ON THE CLASSIFICATION OF HYPERBOLIC ROOT SYSTEMS OF THE RANK THREE. PART II

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Abstract. Here we prove classification results announced in Part I (alg-geom/9711032). We classify maximal hyperbolic root systems of the rank 3 having restricted arithmetic type and a generalized lattice Weyl vector ρ with $\rho^2 \geq 0$ (i.e. of elliptic or parabolic type). We give classification of all reflective of elliptic or parabolic type elementary hyperbolic lattices of the rank three.

We apply the same method (narrow places of polyhedra) which was developed to prove finiteness results on reflective hyperbolic lattices. We also use some additional arithmetic arguments: studying of class numbers of central symmetries.

The same methods permit to get similar results for hyperbolic type. We will consider hyperbolic type in Part III.

These results are important for Theory of Lorentzian Kac–Moody algebras and some aspects of Mirror Symmetry.

0. Introduction

Here we prove results announced in Part I [N14]. We continue numeration of Sections started in Part I. We also keep notations of Part I.

We consider main hyperbolic (i. e. of signature $(1,k)$) lattices (i. e. non-degenerate integral symmetric bilinear forms) S of the rank three and with square-free determinant d. Here "main" means that S should be even for even d. The lattice S is defined uniquely by its determinant $d = \det(S)$ and some additional invariant η (see Sect. 2.2 of Part I or Sect. 3.2 here) where $0 \leq \eta < 2^t$, here t is the number of odd prime divisors of d.

Let $W(S)$ be the reflection group of S (generated by reflections in roots $\alpha \in S$ with $\alpha^2 < 0$) and $\mathcal{M}$ its fundamental polyhedron in hyperbolic space $\mathcal{L}(S)$ defined by S. We denote by $A(\mathcal{M}) = \{\phi \in O^+(S) \mid \phi(\mathcal{M}) = \mathcal{M}\}$ the group of symmetries of $\mathcal{M}$. An involution $u \in A(\mathcal{M})$ is called a *central symmetry* if u acts as a central symmetry in $\mathcal{L}(S)$. Two central symmetries from $A(\mathcal{M})$ are called *equivalent* if they are conjugate in $A(\mathcal{M})$. In Sect. 3, Theorem 3.2.1 we give a formula for the number $h = hnr(S)$ of classes of central symmetries of the lattice S with invariants (d,η). This formula uses the Legendre symbol and class numbers of imaginary quadratic fields. We use this formula to give the list of all the lattices S with $h = hnr(S) \leq 1$ and $d \leq 100000$. This list was announced in Table 3 of Part I and contains 206 invariants (d,η). This list is important for classification of the reflective lattices S.

A hyperbolic lattice S is called *reflective* if there exist a non-zero $\rho \in S$ and a subgroup $A \subset A(\mathcal{M})$ of finite index such that $A(\rho) = \rho$. Here ρ is called a *generalized lattice Weyl vector*. If S has a generalized lattice Weyl vector ρ with

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$\rho^2 > 0$, then S is called *reflective of elliptic type* (or *elliptically reflective*). If S has a generalized lattice Weyl vector ρ with $\rho^2 = 0$ and does not have a generalized lattice Weyl vector ρ with $\rho^2 > 0$, then S is called reflective of *parabolic type*. If S has a generalized lattice Weyl vector ρ with $\rho^2 < 0$ and does not have a generalized lattice Weyl vector ρ with $\rho^2 \geq 0$, then S is called reflective of *hyperbolic type*. It is easy to prove that $h \leq 1$ if S is reflective of elliptic type, $h = 0$ if S is reflective of parabolic type and $h = 0, 2$ if S is reflective of hyperbolic type and $\mathrm{rk}\ S = 3$. Thus, Table 3 contains all elliptically or parabolically reflective main hyperbolic lattices S of the rank 3 and with square-free determinant $d \leq 100000$.

In [N4], [N5], [N11] and [N13] we proved finiteness results about reflective hyperbolic lattices using some geometrical arguments: studying of narrow places of fundamental polyhedra. In Sect. 4 we improve and optimize this method for 2-dimensional case to apply it for classification of reflective hyperbolic lattices of rank 3. In Sect. 4.3 we apply these results to fundamental polygons of the reflection groups of reflective hyperbolic lattices of elliptic or parabolic type and of rank 3. As a result, we get estimates of the determinant and some other invariants of these lattices.

In Sect. 5 we apply results of Sects. 4 and 5 to classify all elliptically or parabolically reflective hyperbolic lattices of the rank 3 and with square-free determinant. The list of these lattices was announced in Tables 1 and 2 of Part I (see [N14]). In particular, we prove that all elliptically or parabolically reflective main hyperbolic lattices of the rank 3 and with square-free determinant are contained in Table 3. To study reflective type of lattices, we use Vinberg's algorithm [V2].

The same methods as developed here permit to classify reflective hyperbolic lattices of the rank 3 of hyperbolic type. We hope to do this in Part III.

3. The number of classes of central symmetries of main hyperbolic lattices with square-free determinant and of the rank 3.

3.1. Reminding of some classical results about binary positive lattices.

Here we remind some classical results about binary positive quadratic forms (e. g. see [B-Sh] and [C]).

We consider binary (i.e. of the rank two) positive definite lattices K with square-free determinant $d = \det K$. They will be called *fundamental*. If d is odd, the fundamental lattice K is unimodular over $\mathbb{Z}_2$. Considering K over $\mathbb{Z}_2$, it is easy to see that K is odd if $d \equiv 1, 2 \mod 4$. If $d \equiv -1 \mod 4$, the lattice K may be odd or even. The number

$$D = \begin{cases} -\det K, & \text{if } K \text{ is even,} \\ -4\det K, & \text{if } K \text{ is odd} \end{cases} \tag{3.1.1}$$

is called *discriminant* of a binary lattice K. Discriminants of fundamental binary lattices are called *fundamental discriminants*. Thus, for a fundamental discriminant D either $D \equiv 1 \mod 4$ or $D \equiv \pm 4, 8 \mod 16$. A binary positive lattice K is called *classical fundamental* if $d = \det K$ is square-free and additionally K is even if $d \equiv -1 \mod 4$. The discriminant D of a classical fundamental binary lattice K is called the *classical fundamental discriminant*. Thus, for a classical fundamental discriminant D either $D \equiv 1 \mod 4$ (when K is even) or $D \equiv -4, 8 \mod 16$.

It is known

Theorem 3.1.1. *The number $h(D)$ of proper classes (i. e. classes of preserving orientation isomorphisms of oriented lattices) of classical fundamental binary lattices K of the discriminant $D<0$ is equal to the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ of the discriminant D and is given by the Dirichlet's formula*

$$h(D)=\frac{w}{2|D|}\sum_{0<r<|D|}\left(\frac{D}{r}\right)r \tag{3.1.2}$$

where

$$w=\begin{cases}6, & \text{if } D=-3,\\ 4, & \text{if } D=-4,\\ 2, & \text{otherwise,}\end{cases} \tag{3.1.3}$$

and $\left(\frac{D}{r}\right)$ is the Kronecker (i.e. generalized Legendre) symbol.

Let K be an odd binary lattice of an odd square-free determinant d where $d\equiv -1 \mod 4$ (i. e. it is fundamental, but it is not classical fundamental). The maximal even sublattice of K is the lattice $T(2)$ where T is the fundamental (even) lattice of the determinant d. Thus, $T(2)\subset K$ is an odd overlattice of index 2 of the lattice $T(2)$. If $d\equiv -1 \mod 8$, then the overlattice K is unique. If $d\equiv -5 \mod 8$, then there are 3 overlattices $T(2)\subset K$ (which we should consider up to proper automorphisms of T to get the number of proper classes of K). We have $\operatorname{discr} K=-4d$ and denote $h(-4d)$ the number of proper classes of lattices K of that discriminant. By considerations above, we get

$$h(4D)=\begin{cases}h(D), & \text{if } D\equiv 1 \mod 8,\\ 3h(D), & \text{if } D\equiv 5 \mod 8 \text{ and } D<-3,\\ 1=h(D), & \text{if } D=-3.\end{cases} \tag{3.1.4}$$

Let K be a fundamental positive binary lattice of the discriminant D. Here $D\equiv 0,1 \mod 4$. If $D\equiv 0 \mod 4$, then $D\equiv \pm 4, 8 \mod 16$. We use the discriminant form technique [N1]. The genus of K is defined by the discriminant form of K (it is quadratic if K is even, and is bilinear if K is odd) which is defined by the map $\mu: p\mapsto \mu_p\in\{0,1\}$, p runs through all odd prime $p|D$. For odd (prime) $p|D$, the discriminant form $b_{K_p}=b^{(p)}_{\theta_p}(p)$ where $\left(\frac{\theta_p}{p}\right)=(-1)^{\mu_p}$. The form b_{K_2} is trivial if either $D\equiv 1 \mod 4$ or $D\equiv 4 \mod 8$. If $D\equiv 8 \mod 16$, then $b_{K_2}=b^{(2)}_1(2)$.

We denote by t *the number of prime divisors of* D. If $D\equiv 1 \mod 4$ (equivalently, K is even), existence of K with the invariants (D,μ) is equivalent to

$$\sum_{p|d}[(1-p)+4\mu_p]\equiv 2 \mod 8, \tag{3.1.5}$$

and there are 2^{t-1} genuses.

If $D\equiv -4 \mod 16$, (equivalently, $d\equiv 1 \mod 4$) existence of K with the invariants (D,μ) is equivalent to $d\equiv 1 \mod 4$ which is given. Thus, number of genuses is equal to 2^{t-1}.

If $D\equiv 4 \mod 16$ (equivalently, $d\equiv -1 \mod 4$ and the lattice K is odd), existence of K is equivalent to existence of the corresponding even lattice of the discriminant $D = -d$ (by the construction above), and number of genuses is equal to 2^{t-2}.

If $D \equiv 8 \mod 16$, there are no conditions on (D, μ) for existence of K, and number of genuses is equal to 2^{t-1}.

Thus, *the number of different genuses of the discriminant D is equal to* $2^{\tau(D)}$ where

$$\tau(D) = \begin{cases} t-1, & \text{if } D \equiv 1 \mod 4, \\ t-1, & \text{if } D \equiv -4,\, 8 \mod 16, \\ t-2, & \text{if } D \equiv 4 \mod 16, \end{cases} \tag{3.1.6}$$

and t is the number of all different prime divisors of D. It is known that each genus contains the same number of classes. Thus, number of classes $h(D)_g$ in a genus of a fundamental lattice K is equal to

$$h(D)_g = \frac{h(D)}{2^{\tau(D)}}. \tag{3.1.7}$$

A fundamental binary lattice K is called *ambiguous* if $O(K)$ contains a reflection (equivalently, the oriented lattice K is proper equivalent to the lattice K with the opposite orientation). If $O(K)$ contains a reflection in $\delta_1 \in K$ where δ_1 is primitive, then it contains reflection in a primitive $\delta_2 \in K$ which is orthogonal to δ_1. We can suppose that $n_1 = \delta_1^2 \le n_2 = \delta_2^2$. It follows that a reflective lattice K is generated by either $\{\delta_1, \delta_2\}$ or $\{\delta_1, \delta_2, (\delta_1 + \delta_2)/2\}$. We say that the type is I if we have the first possibility, and the type is II if we have the second one. It is easy to see (using classification of 2-dimensional reflection groups) that $n_1 \le n_2$ and type are invariants of the ambiguous class K except $K = (1, 1; I) = (2, 2; II)$ of the discriminant -4. Except this case, two ambiguous binary lattices are isomorphic iff they have the same invariants $(n_1, n_2; \text{type})$. We denote by $K(n_1, n_2; \text{type})$ the ambiguous positive lattice with invariants $(n_1, n_2; \text{type})$. A lattice $K(n_1, n_2; I)$ exists for any natural $n_1, n_2 \in \mathbb{N}$. A lattice $K(n_1, n_2; II)$ exists iff $n_1, n_2 \in \mathbb{N}$, $n_1 \equiv n_2 \equiv 0 \mod 2$ and $n_1 + n_2 \equiv 0 \mod 4$.

We have $\det K(n_1, n_2; I) = n_1 n_2$, and the ambiguous binary lattice $K(n_1, n_2; I)$ is fundamental if and only if both n_1 and n_2 are square-free and $(n_1, n_2) = 1$. This lattice is odd of the discriminant $D = -4n_1 n_2$.

We have $\det K(n_1, n_2; II) = n_1 n_2/4$, and $K(n_1, n_2; II)$ is fundamental iff both n_1, n_2 are square-free and $(n_1, n_2) = 2$. The lattice $K(n_1, n_2; II)$ is even iff $n_1 + n_2 \equiv 0 \mod 8$. Thus, we have

$$D(K(n_1, n_2; II)) = \begin{cases} -n_1 n_2/4, & \text{if } n_1 + n_2 \equiv 0 \mod 8 \\ -n_1 n_2, & \text{otherwise.} \end{cases}$$

It follows that the number $hr(D)$ of ambiguous classes (proper or improper, it does not matter because they are improper equivalent to itself) of the discriminant D is equal to

$$hr(D) = 2^{\tau(D)} = \begin{cases} 2^{t-1}, \text{if } D \equiv 1 \mod 4, \\ 2^{t-1}, \text{if } D \equiv -4 \mod 16, \\ 2^{t-2}, \text{if } D \equiv 4 \mod 16, \\ 2^{t-1}, \text{if } D \equiv 8 \mod 16 \end{cases} \tag{3.1.8}$$

This number is equal to the number $2^{\tau(D)}$ of genuses because ambiguous lattices correspond to elements of order 2 in the group A of classes of discriminant D. We denote this group as A_2. For a class $a \in A$, the class $-a \in A$ is the class which is improper equivalent to a. The group of genuses is $A/2A$, and natural homomorphism $A \to A/2A$ is the genus map.

Further we will be especially interesting in genuses which do not contain more than one class of non-ambiguous lattices with respect to the general (i.e. proper or improper) equivalence. A genus is called *ambiguous* if it contains an ambiguous lattice. Otherwise it is called *non-ambiguous*.

Using the genus homomorphism, it is easy to see that a discriminant D contains a non-ambiguous genus iff $2^{\tau(D)+1}|h(D)$. If $2^{\tau(D)+1}|h(D)$, the number of classes of general equivalence (i.e. proper or improper) in a non-ambiguous genus is equal to $\frac{h(D)}{2^{\tau(D)+1}}$. Thus, we have

Lemma 3.1.2. *There exists a non-ambiguous genus of the discriminant D iff $2^{\tau(D)+1}|h(D)$. The number of classes of the general equivalence (i.e. proper or improper) in a non-ambiguous genus is equal to $h(D)/2^{\tau(D)+1}$. In particular, a non-ambiguous genus contains exactly one class of general equivalence iff $h(D) = 2^{\tau(D)+1}$.*

Number of ambiguous classes in an ambiguous genus is equal to $\sharp 2A \cap A_2$. By elementary considerations with finite Abelian groups, we get

Lemma 3.1.3. *An ambiguous genus contains only ambiguous classes iff $h(D) = 2^m$ where $m \le 2\tau(D)$, and the class group $A \cong (\mathbb{Z}/4)^{m-\tau(D)} \oplus (\mathbb{Z}/2)^{2\tau(D)-m}$. Equivalently, $h(D) = 2^m$ where $m \le 2\tau(D)$, and the number of ambiguous classes in the principal genus $2A$ is equal to $2^{m-\tau(D)}$.*

Lemma 3.1.4. *An ambiguous genus contains exactly one non-ambiguous class iff either $h(D) = 2^{\tau(D)} \cdot 3$ or $h(D) = 2^{\tau(D)+2}$ and $A \cong (\mathbb{Z}/8) \oplus (\mathbb{Z}/2)^{\tau(D)-1}$. We remark that if $h(D) = 2^{\tau(D)+2}$, then either D is of the type of Lemma 3.1.3 (when an ambiguous genus contains ambiguous classes only) or $A \cong (\mathbb{Z}/8) \oplus (\mathbb{Z}/2)^{\tau(D)-1}$ and an ambiguous genus contains only one class of general equivalence of non-ambiguous lattices. The last case is characterized by the property that the principal genus $2A$ contains exactly two ambiguous classes.*

Below we calculate numbers $hnr(D,\mu)$, $hr_I(D,\mu)$, $hr_{II}(D,\mu)$ and $hr(D,\mu)$ of non-ambiguous classes of general equivalence, ambiguous classes of type I, ambiguous classes of type II and ambiguous classes respectively of the genus (D,μ). We have

$$hr(-4,\mu) = hr_I(-4,\mu) = hr_{II}(-4,\mu) = 1 \text{ and } hnr(-4,\mu) = 0. \tag{3.1.9}$$

If $D \ne -4$, we have

$$hr(D,\mu) = hr_I(D,\mu) + hr_{II}(D,\mu) \tag{3.1.10}$$

and

$$hnr(D,\mu) = (h(D)/2^{\tau(D)} - hr_I(D,\mu) - hr_{II}(D,\mu))/2. \tag{3.1.11}$$

Below we calculate $hr_I(D,\mu)$ and $hr_{II}(D,\mu)$. Since the principal genus pr is ambiguous, for the number of non-ambiguous classes in an ambiguous genus *ambig* we get

$$hnr(D,ambig) = (h(D)/2^{\tau(D)} - hr_I(D,pr) - hr_{II}(D,pr))/2. \tag{3.1.12}$$

We remind (Lemma 3.1.2) that for a non-ambiguous genus *nambig* we have

$$hnr(D,nambig)=\begin{cases}0, & \text{if } 2^{\tau(D)+1}\not|h(D),\\ \frac{h(D)}{2^{\tau(D)+1}}, & \text{if } 2^{\tau(D)+1}|h(D).\end{cases}\tag{3.1.13}$$

Case $D\equiv 1 \mod 4$. Then the determinant $d=-D\equiv -1\mod 4$. All ambiguous classes of the genus (D,μ) are given by $K(2d_1,2d_2;II)$ such that $d_1d_2=d$ and $\left(\frac{2d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_{II}(D,\mu)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\ \&\ \left(\frac{2d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1\ \&\left(\frac{2(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/d_1)\},\tag{3.1.14}$$

and

$$hr_I(D,\mu)=0.\tag{3.1.15}$$

The principal genus *pr* is given by the ambiguous lattice $K(2,2d;II)$. All ambiguous classes of that genus are $K(2d_1,2d_2;II)$ such that $d_1d_2=d$ and $\left(\frac{2d_1/p}{p}\right)=\left(\frac{2d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=\left(\frac{2d_1d_2/p}{p}\right)$ if $p|d_2$. Thus,

$$hr_{II}(D,pr)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\ \&\ \left(\frac{d/d_1}{p}\right)=1\ \forall p|d_1\ \&\left(\frac{d_1}{p}\right)=1\ \forall p|(d/d_1)\}\tag{3.1.16}$$

and

$$hr_I(D,pr)=0.\tag{3.1.17}$$

Case $D\equiv 4\mod 16$. Then $d=-D/4\equiv -1\mod 4$. All ambiguous classes of the genus (D,μ) are given by $K(d_1,d_2;I)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_I(D,\mu)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\&\ \left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1\ \&\left(\frac{(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/d_1)\},\tag{3.1.18}$$

and

$$hr_{II}(D,\mu)=0.\tag{3.1.19}$$

The principal genus is given by the ambiguous lattice $K(1,d;I)$. All ambiguous classes of that genus are $K(d_1,d_2;I)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_2$. Thus,

$$hr_I(D,pr)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\ \&\ \left(\frac{d/d_1}{p}\right)=1\ \forall p|d_1\ \&\left(\frac{d_1}{p}\right)=1\ \forall p|(d/d_1)\},\tag{3.1.20}$$

and

$$hr_{II}(D,pr)=0. \tag{3.1.21}$$

Case $D\equiv -4 \mod 16$. Then $d=-D/4\equiv 1 \mod 4$. All ambiguous classes of type I of the genus (D,μ) are given by $K(d_1,d_2;I)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_I(D,\mu)=\sharp\{d_1|d \ \mid \ d_1\le d/d_1 \ \& \ \left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1 \ \& \left(\frac{(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/d_1)\}. \tag{3.1.22}$$

All ambiguous classes of the type II of (D,μ) are given by $K(2d_1,2d_2;II)$ such that $d_1d_2=d$ and $\left(\frac{2d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_{II}(D,\mu)=\sharp\{d_1|d \ \mid \ d_1\le d/d_1 \ \& \ \left(\frac{2d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1 \ \& \left(\frac{2(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/d_1)\}. \tag{3.1.23}$$

The principal genus is given by the ambiguous lattice $K(1,d;I)$. All ambiguous classes of the type I of that genus are $K(d_1,d_2;1)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_2$. Thus

$$hr_I(D,pr)=\sharp\{d_1|d \ \mid \ d_1\le d/d_1 \ \& \ \left(\frac{d/d_1}{p}\right)=1\ \forall p|d_1 \& \left(\frac{d_1}{p}\right)=1\ \forall p|(d/d_1)\}. \tag{3.1.24}$$

All ambiguous classes of the type II of the principal genus are $K(2d_1,2d_2;II)$ such that $d_1d_2=d$ and $\left(\frac{2d_1/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_2$. Thus

$$hr_{II}(D,pr)= \sharp\{d_1|d \ \mid \ d_1\le d/d_1 \ \& \ \left(\frac{d/d_1}{p}\right)=\left(\frac{2}{p}\right)\ \forall p|d_1 \ \& \left(\frac{d_1}{p}\right)=\left(\frac{2}{p}\right)\ \forall p|(d/d_1)\}. \tag{3.1.25}$$

Case $D\equiv 8 \mod 16$. Then $d=-D/4\equiv 2 \mod 4$. All ambiguous classes of the genus (D,μ) are given by $K(d_1,2d_2;I)$ such that $d_1d_2=d/2$ and $\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_I(D,\mu)=\sharp\{d_1|d/2 \ \mid \& \ \left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1 \ \& \left(\frac{(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/2d_1)\},$$

(3.1.26)

and

$$hr_{II}(D,\mu)=0. \tag{3.1.27}$$

The principal genus is given by the ambiguous lattice $K(1,d;1)$. All ambiguous classes of that genus are $K(d_1,2d_2;I)$ such that $d_1d_2=d/2$ and $\left(\frac{d_1/p}{p}\right)=\left(\frac{2d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=\left(\frac{2d_1d_2/p}{p}\right)$ if $p|d_2$. Thus,

$$hr_I(D,pr)=\sharp\{d_1|(d/2)\ \mid\ \left(\frac{d/d_1}{p}\right)=1\ \forall p|d_1\ \&\ \left(\frac{d_1}{p}\right)=1\ \forall p|(d/2d_1)\} \tag{3.1.28}$$

and

$$hr_{II}(D,pr)=0. \tag{3.1.29}$$

Using considerations above, In Appendix I, we give Program $h2$ for "GP/PARI" calculator which for a fundamental discriminant $D<0$ and the genus (D,μ) calculates the vector

$$(hr_I(D,\mu),\ hr_{II}(D,\mu),\ hnr(D,\mu)). \tag{3.1.30}$$

The "GP/PARI" calculator uses the Shank's method [Sh] to calculate the class numbers $h(D)$ of the discriminant D. It is very fast: $O(|D|^{1/4})$ operations. We code the invariant μ by the non-negative integer μ having the binary decomposition

$$\mu=\mu_{p_k}\mu_{p_{k-1}}\dots\mu_{p_1} \tag{3.1.31}$$

where $p_1,\ \dots,\ p_k$ are all odd prime divisors of D in increasing order.

3.2. The number h of non-reflective classes of central symmetries of main hyperbolic lattices of the rank 3.

We denote by S a main hyperbolic (i.e. of the signature $(1,k)$) lattice with square-free determinant and of the rank 3. Remind that *main* means that the lattice S is even if the determinant $d=\det(S)$ is even. If d is odd, then S is necessarily odd.

By Proposition 2.2.4, the lattice S is defined by its invariants (d,η) where $d=\det(S)$ and the invariant η is the map $\eta:p\mapsto\eta_p\in\{0,\ 1\}$ of the set of all odd prime divisors of d. The η is defined by the condition

$$b_{S_p}\cong b^{(p)}_{\theta_p}(p),\quad \left(\frac{\theta_p}{p}\right)=(-1)^{\eta_p}. \tag{3.2.1}$$

Here q_S and b_S denote discriminant forms of the lattice S. Here and in what follows we use discriminant forms technique and notations in [N1]. We will especially often using theorems of existence of a lattice with a given discriminant form (Theorems 1.10.1 and 1.16.5 in [N1]).

For even d the lattice S is even and the discriminant quadratic form $q_{S_2}=q^{(2)}_{\theta_2}(2)$ where $\theta_2\equiv\pm1\mod 4$. We denote

$$\theta_2\mod 8\equiv\pm1\mod 8\ \text{ if }\ \theta_2\equiv\pm1\mod 4. \tag{3.2.2}$$

We have

$$\sum[(1-p)+4\eta_p]+\theta_2\mod 8\equiv-1\mod 8.$$

Here sign $q_{\theta_p}^{(p)}(p) \equiv (1-p)+4\eta_p \mod 8$ and sign $q_{\theta_2}^{(2)}(2) \equiv \theta_2 \mod 8$. Since $\theta_2 \mod 8 \equiv \pm 1 \mod 8$, we get

$$\sum_{odd\ p|d} [(1-p)+4\eta_p] \equiv 0, -2 \mod 8 \tag{3.2.3}$$

and the invariant θ_2 is defined by

$$\theta_2 \mod 8 \equiv -\sum_{odd\ p|d} [(1-p)+4\eta_p] - 1 \mod 8. \tag{3.2.4}$$

Here (3.2.3) is the condition of existence of a main hyperbolic lattice S with the invariants (d,η) if d is even. If d is odd, the lattice S is odd, and there are no condition of existence of S. It always does exist. Thus, *for any square-free natural number d and any map η of the set of all odd prime divisors of d into $\{0,1\}$, there exists a main hyperbolic lattice S of the rank 3 with invariants (d,η) if and only if*

$$\sum_{odd\ p|d} [(1-p)+4\eta_p] \equiv 0,\ -2 \mod 8 \tag{3.2.5}$$

if d is even.

We consider primitive elements $f \in S$ such that $n = f^2 > 0$. We consider them up to $\pm f$, thus we can suppose that $f \in V^+(S)$ where $V^+(S)$ is the light-cone of S (see Sect. 1.1). We consider the elements f such that there exists an automorphism u_f of S which is identical on f and is -1 on the orthogonal negative definite lattice $K = f_S^\perp$. Then

$$\text{either } S = \mathbb{Z}f \oplus K \text{ or } S = [\mathbb{Z}f \oplus K,\ (f \oplus k)/2],\ k \in K. \tag{3.2.6}$$

This automorphism is called a *central symmetry* of S. Geometrically, it is the central symmetry at the point $\mathbb{R}_{++}f$ of the hyperbolic space $\mathcal{L}(S) = V^+(S)/\mathbb{R}_{++}$ defined by S. Vice versa, any $\phi \in O^+(S)$ acting as a central symmetry in $\mathcal{L}(S)$, has the form $\phi = u_f$. By considerations over $\mathbb{Z}_2$, one can see (using (3.2.6)) that either n is odd or $n \equiv 2 \mod 4$. (One can also use that $u_f : x \mapsto -x + (2(x,f)/f^2)f$, $x \in S$, and $f^2 | 2(S,f)$.) Let $n = 2^k n_1$ where $(n_1, 2) = 1$, $k = 0, 1$. Then by (3.2.6),

$$n_1 | d, \text{ and } k = 1 \text{ if } d \text{ is even, and } \left(\frac{n/p}{p}\right) = (-1)^{\eta_p}\ \forall \text{ odd } p|n. \tag{3.2.7}$$

The central symmetry u_f is called *reflective* if there exists a reflection s_δ of S in $\delta \perp f$ (i.e., $\delta \in K$). Otherwise, u_f is called *non-reflective*. Geometrically, the central symmetry u_f is reflective if and only if the center $\mathbb{R}_{++}f$ belongs to a mirror of the reflection group $W(S)$. We want to calculate the number $hnr(S)$ of non-reflective central symmetries u_f of S up to conjugation in $O(S)$ (i.e., the number of non-reflective classes of the central symmetries). Since ± 1 belongs to the center of $O(S)$, it is sufficient to consider u_f up to $O^+(S) = \{\phi \in O(S) \mid \phi(V^+(S)) = V^+(S)\}$. Acting by the reflection group $W(S)$, we can always suppose that the center $\mathbb{R}_{++}f \in \mathcal{M}$ where $\mathcal{M}$ is a fundamental polyhedron of $W(S)$ (see Sect. 1.1). If u_f is non-reflective, it is equivalent that $u_f \in A(P(\mathcal{M})_{\text{pr}}) = \{\phi \in O^+(S) \mid \phi(\mathcal{M}) = \mathcal{M}\}$. That two non-reflective central symmetries of S are conjugate by $O^+(S)$ if and only if they are conjugate by $A(P(\mathcal{M})_{\rm pr})$. Thus, the number $h = hnr(S)$ of classes of non-reflective central symmetries of S is the same invariant $h = h(S)$ of S which we have introduced in Sect. 2.3. The central symmetry u_f is defined by the element f (it is defined by u_f up to $\pm f$). Thus, it is sufficient to study all these f up to action of $O(S)$. Obviously, (n, K) are invariants of f. Here $K = f^\perp_S$ is a negative binary lattice which we consider up to isomorphism.

First, we calculate invariants of genus of the central symmetries u_f, $f \in S$ above. Here two elements $f_1 \in S$ and $f_2 \in S$ have the same *genus* if they are conjugate over $\mathbb{Z}_p$ for all prime p. Then we calculate the number of non-reflective classes u_f using class numbers of binary positive lattices. We had considered them in Sect. 3.1.

Case I: $d \equiv 1 \mod 2$, equivalently S is odd. Then there are two cases.

Case (I, II): $K = f^\perp$ is even. Then n is odd because S is unimodular odd over $\mathbb{Z}_2$, and then f is a characteristic element of S (i.e. $(f, x) \equiv (x, x) \mod 2$ for any $x \in S$). By (3.2.6), we then have $S = \mathbb{Z}f \oplus K$. It follows that $n|d$, $\det K = d/n$, and the discriminant quadratic form q_{K_p}, $p|(d/n)$, is equal to

$$q_{K_p} = q_{S_p} = q^{(p)}_{\theta_p}(p), \quad \left(\frac{\theta_p}{p}\right) = (-1)^{\eta_p}, \quad p|(d/n).$$

By discriminant form technique [N1], existence of K is equivalent to

$$\sum_{p|(d/n)} [(1-p) + 4\eta_p] \equiv -2 \mod 8. \tag{3.2.8}$$

Conditions (3.2.7) and (3.2.8) are equivalent to existence of u_f with $f^2 = n$. The lattice $K(-1)$ is an even fundamental positive binary lattice of the discriminant discr $K = -\det K = -d/n$. Thus, $-d/n \equiv 1 \mod 4$. The genus of $K(-1)$ is equal to $(-d/n, \epsilon(p) + \eta_p)$. Here $\left(\frac{-1}{p}\right) = (-1)^{\epsilon(p)}$, and it is known that $\epsilon(p) \equiv (p-1)/2 \mod 2$. To be shorter, here we denote by $\epsilon(p) + \eta_p$ the map $p \mapsto \epsilon(p) + \eta_p$ where p runs through all odd prime $p|(d/n)$. It follows that the number of non-reflective classes of central symmetries $f \in S$ of the type (I, II) is equal to

$$hnr_{II}(S) = \sum_{\substack{n|d \ \& \ (3.2.7) \\ \& \ (3.2.8)}} hnr(-d/n, \epsilon(p) + \eta_p), \tag{3.2.9}$$

where we consider the sum by all n such that $n|d$ and (3.2.7), (3.2.8) are valid.

Case (I, I): $K = f^\perp$ is odd. By (3.2.6), (3.2.7), then $n|2d$ and $\det K = (n,2)^2 d/n$, discr $K = -4(n,2)^2 d/n$. It follows that $K(-1)$ is a fundamental positive binary lattice.

For odd $p|\det K$, we have

$$q_{K_p} = q_{S_p} = q^{(p)}_{\theta_p}(p), \quad \left(\frac{\theta_p}{p}\right) = (-1)^{\eta_p},$$

and

$$b_{K_2} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ b^{(2)}(2), & \text{if } \dots \end{cases}.$$

If n is odd, existence of K is equivalent to

$$-2 \mod 8 \in \sum_{p|(d/n)} [(1-p)+4\eta_p] + 4\omega(d/n) + \{0, \pm 2\} \mod 8.$$

It is equivalent to

$$\sum_{p|(d/n)} [(1-p)+4\eta_p] + 4\omega(d/n) \not\equiv 2 \mod 8. \tag{3.2.10}$$

Here $\omega(k) \equiv (k^2-1)/8 \mod 2$, it is known that $\left(\frac{2}{p}\right) = (-1)^{\omega(p)}$. For even n there are no conditions of existence of K. The lattice K does always exist. The genus of $K(-1)$ is equal to $(-4(n,2)^2 d/n, \epsilon(p)+\eta_p)$.

Thus, this case is characterized by the condition: d is odd, $n|2d$, we have $\mathbb{Z}f \oplus K \subset S$ where K is an odd fundamental binary lattice of the discriminant $-4(n,2)^2 d/n$. If n is odd, $\mathbb{Z}f \oplus K = S$. If n is even, the lattice S is generated by $\mathbb{Z}f \oplus K$ and $u = (f+k)/2$ where $k \in K$ is a primitive element with the property: $(k, K) \equiv 0 \mod 2$. The element $k \mod 2K$, and the overlattice S are defined uniquely. For both these cases, any automorphism of K can be extended to the automorphism of S identical on f. It follows that the number of non-reflective classes of f of the type (I, I) is equal to

$$hnr_I(S) = \sum_{\substack{n|d \ \& \ (3.2.7) \\ \& \ (3.2.10)}} hnr(-4d/n, \epsilon(p)+\eta_p) + \sum_{\substack{n=2n_1|2d \\ \&(3.2.7)}} hnr(-16d/n, \epsilon(p)+\eta_p). \tag{3.2.11}$$

As a result, we get that for the case I (equivalently, when d is odd) the full number of non-reflective classes of central symmetries $f \in S$ is equal to

$$hnr(S) = \sum_{\substack{n|d \ \& \ (3.2.7) \ \& \\ (3.2.8)}} hnr(-d/n, \epsilon(p)+\eta_p)$$
$$+ \sum_{\substack{n|d \ \& \ (3.2.7) \\ \& \ (3.2.10)}} hnr(-4d/n, \epsilon(p)+\eta_p) + \sum_{\substack{n=2n_1|2d \\ \& \ (3.2.7)}} hnr(-16d/n, \epsilon(p)+\eta_p). \tag{3.2.12}$$

Case II: d is even, equivalently, the lattice S is even. Then $n = 2n_1$ where $n_1|(d/2)$. This case is also divided in two cases:

Case $(\mathrm{II}, \mathrm{II})$: the lattice $K = f^\perp$ has an odd determinant. By (3.2.6), we have $S = \mathbb{Z}f \oplus K$. Then $q_{S_2} = q^{(2)}_{n/2}(2)$ and $n_1 = n/2 \equiv \theta_2 \mod 4$. By (3.2.4),

$$n/2 \equiv -\sum_{odd\ p|d} [(1-p)+4\eta_p] - 1 \mod 4. \tag{3.2.13}$$

We have $\det K = d/n$ and, for $p|(d/n)$,

$$q_{K_p} = q_{S_p} = q^{(p)}_{\theta_p}(p), \quad \left(\frac{\theta_p}{p}\right) = (-1)^{\eta_p}.$$

Existence of K is equivalent to $\sum_{p|(d/n)} [(1-p)+4\eta_p] \equiv -2 \mod 8$. This follows from the condition (3.2.2) (or (3.2.3)) of existence of S and (3.2.7), (3.2.13). Thus, condition of existence of K for this case is (3.2.13) together with (3.2.7).

Thus, this case is characterized by the condition: d is even, $n|d$ is even, $S=\mathbb{Z}f\oplus K$, where $K(-1)$ is an even fundamental binary lattice of the genus $(-d/n, \epsilon(p)+\eta_p)$. Thus, the number of non-reflective classes f of the type $(\mathrm{II},\mathrm{II})$ is equal to

$$hnr_{II}(S)=\sum_{\substack{n=2n_1|d\ \&\ (3.2.7)\ \&\\ (3.2.13)}} hnr(-d/n, \epsilon(p)+\eta_p). \tag{3.2.14}$$

Case (II,I): the lattice $K=f^\perp$ has even determinant. The discriminant form $q_{[f]_2}=q^{(2)}_{n/2}(2)$. By (3.2.6), we have

$$q_{K_2}=q^{(2)}_{-n/2}(2)\oplus q^{(2)}_{\theta_2}(2).$$

It follows that the lattice $K(1/2)$ is odd fundamental, K has determinant $4d/n$, and for $p|(d/n)$ one has

$$q_{K_p}=q_{S_p}=q^{(p)}_{\theta_p}(p),\quad \left(\frac{\theta_p}{p}\right)=(-1)^{\eta_p}.$$

Existence of the lattice K is equivalent to existence of the lattice S. The odd fundamental binary lattice $K(-1/2)$ has the determinant d/n, and the discriminant $-4d/n$, and for $p|(d/n)$ one has

$$b_{K(1/2)_p}=b^{(p)}_{2\theta_p}(p),\quad \left(\frac{2\theta_p}{p}\right)=(-1)^{\eta_p+\omega(p)}.$$

Thus, for this case, d is even, $f^2=n|d$ is even, $K=T(-2)$ where T is an odd fundamental positive binary lattice of the discriminant $-4d/n$ and of the genus $(-4d/n, \epsilon(p)+\omega(p)+\eta_p)$. The lattice S is an overlattice $\mathbb{Z}f\oplus K\subset S$ of the index 2 generated by $(f+k)/2$ where $k\in K$ satisfies $(k,k)\equiv -n \mod 8$. If $d/n\equiv -1 \mod 4$, then $k \mod 2K$ and S are unique. If $d/n\equiv 1 \mod 4$, there are exactly two different elements k which give the same lattice S. If $\operatorname{discr} T=-4$, elements $k \mod 2K$ are conjugate by $O(K)$. If $\operatorname{discr} T<-4$, elements $k \mod 2K$ are not conjugate and give different classes of $f\in S$. Further we consider two cases:

Case $d/n\equiv -1 \mod 4$. The number of such non-reflective classes $f\in S$ of the type (II,I) is equal to

$$hnr_{I,-1}(S)=\sum_{\substack{n=2n_1|d\ \&\ (3.2.7)\&\\ n_1\equiv -d/2 \mod 4}} hnr(-4d/n, \epsilon(p)+\omega(p)+\eta_p). \tag{3.2.15}$$

Case $d/n\equiv 1 \mod 4$. Then the class $f\in S$ is non-reflective if and only if the lattice K does not have a reflection which can be extended identically on $\mathbb{Z}f$ to give an automorphism of S. If K is non-ambiguous, then $O(K)=\{\pm 1\}$, and we then get two classes of f corresponding to two different choices of $k \mod 2K$. If $d/n\neq 1$ and T is ambiguous of the type II, then $O(K)$ has order 4, only $\pm 1\in O(K)$ preserve the element $k \mod 2K$, and reflections of K change places two possible different elements $k \mod 2K$. Thus we get exactly one non-reflective class $f\in S$. If $d/n\neq 1$ and T is ambiguous of type I, then reflections of T are identical on $K/2K$ and can be extended on S identically on f. Thus, they give reflective classes

$f \in S$. If $d/n = 1$, the lattice $T = K(1,1;I) = K(2,2;II)$, and it has reflections of both types I and II. This case does not give non-reflective classes $f \in S$. Thus, for this case, the number of non-reflective classes $f \in S$ of type (II, I) is equal to

$$hnr_{I,1}(S) = \sum_{\substack{n=2n_1|d \ \& \ (3.2.7) \ \& \\ n_1 \equiv d/2 \mod 4 \ \& \ n<d}} [2hnr(-4d/n, \epsilon(p)+\omega(p)+\eta_p) + hr_{II}(-4d/n, \epsilon(p)+\omega(p)+\eta_p)]. \tag{3.2.16}$$

As a result, we get for the case II (equivalently, when d is even) that the full number of non-reflective classes of central symmetries $f \in S$ is equal to

$$hnr(S) = \sum_{\substack{n=2n_1|d \ \& \ (3.2.7) \ \& \\ (3.2.13)}} hnr(-d/n, \epsilon(p)+\eta_p)$$
$$+ \sum_{\substack{n=2n_1|d \ \& \ (3.2.7)\& \\ n_1 \equiv -d/2 \mod 4}} hnr(-4d/n, \epsilon(p)+\omega(p)+\eta_p)$$
$$+ \sum_{\substack{n=2n_1|d \ \& \ (3.2.7) \ \& \\ n_1 \equiv d/2 \mod 4 \ \& \ n<d}} [2hnr(-4d/n, \epsilon(p)+\omega(p)+\eta_p) + hr_{II}(-4d/n, \epsilon(p)+\omega(p)+\eta_p)]. \tag{3.2.17}$$

As a final result, we get

Theorem 3.2.1. *Let d be a square-free natural number and $\eta : p \mapsto \{0, 1\}$ a map of all odd prime divisors $p|d$ into $\{0, 1\}$. Then there exists a main hyperbolic lattice S of the rank 3 with the square-free determinant d and the invariant η (see (3.2.1)) if and only if for the even d the congruence (3.2.5) is valid. The number $h = hnr(S) = hnr(d, \eta)$ of classes of non-reflective central symmetries of the lattice S with the invariants (d, η) is given by (3.2.12) for the odd d and by (3.2.17) for the even d.*

Below we will code the invariant η by the non-negative integer η having the binary decomposition

$$\eta = \eta_{p_t} \ldots \eta_{p_1} \tag{3.2.18}$$

where $p_1, \ldots, p_t$ are all odd prime divisors of d in increasing order.

In Appendix: Programs, we give the Program 2: h3 for "GP/PARI" calculator which using Theorem 3.2.1 and Program 1: h2 (see Sect. 3.1) calculates the invariant $h = h(S) = hnr(d, \eta)$ if a main hyperbolic lattice S of the rank 3 with the invariants (d, η) does exist (otherwise, the result will be unreasonable). Using the first statement of the Theorem 3.2.1 and the Program 2, we give Program 3: refh3 which gives all pairs of invariants (d, η) such that $d \le N$, there exists a main hyperbolic lattice of the rank 3 with the invariants (d, η) and the invariant $hnr(d, \eta) \le 1$. Using Program 3, we found all these pairs (d, η) such that $d \le 100000$. The result is given in Table 3 (Part I) and contains 206 lattices. Thus, we get

Theorem 3.2.2. *Table 3 (Part I) gives the complete list (it has 206 lattices) of main hyperbolic lattices S with square-free determinant $d \le 100000$ and of the rank 3 such that the invariant $hnr(S) \le 1$. In Table 3 we give invariants d, η, $h = hnr(S)$, the matrix of the lattice S and the reflective type of the lattice S.*

The greatest d of lattices S of the Table 3 is equal to 4466 in spite we did calculations up to 100000. Thus, it is very likely that we have the statement

Conjecture 3.2.3. *Table 3 (Part I) gives the complete list of main hyperbolic lattices S with square-free determinant and of the rank 3 such that the invariant $hnr(S) \le 1$.*

In Sect. 5, we shall use Theorem 3.2.2 to find all (d, η) corresponding to elliptically or parabolically reflective hyperbolic lattices S since for elliptically or parabolically reflective hyperbolic lattices the invariant $h \le 1$.

Similarly we calculated all pairs (d, η) such that $hnr(d, \eta) = 2$ and $d \le 100000$. (One should change in two places of Program 3 $h \le 1$ by $h = 2$.) The list contains 259 pairs (d, η). The last 10 pairs having the largest d are: $(4290, 1)$, $(4326, 2)$, $(4902, 4)$, $(4991, 7)$, $(5226, 0)$, $(5334, 2)$, $(6006, 2)$, $(7590, 8)$, $(10374, 2)$, $(29526, 2)$. It is very likely that this list also contains all pairs (d, η) with $h = 2$. We shall use the list of these 259 lattices and lattices of Table 3 to find in Part III all hyperbolically reflective main hyperbolic lattices of the rank 3. All of them must have the invariant $h = 0$, 2.

4. Narrow places of elliptic and parabolic convex polygons on hyperbolic plane, types of polygons. Application to reflective lattices

4.1. Narrow places of elliptic convex polygons on the hyperbolic plane. We remind (see Sect. 1.1) that a convex polyhedron in a hyperbolic space is called *elliptic* if it is a convex envelope of a finite set of points (some of them at infinity) and it is non-degenerate. In this section we shall consider only elliptic (i.e. ordinary finite) convex polyhedra and often shall omit the word "elliptic".

Here we follow the general method (of narrow places of polyhedra) suggested in [N4], [N5] for proving finiteness results about arithmetic reflection groups in hyperbolic spaces. On the other hand, we shall prove much more delicate and exact statements, which are important for exact classification. Our estimates in [N4], [N5] were universal, they did not depend on angles of fundamental polyhedra. Here we get estimates which depend on angles of polyhedra which makes the narrow places of polyhedra method much more efficient. Our estimates here are optimal, we belive that one cannot impove them.

We restrict by 2-dimensional case of narrow places of elliptic polygons on the hyperbolic plane, but all results can be easily generalized (like in [N4], [N5]) on elliptic polyhedra of arbitrary dimension in hyperbolic spaces.

We shall often use the following trivial but important for us statement (certainly, it is well-known):

Lemma 4.1.1. $\frac{\sin y}{\sin x} < \frac{y}{x}$ *if* $0 < x \le \pi/2$ *and* $x < y$.

Proof. The function $sinx/x$ is decreasing if $0 \le x \le \pi/2$. In particular, $1 \ge sinx/x \ge 2/\pi$ if $0 \le x \le \pi/2$. It follows the lemma for $0 < x \le y \le \pi/2$. If $0 < x \le \pi/2 \le y$, we get

$$\frac{\sin y}{\sin x} \le \frac{1}{sinx} \le \frac{1}{x(2/\pi)} \le \frac{\pi/2}{x} \le \frac{y}{x}.$$

This proves the statement.

Lemma 4.1.2. *Let (AB) and (CD) are two lines on a hyperbolic plane with terminals A, B, C, D at infinity, and O a point on the hyperbolic plane which does not belong to each line (AB) and (CD) and orientations of the triangles AOB and COD coincide. We consider angles $\theta_1 = AOB$, $\theta_2 = COD$ and $\theta_{12} = BOC$. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that O is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$.*

Then

$$(\delta_1, \delta_2) = 4\frac{\sin\frac{\theta_1+\theta_{12}}{2}\sin\frac{\theta_2+\theta_{12}}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} - 2.$$

As a corollary, we get :

1) If lines (AB) and (CD) do not intersect each other, then

$$2\cosh\rho = (\delta_1, \delta_2) = 4\frac{\sin\frac{\theta_1+\theta_{12}}{2}\sin\frac{\theta_2+\theta_{12}}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} - 2$$

where ρ is the distance between lines (AB) and (CD) (here and in what follows we normalize the curvature $\kappa = -1$).

2) If lines (AB) and (CD) define an angle α containing O, then

$$2\cos\alpha = (\delta_1, \delta_2) = 4\frac{\sin\frac{\theta_1-\theta_{21}}{2}\sin\frac{\theta_2-\theta_{21}}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} - 2$$

where $\theta_{21} = -\theta_{12} = COB$.

Proof. We can correspond to two lines (AB), (CD) and a connected component (containing O) of the complement to these two lines in the hyperbolic plane two invariants up to motions of the hyperbolic plane. First invariant is equal to (δ_1, δ_2) and uses Klein model of the hyperbolic plane. Second invariant is equal to the cross ratio $[A : D : C : B]$ where we suppose that orientations of the triangles ABO and CDO coincide. This invariant uses Poincaré model of the hyperbolic plane. On the other hand, it is clear that any of these two invariants defines the triplet $((AB), (CD)$, the connected component to their complement (containing O)) up to motions of the hyperbolic plane. It follows that there exists a function f(x) such that $(\delta_1, \delta_2) = f([A : D : C : B])$. One can check that $f(x) = 4x - 2$.

Below we shall consider an equation

$$(u - x)(v - x) = auv \tag{4.1.1}$$

where $u, v \geq 0$ and $0 \leq a = \cos^2\frac{\alpha}{2} = (1 + \cos\alpha)/2 \leq 1$. Equivalently we have the equation

$$x^2 - (u + v)x + uv(1 - a) = 0.$$

Its smallest root $x = g(\alpha, u, v)$ equals

$$x = g(\alpha, u, v) = \frac{u + v - \sqrt{a(u + v)^2 + (1 - a)(u - v)^2}}{2} \tag{4.1.2}$$

where $a = a(\alpha) = \cos^2\frac{\alpha}{2} = (1 + \cos\alpha)/2$. In particular, for $u = v$,

$$g(\alpha, u, u) = u(1 - \sqrt{a}) = u(1 - \cos\frac{\alpha}{2}). \tag{4.1.3}$$

The function $g(\alpha, u, v)$ has the following properties (we shall not use them further but they are important):

Proposition 4.1.3. $u \geq g(\alpha,u,v) \geq 0$, $v \geq g(\alpha,u,v) \geq 0$, $g(\alpha,u,v)'_u \geq 0$, $g(\alpha,u,v)'_v \geq 0$, $\left(\frac{u-g(\alpha,u,v)}{u}\right)'_u \geq 0$, $\left(\frac{v-g(\alpha,u,v)}{v}\right)'_v \geq 0$.

Proof. Suppose that $u \geq v$. The expression $(u-x)(v-x)$ equals $uv \geq auv$ if $x=0$ and $(u-x)(v-x)$ equals 0 if $x=u$. Thus the smallest solution $g(a,u,v)$ of the equation $(u-x)(v-x)=auv$ satisfies $v \geq u \geq g(a,u,v)$.

We have $2g(a,u,v)'_u = 1-(a(u+v)+(1-a)(u-v))/\sqrt{a(u+v)^2+(1-a)(u-v)^2} \geq 0$ if $(a(u+v)+(1-a)(u-v))/\sqrt{a(u+v)^2+(1-a)(u-v)^2} \leq 1$. Equivalently, for $-1 \leq t = (u-v)/(u+v) \leq 1$ we should prove that $(a+(1-a)t)/\sqrt{a+(1-a)t^2} \leq 1$. If $-1 \leq t \leq 0$, this is obvious. For $0 \leq t \leq 1$, we have $(a+(1-a)t^2)^{\frac{3}{2}}\left((a+(1-a)t)/\sqrt{a+(1-a)t^2}\right)'_t = (1-a)(a+(1-a)t^2)-(1-a)t(a+(1-a)t) = (1-a)a(1-t) \geq 0$. It follows that $(a+(1-a)t)/\sqrt{a+(1-a)t^2} \leq (a+(1-a))/\sqrt{a+(1-a)} = 1$.

By definition, we have $(u-g(\alpha,u,v))/u = av/(v-g(\alpha,u,v)$. It follows that $((u-g(\alpha,u,v))/u)'_u \geq 0$ because $g(\alpha,u,v)'_u \geq 0$.

It finishes the proof.

Theorem 4.1.4 (about the narrow place of type (I)). *For any elliptic convex polygon $\mathcal{M}$ on a hyperbolic plane there exist its four consecutive vertices A_0, A_1, A_2 and A_3 (where $A_0=A_3$ if $\mathcal{M}$ is a triangle) such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=-2$ one has $(\delta_1,\delta_2)=2\cos\alpha_1$, $(\delta_2,\delta_3)=2\cos\alpha_2$ and*

$$\text{either}\ \ (\delta_1,\delta_3) \leq 2\ \ \text{or}\ \ (\delta_1,\delta_3) < 4(\cos\frac{\alpha_1}{2}+\cos\frac{\alpha_2}{2})^2-2 \leq 14 \tag{4.1.4}$$

where $\alpha_1 = A_0A_1A_2$ and $\alpha_2 = A_1A_2A_3$.

Moreover, the Gram graph of $\{\delta_1, \delta_2, \delta_3\}$ is not connected (i.e. this set is union of two non-empty orthogonal subsets) if and only if $\alpha_1=\alpha_2=\frac{\pi}{2}$.

Proof. To prove Theorem, we take a point O inside of $\mathcal{M}=A_1A_2...A_n$. Let B_{i1} and B_{i2} are terminals at infinity of the line $l_i=(A_{i-1}A_i)$ where B_{i1}, A_{i-1}, A_i and B_{i2} are four consecutive points of the line. We introduce angles $\alpha_i = A_{i-1}A_iA_{i+1}$, $\theta_i = B_{i1}OB_{i2}$ and $\theta_{(i+1)i} = B_{(i+1)1}OB_{i2}$.

By Lemma 4.1.2,

$$2\cos\alpha_i = 4\frac{\sin\frac{\theta_i-\theta_{(i+1)i}}{2}\sin\frac{\theta_{i+1}-\theta_{(i+1)i}}{2}}{\sin\frac{\theta_i}{2}\sin\frac{\theta_{i+1}}{2}}-2.$$

Equivalently,

$$\frac{\sin\frac{\theta_i}{2}\sin\frac{\theta_{i+1}}{2}}{\sin\frac{\theta_i-\theta_{(i+1)i}}{2}\sin\frac{\theta_{i+1}-\theta_{(i+1)i}}{2}} = \frac{2}{1+\cos\alpha_i}. \tag{4.1.5}$$

By Lemma 4.1.1, we get

$$\frac{(\theta_i-\theta_{(i+1)i})(\theta_{i+1}-\theta_{(i+1)i})}{\theta_i\theta_{i+1}} < \frac{1+\cos\alpha_i}{2} = \cos^2\frac{\alpha_i}{2}. \tag{4.1.6}$$

It follows that

(see (4.1.2)).

To prove Theorem 4.1.4, we choose a line l_i with the minimal angle θ_i. Let this line be l_2. Thus

$$\theta_2 = \min_i \theta_i. \tag{4.1.8}$$

It follows that $\theta_1 \ge \theta_2$ and $\theta_3 \ge \theta_2$. We shall then prove the inequalities (4.1.4).

If lines l_1, l_3 intersect, we have $(\delta_1, \delta_3) \le 2$ and (4.1.4) is valid. Suppose that the lines l_1 and l_3 do not intersect.

There exists a line $l_1' = (B_{11}'B_{12}')$ with terminals B_{11}' and B_{12}' at infinity such that the line l_1' is contained in $\mathcal{H}^+_{-\delta_1}$, points B_{11}, B_{11}' and B_{12}, B_{12}' are contained in the same half-planes bounded by l_2, the line l_1' has the same angle $\alpha_1 = B_{11}'A_1'B_{22}$ (as l_1) with the line l_2 (we denote by A_1' their intersection point), and the angle $B_{11}'OB_{12}'$ of l_1' is equal to θ_2. We denote by δ_1' the orthogonal vector to l_1' directed outwards of $\mathcal{M}$ and with $(\delta_1')^2 = -2$, and we denote $\theta_{21}' = B_{21}OB_{12}'$.

Similarly, there exists a line $l_3' = (B_{31}'B_{32}')$ with terminals B_{31}' and B_{32}' at infinity such that the line l_3' is contained in $\mathcal{H}^+_{-\delta_3}$, points B_{31}, B_{31}' and B_{32}, B_{32}' are contained in the same half-planes bounded by l_2, the line l_3' has the same angle $\alpha_2 = B_{21}A_2'B_{32}'$ (as l_3) with the line l_2 (we denote by A_2' their point of intersection), and the angle $B_{31}'OB_{32}'$ of l_3' is equal to θ_2. We denote by δ_3' the orthogonal vector to l_3' directed outwards of $\mathcal{M}$ and with $(\delta_3')^2 = -2$, and we denote $\theta_{32}' = B_{31}'OB_{22}$.

Since the lines l_1 and l_3 do not intersect, by our construction, any interval with terminals at l_1' and l_3' intersects both lines l_1 and l_3. It follows that distance between lines l_1' and l_3' is greater than distance between lines l_1 and l_3. It follows $(\delta_1, \delta_3) \le (\delta_1', \delta_3')$. It is sufficient to prove (4.1.4) for lines l_1' and l_3'. (These geometrical considerations are related with properties in Proposition 4.1.4 of the function $g(\alpha, u, v)$.)

By Lemmas 4.1.1, 4.1.2, and (4.1.7), (4.1.3),

$$(\delta_1,\, \delta_3) \le (\delta_1',\, \delta_3') = 4\frac{\sin\frac{\theta_2+\theta_2-\theta_{21}'-\theta_{32}'}{2}\sin\frac{\theta_2+\theta_2-\theta_{21}'-\theta_{32}'}{2}}{\sin\frac{\theta_2}{2}\sin\frac{\theta_2}{2}} - 2 <$$

$$4\frac{(\theta_2+\theta_2-\theta_{21}'-\theta_{32}')(\theta_2+\theta_2-\theta_{21}'-\theta_{32}')}{\theta_2\theta_2} - 2 \;<$$

$$4\frac{(\theta_2+\theta_2-g(\alpha_1,\theta_2,\theta_2)-g(\alpha_2,\theta_2,\theta_2))(\theta_2+\theta_2-g(\alpha_1,\theta_2,\theta_2)-g(\alpha_2,\theta_2,\theta_2))}{\theta_2\theta_2} - 2 =$$

$$4(\sqrt{a_1}+\sqrt{a_2})^2 - 2 = 4(\cos\frac{\alpha_1}{2}+\cos\frac{\alpha_2}{2})^2 - 2$$

where $a_i = cos^2\frac{\alpha_i}{2}$. It proves (4.1.4).

Elements $\delta_1, \delta_2, \delta_3$ generate the hyperbolic 3-dimensional vector space defining the hyperbolic plane. Otherwise their lines either have a common point or are orthogonal to one line which is not the case. If the Gram graph of these elements is not connected, two of these elements generate a 2-dimensional hyperbolic vector subspace. Elements δ_1 and δ_2 cannot generate a hyperbolic 2-dimensional vector subspace because their orthogonal lines have a common point $\mathbb{R}_{++}h$ where $h^2 \ge 0$ and $h \ne 0$. Then $(h,\, \delta_1) = (h,\, \delta_2) = 0$. The same is valid for δ_2 and δ_3. Thus the Gram graph of $\{\delta_1,\, \delta_2,\, \delta_3\}$ is not connected if and only if $(\delta_1,\, \delta_2) = (\delta_3, \delta_2) = 0$. Equivalently, $\alpha_1 = \alpha_2 = \frac{\pi}{2}$. It finishes the proof of Theorem 4.1.4.

Theorem 4.1.5 (about narrow places of types (II) and (III)). *For any elliptic convex polygon $\mathcal{M}$ having more than 3 vertices (i.e. it is different from a triangle) on a hyperbolic plane, one of two possibilities (II) or (III) below is valid:*

(II) There exist its five consecutive vertices A_0, A_1, A_2, A_3 and A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, one has $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3$, and

$$\text{either } \ (\delta_1, \delta_3) \le 2 \ \text{ or } \ (\delta_1, \delta_3) < 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.1.9}$$

$$(\delta_1, \delta_4) <$$

$$4 \max_{0 \le t \le 1} \frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{a_3 + a_3 t + t^2/4}\right)^2 - \frac{t^2}{4}}{1+t} - 2 =$$
$$4 \max\left(\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2}\right)^2, \frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}}\right)^2 - \frac{1}{4}}{2}\right) - 2 \le 34 \tag{4.1.10}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1,\ 2, 3$, and $a_i = \cos^2\frac{\alpha_i}{2}$. Moreover, the set $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ has a connected Gram graph.

(III). There exist its six consecutive vertices A_0, A_1, A_2, A_3, A_4 and A_5 (where $A_0 = A_5$ if $\mathcal{M}$ is a pentagon) such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ one has $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3, 4$, and

$$\text{either } \ (\delta_1, \delta_3) \le 2 \ \text{ or } \ (\delta_1, \delta_3) < 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.1.11}$$

$$\text{either } \ (\delta_3, \delta_5) \le 2 \ \text{ or } \ (\delta_3, \delta_5) < 4(\cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2})^2 - 2 \le 14, \tag{4.1.12}$$

and

$$(\delta_1, \delta_5) <$$

$$4 \max_{0 \le t \le s \le 1} \Bigg[\Bigg(\Bigg(\sqrt{a_1 + (1-a_1)s^2} + \sqrt{a_2 + (1-a_2)s^2} +$$
$$+ \sqrt{a_3 + a_3(s-t) + \frac{a_3(s-t)^2}{4} + \frac{(1-a_3)(s+t)^2}{4}} + \sqrt{a_4 + (1-a_4)t^2}\Bigg)^2 -$$
$$- \frac{(s-t)^2}{4}\Bigg) / ((1+s)(1+t))\Bigg] - 2 =$$
$$4 \max\left[\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2}\right)^2,\right.$$
$$\left.\frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}} + \cos\frac{\alpha_4}{2}\right)^2 - \frac{1}{4}}{2},\ 4\right] - 2 \ \le \ 62 \tag{4.1.13}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1, 2, 3, 4$, *and* $a_i = \cos^2\frac{\alpha_i}{2}$. *Moreover, the set* $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ *has a connected Gram graph.*

Proof. Like for the proof of Theorem 4.1.4, we take a point O inside of $\mathcal{M}$ and introduce angles θ_i, $\theta_{(i+1)i}$. We choose such a consecutive numeration of vertices of $\mathcal{M}$ that

$$\theta_2 + \theta_3 = \min_i (\theta_i + \theta_{i+1}) \text{ and } \theta_2 \le \theta_3. \tag{4.1.14}$$

Then $\theta_1 \ge \theta_3 \ge \theta_2$ and, like for the proof of Theorem 4.1.4, we get (4.1.9) and (4.1.11) (see considerations below about cases (II) and (III)).

Below we consider two cases:

Case (ii): $\theta_4 \ge (\theta_2 + \theta_3)/2$. We then prove (4.1.10).

For $1 \ge a_1, a_2, a_3 \ge 0$ and $0 \le t$ we introduce a function

$$f_{p2}(a_1, a_2, a_3, t) =$$

$$= \frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{a_3 + a_3 t + t^2/4}\right)^2 - \frac{t^2}{4}}{1+t}. \tag{4.1.15}$$

We prove that

$$(\delta_1, \delta_4) < 4 \max_{0 \le t \le 1} f_{p2}(a_1, a_2, a_3, t) - 2. \tag{4.1.16}$$

If $(\delta_1, \delta_4) \le 2$, it is true because $4f_{p2}(a_1, a_2, a_3, 1) - 2 = 2((2 + \sqrt{2a_3 + 1/4})^2 - 1/4) - 2 \ge 2((2 + 1/2)^2 - 1/4) - 2 = 10$.

Like for the proof of Theorem 4.1.4, we can suppose that $\theta_1 = \theta_3$ (instead of $\theta_1 \ge \theta_3$) and $\theta_4 = (\theta_2 + \theta_3)/2$ (instead of $\theta_4 \ge (\theta_2 + \theta_3)/2$). We denote $c = (\theta_2 + \theta_3)/2$ and $z = (\theta_3 - \theta_2)/2 \ge 0$. Then $\theta_2 = c - z$ and $\theta_3 = c + z$.

Like for the proof of Theorem 4.1.4, we have

$$((\delta_1, \delta_4) + 2)/4 <$$

$$\frac{(\theta_3 + \theta_2 + \theta_3 - \theta_{21} - \theta_{32} - \theta_{43})(c + \theta_2 + \theta_3 - \theta_{21} - \theta_{32} - \theta_{43})}{\theta_3 c} <$$

$$(\theta_3 + \theta_2 + \theta_3 - g(\alpha_1, \theta_3, \theta_2) - g(\alpha_2, \theta_2, \theta_3) - g(\alpha_3, \theta_3, c))\times$$

$$(c + \theta_2 + \theta_3 - g(\alpha_1, \theta_3, \theta_2) - g(\alpha_2, \theta_2, \theta_3) - g(\alpha_3, \theta_3, c))/\theta_3 c =$$

$$(z/2 + \sqrt{a_1c^2 + (1-a_1)z^2} + \sqrt{a_2c^2 + (1-a_2)z^2} + \sqrt{\frac{z^2}{4} + a_3cz + a_3c^2})\times$$

$$(-z/2 + \sqrt{a_1c^2 + (1-a_1)z^2} + \sqrt{a_2c^2 + (1-a_2)z^2} + \sqrt{\frac{z^2}{4} + a_3cz + a_3c^2})/(c+z)c =$$

$$\frac{(\sqrt{a_1c^2 + (1-a_1)z^2} + \sqrt{a_2c^2 + (1-a_2)z^2} + \sqrt{\frac{z^2}{4} + a_3cz + a_3c^2})^2 - z^2/4}{(c+z)c} =$$

$$\frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{\frac{t^2}{4} + a_3t + a_3}\right)^2 - t^2/4}{1+t}$$

where $t = z/c$ and $0 \le t \le 1$. It proves (4.1.16).

If one of angles α_1 or α_2 is not $\pi/2$, Gram graph of $\{\delta_1, \delta_2, \delta_3\}$ is connected. Then Gram graph of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ is connected because $\delta_1, \delta_2, \delta_3$ generate the full 3-dimensional hyperbolic vector space. If $\alpha_1 = \alpha_2 = \pi/2$ (equivalently, $(\delta_1, \delta_2) = (\delta_3, \delta_2) = 0$), then δ_1, δ_3 generate a hyperbolic 2-dimensional subspace and $(\delta_1, \delta_3) \neq 0$. If Gram graph of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ is not connected, $(\delta_1, \delta_4) = (\delta_3, \delta_4) = 0$. It follows that $\delta_4 = \lambda\delta_2$, $\lambda \in \mathbb{R}$. Then sides of $\mathcal{M}$ orthogonal to δ_2 and δ_4 should coincide. We get a contradiction.

Case (ii) gives polygons $\mathcal{M}$ which satisfy the condition (II) of Theorem 4.1.5. We mention that by Theorem 4.1.4, any quadrangle $\mathcal{M}$ satisfies the case (II) of Theorem 4.1.5. Really, for a quadrangle, $(\delta_1, \delta_4) \leq 2$.

Case (iii): $\theta_4 \leq (\theta_2 + \theta_3)/2$ and $\mathcal{M}$ has more than 4 vertices. We denote $(\theta_2 + \theta_3)/2 = c$.

By (4.1.14), $\theta_1 \geq \theta_3$, $\theta_4 + \theta_5 \geq (\theta_2 + \theta_3) = 2c$. It follows, $\theta_5 \geq c$.

Since $\theta_3 \geq c$, $\theta_4 \leq c$ and $\theta_5 \geq c$, like in the proof of Theorem 4.1.4, we get the inequality (4.1.12).

For $1 \geq a_1, a_2, a_3, a_4 \geq 0$ and $0 \leq s, t$, we introduce a function

$$f_{p3}(a_1, a_2, a_3, a_4, s, t) =$$

$$\Bigg(\Bigg(\sqrt{a_1 + (1-a_1)s^2} + \sqrt{a_2 + (1-a_2)s^2} +$$
$$+ \sqrt{a_3 + a_3(s-t) + \frac{a_3(s-t)^2}{4} + \frac{(1-a_3)(s+t)^2}{4}} + \sqrt{a_4 + (1-a_4)t^2}\Bigg)^2 -$$
$$- \frac{(s-t)^2}{4}\Bigg) / ((1+s)(1+t)). \tag{4.1.17}$$

We prove that

$$(\delta_1, \delta_5) < 4 \max_{0 \leq t \leq s \leq 1} f_{p3}(a_1, a_2, a_3, a_4, s, t) - 2. \tag{4.1.18}$$

If $(\delta_1, \delta_5) \leq 2$, it is true because $f_{p3}(a_1, a_2, a_3, a_4, 1, 1) = 4$. If $(\delta_1, \delta_5) > 2$, like for the proof of Theorem 4.1.4, we can assume that $\theta_1 = \theta_3$ (instead of $\theta_1 \geq \theta_3$) and $\theta_5 = 2c - \theta_4$ (instead of $\theta_5 \geq 2c - \theta_4$).

We denote $c = (\theta_2 + \theta_3)/2$, $z = (\theta_3 - \theta_2)/2$ and $w = (\theta_5 - \theta_4)/2$. We have $\theta_1 = \theta_3 = c + z$, $\theta_2 = c - z$, $\theta_4 = c - w$, $\theta_5 = c + w$. By definition, $c \geq 0$, $z \geq 0$, $w \geq 0$. Moreover, $z \geq w$ because $\theta_3 + \theta_4 = 2c + z - w \geq \theta_2 + \theta_3 = 2c$ (we use (4.1.14)).

Like for the proof ot Theorem 4.1.4, using Lemmas 4.1.1, 4.1.2, we get

$$((\delta_1, \delta_5) + 2)/4 <$$

$$\frac{(\theta_3 + \theta_2 + \theta_3 + \theta_4 - \theta_{21} - \theta_{32} - \theta_{43} - \theta_{54})(\theta_5 + \theta_2 + \theta_3 + \theta_4 - \theta_{21} - \theta_{32} - \theta_{43} - \theta_{54})}{\theta_3\theta_5} <$$

$$(\theta_3 + \theta_2 + \theta_3 + \theta_4 - g(\alpha_1, \theta_3, \theta_2) - g(\alpha_2, \theta_2, \theta_3) - g(\alpha_3, \theta_3, \theta_4) - g(\alpha_4, \theta_4, \theta_5))\times$$
$$(\theta_5 + \theta_2 + \theta_3 + \theta_4 - g(\alpha_1, \theta_3, \theta_2) - g(\alpha_2, \theta_2, \theta_3) - g(\alpha_3, \theta_3, \theta_4) - g(\alpha_4, \theta_4, \theta_5))/(\theta_3\theta_5)$$

$$\Big(\Big(\sqrt{a_1c^2+(1-a_1)z^2}+\sqrt{a_2c^2+(1-a_2)z^2}+$$
$$\sqrt{a_3c^2+a_3c(z-w)+\frac{a_3(z-w)^2}{4}+\frac{(1-a_3)(z+w)^2}{4}}+$$
$$\sqrt{a_4c^2+(1-a_4)w^2}\Big)^2-\frac{(z-w)^2}{4}\Big)/((c+z)(c+w)).$$

Denoting $s=z/c$ and $t=w/c$, we get

$$((\delta_1,\delta_5)+2)/4<f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)$$

where $0\le t\le s\le 1$.

Similarly to the case (ii), one can easily prove that Gram graph of $\{\delta_1,\,\delta_2,\,\delta_3,\,\delta_4,\,\delta_5\}$ is connected. Thus, for the case (iii), we get the case (III) of Theorem.

To finish the proof of Theorem, in Lemmas 4.1.6 and 4.1.7 below, we find $\max_{0\le t\le 1} f_{p2}(a_1,\,a_2,\,a_3,\,t)$ and $\max_{0\le t\le s\le 1} f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)$.

Lemma 4.1.6. *For $0\le a_1,\,a_2,\,a_3\le 1$ and $t\ge 0$ the function*

$$f_{p2}(a_1,\,a_2,\,a_3,\,t)=$$
$$=\frac{\Big(\sqrt{a_1+(1-a_1)t^2}+\sqrt{a_2+(1-a_2)t^2}+\sqrt{a_3+a_3t+t^2/4}\Big)^2-\frac{t^2}{4}}{1+t}$$

has the maximum

$$\max_{0\le t\le 1} f_{p2}(a_1,\,a_2,\,a_3,\,t)=\max\left[f_{p2}(a_1,\,a_2,\,a_3,\,0),\;f_{p2}(a_1,\,a_2,\,a_3,\,1)\right]=$$
$$\max\left[(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3})^2,\;\frac{\Big(2+\sqrt{2a_3+\frac14}\Big)^2-\frac14}{2}\right].$$

Lemma 4.1.7. *For $0\le a_1,\,a_2,\,a_3,\,a_4\le 1$ and $s,\,t\ge 0$ the function*

$$f_{p3}(a_1,\,a_2,\,a_3,a_4,\,s,\,t)=$$

$$\Big(\Big(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+$$
$$+\sqrt{a_3+a_3(s-t)+\frac{a_3(s-t)^2}{4}+\frac{(1-a_3)(s+t)^2}{4}}+\sqrt{a_4+(1-a_4)t^2}\Big)^2-$$
$$-\frac{(s-t)^2}{4}\Big)/((1+s)(1+t))$$

has the maximum

$$\max_{0\le t\le s\le 1} f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)=$$
$$\max\left[f_{p3}(a_1,\,a_2,\,a_3,\,a_4,0,\,0),\;f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,1,\,0),\;f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,1,\,1)\right]=$$
$$\max\left[(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})^2,\;\frac{\Big(2+\sqrt{2a_3+\frac14+a_4}\Big)^2-\frac14}{2},\;4\right]$$

Proof of Lemma 4.1.6 for $1 \ge a_1,\, a_2 \ge 0.2723$, *and* $1 \ge a_3 \ge 5/32 = 0.15625$. We shall use Theorem 4.1.5 only for $1 \ge a_i \ge 1/2$ when a polygon $\mathcal{M}$ has acute angles. Therefore, we restrict proving Lemma 4.1.6 for the parameters a_i above.

To prove Lemma 4.1.6 for parameters $a_1,\, a_2,\, a_3$, it is sufficient to show that

$$f_{p2}(a_1,\, a_2,\, a_3,\, t)''_{tt} \ge 0, \quad \text{if } 0 \le t \le 1. \tag{4.1.19}$$

For $1 \ge a \ge 0$ and $0 \le t \le 1$ we consider

$$u = \frac{a + (1-a)t^2}{1+t}. \tag{4.1.20}$$

We have

$$u'_t = \frac{(1-a)t^2 + 2(1-a)t - a}{(1+t)^2} = 1 - a - \frac{1}{(1+t)^2}, \quad u''_{tt} = \frac{2}{(1+t)^3}. \tag{4.1.21}$$

It follows,

$$(u^{\frac{1}{2}})''_{tt} = u^{-\frac{3}{2}}\left(-\frac{1}{4}(u'_t)^2 + \frac{1}{2}u\, u''_{tt}\right) =$$

$$\left(\frac{a + (1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times$$
$$\frac{-(1/4)(1-a)^2t^4 - (1-a)^2t^3 + (3/2)(a-a^2)t^2 + (a-a^2)t + a - a^2/4}{(1+t)^4} \ge$$

$$\left(\frac{a + (1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times$$
$$\frac{-(1/4)(1-a)^2t^2 - (1-a)^2t^2 + (3/2)(a-a^2)t^2 + (a-a^2)t + a - a^2/4}{(1+t)^4} =$$

$$\left(\frac{a + (1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times \frac{(1-a)(\frac{11}{4}a - \frac{5}{4})t^2 + (a-a^2)t + a - \frac{1}{4}a^2}{(1+t)^4} \ge$$

$$\left(\frac{a + (1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times (1+t)^{-4} \times \begin{cases} a - \frac{1}{4}a^2, & \text{if } \frac{1}{3} \le a \le 1 \\ a - \frac{1}{4}a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ 0 \le t \le \frac{4a}{5-11a} \\ -\frac{5}{4} + 6a - 4a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ \frac{4a}{5-11a} \le t \le 1. \end{cases}$$

It follows,

$$(u^{\frac{1}{2}})''_{tt} \ge$$

$$\left(\frac{a + (1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times (1+t)^{-4} \times \begin{cases} a - \frac{1}{4}a^2, & \text{if } \frac{1}{3} \le a \le 1 \\ a - \frac{1}{4}a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ 0 \le t \le \frac{4a}{5-11a} \\ -\frac{5}{4} + 6a - 4a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ \frac{4a}{5-11a} \le t \le 1 \end{cases}$$

$$\ge \left(\frac{a + (1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times \min(a - \frac{1}{4}a^2, -\frac{5}{4} + 6a - 4a^2)/(1+t)^4. \tag{4.1.22}$$

Here $\min\,(a-\frac{1}{4}a^2,\ -\frac{5}{4}+6a-4a^2)\ge 0$ if $a\ge 1/4$. If follows

$$(u^{\frac{1}{2}})''_{tt}\ge \frac{\min(a-\frac{1}{4}a^2,-\frac{5}{4}+6a-4a^2)}{(1+t)^{\frac{5}{2}}}\ge 0 \tag{4.1.23}$$

if $1\ge a\ge \frac{1}{4}$ and $0\le t\le 1$.

For $1\ge a\ge 0$ and $0\le t\le 1$ we consider

$$v=\frac{t^2/4+at+a}{1+t}. \tag{4.1.24}$$

We have

$$v'_t=\frac{t^2/4+t/2}{(1+t)^2}=\frac{1}{4}-\frac{1}{4(1+t)^2},\quad v''_{tt}=\frac{1}{2(1+t)^3}. \tag{4.1.25}$$

It follows

$$(v^{\frac{1}{2}})''_{tt}=\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{-(1/64)t^4-(1/16)t^3+(1/4)at+(1/4)a}{(1+t)^4}\ge$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{-(1/64)t^3-(1/16)t^3+(1/4)at+(1/4)a}{(1+t)^4}=$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{-(5/64)t^3+(1/4)at+(1/4)a}{(1+t)^4}\ge$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\begin{cases}\frac{1}{4}a, & \text{if } \frac{5}{16}\le a\le 1\\ \frac{1}{4}a, & \text{if } 0\le a\le\frac{5}{16}\ \&\ 0\le t\le 4\sqrt{\frac{a}{5}}\\ -\frac{5}{64}+\frac{1}{2}a, & \text{if } 0\le a\le \frac{5}{16}\ \&\ 4\sqrt{\frac{a}{5}}\le t\le 1.\end{cases}$$

It follows

$$(v^{\frac{1}{2}})''_{tt}\ge$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times(1+t)^{-4}\times\begin{cases}\frac{1}{4}a, & \text{if } \frac{5}{16}\le a\le 1\\ \frac{1}{4}a, & \text{if } 0\le a\le\frac{5}{16}\ \&\ 0\le t\le 4\sqrt{\frac{a}{5}}\\ -\frac{5}{64}+\frac{1}{2}a, & \text{if } 0\le a\le \frac{5}{16}\ \&\ 4\sqrt{\frac{a}{5}}\le t\le 1\end{cases}$$

$$\ge\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{\min\,(\frac{1}{4}a,\ -\frac{5}{64}+\frac{1}{2}a)}{(1+t)^4}. \tag{4.1.26}$$

Here $\min\,(\frac{1}{4}a,\ -\frac{5}{64}+\frac{1}{2}a)\ge 0$ if $a\ge\frac{5}{32}$. If follows

$$(v^{\frac{1}{2}})''_{tt}\ge(1/4+2a)^{-\frac{3}{2}}\times\frac{\min\,(\frac{1}{4}a,\ -\frac{5}{64}+\frac{1}{2}a)}{(1+t)^{\frac{5}{2}}} \tag{4.1.27}$$

if $1\ge a\ge 5/32=0.15625$, $0\le t\le 1$.

We have

$$\left(\frac{t^2/4}{1+t}\right)'=\frac{1}{4}-\frac{1}{4(1+t)^2},\quad \left(\frac{t^2/4}{1+t}\right)''=\frac{1}{2(1+t)^3}. \tag{4.1.28}$$

We denote for $1 \ge a_i \ge 0$ and $0 \le t \le 1$,

$$u_1 = \frac{a_1 + (1-a_1)t^2}{1+t}, \quad u_2 = \frac{a_2 + (1-a_2)t^2}{1+t} \quad u_3 = \frac{t^2/4 + a_3 t + a_3}{1+t}$$

and

$$w = \sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3}.$$

We have

$$(w^2)''_{tt} = (w'_t)^2 + 2ww''_{tt} \ge 2ww''_{tt}. \tag{4.1.29}$$

From (4.1.23), (4.1.27), (4.1.28) and (4.1.29), for $1 \ge a_1,\, a_2, \ge 1/4$ and $1 \ge a_3 \ge 5/32$, we get for $f_{p2} = w^2 - (1/4)t^2/(1+t)$,

$$f_{p2}(a_1,\, a_2,\, a_3,\, t)''_{tt} \ge$$

$$\Big[2\left((a_1 + (1-a_1)t^2)^{\frac{1}{2}} + (a_2 + (1-a_2)t^2)^{\frac{1}{2}} + (t^2/4 + a_3 t + a_3)^{\frac{1}{2}}\right) \times \\ \Big(\min\big(a_1 - \frac{1}{4}a_1^2,\; -\frac{5}{4} + 6a_1 - 4a_1^2\big) + \min\big(a_2 - \frac{1}{4}a_2^2,\; -\frac{5}{4} + 6a_2 - 4a_2^2\big) + \\ (1/4 + 2a_3)^{-\frac{3}{2}} \min\big(\frac{1}{4}a_3,\; -\frac{5}{64} + \frac{1}{2}a_3\big)\Big) - \frac{1}{2}\Big] / (1+t)^3 \ge$$

$$[2\,(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3}) \times \\ \Big(\min\big(a_1 - \frac{1}{4}a_1^2,\; -\frac{5}{4} + 6a_1 - 4a_1^2\big) + \min\big(a_2 - \frac{1}{4}a_2^2,\; -\frac{5}{4} + 6a_2 - 4a_2^2\big) + \\ (1/4 + 2a_3)^{-\frac{3}{2}} \min\big(\frac{1}{4}a_3,\; -\frac{5}{64} + \frac{1}{2}a_3\big)\Big) - \frac{1}{2}\Big] / (1+t)^3 \ge 0 \tag{4.1.30}$$

if

$$(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3}) \times \\ \Big(\min\big(a_1 - \frac{1}{4}a_1^2,\; -\frac{5}{4} + 6a_1 - 4a_1^2\big) + \min\big(a_2 - \frac{1}{4}a_2^2,\; -\frac{5}{4} + 6a_2 - 4a_2^2\big) + \\ (1/4 + 2a_3)^{-\frac{3}{2}} \min\big(\frac{1}{4}a_3,\; -\frac{5}{64} + \frac{1}{2}a_3\big)\Big) \ge \frac{1}{4}. \tag{4.1.31}$$

Thus, (4.1.19) is true if $1 \ge a_1,\, a_2 \ge 1/4$, $1 \ge a_3 \ge 5/32$ and (4.1.31) is valid. For $1 \ge a_1,\, a_2 \ge a \ge 1/4$ and arbirary $1 \ge a_3 \ge 5/32$, (4.1.31) is valid if

$$2(2\sqrt{a} + \sqrt{\frac{5}{32}})(-\frac{5}{4} + 6a - 4a^2) \ge \frac{1}{4}. \tag{4.1.32}$$

It is true if $1 \ge a \ge 0.2723$.

It proves Lemma 4.1.6 for $1 \ge a_1,\, a_2 \ge 0.2723$ and $1 \ge a_3 \ge 5/32 = 0.15625$.

Proof of Lemma 4.1.7 for $1 \ge a_1,\, a_2,\, a_3,\, a_4 \ge 0.37646$. We shall use Theorem 4.1.5 only for $1 \ge a_i \ge 1/2$ when a polygon $\mathcal{M}$ has acute angles. Therefore, we restrict proving Lemma 4.1.7 for the parameters a_i above.

To prove Lemma 4.1.7 for parameters a_1, a_2, a_3, a_4, it is sufficient to show that

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, t)''_{tt} \geq 0, \quad \text{if } 0 \leq t \leq s \leq 1, \tag{4.1.33}$$

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, 0)''_{ss} \geq 0, \quad \text{if } 0 \leq s \leq 1; \tag{4.1.34}$$

and

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, s)''_{ss} \geq 0, \quad \text{if } 0 \leq s \leq 1. \tag{4.1.35}$$

We have

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, 0) =$$

$$\frac{\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\sqrt{a_3+a_3 s+\frac{s^2}{4}}+\sqrt{a_4}\right)^2-\frac{s^2}{4}}{1+s}. \tag{4.1.36}$$

Like for the proof of Lemma 4.1.6, it follows for $1 \geq a_1,\, a_2 \geq \frac{1}{4}$, $1 \geq a_3 \geq \frac{5}{32}$, $1 \geq a_4 \geq 0$ that

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, 0)''_{ss} \geq$$

$$\Big[2\left((a_1+(1-a_1)s^2)^{\frac{1}{2}}+(a_2+(1-a_2)s^2)^{\frac{1}{2}}+(s^2/4+a_3 s+a_3)^{\frac{1}{2}}+a_4^{\frac{1}{2}}\right)\times$$
$$\Big(\min\,(a_1-\frac{1}{4}a_1^2,\ -\frac{5}{4}+6a_1-4a_1^2)+\min\,(a_2-\frac{1}{4}a_2^2,\ -\frac{5}{4}+6a_2-4a_2^2)+$$
$$(1/4+2a_3)^{-\frac{3}{2}}\min\,(\frac{1}{4}a_3,\ -\frac{5}{64}+\frac{1}{2}a_3)\Big)-\frac{1}{2}\Big]\,/(1+s)^3 \geq$$

$$[2\,(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\Big(\min\,(a_1-\frac{1}{4}a_1^2,\ -\frac{5}{4}+6a_1-4a_1^2)+\min\,(a_2-\frac{1}{4}a_2^2,\ -\frac{5}{4}+6a_2-4a_2^2)+$$
$$(1/4+2a_3)^{-\frac{3}{2}}\min\,(\frac{1}{4}a_3,\ -\frac{5}{64}+\frac{1}{2}a_3)\Big)-\frac{1}{2}\Big]\,/(1+s)^3 \geq 0 \tag{4.1.37}$$

if

$$(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\Big(\min\,(a_1-\frac{1}{4}a_1^2,\ -\frac{5}{4}+6a_1-4a_1^2)+\min\,(a_2-\frac{1}{4}a_2^2,\ -\frac{5}{4}+6a_2-4a_2^2)+$$
$$(1/4+2a_3)^{-\frac{3}{2}}\min\,(\frac{1}{4}a_3,\ -\frac{5}{64}+\frac{1}{2}a_3)\Big) \geq \frac{1}{4}. \tag{4.1.38}$$

Thus, (4.1.34) is true if $1 \geq a_1,\, a_2 \geq 1/4$, $1 \geq a_3 \geq 5/32$, $1 \geq a_4 \geq 0$ and (4.1.38) is valid.

For $1 \geq a_1,\, a_2 \geq a \geq 1/4$, arbirary $1 \geq a_3 \geq 5/32$ and arbitrary $1 \geq a_4 \geq 0$, we have (4.1.38) if (4.1.32) is valid. It takes place if $1 \geq a \geq 0.2723$.

We have

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, s)$$

$$\frac{\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\sqrt{a_3+(1-a_3)s^2}+\sqrt{a_4+(1-a_4)s^2}\right)^2}{(1+s)^2}. \tag{4.1.39}$$

For $1\ge a\ge 0$ and $0\le s\le 1$ we denote

$$e=\frac{a+(1-a)s^2}{(1+s)^2} \tag{4.1.40}$$

We have

$$\begin{aligned} e'_s&=\frac{-2a+2s(1-a)}{(1+s)^3}=\frac{2s}{(1+s)^3}-\frac{2a}{(1+s)^2},\\ e''_{ss}&=\frac{2+4a+(4a-4)s}{(1+s)^4}=\frac{2-4s}{(1+s)^4}+\frac{4a}{(1+s)^3}. \end{aligned} \tag{4.1.41}$$

It follows,

$$(e^{\frac{1}{2}})''_{ss}=e^{-\frac{3}{2}}\left(-\frac{1}{4}(e'_s)^2+\frac{1}{2}e\,e''_{ss}\right)=$$

$$e^{-\frac{3}{2}}\times\frac{(-2a^2+4a-2)s^3+(-3a^2+3a)s^2+(a^2+a)}{(1+s)^6}\ge$$

$$e^{-\frac{3}{2}}\times\frac{(-2a^2+4a-2)s^2+(-3a^2+3a)s^2+(a^2+a)}{(1+s)^6}=$$

$$\begin{aligned} &e^{-\frac{3}{2}}\times\frac{(-5a^2+7a-2)s^2+(a^2+a)}{(1+s)^6}\ge\\ &e^{-\frac{3}{2}}\times\begin{cases} a^2+a, & \text{if } 1\ge a\ge\frac{2}{5}\\ -4a^2+8a-2, & \text{if } \frac{2}{5}\ge a\ge 0\end{cases}\ \ge\\ &e^{-\frac{3}{2}}\times\frac{\min(a^2+a,\ -4a^2+8a-2)}{(1+s)^6}. \end{aligned} \tag{4.1.42}$$

Here $\min(a^2+a,\ -4a^2+8a-2)\ge 0$ if $a\ge(2-\sqrt{2})/2=0.292893...$. It follows

$$(e^{\frac{1}{2}})''_{ss}\ \ge\ \frac{\min(a^2+a,\ -4a^2+8a-2)}{(1+s)^3}\ge 0\ \text{ if } 1\ge a\ge(2-\sqrt{2})/2. \tag{4.1.43}$$

For $1\ge a_1,\,a_2,\,a_3,\,a_4\ge 0$ we denote

$$e_i=\frac{a_i+(1-a_i)s^2}{(1+s)^2}. \tag{4.1.44}$$

We get

$$f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,s)''_{ss}\ge$$

$$2(\sqrt{e_1}+\sqrt{e_2}+\sqrt{e_3}+\sqrt{e_4})((e_1^{\frac{1}{2}})''_{ss}+(e_2^{\frac{1}{2}})''_{ss}+(e_3^{\frac{1}{2}})''_{ss}+(e_4^{\frac{1}{2}})''_{ss})\ge$$

$$2\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\sqrt{a_3+(1-a_3)s^2}+\sqrt{a_4+(1-a_4)s^2}\right)\times$$
$$\big((\min(a_1^2+a_1,\,-4a_1^2+8a_1-2)+\min(a_2^2+a_2,\,-4a_2^2+8a_2-2)+$$
$$+\min(a_3^2+a_3,\,-4a_3^2+8a_3-2)+\min(a_4^2+a_4,\,-4a_4^2+8a_4-2)\big)/(1+s)^4\geq$$

$$2\,(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\big((\min(a_1^2+a_1,\,-4a_1^2+8a_1-2)+\min(a_2^2+a_2,\,-4a_2^2+8a_2-2)+$$
$$+\min(a_3^2+a_3,\,-4a_3^2+8a_3-2)+\min(a_4^2+a_4,\,-4a_4^2+8a_4-2)\big)/(1+s)^4\geq 0 \qquad (4.1.45)$$

if $1\geq a_i\geq (2-\sqrt{2})/2=0.292893...$.

Let us prove (4.1.33) for $1\geq a_1,\,a_2,\,a_3,\,a_4\geq 0.37235$. We denote

$$f=\frac{a+a(s-t)+\frac{a(s-t)^2}{4}+\frac{(1-a)(s+t)^2}{4}}{t+1}=\frac{a+as+\frac{s^2}{4}+t(-a-as+\frac{s}{2})+\frac{t^2}{4}}{t+1}.$$

We have

$$f'_t=\frac{-2a-2as+\frac{s}{2}-\frac{s^2}{4}+\frac{t}{2}+\frac{t^2}{4}}{(t+1)^2}=\frac{1}{4}-\frac{(1-s)^2+8a(s+1)}{4(t+1)^2},$$

$$f''_{tt}=\frac{(1-s)^2+8a(s+1)}{2(t+1)^3}.$$

For $1\geq a\geq 0$ and $0\leq t\leq s\leq 1$, we have

$$(f^{\frac{1}{2}})''_{tt}=f^{-\frac{3}{2}}\left(-\frac{1}{4}(f'_t)^2+\frac{1}{2}f\,f''_{tt}\right)=$$

$$f^{-\frac{3}{2}}(-\frac{1}{64}t^4-\frac{1}{16}t^3+(\frac{3}{32}s^2+(\frac{3}{4}a-\frac{3}{16})s+\frac{3}{4}a)t^2+$$
$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))/(t+1)^4\geq$$

$$f^{-\frac{3}{2}}(\frac{3}{32}s^2+(\frac{3}{4}a-\frac{3}{16})s+\frac{3}{4}a-\frac{5}{64})t^2+$$
$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))/(t+1)^4.$$

For $1\geq a\geq \frac{1}{4}$ and $0\leq t\leq s\leq 1$ we have

$$(\frac{3}{32}s^2+(\frac{3}{4}a-\frac{3}{16})s+\frac{3}{4}a-\frac{5}{64})t^2+$$
$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))\geq$$

$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))\geq$$

$$\min\left(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a),\right.$$
$$\left.(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+(a^2+\frac{1}{4}a)\right). \tag{4.1.46}$$

Here $\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a)\geq a^2+\frac{1}{4}a$. Let us prove that $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+(a^2+\frac{1}{4}a)\geq\frac{11}{64}a^2+\frac{1}{4}a$. Equivalently, we should prove that $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2\geq 0$.

Assume that $1\geq a\geq\frac{11}{16}$. Then $-\frac{1}{4}a+\frac{11}{64}\leq 0$ and $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2\geq r(a,s)=(-2a^2+\frac{3}{2}a-\frac{5}{64})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2$. We have $r(a,0)\geq 0$ and $r(a,1)=-\frac{267}{64}a^2+\frac{17}{4}a-\frac{5}{64}\geq 0$. It follows that $r(a,s)\geq 0$ if $\frac{1}{2}r''_{ss}=3(-2a^2+\frac{3}{2}a-\frac{5}{64})s+(-3a^2+\frac{9}{4}a)\leq 0$. The last is valid if $\frac{1}{2}r''_{ss}(a,0)=-3a^2+\frac{9}{4}a\leq 0$ and $\frac{1}{2}r''_{ss}(a,1)=3(-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a)\leq 0$. It is true if $1\geq a\geq\frac{3}{4}$ when $-3a^2+\frac{9}{4}a\leq 0$. Suppose that $\frac{3}{4}\geq a\geq\frac{11}{16}$. Then $-3a^2+\frac{9}{4}a\geq 0$. If also $-2a^2+\frac{3}{2}a-\frac{5}{64}\geq 0$, then $r(a,s)\geq 0$. If $-2a^2+\frac{3}{2}a-\frac{5}{64}\leq 0$, we have $r(a,s)\geq((-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a))s^2+\frac{1}{2}as+\frac{53}{64}a^2$. If $(-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a)\leq 0$, we again have that $r(a,s)\geq\min(r(a,0),r(a,1))\geq 0$. If $(-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a)\geq 0$, we obviously have $r(a,s)\geq 0$.

Assume that $\frac{11}{64}\geq a\geq 1/4$. Then $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2\geq r(a,s)=(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2$. Arguing with the function $r(a,s)$ like above, we prove that $r(a,s)\geq 0$. (Here one can get even better result.)

Thus, for $1\geq a\geq\frac{1}{4}$ and $0\leq t\leq s\leq 1$, we have

$$(f^{\frac{1}{2}})''_{tt}\geq$$
$$f^{-\frac{3}{2}}\times\frac{\frac{11}{64}a^2+\frac{1}{4}a}{(t+1)^4} \tag{4.1.47}$$

We have $f\leq(1+(5/4)a)/(t+1)$. It follows

$$(f^{\frac{1}{2}})''_{tt}\geq(1+\frac{5}{4}a)^{-\frac{3}{2}}\times\frac{\frac{11}{64}a^2+\frac{1}{4}a}{(t+1)^{\frac{5}{2}}} \tag{4.1.48}$$

if $1\geq a\geq\frac{1}{4}$ and $1\geq s\geq t\geq 0$.

We have $(t-s)^2/(t+1)=t+1-2(s+1)+(s+1)^2/(t+1)$, it follows

$$\left(\frac{(t-s)^2}{(t+1)(s+1)}\right)''=\frac{2(s+1)^2}{(s+1)(t+1)^3}\leq\frac{8}{(s+1)(t+1)^3}. \tag{4.1.49}$$

Like for the proof of Lemma 4.1.6, from (4.1.23) and (4.1.48), it follows that for $1 \geq a_1,\, a_2 \geq 0,\, 1 \geq a_3,\, a_4 \geq \frac{1}{4}$,

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, t)''_{tt} \geq$$

$$(1+s)^{-1}(1+t)^{-3}\Big[2\Big(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+ + \sqrt{a_3+a_3(s-t)+\frac{a_3(s-t)^2}{4}+\frac{(1-a_3)(s+t)^2}{4}}+\sqrt{a_4+(1-a_4)t^2}\Big)\times \Big((1+\frac{5}{4}a_3)^{-\frac{3}{2}}(\frac{11}{64}a_3^2+\frac{1}{4}a_3)+\min\,(a_4-\frac{1}{4}a_4^2,\; -\frac{5}{4}+6a_4-4a_4^2)\Big)-\frac{(s+1)^2}{2}\Big]\geq$$

$$(1+s)^{-1}(1+t)^{-3}\,[2\,(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times \Big((1+\frac{5}{4}a_3)^{-\frac{3}{2}}(\frac{11}{64}a_3^2+\frac{1}{4}a_3)+\min\,(a_4-\frac{1}{4}a_4^2,\; -\frac{5}{4}+6a_4-4a_4^2)\Big)-2\,]. \tag{4.1.50}$$

Thus, $f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, t)''_{tt} \geq 0$ if $1 \geq a_1,\, a_2 \geq 0,\, 1 \geq a_3,\, a_4 \geq \frac{1}{4}$ and

$$(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times \Big((1+\frac{5}{4}a_3)^{-\frac{3}{2}}(\frac{11}{64}a_3^2+\frac{1}{4}a_3)+\min\,(a_4-\frac{1}{4}a_4^2,\; -\frac{5}{4}+6a_4-4a_4^2)\Big)\;\geq\;1. \tag{4.1.51}$$

For arbitrary $1 \geq a_1,\, a_2,\, a_3,\, a_4 \geq a \geq \frac{1}{4}$ it is valid if

$$\sqrt{a}\,\Big((1+\frac{5}{4}a)^{-\frac{3}{2}}(\frac{11}{64}a^2+\frac{1}{4}a)+a-\frac{1}{4}a^2\Big)\;\geq\frac{1}{4}. \tag{4.1.52}$$

It is true if $a \geq 0.37646$.

It proves Lemma 4.1.7 for $1 \geq a_1,\, a_2,\, a_3,\, a_4 \geq 0.37646$ when all three conditions (4.1.33), (4.1.34) and (4.1.35) are valid.

Using Theorems 4.1.4 and 4.1.5, we can divide all convex elliptic polygons on the hyperbolic plane in three types. This subdivision is very useful for fundamental polygons of finite volume of reflection groups, and we shall use it in Sects. 4.3 and 5.

Theorem 4.1.8. *Let $\mathcal{M}$ be an elliptic convex polygon on a hyperbolic plane. Then $\mathcal{M}$ has one of types (I), (II) or (III) (or the type (I), (II) or (III) of its narrow place) below:*

Type (I): *There exist its four consecutive vertices A_0, A_1, A_2, A_3 (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) with angles $\alpha_1 = A_0A_1A_2$, $\alpha_2 = A_1A_2A_3 \neq \frac{\pi}{2}$ and such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1, \delta_2) = 2\cos\alpha_1$, $(\delta_2, \delta_3) = 2\cos\alpha_2 \neq 0$ and*

$$\textit{either}\;\;(\delta_1,\delta_3) \leq 2\;\;\textit{or}\;\;(\delta_1,\delta_3) < 4(\cos\frac{\alpha_1}{2}+\cos\frac{\alpha_2}{2})^2-2 \leq 14 \tag{4.1.53}$$

The set $\{\delta_1, \delta_2, \delta_3\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph. Any triangle or quadrangle* $\mathcal{M}$ *has the type (I).*

Type (II): *There exist its five consecutive vertices* A_0, A_1, A_2, A_3, A_4 *(where* $A_0 = A_4$ *if* $\mathcal{M}$ *is a quadrangle) with angles* $\alpha_1 = A_0A_1A_2 = \frac{\pi}{2}$, $\alpha_2 = A_1A_2A_3 = \frac{\pi}{2}$, $\alpha_3 = A_2A_3A_4$ *and orthogonal vectors* δ_1, δ_2, δ_3 *and* δ_4 *to lines* (A_0A_1), (A_1A_2), (A_2A_3) *and* (A_3A_4) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$ *such that* $(\delta_1, \delta_2) = (\delta_2, \delta_3) = 0$, $(\delta_3, \delta_4) = 2cos\alpha_3$,

$$(\delta_1, \delta_3) < 6 \tag{4.1.54}$$

and

$$\begin{aligned}(\delta_1, \delta_4) \ &< \ 4 \max \left(\left(\sqrt{2} + \cos\frac{\alpha_3}{2}\right)^2, \frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}}\right)^2 - \frac{1}{4}}{2} \right) - 2 \leq \\ &\leq \ 10 + 8\sqrt{2} = 21.313708... \ . \end{aligned} \tag{4.1.55}$$

Moreover, the set $\{\delta_1, \delta_3, \delta_4\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph.*

Type (III). *There exist its six consecutive vertices* A_0, A_1, A_2, A_3, A_4, A_5 *(where* $A_0 = A_5$ *if* $\mathcal{M}$ *is a pentagon) with right angles* $\alpha_1 = A_0A_1A_2 = \frac{\pi}{2}$, $\alpha_2 = A_1A_2A_3 = \frac{\pi}{2}$, $\alpha_3 = A_2A_3A_4 = \frac{\pi}{2}$, $\alpha_4 = A_3A_4A_5 = \frac{\pi}{2}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3, δ_4 *and* δ_5 *to lines* (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) *and* (A_4A_5) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ *one has* $(\delta_1, \delta_2) = (\delta_2, \delta_3) = (\delta_3, \delta_4) = (\delta_4, \delta_5) = 0$ *and*

$$(\delta_1, \delta_3) < 6, \tag{4.1.56}$$

$$(\delta_3, \delta_5) < 6, \tag{4.1.57}$$

and

$$(\delta_1, \delta_5) < 30. \tag{4.1.58}$$

Moreover, the set $\{\delta_1, \delta_3, \delta_5\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph.*

Proof. If $\mathcal{M}$ is a triangle, at least one angle of $\mathcal{M}$ is not right. It follows that $\mathcal{M}$ has a sequence of consecutive vertices of type (I). If $\mathcal{M}$ is a quadrangle, we consider its narrow place of type (I) from Theorem 4.1.4. If both angles $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, we replace δ_2 by δ_4 with $\delta_4^2 = -2$ which is orthogonal to the line (A_0A_3) and directed outwards of $\mathcal{M}$. Since a quadrangle has at least one non-right angle (we can assume that it is $\alpha_0 = A_3A_0A_1$), then the sequence of vertices A_2, A_3, A_0, A_1 with orthogonal vectors δ_3, δ_4, δ_1 to sides (A_2A_3), (A_3A_0), (A_0A_1) respectively satisfies the condition (I) of Theorem 4.1.8.

We now suppose that $\mathcal{M}$ does not have a sequence of vertices of type (I) (we always suppose that it is a sequence of consecutive vertices). Then $\mathcal{M}$ has more than 4 vertices.

By Theorem 4.1.5, $\mathcal{M}$ has a narrow place of type (II) or (III).

Suppose that $\mathcal{M}$ has the narrow place of type (II). We have $\alpha_1 = \alpha_2 = \frac{\pi}{2}$ since changing the numeration). We then get estimates (4.1.54) and (4.1.55). Obviously, the set $\{\delta_1,\, \delta_2,\, \delta_3\}$ generates the 3-dimensional hyperbolic vector space and has connected components $\{\delta_1,\, \delta_3\}$ and $\{\delta_2\}$ since $\alpha_1 = \alpha_2 = \frac{\pi}{2}$. If the set $\{\delta_1,\, \delta_3,\, \delta_4\}$ is not connected, it then follows that δ_4 is orthogonal to the set $\{\delta_1,\, \delta_3\}$ and $\delta_4 = \lambda\delta_2$ which is impossible since the lines (A_1A_2) and (A_3A_4) are different. It follows that the Gram graph of the set $\{\delta_1,\, \delta_3,\, \delta_4\}$ is connected. This set generates the 3-dimensional hyperbolic vector space because the orthogonal lines (A_0A_1), (A_2A_3), (A_3A_4) do not have a common point and are not orthogonal to one line. Thus, we have proved that the $\mathcal{M}$ has type (II) of Theorem 4.1.8.

Assume that $\mathcal{M}$ has a narrow place of type (III) of Theorem 4.1.5. If the sequence $A_0,\, A_1,\, A_2,\, A_3,\, A_4,\, A_5$ does not have a subsequence of the type (I) of Theorem 4.1.8, then all angles $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{\pi}{2}$ are right. We then get estimates (4.1.56), (4.1.57) and (4.1.58). The same consideration as above, shows that the Gram graph of the $\{\delta_1,\, \delta_3,\, \delta_5\}$ is connected and this set generates the 3-dimensional hyperbolic vector space. Thus, the $\mathcal{M}$ has type (III) of Theorem 4.1.8.

It finishes the proof.

4.2. Narrow places of restricted parabolic convex polygons on the hyperbolic plane.

Compare [N9], [N11].

We use notation of Sect. 1.1. Fix a point $O = \mathbb{R}_{++}c$ at infinity of $\mathcal{L} = \mathcal{L}(\Phi)$. Thus, $c \in \Phi$, $(c,\, c) = 0$ and $(c,\, V^+) > 0$.

Recall that a *horosphere* $\mathcal{E}_O$ with the center O is the set of all lines in $\mathcal{L}$ containing O. The line $l = O\mathbb{R}_{++}h \in \mathcal{E}_O$, $\mathbb{R}_{++}h \in \mathcal{L}$ is the set $l = \{\mathbb{R}_{++}(tc+h) \mid t \in \mathbb{R}$ and $(tc+h,\, tc+h) > 0\}$. Fix a constant $R > 0$. Then there is a unique $\mathbb{R}_{++}h \in l$ such that $(h,\, c) = R$ and $(h, h) = 1$. Given $l_1,\, l_2 \in \mathcal{E}_O$, we denote the corresponding h's by $h_1,\, h_2$ and put

$$\rho(l_1, l_2) = \sqrt{-(h_1 - h_2)^2}. \tag{4.2.1}$$

Endowed with this distance, the horosphere $\mathcal{E}_O$ becomes an affine Euclidean space. If one changes R, the distance ρ is multiplied by a constant. We denote

$$\mathcal{E}_{O,R} = \{\mathbb{R}_{++}h \in \mathcal{L} \mid (h, c) = R \text{ and } (h, h) = 1\}. \tag{4.2.2}$$

The set $\mathcal{E}_{O,R} \cup \{O\}$ is a sphere in $\overline{\mathcal{L}}$, which is tangent to $\mathcal{L}_\infty$ at O. Moreover, the set $\mathcal{E}_{O,R}$ is orthogonal to every line $l \in \mathcal{E}_O$ at the point $\mathbb{R}_{++}h$, $h \in \mathcal{E}_{O,R}$ that corresponds to l. The distance in $\mathcal{L}$ induces Euclidean distance in $\mathcal{E}_{O,R}$ which is homothetic to the distance (4.2.1). The set $\mathcal{E}_{O,R}$ is identified with $\mathcal{E}_O$ and is also called a *horosphere*.

Let $K \subset \mathcal{E}_O$. The set

$$C_K = \bigcup_{\text{line } l \in K} l \tag{4.2.3}$$

is called the *cone with vertex O and base K*.

A non-degenerate convex locally finite polyhedron $\mathcal{M}$ in $\mathcal{L}$ is called *parabolic* (relative to the point $O \in \mathcal{L}_\infty$ if 1) and 2) below are valid:

1) $\mathcal{M}$ is finite at the point O, that is, the set $\{\delta \in P(\mathcal{M}) \mid (c,\, \delta) = 0\}$ is finite;

2) for every elliptic polyhedron $\mathcal{N} \subset \mathcal{E}_O$ (that is $\mathcal{N}$ is the convex hull of a finite subset of $\mathcal{E}_O$), the polyhedron $\mathcal{M} \cap C_{\mathcal{N}}$ is elliptic.

A parabolic polyhedron $\mathcal{M}$ is called *restricted parabolic* if

3) the set

$$r(\mathcal{M}) = \{(c, \delta/\sqrt{-\delta^2}) \mid \delta \in P(\mathcal{M})\} \ \text{ is finite.} \tag{4.2.4}$$

Geometrically this means that all hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, of faces of $\mathcal{M}$ are tangent to a finite set of horospheres with the center O.

We remark that if $O \notin \mathcal{M}$ for a parabolic polyhedron $\mathcal{M}$, then $\mathcal{M}$ is elliptic. Thus, it is only interesting to consider parabolic polyhedra relative to a point $O \in \mathcal{L}_\infty \cap \mathcal{M}$. Moreover, if $\mathcal{M}$ is parabolic and has a finite number of faces (or it has a finite set $P(\mathcal{M})$) then $\mathcal{M}$ is elliptic. Thus, it is only interesting to consider *infinite* (i.e. having infinite number of faces) parabolic polyhedra.

Theorems 4.1.4, 4.1.5 and 4.1.8 can be generalized on restricted parabolic polygons with strong inequalities replaced by non-strong ones.

Theorem 4.2.1 (about the narrow place of type (I)). *For any restricted parabolic convex polygon $\mathcal{M}$ on a hyperbolic plane there exist its four consecutive vertices A_0, A_1, A_2 and A_3 (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1, \delta_2) = 2\cos\alpha_1$, $(\delta_2, \delta_3) = 2\cos\alpha_2$ and*

$$\textit{either}\ \ (\delta_1, \delta_3) \le 2 \ \ \textit{or}\ \ (\delta_1, \delta_3) \le 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14 \tag{4.2.5}$$

where $\alpha_1 = A_0A_1A_2$ and $\alpha_2 = A_1A_2A_3$.

Moreover, the Gram graph of $\{\delta_1, \delta_2, \delta_3\}$ is not connected (i.e. this set is union of two non-empty orthogonal subsets) if and only if $\alpha_1 = \alpha_2 = \frac{\pi}{2}$.

Theorem 4.2.2 (about narrow places of types (II) and (III)). *For any restricted parabolic convex polygon $\mathcal{M}$ having more than 3 vertices (i.e. it is different from a triangle) on a hyperbolic plane, one of two possibilities (II) or (III) below is valid:*

(II) There exist its five consecutive vertices A_0, A_1, A_2 , A_3 and A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, one has $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3$,

$$\textit{either}\ \ (\delta_1, \delta_3) \le 2 \ \ \textit{or}\ \ (\delta_1, \delta_3) \le 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.2.6}$$

$$(\delta_1, \delta_4) \le$$

$$4\max_{0\le t\le 1} \frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{a_3 + a_3 t + t^2/4}\right)^2 - \frac{t^2}{4}}{1+t} - 2 =$$

$$4\max\left(\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2}\right)^2, \frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}}\right)^2 - \frac{1}{4}}{2}\right) - 2 \le 34 \tag{4.2.7}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1,\ 2,\ 3$, and $a_i = \cos^2\frac{\alpha_i}{2}$. Moreover, the set $\{\delta_1$, δ_2, $\delta_3, \delta_4\}$ has a connected Gram graph.

(III). There exist its six consecutive vertices A_0, A_1, A_2, A_3, A_4 and A_5 (where $A_0 = A_5$ if $\mathcal{M}$ is a pentagon) such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ one has $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3, 4$,

$$\text{either} \quad (\delta_1, \delta_3) \leq 2 \quad \text{or} \quad (\delta_1, \delta_3) \leq 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \leq 14, \tag{4.2.8}$$

$$\text{either} \quad (\delta_3, \delta_5) \leq 2 \quad \text{or} \quad (\delta_3, \delta_5) \leq 4(\cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2})^2 - 2 \leq 14, \tag{4.2.9}$$

and

$$(\delta_1, \delta_5) \leq$$

$$4 \max_{0 \leq t \leq s \leq 1} \Bigg[\Bigg(\Bigg(\sqrt{a_1 + (1 - a_1)s^2} + \sqrt{a_2 + (1 - a_2)s^2} +$$
$$+ \sqrt{a_3 + a_3(s - t) + \frac{a_3(s - t)^2}{4} + \frac{(1 - a_3)(s + t)^2}{4}} + \sqrt{a_4 + (1 - a_4)t^2} \Bigg)^2 -$$
$$- \frac{(s - t)^2}{4} \Bigg) / ((1 + s)(1 + t)) \Bigg] - 2 =$$
$$4 \max \Bigg[\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2} \right)^2,$$
$$\frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}} + \cos\frac{\alpha_4}{2}\right)^2 - \frac{1}{4}}{2}, \ 4 \Bigg] - 2 \ \leq \ 62 \tag{4.2.10}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1, 2, 3, 4$, and $a_i = \cos^2\frac{\alpha_i}{2}$. Moreover, the set $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ has a connected Gram graph.

Theorem 4.2.3. *Let $\mathcal{M}$ be a restricted parabolic convex polygon on a hyperbolic plane. Then $\mathcal{M}$ has one of types (I), (II) or (III) (or the type (I), (II) or (III) of its narrow place) below:*

Type (I): *There exist its four consecutive vertices A_0, A_1, A_2, A_3 (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) with angles $\alpha_1 = A_0A_1A_2$, $\alpha_2 = A_1A_2A_3 \neq \frac{\pi}{2}$ and such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1, \delta_2) = 2\cos\alpha_1$, $(\delta_2, \delta_3) = 2\cos\alpha_2 \neq 0$ and*

$$\text{either} \quad (\delta_1, \delta_3) \leq 2 \quad \text{or} \quad (\delta_1, \delta_3) \leq 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \leq 14 \tag{4.2.11}$$

The set $\{\delta_1, \delta_2, \delta_3\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Any triangle or quadrangle $\mathcal{M}$ has the type (I).

Type (II): *There exist its five consecutive vertices A_0, A_1, A_2, A_3, A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) with angles $\alpha_1 = A_0A_1A_2 = \frac{\pi}{2}$, $\alpha_2 = A_1A_2A_3 = \frac{\pi}{2}$, $\alpha_3 = A_2A_3A_4$ and orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$ such that $(\delta_1, \delta_2) = (\delta_2, \delta_3) = 0$, $(\delta_3, \delta_4) = 2cos\alpha_3$,*

$$(\delta, \delta) \leq 6 \tag{4.2.12}$$

and

$$(\delta_1,\,\delta_4)\ \le$$

$$4\ \max\left(\left(\sqrt{2}+\cos\frac{\alpha_3}{2}\right)^2,\ \frac{\left(2+\sqrt{2\cos^2\frac{\alpha_3}{2}+\frac{1}{4}}\right)^2-\frac{1}{4}}{2}\right)-2\le 10+8\sqrt{2}=$$

$$=21.313708...\ . \tag{4.2.13}$$

Moreover, the set $\{\delta_1,\delta_3,\delta_4\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph.*

Type (III). *There exist its six consecutive vertices* A_0, A_1, A_2, A_3, A_4, A_5 *(where* $A_0=A_5$ *if* $\mathcal{M}$ *is a pentagon) with right angles* $\alpha_1=A_0A_1A_2=\frac{\pi}{2}$, $\alpha_2=A_1A_2A_3=\frac{\pi}{2}$, $\alpha_3=A_2A_3A_4=\frac{\pi}{2}$, $\alpha_4=A_3A_4A_5=\frac{\pi}{2}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3, δ_4 *and* δ_5 *to lines* (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) *and* (A_4A_5) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=\delta_5^2=-2$ *one has* $(\delta_1,\,\delta_2)=(\delta_2,\,\delta_3)=(\delta_3,\,\delta_4)=(\delta_4,\,\delta_5)=0$ *and*

$$(\delta_1,\,\delta_3)\le 6, \tag{4.2.14}$$

$$(\delta_3,\,\delta_5)\le 6, \tag{4.2.15}$$

and

$$(\delta_1,\,\delta_5)\le 30. \tag{4.2.16}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_5\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph.*

Proofs of Theorems 4.2.1, 4.2.2, 4.2.3. If $\mathcal{M}$ is parabolic relative to $O\notin\mathcal{M}$, then $\mathcal{M}$ is finite and we can use Theorems 4.1.4, 4.1.5 and 4.1.8. Thus, we can suppose that $O\in\mathcal{M}$ and $\mathcal{M}$ is infinite.

To prove Theorems 4.1.4, 4.1.5 and 4.1.8, we were taken a point O inside of $\mathcal{M}$ and used the formulae of Lemma 4.1.2. To prove Theorems 4.2.1, 4.2.2, 4.2.3, we use the point O at infinity of $\mathcal{M}$ such that $\mathcal{M}$ is restricted parabolic relative to O. We use an analog of Lemma 4.1.2 which uses the infinite point O.

Let $O=\mathbb{R}_{++}c$ where $c^2=0$. For a line (AB) with terminals A and B at infinity and $\delta\in\Phi$ orthogonal to (AB) with $\delta^2=-2$ we introduce an 'angle'

$$\theta(\delta)=\frac{1}{(c,\,\delta)} \tag{4.2.17}$$

where one can replace δ by $-\delta$ according to the orientation of the angle AOB: e. g. one should take δ such that $\mathcal{H}_\delta^+$ contains O if AOB is oriented correctly, and one should take $-\delta$ if not.

The 'angle' $\theta(\delta)$ really behaves like an angle. For three points A, B and C at infinity and three lines (AB), (BC) and (AC) with the corresponding vectors δ_1, δ_2 and δ_3 orthogonal to lines (AB), (BC) and (AC) respectively and with $\delta_1^2=\delta_2^2=\delta_3^2=-2$, one has

$$\theta(\delta_1)+\theta(\delta_2)=\theta(\delta_3). \tag{4.2.18}$$

We leave an elementary proof of (4.2.18) to a reader.

We have the following analog of Lemma 4.1.2.

Lemma 4.2.4. *Let (AB) and (CD) are two lines on a hyperbolic plane with terminals A, B, C, D at infinity, and $O = \mathbb{R}_{++}c$ an infinite point on the hyperbolic plane which does not belong to each line (AB) and (CD) and orientations of the triangles AOB and COD coincide. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that O is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$. Let δ_{12} be the orthogonal vector with square -2 to the line (BC). Then*

$$(\delta_1, \delta_2) = 4\frac{(\theta(\delta_1) + \theta(\delta_{12}))(\theta(\delta_2) + \theta(\delta_{12}))}{\theta(\delta_1)\theta(\delta_2)} - 2.$$

As a corollary, we get :

1) If lines (AB) and (CD) do not intersect each other, then

$$2\cosh\rho = (\delta_1, \delta_2) = 4\frac{(\theta(\delta_1) + \theta(\delta_{12}))(\theta(\delta_2) + \theta(\delta_{12}))}{\theta(\delta_1)\theta(\delta_2)} - 2$$

where ρ is the distance between lines (AB) and (CD) (here and in what follows we normalize the curvature $\kappa = -1$).

2) If lines (AB) and (CD) define an angle α containing O, then

$$2\cos\alpha = (\delta_1, \delta_2) = 4\frac{(\theta(\delta_1) - \theta(\delta_{21}))(\theta(\delta_2) - \theta(\delta_{21}))}{\theta(\delta_1)\theta(\delta_2)} - 2$$

where $\theta(\delta_{21}) = -\theta(\delta_{12}) = \theta(-\delta_{12})$.

Proof of Lemma 4.2.4. One can prove it similarly to Lemma 4.1.2. One can also prove it as follows. Take a finite point O' and move O' to the infinite point O. Lemma 4.2.4 is the limit of the Lemma 4.1.2 applied to O' when O' tends to O. We leave details to a reader.

Now the proof of Theorems 4.2.1, 4.2.2 and 4.2.3 is the same as proof of the corresponding theorems 4.1.4, 4.1.5 and 4.1.8 if one uses the 'angles' (4.2.17) and Lemma 4.2.4. One has more: almost all inequalities become equalities, and the proof is even simpler.

4.3. Description of narrow places of fundamental polygons $\mathcal{M}$ of reflection subgroups $W \subset W(S)$ of elliptic and parabolic type where rk $S = 3$. Application to reflective lattices of elliptic and parabolic type.

Let S be a primitive hyperbolic lattice of rk $S = 3$ and $W \subset W(S)$ its reflection subgroup of elliptic or parabolic type with a fundamental polygon $\mathcal{M}$ (see Sects. 1.3, 1.4). Remind that it means that W and $P(\mathcal{M})_{\rm pr}$ have restricted arithmetic type and $P(\mathcal{M})_{\rm pr}$ has a generalized lattice Weyl vector ρ with $\rho^2 \ge 0$. It is known (e. g. see [N11], [N9]) that the polygon $\mathcal{M}$ is elliptic if $\rho^2 > 0$, and it is parabolic relative to $\mathbb{R}_{++}\rho$ if $\rho^2 = 0$. We remind that the lattice S having reflection subgroups $W \subset W(S)$ of elliptic or parabolic type is called reflective of elliptic or parabolic type. Here we apply Theorem 4.2.3 to describe narrow places of $\mathcal{M}$. Using this description, we shall get a finite list of lattices such that any elliptically or parabolically reflective lattice S belongs to the list. Let $a(S^*/S)$ be the exponent of the discriminant group S^*/S: i. e. $a(S)$ is the least natural a such that $aS^*/S = 0$. We shall get an estimate of $a(S^*/S)$.

Let K be a lattice. We denote by K_0 the primitive lattice defined by K. Thus, $K = K_0(\lambda)$ where $\lambda \in \mathbb{N}$ and K_0 is primitive. If K is generated by elements with the Gram matrix A, then K_0 is generated by the same elements with the Gram matrix A/λ where λ is the greatest common divisor of all elements of A. We denote by $a(A)$ the exponent of a finite Abelian group A: i. e. $a(A)$ is the least natural a such that $aA = 0$.

We have the following useful statement (compare the proof of Theorem 1 in [N5, Appendix]).

Proposition 4.3.1. *Let L be a primitive lattice and $\alpha_1, \dots, \alpha_k$ are primitive roots of L which generate a sublattice $G \subset L$ of a finite index. Let G_0 be the primitive lattice defined by G. Then $a(L^*/L) | 8a(G_0^*/G_0)^2$.*

Proof. We have $G = G_0(\lambda) \subset L$ for some natural $\lambda \in \mathbb{N}$. Since G is generated by roots α_i of L and $\alpha_i^2 | 2(\alpha_i, L)$, we get $2G_0(\lambda) \subset 2L \subset \lambda G_0(\lambda)^*$. Identifying (naturally) modules of the lattices G_0 and $G_0(\lambda)$, we obviously have $\lambda G_0(\lambda)^* = G_0^*$. It follows that

$$L = M(\lambda) \text{ where } G_0 \subset M \subset \frac{1}{2} G_0^*. \tag{4.3.1}$$

Here M is any intermediate module which is invariant with respect to reflections in roots α_i defining the lattice G_0, and the roots α_i should be primitive in M. The $\lambda \in \mathbb{N}$ is the smallest natural number such that $M(\lambda)$ is a lattice (otherwise, the lattice L is not primitive). Using (4.3.1), we get

$$2G_0 \subset M^* \subset G_0^*. \tag{4.3.2}$$

If $tM \subset M^*$, $t \in \mathbb{N}$, then $M(t)$ is a lattice. Really, for any $m_1, m_2 \in M$ we have $t(m_1, m_2) = (tm_1, m_2) \in \mathbb{Z}$ because $tm_1 \in M^*$. Using (4.3.1) and (4.3.2), we get $4aM \subset M^*$ where $a = a(G_0^*/G_0)$. It follows that $\lambda | 4a$. Identifying modules of M and $M(\lambda)$, we have $M(\lambda)^* = (1/\lambda)M^* \subset (1/4a)G_0^*$. It follows that the exponent of $M(\lambda)^*/M(\lambda)$ divides the exponent of $(1/4a)G_0^*/2G_0$ which is equal to $8a^2$. This finishes the proof.

Below we describe narrow places of the fundamental polygons $\mathcal{M}$. According to Theorem 4.2.3, a narrow place of $\mathcal{M}$ is defined by vectors $\delta_1, \dots, \delta_k$ with $\delta_i^2 = -2$ where $k \le 5$. We denote by r_i corresponding primitive roots of S such that

$$\delta_i = \frac{2r_i}{\sqrt{-2r_i^2}}. \tag{4.3.3}$$

with the Gram matrix

$$\Gamma = (\gamma_{ij}) = ((\delta_i, \delta_j)). \tag{4.3.4}$$

Since r_i are primitive roots of S, we have $r_i^2 | 2(S, r_i)$. It then follows that

$$\alpha_{ij} = \gamma_{ij}^2 = (\delta_i,\, \delta_j)^2 = \frac{4(r_i, r_j)^2}{r_i^2 r_j^2} \in \mathbb{Z}_+ \tag{4.3.5}$$

are non-negative integers.

We want to describe all possible matrices

$$B = ((r_i, r_j)) \tag{4.3.6}$$

where for an integral matrix T we denote by T_{pr} the corresponding primitive integral matrix T/t where $t = \text{g.c.d}(T)$ denote the greatest common divisor of all elements of T. We shall make it in three steps.

First we describe all possible symmetric $(k \times k)$-matrices

$$\mathcal{A} = (\alpha_{ij}) = \left((\delta_i,\, \delta_j)^2\right) = \left(\frac{4(r_i, r_j)^2}{r_i^2 r_j^2}\right). \tag{4.3.7}$$

The matrix $\mathcal{A}$ has non-negative integral coefficients, all its diagonal coefficients are equal to 4. All cyclic products

$$\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_{r-1} i_r}\alpha_{i_r i_1} \tag{4.3.8}$$

are perfect squares.

In the second place, we describe all possible $k \times k$ symmetrizable generalized Cartan matrices (see [Kac] about generalized Cartan matrices, but remember that we use the opposite sign)

$$A = (a_{ij}) = \left(\frac{2(r_i, r_j)}{-r_i^2}\right) \tag{4.3.9}$$

using relations

$$a_{ii} = -2,\ a_{ij} \in \mathbb{Z}_+ \text{ if } i \neq j,\ a_{ij}a_{ji} = \alpha_{ij},\ a_{ij} = 0 \text{ iff } a_{ji} = 0, \tag{4.3.10}$$

and

$$a_{i_1 i_2}a_{i_2 i_3}\cdots a_{i_{r-1} i_r}a_{i_r i_1} = a_{i_1 i_r}a_{i_r i_{r-1}}\cdots a_{i_3 i_2}a_{i_2 i_1}. \tag{4.3.11}$$

In the third place, we find a diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_k) \tag{4.3.12}$$

with $\lambda_i \in \mathbb{N}$ such that the matrix

$$B' = A\Lambda \tag{4.3.13}$$

is symmetric. This defines the matrix Λ uniquely up to multiplication by a scalar rational matrix (since the Gram graph of $r_1, \dots, r_k$ is connected and the matrix A is indecomposable). The matrix $\Lambda = s\,\text{diag}(-r_1^2, \dots, -r_k^2)$ where $s \in \mathbb{Q}_{++}$. Then we calculate

$$B = \frac{B'}{\text{g.c.d}(B')} \tag{4.3.14}$$

which gives the matrix $((r_i, r_j))_{\mathrm{pr}}$, see (4.3.6). These procedure gives a finite set of possible matrices B. See the corresponding general considerations in [N5, Appendix])

For the lattice $G_0 = [r_1, \dots, r_k]_{\mathrm{pr}}$ defined by B we calculate the invariant $a(B) = a(G_0^*/G_0)$, the invariant $a_1(B)$ which is the product of all different odd prime divisors of $a(B)$, and the invariant $a_{[illegible]}(B)$ which is the greatest odd prime divisor of $a(B)$. By Proposition 4.3.1, we get an estimate of the similar invariants $a(S)$, $a_1(S)$ and $a_2(S)$:

$$a(S) \le 8a(B)^2,\ a_1(S) \le a_1(B),\ a_2(S) \le a_2(B). \tag{4.3.15}$$

Calculating the invariants $a(B)$, $a_1(B)$ and $a_2(B)$ for all possible matrices B and taking their maximum, we estimate the invariants $a(S)$, $a_1(S)$ and $a_2(S)$ for all elliptically or parabolically reflective lattices S.

Below we describe this procedure for each type of the narrow place of Theorem 4.2.3.

4.3.1. Matrices B of the narrow places of the type (I1). It is a particular case of the type (I) of Theorem 4.2.3 when additionally the angle $\alpha_1 \ne \pi/2$. For this case $k = 3$ and δ_1, δ_2, δ_3 give a bases of the 3-dimensional hyperbolic vector space. We get that the matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 4 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 4 \end{pmatrix} \tag{4.3.1.1}$$

with integral non-negative coefficients where from the condition of narrow places we have (after changing numeration if necessary) that

$$\begin{gathered} 1 \le \alpha_{12} = \alpha_{21} \le 4,\quad \alpha_{12} \le \alpha_{23} = \alpha_{32} \le 4, \\ \alpha_{23} \le \alpha_{13} = \alpha_{31} \le \left[\left(\sqrt{2+\sqrt{\alpha_{12}}} + \sqrt{2+\sqrt{\alpha_{23}}}\right)^2 - 2\right]^2 \end{gathered} \tag{4.3.1.2}$$

(here we also use that $2\cos\frac{\alpha}{2} = \sqrt{2+2\cos\alpha}$), and

$$d = det(\Gamma) = -8 + 2\sqrt{\alpha_{12}\alpha_{23}\alpha_{31}} + 2\alpha_{12} + 2\alpha_{23} + 2\alpha_{13} > 0 \tag{4.3.1.3}$$

where

$$\alpha_{12}\alpha_{23}\alpha_{31} \ \text{ is a perfect square.} \tag{4.3.1.4}$$

It is easy to enumerate the finite set of all matrices $\mathcal{A}$ satisfying (4.3.1.2), (4.3.1.3) and (4.3.1.4). For each $\mathcal{A}$ we then find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & a_{12} & a_{13} \\ a_{21} & -2 & a_{23} \\ a_{31} & a_{32} & -2 \end{pmatrix} \tag{4.3.1.5}$$

using the relations

$$a_{12}a_{21} = \alpha_{12},\ a_{23}a_{32} = \alpha_{23},\ a_{13}a_{31} = \alpha_{13},\ a_{12}a_{23}a_{31} = a_{13}a_{32}a_{21}. \tag{4.3.1.6}$$

The diagonal matrix

$$\Lambda = \text{diag}(a_{13}a_{32},\ a_{23}a_{31},\ a_{31}a_{32}). \tag{4.3.1.7}$$

Finally, we get that

$$B = (\Lambda A) \tag{4.3.1.8}$$

In Appendix, we give the Program 4: fund11.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nI1$ of all the matrices B, and the numbers

$$aI1 = \max_B a(B),\ aI1_1 = \max_B a_1(B),\ aI1_2 = \max_B a_2(B). \tag{4.3.1.9}$$

Calculation using this program gives

$$nI1 = 272,\ aI1 = 3528,\ aI1_1 = 543,\ aI1_2 = 181. \tag{4.3.1.10}$$

4.3.2. Matrices B of the narrow places of the type (I0). It is a particular case of Type (I) of Theorem 4.2.3 when additionally the angle $\alpha_1 = \pi/2$. For this case $k = 3$ and δ_1, δ_2, δ_3 give a bases of the 3-dimensional hyperbolic vector space. We get that the matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} \\ 0 & 4 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 4 \end{pmatrix} \tag{4.3.2.1}$$

with integral non-negative coefficients where from the condition of narrow places we have (after changing numeration if necessary) that

$$1 \le \alpha_{23} = \alpha_{32} \le 4,\quad \alpha_{23} \le \alpha_{13} = \alpha_{31} \le \left[\left(\sqrt{2} + \sqrt{2 + \sqrt{\alpha_{23}}}\right)^2 - 2\right]^2, \tag{4.3.2.2}$$

and

$$d = det(\Gamma) = -8 + 2\alpha_{23} + 2\alpha_{13} > 0. \tag{4.3.2.3}$$

For each $\mathcal{A}$ we then find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} \\ 0 & -2 & a_{23} \\ a_{31} & a_{32} & -2 \end{pmatrix} \tag{4.3.2.4}$$

using the relations

$$a_{23}a_{32} = \alpha_{23},\ a_{13}a_{31} = \alpha_{13}. \tag{4.3.2.5}$$

The diagonal matrix

$$\Lambda = \mathrm{diag}(a_{13}a_{32},\ a_{23}a_{31},\ a_{31}a_{32}). \tag{4.3.2.6}$$

Finally, we get that

$$B = (A\Lambda)_{\mathrm{pr}}. \tag{4.3.2.7}$$

In Appendix, we give the Program 5: fund10.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nI0$ of all the matrices B, and the numbers

$$aI0 = \max_B a(B),\ aI0_1 = \max_B a_1(B),\ aI0_2 = \max_B a_2(B). \tag{4.3.2.8}$$

Calculation using this program gives

$$nI0 = 2998,\ aI0 = 69192,\ aI0_1 = 10209,\ aI0_2 = 89. \tag{4.3.2.9}$$

4.3.3. Matrices B of the narrow places of the type (II1). It is a particular case of Type (II) of Theorem 4.2.3 when additionally the angle $\alpha_3 \neq \pi/2$. For this case $k = 4$ and δ_1, δ_2, δ_3, δ_4 generate the 3-dimensional hyperbolic vector space and any three of them give a bases of the space. We get that the matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 4 & 0 & \alpha_{24} \\ \alpha_{31} & 0 & 4 & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 4 \end{pmatrix} \qquad (4.3.3.1)$$

with integral non-negative coefficients where

$$1 \le \alpha_{34} = \alpha_{43} \le 4, \quad 4 < \alpha_{13} = \alpha_{31} \le 36,$$

$$0 \le \alpha_{14} = \alpha_{41} \le \left[4 \max\left(\left(\sqrt{2} + \sqrt{\frac{\sqrt{\alpha_{34}}}{4} + \frac{1}{2}} \right)^2, \frac{\left(2 + \sqrt{\frac{\sqrt{\alpha_{34}}}{2} + \frac{5}{4}}\right)^2 - \frac{1}{4}}{2} \right) - 2 \right]^2,$$

$$\left[\left(\sqrt{2} + \sqrt{2 + \sqrt{\alpha_{34}}} \right)^2 - 2 \right]^2 < \alpha_{24} = \alpha_{42}. \qquad (4.3.3.2)$$

Here we use the conditions (II) of Theorem 4.2.3 and add some inequalities to avoid repeating of cases we have considered in Sect. 4.3.3. We also have

$$det(\Gamma) = 4(4 - \sqrt{\alpha_{13}\alpha_{34}\alpha_{14}} - \alpha_{14} - \alpha_{34} - \alpha_{13}) + \alpha_{24}\alpha_{13} - 4\alpha_{24} = 0. \qquad (4.3.3.3)$$

where

$$\alpha_{13}\alpha_{34}\alpha_{14} \text{ is a perfect square.} \qquad (4.3.3.4)$$

The condition (4.3.3.3) is equivalent

$$\alpha_{24} = \frac{4(\sqrt{\alpha_{13}\alpha_{34}\alpha_{14}} + \alpha_{14} + \alpha_{34} + \alpha_{13} - 4)}{\alpha_{13} - 4}. \qquad (4.3.3.5)$$

We can easily enumerate the finite set of all possible matrices $\mathcal{A}$ satisfying these conditions. For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} & a_{14} \\ 0 & -2 & 0 & a_{24} \\ a_{31} & 0 & -2 & a_{34} \\ a_{41} & a_{42} & a_{43} & -2 \end{pmatrix} \qquad (4.3.3.6)$$

using relations

$$a_{34}a_{43} = \alpha_{34},\ a_{13}a_{31} = \alpha_{13},\ a_{24}a_{42} = \alpha_{24},\ a_{13}a_{34}a_{41} = \sqrt{\alpha_{13}\alpha_{34}\alpha_{14}},$$
$$a_{14}a_{41} = \alpha_{14},\ a_{14} = a_{41} = 0 \text{ if } \alpha_{14} = 0. \qquad (4.3.3.7)$$

The diagonal matrix

$$\Lambda = diag(\text{[illegible]}) \qquad (4.3.3.8)$$

Finally, we get that

$$B = (A\Lambda)_{\rm pr}. \tag{4.3.3.9}$$

In Appendix, we give the Program 6: fund21.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nII1$ of all the matrices B, and the numbers

$$aII1 = \max_B a(B),\ aII1_1 = \max_B a_1(B),\ aII1_2 = \max_B a_2(B). \tag{4.3.3.10}$$

Calculation using this program gives

$$nII1 = 9818,\ aII1 = 47432,\ aII1_1 = 10965,\ aII1_2 = 487. \tag{4.3.3.11}$$

4.3.4. Matrices B of the narrow places of the type (II0). It is a particular case of Type (II) of Theorem 4.2.3 when additionally the angle $\alpha_3 = \pi/2$. For this case $k = 4$ and δ_1, δ_2, δ_3, δ_4 generate the 3-dimensional hyperbolic vector space and any three of them give a bases of the space. We get that the matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 4 & 0 & \alpha_{24} \\ \alpha_{31} & 0 & 4 & 0 \\ \alpha_{41} & \alpha_{42} & 0 & 4 \end{pmatrix} \tag{4.3.4.1}$$

with integral non-negative coefficients where

$$4 < \alpha_{13} = \alpha_{31} \le 36,\quad 0 < \alpha_{14} = \alpha_{41} \le (8+4\sqrt{5})^2 = 287.108350...,\ \alpha_{13} \le \alpha_{24}. \tag{4.3.4.2}$$

($8+4\sqrt{5} = 16.94427190...$) Here we use the conditions (II) of Theorem 4.2.3 and that $\mathcal{M}$ is a fundamental polygon having at least 4 sides. We also have

$$det(\Gamma) = 4(4-\alpha_{14}-\alpha_{13}) + \alpha_{24}\alpha_{13} - 4\alpha_{24} = 0 \tag{4.3.4.3}$$

which is equivalent

$$(\alpha_{13}-4)(\alpha_{24}-4) = 4\alpha_{14}. \tag{4.3.4.4}$$

It is easy to enumerate the finite set of all possible matrices $\mathcal{A}$ satisfying these conditions. For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} & a_{14} \\ 0 & -2 & 0 & a_{24} \\ a_{31} & 0 & -2 & 0 \\ a_{41} & a_{42} & 0 & -2 \end{pmatrix} \tag{4.3.4.5}$$

using relations

$$a_{13}a_{31} = \alpha_{13},\ a_{24}a_{42} = \alpha_{24},\ a_{14}a_{41} = \alpha_{14}, \tag{4.3.4.6}$$

The diagonal matrix

$$\Lambda = \mathrm{diag}(a_{13}a_{14}a_{42},\ a_{13}a_{41}a_{24},\ a_{31}a_{14}a_{42},\ a_{13}a_{42}a_{41}). \tag{4.3.4.7}$$

Finally, we get that

$$B = (A\Lambda)_{\rm pr}. \tag{4.3.4.8}$$

In Appendix, we give the Program 7: fund20.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nII0$ of all the matrices B, and the numbers

$$aII0 = \max_B a(B),\ \ aII0_1 = \max_B a_1(B),\ \ aII0_2 = \max_B a_2(B). \tag{4.3.4.9}$$

Calculation using this program gives

$$nII0 = 376208,\ \ aII0 = 995316,\ \ aII0_1 = 238569,\ \ aII0_2 = 283. \tag{4.3.4.10}$$

4.3.5. Matrices B of the narrow places of the type (III). For this case $k = 5$ and $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ generate the 3-dimensional hyperbolic vector space, and any three of them give a bases of the space. The matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ 0 & 4 & 0 & \alpha_{24} & \alpha_{25} \\ \alpha_{31} & 0 & 4 & 0 & \alpha_{35} \\ \alpha_{41} & \alpha_{42} & 0 & 4 & 0 \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & 0 & 4 \end{pmatrix} \tag{4.3.5.1}$$

with integral non-negative coefficients where

$$\begin{gathered} 4 < \alpha_{13} = \alpha_{31} \le 36,\ \ \alpha_{31} \le \alpha_{35} = \alpha_{53} \le 36,\ \ 0 \le \alpha_{15} = \alpha_{51} \le 30^2 = 900, \\ 287.108350 < \alpha_{14} = \alpha_{41},\ \ 287.108350 < \alpha_{25} = \alpha_{52}, \end{gathered} \tag{4.3.5.2}$$

Here we use the conditions (III) of Theorem 4.2.3 and that $\mathcal{M}$ is a fundamental polygon having at least 5 sides. The last two inequalities were added to avoid repetition with the previous case. The Gram matrix $\Gamma = \Gamma(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ has the rank 3. Coefficients of the matrix $\mathcal{A}$ are defined by the coefficients α_{13}, α_{15}, α_{35} defining the Gram matrix $\Gamma(\delta_1, \delta_3, \delta_5)$. Let

$$d = 4(\alpha_{13} + \alpha_{35} + \alpha_{15} + \sqrt{\alpha_{13}\alpha_{35}\alpha_{15}} - 4), \tag{4.3.5.3}$$

one can see that $d/2 = det(\Gamma(\delta_1, \delta_3, \delta_5))$. We have

$$\alpha_{14} = \frac{d}{\alpha_{35} - 4},\ \ \alpha_{25} = \frac{d}{\alpha_{13} - 4} \tag{4.3.5.4}$$

and

$$\alpha_{24} = \frac{4(\alpha_{13}\alpha_{35} + 4\sqrt{\alpha_{13}\alpha_{35}\alpha_{15}} + 4\alpha_{15})}{(\alpha_{13} - 4)(\alpha_{35} - 4)} \tag{4.3.5.5}$$

Here (4.3.5.4) follows from $\det(\Gamma(\delta_1, \delta_3, \delta_5, \delta_2)) = \det(\Gamma(\delta_1, \delta_3, \delta_5, \delta_4)) = 0$. To get (4.3.5.5), one should remark that $\delta_2 = \gamma_{25}\delta_5^*$ and $\delta_4 = \gamma_{14}\delta_1^*$. It follows that $\gamma_{24} = \gamma_{25}\gamma_{14}(g^{-1})_{13}$ where $g = \Gamma(\delta_1, \delta_3, \delta_5)$. All together, (4.3.5.4) and (4.3.5.5) are equivalent to rk $(\Gamma(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)) = 3$, or that $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ generate a 3-dimensional hyperbolic form. Moreover, cyclic products

are perfect squares (4.3.5.6)

It is easy to enumerate the finite set of all possible matrices $\mathcal{A}$ satisfying these conditions. For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A=\begin{pmatrix} -2 & 0 & a_{13} & a_{14} & a_{15}\\ 0 & -2 & 0 & a_{24} & a_{25}\\ a_{31} & 0 & -2 & 0 & a_{35}\\ a_{41} & a_{42} & 0 & -2 & 0\\ a_{51} & a_{52} & a_{53} & 0 & -2 \end{pmatrix} \tag{4.3.5.7}$$

using relations

$$\begin{gathered} a_{13}a_{31}=\alpha_{13},\ a_{35}a_{53}=\alpha_{35},\ a_{14}a_{41}=\alpha_{41},\ a_{25}a_{52}=\alpha_{25},\ a_{15}a_{51}=\alpha_{15},\\ a_{15}=a_{51}=0 \text{ if } \alpha_{15}=0,\ a_{24}a_{42}=\alpha_{24},\ a_{25}a_{52}=\alpha_{25},\ a_{35}a_{53}=\alpha_{35},\\ a_{13}a_{35}a_{52}a_{24}a_{41}=\sqrt{\alpha_{13}\alpha_{35}\alpha_{52}\alpha_{24}\alpha_{41}},\ a_{13}a_{35}a_{51}=\sqrt{\alpha_{13}\alpha_{35}\alpha_{51}}, \end{gathered} \tag{4.3.5.8}$$

The diagonal matrix

$$\Lambda=\operatorname{diag}(a_{14}a_{13}a_{42}a_{25},\ a_{41}a_{13}a_{24}a_{25},\ a_{14}a_{31}a_{42}a_{25},\ a_{41}a_{13}a_{42}a_{25},\ a_{41}a_{13}a_{24}a_{52}) \tag{4.3.5.9}$$

Finally, we get that

$$B=(A\Lambda)_{\mathrm{pr}}. \tag{4.3.5.10}$$

In Appendix, we give the Program 8: fund30.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nIII$ of all the matrices B, and the numbers

$$aIII=\max_B a(B),\ aIII_1=\max_B a_1(B),\ aIII_2=\max_B a_2(B). \tag{4.3.5.11}$$

Calculation using this program gives

$$nIII=200539,\ aIII=324900,\ aIII_1=26565,\ aIII_2=907. \tag{4.3.5.12}$$

4.3.6. The global estimate of invariants of primitive reflective hyperbolic lattices of the rank 3 *having elliptic or parabolic type.*

For a lattice L we denote by $a(L^*/L)$ the exponent of the discriminant group L^*/L, we denote by $a_1(L^*/L)$ the product of all different odd prime divisors of $a(L^*/L)$, and we denote by $a_2(L^*/L)$ the greatest prime divisor of $a(L^*/L)$.

Using Proposition 4.3.1 and calculations (4.3.1.10), (4.3.2.9), (4.3.3.11), (4.3.4.10), (4.3.5.12) we get

Theorem 4.3.6.1. *For any primitive reflective hyperbolic lattice S of rk $S=3$ having elliptic or parabolic type we have estimates:*

$$a(S^*/S)\le 8(995316)^2,\quad a_1(S^*/S)\le 238569,\quad a_2(S^*/S)\le 907.$$

Since $\det(S)\le a(S^*/S)^2$ and number of lattices with the fixed rank and determinant is finite (e. g. see [C]), Theorem 4.3.6.1 gives a finite list of lattices which contains all the reflective lattices S.

The estimates of Theorem 4.3.6.1 are very preliminary, and we shall significantly improve them below.

5. Classification of reflective hyperbolic lattices of the rank 3 and of elliptic or parabolic type: proofs

Now we are ready to prove classification results of Sect. 2.

5.1. Proof of Basic Theorem 2.3.2.1.

We first prove

Theorem 5.1.1. *Any elliptically or parabolically reflective main hyperbolic lattice S of rank 3 and with square-free determinant belongs to the list of Table 3 containing all main hyperbolic lattices S of the rank 3 with square-free $d = \det(S) \le 100000$ and $h = hnr(S) \le 1$.*

Proof. Let S be an elliptically or parabolically reflective main hyperbolic lattice of the rank 3 and with square-free determinant $d = \det(S)$. Let $\mathcal{M}$ be a fundamental polygon of $W(S)$.

First we will show that $h(S) \le 1$ (it gives the proof of Lemma 2.3.1.2). If S is elliptically reflective, then $\mathcal{M}$ is an elliptic (finite and of finite volume) polygon. This polygon has ≤ 1 central symmetries. Really, otherwise, a composition of two different central symmetries gives an automorphism of infinite order of $\mathcal{M}$ which is impossible. It follows that $h(S) \le 1$. Assume that S is parabolically reflective and ρ is a generalized lattice Weyl vector for $\mathcal{M}$. Let $A(\mathcal{M}) \subset O^+(S)$ be the group of symmetries of $\mathcal{M}$. By definition of a generalized lattice Weyl vector $\rho \in S$, then ρ is preserved by a subgroup $A \subset A(\mathcal{M})$ of finite index, $\rho^2 = 0$ and $\rho \neq 0$. For any $\phi \in A(\mathcal{M})$, the element $\phi(\rho)$ also has all these properties and is then a generalized lattice Weyl vector for $\mathcal{M}$. If $\phi(\rho) \neq \rho$, the group $A(\mathcal{M})$ has a subgroup of finite index which is trivial on the hyperbolic sublattice $\mathbb{Z}\rho + \mathbb{Z}\phi(\rho) \subset S$. It then follows that $A(\mathcal{M})$ is finite and S is elliptically reflective. We get a contradiction. Thus $\phi(\rho) = \rho$ for any $\phi \in A(\mathcal{M})$. If $u \in A(\mathcal{M})$ is a central symmetry, then the fixed part $S^u = \{x \in S \mid u(x) = x\}$ of u is negative definite and does not have non-zero elements ρ with $\rho^2 = 0$. This shows that $h = h(S) = 0$.

Now we apply to S and $\mathcal{M}$ results of Sect. 4.3. By Proposition 4.3.1 and (4.3.1.10), (4.3.2.9), (4.3.3.11), (4.3.5.12), we have $d \le 100000$ if $\mathcal{M}$ has a narrow place of types $I1$, $I0$, $II1$ or III. Really, for all this cases the invariant $a_1 < 50000$ and then $d \le 2a_1(S) \le 2a_1 < 100000$. Thus, Theorem 5.1.1 is valid for these lattices S.

Only if $\mathcal{M}$ has a narrow place of type $II0$, our estimate (4.3.4.10) of a_1 is not good enough:

$$a_1(S) \le aII0_1 = 238569, \quad a_2(S) \le 238. \tag{5.1.1}$$

Let us consider this case. In Sect. 4.3.4, for all primitive elliptically or parabolically reflective hyperbolic lattices S or rank 3 we considered the primitive Gram matrices $B = \Gamma_{\rm pr}$ where $\Gamma = \Gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the Gram matrix of primitive roots of S corresponding to a narrow place of type $II0$ of $\mathcal{M}$. We now consider them only for main lattices S with square-free $d = \det(S)$. Then we should add some additional conditions to Sect. 4.3.4. We introduce these conditions below.

Let $\beta \in S$ be a primitive root and $K = \delta^\perp$. Then

$$\text{either } S = \mathbb{Z}\beta \oplus K \text{ or } S = [\beta,\ K, (\beta + k)/2] \text{ where } k \in K. \tag{5.1.2}$$

It follows (by a simple consideration over $\mathbb{Z}_2$) that

$$\beta^2 \text{ is square-free.} \tag{5.1.3}$$

If $\beta_1,\,\beta_2 \in S$ are two primitive orthogonal roots and $\mathbb{Z}h = [\beta_1,\,\beta_2]^\perp$, then for any odd p

$$S\otimes \mathbb{Z}_p = \mathbb{Z}_p\beta_1 \oplus \mathbb{Z}_p\beta_2 \oplus \mathbb{Z}_p h \text{ and } h^2 \text{ is square-free.} \tag{5.1.4}$$

It follows

$$\text{g.c.d.}(\beta_1^2, \beta_2^2) \le 2,\ \ \text{g.c.d.}(\beta_1^2, h^2) \le 2,\ \ \text{g.c.d.}(\beta_2^2, h^2) \le 2, \tag{5.1.5}$$

and

$$d = \beta_1^2\beta_2^2h^2/2^t,\ \text{where } t \equiv 0 \mod 2. \tag{5.1.6}$$

It defines the t uniquely because d is square-free. From (5.1.6), for any odd $p|d$, we also have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{\beta_1^2/p}{p}\right) & \text{if } p|\beta_1^2, \\ \left(\frac{\beta_2^2/p}{p}\right) & \text{if } p|\beta_2^2, \\ \left(\frac{h^2/p}{p}\right) = \left(\frac{d\beta_1^2\beta_2^2/p}{p}\right) & \text{if } p|h^2, \end{cases} \tag{5.1.7}$$

where η is the invariant of S. The Gram matrix $\Gamma = \Gamma(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4)$ has three pairs of orthogonal roots: $(\alpha_1,\,\alpha_2) = (\alpha_2,\,\alpha_3) = (\alpha_3, \alpha_4) = 0$. It then follows (it is sufficient to have only one pair) that

$$\text{either } \Gamma = B \text{ or } \Gamma = 2B. \tag{5.1.8}$$

By (5.1.3), in (5.1.8), the second case is possible only if $b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2$. We denote (like in Sect. 4.3) by G_0 the lattice defined by the primitive matrix B. Suppose that $b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2$ and $\det(G_0)) = 2^k m$ where m is odd and $k \equiv 1 \mod 2$. We then have the second case $\Gamma = 2B$ in (5.1.8), because otherwise S is odd but d is even. Thus, for a fixed B, we may have both cases in (5.1.8) only if $b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2$ and $\det(G_0) = 2^k m$ where m is odd and $k \equiv 0 \mod 2$. In all other cases the matrix B prescribes the Gram matrix Γ. For both cases in (5.1.8), we can calculate d using the matrix B since d is square-free. If one of α_i^2 is odd but d is even, then $\Gamma = B$, and this case is impossible for a main S (the lattice S should be even for even d).

Summarizing considerations above, we get the following additional conditions for B and we calculate the invariants (d, η) of the lattice S using the matrix B. We denote by G_0 the lattice with the matrix B. We have

$$b_{11},\, b_{22},\, b_{33},\, b_{44} \text{ are square-free} \tag{5.1.9}$$

and

$$\text{g.c.d}(b_{11}, b_{22}) \le 2,\ \text{g.c.d}(b_{22}, b_{33}) \le 2,\ \text{g.c.d}(b_{33}, b_{44}) \le 2. \tag{5.1.10}$$

We denote by $\nu_p(a)$ the order of p in factorisation of a. We have

$$\text{if odd } p|b_{11}b_{22}b_{33}b_{44},\ \text{then } \nu_p(\det(G_0)) \equiv 1 \mod 2. \tag{5.1.11}$$

We have

$$\Gamma = B,\ \text{if not all } b_{11},\, b_{22},\, b_{33},\, b_{44} \text{ are odd;} \tag{5.1.12}$$

$$\Gamma = 2B,\ \text{if } b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2 \text{ and } \nu_2(det(G_0)) \equiv 1 \mod 2; \tag{5.1.13}$$

$$\Gamma = B \text{ or } \Gamma = 2B, \text{ if } b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2 \text{ and } \nu_2(det(G_0)) \equiv 0 \mod 2. \tag{5.1.14}$$

We denote by G the lattice generated by α_i and defined by the Gram matrix $\Gamma = ((\alpha_i,\, \alpha_j))$. Let d be the product of all p such that $p|\det(G)$ and $\nu_p(det(G)) \equiv 1 \mod 2$. We have

$$\nu_2(d) \equiv 0 \mod 2 \text{ if one of } \alpha_i^2 \text{ is odd.} \tag{5.1.15}$$

For odd $p|d$ we have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{\alpha_1^2/p}{p}\right) & \text{if } p|\alpha_1^2, \\ \left(\frac{\alpha_2^2/p}{p}\right) & \text{if } p|\alpha_2^2, \\ \left(\frac{d\alpha_1^2\alpha_2^2/p}{p}\right) & \text{otherwise.} \end{cases} \tag{5.1.16}$$

This defines the invariant η. At last, we have the most delicate condition which enormously drops the finite number of possibilities:

$$h = hnr(d,\eta) \le 1 \tag{5.1.17}$$

(see Theorem 3.2.1 about $hnr(d,\eta)$). To calculate $hnr(d,\eta)$ using Theorem 3.2.1, we should calculate class-numbers of imaginary quadratic fields of the discriminant D where $-4d \le D < 0$. By (5.1.1), $d \le 2\cdot 238569$ and $0 < -D \le 8\cdot 238569 = 1908552$. Moreover, by (5.1.1), prime divisors of D are not more than 238. Thus, checking the condition (5.1.17), we work with reasonable (not too big) integers. Or, if one wants, we should use a program which correctly calculates class-numbers of discriminants D where $-1908552 \le D < 0$ and D is product of primes $p \le 238$.

In Appendix: Program 9: fund20.main, we give a program for GP/PARI calculator, which enumerates all the matrices B satisfying the conditions of Sect. 4.3 (this part of Program 9 is the same as the Program 7: fund20.gen) and conditions (5.1.9) — (5.1.17), and gives all triplets of invariants (d,η,h) of the matrices B (it gives 132 triplets). We can see that all these triplets belong to the Table 3. The GP/PARI calculator calculates class numbers of negative discriminants D for $|D| < 10^{25}$ (see user's guide to the calculator). This finishes the proof of Theorem 5.1.1.

By Theorem 5.1.1, to prove Theorem 2.3.2.1, we should now check reflective type of all lattices of Table 3 and calculate the sets $P(\mathcal{M})_{\text{pr}}$ and the Gram matrices $G(P(\mathcal{M})_{\text{pr}})$ if they are elliptically or parabolically reflective. In Table 3, for each pair of invariants (d,η) we give a main lattice S with these invariants. If

$$S = U \oplus \langle -d\rangle, \tag{5.1.18}$$

we have

$$(-1)^{\eta_p} = \left(\frac{-d/p}{p}\right), \text{ for any odd } p|d. \tag{5.1.19}$$

If

$$S = \langle n_1\rangle \oplus \langle -n_2\rangle \oplus \langle -n_3\rangle(\epsilon_1/2, \epsilon_2/2, \epsilon_3/2) \tag{5.1.20}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$, we have $d = n_1n_2n_3$ if $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, and $d = n_1n_2n_3/4$ otherwise. Moreover, $n_i \equiv 0 \mod 2$ if $\epsilon_i = 1$; $(n_1\epsilon_1 - n_2\epsilon_2 - n_3\epsilon_3)/4 \in \mathbb{Z}$; $(n_1\epsilon_1 - n_2\epsilon_2 - n_3\epsilon_3)/4 \in 2\mathbb{Z}$ if $d \equiv 0 \mod 2$; for any odd $p|d$ we have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{n_1/p}{p}\right) & \text{if } p|n_1, \\ \left(\frac{-n_2/p}{p}\right) & \text{if } p|n_2, \\ \left(\frac{-n_3/p}{p}\right) & \text{if } p|n_3. \end{cases} \tag{5.1.21}$$

Checking all these conditions, one can prove that our calculation of main lattices S corresponding to the invariants (d, η) of Table 3 are correct. Using known criteria (e. g. see [Se]), it is easy to prove that $S = U \oplus \langle -d \rangle$ if and only if S represents 0. The condition (5.1.19) is equivalent to this property. If (5.1.19) is valid, we always give S in the form $S = U \oplus \langle -d \rangle$.

Thus, fortunately, all our lattices S have one of two forms (5.1.18) or (5.1.20).

We use the Vinberg's algorithm [V2] to calculate the sets $P(\mathcal{M})_{\text{pr}}$ for lattices of the forms (5.1.18) and (5.1.20). Below we describe this algorithm.

First, we remark that the lattice $U \oplus \langle -d \rangle$ is equivariantly equivalent to its maximal even sublattice $U \oplus \langle -4d \rangle$ if d is odd. These lattices have naturally isomorphic groups of automorphisms and the reflection groups. Thus, it is sufficient to consider only the lattices

$$U \oplus \langle -2k \rangle, \quad k \in \mathbb{N}. \tag{5.1.22}$$

For lattices S of the form (5.1.22), we use the isotropic vector $c = (1, 0, 0)$ as the center of Vinberg's algorithm. We find $\mathcal{M}$ which contains $\mathbb{R}_{++}c$ as an infinite vertex and $v_1 = (0, 0, 1)$, $v_2 = (n, 0, -1)$ as roots orthogonal to faces of $\mathcal{M}$ containing $\mathbb{R}_{++}c$ (equivalently, they are roots of the height 0). It is easy to see that v_1, v_2 give the orthogonal primitive roots to the fundamental polyhedron of the stabilizer subgroup of c in the reflection group $W(S)$.

For lattices S of the form (5.1.20) we assume that either $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)/2 = 0$ or $\epsilon_i \neq 0$ at least for two different i. We take $c = (1, 0, 0)$ as the center of Vinberg's algorithm. We take as roots of the height 0
$v_1 = (0, 1, 0)$ and $v_2 = (0, 0, 1)$ if $n_2/n3 \neq 1/3,\ 1,\ 3$;
$v_1 = (0, 1, 0)$, $v_2 = (0, -1, 1)$ if $n_2 = n_3$ and $\epsilon = (0, 0, 0)$, $(1/2, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, -1/2, 1/2)$ if $n_2 = n_3$ and $\epsilon = (0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$ if $n_2 = n_3$ and $\epsilon \neq (0, 0, 0)$, $(1/2, 1/2, 1/2)$, $(0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, -1/2, 1/2)$ if $n_2/n_3 = 3$ and $\epsilon = (0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$ if $n_2/n_3 = 3$ and $\epsilon \neq (0, 1/2, 1/2)$;
$v_1 = (0, 0, 1)$, $v_2 = (0, 1/2, -1/2)$ if $n_2/n_3 = 1/3$ and $\epsilon = (0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$ if $n_2/n_3 = 1/3$ and $\epsilon \neq (0, 1/2, 1/2)$.
One can check that v_1, v_2 are the orthogonal primitive roots to the fundamental polyhedron of the stabilizer subgroup of c in $W(S)$.

Further steps of the Vinberg's algorithm are prescribed canonically (see [V2]). One introduces the height

$$\text{height} = \frac{2(\delta,\, c)^2}{\delta^2} \in \mathbb{Z} \tag{5.1.23}$$

of primitive roots $\delta \in S$. The roots $\{v_1, v_2\}$ above give all roots of $P(\mathcal{M})_{\rm pr}$ of the height 0. If one knows all roots of $P(\mathcal{M})_{\rm pr}^{\le n} \subset P(\mathcal{M})_{\rm pr}$ of the height $\le n$, all primitive roots $P(\mathcal{M})_{\rm pr}^{n+1} \subset P(\mathcal{M})_{\rm pr}$ of the height $n+1$ are given by the condition

$$\frac{2(\delta,\, c)^2}{-\delta^2} = n+1 \text{ and } (\delta,\, P(\mathcal{M})_{\rm pr}^{\le n}) \ge 0. \tag{5.1.24}$$

In Appendix, Program 10: refl0.1, we give this algorithm for lattices (5.1.22). In Appendix, Program 12: refl0.13, we give this algorithm for lattices (5.1.20). We also give Program 11: refl0.12 which calculates only for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ but is faster. These programs calculate a sequence v_i of all elements of the subset $P(\mathcal{M})_{\rm pr}^{\le n} \subset P(\mathcal{M})_{\rm pr}$ (i. e. all roots of the height $\le n$).

After calculation of $P(\mathcal{M})_{\rm pr}^{\le n}$ for a sufficiently large height n, next steps of our algorithm are as follows. A sequence $r = r_1, \dots, r_k$ of different elements from $P(\mathcal{M})_{\rm pr}$ is called a *chain* if

$$0 \le \frac{4(r_i, r_{i+1})^2}{r_i^2 r_{i+1}^2} \le 2, \text{ for any } 1 \le i \le k-1. \tag{5.1.25}$$

Geometrically it means that $\mathcal{H}_{r_1}, \dots, \mathcal{H}_{r_k}$ are lines of consecutive sides (i. e. defining vertices) of the polygon $\mathcal{M}$. We find the maximal chain $e = (e_1, \dots, e_k)$ in $P(\mathcal{M})_{\rm pr}^{\le n}$ containing v_1. Suppose that (for a sufficiently large n) we also have

$$\frac{4(e_1, e_k)^2}{e_1^2 e_k^2} \le 2, \tag{5.1.26}$$

i. e. the lines $\mathcal{H}_{e_k}$ and $\mathcal{H}_{e_1}$ also define a vertex of $\mathcal{M}$. Then the polygon $\mathcal{M}$ is elliptic, $P(\mathcal{M})_{\rm pr} = e$, and the chain e gives the orthogonal primitive roots to consecutive sides of the elliptic polygon $\mathcal{M}$. This situation takes place for 122 invariants (d, η) of the Table 3 marked by *er* (elliptically reflective cases). The result of our calculation of the chain e and the Gram matrix $((e_i, e_j))$, e_i, $e_j \in e$, for these 122 cases is given in Table 1.

For all other $206 - 122 = 84$ cases we should prove that S is not reflective of elliptic or parabolic type.

To prove that, for small determinants d we use the following arguments. For a large height n, the chain $e \subset P(\mathcal{M})_{\rm pr}^{\le n}$ contains a *period:* there exists $1 < q < k$ such that the Gram matrices $\Gamma(e_1,\, e_2)$ and $\Gamma(e_q,\, e_{q+1})$ coincide and there exists $C \in O^+(S)$ such that $C(e_1) = e_q$, $C(e_2) = e_{q+1}$. It follows that $C \in A(\mathcal{M})$. We find all integral (i.e. from S) eigenvectors u of C, and show that all of them have $u^2 < 0$. Clearly, these eigenvectors have the eigenvalue ± 1. It follows that S is not elliptically or parabolically reflective. Really, if S is elliptically reflective, the group $A(\mathcal{M})$ is finite, C has finite order and has an integral eigenvector with the eigenvalue 1 and positive square. If S is parabolically reflective, then A has an eigenvector with the eigenvalue 1 and with zero square (the generalized lattice Weyl vector ρ). See the proof above of Lemma 2.3.1.2. If S is hyperbolically reflective, the eigenvector u defines a generalized lattice Weyl vector ρ.

Let us consider an example of these calculations for $(d, \eta) = (114, 2)$. Then $S = U \oplus \langle -114 \rangle$. Using Program 10 we calculate up to the height $n = 50000$ and find the chain e in $P(\mathcal{M})_{\rm pr}^{\le n}$ which is equal to

$$e_1 = (321, 30, -13),\ e_2 = (28, 2, -1),\ e_3 = (57, 0, -1),\ e_4 = (0, 0, 1),$$

$$e_5 = (-1, 1, 0),\ e_6 = (9, 6, -1),\ e_7 = (292, 122, -25) \tag{5.1.27}$$

with the Gram matrix

$$ge=\begin{pmatrix} -6 & 0 & 228 & 1482 & 291 & 714 & 10872\\ 0 & -2 & 0 & 114 & 26 & 72 & 1150\\ 228 & 0 & -114 & 114 & 57 & 228 & 4104\\ 1482 & 114 & 114 & -114 & 0 & 114 & 2850\\ 291 & 26 & 57 & 0 & -2 & 3 & 170\\ 714 & 72 & 228 & 114 & 3 & -6 & 0\\ 10872 & 1150 & 4104 & 2850 & 170 & 0 & -2 \end{pmatrix}. \qquad (5.1.28)$$

We see that the Gram matrices of e_1, e_2 and e_6, e_7 coincide. We calculate that

$$C=\begin{pmatrix} 2209 & 22800 & 107160\\ 912 & 9409 & 44232\\ -188 & -1940 & -9119 \end{pmatrix} \qquad (5.1.29)$$

belongs to $O^+(S)$ and $C(e_1^t, e_2^t) = (e_6^t, e_7^t)$. It follows that $C \in A(\mathcal{M})$. The matrix C has the only integral eigenvector $w = (95, 19, -6)$, it has the eigenvalue 1. We have $w^2 = -494$. It follows that S is not elliptically or parabolically reflective: the automorphism C has infinite order and does not have integral eigenvectors with non-negative square.

Since for $(d, \eta) = (114, 2)$ the invariant $h = 0$, it is possible that S is hyperbolically reflective with the generalized lattice Weyl vector $\rho = w$. Let us prove that this is the case. We find another chain f in $P(\mathcal{M})_{\text{pr}}^{\le n}$ which is equal to

$$\begin{aligned} &f_1 = (1766, 172, -73),\ f_2 = (6384, 627, -265),\ f_3 = (4560, 456, -191),\\ &f_4 = (283, 29, -12),\ f_5 = (18, 3, -1),\ f_6 = (14, 4, -1),\\ &f_7 = (456, 171, -37),\ f_8 = (2280, 912, -191),\ f_9 = (427, 173, -36) \end{aligned} \qquad (5.1.30)$$

with the Gram matrix

$$gf=\begin{pmatrix} -2 & 0 & 114 & 26 & 72 & 1150 & 72504 & 413250 & 79370\\ 0 & -114 & 114 & 57 & 228 & 4104 & 259806 & 1481658 & 284601\\ 114 & 114 & -114 & 0 & 114 & 2850 & 182058 & 1039566 & 199728\\ 26 & 57 & 0 & -2 & 3 & 170 & 11001 & 62928 & 12094\\ 72 & 228 & 114 & 3 & -6 & 0 & 228 & 1482 & 291\\ 1150 & 4104 & 2850 & 170 & 0 & -2 & 0 & 114 & 26\\ 72504 & 259806 & 182058 & 11001 & 228 & 0 & -114 & 114 & 57\\ 413250 & 1481658 & 1039566 & 62928 & 1482 & 114 & 114 & -114 & 0\\ 79370 & 284601 & 199728 & 12094 & 291 & 26 & 57 & 0 & -2 \end{pmatrix}. \qquad (5.1.31)$$

We see that the Gram matrix of f_1, f_2 is equal to the Gram matrix of f_6, f_7. We calculate that $C(f_1^t, f_2^t) = (f_6^t, f_7^t)$. Moreover, we can see that $(e_i, w) > 0$ and $(f_i, w) < 0$.

It follows that $P(\mathcal{M})_{\text{pr}} = H(e) \cup H(f)$ where $H = [C]$ is the infinite cyclic group generated by the C. The infinite polygon $\mathcal{M}$ contains the line (the axis) $\mathcal{H}_w$ which is preserved by H. All sides of $\mathcal{M}$ define two infinite chains $\mathcal{H}_\delta$, $\delta \in H(e)$ and $\mathcal{H}_\delta$, $\delta \in H(f)$, these two chains are contained in two different half-planes bounded by the line $\mathcal{H}_w$. The polygon $\mathcal{M}$ is finite in every orthogonal cylinder over a compact base in $\mathcal{H}_w$ (it is restricted hyperbolic relative to $\mathcal{H}_w$). The polygon $\mathcal{M}$ has an infinite symmetry group $H = [C]$. It is not the full symmetry group $A(\mathcal{M})$

of $\mathcal{M}$. After considerations above, it is easy to see that $A(\mathcal{M})$ is the cyclic group generated by the glide reflection C_1 with axis $\mathcal{H}_w$ where

$$C_1 = \begin{pmatrix} 48 & 475 & 2280 \\ 19 & 192 & 912 \\ -4 & -40 & -191 \end{pmatrix}. \tag{5.1.32}$$

The vector w is the eigenvector of C_1 with the eigenvalue -1. We have $C_1^2 = C$. The glide reflection C_1 changes places the infinite chains $H(e)$ and $H(f)$. The lattice S is hyperbolically reflective with the generalized lattice Weyl vector $\rho = w$.

For large d following arguments are very useful. We calculate the set $P(\mathcal{M})_{\text{pr}}^{\le n}$ for a sufficiently large height n and find two pairs of elements $r_1, r_2 \in P(\mathcal{M})_{\text{pr}}^{\le n}$, $s_1, s_2 \in P(\mathcal{M})_{\text{pr}}^{\le n}$ such that $4(r_1,r_2)^2/r_1^2r_2^2 \le 2$, $4(s_1,s_2)^2/s_1^2s_2^2 \le 2$ and Gram matrices of these pairs coincide. In many cases it is sufficient to consider pairs with $(r_1,r_2) = (s_1,s_2) = 0$ and $r_1^2 = s_1^2$, $r_2^2 = s_2^2$. Thus, orthogonal sides of these pairs define two vertices of $\mathcal{M}$. We find such pairs that there exists an automorphism $B \in O^+(S)$ such that $B(r_i) = s_i$. Then $B \in A(\mathcal{M})$. Similarly, considering two other pairs in $P(\mathcal{M})_{\text{pr}}^{\le n}$, we find another $C \in A(\mathcal{M})$. If $B^2C^2 \ne C^2B^2$, the lattice S is not reflective of any type: elliptic, parabolic or hyperbolic. In many cases it is sufficient to find only one $B \in A(\mathcal{M})$ and calculate that $B^{12} \ne E$. Really, then B has infinite order and the lattice S cannot be elliptically reflective. If $h = 1$, the lattice S also cannot be parabolically or hyperbolically reflective.

We consider an example of these calculations for $(d,\eta) = (3990, 4)$. Then $S = \langle 30\rangle \oplus \langle -38\rangle \oplus \langle -14\rangle(1/2,1/2,0)$. Using Program 12 from Appendix, we calculate up to the height $n = 500000$. It gives 33 elements $v_i \in P(\mathcal{M})_{\text{pr}}^{\le n}$. They are

$$\begin{aligned}
&v_1 = (0,1,0),\ v_2 = (0,0,1),\ v_3 = (1/2,-1/2,0),\\
&v_4 = (2,0,-3),\ v_5 = (13/2,-9/2,-6),\ v_6 = (7,-3,-9),\\
&v_7 = (57/2,-21/2,-38),\ v_8 = (17/2,-9/2,-10),\ v_9 = (28,-21,-22),\\
&v_{10} = (63/2,-35/2,-36),\ v_{11} = (57,-37,-57),\ v_{12} = (25,-15,-27),\\
&v_{13} = (42,-14,-57),\ v_{14} = (17,-4,-24),\ v_{15} = (76,-58,-57),\\
&v_{16} = (19,-6,-26),\ v_{17} = (52,-15,-72),\ v_{18} = (84,-21,-118),\\
&v_{19} = (1729/2,-1015/2,-950),\ v_{20} = (119/2,-69/2,-66),\\
&v_{21} = (69/2,-43/2,-36),\\
&v_{22} = (73,-57,-51),\ v_{23} = (74,-57,-54),\ v_{24} = (231/2,-147/2,-118),\\
&v_{25} = (2261/2,-1645/2,-950),\ v_{26} = (101,-27,-141),\ v_{27} = (266,-132,-323),\\
&v_{28} = (119,-87,-99),\ v_{29} = (128,-63,-156),\ v_{30} = (399/2,-315/2,-134),\\
&v_{31} = (342,-96,-475),\ v_{32} = (361,-77,-513),\ v_{33} = (238,-119,-288).
\end{aligned} \tag{5.1.33}$$

We look at all the pairs v_i, v_j with $(v_i,v_j) = 0$, and we find that there are five such pairs v_1,v_2; v_7,v_{13}; v_{15},v_9; v_{11},v_{24}; v_{27},v_{33} having squares $v_i^2 = -38$, $v_j^2 = -14$. We calculate a primitive orthogonal element $w \in S$ to each of these five pairs v_i, v_j, and we find that between these five pairs there are two pairs v_{11}, v_{24} and v_{27}, v_{33} such that $(w + v_j)/2 \in S$. We then find the matrix

$$C = \begin{pmatrix} 6863/2 & 5339/2 & 1694 \\ -3345/2 & -2601/2 & -826 \\ 4200 & 3268 & 2073 \end{pmatrix} \tag{5.1.34}$$

such that $C \in O^+(S)$ and $C(v_{11}^t, v_{24}^t) = (v_{27}^t, v_{33}^t)$. It follows that $C \in A(\mathcal{M})$. We have $C^{12} \neq E$. It follows that C has infinite order. Thus, the lattice S is not elliptically reflective. For $(d, \eta) = (3990, 4)$, the invariant $h = 1$, thus the lattice S cannot be also parabolically or hyperbolically reflective. It follows that S is not reflective.

These arguments permit to prove that the rest 84 cases of Table 3 which are not contained in the Table 1, are not reflective of elliptic or parabolic type. It finishes the proof.

Remark 5.1.2. Here we want to outline another way which helps to study reflective type of lattices of Table 3. We can write similar programs as Program 9: fund20.main for all types of narrow places of $\mathcal{M}$ of main lattices S having the invariant $h \leq 1$. They would be programs fund11.main, fund10.main, fund21.main and fund 30.main specializing the programs fund11.gen, fund10.gen, fund21.gen and fund30.gen (one can write them similarly to Program 9: fund20.main). Together with Program 9: fund20.main, they give a list L of invariants (d, η, h) containing in Table 3. All invariants (d, η, h) of Table 3 which are not in the list L cannot be elliptically or parabolically reflective. If $h = 1$, they cannot be hyperbolically reflective either.

We did this calculations and we found that the list L is much smaller than Table 3 (e. g. Program 9: fund20.main gives only 132 triplets (d, η, h)). It shows that the narrow places of polyhedra arguments are sometimes stronger than arithmetic arguments of studying the invariant h (even if we forget about the problem with infinity). Both these arguments surprisingly fit together.

Here is the list of invariants (d, η, h) which are contained in Table 3 but are not in the list L (39 pairs):

$$
\begin{aligned}
&(d, \eta, h) = \\
&(57,1,1);\ (65,3,1);\ (71,0,1);\ (119,3,1);\ (161,1,1);\ (182,3,1); \\
&(194,0,1);\ (246,0,1);\ (259,3,1);\ (266,0,1);\ (266,3,1);\ (285,1,1); \\
&(299,3,1);\ (326,0,1);\ (335,0,1);\ (354,2,1);\ (386,0,1);\ (407,0,1); \\
&(506,0,1);\ (530,0,1);\ (534,0,1);\ (546,2,0);\ (602,3,1);\ (645,6,1); \\
&(714,6,1);\ (777,6,1);\ (854,3,1);\ (897,5,1);\ (897,7,1);\ (935,6,1); \\
&(966,2,1);\ (1106,1,1);\ (1254,4,1);\ (1394,3,1);\ (1659,2,1);\ (2210,6,1); \\
&(3311,1,1);\ (3990,4,1);\ (4466,1,1). \qquad (5.1.35)
\end{aligned}
$$

These (d, η, h) give lattices S which are not elliptically or parabolically reflective: their fundamental polygon $\mathcal{M}$ does not have a narrow place satisfying Theorem 4.2.3. Of course, the list (5.1.35) is in complete agreement with calculations above using Vinberg's algorithm.

5.2. Proof of Theorem 2.3.3.1.

We should check reflective type of non-main lattices $\widetilde{S}$ corresponding to main lattices S of Table 1 with odd d (there are 97 these cases). See Proposition 2.2.6. Let S be one of lattices of Table 1 (or of Table 3 marked by er) with invariants (d, η) where d is odd. Then $\widetilde{S}$ has invariants $(2d, odd, \eta + \omega(p))$. If S and $\widetilde{S}$ are equivariantly equivalent, i. e.

$$\sum (1 - p + 4\eta_p + 4\omega(p)) \equiv 0 \text{ or } 6 \mod 8, \qquad (5.2.1)$$

then lattices S and $\widetilde{S}$ have the same reflective type and calculation of $P(\mathcal{M})_{\rm pr}$ and its Gram matrix for the lattice $\widetilde{S}$ follows from similar calculation for S (see Remark 2.3.3.2). There are 21 these cases. For example if $S=U\oplus\langle -d\rangle$, where d is odd, then $\widetilde{S}=\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -2d\rangle$, and these lattices are equivariantly equivalent. Thus, we need to study only cases when (5.2.1) is not valid. There are $97-21=76$ these cases. Let S be one of these lattices.

We have: if $S=\langle n_1\rangle\oplus\langle -n_2\rangle\oplus\langle -n_3\rangle$, where all n_i are odd, then $\widetilde{S}=\langle 2n_1\rangle\oplus\langle -2n_2\rangle\oplus\langle -2n_3\rangle(\epsilon_1,\epsilon_2,\epsilon_3)$ where one of ϵ_i is equal to 0, two of them are 1/2 and $(\epsilon_1n_1-\epsilon_2n_2-\epsilon_3n_3)/2$ is odd. If $S=\langle n_1\rangle\oplus\langle -n_2\rangle\oplus\langle -n_3\rangle(\epsilon_1,\epsilon_2,\epsilon_3)$ where for example $\epsilon_1=\epsilon_2=1/2$ and $\epsilon_3=0$ (it follows that $n_1\equiv n_2\equiv 2\mod 4$ and n_3 is odd), then $\widetilde{S}=\langle n_1/2\rangle\oplus\langle -n_2/2\rangle\oplus\langle -2n_3\rangle$. We see that the lattice $\widetilde{S}$ which we should check for reflective type, has the form (5.1.20). Thus, we need to make similar calculations as in Sect. 5.1. They are of the same difficulty. These calculations finish the proof.

6. Appendix: Programs for GP/PARI calculator

Program 1: h2

```
\\hclass(d,muu) calculates h=(hrI(d,muu),hrII(d,muu),hnr(d,muu))
\\here d<0 and 0\le muu<2^k are integers,
\\k is the number of all odd prime divisors of d.
\\Assume d is a fundamental discriminant of fundamental binary
\\positive lattices (i.e. with a square free determinant),
\\then d\equiv 1 \mod 4 or
\\d \equiv \pm 4, 8\mod 16.
\\ Assume (d,muu) is the genus of them. Then
\\hrI(d,muu), hrII(d,muu), hnr(d,muu) are numbers of ambiguous
\\classes of the types I, II and
\\hnr(d,muu)=(h(d)/2^{tau(d)}- hrI(d,muu)-hrII(d,muu))/2
\\of non-ambiguous classes of the general
\\equivalence respectively of the genus (d,muu);
\\hclass(d,muu)=[0,0,0] if
\\d is not a fundamental discriminant.
\\if d is a fundamental discriminant but 0\le muu <2^k
\\does not correspond to a
\\genus, then hnr(d,muu)=[0,0,h(d)/2^{tau(d)+1}]
hclass(d,muu,dd,fdd,dd1,beta,alpha,k,hr,hrI,hrII,t,h)=\
h=[0,0,0];hr=0;hrI=0;hrII=0;\
if(mod(d,4)!=mod(1,4),,dd=-d;\
fdd=factor(dd);\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(2*dd1/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1), \
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro(2*(dd/dd1)/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hr=hr+beta));t=matsize(fdd)[1]-1;\
```

```
h=[0,hr,(classno(d)/2^t–hr)/2]);\
if(mod(d,16)!=mod(4,16),,dd=–d/4;\
fdd=factor(dd);\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(dd1/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro((dd/dd1)/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1), \
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hr=hr+beta));t=matsize(fdd)[1]–1;\
h=[hr,0,(classno(d)/2^t–hr)/2]);\
if(mod(d,16)!=mod(8,16),,dd=–d/4;\
if(dd/2==1,hr=1;h=[1,0,0],fdd=factor(dd/2);\
fordiv(dd/2,dd1,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(dd1/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro((dd/dd1)/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hr=hr+beta);t=matsize(fdd)[1];\
h=[hr,0,(classno(d)/2^t–hr)/2]));\
if(mod(d,16)!=mod(–4,16),,dd=–d/4;\
if(dd==1,h=[1,1,0],fdd=factor(dd);\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(dd1/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro((dd/dd1)/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hrI=hrI+beta));\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(2*dd1/fdd[k,1],fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro(2*(dd/dd1)/fdd[k,1], fdd[k,1])==(–1)^bittest(muu,k–1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hrII=hrII+beta));\
hr=hrI+hrII;t=matsize(fdd)[1];\
h=[hrI,hrII,(classno(d)/2^t–hr)/2]));h;
```

Program 3: h3

```
\\the program h3
\\hnr(d,et) calculates the number of classes of
\\non-reflective central symmetries of a 3-dimensional main
\\hyperbolic lattice with the square-free determinant
\\d and the invariant et (a non-negative integer)
\\whose binary decomposition et_{p_k}....et_{p_1}
\\gives the map of all odd prime divisors p_1,...,p_k
\\of the d in increasing order to {0,1}

\r h2

\\checking the condition (5)=(3.2.7)
beta5(d,et,n,fd,sfd,alpha,k,b,etap)=\
fd=factor(d);sfd=matsize(fd)[1];alpha=1;k=1;\
while(alpha,\
if(k>sfd,b=1;alpha=0,\
if(type(n/fd[k,1])!=1,k=k+1,\
if(fd[k,1]==2,k=k+1,\
if(fd[1,1]==2,etap=bittest(et,k–2),etap=bittest(et,k–1));\
if(kro(n/fd[k,1],fd[k,1])==(–1)^etap,k=k+1,b=0;alpha=0)))));b;

\\checking the condition (6)=(3.2.8)
beta6(d,et,n,fd,sfd,u,k,b)=\
fd=factor(d);sfd=matsize(fd)[1];\
u=mod(0,8);\
for(k=1,sfd,\
if(type((d/n)/fd[k,1])!=1,,u=u+mod(1–fd[k,1]+4*bittest(et,k–1),8)));\
if(u!=mod(–2,8),b=0,b=1);b;

\\checking the condition (8)=(3.2.10)
beta8(d,et,n,fd,sfd,u,k,b)=\
fd=factor(d);sfd=matsize(fd)[1];\
u=mod(0,8);\
for(k=1,sfd,\
if(type((d/n)/fd[k,1])!=1,,u=u+mod(1–fd[k,1]+ 4*bittest(et,k–1),8)));\
u=u+mod(((d/n)^2–1)/2,8);\
if(u!=mod(2,8),b=1,b=0);b;

\\checking the condition (11)=(3.2.13)
beta11(d,et,n,fd,sfd,u,k,b)=\
fd=factor(d);sfd=matsize(fd)[1];\
u=mod(0,4);\
for(k=2,sfd,u=u+mod(1–fd[k,1]+4*bittest(et,k–2),4));\
u=–u–mod(1,4);\
if(u!=mod(n/2,4),b=0,b=1);b;

\\calculation of the numbers et_p+epsilon(p) if odd p|d
eps(d,et,e,fd,sfd,sfd1,e1,et1,eet1,k)=\
if(d==1,e=0,\
fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]>2,sfd1=sfd,sfd1=sfd–1);\
e1=vector(sfd1,k,mod((fd[sfd–k+1,1]–1)/2,2));\
et1=vector(sfd1,k,mod(bittest(et,sfd1–k),2));\
```

```
eet1=e1+et1;eet1=lift(eet1);\
e=0;for(k=0,sfd1–1,e=e+2^k*eet1[sfd1–k]));e;
\\calculation of the numbers et_p+epsilon(p)+omega(p) if odd p|d
epsomeg(d,et,e,fd,sfd,sfd1,e1,et1,eet1,k)=\
if(d==1,e=0,\
fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]>2,sfd1=sfd,sfd1=sfd–1);\
e1=vector(sfd1,k,mod((fd[sfd–k+1,1]–1)/2+(fd[sfd–k+1,1]^2–1)/8,2));\
et1=vector(sfd1,k,mod(bittest(et,sfd1–k),2));\
eet1=e1+et1;eet1=lift(eet1);\
e=0;for(k=0,sfd1–1,e=e+2^k*eet1[sfd1–k]));e;
\\calculation of the numbers mu_p=eta_p if odd p|t|2d
muuu(d,et,t,m,fd,sfd,sfd1,k1,k)=\
if(d¡=2,m=0,m=0;fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]>2,sfd1=1,sfd1=2);k1=0;\
for(k=0,sfd–sfd1,\
if(type(t/fd[k+sfd1,1])==1,m=m+2^k1*bittest(et,k);k1=k1+1,)));m;
\\number of classes of non-reflective central symmetries
hnr(d,et,h,n,n1)=\
h=0;\
if(mod(d,2)==mod(1,2),\
fordiv(d,n,\
if(beta5(d,et,n)==1&&beta6(d,et,n)==1,\
h=h+hclass(–d/n,muuu(d,eps(d,et),d/n))[3],));\
fordiv(d,n,\
if(beta5(d,et,n)==1&&beta8(d,et,n)==1,\
h=h+hclass(–4*d/n,muuu(d,eps(d,et),d/n))[3],));\
fordiv(d,n1,\
n=2*n1;\
if(beta5(d,et,n)==1,\
h=h+hclass(–16*d/n,muuu(d,eps(d,et),d/n1))[3],)),\
fordiv(d/2,n1,\
n=2*n1;\
if(beta5(d,et,n)==1&&beta11(d,et,n)==1,\
h=h+hclass(–d/n,muuu(d,eps(d,et),d/n))[3],));\
fordiv(d/2,n1,\
n=2*n1;\
if(beta5(d,et,n)==1&&mod(n1,4)==mod(–d/2,4),\
h=h+hclass(–4*d/n,muuu(d,epsomeg(d,et),d/n))[3],));\
fordiv(d/2,n1,\
n=2*n1;\
if(beta5(d,et,n)==1&&mod(n1,4)==mod(d/2,4)&&n¡d,\
h=h+2*hclass(–4*d/n,muuu(d,epsomeg(d,et),d/n))[3]+\
hclass(–4*d/n,muuu(d,epsomeg(d,et),d/n))[2],)));h;
```

Program 3: refh3

```
\\refh3(N) gives the list of invariants (d,et) of
\\main hyperbolic lattices with the square-free
\\determinant d\le N and of the rank 3 having the
```

```
\\number h=hnr(d,et)\le 1 of classes of non-reflective
\\central symmetries. Here et is a non-negative integer
\\having the binary decomposition et=p_k,...,p_1 where
\\p_1,...,p_k are all odd prime divisors of d in icreasing order
\r h2
\r h3
refh3(m,n,d,fd,sfd,sfd1,sig,n,et,h)=\
n=0;\
for(d=1,m,\
if(issqfree(d)!=1,,\
if(d<=2,n=n+1;h=0;et=0;pprint("n=",n," d=",d," et=",et," h=",h),\
fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]==2,sfd1=sfd-1,sfd1=sfd);\
for(et=0,2^sfd1-1,\
if(fd[1,1]!=2,,\
sig=mod(0,8);for(k=2,sfd,sig=sig+mod(1-fd[k,1]+4*bittest(et,k-2),8)));\
if(fd[1,1]==2&&(sig==mod(0,8)||sig==mod(-2,8)),\
h=hnr(d,et);if(h<=1,n=n+1;\
pprint("n=",n," d=",d," et=",et," h=",h),),\
if(fd[1,1]!=2,h=hnr(d,et);\
if(h<=1,n=n+1;pprint("n=",n," d=",d," et=",et," h=",h),),))))))
```

Program 4: fund11.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
for(alpha12=1,4,for(alpha23=alpha12,4,\
u=((sqrt(2+sqrt(alpha12))+sqrt(2+sqrt(alpha23)))^2- 2)^2+0.000001;\
for(alpha13=alpha23,u,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);\
fordiv(alpha[1,2],a12,fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a21=alpha[1,2]/a12;a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;\
db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
)))))));pprint("nI1= ",n);pprint("eI1= ",epsilon);\
```

```
pprint("aI1_1=",epsilon1);pprint("aI1_2=",epsilon2);
```

Program 5: fund10.gen

```
\l;\ epsilon=1;epsilon1=1;epsilon2=1;\
n=0;\
alpha12=0;for(alpha23=1,4,\
w=((sqrt(2)+sqrt(2+sqrt(alpha23)))^2-2)^2+0.000001;\
for(alpha13=alpha23,w,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);a12=0;a21=0;\
fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;\
db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))));pprint("nI0=",n);pprint("aI0=",epsilon);\
pprint("aI0_1=",epsilon1);pprint("aI0_2=",epsilon2);
```

Program 6: fund21.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
alpha12=alpha23=0;\
for(alpha34=1,4,\
w=(4*max((sqrt(2)+sqrt(sqrt(alpha34)/4+1/2))^2,\
((2+sqrt(sqrt(alpha34)/2+5/4))^2-1/4)/2)-2)^2+0.000001;\
for(alpha14=0,w,\
for(alpha13=5,36,\
if(issquare(u=alpha13*alpha34*alpha14)!=1,,\
if(type(alpha24=4+4*(alpha14+alpha34+isqrt(u))/(alpha13-4))!=1||\
alpha24<=((sqrt(2)+sqrt(2+sqrt(alpha34)))^2-2)^2-0.0000001,,\
alpha=4*idmat(4);\
alpha[1,3]=alpha[3,1]=alpha13;alpha[3,4]=alpha[4,3]=alpha34;\
alpha[2,4]=alpha[4,2]=alpha24;alpha[1,4]=alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,\
fordiv(alpha[1,3],a13,\
fordiv(alpha[2,4],a24,\
if(type(a41=isqrt(u)/(a13*a34))!=1,,\
```

```
a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[4,1]=a41;if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);\
if(type(a[1,4])!=1,,\
diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))))));pprint("nII1=",n);pprint("aII1=",epsilon);\
pprint("aII1_1=",epsilon1);pprint("aII1_2=",epsilon2);
```

Program 7: fund20.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
alpha12=0;alpha23=0;alpha34=0;\
for(alpha14=1,287.10,\
fordiv(4*alpha14,aa,\
if(aa^2>4*alpha14,,alpha13=4+aa;alpha24=4*alpha14/aa+4;\
if(alpha13>36,,\
alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,fordiv(alpha[1,4],a14,\
a=-2*idmat(4);a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[1,4]=a14;\
a[4,1]=alpha[1,4]/a14;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[4,2];\
diag[2,2]=a[1,3]*a[4,1]*a[2,4];diag[3,3]=a[3,1]*a[1,4]*a[4,2];\
diag[4,4]=a[1,3]*a[4,2]*a[4,1];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
)))))));pprint("nII0=",n);pprint("aII0=",epsilon);\
pprint("aII0_1=",epsilon1);pprint("aII0_2=",epsilon2);
```

Program 8: fund30.gen

```
\l;\
```

```
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
al12=al23=al34=al45=0;\
for(al15=0,900,for(al13=5,36,for(al35=al13,36,\
if(issquare(q=al13*al35*al15)==0,,d=(al13+al35+al15-4+isqrt(q))*4;\
if(type(al14=d/(al35-4))!=1||al14<=287.108350||type(al25=d/(al13-4))!=1||\
al25<=287.108350||\
type(al24=(al13*al35+4*al15+4*isqrt(q))*4/((al35-4)*(al13-4)))!=1||\
issquare(q1=al13*al35*al25*al24*al14)==0,,\
al=idmat(5)*4;al[1,5]=al[5,1]=al15;al[1,3]=al[3,1]=al13;\
al[1,4]=al[4,1]=al14;al[2,4]=al[4,2]=al24;al[2,5]=al[5,2]=al25;\
al[3,5]=al[5,3]=al35;\
fordiv(al13,a13,\
fordiv(al35,a35,a51=isqrt(q)/a13/a35;\
if(a51==0,a15==0,if(type(a15=al15/a51)!=1,,\
fordiv(al14,a14,\
fordiv(al24,a24,a52=isqrt(q1)/a13/a35/a24*a14/al14;\
if(type(a25=al25/a52)!=1,,\
a=idmat(5)*-2;\
a[1,3]=a13;a[3,1]=al13/a13;\
a[1,4]=a14;a[4,1]=al14/a14;\
a[1,5]=a15;a[5,1]=a51;\
a[2,4]=a24;a[4,2]=al24/a24;\
a[2,5]=a25;a[5,2]=a52;\
a[3,5]=a35;a[5,3]=al35/a35;\
diag=idmat(5);diag[1,1]=a[1,4]*a[1,3]*a[4,2]*a[2,5];\
diag[2,2]=a[4,1]*a[1,3]*a[2,4]*a[2,5];\
diag[3,3]=a[1,4]*a[3,1]*a[4,2]*a[2,5];\
diag[4,4]=a[4,1]*a[1,3]*a[4,2]*a[2,5];\
diag[5,5]=a[4,1]*a[1,3]*a[2,4]*a[5,2];\
b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[3];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],);\
)))))))))))));pprint("nIII0=",n);\
pprint("aIII0=",epsilon);pprint("aIII0_1=",epsilon1);\
pprint("aIII0_2=",epsilon2);
```

Program 9: fund20.main

```
\r h2
\r h3
\l;\
n=0;\
alpha12=0;alpha23=0;alpha34=0;\
for(alpha14=1,287.10,\
fordiv(4*alpha14,aa,\
```

```
if(aa^2>4*alpha14,,alpha13=4+aa;alpha24=4*alpha14/aa+4;\
alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,fordiv(alpha[1,4],a14,\
a=-2*idmat(4);a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[1,4]=a14;\
a[4,1]=alpha[1,4]/a14;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[4,2];\
diag[2,2]=a[1,3]*a[4,1]*a[2,4];diag[3,3]=a[3,1]*a[1,4]*a[4,2];\
diag[4,4]=a[1,3]*a[4,2]*a[4,1];b=a*diag;b=b/content(b);\
m=m+1;db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[2,2],-b[3,3]])>2||content([-b[3,3],-b[4,4]])>2,,\
detb=db[2]*db[3]*db[4];\
if(detb==1,,\
fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;\
for(j=1,matsize(fdetb)[1],\
if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));\
if(gam==1,,\
for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1=1,,\
detb=db[2]*db[3]*db[4];fdetb=factor(detb);\
d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));\
if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],\
if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,i,k,0);u[1,]=[d,et,hhh],\
```

```
aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);\
for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],\
u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1));\
)))))))))))));pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 10: refl0.1

```
\\main(n,h) calculates for U+<-2n> (it is given by the matrix g)
\\the vectors v of the height \le h from P(\M)_{pr}
\\and calculates chains e and f and their Gram matrices ge, gf
refl(n,h,m,m1,j,k,v1,y1,y2,y3,z)=\
g=[0,1,0;1,0,0;0,0,-2*n];\
m=3;v=matrix(m,3,j,k,0);v[1,]=[0,0,1];v[2,]=[n,0,-1];v[3,]=[-1,1,0];\
for(h1=2,h,\
fordiv(h1,y2,if(type(y2^2/h1)!=1||type(n*h1/y2^2)!=1,,\
d=-2*y2^2/h1;\
for(z=floor(sqrt((2*y2^2-d)/(2*n))),y2,\
y3=-z;y1=(2*n*y3^2+d)/(2*y2);\
if(type(2*y1/d)!=1||content([y1,y2,y3])!=1,,\
u=[y1,y2,y3];\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u~>=0,m1=m1+1,alpha=0));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
)))))));v;
\\**
\\getting e1-matrix from v
e1fromv(v,s,s1,alpha,s2,ex)=\
s=matsize(v)[1];\
e1=matrix(s,3,j,k,0);e1[1,]=v[2,];e1[2,]=v[1,];\
alpha=1;s1=2;\
while(alpha,\
if(s1==s,alpha=0,\
for(j=1,s,\
if(v[j,]==e1[s1-1,]||v[j,]==e1[s1,]||\
(v[j,]*g*e1[s1,]~)^2/((v[j,]*g*v[j,]~)*(e1[s1,]*g*e1[s1,]~))>1,,\
s2=s1+1;e1[s2,]=v[j,]));if(s2>s1,s1=s2,alpha=0)));\
ex=matrix(s1,3,j,k,e1[j,k]);e1=ex;
\\getting e2-matrix from v
e2fromv(v,s,s1,alpha,s2,ex)=\
s=matsize(v)[1];\
e2=matrix(s,3,j,k,0);e2[1,]=v[1,];e2[2,]=v[2,];\
alpha=1;s1=2;\
while(alpha,\
if(s1==s,alpha=0,\
for(j=1,s,\
```

```
if(v[j,]==e2[s1-1,]||v[j,]==e2[s1,]||\
(v[j,]*g*e2[s1,]~)^2/((v[j,]*g*v[j,]~)*(e2[s1,]*g*e2[s1,]~))>1,,\
s2=s1+1;e2[s2,]=v[j,]));if(s2>s1,s1=s2,alpha=0)));\
ex=matrix(s1,3,j,k,e2[j,k]);e2=ex;
\\getting e-matrix from v
efromv(v,e1,e2,s,s1,s2)=\
s=matsize(v)[1];s1=matsize(e1)[1];s2=matsize(e2)[1];\
if(s1==s&&s2==s,e=e1,\
e=matrix(s1+s2-2,3,j,k,0);\
for(j=1,s2,e[j,]=e2[s2+1-j,]);\
for(j=1,s1-2,e[s2+j,]=e1[2+j,]));\
e;ge=e*g*e~;
\\getting f1-matrix from v
f1fromv(v,e,s,s1,s2,q,alpha,f1x)=\
s=matsize(v)[1];s1=matsize(e)[1];\
if(s==s1,f1=0;v1=0,\
f1=matrix(s-s1,3,j,k,0);v1=v;\
for(j=1,s,\
for(k=1,s1,\
if(v1[j,]!=e[k,],,v1[j,]=[0,0,0])));\
q=1;s2=0;alpha=1;\
while(alpha,\
if(q>s,alpha=0,\
if(v1[q,]==[0,0,0],q=q+1,s2=s2+1;f1[s2,]=v1[q,];\
v1[q,]=[0,0,0];alpha=0)));\
if(s2==s-s1,,\
for(t=1,s,\
for(q=1,s,\
if(v1[q,]==[0,0,0],,\
if((v1[q,]*g*f1[s2,]~)^2/((v1[q,]*g*v1[q,]~)*(f1[s2,]*g*f1[s2,]~))>1,,\
s2=s2+1;f1[s2,]=v1[q,];v1[q,]=[0,0,0]))))));\
if(s2==0,,f1x=matrix(s2,3,j,k,f1[j,k]);f1=f1x);f1);
\\getting f2-matrix from v
f2fromv(f1,s,s1,s2,s3,q,f2x)=\
if(f1==0,f2=0;s3=0,\
s=matsize(v1)[1];s1=matsize(e)[1];s2=matsize(f1)[1];\
f2=matrix(s-s1-s2+1,3,j,k,0);\
s3=1;f2[1,]=f1[1,];\
for(t=1,s,\
for(q=1,s,\
if(v1[q,]==[0,0,0],,\
if((v1[q,]*g*f2[s3,]~)^2/((v1[q,]*g*v1[q,]~)*(f2[s3,]*g*f2[s3,]~))>1,,\
s3=s3+1;f2[s3,]=v1[q,];v1[q,]=[0,0,0])))));\
if(s3==0,,f2x=matrix(s3,3,j,k,f2[j,k]);f2=f2x));f2;
\\getting f-matrix from v
ffromv(f1,f2,s1,s2)=\
if(f2==0,f=f1,\
s1=matsize(f1)[1];s2=matsize(f2)[1];\
if(s2==0,f=f1,\
```

```
f=matrix(s1+s2-1,3,j,k,0);\
for(j=1,s2,f[j,]=f2[s2+1-j,]);\
for(j=1,s1-1,f[s2+j,]=f1[1+j,])));\
if(f==0,gf=0,gf=f*g*f~);
\\***
\l;\
main(n,h)=refl(n,h);e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

Program 11: refl0.12

```
\\main(n1,n2,n3,h) calculates for <n1>\oplus <-n2>\oplus <-n3>
\\(it is given by the matrix g)
\\the vectors v of the height \le h from P(\M)_{pr}
\\and calculates chains e and f and their Gram matrices ge, gf
refl(n1,n2,n3,h,m,m1,j,k,v1,y1,y2,y3,z,d,dd,h1,u,u1,w,w1)=\
g=[n1,0,0;0,-n2,0;0,0,-n3];\
m=2;v=matrix(m,3,j,k,0);v[1,]=[0,1,0];\
if(n2==n3,v[2,]=[0,-1,1],v[2,]=[0,0,1]);\
for(h1=1,h,\
fordiv(h1,y1,dd=gcd(2*lcm(lcm(n1,n2),n3),2*n1*y1);\
fordiv(dd,d,if(h1!=2*n1*y1^2/d,,\
for(z=0,floor(sqrt((w=n1*y1^2+d)/n2)+0.000001),\
if(type(2*n2*z/d)!=1||type(w1=(w-n2*z^2)/n3)!=1,,\
if(issquare(w1)!=1,,\
y2=-z;y3=-isqrt(w1);if(type(2*n3*y3/d)!=1||content([y1,y2,y3])!=1,,\
u=[y1,y2,y3];\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u~>=0,m1=m1+1,alpha=0)));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
))))))))));v;
\\**
This part is the same as in Program 10: refl0.1 between \\** and \\***
\\***
\l;\
main(n1,n2,n3,h)=\
refl(n1,n2,n3,h);e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

Program 12: refl0.13

```
\\main(n1,n2,n3,eps1,eps2,eps3,h) calculates for
\\<n1>\oplus <-n2>\oplus <-n3>(eps1/2,eps2/2,eps3/2)
\\(it is given by the matrix g) where ni\ge 0 and epsi=0 or 1 and
\\either all of them are 0 or at least two of them are not zero,
\\the vectors v of the height \le h from P(\M)_{pr}
\\and calculates chains e and f and their Gram matrices ge, gf
refl(n1,n2,n3,eps1,eps2,eps3,h,m,m1,j,k,v1,y1,y2,y3,y1t,y2t,y3t,\
z,d,dt,dd,h1,u,u1,w,w1)=\
g=[n1,0,0;0,-n2,0;0,0,-n3];eps=[eps1,eps2,eps3]/2;\
m=2;v=matrix(m,3,j,k,0);\
```

```
if(n2/n3!=1&&n2/n3!=3&&n2/n3!=1/3,v[1,]=[0,1,0];v[2,]=[0,0,1],\
if(n2==n3,if(eps==[0,0,0]||eps==[1/2,1/2,1/2],v[1,]=[0,1,0];v[2,]=[0,-1,1],\
if(eps==[0,1/2,1/2],v[1,]=[0,1,0];v[2,]=[0,-1/2,1/2],\
v[1,]=[0,1,0];v[2,]=[0,0,1])),\
if(n2/n3==3,if(eps==[0,1/2,1/2],v[1,]=[0,1,0];v[2,]=[0,-1/2,1/2],\
v[1,]=[0,1,0];v[2,]=[0,0,1]),\
if(n2/n3==1/3,if(eps==[0,1/2,1/2],v[1,]=[0,0,1];v[2,]=[0,1/2,-1/2],\
v[1,]=[0,1,0];v[2,]=[0,0,1]),))));\
for(h1=1,h,\
fordiv(h1,y1t,dd=gcd(2*lcm(lcm(n1,n2),n3),n1*y1t);\
fordiv(dd,d,if(d*h1!=n1*y1t^2,,dt=4*d;\
for(z=0,floor(sqrt((w=n1*y1t^2+dt)/n2)+0.000001),\
if(type(n2*z/d)!=1||type(w1=(w-n2*z^2)/n3)!=1,,\
if(issquare(w1)!=1,,\
y2t=-z;y3t=-isqrt(w1);if(type(n3*y3t/d)!=1,,\
y1=y1t/2;y2=y2t/2;y3=y3t/2;u=[y1,y2,y3];\
if(type(2*u*g*eps~/d)!=1,,\
if(mod(2*u,2)!=mod([0,0,0],2)&&mod(2*u,2)!=mod(2*eps,2),,\
if((mod(2*u,2)==mod([0,0,0],2)&&(content(u)>1||\
mod(2*u,4)==mod(4*eps,4)))||\
(eps!=[0,0,0]&&mod(2*u,2)==mod(2*eps,2)&&content(2*u)>1),,\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u~>=0,m1=m1+1,alpha=0));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
)))))))))))));v;
\\**
```

This part is the same as in Program 10: refl0.1 between ** and ***

```
\\***
\l;\
main(n1,n2,n3,eps1,eps2,eps3,h)=\
refl(n1,n2,n3,eps1,eps2,eps3,h);\
e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

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