

FLOWS FOR SINGULAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH UNBOUNDED DRIFTS

OLIVIER MENOUKEU-PAMEN

*African Institute for Mathematical Sciences, Ghana,
University of Ghana, Ghana
Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences,
University of Liverpool, L69 7ZL, United Kingdom*

SALAH E. A. MOHAMMED *

*formally at the Department of Mathematics, Southern Illinois University, Carbondale, Illinois
62901, USA*

ABSTRACT. In this paper, we are interested in the following singular stochastic differential equation (SDE)

$$dX_t = b(t, X_t)dt + dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}^d,$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel measurable, possibly unbounded and has spatial linear growth. The driving noise B_t is a d -dimensional Brownian motion. The main objective of the paper is to establish the existence and uniqueness of a strong solution and a Sobolev differentiable stochastic flow for the above SDE. Malliavin differentiability of the solution is also obtained (cf.[21, 24]). Our results constitute significant extensions to those in [33, 32, 14, 21, 24] by allowing the drift b to be unbounded. We employ methods from white-noise analysis and the Malliavin calculus. As application, we prove existence of a unique strong Malliavin differentiable solution to the following stochastic delay differential equation

$$dX(t) = b(X(t-r), X(t, 0, (v, \eta)))dt + dB(t), \quad t \geq 0, \quad (X(0), X_0) = (v, \eta) \in \mathbb{R}^d \times L^2([-r, 0], \mathbb{R}^d),$$

with the drift coefficient $b : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel-measurable function bounded in the first argument and has linear growth in the second argument.

1. INTRODUCTION

The purpose of this paper is twofold: First, it aims at studying existence, uniqueness and Malliavin regularity of solutions of stochastic differential equations with singular drift coefficients satisfying *linear growth condition*. Second, it studies existence of Sobolev differentiable flows for this class of SDEs. More specifically, we consider the following SDE

$$dX_t = b(t, X_t)dt + dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}^d, \quad (1.1)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable function and B_t is a d -dimensional Brownian motion on a probability space (Ω, \mathcal{F}, P) .

E-mail addresses: menoukeu@liv.ac.uk.

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When the drift coefficient b in (1.1) is globally Lipschitz continuous and of linear growth, it is known that the SDE (1.1) admits a unique strong solution. However, the Lipschitz continuity condition is not always satisfied in many interesting SDEs used in practice. Consequently, the study of SDEs with non-Lipschitz (singular) drift coefficients has received a lot of attention in recent years.

Assuming that the drift coefficient b is bounded and measurable, Zvonkin [33] proved existence of a unique strong solution of (1.1) in the one dimensional case. It should be noted that, under the above condition, the one-dimensional deterministic ordinary differential equation

$$dX_t = b(t, X_t)dt, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}^d$$

may not have a *unique* solution even when one exists. Zvonkin's result was generalised to the d -dimensional case by Veretenikov [32]. For other results in this direction, the reader may consult Gyöngy, Krylov [9] or Gyöngy, Martínez [10], Krylov, Röckner [14] and Portenko [28]. The proofs of the latter results are based either on estimates of solutions of parabolic partial differential equations, the Yamada-Watanabe principle, the Skorohod embedding or a technique due to Portenko [28]. See also [20], where the authors deal with pathwise uniqueness for some particular one-dimensional reflected SDEs.

Recently Meyer-Brandis and Proske [22] developed a new technique for the construction of strong solutions of SDEs with singular drift coefficients. This method is based on Malliavin calculus and white noise analysis. Assuming that the drift coefficient in (1.1) is bounded and measurable and satisfies a certain symmetry condition (see [22, Definition 3]), they showed the existence of a unique strong solution. The argument in [22] is not based on a pathwise uniqueness argument but it rather gives a direct construction of a strong solution of the SDE. Menoukeu-Pamen et al [21] further developed this approach and derived the results obtained in [22] by relaxing the symmetry condition on the drift coefficient. A striking fact when using this technique is that the strong solution of the SDE with a bounded and measurable drift coefficient is Malliavin differentiable.

The first objective of this paper is to study existence and uniqueness of the solution of the SDE (1.1) when the drift coefficient is Borel measurable and satisfies a linear growth condition. This problem was studied by Engelbert and Schmidt [7] in the one-dimensional time autonomous case. See also Nilssen [25] for the time dependent case. The proofs of the results in [7] use occupation time formula combined with the Yamada-Watanabe principle. This technique is a purely measure theoretical and the dependence on the initial parameter is not obvious whereas Nilssen in [25] uses the more direct method based on Malliavin calculus and the probability probabilistic estimates on the weak derivative with respect to the initial condition of the solution to the SDE (1.1). The main difference between the latter work and this paper is that there is an explicit representation of the Malliavin derivative (respectively the weak derivative with respect to the initial condition) in the one dimensional case, and using local time space integration, the $L^p(P)$, $p \geq 1$ estimates of the above processes are obtained. These representations are not valid in the multi dimensional case resulting to an infinite sum representation of the derivatives. The latter makes the moment estimates of the derivatives more challenging.

In this article, we extend the results obtained in [7, 25] to the multidimensional case assuming that the drift coefficient is time dependent and has spatial linear growth. This result constitute a notable extension to the unbounded case of existing results. We develop estimates on the Malliavin derivatives for solutions to the SDE (1.1) when the drift b is smooth and has linear growth. These estimates are obtained on a time interval that depends only on the linear growth rate of the drift b and is independent of the initial condition. A continuation argument (that is repeating the argument starting with the old final time as a new initial time) together with uniqueness of the solution is then used to show that the estimates still hold for any finite time horizon. See Lemma 4.1 and Proposition 4.3 (cf. [21, Lemma 3.5]). Using successive integration by parts, we are able to show that *the estimates are independent of the spatial derivative of the drift coefficient*.

Let us mention that the particular structure of the SDE (1.1) and the Brownian motion enables to study the notion of path by path uniqueness of solution introduced by Davie in [5]. This result was generalised in [3] to the case of fractional Brownian motion with Hurst parameter $H \in (0, 1)$. In particular, the authors show the path by path uniqueness of the solution when the drift belong

to the Besov-Sobolev space $B_{\infty,\infty}^{\alpha+1}$, $\alpha \in \mathbb{R}$. Under some conditions on α , they showed that the solution has a Lipschitz flow. It is also worth mentioning that the distributional drift case ($\alpha < 0$) was also studied in the case $H < \frac{1}{2}$.

A Malliavin calculus based method combined with local time variational calculus was used in [1] to study the SDE (1.1) when the driving noise is a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. Assuming that the drift $b \in L^1(\mathbb{R}^d, L^\infty([0, T], \mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d, L^\infty([0, T], \mathbb{R}^d))$, the authors prove that the unique solution has a higher order Fréchet differentiable stochastic flows for H small.

The second objective of the paper is to establish spatial Sobolev regularity of the strong solution of the SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x}) du + B_t - B_s, \quad s, t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d, \quad (1.2)$$

when the drift coefficient $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable function with spatial linear growth. More specifically, we obtain a two-parameter stochastic flow for the SDE (1.2) defined by

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \mapsto \phi_{s,t}(x) := X_t^{s,x} \in \mathbb{R}^d$$

which is pathwise Sobolev differentiable in the spatial variable x . Moreover, each flow map

$$\mathbb{R}^d \ni x \mapsto \phi_{s,t}(x) \in \mathbb{R}^d$$

is a Sobolev diffeomorphism in the sense that

$$\phi_{s,t}(x) \text{ and } \phi_{s,t}^{-1}(x) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, \mathbf{p}))$$

for all $s, t \in \mathbb{R}$ and all $p \in (0, \infty)$. The symbol $W^{1,p}(\mathbb{R}^d, \mathbf{p})$ stands for the weighted Sobolev space with the weight function \mathbf{p} having exponential of the second moment with respect to Lebesgue measure on \mathbb{R}^d . The above result is an extension to the case of unbounded drifts of the results in [24]; see also [15]

As application of our results, we study the following stochastic delay differential equation

$$dX(t) = b(X(t-r), X(t, 0, (v, \eta)))dt + dB(t), \quad t \geq 0, \quad (X(0), X_0) = (v, \eta) \in \mathbb{R}^d \times L^2([-r, 0], \mathbb{R}^d), \quad (1.3)$$

when the drift coefficient $b : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel-measurable function bounded in the first argument and has linear growth in the second argument. We prove that there exists a unique strong Malliavin differentiable solution to the above delay equation.

The paper is organized as follows: In Section 2, we recall some basic definitions and facts on Malliavin calculus, Gaussian white noise theory and stochastic flows. We also give some preliminary results in this section. In Section 3, we present the main results of the paper. Section 4 is devoted to the proof of the main results.

2. PRELIMINARY BACKGROUND

In this section we briefly review some facts from Gaussian white noise analysis, Malliavin calculus and stochastic dynamical systems. These facts will be used in establishing our results in the forthcoming sections. We refer the reader to [11, 27, 16] for more information on white noise theory. For the Malliavin calculus the reader may consult [26, 18, 6]. For stochastic flow theory, see [15].

2.1. Basic facts of Gaussian white noise theory. In this section, we recall the definition of the Hida distribution space; See T. Hida et al. [11]. This space plays a crucial role in our proof of the construction of a unique strong solution for the singular SDE (1.1).

Let us fix a time horizon $0 < T < \infty$. Consider a positive self-adjoint operator A on $L^2([0, T])$ with discrete eigenvalues greater than 1.

Denote by $\mathcal{S}([0, T])$ the standard nuclear countably Hilbertian subspace of $(L^2([0, T]), A)$ constructed in [27]. Denote by $\mathcal{S}'([0, T])$ its topological dual. Denote by $\mathcal{B}(\mathcal{S}'([0, T]))$ the Borel

σ -algebra of $\mathcal{S}'([0, T])$. It follows from the Bochner-Minlos theorem that there exists a unique probability measure π on $\mathcal{B}(\mathcal{S}'([0, T]))$ such that

$$\int_{\mathcal{S}'([0, T])} e^{i\langle \omega, \phi \rangle} \pi(d\omega) = e^{-\frac{1}{2} \|\phi\|_{L^2([0, T])}^2}$$

is valid for all $\phi \in \mathcal{S}([0, T])$, where $\langle \omega, \phi \rangle$ is the action of $\omega \in \mathcal{S}'([0, T])$ on $\phi \in \mathcal{S}([0, T])$. The d -dimensional white noise probability measure P is defined as the product measure

$$P = \bigotimes_{i=1}^d \pi \quad (2.1)$$

on the measurable space

$$(\Omega, \mathcal{F}) := \left(\prod_{i=1}^d \mathcal{S}'([0, T]), \bigotimes_{i=1}^d \mathcal{B}(\mathcal{S}'([0, T])) \right). \quad (2.2)$$

For $\omega = (\omega_1, \dots, \omega_d) \in (\mathcal{S}'([0, T]))^d$ and $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}([0, T]))^d$ let

$$\tilde{e}(\phi, \omega) = \exp \left(\langle \omega, \phi \rangle - \frac{1}{2} \|\phi\|_{L^2([0, T]; \mathbb{R}^d)}^2 \right),$$

be the stochastic exponential. Here $\langle \omega, \phi \rangle := \sum_{i=1}^d \langle \omega_i, \phi_i \rangle$.

In what follows, denote by $((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$ the n -th complete symmetric tensor product of $(\mathcal{S}([0, T]))^d$ with itself. Since $\tilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero, there exist generalized Hermite polynomials $H_n(\omega) \in \left(((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n} \right)'$ satisfying

$$\tilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \langle H_n(\omega), \phi^{\otimes n} \rangle \quad (2.3)$$

for ϕ in a neighbourhood of zero in $(\mathcal{S}([0, T]))^d$. One can prove that

$$\left\{ \langle H_n(\omega), \phi^{(n)} \rangle : \phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}, n \in \mathbb{N}_0 \right\} \quad (2.4)$$

is a total subset of $L^2(P)$. Moreover, for all $n, m \in \mathbb{N}_0$, $\phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$, $\psi^{(m)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} m}$, we have the following orthogonality relation

$$\int_{\mathcal{S}'} \langle H_n(\omega), \phi^{(n)} \rangle \langle H_m(\omega), \psi^{(m)} \rangle P(d\omega) = \delta_{n,m} n! \langle \phi^{(n)}, \psi^{(n)} \rangle_{L^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})}. \quad (2.5)$$

Let $\widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ be the space of square integrable symmetric functions $f : [0, t]^n \rightarrow (\mathbb{R}^d)^{\otimes n}$ with values in $(\mathbb{R}^d)^{\otimes n}$. It follows from (2.5) that we can uniquely extend the action $\langle H_n(\omega), \phi^{(n)} \rangle$ of $H_n \in \left(\left((\mathcal{S}([0, T])^d \right)^{\widehat{\otimes} n} \right)'$ on $\phi^{(n)} \in \left((\mathcal{S}([0, T])^d \right)^{\widehat{\otimes} n}$ to the action of $H_n \in \left(\left((\mathcal{S}([0, T])^d \right)^{\widehat{\otimes} n} \right)'$ on $\phi^{(n)} \in \widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ for ω a.e and for all n . Denote by I_n this extension, then $I_n(\phi^{(n)})$ can be viewed as n -fold iterated Itô integral of $\phi^{(n)} \in \widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a d -dimensional Wiener process

$$B_t = \left(B_t^{(1)}, \dots, B_t^{(d)} \right) \quad (2.6)$$

on the white noise probability space

$$(\Omega, \mathcal{F}, P). \quad (2.7)$$

Using (2.3) and (2.5), we deduce the Wiener-Itô decomposition of square integrable Brownian functional that is

$$L^2(P) = \bigoplus_{n \geq 0} I_n(\widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})). \quad (2.8)$$

The construction of the Hida stochastic test function and distribution space is based on the decomposition (2.8). To this end let

$$A^d := (A, \dots, A), \quad (2.9)$$

with A being the self-adjoint operator introduced earlier. It follows from a second quantization argument that the *Hida stochastic test function space* (\mathcal{S}) can be defined as the space of all $f = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle \in L^2(P)$ such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| ((A^d)^{\otimes n})^p \phi^{(n)} \right\|_{L^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})}^2 < \infty \quad (2.10)$$

for all $p \geq 0$. Endowed with the seminorm $\|\cdot\|_{0,p}$, $p \geq 0$, the space (\mathcal{S}) is a nuclear Fréchet algebra with respect to multiplication of functions. One sees from (2.3) that

$$\tilde{e}(\phi, \omega) \in (\mathcal{S}) \quad (2.11)$$

for all $\phi \in (\mathcal{S}([0, T]))^d$.

The *Hida stochastic distribution space* $(\mathcal{S})^*$ is the topological dual of (\mathcal{S}) . Hence we get the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(P) \hookrightarrow (\mathcal{S})^*.$$

By construction, the Hida distribution space $(\mathcal{S})^*$ contains the the time derivatives of the d -dimensional Wiener process B_t i.e.,

$$W_t^i := \frac{d}{dt} B_t^i \in (\mathcal{S})^*, \quad i = 1, \dots, d. \quad (2.12)$$

We will also need the definition of the *S-transform*, see [29]. Let $S(\Phi)$ be the S -transform of $\Phi \in (\mathcal{S})^*$ then $S(\Phi)$ is defined by the following dual pairing

$$S(\Phi)(\phi) := \langle \Phi, \tilde{e}(\phi, \omega) \rangle \quad (2.13)$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$ ($\mathcal{S}_{\mathbb{C}}([0, T])$ denotes the complexification of $\mathcal{S}([0, T])$.) The S -transform is an injective map from $(\mathcal{S})^*$ to \mathbb{C} i.e., if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*,$$

then

$$\Phi = \Psi.$$

Furthermore,

$$S(W_t^i)(\phi) = \phi^i(t), \quad i = 1, \dots, d \quad (2.14)$$

for $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$.

Finally, we recall the concept of the *Wick-Grassmann product*. Let $\Phi, \Psi \in (\mathcal{S})^*$ be two distribution then the Wick product $\Phi \diamond \Psi$ of Φ and Ψ is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi) S(\Psi)(\phi) \quad (2.15)$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$. As an example, we have

$$\langle H_n(\omega), \phi^{(n)} \rangle \diamond \langle H_m(\omega), \psi^{(m)} \rangle = \langle H_{n+m}(\omega), \phi^{(n)} \hat{\otimes} \psi^{(m)} \rangle \quad (2.16)$$

for $\phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\hat{\otimes} n}$ and $\psi^{(m)} \in ((\mathcal{S}([0, T]))^d)^{\hat{\otimes} m}$. Using (2.16) and (2.3), we get

$$\tilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \quad (2.17)$$

for $\phi \in (\mathcal{S}([0, T]))^d$. Here the Wick exponential $\exp^{\diamond}(X)$ of an element $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \quad (2.18)$$

where $X^{\diamond n} = X \diamond \dots \diamond X$, assuming that the sum on the right side of (2.18) converges in $(\mathcal{S})^*$.

2.2. The Malliavin Derivative. Here we recall the definitions of the Malliavin derivative within the context of white noise theory. Without loss of generality we assume that $d = 1$. Let $F \in L^2(P)$. Then using (2.8), there exists a unique sequence $\phi^{(n)} \in \hat{L}^2([0, T]^n)$, $n \geq 1$ such that

$$F = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle. \quad (2.19)$$

Suppose that

$$\sum_{n \geq 1} nn! \left\| \phi^{(n)} \right\|_{L^2([0, T]^n)}^2 < \infty \quad (2.20)$$

and denote by $D_t F$ the Malliavin derivative of F in the direction of the Brownian motion. Then

$$D_t F := \sum_{n \geq 1} n \left\langle H_{n-1}(\cdot), \phi^{(n)}(\cdot, t) \right\rangle. \quad (2.21)$$

Denote by $\mathbb{D}_{1,2}$ the family of all $F \in L^2(P)$ which are Malliavin differentiable. Define the norm $\|\cdot\|_{1,2}$ on $\mathbb{D}_{1,2}$ by

$$\|F\|_{1,2}^2 := \|F\|_{L^2(P)}^2 + \|D \cdot F\|_{L^2([0, T] \times \Omega, \lambda \times P)}^2, \quad (2.22)$$

where λ stands for the Lebesgue measure. Endowed with this norm, $\mathbb{D}_{1,2}$ is a Hilbert space and the following chain of continuous inclusions are satisfied:

$$(S) \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(P) \hookrightarrow \mathbb{D}_{-1,2} \hookrightarrow (S)^*, \quad (2.23)$$

where $\mathbb{D}_{-1,2}$ is the dual of $\mathbb{D}_{1,2}$.

2.3. Some basic facts on stochastic flows. In this section we state some basic facts needed to describe the Sobolev differentiable flow generated by the singular SDE (1.1).

Definition 2.1. *The P -preserving (ergodic) Wiener shift $\theta(t, \cdot) : \Omega \mapsto \Omega$ is defined by*

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \omega \in \Omega, t, s \in \mathbb{R}. \quad (2.24)$$

Note that the Brownian motion satisfies the following *perfect helix property*:

$$B_{t_1+t_2}(\omega) - B_{t_1}(\omega) = B_{t_2}(\theta(t_1, \omega)). \quad (2.25)$$

The above perfect helix property expresses, in a pathwise manner, the fact that the Brownian motion B has stationary ergodic increments.

Next, consider the SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u) du + B_t - B_s, \quad s, t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d, \quad (2.26)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable function satisfying a linear growth condition. The following definition describes the dynamics of the SDE (2.26):

Definition 2.2. *A stochastic flow of homeomorphisms for the SDE (2.26) is a map*

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d &\rightarrow \mathbb{R}^d \times \Omega \\ (s, t, x, \omega) &\mapsto \phi_{s,t}(x, \omega) \end{aligned}$$

with a universal set $\Omega^* \in \mathcal{F}$ of full Wiener measure such that, for all $\omega \in \Omega^*$, the following statements hold:

- (1) For any $x \in \mathbb{R}^d$, the process $\phi_{s,t}(x, \omega)$, $s, t \in \mathbb{R}$, is a global strong solution to the SDE (2.26).
- (2) $\phi_{s,t}(x, \omega)$ is continuous in $(s, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.
- (3) $\phi_{s,t}(\cdot, \omega) = \phi_{u,t}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega)$ for all $s, u, t \in \mathbb{R}$.
- (4) $\phi_{t,t}(x, \omega) = x$ for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$.
- (5) $\phi_{s,t}(\cdot, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are homeomorphisms for all $s, t \in \mathbb{R}$.

We next define a class of weighted Sobolev spaces. Let $\mathbf{p} : \mathbb{R}^d \rightarrow (0, \infty)$ be a Borel measurable function satisfying

$$\int_{\mathbb{R}^d} e^{|x|^2} \mathbf{p}(x) dx < \infty. \quad (2.27)$$

Denote by $L^p(\mathbb{R}^d, \mathbf{p})$ the Banach space of all Borel measurable functions $u = (u_1, \dots, u_d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} |u(x)|^p \mathbf{p}(x) dx < \infty \quad (2.28)$$

and equipped with the norm

$$\|u\|_{L^p(\mathbb{R}^d, \mathbf{p})} := \left[\int_{\mathbb{R}^d} |u(x)|^p \mathbf{p}(x) dx \right]^{\frac{1}{p}}. \quad (2.29)$$

Moreover, denote by $W^{1,p}(\mathbb{R}^d, \mathbf{p})$ the space of functions $u \in L^p(\mathbb{R}^d, \mathbf{p})$ with weak partial derivatives $\nabla_j u \in L^p(\mathbb{R}^d, \mathbf{p})$ for $j = 1, \dots, d$. For each $u \in W^{1,p}(\mathbb{R}^d, \mathbf{p})$, define its norm $\|u\|_{1,p,\mathbf{p}}$ as follows

$$\|u\|_{1,p,\mathbf{p}} := \|u\|_{L^p(\mathbb{R}^d, \mathbf{p})} + \sum_{i,j=1}^d \|\nabla_j u_i\|_{L^p(\mathbb{R}^d, \mathbf{p})}. \quad (2.30)$$

Equipped with the norm (2.30), the space $W^{1,p}(\mathbb{R}^d, \mathbf{p})$ is a Banach space.

Definition 2.3. A stochastic flow $\phi_{s,t}(\cdot, \omega)$ of homeomorphisms is said to be Sobolev-differentiable if for all $s, t \in \mathbb{R}$, the maps $\phi_{s,t}(\cdot, \omega)$ and $\phi_{s,t}^{-1}(\cdot, \omega)$ belong to $W^{1,p}(\mathbb{R}^d, \mathbf{p})$.

Definition 2.4. Let $\theta(t, \cdot) : \Omega \rightarrow \Omega$ be a P -preserving Wiener shift for each $t \in \mathbb{R}$. A stochastic flow is a perfect Sobolev-differentiable cocycle $\phi_{0,t}(\cdot, \theta(t, \cdot))$ if it satisfies the following property

$$\phi_{0,t_1+t_2}(\cdot, \omega) = \phi_{0,t_2}(\cdot, \theta(t_1, \omega)) \circ \phi_{0,t_1}(\cdot, \omega) \quad (2.31)$$

for all $\omega \in \Omega$ and $t_1, t_2 \in \mathbb{R}$.

In the next section, we recall some preliminary facts results which will be useful in the proof of our main results.

2.4. Preliminary Facts. Our method of constructing the strong solution for the singular SDE employs the following result which is an extension of the result in [17].

Proposition 2.5. Let $T > 0$ be sufficiently small. Assume that the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ in (1.1) is Lipschitz continuous and of linear growth. Then the unique strong solution $X_t = (X_t^1, \dots, X_t^d)$ of (1.1) has the following representation

$$\varphi(t, X_t^i(\omega)) = E_{\tilde{P}} \left[\varphi(t, \tilde{B}_t^i(\tilde{\omega})) \mathcal{E}_T^\diamond(b) \right] \quad (2.32)$$

for all $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(t, B_t^i) \in L^2(P)$ for all $0 \leq t \leq T$, $i = 1, \dots, d$. The symbol $\mathcal{E}_T^\diamond(b)$ stands for

$$\begin{aligned} \mathcal{E}_T^\diamond(b)(\omega, \tilde{\omega}) := & \exp^\diamond \left(\sum_{j=1}^d \int_0^T \left(W_s^j(\omega) + b^j(s, \tilde{B}_s(\tilde{\omega})) \right) d\tilde{B}_s^j(\tilde{\omega}) \right. \\ & \left. - \frac{1}{2} \int_0^T \left(W_s^j(\omega) + b^j(s, \tilde{B}_s(\tilde{\omega})) \right)^{\diamond 2} ds \right). \end{aligned} \quad (2.33)$$

$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), (\tilde{B}_t)_{t \geq 0}$ is a copy of the quadruple $(\Omega, \mathcal{F}, P), (B_t)_{t \geq 0}$ in (3.1). Moreover $E_{\tilde{P}}$ denotes the Pettis integral of random elements $\Phi : \tilde{\Omega} \rightarrow (\mathcal{S})^*$ with respect to the measure \tilde{P} . The Wick product \diamond in the Wick exponential of (2.33) is taken with respect to P and W_t^j is the white noise derivative of B_t^j in the Hida space $(\mathcal{S})^*$. The stochastic integrals $\int_0^T \phi(t, \tilde{\omega}) d\tilde{B}_s^j(\tilde{\omega})$ in (2.33) are defined for predictable integrands ϕ with values in the conuclear space $(\mathcal{S})^*$, and the second integral in (2.33) is in the sense of Pettis.

Proof. It follows as in [17] using Beněs theorem (see Lemma A.4). \square

Remark 2.6. *It is worth mentioning one needs to construct the stochastic integral in $(\mathcal{S})^*$ with respect to the Brownian motion \tilde{B}_t on the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, $(\tilde{\mathcal{F}}_t)_{t \geq 0}$, where $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ is the \tilde{P} -completed filtration generated by the \tilde{B}_t . The existence of \tilde{P} comes from the Bochner-Minlo's Theorem. We refer the reader to [12] for more on stochastic integral in conuclear space.*

In the following, we denote by $Y_t^{i,b}$ the expectation on the right side of (2.32) for $\varphi(t, x) = x$, that is

$$Y_t^{i,b} := E_{\tilde{P}} \left[\tilde{B}_t^{(i)} \mathcal{E}_T^\circ(b) \right]$$

for $i = 1, \dots, d$. We set

$$Y_t^b := (Y_t^{1,b}, \dots, Y_t^{d,b}). \quad (2.34)$$

The next result gives an exponential estimate of the square of solution to the SDE (1.1) and plays a vital part in the proof of our main results. Its proof uses Fernique's Theorem.

Lemma 2.7. *Let $t_0 \in [0, 1]$ and $Y : \Omega \rightarrow \mathbb{R}^d$ be a \mathcal{F}_{t_0} -measurable random variable independent of the P -augmented filtration generated by the Brownian motion B_t . Let $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth coefficient with compact support satisfying a global linear growth condition; that is*

$$k := \text{ess sup} \left\{ \frac{|b(t, z)|}{1 + |z|} : t \in [0, T], z \in \mathbb{R}^d \right\} < \infty.$$

Denote by $X_t^{t_0, Y}$ the unique strong solution (if it exists) to the SDE (1.1) starting at Y and with drift coefficient b . Then there exists a positive number δ_0 independent of t_0 and Y (but may depend on k) such that

$$E \exp\{\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, Y}|^2\} \leq C_1 E \exp\{C_2 \delta_0 |Y|^2\}, \quad (2.35)$$

where C_1, C_2 are positive constants independent of Y , but may depend on k . In addition, C_1 may depend on δ_0 and d . This expectation is finite provided that the right side of (2.35) is finite.

Proof. Start with the almost sure (a.s.) relation

$$X_t^{t_0, Y} = Y + \int_{t_0}^t b(u, X_u^{t_0, Y}) du + B_t - B_{t_0}, \quad t_0 \leq t \leq 1. \quad (2.36)$$

Using the above equation, we get the following a.s. estimates using Hölder's inequality

$$\begin{aligned} |X_t^{t_0, Y}|^2 &\leq 3|Y|^2 + 3 \left| \int_{t_0}^t b(u, X_u^{t_0, Y}) du \right|^2 + 3|B_t - B_{t_0}|^2 \\ &\leq 3|Y|^2 + 3 \left(\int_{t_0}^t k(1 + |X_u^{t_0, Y}|) du \right)^2 + 3|B_t - B_{t_0}|^2 \\ &\leq 3|Y|^2 + 3 \left(\int_{t_0}^t k(1 + |X_u^{t_0, Y}|) du \right)^2 + 3|B_t - B_{t_0}|^2 \\ &\leq 3|Y|^2 + 6k^2(t - t_0)^2 \int_{t_0}^t \{1 + |X_u^{t_0, Y}|^2\} du + 3|B_t - B_{t_0}|^2 \\ &\leq 3|Y|^2 + 6k^2(t - t_0)^3 + 6k^2(t - t_0)^2 \int_{t_0}^t |X_u^{t_0, Y}|^2 du + 3|B_t - B_{t_0}|^2. \end{aligned} \quad (2.37)$$

Taking the supremum on both sides and multiplying by δ_0 , we have

$$\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, Y}|^2 \leq 3\delta_0 |Y|^2 + 6k^2\delta_0 + 3\delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 + 6k^2 \int_{t_0}^1 \delta_0 \sup_{0 \leq u \leq s} |X_u^{t_0, Y}|^2 ds, \quad \text{a.s.} \quad (2.38)$$

Applying Gronwall's lemma to the last inequality, we get

$$\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, Y}|^2 \leq \left[3\delta_0 |Y|^2 + 6k^2 \delta_0 + 3\delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 \right] e^{6k^2}, \text{ a.s.} \quad (2.39)$$

Denote $C_2 := 3e^{6k^2}$. Then from the above inequality, we get

$$\exp\{\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, Y}|^2\} \leq \exp\{2C_2 \delta_0 k^2\} \cdot \exp\{\delta_0 C_2 |Y|^2\} \cdot \exp\{C_2 \delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2\}, \text{ a.s.} \quad (2.40)$$

Taking expectations in the above inequality and using the fact that Y and $|B_u - B_{t_0}|$, $u \geq t_0$, are independent, we obtain

$$E \exp\{\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, Y}|^2\} \leq \exp\{2C_2 k^2 \delta_0\} \cdot E \exp\{C_2 \delta_0 |Y|^2\} \cdot E \exp\{C_2 \delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2\}. \quad (2.41)$$

The result follows if we can find δ_0 independent of Y and t_0 such that

$$E \exp\{C_2 \delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2\} < \infty. \quad (2.42)$$

The above inequality follows from Fernique's Theorem. For the sake of completeness we provide the details.

We use the exponential series expansion of the left side followed by Doob's maximal inequality to obtain the following estimates:

$$\begin{aligned} E \exp\{C_2 \delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2\} &\leq 1 + \sum_{n=1}^{\infty} \frac{C_2^n \delta_0^n}{n!} E \left[\sup_{t_0 \leq u \leq t_0 + (1-t_0)} |B_u - B_{t_0}|^{2n} \right] \\ &= 1 + \sum_{n=1}^{\infty} \frac{C_2^n d^n \delta_0^n}{n!} \left(\frac{2n}{2n-1} \right)^{2n} \frac{(2n)!}{2^n n!} (1-t_0)^n \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{C_2^n d^n \delta_0^n}{n!} \left(\frac{2n}{2n-1} \right)^{2n} \frac{(2n)!}{2^n n!}. \end{aligned}$$

Apply the ratio test to the above series $\sum_{n=1}^{\infty} a_n$, where $a_n := \frac{C_2^n d^n \delta_0^n}{n!} \left(\frac{2n}{2n-1} \right)^{2n} \frac{(2n)!}{2^n n!}$, $n \geq 1$ to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{C_2^{n+1} d^{n+1} \delta_0^{n+1} [2(n+1)]!}{(n+1)! 2^{(n+1)} (n+1)!} \frac{n! 2^n n!}{C_2^n d^n \delta_0^n (2n)!} \left(\frac{2(n+1)}{2(n+1)-1} \right)^{2(n+1)} \left(\frac{2n}{2n-1} \right)^{-2n} \\ &= \lim_{n \rightarrow \infty} \frac{C_2^{n+1} d^{n+1} \delta_0^{n+1} [2(n+1)]!}{(n+1)! 2^{(n+1)} (n+1)!} \frac{n! 2^n n!}{C_2^n d^n \delta_0^n (2n)!} \\ &= \lim_{n \rightarrow \infty} \frac{C_2 d \delta_0}{2} \frac{(2n+2)(2n+1)}{(n+1)^2} \\ &= 2dC_2 \delta_0. \end{aligned} \quad (2.43)$$

By the ration test, it follows that the series $\sum_{n=1}^{\infty} a_n$ converges for $\delta_0 < \frac{1}{2dC_2}$ (e.g. for $\delta_0 := \frac{1}{12dC_2}$).

With this choice of δ_0 , take

$$C_1 := \exp\{2C_2 k^2 \delta_0\} \cdot E \exp\{C_2 \delta_0 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2\} < \infty. \quad (2.44)$$

Therefore (2.41) gives

$$E \exp\{\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, Y}|^2\} \leq C_1 E e^{C_2 \delta_0 |Y|^2}. \quad (2.45)$$

Note that C_1, C_2 and δ_0 are independent of Y and t_0 (but may depend on k and d). Thus the claim (2.35) holds for the above choice of δ_0 . \square

Remark 2.8. Note that for Y deterministic, the above expectation is finite. If Y is for example independent drifted Brownian motion, the above expression is also finite given the choice of δ_0 .

3. STATEMENTS OF THE MAIN RESULTS

3.1. Existence and uniqueness of the strong solution. Let B_t be a d -dimensional Brownian motion with respect to the stochastic basis

$$(\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{0 \leq t \leq T} \quad (3.1)$$

with $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ the P -augmented filtration generated by B_t . In this section, we extend the results in [22] to cover singular drifts with linear growth. In particular, we establish the existence and uniqueness of a strong solution of the singular SDE

$$dX_t = b(t, X_t)dt + dB_t, \quad 0 \leq t \leq 1, \quad X_0 = x \in \mathbb{R}^d, \quad (3.2)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable function which has linear growth; that is

$$k := \text{ess sup} \left\{ \frac{|b(t, z)|}{1 + |z|} : t \in [0, T], z \in \mathbb{R}^d \right\} < \infty. \quad (3.3)$$

One of the main results of this paper is the following:

Theorem 3.1. Suppose that the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ in the SDE (3.2) is a Borel-measurable function such that $k := \text{ess sup} \left\{ \frac{|b(t, z)|}{1 + |z|} : t \in [0, T], z \in \mathbb{R}^d \right\} < \infty$. Then there exists a unique global strong solution X to the SDE (3.2) adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. Furthermore, the solution X_t is Malliavin differentiable for all $0 \leq t \leq T$.

Remark 3.2. In the one dimensional autonomous case, existence and uniqueness for the solution of (3.2) was first obtain by Engelbert and Schmidt [7]. existence, uniqueness and Malliavin differentiability of the solution in the one dimensional case was obtained by Nilssen in [25] for small time intervals. Thus Theorem 3.1 extends the above results to the multidimensional case on any time horizon.

Theorem 3.1 can further be generalized to cover a class of non-degenerate d -dimensional Itô-diffusions as follows:

Theorem 3.3. Consider the autonomous \mathbb{R}^d -valued SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x \in \mathbb{R}^d, \quad 0 \leq t \leq T, \quad (3.4)$$

where the coefficients $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Suppose that there exists a bijection $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$, which is twice continuously differentiable and satisfies the following requirements. Let $\Lambda_x : \mathbb{R}^d \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ and $\Lambda_{xx} : \mathbb{R}^d \rightarrow L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ be the corresponding derivatives of Λ and assume that

$$\Lambda_x(y)\sigma(y) = id_{\mathbb{R}^d} \text{ for } y \text{ a.e.}$$

and

$$\Lambda^{-1} \text{ is Lipschitz continuous.}$$

Suppose that the function $b_* : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$\begin{aligned} b_*(x) &:= \Lambda_x(\Lambda^{-1}(x)) [b(\Lambda^{-1}(x))] \\ &\quad + \frac{1}{2} \Lambda_{xx}(\Lambda^{-1}(x)) \left[\sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i] \right] \end{aligned}$$

satisfies the conditions of Theorem 3.1, where $e_i, i = 1, \dots, d$, is a basis of \mathbb{R}^d . Then the SDE (3.4) has a stochastic flow $\phi_{s,t} \in L^2(\Omega, W^{1,p})$ for all $p > 1$. Furthermore, each map $\phi_{s,t} : \Omega \rightarrow W^{1,p}$ is Malliavin differentiable.

Proof. The proof can be directly obtained from Itô's Lemma. See [22]. \square

3.2. Existence of a Sobolev differentiable stochastic flow. In this section, we aim at showing existence of a Sobolev differentiable stochastic flow for the following d -dimensional SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u) du + B_t - B_s, \quad s, t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d \quad (3.5)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measurable function satisfying $\|\tilde{b}\|_\infty < \infty$, where $\tilde{b}(t, z) := \frac{b(t, z)}{1 + |z|}$, $t \in [0, T]$, $z \in \mathbb{R}^d$.

By Theorem 3.1, the SDE (3.5) has a unique strong solution which we will denote by $X^{s,x}$. The following theorem gives the existence of a Sobolev differentiable flow for the SDE (3.5), and is the main result of this section.

Theorem 3.4. *Assume that the drift coefficient b in the SDE (3.5) is Borel-measurable and has linear growth. Then the SDE (3.5) has a Sobolev differentiable stochastic flow $\phi_{t,s} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $s, t \in \mathbb{R}$; that is*

- (1) $\phi_{t,s}(\cdot)$ and $\phi_{t,s}^{-1}(\cdot) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d; \mathbf{p}))$ for all $s, t \in \mathbb{R}$ and $p \in (1, \infty)$;
- (2) $\phi_{t,u}(\cdot, \omega) = \phi_{t,s}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega)$ for all $t \leq s \leq u$ a.s.;
- (3) $\phi_{t,t}(\cdot, \omega) = id_{\mathbb{R}^d}$ for all $t \in \mathbb{R}$ a.s.

We also have the following cocycle property in the autonomous case.

Corollary 3.5. *Consider the autonomous SDE*

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x}) du + B_t - B_s, \quad s, t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (3.6)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel measurable and has linear growth. Then the stochastic flow of the SDE (3.6) has a version which generates a perfect Sobolev-differentiable cocycle $(\phi_{0,t}, \theta(t, \cdot))$ where $\theta(t, \cdot) : \Omega \rightarrow \Omega$ is the P -preserving Wiener shift. More specifically, the following perfect cocycle property holds for all $\omega \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$

$$\phi_{0,t_1+t_2}(\cdot, \omega) = \phi_{0,t_2}(\cdot, \theta(t_1, \omega)) \circ \phi_{0,t_1}(\cdot, \omega). \quad (3.7)$$

Remark 3.6. *Similar results were proved in [24], assuming that the drift coefficient b is measurable and globally bounded. Considering unbounded drift coefficients, Nilssen in [25] proves similar result in one dimension and only for small time interval. Note however that the technique used in [25], which is based on the Itô's-Tanaka formula, cannot be applied here since the local time for multidimensional Brownian is not defined.*

Theorem 3.7. *Consider the autonomous d -dimensional SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x \in \mathbb{R}^d, \quad 0 \leq t \leq T, \quad (3.8)$$

where the coefficients $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Assume that $\sigma(x)$ has an inverse $\sigma^{-1}(x)$ for all $x \in \mathbb{R}^d$. Moreover, suppose that $\sigma^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ is in $C^1(\mathbb{R}^d)$ and

$$\frac{\partial}{\partial x_k} \sigma_{lj}^{-1} = \frac{\partial}{\partial x_j} \sigma_{lk}^{-1}$$

for all $l, k, j = 1, \dots, d$. Furthermore, suppose that the function $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

$$\Lambda(x) := \int_0^1 \sigma^{-1}(tx) \cdot x dt$$

has a Lipschitz continuous inverse $\Lambda^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Suppose Λ is C^2 with derivatives $D\Lambda : \mathbb{R}^d \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ and $D^2\Lambda : \mathbb{R}^d \rightarrow L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$. Suppose that the function $b_* : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$\begin{aligned} b_*(x) &:= D\Lambda(\Lambda^{-1}(x)) [b(\Lambda^{-1}(x))] \\ &+ \frac{1}{2} D^2\Lambda(\Lambda^{-1}(x)) \left[\sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i] \right] \end{aligned}$$

satisfies the conditions of Theorem 3.4, where $e_i, i = 1, \dots, d$, is a basis of \mathbb{R}^d . Then there exists a stochastic flow $(s, t, x) \mapsto \phi_{s,t}(x)$ of the SDE (3.8) such that

$$\phi_{s,t}(x) \in L^2(\Omega, W^p(\mathbb{R}^d, \mathbf{p}))$$

for all $0 \leq s \leq t \leq 1$ and for all $p > 1$.

Proof. The proof can be directly obtained from Itô's Lemma. See [24]. \square

We also have the following proposition which is essential for the proof of Theorem 3.4.

Proposition 3.8. *Let $b : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable and has linear growth. Let U be an open a bounded subset of \mathbb{R}^d . Then for each $t \in \mathbb{R}$ and $p > 1$ we have*

$$X_t \in L^2(\Omega; W^{1,p}(U)).$$

4. PROOFS OF THE MAIN RESULTS

4.1. Proof of existence and uniqueness of the strong solution. We first prove Theorem 3.1 on a small time interval $[0, t_1]$. Since t_1 is necessarily independent of the initial point, the result will then follow by a continuation argument on successive intervals of length t_1 (using random non-anticipative initial conditions and Lemma 2.7).

The proof of Theorem 3.1 is performed in two steps. In first step, we consider a sequence $b_n : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, n \geq 1$ of smooth coefficients with compact support satisfying a global linear growth condition and converging a.e. to b . Using the relative compactness criteria (Corollary A.3), we prove that for each $0 \leq t \leq t_1$ the sequence of corresponding strong solutions $X_t^n = Y_t^{b_n}, n \geq 1$, of the SDEs

$$dX_t^n = b_n(t, X_t^n)dt + dB_t, \quad 0 \leq t \leq t_1, \quad X_0^n = x \in \mathbb{R}^d, \quad n \geq 1, \quad (4.1)$$

is relatively compact in $L^2(P; \mathbb{R}^d)$.

In the second step of the argument, we show that for a measurable drift coefficient b satisfying linear growth condition, the solution $Y_t^b, 0 \leq t \leq t_1$, of (3.2) is a generalized process in the Hida distribution space. Then using the S -transform (2.13), we show that for a sequence $\{b_n\}_{n=1}^\infty$ of a.e. compactly supported smooth coefficients approximating b and satisfying a uniform global linear growth condition, there exists a convergent subsequence of the corresponding strong solutions $X_{n_j, t} = Y_t^{b_{n_j}}$ satisfying

$$Y_t^{b_{n_j}} \rightarrow Y_t^b \text{ in } L^2(P, \mathbb{R}^d)$$

for $0 \leq t \leq t_1$. We then check that the limiting process Y_t^b is a strong solution of the SDE (3.2), using a transformation property for Y_t^b .

The following lemma is an essential part of the first step of our procedure.

Lemma 4.1. *In the SDE (3.2), let $b : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth function with compact support. Then the corresponding strong solution X of (3.2) satisfies*

$$E[|D_t X_s - D_{t'} X_s|^2] \leq C_d \left(\text{ess sup}_{(u,z) \in [0, t_1] \times \mathbb{R}^d} \frac{|b(u, z)|}{1 + |z|} \right) |t - t'|^\alpha$$

for $0 \leq t' \leq t \leq t_1, \alpha = \alpha(s) > 0$ with t_1 small and

$$\sup_{0 \leq t \leq t_1} E[|D_t X_s|^2] \leq C_d \left(\text{ess sup}_{(u,z) \in [0, t_1] \times \mathbb{R}^d} \frac{|b(u, z)|}{1 + |z|} \right),$$

where $C_{d, t_1} : [0, \infty) \rightarrow [0, \infty)$ is an increasing, continuous function, $|\cdot|$ a matrix-norm on $\mathbb{R}^{d \times d}$.

Combination of Lemma 4.1 and Corollary A.3 yields the following result

Corollary 4.2. *Let $b_n : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, n \geq 1$, be a sequence of smooth coefficients with compact support and with $\sup_{n \geq 1} \text{ess sup}_{(u,z) \in [0, t_1] \times \mathbb{R}^d} \frac{|b_n(u, z)|}{1 + |z|} < \infty$. Then for each $0 \leq t \leq t_1$ the sequence of corresponding strong solutions $X_{n, t} = Y_t^{b_n}, n \geq 1$, is relatively compact in $L^2(P; \mathbb{R}^d)$.*

The following crucial estimate which generalises those in [5, 21] and plays an important role in deriving our results:

Proposition 4.3. *Let B be a d -dimensional Brownian motion starting from the origin and b_1, \dots, b_n be compactly supported continuously differentiable functions $b_i : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$. Let $\alpha_i \in \{0, 1\}^d$ be a multiindex such that $|\alpha_i| = 1$ for $i = 1, 2, \dots, n$. Then there exists a universal constant C (independent of $\{b_i\}$, n , and $\{\alpha_i\}$) such that*

$$\left| E \left[\int_{t_0 < t_1 < \dots < t_n < t} \left(\prod_{i=1}^n D^{\alpha_i} b_i(t_i, B(t_i)) \right) dt_1 \dots dt_n \right] \right| \leq \frac{C^n \prod_{i=1}^n \|\tilde{b}_i\|_\infty (t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \quad (4.2)$$

where Γ is the Gamma-function and $\tilde{b}(t, z) := \frac{b(t, z)}{1 + |z|}$, $(t, z) \in [0, 1] \times \mathbb{R}^d$ satisfies $\|\tilde{b}(t, z)\| \leq 1$. Here D^{α_i} denotes the partial derivative with respect to the j th space variable, where j is the position of the 1 in α_i .

Proof. See Appendix B □

We are now ready to give the proof of Lemma 4.1.

Proof of Lemma 4.1. The chain-rule for the Malliavin derivatives (see [26, Proposition 1.2.2] and [26, Theorem 2.2.1]) yields

$$D_t X_s = \mathcal{I}_d + \int_t^s b'(u, X_u) D_t X_u du, \quad (4.3)$$

P -a.e., for all $t \leq s \leq t_1$. Here \mathcal{I}_d is the $d \times d$ identity matrix and $b' = \left(\frac{\partial}{\partial x_i} b^{(j)}(t, x) \right)_{1 \leq i, j \leq d}$ is the spatial Jacobian derivative of b . Then, we get for $t_1 \geq s \geq t \geq t'$,

$$\begin{aligned} D_{t'} X_s - D_t X_s &= \int_{t'}^s b'(u, X_u) D_{t'} X_u du - \int_t^s b'(u, X_u) D_t X_u du \\ &= \int_{t'}^t b'(u, X_u) D_{t'} X_u du + \int_t^s b'(u, X_u) (D_{t'} X_u - D_t X_u) du \\ &= D_{t'} X_t - \mathcal{I}_d + \int_t^s b'(u, X_u) (D_{t'} X_u - D_t X_u) du. \end{aligned} \quad (4.4)$$

Iterating (4.4) leads to

$$\begin{aligned} D_{t'} X_s - D_t X_s &= \left(\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, X_{s_1}) : \dots : b'(s_n, X_{s_n}) ds_1 \dots ds_n \right) (D_{t'} X_t - \mathcal{I}_d) \end{aligned} \quad (4.5)$$

in $L^2(P)$, where “ $:$ ” denotes matrix multiplication. On the other hand we also have that

$$D_{t'} X_t - \mathcal{I}_d = \sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, X_{s_1}) : \dots : b'(s_n, X_{s_n}) ds_1 \dots ds_n. \quad (4.6)$$

Let $|\cdot|$ be the maximum norm on $\mathbb{R}^{d \times d}$. Applying Girsanov’s theorem, Hölder’s inequality and the Benčs Theorem (see Lemma A.4) in relation to (4.5) and (4.6), we get

$$\begin{aligned}
& E\left[|D_{t'}X_s - D_tX_s|^2\right] \\
&= E\left[\left(\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n\right)\right. \\
&\quad \times \left.\left(\sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n\right)\right]^2 \\
&\quad \times \exp\left(\sum_{j=1}^d \int_0^{t_1} b^{(j)}(u, B_u) dB_u^{(j)} - \sum_{j=1}^d \int_0^{t_1} (b^{(j)}(u, B_u))^2 du + \frac{1}{2} \sum_{j=1}^d \int_0^{t_1} (b^{(j)}(u, B_u))^2 du\right) \\
&\leq E\left[\exp\left(2 \sum_{j=1}^d \int_0^{t_1} (b^{(j)}(u, B_u))^2 du\right)\right]^{\frac{1}{4}} \times E\left[\exp\left(2 \sum_{j=1}^d \int_0^{t_1} b^{(j)}(u, B_u) dB_u^{(j)} - 2 \sum_{j=1}^d \int_0^{t_1} (b^{(j)}(u, B_u))^2 du\right)\right]^{\frac{1}{2}} \\
&\quad \times \left\|\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n\right\|_{L^{16}(P; \mathbb{R}^{d \times d})}^2 \\
&\quad \times \left\|\sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n\right\|_{L^{16}(P; \mathbb{R}^{d \times d})}^2. \tag{4.7}
\end{aligned}$$

The term $E\left[\exp\left(2 \sum_{j=1}^d \int_0^{t_1} b^{(j)}(u, B_u) dB_u^{(j)} - 2 \sum_{j=1}^d \int_0^{t_1} (b^{(j)}(u, B_u))^2 du\right)\right]$ is equal to one, by Benčs Theorem (see Lemma A.4) applied to the martingale $2 \sum_{j=1}^d \int_0^{t_1} b^{(j)}(u, B_u) dB_u^{(j)}$. In addition, the term $E\left[\exp\left(2 \sum_{j=1}^d \int_0^{t_1} (b^{(j)}(u, B_u))^2 du\right)\right]$ is also finite for small t_1 (see for example proof of Lemma 4.4). Hence, there is a positive constant C_{t_1} such that for all $t' \leq t \leq s \leq t_1$, we have

$$\begin{aligned}
& E\left[|D_{t'}X_s - D_tX_s|^2\right] \\
&\leq C_{t_1} \left\|\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n\right\|_{L^{16}(P; \mathbb{R}^{d \times d})}^2 \\
&\quad \times \left\|\sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n\right\|_{L^{16}(P; \mathbb{R}^{d \times d})}^2 \\
&\leq C_{t_1} \left(1 + \sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\|\int_{t < s_1 < \dots < s_n < s} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \right.\right. \\
&\quad \left.\dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n\right\|_{L^{16}(P; \mathbb{R})}^2 \\
&\quad \times \left(\sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\|\int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \right.\right. \\
&\quad \left.\dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n\right\|_{L^{16}(P; \mathbb{R})}^2 \Big). \tag{4.8}
\end{aligned}$$

Consider the following the expression

$$A := \int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \frac{\partial}{\partial x_{l_n}} b^{(l_n)}(s_n, B_{s_n}) ds_1 \dots ds_n. \tag{4.9}$$

It follows from repeated use of (deterministic) integration by parts that A^2 is equal to a sum of at most 2^{2n} summands of the form

$$\int_{t' < s_1 < \dots < s_{2n} < t} g_1(s_1) \dots g_{2n}(s_{2n}) ds_1 \dots ds_{2n}, \tag{4.10}$$

where $g_l \in \left\{ \frac{\partial}{\partial x_j} b^{(i)}(\cdot, B) : 1 \leq i, j \leq d \right\}$, $l = 1, 2, \dots, 2n$. Since $A^4 = A^2 A^2$, then by a similar argument there are *at most* 2^{8n} such summands (of length $4n$). Applying the same concept to A^{16} , one gets that it can be represented as a sum of at most 2^{128n} terms of the form (4.10) with length $16n$.

Using this fact and Proposition 4.3 we obtain

$$\begin{aligned} & \left\| \int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^{16}(\mu; \mathbb{R})} \\ & \leq \left(\frac{2^{128n} C^{16n} \|\tilde{b}\|_\infty^{16n} |t - t'|^{8n}}{\Gamma(8n + 1)} \right)^{1/16} \\ & \leq \frac{2^{8n} C^n \|\tilde{b}\|_\infty^n |t - t'|^{n/2}}{(8n!)^{1/16}}. \end{aligned} \quad (4.11)$$

Combining (4.8) and (4.11) we get

$$\begin{aligned} E[|D_t X_s - D_{t'} X_s|^2] & \leq C_{t_1} \left(1 + \sum_{n=1}^{\infty} \frac{d^{n+2} 2^{8n} C^n \|\tilde{b}\|_\infty^n |t - s|^{n/2}}{(8n!)^{1/16}} \right)^2 \\ & \quad \times \left(\sum_{n=1}^{\infty} \frac{d^{n+2} 2^{8n} C^n \|\tilde{b}\|_\infty^n |t - t'|^{(n-1)/2}}{(4n!)^{1/16}} \right)^2 |t - t'| \\ & \leq C_{d, t_1} (\|\tilde{b}\|_\infty) |t - t'| \end{aligned}$$

for $t' \leq t \leq s \leq t_1$ and a function C_{d, t_1} as claimed in the theorem.

Similarly, we get the estimate for $\sup_{0 \leq t \leq s} E[|D_t X_s|^2]$. This completes the proof of Lemma 4.1. \square

The next step based on White-noise analysis will be used to identify the process Y^b defined by (2.34) as the Malliavin differentiable solution to (3.2). Assume that b is Borel measurable function satisfying the linear growth condition. We first show that Y_t^b is a well-defined element in the Hida distribution space $(\mathcal{S})^*$, $0 \leq t \leq t_1$. Then we show that for a.e. approximating sequence of smooth coefficients b_n with compact support and satisfying a uniform linear growth condition, a subsequence of the corresponding strong solutions $X_{n_j, t} = Y_t^{b_{n_j}}$, satisfies $Y_t^{b_{n_j}} \rightarrow Y_t^b$ in $L^2(P)$ for $0 \leq t \leq 1$. Finally, we identify Y_t^b as a strong solution to (3.2) by applying a transformation property for Y_t^b .

Lemma 4.4. *Assume that the sequence $\{b_n\}_{n=1}^\infty$ satisfies conditions of Corollary 4.2. Then the expectation $E\left[\exp\left(512 \int_0^{t_1} |b_n(u, B_u)|^2 du\right)\right]$ is finite for a small time t_1 independent of n .*

Proof.

$$\begin{aligned} E\left[\exp\left(512 \int_0^T |b_n(u, B_u)|^2 du\right)\right] & \leq E\left[\exp\left(512 \int_0^T k^2(1 + |B_u|^2) du\right)\right] \\ & \leq \exp\left(512k^2T\right) \times E\left[\exp\left(512k^2T \max_{0 \leq u \leq T} |B_u|^2\right)\right]. \end{aligned}$$

The process $|B_u|^2$ is a non negative submartingale. Using the Doob's martingale inequality and the exponential expansion, one can show as in the proof of Lemma 2.7 that for $T = t_1 \leq \frac{1}{32\sqrt{2}dk^2}$ the expectation is finite. \square

From now on, we assume that $t_1 \leq \frac{1}{32\sqrt{2}dk^2}$. The next lemma gives a condition under which the process Y_t^b belongs to the Hida distribution space.

Lemma 4.5. *Suppose that the drift $b : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel measurable and satisfies the linear growth condition (3.3). Then the coordinates of the process Y_t^b defined in (2.34), by*

$$Y_t^{i,b} := E_{\tilde{P}} \left[\tilde{B}_t^{(i)} \mathcal{E}_{t_1}^\circ(b) \right], \quad (4.12)$$

belong to the Hida distribution space.

Proof. Since $t_1 \leq \frac{1}{32\sqrt{2}dk^2}$, using the same argument as in the proof of Lemma 4.4, the result follows in a similar way as in [22, Lemma 11]. \square

Lemma 4.6. *Let $b_n : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a sequence of Borel measurable functions with $b_0 = b$ and satisfying $\sup_{n \geq 1} \text{ess sup}_{(u,z) \in [0,t_1] \times \mathbb{R}^d} \frac{|b_n(u,z)|}{1+|z|} < \infty$. Then the following estimate holds*

$$\left| S(Y_t^{i,b_n} - Y_t^{i,b})(\phi) \right| \leq C \cdot E[J_n]^{\frac{1}{2}} \cdot \exp(34 \int_0^{t_1} |\phi(s)|^2 ds), \quad t \in [0, T]$$

for all $\phi \in (S_C([0, t_1]))^d$, $i = 1, \dots, d$, where J_n is given by

$$\begin{aligned} J_n := & \sum_{j=1}^d \left(2 \int_0^{t_1} \left(b_n^{(j)}(u, B_u) - b^{(j)}(u, B_u) \right)^2 du \right. \\ & \left. + \left(\int_0^{t_1} \left| (b_n^{(j)}(u, B_u))^2 - (b^{(j)}(u, B_u))^2 \right| du \right)^2 \right), \quad n \geq 1. \end{aligned} \quad (4.13)$$

More specifically, if b_n approximates b in the following sense

$$E[J_n] \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (4.14)$$

then

$$Y_t^{b_n} \rightarrow Y_t^b \text{ in } (\mathcal{S})^* \text{ as } n \rightarrow \infty$$

for all $0 \leq t \leq t_1$, $i = 1, \dots, d$.

Proof. The proof follows in a similar way as in [22, Lemma 11] and the conclusion follows since t_1 is sufficiently small, using the same reasoning as in the proof of Lemma 4.4. \square

Lemma 4.7. *Let $b_n : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a sequence of Borel-measurable, smooth functions with compact support such that $\lim_{n \rightarrow \infty} \text{ess sup}_{(t,z) \in [0,T] \times \mathbb{R}^d} \frac{|b_n(t,z)|}{1+|z|} < \infty$ which approximates a Borel-measurable, coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ (satisfying $\text{ess sup}_{(t,z) \in [0,T] \times \mathbb{R}^d} \frac{|b(t,z)|}{1+|z|} < \infty$). Then there exists a subsequence of the corresponding strong solutions $X_{n_j,t} = Y_t^{b_{n_j}}$, $j = 1, 2, \dots$, such that*

$$Y_t^{b_{n_j}} \rightarrow Y_t^b, \quad 0 \leq t \leq t_1,$$

in $L^2(\mu)$ as $j \rightarrow \infty$. In particular, $Y_t^b \in L^2(P)$ for $0 \leq t \leq t_1$.

Proof. Corollary 4.2 guarantees existence of a subsequence $Y_t^{b_{n_j}}$, $j = 1, 2, \dots$, converging in $L^2(P)$. Hence, using the uniform linear growth property of the b_n 's, it follows by the dominated convergence theorem that $E[J_{n_j}] \rightarrow 0$ in (4.14), and hence $Y_t^{b_{n_j}} \rightarrow Y_t^b$ in $(\mathcal{S})^*$. Therefore, it follows by the uniqueness of the limit that $Y_t^{b_{n_j}} \rightarrow Y_t^b$ in $L^2(P)$ for all $t \in [0, t_1]$. \square

Lemma 4.8. *Assume that $b : [0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel-measurable and satisfies $\text{ess sup}_{(t,z) \in [0,T] \times \mathbb{R}^d} \frac{|b(t,z)|}{1+|z|} < \infty$. Then*

$$\varphi^{(i)}(t, Y_t^b) = E_{\tilde{P}} \left[\varphi^{(i)}(t, \tilde{B}_t) \mathcal{E}_{t_1}^\circ(b) \right], \text{ a.e.} \quad (4.15)$$

for all $0 \leq t \leq t_1$, $i = 1, \dots, d$ and $\varphi = (\varphi^{(1)}, \dots, \varphi^{(d)})$ such that $\varphi(B_t) \in L^2(P; \mathbb{R}^d)$.

Proof. Assuming that t_1 is small, the proof follows in a similar way as in [30, Lemma 16]. \square

We now turn out to the proof of Theorem 3.1. We first prove the result for a particular $T \leq t_1$

Proof of Theorem 3.1.

Case 1: We assume first that $T = \frac{\delta_0}{64\sqrt{2}dk^2} \leq t_1$, where δ_0 is given in Lemma 2.7. We will use (4.15) of Lemma 4.8 to verify that Y_t^b is a unique strong solution of the SDE (3.2). Set $\int_0^t \varphi(s, \omega) dB_s := \sum_{j=1}^d \int_0^t \varphi^{(j)}(s, \omega) dB_s^{(j)}$ and $x = 0$. Suppose the sequence $\{b_n\}_{n=1}^\infty$ satisfies the conditions of Lemma 4.7. We first show that Y_t^b has a continuous modification. Since each $Y_t^{b_n}$ is a strong solution of the SDE (3.2) with $b = b_n$, we obtain, from Girsanov's theorem using Beněs condition (see [2] or Lemma A.4) and our assumptions, that

$$\begin{aligned} E_P \left[\left(Y_t^{i, b_n} - Y_u^{i, b_n} \right)^4 \right] &= E_{\tilde{P}} \left[\left(\tilde{B}_t^{(i)} - \tilde{B}_u^{(i)} \right)^4 \mathcal{E} \left(\int_0^T b_n(s, \tilde{B}_s) d\tilde{B}_s \right) \right] \\ &\leq C \cdot |t - u|^2 \end{aligned}$$

for all $0 \leq u, t \leq T$, $n \geq 1$, $i = 1, \dots, d$. Although the sequence $\left\{ \mathcal{E} \left(\int_0^T b_n(s, \tilde{B}_s) d\tilde{B}_s \right) \right\}_{n \geq 1}$ may not be bounded in $L^2(\tilde{P}, \mathbb{R}^d)$ for all $T > 0$, we can still show that it is uniformly bounded for $T = \frac{\delta_0}{64\sqrt{2}dk^2} \leq t_1$ (as in (4.7)), using the supermartingale property of the Doléans-Dade exponential and the proof of Lemma 4.4.

From Lemma 4.7, we have that

$$Y_t^{b_{n_j}} \longrightarrow Y_t^b \text{ in } L^2(P; \mathbb{R}^d), \quad 0 \leq t \leq T,$$

for a subsequence of $\{Y_t^{b_n}\}_{n=1}^\infty$ and hence almost sure convergence holds for a further subsequence, $0 \leq t \leq T$. Therefore Fatou's Lemma yields

$$E_P \left[\left(Y_t^{i, b} - Y_u^{i, b} \right)^4 \right] \leq C \cdot |t - u|^2 \quad (4.16)$$

for all $0 \leq u, t \leq T$, $i = 1, \dots, d$. Thus Y_t^b has a continuous modification from the Kolmogorov's continuity theorem.

Now \tilde{B}_t is a weak solution of (3.2) with respect to the measure $dP^* = \mathcal{E} \left(\int_0^T (b(s, \tilde{B}_s) + \phi(s)) d\tilde{B}_s \right) dP$ when the drift coefficient $b(s, x)$ is replaced by $b(s, x) + \phi(s)$. Thus we get

$$\begin{aligned} S(Y_t^{i, b})(\phi) &= E_{\tilde{P}} \left[\tilde{B}_t^{(i)} \mathcal{E} \left(\int_0^T (b(s, \tilde{B}_s) + \phi(s)) d\tilde{B}_s \right) \right] \\ &= E_{P^*} \left[\tilde{B}_t^{(i)} \right] \\ &= E_{P^*} \left[\int_0^t (b^{(i)}(s, \tilde{B}_s) + \phi^{(i)}(s)) ds \right] \\ &= \int_0^t E_{\tilde{P}} \left[b^{(i)}(s, \tilde{B}_s) \mathcal{E} \left(\int_0^T (b(u, \tilde{B}_u) + \phi(u)) d\tilde{B}_u \right) \right] ds + S(B_t^{(i)})(\phi) \end{aligned}$$

for $i = 1, 2, \dots, d$. Applying the transformation property (4.15) to b , we get

$$S(Y_t^{i, b})(\phi) = S \left(\int_0^t b^{(i)}(u, Y_u^{i, b}) du \right) (\phi) + S(B_t^{(i)})(\phi), \quad i = 1, 2, \dots, d.$$

Since S is injective, we get

$$Y_t^b = \int_0^t b(s, Y_s^b) ds + B_t.$$

The Malliavin differentiability of Y_t^b follows from the fact that

$$\sup_{n \geq 1} \|Y_t^{i, b_n}\|_{1,2} \leq M < \infty$$

for all $i = 1, \dots, d$ and $0 \leq t \leq T$. (See e.g. [26]). Indeed, recall that:

$$\|Y_t^{i, b_n}\|_{1,2}^2 := \|Y_t^{i, b_n}\|_{L^2(P)}^2 + \|D \cdot Y_t^{i, b_n}\|_{L^2([0, T] \times \Omega, \lambda \times P)}^2.$$

Since b_n are of linear growth, using Gronwall Lemma, one shows that the first term is uniformly bounded. The uniform boundedness of the second term follows from Girsanov theorem and Proposition 4.3.

On the other hand, using Benčs theorem, we can apply Girsanov's theorem to any other strong solution. Then the proof of Proposition 2.5 (see e.g. [30, Proposition 1]) implies that any other solution must be equal to Y_t^b .

The result is given for $T = \frac{\delta_0}{64\sqrt{2}dk^2}$ independent of the initial condition. We will use induction and a continuation method with random non-anticipative initial conditions to iterate the above argument on successive intervals of lengths $\frac{\delta_0}{64\sqrt{2}dk^2}$ and show that the result is valid for all $0 \leq T \leq 1$.

Case 2: We prove by induction that the unique solution to (3.2) is Malliavin differentiable. Since there exists a unique strong solution to (3.2) in small time interval, combination of Gronwall Lemma and linear growth condition guarantee non explosion of the unique strong solution. The result will follow if we show that its Malliavin derivative is finite in the $L^2(P)$ norm. We will prove this by induction.

Choose once more δ_0 as in Lemma 2.7 and set $\tau := \frac{\delta_0^2}{64\sqrt{2}dk^2}$, $s_i = i\tau$, $x_i := X_{s_i}^{n,0,x}$, $\frac{\partial}{\partial x_i} X_{s_{i+1}}^{n,s_i,x_i} := \frac{\partial}{\partial x} X_{s_{i+1}}^{n,s_i,x} \Big|_{x=x_i}$, $i \geq 1$. For $0 \leq t \leq s_1$, the result is true. Assume that there exists a Malliavin differentiable solution $\{X_t, 0 \leq t \leq s_m\}$. Set t such that $s_m \leq t < s_{m+1}$.

Let X_t^n , $n \geq 1$ be the approximating sequence defined by (4.1) and let start with the almost sure (a.s.) relation

$$X_t^n = X_{s_m}^n + \int_{s_m}^t b_n(u, X_u^n) du + B_t - B_{s_m}, \quad s_m \leq t \leq s_{m+1}.$$

Using once more the chain-rule for the Malliavin derivatives, it follows that

$$D_s X_t^n = \begin{cases} D_s X_{s_m}^n + \int_{s_m}^t b'_n(u, X_u) D_s X_u^n du & \text{if } s \leq s_m \\ \mathcal{I}_d + \int_s^t b'_n(u, X_u) D_s X_u^n du & \text{if } s > s_m \end{cases} \quad (4.17)$$

P -a.e., for all $0 \leq s \leq t$, that is

$$D_s X_t^n = \begin{cases} \left(\mathcal{I}_d + \sum_{q=1}^{\infty} \int_{s_m < u_1 < \dots < u_q < t} b'_n(u_1, X_{u_1}^n) : \dots : b'_n(u_q, X_{u_q}^n) du_1 \dots du_q \right) D_s X_{s_m}^n & \text{if } s \leq s_m \\ \mathcal{I}_d + \sum_{q=1}^{\infty} \int_{s < u_1 < \dots < u_q < t} b'_n(u_1, X_{u_1}^n) : \dots : b'_n(u_q, X_{u_q}^n) du_1 \dots du_q & \text{if } s > s_m \end{cases} \quad (4.18)$$

P -a.e., for all $0 \leq s \leq t$. We only consider the first term in (4.18).

$$\begin{aligned} & E \left[|D_s X_t^n|^2 \right] \\ &= E \left[E \left[\left(\mathcal{I}_d + \sum_{q=1}^{\infty} \int_{s_m < u_1 < \dots < u_q < t} b'_n(u_1, X_{u_1}^n) : \dots : b'_n(u_q, X_{u_q}^n) du_1 \dots du_q \right) D_s X_{s_m}^n \right]^2 \middle| \mathcal{F}_{s_m} \right] \\ &= E \left[E \left[\left(\mathcal{I}_d + \sum_{q=1}^{\infty} \int_{s_m < u_1 < \dots < u_q < t} b'_n(u_1, X_{u_1}^n) : \dots : b'_n(u_q, X_{u_q}^n) du_1 \dots du_q \right)^2 \middle| \mathcal{F}_{s_m} \right] \left| D_s X_{s_m}^n \right|^2 \right] \\ &\leq C_1 E \left[\left(1 + \sum_{q=1}^{\infty} \sum_{j=1}^d \sum_{l_1, \dots, l_{q-1}=1}^d \left\| \int_{s_m < u_1 < \dots < u_q < t} \frac{\partial}{\partial x_{l_1}} b_n^{(i)}(u_1, X_{s_m} + B_{u_1}) \frac{\partial}{\partial x_{l_2}} b_n^{(l_1)}(u_2, X_{s_m} + B_{u_2}) : \dots \right. \right. \right. \\ &\quad \left. \left. \left. \dots : \frac{\partial}{\partial x_j} b_n^{(l_{q-1})}(u_q, X_{s_m} + B_{u_q}) du_1 \dots du_q \right\|_{L^{16}(P, \mathbb{R}^d, \mathcal{F}_{s_m})} \right)^2 \times \exp \left(12k^2 \tau (1 + |X_{s_m}|^2) \right) \left| D_s X_{s_m}^n \right|^2 \right]. \end{aligned}$$

The last inequality follows from Girsanov transform and Hölder inequality. One can show as in Proposition 4.11 (see also (4.29)) that there exist a positive constant and an increasing and

continuous function $\tilde{C}_d : [0, \infty) \rightarrow [0, \infty)$ such that

$$\left(1 + \sum_{q=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\| \int_{s_m < u_1 < \dots < u_q < t} \frac{\partial}{\partial x_{l_1}} b_n^{(i)}(u_1, X_{s_m} + B_{u_1}) \frac{\partial}{\partial x_{l_2}} b_n^{(l_1)}(u_2, X_{s_m} + B_{u_2}) : \dots \right. \right. \\ \left. \left. \dots : \frac{\partial}{\partial x_j} b_n^{(l_{q-1})}(u_q, X_{s_m} + B_{u_q}) du_1 \dots du_n \right\|_{L^{16}(P, \mathbb{R}^d, \mathcal{F}_{s_m})} \right)^2 \leq C \tilde{C}_d(|X_{s_m}|, \|\tilde{b}\|_{\infty}). \quad (4.19)$$

Therefore, using Cauchy-Schwartz inequality, we obtain that there exists $C > 0$, which may change from one line to the other and such that

$$\begin{aligned} E[|D_s X_t^n|^2] &\leq C E[\tilde{C}_d(|X_{s_m}|, \|\tilde{b}\|_{\infty}) \exp(12k^2\tau(1 + |X_{s_m}|^2)) |D_s X_{s_m}^n|^2] \\ &\leq C E[\tilde{C}_d^2(|X_{s_m}|, \|\tilde{b}\|_{\infty})]^{1/4} E[\exp(48k^2\tau(1 + |X_{s_m}|^2))]^{1/4} E[|D_s X_{s_m}^n|^4]^{1/2} \\ &\leq C_1 \exp\{C_2 \delta_0 |x|^2\} E[|D_s X_{s_m}^n|^4] \leq C. \end{aligned} \quad (4.20)$$

Let us notice that $E[|D_s X_{s_m}^n|^4] < C$. In fact, one can show (see for example Propositions 4.3 and 4.11) that $E[|D_s X_{s_1}^n|^p] < C$ for all $p \geq 1$. Hence using continuation argument on the successive intervals and Lemma 2.7, the fourth moment of the Malliavin derivative $D_s X_{s_m}^n$ can be controlled. \square

Remark 4.9. *One limitation in our argument is the use of Girsanov transform (via Beněs Theorem) in the sense that we cannot cover SDEs with superlinear growth drift coefficients at the moment.*

4.2. Proof of existence of a Sobolev differentiable stochastic flow. In order to prove Theorem 3.4, we first need to prove Proposition 3.8.

We first consider a smooth drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ with compact support. We estimate the norm of $X_t \in L^2(\Omega; W^{1,p}(U))$ in terms of $\|\tilde{b}\|_{\infty}$, where U is a bounded open subset of \mathbb{R}^d .

Second, we consider a sequence of smooth functions $b_n : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ with compact support and such that $b_n(t, x) \rightarrow b(t, x)$ $dt \times dx$ -a.e. and there is a positive k with

$$\sup_{n \geq 1} |b_n(t, x)| \leq k(1 + |x|)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Let $\{X_n^{n,s,x}\}_{n=1}^{\infty}$ be the sequence of solutions of (3.5) when b is replaced by b_n , $n \geq 1$. We will show that $\{X_n^{n,s,x}\}_{n=1}^{\infty}$ is relatively compact in $L^2(\Omega)$ when integrated against test functions on \mathbb{R}^d .

One can show (see [15]) that when b is continuously differentiable with a bounded derivative, the process X_s^x is differentiable in x and its spatial derivative is given by the following linear ODE

$$\frac{\partial}{\partial x} X_s^x = \mathcal{I}_d + \int_0^s b'(u, X_u^x) \frac{\partial}{\partial x} X_u^x du. \quad (4.21)$$

Observe that (4.21) is analogous to (4.3) when $t = 0$ with the Malliavin derivative of X_s^x replaced by the spatial derivative $\frac{\partial}{\partial x} X_s^x$. Therefore, similar assertions to Lemmas B.1-B.4 also hold for the spatial derivative $\frac{\partial}{\partial x} X_s^x$. Using this observation and the proof of Proposition 4.3, we get the following proposition.

Proposition 4.10. *Let B be a d -dimensional Brownian motion starting from the origin and b_1, \dots, b_n be compactly supported continuously differentiable functions $b_i : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$. Let $\alpha_i \in \{0, 1\}^d$ be a multiindex such that $|\alpha_i| = 1$ for $i = 1, 2, \dots, n$. Then there exists a universal constant C (independent of $\{b_i\}_i$, n , and $\{\alpha_i\}_i$) such that*

$$\left| E \left[\int_{t_0 < t_1 < \dots < t_n < t} \left(\prod_{i=1}^n D^{\alpha_i} b_i(t_i, x + B(t_i)) \right) dt_1 \dots dt_n \right] \right| \leq \frac{C^n |x|^n \prod_{i=1}^n \|\tilde{b}_i\|_{\infty} (t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \quad (4.22)$$

where Γ is the Gamma-function and $\tilde{b}(t, z) := \frac{b(t, z)}{1+|z|}, (t, z) \in [0, 1] \times \mathbb{R}^d$. Here D^{α_i} denotes the partial derivative with respect to the j 'th space variable, where j is the position of the 1 in α_i .

The following result gives a key estimate on the spatial weak derivative in terms of $\|\tilde{b}\|_\infty$.

Proposition 4.11. *Let $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth function with compact support. Then for any $p \in [1, \infty)$ and $t \in [0, T]$, the following estimate holds*

$$E \left[\left\| \frac{\partial}{\partial x} X_t^x \right\|^p \right] \leq \tilde{C}_{d,p}(|x|, \|\tilde{b}\|_\infty), \quad (4.23)$$

where $\tilde{C}_{p,d} : [0, \infty) \rightarrow [0, \infty)$ are increasing, continuous functions, $|\cdot|$ a matrix-norm on $\mathbb{R}^{d \times d}$, $\|\cdot\|_\infty$ the supremum norm and C_1 is a constant.

Proof. We prove this in two steps. We first prove the result for $T \in [0, 1]$ sufficiently small. Then we show that there does not exist a maximal interval $[0, T_1] \subset [0, 1]$ for which estimate (4.23) holds.

Let $t \in [0, T]$ for T sufficiently small. Iterating (4.21) gives

$$\frac{\partial}{\partial x} X_t^x = \mathcal{I}_d + \sum_{i=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, X_{s_1}^x) : \dots : b'(s_n, X_{s_n}^x) ds_1 \dots ds_n, \quad (4.24)$$

$L^p(P)$, where b' is the spatial Jacobian matrix of b . Applying Beněš' Theorem to the martingale $2 \sum_{j=1}^d \int_0^t b^{(j)}(u, B_u) dB_u^{(j)}$, and using Girsanov's theorem and Hölder's inequality, we have

$$\begin{aligned} E \left[\left\| \frac{\partial}{\partial x} X_t^x \right\|^p \right] &= E \left[\left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|^p \right. \\ &\quad \times \mathcal{E} \left(\int_0^T b(u, x + B_u) dB_u \right) \Big] \\ &= E \left[\left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|^p \right. \\ &\quad \times \exp \left(\sum_{j=1}^d \int_0^T b^{(j)}(u, x + B_u) dB_u^{(j)} - \sum_{j=1}^d \int_0^T (b^{(j)}(u, x + B_u))^2 du \right. \\ &\quad \left. \left. + \frac{1}{2} \sum_{j=1}^d \int_0^T (b^{(j)}(u, x + B_u))^2 du \right) \right] \\ &\leq C \left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{4p}(P, \mathbb{R}^{d \times d})}^p \\ &\quad \times E \left[\exp \left(2 \sum_{j=1}^d \int_0^T (b^{(j)}(u, x + B_u))^2 du \right) \right]^{\frac{1}{4}} \end{aligned}$$

$$\begin{aligned}
 &\leq C \left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{4p}(\Omega, \mathbb{R}^{d \times d})}^p \\
 &\quad \times \exp \left(12k^2 T(1 + |x|^2) \right) E \left[\exp \left(12k^2 T \max_{0 \leq u \leq T} |B_u|^2 \right) \right] \\
 &\leq C_1 \left(1 + \sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\| \int_{0 < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_1, x + B_{s_2}) : \dots \right. \right. \\
 &\quad \left. \left. \dots : \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{4p}(P, \mathbb{R}^d)} \right)^p \times \exp \left(12k^2 T(1 + |x|^2) \right) \\
 &\leq C_1 \left(1 + \sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\| \int_{0 < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) : \dots \right. \right. \\
 &\quad \left. \left. \dots : \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{4p}(P, \mathbb{R}^d)} \right)^p \times \exp \left(12k^2 T(1 + |x|^2) \right), \tag{4.25}
 \end{aligned}$$

where the positive constant C_1 is obtained in a similar way with the choice of T small as in Lemma 4.4. The last inequality follows from Hölder's inequality since $4p \leq 4^p$.

As in the proof of Lemma 4.1, we consider the expression

$$A := \int_{0 < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \dots \frac{\partial}{\partial x_{l_n}} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n. \tag{4.26}$$

Repeated use of deterministic integration by part shows that A^2 can be written as a sum of at most 2^{2n} summands of the form

$$\int_{0 < s_1 < \dots < s_{2n} < t} g_1(s_1) \dots g_{2n}(s_{2n}) ds_1 \dots ds_{2n}, \tag{4.27}$$

where $g_l \in \left\{ \frac{\partial}{\partial x_j} b^{(i)}(\cdot, B) : 1 \leq i, j \leq d \right\}$, $l = 1, 2, \dots, 2n$. We deduce by induction that A^{4^p} is the sum of at most 4^{p4^n} terms of the form (4.27) with length $4^p n$. Using this fact and Proposition 4.10, we have that

$$\begin{aligned}
 &\left\| \int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{4p}(\mu; \mathbb{R})} \\
 &\leq \left(\frac{4^{p4^n} C^{4^p n} |x|^{4^p n} \|\tilde{b}\|_{\infty}^{4^p n} |t|^{4^p - 1}}{\Gamma(\frac{4^p}{2} n + 1)} \right)^{4^{-p}} \leq \frac{4^{pn} C^n |x|^n |T|^{n/4} \|\tilde{b}\|_{\infty}^n}{((2^{2p-1} n)!)^{4^{-p}}}, \tag{4.28}
 \end{aligned}$$

from which we get

$$\begin{aligned}
 E \left[\left| \frac{\partial}{\partial x} X_t^x \right|^p \right] &\leq C_1 \exp \left(12k^2 T(1 + |x|^2) \right) \left(1 + \sum_{n=1}^{\infty} \frac{d^{n+2} 4^{pn} C^n |T|^{n/4} |x|^n \|\tilde{b}\|_{\infty}^n}{((2^{2p-1} n)!)^{4^{-p}}} \right)^p \\
 &\leq C_1 \exp \left(12k^2 T(1 + |x|^2) \right) C_{d,p} (\|\tilde{b}\|_{\infty}, |x|). \tag{4.29}
 \end{aligned}$$

Hence for T sufficiently small, the above expectation is finite. Next, assume that there exists a maximal interval $[0, T_1]$ for which (4.23) holds and let show that there exists $1 \geq T_2 > T_1$ such that conclusion of Proposition 4.11 is valid.

Choose δ_0 as in Lemma 2.7 and set $\tau := \frac{\delta_0}{64\sqrt{2}dk^2}$, $s_i = i\tau$, $x_i := X_{s_i}^{n,0,x}$, $\frac{\partial}{\partial x_i} X_{s_{i+1}}^{n,s_i,x_i} := \frac{\partial}{\partial x} X_{s_{i+1}}^{n,s_i,x} \Big|_{x=x_i}$, $i \geq 1$. There exists m positive integer such that $m\tau \leq T_1 < (m+1)\tau$. Let show that the conclusion of Proposition 4.11 is valid for $T_2 = (m+1)\tau$.

Consider the times $0 < s_m < t \leq s_{m+1}$. Then by the flow property

$$X_t^{n,0,x} = X_t^{n,s_m,\cdot} (X_{s_m}^{n,0,x})$$

a.s. By the chain rule

$$\frac{\partial}{\partial x} X_t^{n,0,x} = \frac{\partial}{\partial x_m} X_t^{n,s_m,x_m} \cdot \frac{\partial}{\partial x} X_{s_m}^{n,0,x}$$

a.s., where \cdot denotes matrix multiplication for the Jacobian derivatives.

Let $\{\mathcal{F}_t\}_{t \geq 0}$ denote the filtration generated by the driving Brownian motion B . Taking conditional expectation w.r.t to \mathcal{F}_{s_m} in the above equality, using (4.29) and Lemma 2.7, we get

$$\begin{aligned} E \left| \frac{\partial}{\partial x} X_t^{n,0,x} \right|^p &= E \left(E \left[\left| \frac{\partial}{\partial x_m} X_t^{n,s_m,x_m} \cdot \frac{\partial}{\partial x} X_{s_m}^{n,0,x} \right|^p \middle| \mathcal{F}_{s_m} \right] \right) \\ &= E \left(E \left[\left| \frac{\partial}{\partial x_m} X_t^{n,s_m,x_m} \right|^p \middle| \mathcal{F}_{s_m} \right] \cdot \left| \frac{\partial}{\partial x} X_{s_m}^{n,0,x} \right|^p \right) \\ &\leq C E \left(C_p(|x_m|, \|\tilde{b}\|_\infty) e^{6k^2 \tau |x_m|^2} \left| \frac{\partial}{\partial x} X_{s_m}^{n,0,x} \right|^p \right) \\ &\leq C E \left(C_p^2(|x_m|, \|\tilde{b}\|_\infty) \right)^{1/4} \left(E e^{\delta_0 |x_m|^2} \right)^{1/4} \left(E \left[\left| \frac{\partial}{\partial x} X_{s_m}^{n,0,x} \right|^{2p} \right] \right)^{1/2} \\ &\leq C \left(E e^{\delta_0 |x_m|^2} \right)^{1/2} \left(E \left[\left| \frac{\partial}{\partial x} X_{s_m}^{n,0,x} \right|^{2p} \right] \right)^{1/2} \\ &\leq C \left(C_1 e^{C_2 \delta_0 |x|^2} \right)^{1/2} \cdot C_d(|x|^2) C_{d,p}(\|\tilde{b}\|_\infty, |x|) \\ &\leq C_{1,\delta_0} C_{d,p}(\|\tilde{b}\|_\infty, |x|, |x^2|), \end{aligned} \tag{4.30}$$

where C_1, C_2 are positive constants independent of x , but may depend on k . \square

This complete the first step of the proof of Theorem 3.4.

We now fix a measurable coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying the linear growth condition. By Theorem 3.1, the SDE (3.5) admits a (Malliavin differentiable) unique strong solution. Let $X^{s,x}$ be that solution. Consider the approximating sequence b_n introduced at the beginning of the section and denote by $X_t^{n,s,x}$ the corresponding sequence of solutions of the SDE. We then have the following result

Lemma 4.12. *Fix $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Then the sequence $X_t^{n,s,x}$ converges weakly in $L^2(\Omega; \mathbb{R}^d)$ to $X_t^{s,x}$.*

Proof. We assume here without loss of generality that $d = 1$ and $s = 0$. We know that the set

$$\left\{ \mathcal{E} \left(\int_0^T h(u) dB_u \right) : h \in C_b^1(\mathbb{R}) \right\}$$

generates a dense subspace of $L^2(\Omega; \mathbb{R}^d)$. Hence it is enough to show that $E \left[X_t^{n,x} \mathcal{E} \left(\int_0^T h(u) dB_u \right) \right] \rightarrow E \left[X_t^{x,x} \mathcal{E} \left(\int_0^T h(u) dB_u \right) \right]$. Noting that the function $(t, x) \mapsto b(t, x) + h'(t)$ is of linear growth in x , it follows by the Cameron-Martin theorem and the uniqueness in law for the SDE (3.5) that

$$\begin{aligned} &E \left[X_t^{n,x} \mathcal{E} \left(\int_0^T h(u) dB_u \right) \right] - E \left[X_t^{x,x} \mathcal{E} \left(\int_0^T h(u) dB_u \right) \right] \\ &= E \left[(x + B_t) \left\{ \mathcal{E} \left(\int_0^T b_n(u, x + B_u) + h'(u) dB_u \right) - \mathcal{E} \left(\int_0^T b(u, x + B_u) + h'(u) dB_u \right) \right\} \right] \\ &= E \left[(x + B_t) \left\{ \exp \left(\int_0^T b_n(u, x + B_u) + h'(u) dB_u - \frac{1}{2} \int_0^T (b_n(u, x + B_u) + h'(u))^2 du \right) \right. \right. \\ &\quad \left. \left. - \exp \left(\int_0^T b(u, x + B_u) + h'(u) dB_u - \frac{1}{2} \int_0^T (b(u, x + B_u) + h'(u))^2 du \right) \right\} \right]. \end{aligned}$$

Using the inequality $|e^a - e^b| \leq |e^a + e^b||a - b|$, we get

$$\begin{aligned}
 & E\left[X_t^{n,x}\mathcal{E}\left(\int_0^T h(u)dB_u\right)\right] - E\left[X_t^x\mathcal{E}\left(\int_0^T h(u)dB_u\right)\right] \\
 & \leq E\left[(x+B_t)\left\{\exp\left(\int_0^T b_n(u,x+B_u)+h'(u)dB_u - \frac{1}{2}\int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right.\right. \\
 & \quad \left.+\exp\left(\int_0^T b(u,x+B_u)+h'(u)dB_u - \frac{1}{2}\int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}\times \\
 & \quad \left|\int_0^T b_n(u,x+B_u)-b(u,x+B_u)dB_u + \frac{1}{2}\int_0^T \left((b(u,x+B_u)+h'(u))^2 - (b_n(u,x+B_u)+h'(u))^2\right)du\right|.
 \end{aligned}$$

Using Hölder inequality, we get

$$\begin{aligned}
 & E\left[X_t^{n,x}\mathcal{E}\left(\int_0^T h(u)dB_u\right)\right] - E\left[X_t^x\mathcal{E}\left(\int_0^T h(u)dB_u\right)\right] \\
 & \leq E\left[\left\{\exp\left(\int_0^T b_n(u,x+B_u)+h'(u)dB_u - \frac{1}{2}\int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right.\right. \\
 & \quad \left.+\exp\left(\int_0^T b(u,x+B_u)dB_u - \frac{1}{2}\int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^{\frac{3}{2}}\right]^{\frac{2}{3}} \times E\left[(x+B_t)^{12}\right]^{\frac{1}{12}} \\
 & \quad \times E\left[\left\{\int_0^T b_n(u,x+B_u)-b(u,x+B_u)dB_u\right.\right. \\
 & \quad \left.+\frac{1}{2}\int_0^T \left((b(u,x+B_u)+h'(u))^2 - (b_n(u,x+B_u)+h'(u))^2\right)du\right\}^4\right]^{\frac{1}{4}}.
 \end{aligned}$$

It follows from Cauchy and Burkholder-Davis-Gundy inequality that there exists a constant C such that

$$\begin{aligned}
& E\left[X_t^{n,x}\mathcal{E}\left(\int_0^T h(u)dB_u\right)\right] - E\left[X_t^x\mathcal{E}\left(\int_0^T h(u)dB_u\right)\right] \\
& \leq CE\left[\left\{\exp\left(\int_0^T b_n(u,x+B_u)+h'(u)dB_u - \int_0^T (b_n(u,x+B_u)+h'(u))^2du\right.\right.\right. \\
& \quad \left.\left.+\frac{1}{2}\int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right\}^{\frac{3}{2}} \\
& \quad +\left\{\exp\left(\int_0^T b(u,x+B_u)dB_u - \int_0^T (b(u,x+B_u)+h'(u))^2du\right.\right. \\
& \quad \left.\left.+\frac{1}{2}\int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^{\frac{3}{2}}\right]^{\frac{2}{3}} \\
& \quad \times E\left[\left\{\int_0^T \left(b_n(u,x+B_u)-b(u,x+B_u)\right)^2du\right\}^2\right. \\
& \quad \left.+\left\{\int_0^T \left((b(u,x+B_u)+h'(u))^2-(b_n(u,x+B_u)+h'(u))^2\right)du\right\}^4\right]^{\frac{1}{4}} \\
& \leq C\left(E\left[\left\{\exp\left(\int_0^T b_n(u,x+B_u)+h'(u)dB_u - \int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right\}^2\right]^{\frac{1}{2}}\right. \\
& \quad \times E\left[\left\{\exp\left(\frac{1}{2}\int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right\}^6\right]^{\frac{1}{6}} \\
& \quad +E\left[\left\{\exp\left(\int_0^T b(u,x+B_u)+h'(u)dB_u - \int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^2\right]^{\frac{1}{2}} \\
& \quad \times E\left[\left\{\exp\left(\frac{1}{2}\int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^6\right]^{\frac{1}{6}} \\
& \quad \times E\left[\left\{\int_0^T \left(b_n(u,x+B_u)-b(u,x+B_u)\right)^2du\right\}^2\right. \\
& \quad \left.+\left\{\int_0^T \left((b(u,x+B_u)+h'(u))^2-(b_n(u,x+B_u)+h'(u))^2\right)du\right\}^4\right]^{\frac{1}{4}}.
\end{aligned}$$

The terms $E\left[\left\{\exp\left(\int_0^T b_n(u,x+B_u)+h'(u)dB_u - \int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right\}^2\right]$ and $E\left[\left\{\exp\left(\int_0^T b(u,x+B_u)+h'(u)dB_u - \int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^2\right]$ are finite and equal to one by the Benès theorem applied to the stochastic integrals with drift $2b$ or $2b_n$.

Moreover, the terms $E\left[\left\{\exp\left(\frac{1}{2}\int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^6\right]$ and $E\left[\left\{\exp\left(\frac{1}{2}\int_0^T (b_n(u,x+B_u)+h'(u))^2du\right)\right\}^6\right]$ are finite for small time T . In fact

$$\begin{aligned}
& E\left[\left\{\exp\left(\frac{1}{2}\int_0^T (b(u,x+B_u)+h'(u))^2du\right)\right\}^6\right] \\
& \leq E\left[\exp\left(6\int_0^T (k(1+|x+B_u|)^2+h'(u)^2)du\right)\right] \\
& \leq C\exp(12k^2T(1+|x|^2))E\left[\exp\left(12k^2T\max_{0\leq u\leq T}|B_u|^2\right)\right].
\end{aligned}$$

This expectation is finite for T sufficiently small and independent on the initial condition of the solution. We can once more use conditioning, induction and a continuation argument with random non-anticipative initial conditions to iterate the above argument on successive intervals of lengths τ . Hence the above result holds for all $T > 0$. \square

As a consequence of the compactness criteria, we have by combining Corollary A.3 and Proposition 4.10, the following result

Theorem 4.13. *For any fixed $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, the sequence $\{X_t^{n,s,x}\}_{n=1}^\infty$ converges strongly in $L^2(\Omega, \mathbb{R}^d)$ to $X_t^{s,x}$.*

The next result is a consequence of Proposition 4.11.

Corollary 4.14. *Let $X^{s,x}$ be the unique strong solution to the SDE (1.1) and $q > 1$ an integer. Then there exists a constant $C = C(d, k, q) < \infty$ independent of x_1, x_2 in every bounded subset of \mathbb{R}^d such that*

$$E\left[\left|X_{t_1}^{s_1, x_1} - X_{t_2}^{s_2, x_2}\right|^q\right] \leq C\left(|s_2 - s_1|^{q/2} + |t_2 - t_1|^{q/2} + |x_2 - x_1|^q\right) \quad (4.31)$$

for all $s_2, s_1, t_2, t_1, x_2, x_1$.

In particular, there exists a locally Hölder continuous version of the random field $(s, t, x) \mapsto X_t^{s,x}$ with Hölder constant $\alpha < \frac{1}{2}$ in s, t and $\alpha < 1$ in x .

Proof. Assume that the above condition holds. Moreover, without loss of generality, assume that $0 \leq s_1 < s_2 < t_1 < t_2$. Then

$$\begin{aligned} X_{t_1}^{n, s_1, x_1} - X_{t_2}^{n, s_2, x_2} &= x_1 - x_2 + \int_{s_1}^{t_1} b_n(u, X_u^{n, s_1, x_1}) du - \int_{s_2}^{t_2} b_n(u, X_u^{n, s_2, x_2}) du \\ &\quad + (B_{t_1} - B_{s_1}) - (B_{t_2} - B_{s_2}) \\ &= x_1 - x_2 + \int_{s_1}^{s_2} b_n(u, X_u^{n, s_1, x_1}) du - \int_{t_1}^{t_2} b_n(u, X_u^{n, s_2, x_2}) du \\ &\quad + \int_{s_2}^{t_1} (b_n(u, X_u^{n, s_1, x_1}) - b_n(u, X_u^{n, s_1, x_2})) du \\ &\quad + \int_{s_2}^{t_1} (b_n(u, X_u^{n, s_1, x_2}) - b_n(u, X_u^{n, s_2, x_2})) du \\ &\quad + (B_{t_2} - B_{t_1}) - (B_{s_2} - B_{s_1}). \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} E\left[\left|X_{t_1}^{n, s_1, x_1} - X_{t_2}^{n, s_2, x_2}\right|^q\right] &\leq 7^{q-1} \left(|x_1 - x_2|^q + E\left[\left|\int_{s_1}^{s_2} b_n(u, X_u^{n, s_1, x_1}) du\right|^q\right]\right. \\ &\quad + E\left[\left|\int_{t_1}^{t_2} b_n(u, X_u^{n, s_2, x_2}) du\right|^q\right] \\ &\quad + E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n, s_1, x_1}) - b_n(u, X_u^{n, s_1, x_2})) du\right|^q\right] \\ &\quad + E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n, s_1, x_2}) - b_n(u, X_u^{n, s_2, x_2})) du\right|^q\right] \\ &\quad + E\left[|B_{s_2} - B_{s_1}|^q\right] + E\left[|B_{t_2} - B_{t_1}|^q\right]\Big) \\ &\leq 7^{q-1} \frac{(2q)!}{2^q q!} \left(|x_1 - x_2|^q + |t_2 - t_1|^{q/2} + |s_2 - s_1|^{q/2}\right. \\ &\quad + E\left[\left|\int_{s_1}^{s_2} b_n(u, X_u^{n, s_1, x_1}) du\right|^q\right] + E\left[\left|\int_{t_1}^{t_2} b_n(u, X_u^{n, s_2, x_2}) du\right|^q\right] \\ &\quad + E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n, s_1, x_1}) - b_n(u, X_u^{n, s_1, x_2})) du\right|^q\right] \\ &\quad + E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n, s_1, x_2}) - b_n(u, X_u^{n, s_2, x_2})) du\right|^q\right]\Big). \quad (4.32) \end{aligned}$$

The assumption on b_n and Hölder's inequality yield

$$\begin{aligned}
E \left[\left| \int_{s_1}^{s_2} b_n(u, X_u^{n,s_1,x_1}) du \right|^q \right] &\leq \int_{s_1}^{s_2} E \left[\left| b_n(u, X_u^{n,s_1,x_1}) \right|^q \right] du \times |s_1 - s_2|^{q-1} \\
&\leq \int_{s_1}^{s_2} E \left[2^{q-1} k^q + 2^{q-1} k^q |X_u^{n,s_1,x_1}|^q \right] du \times |s_1 - s_2|^{q-1} \\
&= 2^{q-1} k^q |s_1 - s_2|^{q-1} \left(|s_1 - s_2| + \int_{s_1}^{s_2} E \left[|X_u^{n,s_1,x_1}|^q \right] du \right).
\end{aligned}$$

Using again Hölder's inequality and Gronwall's Lemma, we get

$$\begin{aligned}
\int_{s_1}^{s_2} E \left[|X_u^{n,s_1,x_1}|^q \right] du &\leq \int_{s_1}^{s_2} |x_1|^q du + \int_{s_1}^{s_2} E \left| \int_{s_1}^u b_n(r_1, X_{r_1}^{n,s_1,x_1}) dr_1 \right|^q du \\
&\quad + \int_{s_1}^{s_2} E \left[|B_u - B_{s_1}|^q \right] du \\
&\leq |x_1|^q |s_1 - s_2| + \int_{s_1}^{s_2} \int_{s_1}^u E \left| b_n(r_1, X_{r_1}^{n,s_1,x_1}) \right|^q dr_1 (u - s_1)^{q-1} du \\
&\quad + \frac{(2q)!}{2^q q!} \int_{s_1}^{s_2} |u - s_1|^{q/2} du \\
&\leq |x_1|^q |s_2 - s_1| + \frac{2^{q-1} k^q}{q} |s_2 - s_1|^{q+1} + \frac{(2q)!}{2^q q! (q/2 + 1)} |s_2 - s_1|^{q/2+1} \\
&\quad + 2^{q-1} k^q \int_{s_1}^{s_2} (u - s_1)^{q-1} \left(\int_{s_1}^u E \left[|X_{r_1}^{n,s_1,x_1}|^q \right] dr_1 \right) du \\
&\leq \left(|x_1|^q |s_2 - s_1| + \frac{2^{q-1} k^q}{q} |s_2 - s_1|^{q+1} + \frac{(2q)!}{2^q q! (q/2 + 1)} |s_2 - s_1|^{q/2+1} \right) \\
&\quad \times \exp \left\{ 2^{q-1} k^q \int_{s_1}^{s_2} (u - s_1)^{q-1} du \right\} \\
&\leq C_q |s_2 - s_1| \left(|x_1|^q + \frac{2^{q-1} k^q}{q} |s_2 - s_1|^q + \frac{(2q)!}{2^q q! (q/2 + 1)} |s_2 - s_1|^{q/2} \right) \\
&\quad \times \exp \left\{ \frac{2^{q-1} k^q}{q} |s_2 - s_1|^q \right\}.
\end{aligned}$$

Thus

$$E \left[\left| \int_{s_1}^{s_2} b_n(u, X_u^{n,s_1,x_1}) du \right|^q \right] \leq |s_1 - s_2|^{q/2} C(q, k, |x_1|), \quad (4.33)$$

where

$$\begin{aligned}
C(q, k, |x_1|) &= \left(1 + C_q \left\{ |x_1|^q + \frac{2^{q-1} k^q}{q} |s_2 - s_1|^q + \frac{(2q)!}{2^q q! (q/2 + 1)} |s_2 - s_1|^{q/2} \right\} \exp \left\{ \frac{2^{q-1} k^q}{q} |s_2 - s_1|^q \right\} \right) \\
&\quad \times 2^{q-1} k^q |s_1 - s_2|^{q/2}.
\end{aligned}$$

A similar bound holds for $E \left[\left| \int_{t_1}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \right|^q \right]$.

By the Mean Value Theorem and Proposition 4.3, we have

$$\begin{aligned}
 & E \left[\left| \int_{s_2}^{t_1} \left(b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2}) \right) du \right|^q \right] \\
 &= |x_2 - x_1|^q E \left[\left| \int_{s_2}^{t_1} \int_0^1 b'_n(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}) \frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)} du d\tau \right|^q \right] \\
 &\leq |x_2 - x_1|^q \int_0^1 E \left[\left| \int_{s_2}^{t_1} b'_n(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}) \frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)} du \right|^q \right] d\tau \\
 &= |x_2 - x_1|^q \int_0^1 E \left[\left| \frac{\partial}{\partial x} X_{t_1}^{n,s_1,x_1+\tau(x_2-x_1)} - \frac{\partial}{\partial x} X_{s_2}^{n,s_1,x_1+\tau(x_2-x_1)} \right|^q \right] d\tau \\
 &\leq C_q |x_2 - x_1|^q \sup_{t \in [s_1, 1], x \in B(0, |x_1| + |x_2|)} E \left[\left| \frac{\partial}{\partial x} X_{t_1}^{n,s_1,x} \right|^q \right] \\
 &\leq C(k, q, d) |x_2 - x_1|^q.
 \end{aligned} \tag{4.34}$$

Using the Markov property, we obtain

$$\begin{aligned}
 & E \left[\left| \int_{s_2}^{t_1} \left(b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2}) \right) du \right|^q \right] \\
 &\leq \int_{s_2}^{t_1} E \left[\left| \left(b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2}) \right) \right|^q \right] du \\
 &= \int_{s_2}^{t_1} E \left[E \left[\left| \left(b_n(u, X_u^{n,s_2,y}) - b_n(u, X_u^{n,s_2,x_2}) \right) \right|^q \right] \Big|_{y=X_{s_2}^{n,s_1,x_2}} \right] du \\
 &\leq C E \left[\left| X_{s_2}^{n,s_1,x_2} - x_2 \right|^q \right] \\
 &\leq C(q, k) |s_2 - s_1|^{q/2}.
 \end{aligned} \tag{4.35}$$

Combining (4.32)-(4.35), we get (4.31) with X replaced by X^n .

Since $X_{t_2}^{n,s_2,x_2} \rightarrow X_{t_2}^{s_2,x_2}$ and $X_{t_1}^{n,s_1,x_1} \rightarrow X_{t_1}^{s_1,x_1}$ strongly in $L^2(\Omega; \mathbb{R}^d)$ as $n \rightarrow \infty$, it follows that there exists subsequences of $\{X_{t_2}^{n,s_2,x_2}\}_{n \geq 1}$ and $\{X_{t_1}^{n,s_1,x_1}\}_{n \geq 1}$ that converge almost everywhere. The result then follows from Fatou's lemma. \square

In the next Lemma, we prove that the sequence defined by $\{X_t^{n,x}\}_{n \geq 1} := \{X_t^{n,0,x}\}_{n \geq 1}$ converges to $X_t^x := X_t^{0,x}$.

Lemma 4.15. *For any $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $t \in [0, T]$, $T > 0$ the sequence*

$$\langle X_t^n, \varphi \rangle = \int_{\mathbb{R}^d} \langle X_t^{n,x}, \varphi(x) \rangle_{\mathbb{R}^d} dx$$

converges to $\langle X_t, \varphi \rangle$ in $L^2(\Omega, \mathbb{R}^d)$.

Proof. Let D_s be the Malliavin derivative and let U be the compact support of φ . Then we have

$$E \left[|D_s \langle X_t^n, \varphi \rangle| \right] \leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2 |U| \sup_{x \in U} E \left[|D_s X_t^{n,x}|^2 \right]$$

and

$$E \left[|D_{s_1} \langle X_t^n, \varphi \rangle - D_{s_2} \langle X_t^n, \varphi \rangle| \right] \leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2 |U| \sup_{x \in U} E \left[|D_{s_1} X_t^{n,x} - D_{s_2} X_t^{n,x}|^2 \right].$$

Using the compactness criteria (Corollary A.3), there exists a subsequence $\langle X_t^{n(k)}, \varphi \rangle$ converging in $L^2(\Omega, \mathbb{R}^d)$ as $k \rightarrow \infty$ to a limit $Y(\varphi)$. Hence as in the proof of Lemma 4.12, one can show that $E \left[\langle X_t^n, \varphi \rangle \mathcal{E} \left(\int_0^T h(u) dB_u \right) \right]$ converges to $E \left[\langle X_t, \varphi \rangle \mathcal{E} \left(\int_0^T h(u) dB_u \right) \right]$ for all $h \in C_b^1(\mathbb{R}; \mathbb{R}^d)$. Therefore $\langle X_{t,n}, \varphi \rangle$ converges weakly to $\langle X_t, \varphi \rangle$ and the uniqueness of the limits implies that

$$Y(\varphi) = \langle X_t, \varphi \rangle.$$

Using contradiction, suppose that the full sequence does not converge. Then there exist $\varepsilon_0 > 0$ and a subsequence $\langle X_{t,n(k)}, \varphi \rangle$ for which

$$\|\langle X_{t,n(k)}, \varphi \rangle - \langle X_{t,n}, \varphi \rangle\| \geq \varepsilon_0 \quad (4.36)$$

for every k . But using once more Lemma 4.12, one can show that there exist a further subsequence $\langle X_{t,n(k_i)}, \varphi \rangle$ of $\langle X_{t,n(k)}, \varphi \rangle$ that converges to $\langle X_t, \varphi \rangle$. This is a contradiction to (4.36). \square

We are now ready to conclude the proof of Proposition 3.8.

Proof Proposition 3.8. For each bounded set B of \mathbb{R}^d , we have

$$\sup_n \sup_{x \in B} E \left[\left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p \right] < \infty.$$

From this, it follows that there exists a subsequence $\frac{\partial}{\partial x} X_t^{n(k),x}$ that converges to an element Y in the weak topology of $L^2(\Omega, L^p(U))$. Hence for any $A \in \mathcal{F}$ and $\varphi \in C_0^\infty(U; \mathbb{R}^d)$

$$\begin{aligned} E[1_A \langle X_t, \varphi' \rangle] &= \lim_{k \rightarrow \infty} E[1_A \langle X_t^{n(k)}, \varphi' \rangle] \\ &= - \lim_{k \rightarrow \infty} E[1_A \langle \frac{\partial}{\partial x} X_t^{n(k)}, \varphi \rangle] \\ &= - E[1_A \langle X_t, \varphi \rangle]. \end{aligned}$$

Hence for any $\varphi \in C_0^\infty(U; \mathbb{R}^d)$,

$$E[1_A \langle X_t, \varphi' \rangle] = -E[1_A \langle X_t, \varphi \rangle] \quad P\text{-a.s.} \quad (4.37)$$

We now construct a measurable set $\Omega_0 \subset \Omega$ of full measure such that X_t has a weak derivative $\frac{\partial}{\partial x} X_t^x$ on this subset. Let $\{\varphi_n\}$ be a sequence in $C_0^\infty(U; \mathbb{R}^d)$ dense in $W_0^{1,2}(U; \mathbb{R}^d)$. Replace φ by φ_n and choose a measurable subset Ω_n of Ω with full measure such that (4.37) holds on Ω_n . Set $\Omega_0 = \bigcap_{n \geq 1} \Omega_n$, then Ω_0 satisfies the required property. \square

We get the following result for a weighted Sobolev space.

Lemma 4.16. *For all $p \in (1, \infty)$, we have*

$$X_t \in L^2(\Omega, W^{1,p}(\mathbb{R}^d; \mathbf{p}(x)dx)).$$

Proof. Without loss of generality, we assume that $d = 1$. We first show that

$$E \left[\left(\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x} X_t^x \right|^p \mathbf{p}(x) dx \right)^{2/p} \right] < \infty. \quad (4.38)$$

We prove this by successive conditioning on the filtration generated by the Brownian motion and successive use of the flow property and Lemma 2.7. Let $X_t^{n,x}$ be the sequence as defined in Lemma 4.15.

Choose once more δ_0 as in Lemma 2.7 and set $\tau := \frac{\delta_0}{64\sqrt{2dk^2}}$, $s_i = i\tau$, $x_i := X_{s_i}^{n,0,x}$, $\frac{\partial}{\partial x_i} X_{s_{i+1}}^{n,s_i,x_i} := \frac{\partial}{\partial x} X_{s_{i+1}}^{n,s_i,x} \Big|_{x=x_i}$, $i \geq 1$. There exists m positive integer such that $s_m \leq t < s_{m+1}$.

For $p \geq 2$, using Hölder inequality with respect to the measure P and Fubini Theorem, we have

$$\begin{aligned} E \left[\left(\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p \mathbf{p}(x) dx \right)^{2/p} \right] &\leq \left(E \left[\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p \mathbf{p}(x) dx \right] \right)^{2/p} \\ &\leq \left(\int_{\mathbb{R}^d} E \left[\left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p \mathbf{p}(x) \right] dx \right)^{2/p}. \end{aligned}$$

Using the flow property, the chain rule and Proposition 4.11, we have

$$\begin{aligned}
 E\left[\left|\frac{\partial}{\partial x}X_t^{n,0,x}\right|^p\right] &= E\left(E\left[\left|\frac{\partial}{\partial x_m}X_t^{n,s_m,x_m} \cdot \frac{\partial}{\partial x_{m-1}}X_{s_m}^{n,s_{m-1},x_{m-1}} \dots \frac{\partial}{\partial x}X_{s_1}^{n,0,x}\right|^p \middle| \mathcal{F}_{s_m}\right]\right) \\
 &= E\left(E\left[\left|\frac{\partial}{\partial x_m}X_t^{n,s_m,x_m}\right|^p \middle| \mathcal{F}_{s_m}\right] \cdot \left|\frac{\partial}{\partial x_{m-1}}X_{s_m}^{n,s_{m-1},x_{m-1}} \dots \frac{\partial}{\partial x}X_{s_1}^{n,0,x}\right|^p\right) \\
 &\leq CE\left(e^{6k^2\tau|x_m|^2}\left|\frac{\partial}{\partial x_{m-1}}X_{s_m}^{n,s_{m-1},x_{m-1}} \dots \frac{\partial}{\partial x}X_{s_1}^{n,0,x}\right|^p\right) \\
 &\leq C\left(Ee^{\delta_0|x_m|^2}\right)^{1/2}\left[E\left\{\left|\frac{\partial}{\partial x_{m-1}}X_{s_m}^{n,s_{m-1},x_{m-1}} \dots \frac{\partial}{\partial x}X_{s_1}^{n,0,x}\right|^{2p}\right\}\right]^{1/2} \\
 &\leq C\left(C_1e^{C_2\delta_0|x|^2}\right)^{1/2}\left[E\left\{\left|\frac{\partial}{\partial x_{m-1}}X_{s_m}^{n,s_{m-1},x_{m-1}} \dots \frac{\partial}{\partial x}X_{s_1}^{n,0,x}\right|^{2p}\right\}\right]^{1/2}, \quad (4.39)
 \end{aligned}$$

where the last inequality comes from Lemma 2.7 with C_1, C_2 positive constants independent of x , but may depend on k . Successive application of the previous step on the second term of the right side of (4.39) gives

$$\begin{aligned}
 E\left[\left|\frac{\partial}{\partial x}X_t^{n,0,x}\right|^p\right] &\leq C_{m,k,\delta_0}\left(e^{C_2\delta_0|x|^2}\right)^{1/2}\left(e^{C_2\delta_0|x|^2}\right)^{1/4} \dots \left(e^{C_2\delta_0|x|^2}\right)^{1/2m}\left[E\left\{\left|\frac{\partial}{\partial x}X_{s_1}^{n,0,x}\right|^{2mp}\right\}\right]^{1/2m} \\
 &\leq C_{m,k,\delta_0}\left(e^{C_2\delta_0|x|^2}\right)^{1/2}\left(e^{C_2\delta_0|x|^2}\right)^{1/4} \dots \left(e^{C_2\delta_0|x|^2}\right)^{1/2m}\left(e^{C_2\delta_0|x|^2}\right)^{1/2m} \\
 &\leq C_{m,k,\delta_0}e^{C'_{m,k,\delta_0}|x|^2}. \quad (4.40)
 \end{aligned}$$

Thus

$$E\left[\left(\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x}X_t^{n,x}\right|^p\mathbf{p}(x)dx\right)^{2/p}\right] \leq C_{m,k,p,\delta_0}\left(\int_{\mathbb{R}^d}e^{C'_{m,k,\delta_0}|x|^2}\mathbf{p}(x)dx\right)^{2/p}.$$

It follows that (4.38) is satisfied for $p \geq 2$.

Now, choose $1 \leq p \leq 2$, it follows from the Hölder's inequality with respect to the measure $\mathbf{p}(x)dx$ that

$$E\left[\left(\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x}X_t^{n,x}\right|^p\mathbf{p}(x)dx\right)^{2/p}\right] \leq \left(\int_{\mathbb{R}^d}\mathbf{p}(x)dx\right)^{\frac{2-p}{2}}\left(E\left[\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x}X_t^{n,x}\right|^p\mathbf{p}(x)dx\right]\right).$$

From this, we get that (4.38) holds for $1 \leq p \leq 2$.

Independently of the choice of p , there exists a subsequence converging to an object $Y \in L^2(\Omega, L^q(\mathbb{R}^d; \mathbf{p}(x)dx))$ in the weak topology. More specifically, for every $A \in \mathcal{F}$ and $f \in L^q(\mathbb{R}^d; \mathbf{p}(x)dx)$ (with q such that $\frac{1}{p} + \frac{1}{q} = 1$), such that $f\mathbf{p} \in L^q(\mathbb{R}^d; dx)$, we have

$$\lim_{n \leftarrow \infty} E\left[1_A \int_{\mathbb{R}^d} \frac{\partial}{\partial x}X_t^{n(k),x} f(x)\mathbf{p}(x)dx\right] = E\left[1_A \int_{\mathbb{R}^d} Y(x)f(x)\mathbf{p}(x)dx\right].$$

Hence Y must be equal to the weak derivative of X_t^x and the results follows. \square

Proof of Theorem 3.4. Let $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \mapsto \phi_{s,t}(x) \in \mathbb{R}^d$ be the continuous version of the map $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \mapsto X_t^{s,x}$ given in Proposition 4.14. Denote by Ω^* the set of all $\omega \in \Omega$ for which there exists a unique spatially Sobolev differentiable family of solutions to equation (3.5). Since (Ω, \mathcal{F}, P) is a complete probability space, we get that $\Omega^* \in \mathcal{F}$ and $P(\Omega^*) = 1$. Moreover, the uniqueness of solutions to the SDE (3.5), implies the following two-parameter group property

$$\phi_{s,t}(\cdot, \omega) = \phi_{u,t}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega), \quad \phi_{s,s}(\cdot, \omega) = x, \quad (4.41)$$

is satisfied for all $s, u, t \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and all $\omega \in \Omega^*$. In fact, without loss of generality, we can assume that $u < s < t$ and for $s, t \in \mathbb{R}$, there exists an integer m such that $s_m < t - s \leq s_{m+1}$ and one can verify that the flow property (4.41) holds in this case. Repeated use of (4.41) in small intervals of length τ and the uniqueness of the solution gives the above two-parameter group property for all $s, u, t \in \mathbb{R}$.

The proof of the theorem is completed by applying Lemma 4.16 and using the relation $\phi_{s,t}(\cdot, \omega) = \phi_{t,s}^{-1}(\cdot, \omega)$. \square

Proof of Corollary 3.5. Let Ω^* be as in the proof of Theorem 3.4. We will show that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ and let $\omega \in \Omega^*$. The relation (3.6) yields

$$X_{t+t_1}^{t_1, x}(\omega) = x + \int_{t_1}^{t+t_1} b(X_r^{t_1, x}(\omega)) dr + B_{t+t_1}(\omega) - B_{t_1}(\omega), \quad t_1, t \in \mathbb{R}. \quad (4.42)$$

Using the helix property of B and a change of variable we get from (4.42)

$$X_{t+t_1}^{t_1, x}(\omega) = x + \int_0^t b(X_{r+t_1}^{t_1, x}(\omega)) dr + B_t(\theta(t_1, \omega)), \quad t \in \mathbb{R}. \quad (4.43)$$

Substituting ω by $\theta(t_1, \omega)$, the relation (4.43) suggests that the SDE (3.5) has a Sobolev differentiable family of solutions. Therefore, $\theta(t_1, \omega) \in \Omega^*$ and hence $\theta(t, \cdot)(\Omega^*) \subseteq \Omega^*$. Since this holds for arbitrary $t_1 \in \mathbb{R}$, we have $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$.

Moreover, the uniqueness of solutions of the integral equation (4.42) gives

$$X_{t+t_1}^{t_1, x}(\omega) = X_{t_2}^{0, x}(\theta(t_1, \omega)) \quad (4.44)$$

for all $t_1, t_2 \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and all $\omega \in \Omega^*$.

To prove the perfect cocycle property (3.7), note that (4.44) can be rewritten as

$$\phi_{t_1, t+t_1}(x, \omega) = \phi_{0, t_2}(x, \theta(t_1, \omega)), \quad t_1, t_2 \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega^*. \quad (4.45)$$

The perfect cocycle property (3.7) now follows by replacing x in the relation (4.45) by $\phi_{0, t_1}(x, \omega)$ and using the two parameter flow property (4.41). \square

5. APPLICATION TO STOCHASTIC DELAY DIFFERENTIAL EQUATION

In this section we consider the following stochastic delay differential equation

$$\begin{cases} dX(t) = b(X(t-r), X(t, 0, (v, \eta))) dt + dB(t), & t \geq 0 \\ (X(0), X_0) = (v, \eta) \in M_2 := \mathbb{R}^d \times L^2([-r, 0], \mathbb{R}^d) \end{cases} \quad (5.1)$$

Theorem 5.1. *Suppose that the drift coefficient $b : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ in the SDE (5.1) is a Borel-measurable function bounded in the first argument and has linear growth in the second argument. Then there exists a unique global strong solution X to the SDE (5.1) adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. Furthermore, the solution X_t is Malliavin differentiable for all $0 \leq t \leq T$.*

The proof of Theorem 5.1 also uses the relative compactness criteria. Let consider a sequence $b_n : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n \geq 1$ of smooth coefficients with compact support such that $b_n(v_1, v_2)$ is Borel measurable, bounded in $v_1 \in \mathbb{R}^d$ and has linear growth in $v_2 \in \mathbb{R}^d$. We will view the semiflow $\tilde{X}^n(t, 0, (v, \eta)) := (X^n(t), X_t^n)$ as a process with values in the Hilbert state space M_2 in order to use Malliavin calculus. The symbol $X_t^n \in L^2([-r, 0], \mathbb{R}^d)$ stands for the segment $X_t^n(s) := X^n(t+s)$, $s \in [-r, 0]$ and $r > 0$ is the delay. For brevity, we will often denote $z := (v, \eta) \in M_2$ and by $p_1 : M_2 \rightarrow \mathbb{R}^d$, $p_2 : M_2 \rightarrow L^2([-r, 0], \mathbb{R}^d)$ the natural projections onto \mathbb{R}^d and $L^2([-r, 0], \mathbb{R}^d)$. For any $t_1 > 0$ denote by $X^n(\cdot, t_1, z)$ the solution of the following approximating SDDE starting at t_1 :

$$X^n(t, t_1, z) := \begin{cases} p_1(z) + \int_{t_1}^t b_n(X^n(u-r, t_1, z), X^n(u, t_1, z)) du + B(t) - B(t_1) & \text{for } t_1 \leq t, \\ p_2(z)(t-t_1) & \text{for } t \in [t_1-r, t_1]. \end{cases} \quad (5.2)$$

By the continuation property, we have

$$X^n(t, 0, (v, \eta)) := X^n(t, t_1, \tilde{X}^n(t_1, 0, (v, \eta))) \quad t \geq t_1, (v, \eta) \in M_2. \quad (5.3)$$

Apply the chain rule (for Malliavin derivatives) to the above relation and get:

$$D_s X^n(t, 0, (v, \eta)) = D_s X^n(t, t_1, z) \Big|_{z=\tilde{X}^n(t_1, 0, (v, \eta))} + D X^n(t, t_1, z) \Big|_{z=\tilde{X}^n(t_1, 0, (v, \eta))} \cdot D_s \tilde{X}^n(t_1, 0, (v, \eta)) \quad (5.4)$$

for $0 < s < t$ and each deterministic initial data $(v, \eta) \in M_2$.

Let t_1 be the supremum of all $t \geq r$ such that there exists $C_d > 0$ and the following two estimates hold:

$$\begin{cases} \sup_{n \geq 1} E \|D_s X^n(t, 0, (v, \eta)) - D_{s'} X^n(t, 0, (v, \eta))\|^2 \leq C_d |s - s'| & \text{for } 0 \leq s, s' \leq t, \\ \sup_{n \geq 1} \sup_{0 \leq s \leq t} E \|D_s X^n(t, 0, (v, \eta))\|^2 \leq C_d \end{cases} \quad (5.5)$$

for all deterministic initial data $(v, \eta) \in M_2$.

Claim: We claim that $t_1 = \infty$.

Proof. Use contradiction: Suppose $t_1 < \infty$. We will show by continuation and a conditioning argument that there exists $t_2 \in (t_1, t_1 + r)$ (with $t_2 - t_1$ possibly small) such that the inequalities (5.5) hold for $t = t_2$. This will contradict the choice of t_1 as a supremum.

Consider the following cases:

Case 1: $t_1 \leq s \leq t \leq t_1 + r$:

For deterministic $z \in M_2$, take the Malliavin derivative D_s in (5.2) to obtain

$$D_s X^n(t, t_1, z) = \int_s^t D_2 b_n(p_2 z(u - r), X^n(u, t_1, z)) D_s X^n(u, t_1, z) du + I. \quad (5.6)$$

Using successive iterations in the above integral equation, the Girsanov theorem and integrations by parts, we obtain $t_2 > t_1$ with $t_2 - t_1$ possibly small and a positive constant C_d (independent of $z \in M_2, n \geq 1$) such that

$$\sup_{t_1 \leq s \leq t} E \|D_s X^n(t, t_1, z)\|^2 \leq C_d. \quad (5.7)$$

Next, we use (5.4), conditioning on $z = \tilde{X}^n(t_1, (v, \eta))$, and (5.7) to get the following:

$$E \|D_s X^n(t, 0, (v, \eta))\|^2 = E \left(E \left(\|D_s X^n(t, t_1, z)\|^2 \middle|_{z=\tilde{X}^n(t_1, 0, (v, \eta))} \middle| \tilde{X}^n(t_1, 0, (v, \eta)) \right) \right) \leq C_d \quad (5.8)$$

for $t_1 < s \leq t \leq t_2$.

Therefore, from the above inequality, we get

$$\sup_{n \geq 1} \sup_{t_1 \leq s \leq t} E \|D_s X^n(t, 0, (v, \eta))\|^2 \leq C_d \quad (5.9)$$

for $t_1 \leq t \leq t_2$.

Case 2: $0 \leq s \leq t_1 \leq t \leq t_2$:

Taking Malliavin derivatives D_s for $s \leq t_1$ in the integral equation (5.2), we get

$$D_s X^n(t, t_1, z) = \int_{t_1}^t D_2 b_n(p_2 z(u - r), X^n(u, t_1, z)) D_s X^n(u, t_1, z) du. \quad (5.10)$$

The above linear integral equation implies that

$$D_s X^n(t, t_1, z) = 0, \text{ a.s., } s \leq t_1 \leq t, n \geq 1 \quad (5.11)$$

for any deterministic $z \in M_2$.

As before, we apply the chain rule in (5.4) followed by conditioning with respect to $\tilde{X}^n(t_1, 0, (v, \eta))$ together with the above equality, to get

$$\begin{aligned} E \|D_s X^n(t, 0, (v, \eta))\|^2 &\leq 2E \left(E \left(\|D_s X^n(t, t_1, z)\|^2 \middle|_{z=\tilde{X}^n(t_1, 0, (v, \eta))} \middle| \tilde{X}^n(t_1, 0, (v, \eta)) \right) \right) \\ &\quad + 2E \left(E \left(\|DX^n(t, t_1, z)\|^2 \middle|_{z=\tilde{X}^n(t_1, 0, (v, \eta))} \right) \cdot \|D_s \tilde{X}^n(t_1, 0, (v, \eta))\|^2 \middle| \tilde{X}^n(t_1, 0, (v, \eta)) \right) \\ &= 2E \left(E \left(\|D_s X^n(t, t_1, z)\|^2 \middle|_{z=\tilde{X}^n(t_1, 0, (v, \eta))} \middle| \tilde{X}^n(t_1, 0, (v, \eta)) \right) \right) \\ &= 0 \end{aligned} \quad (5.12)$$

for $s \leq t_1 \leq t \leq t_2$ and all deterministic $(v, \eta) \in M_2$.

Now combine the second estimate in (5.5) (for $t = t_1$) with (5.9) and (5.12) to get

$$\sup_{n \geq 1} \sup_{0 \leq s \leq t} E \|D_s X^n(t, 0, (v, \eta))\|^2 \leq C_d \quad (5.13)$$

for $t_1 < t \leq t_2$ with a positive constant C_d (denoted by the same symbol). This shows that the second estimate in (5.5) still holds for $t_2 \geq t \geq t_1$ and therefore contradicts the maximality of t_1 if $t_1 < \infty$.

It remains to show that the first estimate in (5.5) also holds for $t_1 \leq t \leq t_2$. To do this consider the following cases:

Case 3: $t_1 < s' < s \leq t \leq t_2$:

Using the chain rule in (5.4) gives

$$D_s X^n(t, 0, (v, \eta)) - D_{s'} X^n(t, 0, (v, \eta)) = [D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z)] \Big|_{z=\tilde{X}^n(t_1, 0, (v, \eta))}. \quad (5.14)$$

For any fixed $z \in M_2$, consider the expression

$$\begin{aligned} & D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z) \\ &= \int_{t_1}^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) [D_s X^n(u, t_1, z) - D_{s'} X^n(u, t_1, z)] du. \end{aligned} \quad (5.15)$$

The above linear integral equation implies that

$$[D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z)] = 0 \quad (5.16)$$

for any fixed $z \in M_2$.

Now taking $E(\|\cdot\|^2)$ on both sides of (5.14), conditioning with respect to $X^n(t_1, 0, (v, \eta))$ and using (5.16), gives

$$E \|D_s X^n(t, 0, (v, \eta)) - D_{s'} X^n(t, 0, (v, \eta))\|^2 = 0 \leq C_d |s - s'|, \quad t_1 < s' < s \leq t \leq t_2 \quad (5.17)$$

for any $(v, \eta) \in M_2$.

Case 4: $s' < t_1 < s \leq t \leq t_2$:

Fix any $z \in M_2$ and use (5.6) and (5.10) to get

$$\begin{aligned} & D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z) \\ &= \int_s^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) D_s X^n(u, t_1, z) du + I \\ &\quad - \int_{t_1}^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) D_{s'} X^n(u, t_1, z) du \\ &= \int_s^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) [D_s X^n(u, t_1, z) - D_{s'} X^n(u, t_1, z)] du + I \\ &\quad + \int_s^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) D_{s'} X^n(u, t_1, z) du \\ &\quad - \int_{t_1}^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) D_{s'} X^n(u, t_1, z) du \\ &= I - D_{s'} X^n(s, t_1, z) \\ &\quad + \int_s^t D_2 b_n(p_2 z(u-r), X^n(u, t_1, z)) [D_s X^n(u, t_1, z) - D_{s'} X^n(u, t_1, z)] du. \end{aligned} \quad (5.18)$$

Using successive iterations in the last equality above, the Girsanov Theorem and successive integrations by parts, we obtain $t_2 > t_1$ (with $(t_2 - t_1)$ possibly small) and a positive constant $\tilde{C}_d = \tilde{C}_d(\|\tilde{b}\|_\infty)$, independent of $z \in M_2$ and $n \geq 1$ such that

$$\sup_{n \geq 1} E \|D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z)\|^2 \leq \tilde{C}_d |s - s'| \quad \text{for } s' < t_1 < s < t < t_2. \quad (5.19)$$

Next, we use the above estimate, the chain rule (as in (5.12)), and conditioning on $z = \tilde{X}^n(t_1, (v, \eta))$, to get

$$\begin{aligned} & E \|D_s X^n(t, 0, (v, \eta)) - D_{s'} X^n(t, 0, (v, \eta))\|^2 \\ & \leq 2E \left(E \left(\| [D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z)] \|^2 \middle| \tilde{X}^n(t_1, (v, \eta)) \right) \right) \\ & \leq C_d |s - s'| \text{ for } s' < t_1 < s < t < t_2, \end{aligned} \quad (5.20)$$

where $C_d = 2\tilde{C}_d$.

Case 5: $s' < s < t_1 < t \leq t_2$:

In this case, for fixed $z \in M_2$, we have

$$\begin{aligned} & [D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z)] \\ & = \int_{t_1}^t D_2 b_n(p_2 z(u - r), X^n(u, t_1, z)) [D_s X^n(u, t_1, z) - D_{s'} X^n(u, t_1, z)] du, \end{aligned} \quad (5.21)$$

which implies that

$$[D_s X^n(t, t_1, z) - D_{s'} X^n(t, t_1, z)] = 0, \text{ a.s.} \quad (5.22)$$

Therefore, using similar arguments as before (chain rule and conditioning), we obtain

$$E \|D_s X^n(t, 0, (v, \eta)) - D_{s'} X^n(t, 0, (v, \eta))\|^2 = 0 \text{ for } s' < s < t_1 < t < t_2. \quad (5.23)$$

Finally, putting together the first estimate in (5.5) with $t = t_1$, (5.17), (5.20) and (5.23), it follows that the first estimate in (5.5) holds for $t_1 < t < t_2$. This contradicts the maximal choice of t_1 and completes the proof of the proposition. Thus $t_1 = \infty$. \square

Further applications of the results in Section 3 are given in [19]

APPENDIX A. COMPACTNESS CRITERIA

The proposed construction of the strong solution and the stochastic flow for the SDE (1.1) is based on the following relative compactness criteria from Malliavin calculus due to [4].

Theorem A.1. *Let $\{(\Omega, \mathcal{A}, P); H\}$ be a Gaussian probability space, that is (Ω, \mathcal{A}, P) is a probability space and H a separable closed subspace of Gaussian random variables in $L^2(\Omega)$, which generate the σ -field \mathcal{A} . Denote by \mathbf{D} the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \dots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \dots, h_n) h_i, \quad h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega; H)}.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1} \mathbf{D}G\|_{L^2(\Omega; H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

The relative compactness criteria in our setting required the subsequent result (see [4, Lemma 1]).

Lemma A.2. *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0, 1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0, 1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt dt' \right)^{1/2} \right\}.$$

The next compactness criteria which plays a key role in the proof of our results is a direct consequence of Theorem A.1 and Lemma A.2.

Corollary A.3. *Let $X_n \in \mathbf{D}_{1,2}$, $n = 1, 2, \dots$, be a sequence of \mathcal{F}_1 -measurable random variables such that there exist constants $\alpha > 0$ and $C > 0$ with*

$$\sup_n E [|D_t X_n - D_{t'} X_n|^2] \leq C |t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$, and

$$\sup_n \sup_{0 \leq t \leq 1} E [|D_t X_n|^2] \leq C.$$

Then the sequence X_n , $n = 1, 2, \dots$, is relatively compact in $L^2(\Omega)$.

Let us recall the following result is due to Benes (see [2] and [13, Corollary 5.16]).

Lemma A.4. *Let B_t be a d -dimensional Brownian motion on a probability space (Ω, \mathcal{F}, P) . Let the vector $b = (b_1, \dots, b_d)$ of progressively measurable functional on $C[0, \infty)^d$ satisfy for each $0 \leq T < \infty$ and some $K_T > 0$ depending on T , the condition*

$$|b(t, x)| \leq K_T (1 + \max_{0 \leq s \leq t} |x(s)|)$$

then the process $Z(b(\cdot, B(\cdot)))$ defined by

$$Z(b(t, B(t))) = \exp \left(\sum_{j=1}^d \int_0^T b^{(j)}(u, B_u) dB_u^{(j)} - \frac{1}{2} \sum_{j=1}^d \int_0^T (b^{(j)}(u, B_u))^2 du \right)$$

is a martingale.

APPENDIX B. PROOF OF PROPOSITION 4.3

Here we give the proof of Proposition 4.3. Before we proceed, we need some notations and intermediate results.

Without loss of generality, assume that $\|\tilde{b}_i\|_\infty \leq 1$ for $i = 1, 2, \dots, n$. Let $z = (z^{(1)}, \dots, z^{(d)})$ be a generic element of \mathbb{R}^d and $|\cdot|$ be the Euclidean norm. Let $P(t, z) = (2\pi t)^{-d/2} e^{-|z|^2/2t}$ be the Gaussian kernel, then the left side of (4.2) can be written as

$$\left| \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n \right|.$$

Define

$$J_n^\alpha(t_0, t, z_0) := \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n,$$

with $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^{nd}$. To prove the proposition, it is enough to prove that $t |J_n^\alpha(t_0, t, 0)| \leq C^n (t - t_0)^{n/2} / \Gamma(n/2 + 1)$.

To this end, we shift the derivatives from the b_i 's onto P by using the integration by part. This is done by introducing the alphabet $\mathcal{A}(\alpha) = \{P, D^{\alpha_1} P, \dots, D^{\alpha_n} P, D^{\alpha_1} D^{\alpha_2} P, \dots, D^{\alpha_{n-1}} D^{\alpha_n} P\}$. Here D^{α_i} , $D^{\alpha_i} D^{\alpha_{i+1}}$ stands for the derivative of $P(t, z)$ with respect to the space.

Choose a string $S = S_1 \dots S_n$ in $\mathcal{A}(\alpha)$ and define

$$I_S^\alpha(t_0, t, z_0) := \int_{t_0 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n b_i(t_i, z_i) S_i(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n.$$

In the following, we say that a string is *allowed* if, when all the $D^{\alpha_i} P$'s are taken out from the string, a string of type $P \cdot D^{\alpha_s} D^{\alpha_{s+1}} P \cdot P \cdot D^{\alpha_{s+1}} D^{\alpha_{s+2}} P \dots P \cdot D^{\alpha_r} D^{\alpha_{r+1}} P$ for $s \geq 1$, $r \leq n - 1$

remains. Moreover, assume that the first derivatives $D^{\alpha_i}P$ are written in an increasing order with respect to i .

Lemma B.1. *The following representation holds*

$$J_n^\alpha(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} \epsilon_j I_{S^j}^\alpha(t_0, t, z_0),$$

where each ϵ_j is either -1 or 1 and each S^j is an allowed string in $\mathcal{A}(\alpha)$.

Proof. We use induction on $n \geq 1$. Clearly, the representation holds for $n = 1$. Now assume that it holds for $n \geq 1$, and let b_0 be another function satisfying the assumptions of the proposition. as well as α_0 . Then

$$\begin{aligned} J_{n+1}^{(\alpha_0, \alpha)}(t_0, t, z_0) &= \int_{t_0}^t \int_{\mathbb{R}^d} D^{\alpha_0} b_0(t_1, z_1) P(t_1 - t_0, z_1 - z_0) J_n^\alpha(t_1, t, z_1) dz_1 dt_1 \\ &= - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1, z_1) D^{\alpha_0} P(t_1 - t_0, z_1 - z_0) J_n^\alpha(t_1, t, z_1) dz_1 dt_1 \\ &\quad - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1, z_1) P(t_1 - t_0, z_1 - z_0) D^{\alpha_0} J_n^\alpha(t_1, t, z_1) dz_1 dt_1. \end{aligned}$$

Moreover

$$D^{\alpha_0} I_S^\alpha(t_1, t, z_1) = -I_{\tilde{S}}^{(\alpha_0, \alpha)}(t_1, t, z_1),$$

with

$$\tilde{S} = \begin{cases} D^{\alpha_0} P \cdot S_2 \cdots S_n & \text{if } S = P \cdot S_2 \cdots S_n \\ D^{\alpha_0} D^{\alpha_1} P \cdot S_2 \cdots S_n & \text{if } S = D^{\alpha_1} P \cdot S_2 \cdots S_n. \end{cases}$$

Clearly, \tilde{S} is not an allowed string in $\mathcal{A}(\alpha)$. Using the induction hypothesis $D^{\alpha_0} J_n^\alpha(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{\tilde{S}}^{(\alpha_0, \alpha)}(t_0, t, z_0)$ and we have

$$J_{n+1}^{(\alpha_0, \alpha)} = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{D^{\alpha_0} P \cdot S^j}^{(\alpha_0, \alpha)} + \sum_{j=1}^{2^{n-1}} \epsilon_j I_{P \cdot \tilde{S}^j}.$$

One verifies that both $D^{\alpha_0} P \cdot S^j$ and $P \cdot \tilde{S}^j$ are allowed strings in $\mathcal{A}(\alpha_0, \alpha)$ whenever S^j is an allowed string in $\mathcal{A}(\alpha)$. \square

Assuming that S is a allowed string, we will give an upper bound of I_S^α , thus the proof of Proposition 4.3 will be completed using the above representation.

Lemma B.2. *Let $\phi, h : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable functions satisfying $\frac{|\phi(s, z)|}{1+|z|} \leq e^{-|z|^2/3s}$ and $\|\tilde{h}\|_\infty \leq 1$, with $\tilde{h}(s, y) := \frac{h(s, y)}{1+|y|}$. Let $\alpha, \beta \in \{0, 1\}^d$ be multiindices satisfying $|\alpha| = |\beta| = 1$. Then one can find a universal constant C (independent of ϕ, h, α and β) satisfying*

$$|I| \leq C,$$

where

$$I = \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s, z) h(t, y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt. \quad (\text{B.1})$$

Proof. Let $l, m \in \mathbb{Z}^d$ and define $[l, l+1] := [l^{(1)}, l^{(1)}+1] \times \cdots \times [l^{(d)}, l^{(d)}+1]$ and similarly for $[m, m+1]$. Moreover, define $\phi_l(s, z) := \phi(s, z) 1_{[l, l+1]}(z)$ and $h_m(t, y) := h(t, y) 1_{[m, m+1]}(y)$.

We denote by $I_{l,m}$ the integral defined in (B.1) when ϕ, h are replaced by ϕ_l, h_m . Then $I = \sum_{l,m \in \mathbb{Z}^d} I_{l,m}$. In the following, C denotes a constant that may change from one line to the other.

We rewrite $I_{l,m}$ as:

$$\begin{aligned}
I_{l,m} &:= \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_l(s, z) h_m(t, y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \\
&\leq \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi_l(s, z)|}{1+|z|} (1+|z|) \frac{|h_m(t, y)|}{1+|y|} (1+|y-z|+|z|) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \\
&\leq I_{l,m}^1 + I_{l,m}^2,
\end{aligned} \tag{B.2}$$

where

$$\begin{aligned}
I_{l,m}^1 &:= \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi_l(s, z)|}{1+|z|} (1+|z|) \frac{|h_m(t, y)|}{1+|y|} (1+|y-z|) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt, \\
I_{l,m}^2 &:= \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi_l(s, z)|}{1+|z|} (1+|z|)^2 \frac{|h_m(t, y)|}{1+|y|} D^\alpha D^\beta P(t-s, y-z) dy dz ds dt.
\end{aligned}$$

We only give an estimate of $I_{l,m}^1$. The corresponding estimate for $I_{l,m}^2$ follows similarly.

suppose that $\|l-m\|_\infty := \max_i |l^{(i)} - m^{(i)}| \geq 2$. For $z \in [l, l+1)$ and $y \in [m, m+1)$ we have $|z-y| \geq \|l-m\|_\infty - 1$.

Assume $\alpha \neq \beta$, then

$$\begin{aligned}
|(1+|y-z|) D^\alpha D^\beta P(t-s, z-y)| &= \left| \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t-s)^2} \right| (1+|y-z|) P(t-s, y-z) \\
&\leq C \left| \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t-s)^2} \right| \tilde{P}(t-s, y-z)
\end{aligned}$$

for appropriate choice of i, j , where $\tilde{P}(t, z) = (2\pi t)^{-d/2} e^{-|z|^2/4t}$. Then there exists a positive constant C such that

$$|(1+|y-z|) D^\alpha D^\beta P(t-s, z-y)| \leq C e^{-(\|l-m\|_\infty - 2)^2/8}.$$

Assume $\alpha = \beta$, then

$$(1+|y-z|) (D^\alpha)^2 P(t-s, y-z) = \left(\frac{(y^{(i)} - z^{(i)})^2}{t-s} - 1 \right) (1+|y-z|) \frac{P(t-s, y-z)}{t-s}$$

and similarly there exists a positive constant C such that

$$|(1+|y-z|) (D^\alpha)^2 P(t-s, y-z)| \leq C e^{-(\|l-m\|_\infty - 2)^2/8}.$$

Using once more the fact that $(1+|z|)e^{-|z|^2/3s}$ is bounded by $Ce^{-|z|^2/6s}$ for $s \in [0, 1]$, we get in both cases $|I_{l,m}^1| \leq C e^{-\|l\|^2/16} e^{-(\|l-m\|_\infty - 2)^2/8}$; thus

$$\sum_{\|l-m\|_\infty \geq 2} |I_{l,m}^1| \leq C.$$

Suppose that $\|l-m\|_\infty \leq 1$ and denote by $\hat{\phi}_l(s, u)$ and $\hat{h}_m(t, u)$ the Fourier transform of ϕ and h in the second variable. Then

$$\hat{h}_m(t, u) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} h_m(t, x) e^{-i(u, x)} dx,$$

and similarly for $\hat{\phi}_l(s, u)$. It follows from the Plancherel theorem that

$$\begin{aligned} \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 du &= \int_{\mathbb{R}^d} \phi_l(s, z)^2 dz \\ &= \int_{\mathbb{R}^d} \frac{\phi_l(s, z)^2}{(1 + |z|)^2} (1 + |z|)^2 dz \\ &\leq \int_{\mathbb{R}^d} e^{-2|z|^2/3s} (1 + |z|)^2 1_{[l, l+1)}(z) dz \\ &\leq C \int_{\mathbb{R}^d} e^{-|z|^2/6s} 1_{[l, l+1)}(z) dz \\ &\leq C e^{-\|l\|^2/24} \end{aligned}$$

for all $s \in [0, 1]$ and

$$\int_{\mathbb{R}^d} \hat{h}_m(t, u)^2 du = \int_{\mathbb{R}^d} h_m(t, y)^2 dy.$$

We have

$$I_{l,m} = \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u) \hat{h}_m(t, -u) u^{(i)} u^{(j)} (t-s) e^{-(t-s)|u|^2/2} du ds dt. \quad (\text{B.3})$$

This can be seen by applying the Fubini's theorem to the right side of (B.3) to get

$$\begin{aligned} &\int_{\mathbb{R}^d} \hat{h}_m(t, -u) \hat{\phi}_l(s, u) u^i u^j (t-s) e^{-(t-s)|u|^2/2} du \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, x) e^{i(u,x)} \phi_l(s, y) e^{-i(u,y)} u^i u^j (t-s) e^{-(t-s)|u|^2/2} du dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, x) \phi_l(s, y) (t-s) \left[(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(u,x-y)} u^i u^j e^{-(t-s)|u|^2/2} du \right] dx dy. \end{aligned}$$

Consider the expression in the square brackets. Substituting $v = \sqrt{t-s}u$, we get

$$\begin{aligned} &(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(u,x-y)} u^i u^j e^{-(t-s)|u|^2/2} du \\ &= (2\pi)^{-d} (t-s)^{-d/2} \int_{\mathbb{R}^d} e^{i(\frac{v}{\sqrt{t-s}}, x-y)} \frac{v^i}{\sqrt{t-s}} \frac{v^j}{\sqrt{t-s}} e^{-|v|^2/2} dv \\ &= (2\pi)^{-d} (t-s)^{-d/2} (t-s)^{-1} \int_{\mathbb{R}^d} e^{i(v, \frac{x-y}{\sqrt{t-s}})} v^i v^j e^{-|v|^2/2} dv. \end{aligned}$$

Set $f(v) = e^{-|v|^2/2}$ and $p(v) = v^{(i)} v^{(j)}$. The properties of the Fourier transform yield $\widehat{pf} = D^\alpha D^\beta \hat{f}$ and $\hat{f} = f$. Hence, the above expression can be written by

$$(2\pi)^{-d/2} (t-s)^{-d/2} (t-s)^{-1} D^\alpha D^\beta f \left(\frac{x-y}{\sqrt{t-s}} \right) = (t-s)^{-1} D^\alpha D^\beta P(t-s, x-y),$$

from which we obtain (B.3).

Using the inequality $ab \leq \frac{1}{2}a^2c + \frac{1}{2}b^2c^{-1}$ with $a = \hat{\phi}_l(s, u)u^{(i)}$, $b = \hat{h}_m(t, -u)u^{(j)}$ and $c = e^{\|l\|^2/48}$, we have

$$\begin{aligned} |I_{l,m}| &\leq \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 (u^{(i)})^2 e^{\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \\ &\quad + \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 (u^{(j)})^2 (t-s)^2 e^{-\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \\ &\leq \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 |u|^2 e^{\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \\ &\quad + \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 |u|^2 (t-s)^2 e^{-\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt. \end{aligned}$$

Integrating the first term of the right side in the above inequality with respect to t gives

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 |u|^2 e^{\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \leq C e^{-\|l\|^2/48}.$$

Now let assume $m = 0$ that is $m^{(i)} = 0$ for $i = 1, \dots, d$ then we have $h_m = h_0$ and

$$\int_{\mathbb{R}^d} \hat{h}_0(t, u)^2 du = \int_{\mathbb{R}^d} h_0(t, y)^2 dy \leq C(1+d).$$

Hence, integrating the second term with respect to s gives

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_0(t, -u)^2 |u|^2 (t-s)^2 e^{-\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \leq C(1+d) e^{-\|l\|^2/48}$$

Let now assume that $\|m\| \neq 0$, then $\|m\| \geq 1$ and we have

$$\begin{aligned} &\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 |u|^2 (t-s)^2 e^{-\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \\ &= \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 (t-s) \left(|u| \sqrt{(t-s)} \right)^2 e^{-\|l\|^2/48} e^{-(t-s)|u|^2/2} du ds dt \\ &\leq C e^{-\|l\|^2/48} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 (t-s) e^{-(t-s)|u|^2/8} du ds dt \\ &\leq C e^{-\|l\|^2/48} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} h_m(t, -u) * h_m(t, -u) \frac{2^d}{(t-s)^{d/2}} e^{-2|u|^2/(t-s)} du ds dt \\ &= C e^{-\|l\|^2/48} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, v) h_m(t, v-u) \frac{2^d}{(t-s)^{d/2}} e^{-2|u|^2/(t-s)} dv du ds dt \\ &= C e^{-\|l\|^2/48} \int_{1/2}^1 \int_{t/2}^t \int_{[m, m+1]^d} \int_{[m, m+1]^d} (1+|v|)(1+|u-v|) \frac{2^d}{(t-s)^{d/2}} e^{-2|u|^2/(t-s)} dv du ds dt \\ &\leq C e^{-\|l\|^2/48} (1+\|m+1\|^2) 2^d \int_{1/2}^1 \int_{t/2}^t \frac{1}{(t-s)^{d/2}} e^{-2\|m\|^2/(t-s)} ds dt. \end{aligned}$$

We show that when we integrate first with respect to s , the integral is bounded above. We split the proof in several cases.

- For $d = 1$

$$\begin{aligned} \int_{t/2}^t \frac{1}{(t-s)^{1/2}} e^{-2\|m\|^2/(t-s)} ds &= \left[-2\sqrt{t-s} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t - \int_{t/2}^t \frac{4\|m\|^2}{(t-s)^{3/2}} e^{-2\|m\|^2/(t-s)} ds \\ &\leq \left[-2\sqrt{t-s} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t \leq 2e^{-4\|m\|^2}. \end{aligned}$$

- For $d = 2$, we have

$$\begin{aligned} \int_{t/2}^t \frac{1}{(t-s)} e^{-2\|m\|^2/(t-s)} ds &= \left[\frac{-(t-s)}{2\|m\|^2} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t - \frac{1}{2\|m\|^2} \int_{t/2}^t e^{-2\|m\|^2/(t-s)} ds \\ &\leq \left[\frac{-(t-s)}{2\|m\|^2} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t \leq \frac{1}{4\|m\|^2} e^{-4\|m\|^2}. \end{aligned}$$

- For $d = 3$, we have

$$\begin{aligned} \int_{t/2}^t \frac{1}{(t-s)^{3/2}} e^{-2\|m\|^2/(t-s)} ds &= \int_{t/2}^t \frac{(t-s)^{1/2}}{(t-s)^2} e^{-2\|m\|^2/(t-s)} ds \\ &\leq \int_{t/2}^t \frac{1}{(t-s)^2} e^{-2\|m\|^2/(t-s)} ds \\ &= \left[\frac{-1}{2\|m\|^2} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t \\ &\leq \frac{1}{4\|m\|^2} e^{-4\|m\|^2}. \end{aligned}$$

- For $d = 4$, we have

$$\begin{aligned} \int_{t/2}^t \frac{1}{(t-s)^2} e^{-2\|m\|^2/(t-s)} ds &= \left[\frac{-1}{2\|m\|^2} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t \\ &\leq \frac{1}{4\|m\|^2} e^{-4\|m\|^2}. \end{aligned}$$

- For $d > 4$, we use induction. Let $J_d = \int_{t/2}^t \frac{1}{(t-s)^{d/2}} e^{-2\|m\|^2/(t-s)} ds$. Assume that $J_{d-1} \leq \frac{C}{4\|m\|^2} e^{-4\|m\|^2}$. We will show that $J_d \leq \frac{C}{4\|m\|^2} e^{-4\|m\|^2}$.

$$\begin{aligned} J_d &= \int_{t/2}^t \frac{1}{(t-s)^{d/2-2+2}} e^{-2\|m\|^2/(t-s)} ds \\ &= \left[\frac{-(t-s)^{-d/2+2}}{2\|m\|^2} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t + \frac{(d/2-2)}{2\|m\|^2} \int_{t/2}^t \frac{1}{(t-s)^{d/2-1}} e^{-2\|m\|^2/(t-s)} ds \\ &\leq \left[\frac{-(t-s)^{-d/2+2}}{2\|m\|^2} e^{-2\|m\|^2/(t-s)} \right]_{t/2}^t + (d/2-2) \int_{t/2}^t \frac{1}{(t-s)^{(d-1)/2}} e^{-2\|m\|^2/(t-s)} ds \\ &\leq \frac{C}{4\|m\|^2} e^{-4\|m\|^2}, \end{aligned}$$

which gives

$$\begin{aligned} |I_{l,m}| &\leq C(1 + \|m + 1\|^2) e^{-\|l\|^2/48} e^{-4\|m\|^2} \\ &\leq C(1 + \|m\|^2 + d^2) e^{-\|l\|^2/48} e^{-4\|m\|^2}. \end{aligned}$$

In any cases, we have $|I_{l,m}| \leq C(1 + \|m\|^2 + d^2) e^{-\|l\|^2/48} e^{-4\|m\|^2}$. Thus

$$\sum_{\|l-m\|_\infty \leq 1} |I_{l,m}| \leq C.$$

□

Corollary B.3. *Let $g, h : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable functions satisfying $\|g\|_\infty \leq 1, \|\tilde{h}\|_\infty \leq 1$ where $\tilde{g}(s, y) := \frac{g(s, y)}{1+|y|}$ and $\tilde{h}(s, y) := \frac{h(s, y)}{1+|y|}$ for $(s, y) \in [0, 1] \times \mathbb{R}^d$. Then there is a positive constant C such that*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \right| \leq C$$

and

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) D^\gamma P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) dy dz ds dt \right| \leq C.$$

Furthermore, $\int_{\mathbb{R}^d} P(t, z) dz = 1$ and

$$\int_{\mathbb{R}^d} |D^\alpha P(t, z)| dz \leq C t^{-1/2}, \quad (\text{B.4})$$

$$\int_{\mathbb{R}^d} |D^\alpha D^\beta P(t, z)| dz \leq C t^{-1}. \quad (\text{B.5})$$

Lemma B.4. *We can find an absolute constant C such that for every Borel-measurable functions g and h such that \tilde{h} and \tilde{g} are bounded by 1, and $r \geq 0$*

$$\begin{aligned} & \left| \int_{t_0}^t \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(t_2, z) P(t_2 - t_0, z) h(t_1, y) D^\alpha D^\beta P(t_1 - t_2, y - z) (t - t_1)^r dy dz dt_1 dt_2 \right| \\ & \leq C(1 + r)^{-1} (t - t_0)^{r+1} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{t_0}^t \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(t_2, z) D^\gamma E(t_2 - t_0, z) h(t_1, y) D^\alpha D^\beta P(t_1 - t_2, y - z) (t - t_1)^r dy dz dt_1 dt_2 \right| \\ & \leq C(1 + r)^{-1/2} (t - t_0)^{r+1/2}. \end{aligned}$$

Proof. We first prove the estimate for $t = 1, t_0 = 0$. Using Corollary B.3 it follows that for each $k \geq 0$

$$\left| \int_{2^{-k-1}}^{2^{-k}} \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \leq C(1 - 2^{-k-1})^r 2^{-k}.$$

In fact, put $t' = 2^k t$ and $s' = 2^k s$ and use the fact that $P(at, z) = a^{-d/2} P(t, a^{-1/2} z)$, then substitute $z' = 2^{k/2} z$ and $y' = 2^{k/2} y$. The result follows by sing $h_1(t, y) := \frac{(1-t)^r}{(1-2^{-k-1})^r} h(t, y)$ in Corollary B.3.

Summing the above inequalities over k gives

$$\left| \int_0^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \leq C(1 + r)^{-1}.$$

Using the bound (B.5), we get

$$\begin{aligned} & \int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \\ & \leq \int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(s, z)|}{1 + |z|} (1 + |z|) P(s, z) \frac{|h(t, y)|}{1 + |y|} (1 + |y|) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \\ & \leq 2 \left(\int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |z|^2) P(s, z) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right. \\ & \quad \left. + \int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |z|) P(s, z) (1 + |y - z|) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right) \\ & \leq C \int_0^1 \int_0^{t/2} (t - s)^{-1} (1 - t)^r ds dt \leq C(1 + r)^{-1}, \end{aligned} \quad (\text{B.6})$$

where in the third inequality, we used the following estimates:

- $(1 + |z|) P(s, z) \leq C \tilde{P}(s, z) = C(2\pi s)^{-d/2} e^{-|z|^2/4s}$,
- $(1 + |z|^2) P(s, z) \leq C P_1(s, z) = C(2\pi s)^{-d/2} e^{-|z|^2/8s}$,
- $(1 + |y - z|) D^\alpha D^\beta P(t - s, y - z) \leq C D^\alpha D^\beta \tilde{P}(t - s, y - z)$ and one can also show that $\int_{\mathbb{R}^d} |D^\alpha D^\beta \tilde{P}(t, z)| dz \leq C t^{-1}$.

Combination of these bounds gives the first assertion for $t = 1, t_0 = 0$. For general t and t_0 use the change of variables $t'_1 = \frac{t_1 - t_0}{t - t_0}$, $t_2 = \frac{t_2 - t_0}{t - t_0}$, $y' = (t - t_0)^{-1/2}y$ and $z' = (t - t_0)^{-1/2}z$.

The second assertion of the lemma follows similarly. \square

We are now ready to give a proof of Proposition 4.3.

Proof of Proposition 4.3. : We prove that there is a constant M such that for each allowed string S in the alphabet $\mathcal{A}(\alpha)$ we have

$$I_S^\alpha(t_0, t, z_0) \leq \frac{M^n(t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

We use induction on n . The case $n = 0$ is straightforward. Assume $n > 0$ and that the above inequality is valid for all allowed strings of length less than n . There are three possibilities.

- (1) $S = D^{\alpha_1}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_2, \dots, \alpha_n)$
- (2) $S = P \cdot D^{\alpha_1}D^{\alpha_2}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_3, \dots, \alpha_n)$
- (3) $S = P \cdot D^{\alpha_1}P \dots D^{\alpha_m}P \cdot D^{\alpha_{m+1}}D^{\alpha_{m+2}}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_{m+3}, \dots, \alpha_n)$.

In each possibility, S' is an allowed string in the given alphabet.

- (1) Use the inductive hypothesis to get an upper bound for $I_{S'}^{\alpha'}(t_1, t, z_1)$ and using (B.4), we have

$$\begin{aligned} |I_S^\alpha(t_0, t, z_0)| &= \left| \int_{t_0}^t \int_{\mathbb{R}^d} b_1(t_1, z_1) D^{\alpha_1}P(t_1 - t_0, z_1 - z_0) I_{S'}^{\alpha'}(t_1, t, z_1) dz_1 dt_1 \right| \\ &\leq \int_{t_0}^t \int_{\mathbb{R}^d} \frac{|b_1(t_1, z_1)|}{1 + |z_1|} (1 + |z_1|) D^{\alpha_1}P(t_1 - t_0, z_1 - z_0) I_{S'}^{\alpha'}(t_1, t, z_1) dz_1 dt_1 \\ &\leq \frac{M^{n-1}}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} \int_{\mathbb{R}^d} (1 + |z_1|) |D^{\alpha_1}P(t_1 - t_0, z_1 - z_0)| dz_1 dt_1 \\ &\leq \frac{M^{n-1}}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} \left(\int_{\mathbb{R}^d} (1 + |z_0|) |D^{\alpha_1}P(t_1 - t_0, z_1 - z_0)| dz_1 dt_1 \right. \\ &\quad \left. + \int_{\mathbb{R}^d} (1 + |z_1 - z_0|) |D^{\alpha_1}P(t_1 - t_0, z_1 - z_0)| dz_1 \right) dt_1 \\ &\leq \frac{M^{n-1}C}{\Gamma(\frac{n+1}{2})} (1 + |z_0|) \int_{t_0}^t (t - t_1)^{(n-1)/2} (t_1 - t_0)^{-1/2} dt_1 \\ &= \frac{M^{n-1}C\sqrt{\pi}(t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)} (1 + |z_0|), \end{aligned}$$

where the last inequality is similar to (B.4) and follows as in (B.6). The result follows if M is large enough.

- (2) Here, write

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0}^t \int_{t_1}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b_1(t_1, z_1) b_2(t_2, z_2) \\ &\quad \times P(t_1 - t_0, z_1 - z_0) D^{\alpha_1}D^{\alpha_2}P(t_2 - t_1, z_2 - z_1) I_{S'}^{\alpha'}(t_2, t, z_2) dz_1 dz_2 dt_2 dt_1. \end{aligned}$$

Define $h(t_2, z_2) := b_2(t_2, z_2) I_{S'}^{\alpha'}(t_2, t, z_2) (t - t_2)^{1-n/2}$. Hence it follows from the inductive hypothesis that

$$\frac{|h(t_2, z_2)|}{1 + |z_2|} \leq M^{n-2}/\Gamma(n/2).$$

Use the above bound in the first part of Lemma B.4 with $g = b_1$ and integrating first with respect to t_2 , we get

$$|I_S^\alpha(t_0, t, z_0)| \leq \frac{CM^{n-2}(t - t_0)^{n/2}}{n\Gamma(n/2)}$$

and the result follows if M is large enough.

(3) We have

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0 < \dots < t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^{m+2} b_j(t_j, z_j) \\ &\quad \times \prod_{j=2}^{m+1} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) \\ &\quad \times I_{S'}^{\alpha'}(t_{m+2}, t, z_{m+2}) dz_1 \dots dz_{m+2} dt_1 \dots dt_{m+2}. \end{aligned}$$

Define $h(t_{m+2}, z_{m+2}) = b_{m+2}(t_{m+2}, z_{m+2}) I_{S'}^{\alpha'}(t_{m+2}, t, z)(t - t_{m+2})^{(2+m-n)/2}$. It follows from the induction hypothesis that

$$\frac{|h(t_{m+2}, z_{m+2})|}{1 + |z_{m+2}|} \leq CM^{n-m-2} / \Gamma((n-m)/2).$$

Let

$$\begin{aligned} \Omega(t_m, z_m) &:= \int_{t_m}^t \int_{t_{m+1}}^t \int_{\mathbb{R}^{2d}} b_{m+1}(t_{m+1}, z_{m+1}) h(t_{m+2}, z_{m+2}) \\ &\quad \times (t - t_{m+2})^{(n-m-2)/2} D^{\alpha_m} P(t_{m+1} - t_m, z_{m+1} - z) \\ &\quad \times D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) dz_{m+1} dz_{m+2} dt_{m+1} dt_{m+2}. \end{aligned}$$

Using Lemma B.4, we have

$$|\Omega(t_m, z_m)| \leq \frac{C(n-m)^{-1/2} M^{n-m-2} (t - t_m)^{(n-m-1)/2}}{\Gamma(\frac{n-m}{2})}.$$

Using the above bound in

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0 < \dots < t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^m b_j(t_j, z_j) \\ &\quad \times \prod_{j=1}^{m-1} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) \Omega(t_m, z_m) dz_1 \dots dz_m dt_1 \dots dt_m, \end{aligned}$$

and using (B.4) repeatedly, we get

$$\begin{aligned} |I_S^\alpha(t_0, t, z_0)| &\leq C^{m+1} (n-m)^{-1/2} \frac{M^{n-m-2}}{\Gamma((n-m)/2)} \\ &\quad \times \int_{t_0 < \dots < t_m < t} (t_2 - t_1)^{-1/2} \dots (t_m - t_{m-1})^{-1/2} (t - t_m)^{(n-m-1)/2} dt_1 \dots dt_m \\ &= C^{m+1} (n-m)^{-1/2} \frac{M^{n-m-2} \pi^{(m-1)/2} \Gamma(\frac{n-m+1}{2})}{\Gamma(\frac{n-m}{2}) \Gamma(\frac{n}{2} + 1)} (t - t_0)^{n/2} \end{aligned}$$

and the result follows when M is large enough, thus completing the induction argument. The proof of Proposition 4.3 is now completed. \square

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