \downarrow F_* & & \downarrow F_* & & \cong \downarrow F_* \\
 \dots & \longrightarrow & \pi_i(E_0^{-1}(F(x)) \cap E_1^{-1}(F(x'))) & \longrightarrow & \pi_i(E_0^{-1}(F(x))) & \xrightarrow{(E_1)_*} & \pi_i(Y_{f(q)}) \longrightarrow \dots
 \end{array}$$

 Figure 9.20: The long exact sequence of homotopy groups arising from the Serre fibration E_1 .

Consider the long exact sequence arising from the E_1 Serre fibration shown in Figure 9.20; this time we have an isomorphism of homotopy groups on the left hand side and the right hand side. An application of the Five Lemma shows us that we have an isomorphism between the i -th homotopy groups of $E_0^{-1}(x)$ and $E_0^{-1}(F(x))$ for all $i \geq 1$. It is automatic for $i = 0$ that we have an injective map of homotopy groups; this follows from the proof of injectivity contained within the Five Lemma, however we are not guaranteed that this map will be surjective. If however for any $p \in P$, the stratified path space $E_0^{-1}(x)$ was not empty, then we would.

We now construct the long exact sequence of homotopy groups associated to the E_0 Serre fibration, giving us the long exact sequence pictured in Figure 9.21. Note that on the left hand side we have indicated isomorphisms induced by F between $\pi_i(E_0^{-1}(x))$ and $\pi_i(E_0^{-1}(F(x)))$; this is only true for $i \geq 1$, and for $i = 0$, we have an injective map of pointed sets.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \pi_i(E_0^{-1}(x)) & \longrightarrow & \pi_i(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}) & \xrightarrow{(E_0)_*} & \pi_i(X_p) \longrightarrow \dots \\
 & & \cong \downarrow F_* & & \downarrow F_* & & \cong \downarrow F_* \\
 \dots & \longrightarrow & \pi_i(E_0^{-1}(F(x))) & \longrightarrow & \pi_i(\mathbf{strat}(\|\Delta^1\|, Y)_{f(p),f(q)}) & \xrightarrow{(E_0)_*} & \pi_i(Y_{f(p)}) \longrightarrow \dots
 \end{array}$$

 Figure 9.21: The long exact sequence of homotopy groups arising from the Serre fibration E_0 .

Again we apply the Five Lemma which shows that F induces isomorphisms between the homotopy groups of $\mathbf{strat}(\|\Delta^1\|, X)_{p,q}$ and $\mathbf{strat}(\|\Delta^1\|, Y)_{f(p),f(q)}$ for $i \geq 2$. Consider the long exact sequence stage where $i = 1$ and the induced map on fundamental groups:

$$F_*: \pi_1(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}) \rightarrow \pi_1(\mathbf{strat}(\|\Delta^1\|, Y)_{f(p),f(q)}).$$

In this case, we also have an isomorphism between the fundamental groups, because for $i = 0$ we have an injective map of pointed sets $F_*: \pi_0(E_0^{-1}(x)) \rightarrow \pi_0(E_0^{-1}(F(x)))$ and this is all that is required to apply the Five Lemma.

Consider now the case of $i = 0$; we are left with showing that we have a bijection induced by F of the form:

$$F_*: \pi_0(\mathbf{strat}(\|\Delta^1\|, X)_{p,q}) \rightarrow \pi_0(\mathbf{strat}(\|\Delta^1\|, Y)_{f(p),f(q)}).$$

Importantly, we note that the underlying map on posets f is a bijection, and we have required

that the strata of stratified spaces be path connected. To show that F_* is surjective, consider a path component of $\mathbf{strat}(\|\Delta^1\|, Y)_{f(p), f(q)}$ with representative $\gamma: \|\Delta^1\| \rightarrow Y$. We can use the fact that $SS(F)$ is essentially surjective to find points $F(x)$ and $F(x')$ in the image of X under F , such that we can extend γ to an exit path in Y defined as the composite of a stratum preserving path from $F(x)$ to $\gamma(0)$, then carrying out γ ending at $\gamma(1)$, and finally extending this by a stratum preserving path to $F(x')$. Using the fact that Y is fibrant, we can fill a horn $\|\Lambda_1^2\|$ defined sending the $\langle 0, 1 \rangle$ edge to by the path from $F(x)$ to $\gamma(0)$ and the $\langle 1, 2 \rangle$ edge to the extension of the path γ to $F(x')$. The filler $\langle 0, 2 \rangle$ face give us a path γ' from $F(x)$ to $F(x')$, which lies in the same path component as γ of $\mathbf{strat}(\|\Delta^1\|, Y)_{f(p), f(q)}$. In Figure 9.22, we picture the interior of the filler simplex as being spanned by the dashed lines and filler face is the path γ' . The diagram is drawn in this way, so that it evidently indicates a stratum preserving homotopy from γ to γ' .

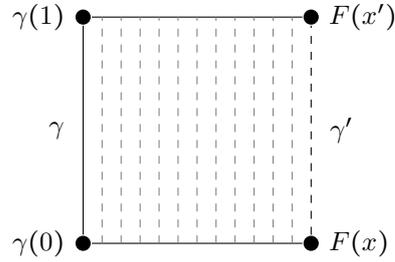


Figure 9.22: The filler gives a path from γ to γ' .

We have assumed that F is a stratified weak equivalence, hence there is a homotopy equivalence of Kan Complexes $SS(F): SS(X)(x, x') \rightarrow SS(Y)(F(x), F(x'))$. We can apply the homotopy inverse of $SS(F)$ to γ' giving a path $\overline{\gamma'}$ in $SS(X)(x, x')$, and then $SS(F)$ to the path $\overline{\gamma'}$ to get a path $\overline{\overline{\gamma'}}$ which is homotopic to γ' in $SS(Y)(F(x), F(x'))$. This means that we can find a stratum preserving homotopy from γ' to $\overline{\overline{\gamma'}}$, so γ and $\overline{\overline{\gamma'}}$ lie in the same path component of $\mathbf{strat}(\|\Delta^1\|, Y)_{f(p), f(q)}$. In particular, we have found an element $\overline{\overline{\gamma'}}$ of $\mathbf{strat}(\|\Delta^1\|, X)_{p, q}$ whose path component is sent to that of γ under F_* .

Finally we need to show that F_* is injective on path components. To do this, consider two representatives $f, g: \|\Delta^1\| \rightarrow X$ of distinct path components of $\mathbf{strat}(\|\Delta^1\|, X)_{p, q}$. As X is fibrant we may assume that $f(0) = g(0)$ and $f(1) = g(1)$. Assume that both f and g are sent to the same path component of $\mathbf{strat}(\|\Delta^1\|, Y)_{f(p), f(q)}$ under $SS(F)$; this implies that there is a stratum preserving homotopy between their images in Y . When we apply the homotopy inverse to $SS(F)$ we arrive at two paths which are stratum preserving homotopic to f and g respectively. However the stratum preserving homotopy between the images of f and g in Y gives a homotopy after applying the homotopy inverse to $SS(F)$; this shows that f and g must be contained in the same path component of $\mathbf{strat}(\|\Delta^1\|, X)_{p, q}$.

Therefore if F is a weak equivalence between fibrant stratified spaces in \mathbf{Strat} and the underlying poset map is a bijection, then F induces weak homotopy equivalences between

the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and of $\mathbf{strat}(\|\Delta^1\|, Y)$. ■

We now provide a number of examples to show the importance of the criteria placed on the map F , to ensure that it is a stratified weak equivalence. We also show that the map F may still be a weak equivalence, even if the underlying poset map is not an isomorphism.

Example 9.4.0.8. Consider two stratified spaces; $A = * \amalg *$ where the coproduct is formed in \mathbf{Strat} , and B where the underlying topological space is $* \amalg *$ but is stratified over $[1]$. Clearly both spaces are fibrant; this follows because any stratified inner horn in A must be constant, and the same is true in B . The map $F: A \rightarrow B$ which is the identity map on underlying sets is a stratified weak equivalence, because $SS(A) = SS(B)$. The underlying poset map of F is not an isomorphism on posets, however F is still a stratified weak equivalence.

Example 9.4.0.9. Consider the inclusion $\|i\|: \|\partial\Delta^2\| \hookrightarrow \|\Delta^2\|$; both spaces are cofibrant and $\|\partial\Delta^2\|$ is not fibrant. This inclusion is an isomorphism on underlying posets, and all strata and holinks are contractible, in both $\|\partial\Delta^2\|$ and $\|\Delta^2\|$. Hence $\|i\|$ satisfies the conditions of Theorem 9.4.0.5, except for fibrancy in the source. However, it is easy to see that $\|i\|$ is not a stratified weak equivalence; we apply the forgetful functor to \mathbf{Top} (which is a left Quillen adjoint), and thus preserves weak equivalences between cofibrant stratified spaces. However when applied to I , we do not arrive at a weak homotopy equivalence, so I cannot be a stratified weak equivalence.

Example 9.4.0.10. We now want to give an example of a stratified weak equivalence between fibrant stratified spaces which is not a stratum-preserving homotopy equivalence. In particular this implies that we need a stratified space which is not cofibrant.

Consider the stratified morphism $F: \|\Delta^1\| \rightarrow [1]$ where the topological space $Y = [1]$ is the poset topology given to the poset $0 \leq 1$ and stratified over itself. It is straightforward to check that $[1]$ is fibrant, and that $X = \|\Delta^1\|$ is cofibrant-fibrant. The continuous map of topological spaces F is defined by $F(0) = 0$ and $F((0, 1]) = 1$.

It is clear that F is an isomorphism on posets, and F induces weak homotopy equivalences between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and the strata of $\mathbf{strat}(\|\Delta^1\|, Y)$ because all strata and holinks are contractible; hence F is a weak equivalence of fibrant stratified spaces. However the map F is not a stratum preserving homotopy equivalence because we are unable to find a stratum preserving homotopy inverse to F ; this follows because for a map $[1] \rightarrow \|\Delta^1\|$ to be continuous, it must be a constant map.

Example 9.4.0.11. More generally, thinking of $[n]$ as a stratified space via the identity map to itself, then we have a weak categorical equivalence $SS([n]) \cong \Delta^n$ (this can be seen by the Fundamental Theorem of Quasi-Categories). Moreover, the counit $\epsilon_{[n]}: \|SS([n])\| \rightarrow [n]$ is a weak equivalence by the same logic as Example 9.4.0.10.

Remark 9.4.0.12. An arbitrary poset can be stratified over itself using the identity map, from which it follows that any poset is a fibrant stratified space which has contractible strata and

holinks. This is true because the transitivity of a partial order on a set implies inner stratified horns always have (unique) fillers. It is interesting to note that thinking of a poset P as a stratified space stratified over itself, the stratified singular simplicial set $SS(P)$ is isomorphic to the nerve of the poset thought of as a category $N(P)$.

We also can consider $\|SS(P)\|$, which is stratified over P via the counit $\epsilon_P: \|SS(P)\| \rightarrow P$. This stratified space is also fibrant; to see this consider an inner horn $F: \|\Delta_k^n\| \rightarrow \|SS(P)\|$. Labelling the vertices of the horn as $0, 1, \dots, n$, we see the horn is contained in the union of the $F(0) \leq F(1) \leq \dots \leq F(n)$ strata of $\|SS(P)\|$. By the transitivity of a partial order and fibrancy of the n -simplex, it follows that there is a continuous map $\|\Delta^n\| \rightarrow P$ where each vertex v of the n -simplex is mapped to $F(v) \in P$. Therefore in $\|SS(P)\|$ there is a stratified n -simplex which extends the stratified inner horn F .

Corollary 9.4.0.13. *Let $X \rightarrow P$ be a fibrant stratified space which has contractible strata and holinks (equivalently the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ are contractible). Then the stratified morphism $X \rightarrow P$, thinking of the poset P as a stratified over itself via the identity map, is a stratified weak equivalence.*

Proof. An application of Theorem 9.4.0.5, noting that $S_P = \mathbb{1}_P: P \rightarrow P$ is a fibrant stratified space with contractible strata and holinks. ■

Remark 9.4.0.14. In general, the stratified weak equivalence $X \rightarrow P$ will not be a stratum preserving homotopy equivalence because there is a very limited selection of continuous maps $P \rightarrow X$. The stratified space P is rarely cofibrant, so we wouldn't expect the stratified weak equivalence $X \rightarrow P$ to be a stratum preserving homotopy equivalence.

We are able to strengthen the weak equivalence $\|SS([n])\| \rightarrow [n]$, by switching the stratified space $[n]$ for an arbitrary poset P .

Corollary 9.4.0.15. *The counit $\epsilon_P: \|SS(P)\| \rightarrow P$ provides a cofibrant-fibrant replacement for any poset P .* ■

As a consequence of Theorem 9.4.0.5 we also have the following result. While this result appears much closer to David Miller's theorem (see [Mil13, Theorem 6.3], stated as Theorem 5.1.0.28), the work motivated both results.

Corollary 9.4.0.16. *Consider a stratified morphism $F: X \rightarrow Y$ where X and Y are cofibrant-fibrant stratified spaces. The map F is a stratum preserving homotopy equivalence if and only if the underlying poset map $f: P \rightarrow Q$ is a bijection and F induces weak homotopy equivalences between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and the strata of $\mathbf{strat}(\|\Delta^1\|, Y)$.*

Proof. Since X and Y are cofibrant-fibrant, the morphism F is a stratified weak equivalence if and only if it is a stratum preserving homotopy equivalence. In addition, a stratum preserving homotopy equivalence must be an isomorphism on posets, hence the corollary follows from Theorem 9.4.0.5. ■

Example 9.4.0.17. This example is taken from [Mil13, Example 1], and will consist of a stratified map from a cofibrant to a cofibrant-fibrant stratified space, which induces weak homotopy equivalences between the corresponding strata and holinks of X and Y , but is not a stratified weak equivalence. This will illustrate that it is vital that X is a fibrant stratified space. Both of our stratified spaces X and Y will be stratifications of \mathbb{R}^2 over the poset $[1]$. Define X to be \mathbb{R}^2 stratified by sending $\{(x, 0) \mid -1 \leq x \leq 1\}$ to $0 \in [1]$ and everything else to $1 \in [1]$. The space Y is defined by sending the point $(1, 0)$ to $0 \in [1]$ and everything else to $1 \in [1]$. Both X and Y are cofibrant because there are many choices of compatible triangulations, and Corollary 9.2.0.7.

We can show that the stratified space X is not fibrant; we will show this geometrically first. Consider the horn $\|\Lambda_2^3\| \rightarrow X$ defined by sending the 2-simplex spanned by 0, 1 and 2 to the 0 stratum of X , defined by sending 2 to $(-1, 0)$ and the 1-simplex spanned by 0 and 1 to the point $(1, 0)$ and then linearly extending along the simplex. The vertex $3 \in \|\Lambda_2^3\|$ is sent to $(-3, 0)$ and the 1-simplex spanned by 2 and 3 is sent to the interval from $(-3, 0)$ to $(-1, 0)$, the 1-simplex spanned by 0 and 3 is sent to the semi-circle between $(-3, 0)$ and $(1, 0)$ with negative y coordinate, and the 1-simplex spanned by 1 and 3 to the semi-circle between $(-3, 0)$ and $(1, 0)$ with positive y coordinate. The two remaining 2-simplices span the interiors of the two disks cut out by the semi-circles and the $(-3, 0)$ and $(1, 0)$ interval. There is no filler that can be found for this inner horn in X and therefore X is not a fibrant stratified space. Alternatively, note that the map $E_0: \text{Holink}(X, X_0) \rightarrow X_0$ is not a Serre fibration because the fibers at $(\pm 1, 0)$ are connected but those at $(t, 0)$ for $-1 < t < 1$ have two connected components.

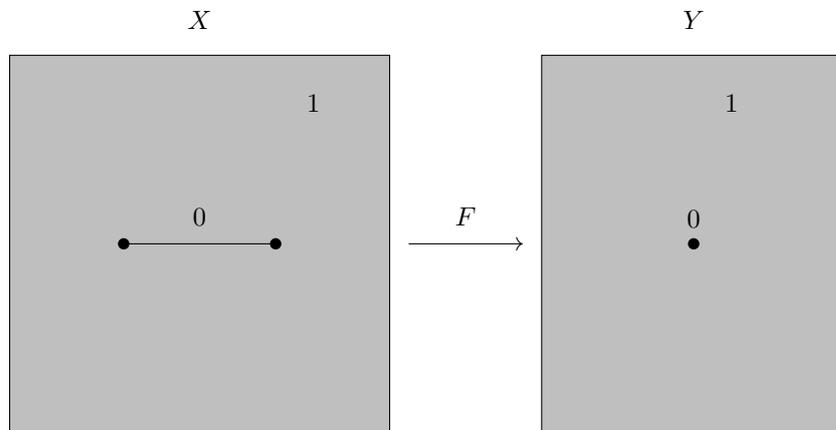


Figure 9.23: The stratified map F has no stratum preserving homotopy inverse.

Consider the stratified morphism $F: X \rightarrow Y$ depicted in Figure 9.23, which is defined coordinate-wise by:

$$F(x, y) = \begin{cases} (x, y) & \text{if } x \geq 1 \\ (1, y) & \text{if } -1 \leq x \leq 1 \\ (x + 2, y) & \text{if } x \leq -1. \end{cases}$$

The map F induces weak homotopy equivalences on the strata and moreover between the strata of $\mathbf{strat}(\|\Delta^1\|, X)$ and $\mathbf{strat}(\|\Delta^1\|, Y)$, so satisfies the criteria of Corollary 9.4.0.16. However it is not a stratum preserving homotopy equivalence because is no possible choice of stratum preserving homotopy inverse to F .

Reflecting on Theorem 9.4.0.5 and Corollary 9.4.0.16, it is key to notice that a stratum preserving homotopy equivalence between cofibrant-fibrant stratified spaces, or a stratified weak equivalence between fibrant stratified spaces, are detected by the homotopy groups of each stratum and the homotopy groups of the holinks of each pair of strata. This motivates the definition in the next section given of the $\mathbb{N}_{\geq 1}$ -indexed homotopy categories of a stratified space.

Part III

Stratified Homotopy Theory

Chapter 10

Based Stratified Spaces

In this chapter, we introduce a notion of basing for a stratified space, and explore the extent to which this allows us to study the homotopy theory of stratified spaces. In particular, the notion we use will allow us to construct new homotopy invariants of a stratified space, which behave analogously to the homotopy groups of a topological space.

10.1 Category of Based Stratified Spaces

We introduce our definition of basing of a stratified space, and the associated category of based stratified spaces. We explain and illustrate how the requirement of a basing provides a genuine constraint on a stratified space.

Definition 10.1.0.1. A *based stratified space* is a stratified space $S_X: X \rightarrow P$ with a choice of continuous map $B_X: \|SS(P)\| \rightarrow X$, such that the counit $\epsilon_P: \|SS(P)\| \rightarrow P$ factors through S_X as $\epsilon_P = S_X \circ B_X$.

Remark 10.1.0.2. For a stratified space $X \rightarrow P$, the stratified realisation $\|SS(P)\|$ lives over the poset P via the counit on P . Hence we can equivalently describe a basing of a stratified space as a choice of stratified morphism $B_X: \|SS(P)\| \rightarrow X$ which lives over the identity map on P . Equivalently, a basing is a choice of continuous map B_X , which makes Figure 10.1 commute.

$$\begin{array}{ccc} \|SS(P)\| & \overset{B_X}{\dashrightarrow} & X \\ & \searrow \epsilon_P & \swarrow S_X \\ & & P \end{array}$$

Figure 10.1: A choice of basing B_X for X .

Remark 10.1.0.3. In a based stratified space $X \rightarrow P$, if there is a relation $p \leq q$ in P , there is a stratified path $\gamma: \|\Delta^1\| \rightarrow X$ such that $\gamma(\{0\}) \in X_p$ and $\gamma(\{1\}) \in X_q$. This is not necessarily true in an arbitrary stratified space; for example if we consider $* \coprod * \rightarrow [1]$ (where the underlying space has the discrete topology), we see that there is no path from the 0-stratum to the 1-stratum.

Remark 10.1.0.4. For a stratified space $X \rightarrow P$, a basing B_X consists of a choice of basepoint x_p in each stratum of X , along with a compatible choice of stratified path (a basepath) $\gamma_{p,q}$ from x_p to x_q , for any relation $p \leq q$ in P . The higher dimensional simplices of $\|SS(P)\|$ encode the compatibility and higher dimensional composite relations between choices of basepaths. For example, the compatibility encoded by a 2-simplex says that the concatenation of the basepaths from p to q and q to r is homotopic through a specified family of elementary exit paths to the basepath from p to r .

Example 10.1.0.5. If a stratified space X is trivially stratified, then a basing of X is a choice of basepoint $x_0: * \cong \|SS(*)\| \rightarrow X$.

Example 10.1.0.6. The *natural stratification* of S^1 over $[1]$ is defined by sending the point $(-1, 0)$ to $\{0\} \in [1]$ and $S^1 \setminus (-1, 0)$ to $\{1\} \in [1]$. A basing $B_{S^1}: \|\Delta^1\| \rightarrow S^1$ is given by $B_{S^1}(0) = (-1, 0)$ and any choice of stratified path such that $B_{S^1}(t) \neq (-1, 0)$ for all $t > 0$.

Not every stratified space can be based. We illustrate this by a variety of examples, showing that there are even cofibrant-fibrant stratified spaces which cannot be based.

Example 10.1.0.7. Consider the stratification of two points given the discrete topology stratified over the poset $[1]$. In this situation, there is a relation $0 \leq 1$ in the poset $P = [1]$ which gives a 1-simplex in $\|SS(P)\|$. However, there can be no path between the two points in the underlying topological space, because any path must be constant. This is an example of a fibrant stratified space which is not cofibrant (due to the choice of stratification), that cannot be based.

Example 10.1.0.8. To provide an example of a cofibrant stratified space that cannot be based, consider $\|\Lambda_1^2\|$. Note that $\|\Lambda_1^2\|$ is not fibrant. In this case $\|SS([2])\| = \|\Delta^2\|$, and there is no basing map $B_X: \|SS([2])\| \rightarrow \|\Lambda_1^2\|$ such that $\epsilon_P = S_X \circ B_X$, because there is no elementary exit path from a point of the 0-stratum of $\|\Lambda_1^2\|$ to any point in the 2-stratum.

Example 10.1.0.9. Consider the fibered coproduct:

$$X = \|\Delta^2\| \coprod_{\|\{0\}\| \coprod \|\{2\}\|} \|\Delta^2\|,$$

constructed by taking the disjoint union of two 2-simplices and identifying them via the 0-simplices of the 0 and 2-strata, as depicted in Figure 10.2, where the 2-stratum is indicated in grey. The cofibrant stratified space X is also fibrant.

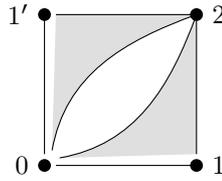


Figure 10.2: A cofibrant-fibrant stratified space which cannot be based.

The space X lives over the poset P which is similar to the poset [2] but with two distinct copies of the element 1, denoted by 1 and $1'$. The stratified space $\|SS(P)\| = \|\Delta^2 \amalg_{\langle 0,2 \rangle} \Delta^2\|$ where the fibered coproduct is taken along the $\langle 0, 2 \rangle$ edge of both 2-simplices. Again, it is clear that there is no possible choice of basing $B_X: \|SS(P)\| \rightarrow X$.

Example 10.1.0.9 cannot be based, because $\text{holink}(X_0 \cup X_2, X_0)$ is disconnected. Therefore, one may hope that any fibrant stratified space stratified over a finite poset, with path-connected strata and holinks can be based. One may hope to start by taking a basepoint $x_p \in X_p$ for every $p \in P$, which defines a map $\|SS(P)_0\| \rightarrow X$. For every $p \leq q$ in P such that there is no $r \in P$ with $p < r < q$, pick a basepath $\gamma_{p,q}$ in X from x_p to x_q . It is not possible however to use the fibrancy of X to pick the composites for pairs of stratified paths compatibly, which we illustrate by constructing another stratified space which cannot be based.

Example 10.1.0.10. We construct an example of a cofibrant-fibrant stratified space $\|A\| \rightarrow P$ with connected strata and holinks, which cannot be based. This space will be constructed as the stratified geometric realisation of nine copies of Δ^2 identified so that the stratified geometric realisation is pictured in Figure 10.3. The grey shaded region depicts the top 2-stratum, making the orientations on the six outermost 2-simplices evident. There are choices to be made for the orientations on the innermost 2-simplices. So that the poset P has a unique top element, orient these simplices as indicated by the red numbers on the interiors of these simplices.

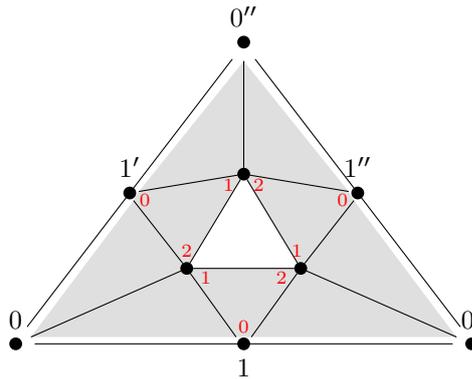


Figure 10.3: A cofibrant-fibrant stratified space which cannot be based.

The poset P over which $\|A\|$ is stratified, is pictured in Figure 10.4. From this geometric

illustration of P , we see there is no possible choice of continuous map $B_{\|A\|}: \|SS(P)\| \rightarrow \|A\|$ which would define a basing of $\|A\|$.

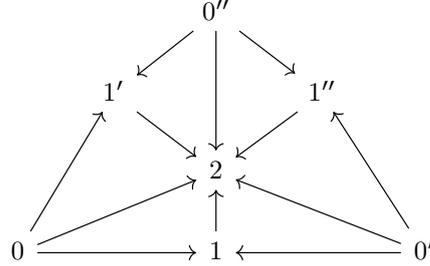


Figure 10.4: The poset P over which $\|A\|$ is stratified.

Remark 10.1.0.11. The reader may feel that the definition of a based stratified space given in Definition 10.1.0.1 is a little unnatural. To justify this definition, we consider the slice category $\mathbf{sSet}/SS(P)$ for a fixed poset P . In this slice category, the choice of a basepoint for an object $S_A: A \rightarrow SS(P)$ is a simplicial morphism $B_A: SS(P) \rightarrow A$ such that the composite $S_A \circ B_A$ is the identity on $SS(P)$. This generalises the concept of a basepoint for a simplicial set (or a pointed simplicial set); if $P = *$, then a map $SS(P) = * \rightarrow A$ is a choice of basepoint in A .

Considering a stratified space $X \rightarrow P$, we see that a basing of $SS(S_X): SS(X) \rightarrow SS(P)$ is a choice of appropriate map $B_{SS(X)}: SS(P) \rightarrow SS(X)$. Applying the adjunction (and naturality), a basing of $SS(X)$ exists if and only if the counit on P factors as the adjunct morphism $B_X: \|SS(P)\| \rightarrow X$ post-composed with the stratification of X . This is precisely the definition of a based stratified space given in Definition 10.1.0.1. Moreover, this argument shows that a stratified space $X \rightarrow P$ can be based if and only if $SS(X) \rightarrow SS(P)$ can be based.

We now proceed to introduce the definition of a morphism between based stratified spaces.

Definition 10.1.0.12. A morphism of based stratified spaces $F: X \rightarrow Y$ is a stratified morphism of the underlying stratified spaces which is compatible with the basing of X and Y , meaning that F fits into a commutative diagram in $k\mathbf{Top}$ pictured in Figure 10.5.

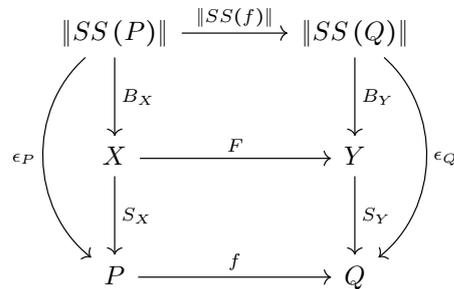


Figure 10.5: A morphism of based stratified spaces $F: X \rightarrow Y$.

Explicitly, the stratified morphism F is a morphism of based spaces if it carries each basepoint and basepath of X to a basepoint or basepath respectively of Y . We denote the category of based stratified spaces with based stratified morphisms between them as **BStrat**.

Proposition 10.1.0.13. *The forgetful functor $Fo(-): \mathbf{BStrat} \rightarrow \mathbf{Strat}$ has a left adjoint, the free basing functor $(-)_+$ shown in Figure 10.6.*

$$\begin{array}{ccc} & (-)_+ & \\ \mathbf{BStrat} & \xleftarrow{\quad} & \mathbf{Strat} \\ & \perp & \\ & Fo(-) & \end{array}$$

Figure 10.6: Adjoint functors between **Strat** and **BStrat**.

Proof. The free basing functor is defined by taking $S_X: X \rightarrow P$ to $(S_X, \epsilon_P): X \amalg \|SS(P)\| \rightarrow P$. The basing $B_{X \amalg \|SS(P)\|}: \|SS(P)\| \rightarrow X \amalg \|SS(P)\|$ is the identity map on $\|SS(P)\|$.

For $X \rightarrow P \in \mathbf{Strat}$ and $Y \rightarrow Q \in \mathbf{BStrat}$, a stratified morphism $X \rightarrow Fo(Y)$ determines a canonical stratified morphism $X \amalg \|SS(P)\| \rightarrow Y$ of based spaces defined by the coproduct of the stratified maps $X \rightarrow F(Y)$ and the map $\|SS(P)\| \rightarrow Y$ given by the composite morphism $B_Y \circ \|SS(F)\|: \|SS(P)\| \rightarrow \|SS(Q)\| \rightarrow Y$. By construction of the basing of $X \amalg \|SS(P)\|$, it follows that the stratified morphism $X \amalg \|SS(P)\| \rightarrow Y$ is compatible with the map $\|SS(f)\|: \|SS(P)\| \rightarrow \|SS(Q)\|$. It is clear that this assignment defines a bijection of sets $\mathbf{BStrat}(X_+, Y) \cong \mathbf{Strat}(X, Fo(Y))$, which is natural with respect to X and Y . ■

10.2 Stratified Loops and Reduced Suspension Adjunction

We introduce stratified notions of reduced suspension and loop space of a stratified space, and show that these define adjoint functors, analogous to the topological situation.

Definition 10.2.0.1. For a based stratified space $X \rightarrow P$, define the *stratified loop space* of X to be the stratified space ΩX constructed via the pullback in **Strat**, shown in Figure 10.7.

$$\begin{array}{ccccc} & & & & \text{co}B_X \\ & & & & \curvearrowright \\ \|SS(P)\| & & & & \\ & \searrow^{B_{\Omega X}} & & & \\ & & \Omega X & \xrightarrow{r_X} & \mathbf{strat}(\|\Delta^1\|, X) \\ & & \downarrow r & & \downarrow (E_0, E_1) \\ & & \|SS(P)\| & \xrightarrow{\Delta \circ B_X} & X \times X \end{array}$$

Figure 10.7: The stratified loop space ΩX of a stratified space X .

The morphism (E_0, E_1) is start point and end point evaluation of stratified paths in X , and the map $\Delta \circ B_X$ is the diagonal map applied to the basing of X . The stratified space ΩX is stratified over the poset P , because the map $\|SS(P)\| \rightarrow X \times X$ is an isomorphism on posets. The canonical basing of ΩX is constructed via the unique map $\|SS(P)\| \rightarrow \Omega X$, given as the induced map out of the cone defined by the identity map on $\|SS(P)\|$, and the map $c \circ B_X$ in which c is the stratified map defined by constant paths at each point in the basing of X .

Explicitly, the stratified loop space of X is the stratified space of stratum preserving paths in X which start and end at the same point in the basing of X (a point in the basing of X is a point in $B_X(\|SS(P)\|) \subseteq X$). The stratified loop space is stratified by the stratification of X .

Remark 10.2.0.2. If the stratified space X is fibrant, then the stratified loop space of X is also fibrant. To see this, note that for any poset P , the stratified space $\|SS(P)\|$ is fibrant, and that when X is fibrant, the evaluation morphism (E_0, E_1) is a stratified fibration. The fact that fibrations are stable under pullback shows that the pullback $\Omega X \rightarrow \|SS(P)\|$ is a fibration, hence ΩX is a fibrant stratified space.

Remark 10.2.0.3. We would like the map $\Omega(SS(X)) \rightarrow SS(\Omega(X))$ to be a weak categorical equivalence, however this does not follow because the Joyal model structure in simplicial sets is not right proper.

To introduce the notion of reduced suspension for a stratified space, we first need the notion of a cone on a stratified space. To define the cone of a stratified space X using the trivially stratified interval, we would like to consider $X \times [0, 1]$ and quotient $X \times \{0\}$ by the map $X \rightarrow *$. However for any X , we will arrive at the trivially stratified cone on X . Therefore to define the stratified cone on a stratified space we need to change the map $X \rightarrow *$ that we use.

Definition 10.2.0.4. A *well-based stratified space* is a based stratified space X with a stratified morphism $r_X: X \rightarrow \|SS(P)\|$, such that the composite $r_X \circ B_X: \|SS(P)\| \rightarrow X \rightarrow \|SS(P)\|$ is stratum preserving homotopic to the identity map.

Remark 10.2.0.5. Notice that the map $r_X \circ B_X$ is a map from a cofibrant(-fibrant) stratified space to a (cofibrant-)fibrant stratified space, hence the notions of left and right homotopy coincide, so we are justified in using the notion of stratum preserving homotopy.

The intuitive idea behind the definition of a well-based stratified space is that we will replace the map $X \rightarrow *$ in the construction of the cone on a stratified space by the map r_X . The requirement that $r_X \circ B_X$ is homotopic to the identity map means that r_X will behave well with respect to the stratification of X .

Example 10.2.0.6. For a based stratified space X , it follows that the stratified loop space of X is well-based. This is true because $\|SS(P)\|$ is a retract of ΩX , which is demonstrated by the map r_X depicted in Figure 10.7.

Example 10.2.0.7. We will provide an example of a basing in which $r_x \circ B_X$ is homotopic but not equal to the identity on $\|SS(P)\|$. To construct a basing of the stratified 1-simplex, we first note that $\|SS([1])\| = \|\Delta^1\|$. Consider the stratified map $B_{\|\Delta^1\|}: \|\Delta^1\| \rightarrow \|\Delta^1\|$ defined by:

$$B_{\|\Delta^1\|}(x) = \begin{cases} x & 0 \leq x \leq 0.5 \\ 0.5 & 0.5 \leq x \leq 1. \end{cases}$$

In this case, there is the identity map which we can take as r_X , however this will clearly not satisfy $r_X \circ B_X = \mathbb{1}_{\|SS([1])\|}$ but there is an evident homotopy from this composite to the identity map on $\|SS([1])\|$.

The reason we ask that $r_X \circ B_X$ is homotopic to the identity is that with this definition, any based cofibrant stratified space is well-based.

Proposition 10.2.0.8. *Any retract of a stratified geometric realisation $X \rightarrow P$ has a canonical map $r_X: X \rightarrow \|SS(P)\|$. Moreover, for any based stratified geometric realisation $\|A\|$, the map $r_{\|A\|}$ can be used to show that any basing of $\|A\|$ is well-based. Therefore, any stratified space X which is obtained as a retract of a based stratified geometric realisation, is also well-based.*

Proof. Consider a cofibrant stratified space which we initially assume to be a stratified realisation $X = \|A\|$, with basing $B_X: \|SS(P)\| \rightarrow X$. To construct the map r_X consider the stratification morphism $S_X: \|A\| \rightarrow P$ as a stratified morphism by letting the poset P be stratified over itself via the identity map. Let r_X be the stratified realisation of the adjunct $A \rightarrow SS(P)$ of S_X .

If the cofibrant space $X \rightarrow Q$ is obtained as a retract of a stratified realisation $\|A\| \rightarrow P$, then we can extend the canonical map $r_{\|A\|}$ to a canonical map r_X constructed as in Figure 10.8.

$$\begin{array}{ccccc}
 & & \mathbb{1}_X & & \\
 & \swarrow & \text{---} & \searrow & \\
 X & \xrightarrow{F} & \|A\| & \xrightarrow{G} & X \\
 \downarrow r_X = \|SS(G)\| \circ r_{\|A\|} \circ F & & \downarrow r_{\|A\|} & & \downarrow r_X = \|SS(G)\| \circ r_{\|A\|} \circ F \\
 \|SS(Q)\| & \xrightarrow{\|SS(F)\|} & \|SS(P)\| & \xrightarrow{\|SS(G)\|} & \|SS(Q)\| \\
 & \searrow & \text{---} & \swarrow & \\
 & & \|SS(\mathbb{1}_Q)\| = \mathbb{1}_{\|SS(Q)\|} & &
 \end{array}$$

Figure 10.8: Constructing the map $r_X: X \rightarrow \|SS(Q)\|$.

To prove the second statement, consider a based stratified realisation $X = \|A\| \rightarrow P$. To show that $r_X \circ B_X$ is stratum preserving homotopic to the identity map on $\|SS(P)\|$, note that any point of $r_X \circ B_X(\|SS(P)_0\|)$ has a canonical, linear, stratum preserving homotopy within $\|SS(P)\|$ to an element of $\|SS(P)_0\|$. This follows because r_X is a stratified morphism,

and there is only one point in $\|SS(P)_0\|$ for each stratum of $\|SS(P)\|$. Within each simplex of $\|SS(P)\|$ we extend the homotopy linearly from the vertex map to define a homotopy of $\|SS(P)\|$, which by construction will be a stratum preserving homotopy (this is easily shown using barycentric coordinates within a simplex of $\|SS(P)\|$).

To complete the proof, let X denote a stratified space which is obtained as a retract of a based stratified realisation $\|A\|$. The previous paragraph shows that $r_{\|A\|}$ is a well-basing of $\|A\|$, which we need to extend to a well-basing r_X of X . To do this, we need to construct a basing of X using the basing of $\|A\|$, which is done as indicated in Figure 10.9.

$$\begin{array}{ccccc}
 & & \text{\scriptsize } \|SS(\mathbb{1}_Q)\| = \mathbb{1}_{\|SS(Q)\|} & & \\
 & \text{\scriptsize } \|SS(Q)\| & \xrightarrow{\text{\scriptsize } \|SS(F)\|} & \text{\scriptsize } \|SS(P)\| & \xrightarrow{\text{\scriptsize } \|SS(G)\|} & \text{\scriptsize } \|SS(Q)\| \\
 & \downarrow \text{\scriptsize } B_X = G \circ B_{\|A\|} \circ \|SS(F)\| & & \downarrow \text{\scriptsize } B_{\|A\|} & & \downarrow \text{\scriptsize } B_X = G \circ B_{\|A\|} \circ \|SS(F)\| \\
 & X & \xrightarrow{\text{\scriptsize } F} & \|A\| & \xrightarrow{\text{\scriptsize } G} & X \\
 & & & \text{\scriptsize } \mathbb{1}_X & &
 \end{array}$$

Figure 10.9: Constructing a basing of X from a basing of $\|A\|$.

The counit on Q factors as $S_X \circ B_X$, because the counit is functorial (applied to X as a retract of $\|A\|$). The composite $r_X \circ B_X$ is stratum preserving homotopic to the identity map on $\|SS(Q)\|$, by the same argument as above. ■

Remark 10.2.0.9. In the above proof, it is vital that in the definition of being well-based, we require that $r_X \circ B_X$ is homotopic to the identity map rather than equal to the identity map. If we instead require that the basing map was injective, then we could strengthen the definition of well-based so that $r_X \circ B_X$ is the identity map.

Example 10.2.0.10. If $A \rightarrow SS(P)$ is based in $\mathbf{sSet}/SS(P)$, then $\|A\| \rightarrow P$ is well-based. This follows from an application of the stratified adjunction.

Example 10.2.0.11. If we have a based stratified space $S_X: X \rightarrow P$, then it follows that the stratified space $\|SS(S_X)\|: \|SS(X)\| \rightarrow P$ is well-based. To see this, apply $\|SS(-)\|$ to the basing of X and stratify $\|SS(X)\|$ by post-composition of $\|SS(S_X)\|$ with the counit at P . The basing of $\|SS(X)\|$ is given by pre-composing the map $\|SS(B_X)\|: \|SS(\|SS(P)\|)\| \rightarrow \|SS(X)\|$ with the counit $\|\eta_{SS(P)}\|: \|SS(P)\| \rightarrow \|SS(\|SS(P)\|)\|$. This gives a diagram defining a well-basing of $\|SS(X)\|$, because of the triangle identities satisfied by the counit and unit.

Example 10.2.0.12. There are examples of based stratified spaces that cannot be well-based. The simplest example is the stratified space $[1] \rightarrow [1]$, which only has one possible choice of basing. This space cannot be well-based because there is no choice of map $r_{[1]}: [1] \rightarrow \|\Delta^1\|$

which could define a well-basing because any continuous map of this form must be a constant map.

Notation. Denote the category of well-based stratified spaces by **WBStrat**, where morphisms between well-based spaces are maps between the underlying based spaces.

Remark 10.2.0.13. When considering morphisms between well-based stratified spaces, it appears natural to also ask for compatibility of morphisms with r_X and r_Y . We do not need this extra compatibility, but for applications elsewhere this may be necessary.

Definition 10.2.0.14. For a well-based stratified spaces X and based stratified space Y , we can define the *based stratified mapping space* $\mathbf{bstrat}(X, Y)$ as a basing of stratified space $\mathbf{strat}(X, Y)$. The stratified mapping space is based by the map $\|SS(\mathbf{poset}_{\mathbf{Strat}}(P, Q))\| \rightarrow \mathbf{bstrat}(X, Y)$ defined by sending the n -simplex arising from a map $f: P \times [n] \rightarrow Q$ to the composite:

$$B_Y \circ \|SS(f)\| \circ (r_X \times \mathbb{1}_{\|\Delta^1\|}) : X \times \|\Delta^n\| \rightarrow \|SS(P)\| \times \|\Delta^n\| \cong \|SS(P \times [n])\| \rightarrow \|SS(Q)\| \rightarrow Y.$$

We are able to define the stratified cone on a well-based stratified space.

Definition 10.2.0.15. For a well-based stratified space $X \rightarrow P$, the *stratified cone on X* is defined as the pushout depicted in Figure 10.10, and denoted by CX .

$$\begin{array}{ccc}
 X & \xrightarrow{r_X} & \|SS(P)\| \\
 \downarrow \mathbb{1}_X \times \{0\} & & \downarrow b_{CX} \\
 X \times [0, 1] & \xrightarrow{\alpha} & CX \\
 & & \downarrow r_{CX} \\
 & & \|SS(P)\|
 \end{array}$$

Figure 10.10: The cone on a well-based stratified space X .

By construction, the stratified cone CX is naturally stratified over the poset P , based via the composite map $B_{CX} = b_{CX} \circ r_X \circ B_X$ and well-based by the map r_{CX} . The counit at P factors through B_{CX} because the map $X \rightarrow \|SS(P)\| \rightarrow CX$ lives over the identity on P . The map r_{CX} gives a well-basing of CX because the stratum preserving homotopy required is precisely the same as the stratum preserving homotopy which is given for the well-basing of X . This follows because the composite $r_{CX} \circ b_{CX}$ is the identity map on the image under r_{CX} of the basing of X ; explicitly this states that $r_X \circ B_X(\|SS(P)\|) = r_{CX} \circ b_{CX} \circ r_X \circ B_X(\|SS(P)\|)$.

Proposition 10.2.0.16. For any based stratified space X , the basing $B_{CX}: \|SS(P)\| \rightarrow CX$ provides a stratum preserving homotopy equivalence between $\|SS(P)\|$ and CX .

Remark 10.2.0.17. This result is analogous to the standard result that the unstratified cone on a topological space is contractible (where by contractible we mean that the cone is homotopy equivalent to a point).

Proof. We claim that the morphism r_{CX} is a stratum preserving inverse to B_{CX} ; explicitly the map r_{CX} is given at a point $(x, t) \in CX$ for $x \in X$ and $t \in [0, 1]$ by $r_{CX}(x, t) = r_X(x)$. Because the map r_{CX} gives a well-basing, $r_{CX} \circ B_{CX}$ is stratum preserving homotopic to the identity map. The composite $B_{CX} \circ r_{CX}$ is stratum preserving homotopic to the identity, via $H((x, t), T): CX \times [0, 1] \rightarrow CX$ given by:

$$H((x, t), T) = \begin{cases} (r_X(x), 0) & \text{when } T = 0 \\ (x, Tt) & \text{when } T > 0. \end{cases}$$

The homotopy H is continuous by the construction of the stratified cone CX . By construction we see that at $T = 0$ the homotopy is the composite $r_{CX} \circ B_{CX}$ and at $T = 1$ it is the identity map on CX . Therefore H defines the required stratum preserving homotopy. ■

Remark 10.2.0.18. The underlying stratified space of CX is fibrant when X is fibrant, because any stratified horn $\|\Lambda_k^n\| \rightarrow CX$ is stratum preserving homotopic to a horn in $X \times \{0\} \subseteq CX$, which has a filler because $\|SS(P)\|$ is fibrant. The filler can be homotoped along the inverse stratum preserving homotopy, because X is fibrant, to provide a filler of the stratified horn in CX .

Definition 10.2.0.19. For a well-based stratified space $X \rightarrow P$, the *stratified unreduced suspension* of X is denoted by SX and is defined by the pushout depicted in Figure 10.11.

$$\begin{array}{ccc} X & \xrightarrow{r_X} & \|SS(P)\| \\ \downarrow \mathbb{1}_{X \times \{1\}} & & \downarrow B_{SX} \\ CX & \xrightarrow{\beta} & SX \end{array}$$

Figure 10.11: Construction of the unreduced suspension of X .

The stratified suspension of X can equivalently be defined as the quotient of CX by the relation $(x, 1) \sim (r_X(x), 1)$ for all $x \in X$.

As with the construction of a stratified cone, the stratified suspension is stratified over P , based by post-composing B_{CX} with the induced map $\beta: CX \rightarrow SX$, and comes with a well-based morphism r_{SX} induced from r_{CX} . If the underlying stratified space X is cofibrant, then so is SX . We could alternatively define the basing of SX to be the map $b_{SX} \circ r_X \circ B_X$; the difference is between choice of ends (either $\{0\}$ or $\{1\}$ of $[0, 1]$ in $X \times [0, 1]$) of the unreduced suspension. Both choices give rise to isomorphic well-based stratified spaces because $[0, 1]$ is trivially stratified.

Definition 10.2.0.20. For a well-based stratified space $X \rightarrow P$, the *stratified reduced suspension* of X is denoted by ΣX and is defined by the pushout depicted in Figure 10.12.

$$\begin{array}{ccc} \|SS(P)\| \times [0, 1] & \xrightarrow{pr_1} & \|SS(P)\| \\ \downarrow B_{SX \circ pr_1} & & \downarrow B_{\Sigma X} \\ SX & \dashrightarrow & \Sigma X \end{array}$$

Figure 10.12: Constructing the reduced suspension of X .

The reduced suspension of X can equivalently be defined as the quotient of SX by identifying $(x, s) \sim (x, t)$ for any $x \in B_X(\|SS(P)\|)$ and any $s, t \in [0, 1]$.

Regardless of which choice of the two natural basings we take for SX , we arrive at the same basing $B_{\Sigma X}$ of ΣX ; this follows because the entirety of $\|SS(P)\| \times [0, 1]$ contained within SX is collapsed down onto $B_X(\|SS(P)\|)$. We construct the map $r_{\Sigma X}$ in the same fashion as for CX or SX , induced by r_{SX} . If the underlying stratified space X is cofibrant, then so is ΣX .

Example 10.2.0.21. For every $n \geq 1$, define the *natural stratification of the n -sphere* over the poset $[1]$ by the map $S_{S^n}: S^n \rightarrow [1]$ by sending the point $(-1, 0, \dots, 0)$ to $\{0\} \in [1]$, and the remainder of S^n to $\{1\} \in [1]$. In this case, the unreduced suspension of S^n is stratum preserving homotopy equivalent to S^{n+1} ; this follows because the difference between the two stratified spaces is that the 0-stratum in the suspension is the south pole times $[0, 1]$ but in S^{n+1} is only one point. In particular, the reduced suspension of S^n is the naturally stratified sphere S^{n+1} .

Example 10.2.0.22. For any choice of well-basing map, the reduced suspension of the stratified 1-simplex with a freely adjoined basing is isomorphic to $S^1 \times \|\Delta^1\|$, where S^1 is trivially stratified. Moreover, quotienting out the $S^1 \times \{0\}$ and $S^1 \times \{1\}$ end points of $S^1 \times \|\Delta^1\|$ gives the naturally stratified 2-sphere.

Remark 10.2.0.23. To check that our definition of reduced suspension is correct, we would like to introduce the smash product. The most obvious way of attempting this would be to define:

$$X \wedge Y = X \times Y / X \vee Y.$$

We have attempted to make sense of the wedge product of stratified spaces, however for this to be meaningful, we need to require the basing maps B_X and B_Y to be injective.

In the unstratified case, the reduced suspension of a space can be described as the smash product with S^1 . The smash product in the category of pointed topological spaces X and Y (with basepoints x_0 and y_0 respectively) is defined by:

$$X \wedge Y = X \times Y / \sim,$$

where we define the equivalence relation by $(x, y_0) \sim (x', y_0)$ and $(x_0, y) \sim (x_0, y')$ for all $x, x' \in X$ and $y, y' \in Y$. Our definition of the smash product in **Strat** will directly mirror this construction, and it is easy to see that in the case of trivially stratified spaces the two constructions are identical.

Definition 10.2.0.24. Let $X \rightarrow P$ and $Y \rightarrow Q$ be two well-based stratified spaces. The *stratified smash product* of X and Y , denoted $X \wedge Y$, is the quotient:

$$X \wedge Y = X \times Y / \sim,$$

with equivalence relation defined by $(x, B_Y(q)) \sim (x', B_Y(q))$ and $(B_X(p), y) \sim (B_X(p), y')$ for all $x, x' \in X$, $y, y' \in Y$, $p \in \|SS(P)\|$ and $q \in \|SS(Q)\|$.

Now consider the trivially stratified circle, which with a choice of basepoint is a well based stratified space. For any well-based stratified spaces $X \rightarrow P$ we form the product $X \times S^1 \rightarrow P$. To construct the smash product we quotient out by the terminal map $S^1 \rightarrow *$ and well-basing map $r_X: X \rightarrow \|SS(P)\|$. This construction gives precisely the stratified reduced suspension of X that was constructed in Definition 10.2.0.20.

Theorem 10.2.0.25. For a well-based stratified space X and based stratified space Y , there is a bijection $\mathbf{BStrat}(\Sigma X, Y) \cong \mathbf{BStrat}(X, \Omega Y)$ which is natural in X and Y .

Proof. Taking well-based stratified spaces X and Y , we will use $(x, t) \in \Sigma X$ for $x \in X$ and $t \in [0, 1]$, to denote a point in the reduced suspension of X . Maps $F \in \mathbf{WBStrat}(\Sigma X, Y)$ and $G \in \mathbf{WBStrat}(X, \Omega Y)$ correspond if $F(x, t) = G(x)(t)$. Note that if $p \in \|SS(P)\|$, then $G(B_X(p))(t) = F(B_X(p), t) = B_Y(f(p))$ for all $t \in [0, 1]$, where $f: P \rightarrow Q$ is the underlying poset map of F . Therefore $G(B_X(p))(-)$ is the constant loop at $B_Y(f(p))$. Moreover, $F(x, t) = F(B_X \circ r_X(x), 0) = B_Y \circ \|SS(f)\| \circ r_X(x)$ for $t = 0, 1$, so that $G(x)(-)$ is a stratum preserving loop at the point $B_Y \circ \|SS(f)\| \circ r_X(x)$. ■

Remark 10.2.0.26. In particular, Theorem 10.2.0.25 implies that the reduced suspension and loop space functor define an adjunction when restricting to the category of well-based stratified space (defining an adjunction $\Sigma: \mathbf{WBStrat} \rightleftarrows \mathbf{WBStrat} : \Omega$).

Definition 10.2.0.27. Consider based fibrant stratified spaces X, Y so that X is also a cofibrant stratified space and consider two stratified maps $f, g: X \rightarrow Y$. We say that f and g are *based homotopic* if there is a homotopy between f and g as stratified maps (which necessarily must be stratum preserving), such that each map throughout the stratum preserving homotopy is a based map.

Using the well-based stratified mapping space, and the notion of based homotopy, we are able to strengthen the stratified suspension-stratified loops adjunction.

Corollary 10.2.0.28. *The natural isomorphism of Theorem 10.2.0.25 actually provides a based stratified isomorphism $\underline{\mathbf{bstrat}}(\Sigma X, Y) \cong \underline{\mathbf{bstrat}}(X, \Omega Y)$. Furthermore, there is a bijection between the based stratum preserving homotopy classes of maps $[\Sigma X, Y] \cong [X, \Omega Y]$.*

Proof. The first statement is true by restriction of the homeomorphism in $k\mathbf{Top}$ of the form $k\mathbf{top}(\Sigma X, Y) \cong k\mathbf{top}(X, \Omega Y)$ to based stratified maps. The second statement holds because a stratum preserving homotopy between maps $\Sigma X \rightarrow Y$ determines a stratum preserving homotopy between maps $X \rightarrow \Omega Y$, and vice versa. ■

10.3 Homotopy Categories of a Fibrant Stratified Space

We now return to considering based stratified spaces, rather than well-based. The notion of a based stratified space being well-based was introduced to make sense of the reduced suspension functor for stratified spaces. We will restrict ourselves to fibrant stratified spaces when constructing the homotopy category of a stratified space; this is to ensure that we actually get a category. Importantly, we will use the fact that for a fibrant stratified space, the stratified loop space is fibrant (see Remark 10.2.0.2), and that for a based stratified space, the stratified loop space is well-based (see Example 10.2.0.6).

Definition 10.3.0.1. For a based fibrant stratified space X , define the *stratified fundamental n -groupoid* as $\Pi_n(X) := \tau_1(SS(\Omega^{n-1}X))$ for $n \geq 1$.

Remark 10.3.0.2. The stratified fundamental n -groupoid is only a groupoid if X is trivially stratified (and in this case it coincides with the fundamental groupoid). We have chosen this rather unfortunate name, because we will pick stratified homotopy invariants out from the stratified fundamental groupoid, in a similar manner to extracting the fundamental group from the fundamental groupoid.

When X is fibrant, the fundamental category τ_1 preserves the composition of exit paths in the stratified space X . In this case, $\Pi_n(X)$ is a category with a canonical surjective on objects functor to P , induced by the stratification of X .

Example 10.3.0.3. For a fibrant stratified space X , we discussed $\Pi_1(X)$ in Example 7.1.0.6. The category $\Pi_2(X) = \tau_1(SS(\Omega X))$ has objects given by $SS(\Omega X)_0$, i.e. stratum preserving loops in X which are based on the image of $B_X: \|SS(P)\| \rightarrow X$. Morphisms in $\Pi_2(X)$ are given by stratum preserving homotopy classes of $\|\Delta^1\|$ -paths of such loops.

Definition 10.3.0.4. For a based fibrant stratified space X , define the n -th homotopy category $\pi_n(X)$ for $n \geq 1$ as the full subcategory of Π_n on constant loops at the basepoints of strata in X (points in the image of $B_X: \|SS(P)_0\| \rightarrow X$).

Remark 10.3.0.5. For each $i \in \mathbb{N}$, the restriction of a based stratified morphism $F: X \rightarrow Y$ to strata induces map from the objects of $\pi_i(X)$ to the objects of $\pi_i(Y)$. The stratified morphism

F restricts to give continuous maps between strata and holinks, hence induces group homomorphisms from $\pi_i(X)(x_p, x_q) \rightarrow \pi_i(Y)(F(x_p), F(x_q))$ for all $p \leq q \in P$. Importantly, this means that the homotopy categories of based fibrant stratified spaces are functorial with respect to based stratified maps.

Remark 10.3.0.6. If the based stratified space X is path connected and trivially stratified, then $\pi_i(X)$ is the i -th homotopy group of X , interpreted as a category. In this sense, the homotopy categories associated to a based fibrant stratified space generalise the homotopy groups of a topological space.

We wish to geometrically interpret the homotopy category of a based fibrant stratified space X . By this, we mean that we wish to understand the arrows in the homotopy category in relation to homotopy groups of strata and holinks in X .

Proposition 10.3.0.7. *For a based fibrant stratified space X , morphisms of $\pi_n(X)$ for $n \geq 2$ are represented by maps $S^n \rightarrow X$, from the naturally stratified n -sphere.*

Proof. For $n \geq 2$, morphisms from x_p to x_q in $\pi_n(X)$ are homotopy classes of elements of $\mathbf{sSet}(\Delta^1, SS(\Omega^{n-1}X(x_p, x_q)))$ where x_p and x_q are considered as constant loops at the basepoints x_p and x_q respectively. Using the results we have proven so far, we have the following chain of isomorphisms:

$$\begin{aligned} \mathbf{sSet}(\Delta^1, SS(\Omega^{n-1}X))(x_p, x_q) &\cong \mathbf{Strat}(\|\Delta^1\|, \Omega^{n-1}X)(x_p, x_q) \\ &\cong \mathbf{BStrat}(\|\Delta^1\|_+, \Omega^{n-1}X)(x_p, x_q) \\ &\cong \mathbf{BStrat}(\Sigma\|\Delta^1\|_+, \Omega^{n-2}X)(x_p, x_q) \\ &\cong \mathbf{BStrat}(S^2, \Omega^{n-2}X)(x_p, x_q) \\ &\cong \mathbf{BStrat}(\Sigma^{n-2}S^2, X)(x_p, x_q) \\ &\cong \mathbf{BStrat}(S^n, X)(x_p, x_q) \end{aligned}$$

using the stratified adjunction, Proposition 10.1.0.13, Theorem 10.2.0.25, and Example 10.2.0.22. In particular, we see that $\Sigma\|\Delta^1\|_+ \cong S^2$ in our setting, because as well as $\Sigma\|\Delta^1\|_+ \cong S^1 \times \|\Delta^1\|$ we also have that $S^1 \times \{0\}$ mapping to the constant path at x_p and $S^1 \times \{1\}$ mapping to the constant path at x_q , so can apply the logic of Example 10.2.0.22. Therefore morphisms of $\pi_n(X)$ are represented by homotopy classes of maps $S^n \rightarrow X$ sending the basepoints of S^n to x_p and x_q , and the basepath of S^n to $\gamma_{p,q}$. \blacksquare

Remark 10.3.0.8. For a based fibrant stratified space with connected holinks $X \rightarrow P$, let x_p denote the basepoint of the stratum X_p and $\gamma_{p,q}$ denote the basepath between x_p and x_q . Then the $\pi_i(X)$ is the category with one object for each basepoint x_p , and with hom-sets defined as

follows for $p \leq q$ in P :

$$\begin{cases} \pi_i(X)(x_p, x_p) &= \pi_i(X_p, x_p), \\ \pi_i(X)(x_p, x_q) &= \pi_{i-1}(E_0^{-1}(x_p) \cap E_1^{-1}(x_q), \gamma_{p,q}). \end{cases}$$

The basing of X allows us to give this explicit description of the homotopy category, and the fact that X is fibrant ensures that we are able to compose arrows (the idea is that the composite of two arrows defines a homotopy class of exit paths in X , and fibrancy allows us to replace this by a homotopy class of elementary exit paths in X

Remark 10.3.0.9. In first approaching this construction of the homotopy categories of a stratified space, it appeared that a basing should be a map of the form $\|SS(P)_1\| \rightarrow X$ which factors the counit on P . This is enough to naïvely define the i -homotopy category of a stratified space, however the lack of compatibility between basepaths means that we may not have a way of composing arrows, and hence may not have a category for $i \geq 2$.

Remark 10.3.0.10. Notice that the homotopy categories of a based fibrant space $X \rightarrow P$ are P -pointed in the category \mathbf{Cat}/P , so it follows that P is a retract of $\pi_n(X)$. To see this, note that a basing of X induces a basing on $\Omega^{n-1}X$. In particular, applying $SS(-)$ to the basing of $\Omega^{n-1}X$ gives Figure 10.13.

$$\begin{array}{ccc} SS(\|SS(P)\|) & \xrightarrow{SS(B_{\Omega^{n-1}X})} & SS(\Omega^{n-1}X) \\ & \searrow \epsilon_{SS(P)} & \downarrow SS(S_{\Omega^{n-1}X}) \\ & & SS(P) \end{array}$$

Figure 10.13: Applying $SS(-)$ to the basing of $\Omega^{n-1}X$.

To Figure 10.13 we apply τ_1 and arrive at Figure 10.14.

$$\begin{array}{ccc} P \simeq \tau_1(SS(P)) & \xrightarrow{\tau_1(\eta_{SS(P)})} & \tau_1(SS(\|SS(P)\|)) \xrightarrow{\tau_1(SS(B_{\Omega^{n-1}X}))} \tau_1(SS(\Omega^{n-1}X)) = \Pi_n(X) \\ & \searrow \tau_1(SS(\epsilon_P)) & \downarrow \tau_1(SS(S_{\Omega^{n-1}X})) \\ & & \tau_1(SS(P)) \simeq P \end{array}$$

$\mathbb{1}_{\tau_1(SS(P))}$

Figure 10.14: Exhibiting P as a retract of $\Pi_n(X)$.

As a category, P is equivalent to $\tau_1(SS(P))$ and hence we see that P is a retract of $\Pi_n(X)$. In particular, because $\pi_n(X)$ is defined as a full subcategory of $\Pi_n(X)$ on the basing of $\Omega^{n-1}X$,

it follows that $\pi_n(X)$ is also P -pointed.

Notation. Noticing that each homotopy category of a based fibrant stratified space $\pi_n(X)$ is P -pointed justifies calling a homotopy category *trivial* if $\pi_n(X) \cong P$.

Remark 10.3.0.11. For a cofibrant-fibrant based stratified space with connected holinks, the homotopy categories will be independent of choice of basing, because in a cofibrant stratified space the strata are path-connected.

The homotopy categories of a based fibrant stratified space define an $\mathbb{N}_{\geq 2}$ -indexed family of functors $\pi_i: \mathbf{BStrat}_{\mathcal{F}} \rightarrow \mathbf{GrpCat}$ from the category of based fibrant stratified spaces to the category of group-enriched small categories. Moreover, for $i \geq 3$, the categories $\pi_i(X)$ are enriched over the category of abelian groups, because homotopy groups π_i are abelian for $i \geq 2$. In the case that $i = 1$, we have categories enriched over the category of pointed sets (where the monoidal product is given as the smash product of pointed sets). This is because for $p \neq q$, the arrows between x_p and x_q are given by $\pi_0(E_0^{-1}(x_p) \cap E_1^{-1}(x_q), \gamma_{p,q})$, which is only a pointed set and not a group.

Example 10.3.0.12. Consider a naturally stratified 2-sphere glued to a $\|\Delta^2\|$ by attaching the point $(-1, 0, 0) \in S^2$ which lives over $\{0\}$ in the poset $[1]$ of S^2 to the vertex $\{0\} \in \|\Delta^2\|$ and attaching $(1, 0, 0)$ to the vertex $2 \in \|\Delta^2\|$. This fibered coproduct is stratified over the poset $[2]$, and will be denoted by X . Take the coproduct of X with $\|\Delta^2\|$, but alter the stratification. We chose to stratify $X \sqcup \|\Delta^2\|$ by the projection map $X \sqcup \|\Delta^2\| \rightarrow [2] \sqcup [2] \rightarrow [2]$. A basing of this space corresponds to a map $B_{X \sqcup \|\Delta^2\|}: \|\Delta^2\| \rightarrow X \sqcup \|\Delta^2\|$.

If we take the basing as the identity map to $\|\Delta^2\|$, then we do not detect the copy of S^2 contained in X . If we take the map to be the identity map onto $\|\Delta^2\| \subseteq X$, then we detect the existence of the path-component of S^2 in π_1 , but cannot detect anything further to identify that there is $S^2 \subseteq X$.

For the homotopy categories of a fibrant stratified space to be a homotopically meaningful notion, we need to know that weak equivalences of stratified spaces are reflected as isomorphisms of homotopy categories.

Corollary 10.3.0.13. *A stratified morphism between based fibrant stratified spaces which induces a bijection on posets is a stratified weak equivalence if and only if it induces an isomorphism on homotopy categories.*

Proof. Follows from Theorem 9.4.0.5 and the construction of the homotopy categories. ■

To understand the notion of isomorphism between homotopy categories and how this relates from the notion of categorical equivalence, we have the following result.

Proposition 10.3.0.14. *A group-enriched functor between homotopy categories $F: \pi_i(X) \rightarrow \pi_j(Y)$ which is an equivalence of categories is an isomorphism.*

Remark 10.3.0.15. By a group-enriched functor, we mean a functor which is a group homomorphism on all Hom-groups. In particular, this applies to any functorially induced morphism between homotopy categories from a stratified morphism between based stratified spaces.

Proof. The functor F is an equivalence of categories if and only if F is fully faithful and essentially surjective on objects. Because of the construction of $\pi_i(X)$ and $\pi_j(Y)$, it is not possible for two objects to be isomorphic unless they are the same. Therefore an equivalence between homotopy categories is a map which is fully faithful and is surjective on objects.

To complete the proof, we also need to show that F is injective on objects; to do this, consider two objects $x_p \neq x_q \in \pi_i(X)$ and assume that $F(x_p) = F(x_q)$. Consider the Hom group $\pi_i(Y)(F(x_p), F(x_q))$; this group has at least one element (the homotopy class of the constant map), and by assumption both $\pi_i(X)(x_p, x_q)$ and $\pi_i(X)(x_q, x_p)$ fully faithfully map onto it by F . However this is impossible because we would need both $\pi_i(X)(x_p, x_q)$ and $\pi_i(X)(x_q, x_p)$ to be non-empty, but if $\pi_i(X)(x_p, x_q)$ was non-empty then this forces $\pi_i(X)(x_q, x_p)$ to be empty, and vice-versa because we cannot have elementary exit paths from X_p to X_q and from X_q to X_p . ■

The following proposition relates loop spaces and homotopy categories, which parallels the isomorphism of homotopy groups $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ for a connected topological space X .

Proposition 10.3.0.16. *For a based fibrant space X , there is an isomorphism of homotopy categories $\pi_i(\Omega X) \cong \pi_{i+1}(X)$.*

Proof. Consider the i -homotopy category of ΩX . By Proposition 10.3.0.7, an arrow in this category is given by a stratum preserving homotopy class of maps $S^i \rightarrow \Omega X$, such that the basing of S^i is sent to a basepoint of ΩX . Corollary 10.2.0.28 tells us that this class is in bijection with a stratum preserving homotopy class $\Sigma S^i \rightarrow X$, which by Example 10.2.0.21 is the same as a stratum preserving homotopy class of maps $S^{i+1} \rightarrow X$. ■

Chapter 11

Homotopy Theoretic Constructions

11.1 Long Exact Sequence of a Stratified Fibration

The goal of this section is to construct the long exact sequence associated to a stratified fibration between fibrant stratified spaces. In order to do this, we need a notion of the stratified fiber of a stratified fibration, and for such a notion to give a long exact sequence of homotopy categories, we need to rectify the issue that the stratified spaces involved in a stratified fibration may be stratified over different posets.

Consider a stratified fibration $F: E \rightarrow B$ where $S_E: E \rightarrow P$ and $S_B: B \rightarrow Q$ are based fibrant stratified spaces. To fix the issue that E and B are stratified over different posets, we change the poset over which B is stratified. We construct the pullback in $k\mathbf{Top}$, shown in Figure 11.1, where f is the underlying poset map of F , and B_P is stratified over the poset P . The intuitive idea is that we want to stratify B over P , taking into account when multiple elements of P are mapped to the same element of Q . Consequently, we take multiple copies of each stratum of B to which multiple strata of E map, allowing us to index B instead by the poset P .

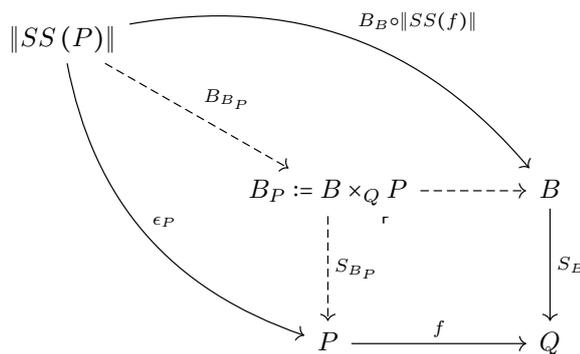


Figure 11.1: Changing the poset over which B is stratified.

The pullback is the set $B_P := B \times_Q P = \{(b, p) \mid S_B(b) = f(p) \text{ in } Q\}$, equipped with the coarsest topology such that the projection maps to B and P are continuous. The stratification map S_{B_P} is surjective because S_B is surjective, and B_P is based via the map B_{B_P} .

Remark 11.1.0.1. In constructing B_{B_P} there are two natural choices of map $\|SS(P)\| \rightarrow B$; either $B_B \circ \|SS(f)\|$ or $F \circ B_E$. However, by definition of a based morphism, it follows that $B_B \circ \|SS(f)\| = F \circ B_E$ and therefore it does not matter which we use.

If E and B are both trivially stratified, then $B_P = B$. If f is an isomorphism of posets then $B_P \cong B$, because isomorphisms are stable under pullback. Note also that if B is fibrant, then Proposition 9.1.0.1 shows the map $S_B: B \rightarrow Q$ is a fibration. Fibrations are stable under pullback so the map S_{B_P} is also a fibration over the fibrant stratified space P , which implies that B_P is also fibrant.

We would like to replace B in the fibration F by the space B_P ; to do this however, we need to also know that the map F' indicated in Figure 11.2 is a stratified fibration.

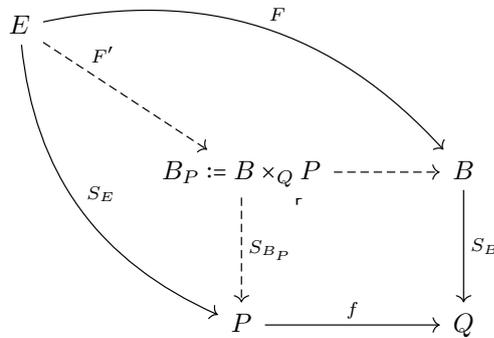


Figure 11.2: The morphism F' is induced by F .

To do this, assume that the stratified fibration F is between fibrant stratified spaces E and B . By the previous paragraph, the assumption that B is fibrant implies that B_P is also fibrant. To show that F' is a stratified fibration, we need to show that it lifts on the right against inner horn inclusions and the inclusion $\|*\| \hookrightarrow \mathcal{J}$. Consider an inner horn inclusion as in Figure 11.3.

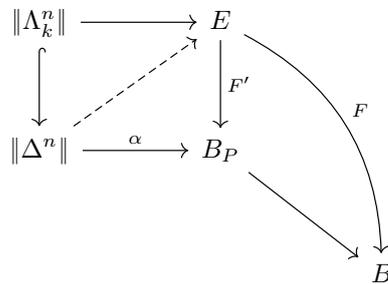


Figure 11.3: We wish to show that F' is a stratified fibration.

Note that we can extend the stratified simplex in B_P to a stratified simplex in B . By assumption, F is a stratified fibration between fibrant stratified spaces, hence we can construct the dashed filler indicated in Figure 11.3 for the inner horn inclusion lifting against F . The constructed dashed filler will automatically make the upper triangle commute, and we need to show that it also makes the bottom triangle commute. To do this, recall that we defined F' as a unique map induced by F and S_E ; we can pre-compose these maps with the dashed filler to construct the composite arrow to B_P , which will have a unique induced map $\|\Delta^n\| \rightarrow B_P$. By universality, this unique map is equal to the given map $\alpha: \|\Delta^n\| \rightarrow B_P$ in Figure 11.3. This shows that the dashed filler also makes the bottom square commute. Exactly the same logic shows that we can construct a filler for the inclusion $\|\ast\| \hookrightarrow \mathcal{J}$, which shows that $F': E \rightarrow B_P$ is a stratified fibration.

Notation. From now onwards, we will also denote the map $F': E \rightarrow B_P$ simply by F .

Definition 11.1.0.2. Let $F: E \rightarrow B$ be a stratified fibration between fibrant based stratified spaces. Define the *stratified fiber of F* , denoted by $F^{-1}(\|SS(P)\|)$, to be the pullback in **Strat**, depicted in Figure 11.4.

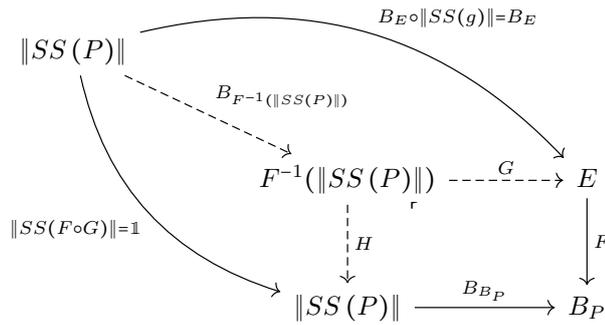


Figure 11.4: The fiber $F^{-1}(\|SS(P)\|)$ of the stratified fibration F .

In the unstratified case, the fiber of a fibration is taken over a basepoint of the base space; the stratified definition is analogous in the sense that it considers the fiber of the entire basing of B_P . Note that the underlying poset map of F is the identity on P , which in particular implies that the stratified fiber is also stratified over the poset P . The stratified fiber of F is based by the stratified morphism $B_{F^{-1}(\|SS(P)\|)}$ depicted as the dashed morphism induced to the pullback, shown in Figure 11.4. The underlying poset map of $B_{F^{-1}(\|SS(P)\|)}$ is the identity on P , which implies that the counit at P (which provides the stratification map of $\|SS(P)\|$), factors through the basing.

Remark 11.1.0.3. Note that the stratified fiber is constructed using the stratified fibration over B_P rather than B , and that the two constructions are not equivalent.

To move towards the induced long exact sequence of homotopy categories associated to

a stratified fibration, we need to understand what it means to have a long exact sequence between group-enriched categories.

Definition 11.1.0.4. Consider the following diagram where each A_i is a group-enriched category, and the arrows $\mathcal{F}_i: A_i \rightarrow A_{i-1}$ are group-enriched functors as depicted:

$$\dots \rightarrow A_{i+1} \xrightarrow{\mathcal{F}_{i+1}} A_i \xrightarrow{\mathcal{F}_i} A_{i-1} \xrightarrow{\mathcal{F}_{i-1}} \dots$$

Then we say we have a *long exact sequence of group-enriched categories* if the functors \mathcal{F}_i satisfy the following two conditions:

1. Every functor \mathcal{F}_i is an isomorphism on objects;
2. The group-enriched functors \mathcal{F}_i must be exact with respect to each Hom-group. Explicitly this says that for all $X, Y \in A_{i+1}$ and $i \in \mathbb{N}$:

$$\text{Im}(\mathcal{F}_{i+1}(\text{Hom}_{A_{i+1}}(X, Y))) = \text{Ker}(\mathcal{F}_i(\text{Hom}_{A_{i-1}}(\mathcal{F}_{i+1}(X), \mathcal{F}_{i+1}(Y)))).$$

Theorem 11.1.0.5. For a stratified fibration $F: E \rightarrow B$ between based fibrant stratified spaces $E \rightarrow P$ and $B \rightarrow Q$, we can construct a long exact sequence of homotopy categories arising from the triple:

$$F^{-1}(\|SS(P)\|) \hookrightarrow E \rightarrow B_P.$$

Proof. The restriction of the stratified fibration $F: E \rightarrow B_P$ to each stratum or holink gives a Serre fibration, which allows us to define a boundary functor ∂ as the identity on objects and in the obvious way on morphisms. Considering the induced functors between homotopy categories, we have an induced long sequence of homotopy categories, of the form:

$$\dots \xrightarrow{\partial} \pi_i(F^{-1}(\|SS(P)\|)) \xrightarrow{G_*} \pi_i(E) \xrightarrow{F_*} \pi_i(B_P) \xrightarrow{\partial} \pi_{i-1}(F^{-1}(\|SS(P)\|)) \xrightarrow{G_*} \dots \rightarrow \pi_2(B_P).$$

By construction the functors ∂ , F_* and G_* induce a bijection on objects of the homotopy categories, because F and G induce isomorphisms on posets.

We explain the construction of ∂ explicitly; we know that B_P and $F^{-1}(\|SS(P)\|)$ are stratified over isomorphic posets, so need to construct a group-enriched functor ∂ which realises this bijection on objects of the respective homotopy categories. We can restrict the stratified fibration F to some $g(p) \in P$, giving a stratified map between strata $F_{g(p)}: E_{g(p)} \rightarrow B_{P_{f(g(p))}}$, which we want to show is a Serre fibration. Consider the lifting problem of any inner horn or the inclusion $\|*\| \hookrightarrow \|\mathcal{J}\|$ against $F_{g(p)}$; we know that F is a stratified fibration therefore lifts exist against all of these maps, and the restriction to the $g(p)$ -stratum means that the image must entirely be contained in one stratum of E and B_P . Therefore the lift is in fact a lift into $E_{g(p)}$, showing that $F_{g(p)}$ is a stratified fibration. The fact that $F_{g(p)}$ is a stratified fibration implies that $SS(F_{g(p)})$ is a quasi-fibration between Kan complexes (because the restriction is between single stratum stratified spaces), therefore by [Rie08, Proposition 5.10] is a Kan fibration. Ap-

plication of the unstratified adjunction, and noticing that $SS(-)$ of a trivially stratified space is the same as $\text{Sing}(-)$, shows that $F_{g(p)}$ lifts against any inclusion $|\Lambda_k^n| \hookrightarrow |\Delta^n|$ for any $0 \leq k \leq n$, and hence is a Serre fibration. Associated to the Serre fibration, we have a long exact sequence of homotopy groups with a boundary morphism $\partial: \pi_i(B_{P_{fg(p)}}) \rightarrow \pi_{i-1}(F^{-1}(\|SS(P)\|)_p)$. Because both f and g are isomorphisms on posets, we assign the object of $\pi_i(B_P)$ corresponding to $fg(p)$ to the object p in $\pi_{i-1}(F_p)$. This assignment defines a map on objects (which is clearly a bijection) between homotopy categories $\partial: \pi_i(B_P) \rightarrow \pi_{i-1}(F^{-1}(\|SS(P)\|))$ and the boundary morphism arising from the Serre fibration defines ∂ on the endomorphisms of $\pi_i(B_P)$. To complete the proof, consider the group of arrows between any two objects in $\pi_i(B_P)$; this corresponds to the $(i-1)$ st homotopy group of the holink between the corresponding basepoints. The same proof shows that the induced map on holinks is also a Serre fibration (noting that holink spaces are trivially stratified within the path space and hence under $SS(-)$ they give Kan complexes), from which we define $\partial: \pi_i(B_P) \rightarrow \pi_{i-1}(F^{-1}(\|SS(P)\|))$ on the remaining morphisms.

It is immediate from the construction of ∂ and the restrictions of F_* and G_* to strata and holinks, that we have an induced exact sequence with respect to Hom-groups, ending at $\pi_2(B_P)$. \blacksquare

Informally, we are able to extend the long exact sequence of homotopy groups through the following π_1 homotopy categories:

$$\dots \xrightarrow{\partial} \pi_2(F^{-1}(\|SS(P)\|)) \xrightarrow{G_*} \pi_2(E) \xrightarrow{F_*} \pi_2(B_P) \rightarrow \pi_1(F^{-1}(\|SS(P)\|)) \rightarrow \pi_1(E) \rightarrow \pi_1(B_P).$$

To make sense of this extension, we drop the requirements that the induced maps on arrows must be group homomorphisms. Exactness still holds for the extended long exact sequence, where we consider the arrows arising from 0-homotopy groups as pointed sets. The neutral element of $\pi_0(E_0(x_p) \cap E_1^{-1}(x_q), \gamma_{p,q})$ is the element representing the distinguished basepath $\gamma_{p,q}$. This follows in precisely the same way that we can extend the long exact sequence of a fibration to the π_0 level, where we drop the requirement that the induced maps must be group homomorphisms.

11.2 A Possible Construction of Stratified Postnikov Towers

In this section, we sketch an idea for how the Postnikov Tower of a stratified space could be constructed. We are currently unable to show that it satisfies all the required properties, but will indicate what we need to complete the proof. One consequence that we hope to explore further (although not in this thesis) is that from this construction there is a stratified version of Eilenberg-Mac Lane spaces, which appear as the fibers over each map f_k . Recall that for a stratified space $X \rightarrow P$, we say that $\pi_i(X)$ is trivial if $\pi_i(X) \cong P$ where we think of the poset P as a category.

Notation. The natural stratification of S^n which we introduced in Example 10.2.0.21 will be vital here. We can extend the natural stratification of the n -sphere to the natural stratification of an $n + 1$ -disk by defining $S_{D^{n+1}}: D^{n+1} \rightarrow [1]$ by sending the point $(-1, 0, \dots, 0)$ to $0 \in [0]$ with the remainder of D^n sent to $1 \in [1]$.

Lemma 11.2.0.1. *Consider a based fibrant stratified space $X \rightarrow P$, and glue a naturally stratified $(n + 1)$ -disk onto X along a naturally stratified n -sphere. Then the inclusion $\pi_i(X) \rightarrow \pi_i(X \amalg D^n)$ is an isomorphism of homotopy categories for all $i < n$ and full as a functor between categories for $i = n$.*

Proof. On strata, if the poset map of the n -sphere factors through $*$, then gluing on a filler we may kill a homotopy class in $\pi_n(X_p, x_p)$ for $p \in P$ the image of the poset map. If the poset map does not factor through $*$, then the filler D^n glued on may kill a homotopy class in $\pi_{n-1}(E_0^{-1}(x_p) \cap E_1^{-1}(x_q), \gamma_{p,q})$ where $p \leq q \in P$ is the image of the underlying poset map. ■

By iterating this process, we are able to kill the homotopy groups of a space; applying this ideology to stratified maps of stratified spaces, we are able to kill off non-trivial arrows in the homotopy category. Using this, we arrive at the following result.

Corollary 11.2.0.2. *For a based fibrant stratified space $X \rightarrow P$, we can construct a relative complex $X \hookrightarrow Y$ by attaching stratified $(n + 1)$ -disks along boundary stratified n -spheres so that $\pi_i(X) \rightarrow \pi_i(Y)$ is an isomorphism for all $i < n$ and such that $\pi_n(Y) \cong P$.*

Proof. Repeated application of Lemma 11.2.0.1 to generators of the appropriate homotopy groups, kills off all non-trivial arrows in the n -homotopy category of X . ■

Remark 11.2.0.3. As in the topological scenario, attaching disks via spheres can have undesirable effects on higher homotopy categories. Corollary 11.2.0.2 parallels the topological situation, where attachment of disks can be understood on homotopy invariants upto the dimension of the disks attached.

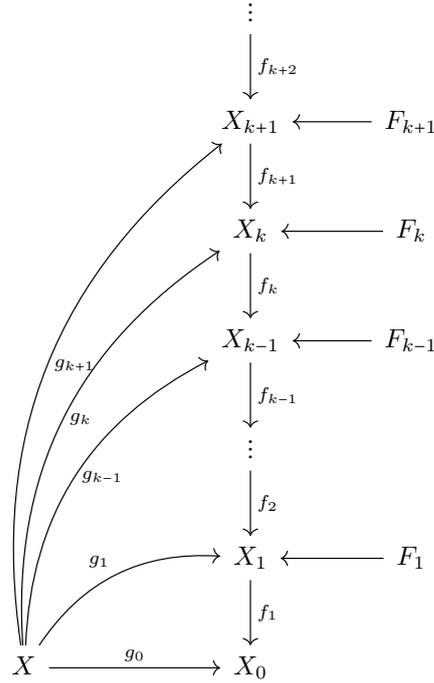
Conjecture 3 (Stratified Postnikov Towers). *For a based fibrant stratified space $X \rightarrow P$ with path-connected strata, we can construct a tower of stratified fibrations:*

$$\dots \xrightarrow{f_{k+1}} X_k \xrightarrow{f_k} X_{k-1} \xrightarrow{f_{k-1}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0,$$

equipped with compatible maps $g_i: X \rightarrow X_i$ (compatible in this context means that $g_i = f_{i+1} \circ g_{i+1}$), such that the following three conditions hold:

1. *for $0 < i < j$, the homotopy categories $\pi_j(X_i)$ are trivial, and for $i = 0$ we have $X_0 \simeq \|SS(P)\|$;*
2. *each map f_i induces an isomorphism on π_j for $j \leq i$;*
3. *the stratified fiber over f_i , which we denote by F_i , has $\pi_i(F_i) \cong \pi_i(X)$, and $\pi_j(F_i)$ trivial whenever $i \neq j$.*

The Postnikov tower for such a based fibrant stratified space X is illustrated in Figure 11.5.


 Figure 11.5: Illustrating the stratified Postnikov tower of X .

Outline of Proof of Conjecture 3. We construct a family of stratified spaces \widehat{X}_k which are obtained from X by attaching an i -disk along its boundary sphere, corresponding to every based stratified map $s_i: S^{i-1} \rightarrow X$ for $i > k$. This immediately implies that \widehat{X}_0 is stratum preserving homotopic to $\|SS(P)\|$. Explicitly, the stratified morphisms f_i are described as a transfinite composition of i -disk attachments, where we attach a stratified i -ball along its boundary for any stratified sphere in X , given by each map s_i . The stratified morphisms $g_i: X \rightarrow \widehat{X}_i$ are the canonical inclusion of X as a subspace of \widehat{X}_i .

The effect of attaching a stratified disk along a stratified sphere is explained in the proof of Lemma 11.2.0.1. A consequence of this construction is that we have a tower of stratified spaces with inclusion maps as demonstrated below:

$$\dots \xrightarrow{\widehat{f}_{k+2}} \widehat{X}_{k+1} \xrightarrow{\widehat{f}_{k+1}} \widehat{X}_k \xrightarrow{\widehat{f}_k} \widehat{X}_{k-1} \xrightarrow{\widehat{f}_{k-1}} \dots \xrightarrow{\widehat{f}_2} \widehat{X}_1 \xrightarrow{\widehat{f}_1} \widehat{X}_0 \simeq \|SS(P)\|.$$

There is a difference at the moment between our construction and the construction of a Postnikov Tower; currently each of our maps \widehat{f}_i are cofibrations rather than fibrations, and we do not know that each space \widehat{X}_i is fibrant.

We would like to fix this issue by applying the model structure on **Strat**, in particular the factorisation axiom of any arrow. The idea is to apply the factorisation to give a Postnikov tower of fibrations, with the properties we desire. The stratified space \widehat{X}_0 is stratified homo-

topy equivalent to $\|SS(P)\|$, and is therefore fibrant, so we set $X_0 = \widehat{X}_0$.

Consider the stratified morphism $\widehat{f}_1: \widehat{X}_1 \rightarrow \widehat{X}_0 \xrightarrow{\cong} X_0$; we would like to use the factorisation of any morphism as relative \mathcal{J} -cell complex followed by a fibration, given by applying the small object argument to the set of stratified inner horn inclusions and the inclusion $\|*\| \hookrightarrow \|\mathcal{J}\|$. The factorisation gives $\widehat{f}_1 = f_1 \circ j_1: \widehat{X}_1 \rightarrow X_1 \rightarrow X_0$. If we instead used the inexplicit generating set of acyclic cofibrations, it is unclear that X_1 is fibrant. We would like X_1 to have homotopy groups of strata and of holinks defined by \widehat{X}_1 . By construction of the factorisation j_1 , we can see that $\widehat{X}_1 \rightarrow X_1$ induces homotopy equivalences on strata, however it is not clear that it induces a weak homotopy equivalence on holinks. The construction of the stratified morphism f_1 implies that X_1 is fibrant. If we could prove that a relative $\|\mathcal{J}\|$ -cell complex was a weak homotopy equivalence on holinks, then the idea of the proof would be to carry this procedure; so at the next stage we factorise the morphism $\widehat{X}_2 \xrightarrow{\widehat{f}_1} \widehat{X}_1 \xrightarrow{\sim} X_1$ to give a fibrant stratified space X_2 which is weakly equivalent to \widehat{X}_2 and stratified fibration f_2 . The final definition we make is that the map g_k is defined as the composition of \widehat{g}_k with the map that would be an acyclic cofibration $\widehat{X}_k \xrightarrow{\sim} X_k$.

We end the thesis with an open question, which would be interesting to investigate if we were able to complete the construction of the stratified Postnikov tower.

Open Question 1. *We expect a notion of Eilenberg-Mac Lane stratified spaces; intuitively these should be constructed as the fiber over each map $X^i \rightarrow X^{i-1}$, indicated by the stratified spaces F_i in Figure 11.5. If these exist, one could ask whether an appropriate notion of stratified EM-spectra co-represents an analogous theory to cohomology, in this context.*

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