R&D policy in the economy with structural change and heterogeneous spillovers. Online Appendices

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A Households

Households are modelled in a standard way. The amount of labour is constant and distributed across the range of final sectors, which are in existence.

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Assume zero depreciation rate of capital for simplicity. Positive depreciation will not essentially change the results of the paper. Denote consumption expenditures by E:

$$E = \int_{N_{min}}^{N_{max}} P_i C_i di \ , \tag{A.1}$$

along the same range of existing sectors to condense notation.

Consumption of the individual good i is given by

$$C_i = E \frac{P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} \ . \tag{A.2}$$

with E being total consumption expenditures.

The standard Euler equation implies that the optimal growth rate for expenditures is given by

$$\frac{\dot{E}}{E} = (r - T) - \rho \ , \tag{A.3}$$

B Goods Producers

Goods producers employ labour and buy technology from the R&D sector. With these inputs they produce the goods which they sell to the consumer. Output of good i is given by:

$$\forall i \in [N_{min}(t); N_{max}(t)] : \ Y_i = A_i^{\alpha} L_i \ , \tag{B.1}$$

The only use for output of all goods i is consumption, so that $C_i = Y_i$. The only product used for investments is financial capital a which is excluded from this spectrum.

Firm i, therefore, sets its price to

$$P_i = \frac{\varepsilon}{\varepsilon - 1} A_i^{-\alpha} \ . \tag{B.2}$$

This is the price defined only for the products in the range $N_{max} - N_{min}$. All products out of the range $N_{max} - N_{min}$ have a price of zero:

$$P_i = \begin{cases} 0,\ t < \tau_{max}(i), \tau_{max}(i) : \Pi_i = 0, \dot{\Pi}_i > 0, \\ \frac{\varepsilon}{\varepsilon-1} A_i^{-\alpha},\ \tau_{max}(i) < t \leq \tau_{min}(i), \tau_{min}(i) : \Pi_i = 0, \dot{\Pi}_i < 0, \\ 0, t > \tau_{min}(i). \end{cases} \tag{B.3}$$

Here and throughout the paper the following notation is used:

- $\tau_{min} = N_{min}^{-1}(i)$, time when product (technology) i becomes out-dated and profit of manufacturing decreases below zero;
- $\tau_{max} = N_{max}^{-1}(i)$, time when product (technology) i becomes profitable and manufacturing sector starts production of positive amounts;
- $\tau_0 = N^{-1}(i)$, time when technology i is invented through horizontal innovations process.

Inserting (A.2) and (B.2) into (B.1) yields labour demand

$$L_i^D = \frac{\epsilon - 1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} \ . \tag{B.4}$$

Labour employed in sector i is thus a function of the relative productivity of labour in sector i. Repeating the arguments made with respect to the price formation, we get a piecewise-defined labour demand:

$$L^D(i) = \begin{cases} 0,\ t < \tau_{max}(i), \tau_{max}(i) : \Pi_i = 0, \dot{\Pi}_i > 0, \\ \frac{\epsilon-1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj},\ \tau_{max}(i) < t \leq \tau_{min}(i), \tau_{min}(i) : \Pi_i = 0, \dot{\Pi}_i < 0, \\ 0, t > \tau_{min}(i). \end{cases} \tag{B.5}$$

The technology is acquired by the goods producers in the form of a patent and the pricing for this patent follows Nordhaus (1967), Romer (1990) and Grimaud and Rouge (2004). The price of the patent (blueprint) equals the total value of profits which can be derived from it. The manufacturing firm can extract positive profits only for a limited period of time. Thus the patent price is defined as:

$$p_A(i) \stackrel{def}{=} \int\limits_{\tau_{max}}^{\tau_{min}} \mathrm{e}^{-r(t-\tau_0)} \Pi_i dt. \tag{B.6}$$

The date at which patent i starts, τ_{max}, is endogenously determined by the productivity threshold necessary to gain positive profits, while the effective duration of the patent is endogenously determined from the demand for the manufactured product i, by the point in time, τ_{min}, when the final producer can no longer earn positive profits. Thus, the duration of the patent is determined by two zero-profits conditions.

Further, the patent price is independent of time. It only depends on the ratio of productivity in sector i at time points τ_{max}, τ_{min}. This observation directly follows the benchmark model.

C R&D Sector

C.1 Horizontal innovations

The incentive for horizontal innovations is the potential profit from selling the technology to manufacturing firms, as given in the main text:

$$\pi^R(i) = p_A(i) - \frac{1}{2} \int_{\tau_0}^{\tau_{min}} \mathrm{e}^{-r(t-\tau_0)} g^2(i,t) dt, \tag{C.1}$$

Assume that the horizontal R&D firm which invents technology i later develops it through vertical innovations. The two-step sequential optimization is equivalent to the joint optimization in this setup, see Bondarev (2016) for example.

Next observation concerns the free entry of R&D firms:

Lemma C.1. *Under the free entry of firms into R&D sector (no strategic behavior of incumbents) the potential profit for each new technology is constant,* $\pi^R(i) = const.$

Proof. Free-entry implies that any firm may enter the horizontal innovations process at any stage. If some technology i^* has higher potential profit, the potential entrant would wait until this technology would become available for research (N approaches this i^*). But then all potential entrants would do the same for any technology with higher potential profit. This implies in the limit all technologies would have the same potential profit. □

This immediately implies that independently of the form of the cross-technologies interactions the horizontal technologies expansion remains linear as in the benchmark case:

Corollary 1. *The horizontal expansion of technologies' range has constant speed*

$$\dot{N} = \pi^R(N) = const \tag{C.2}$$

Proof. Construction of HJB (Hamilton-Jacobi-Bellman) equation for the problem (C.3)-(C.4):

$$\dot{N} = u(t) \ , \tag{C.3}$$

$$V_N = \max_{u(\bullet)} \int_0^\infty \mathrm{e}^{-rt} \left(\pi^R(i)|_{i=N} u(t) - \frac{1}{2} u^2(t) \right) dt. \tag{C.4}$$

yields optimal investments as a function of $\pi^R(N)$ and marginal expansion value $\frac{\partial V_N}{\partial N}$. It then follows that once $\pi^R(N) = const$ via Lemma C.1, the only value function which satisfies this HJB equation is a constant one, implying $\frac{\partial V_N}{\partial N} = 0$ and thus $\dot{N} = u = \pi^R(N) = const$. □

We then may define the resources available for vertical innovations in the same way as for the benchmark model:

$$\pi^R + \int_{N_{min}(t)}^{N(t)} g(i,t) di = a^D(t);$$

$$G(t) \overset{def}{=} \int_{N_{min}(t)}^{N(t)} g(i,t) di = a(t) - \pi^R \tag{C.5}$$

where $G(t)$ denotes the financial resources available for vertical innovations at time t.

C.2 Vertical innovations

Productivity-improving innovations (vertical innovations) lead to a rise in efficiency of technologies that have zero productivity upon their invention. This productivity can be developed through specific investments for every product.

Profits in R&D results from sales of blueprints to manufacturing firms. These sales come in the form of patents for each new technology i and all of the investments into the development of each new technology (vertical innovations) are financed from this patent payment. Costs of R&D are costs of development of the productivity through technology-specific investments g_i. These investments are financed from household assets a forming part of assets demand a^D in (7):

$$u(t) + \int\limits_{N_{min}(t)}^{N(t)} g(i,t)di = a^D(t) \ , \tag{C.6}$$

We first observe that the productivity of any technology is zero before it is invented and after it is not used in manufacturing:

Lemma C.2. *For all non-operational technologies the individually optimal productivity investments are zero*

$$\forall i \notin [N_{min}(t), N(t)] : \ g^P(i,t) = 0 \tag{C.7}$$

Proof. As soon as $i > N(t)$ the technology is not yet invented and thus there is no associated manufacturing sector demand for it. Thus profit incentive is zero and investments are zero. Once $i < N_{min}(t)$ the manufacturing sector associated with technology i does no longer generate positive profit and again investment incentives are zero. $\square$

Apart from individual investments there exist externalities experienced by each R&D firm from all other existing R&D firms. This impact is described by the interactions operator Θ such that:

$$\dot{A}^{SP}(i,t) = \Theta A(j,t) \stackrel{def}{=} \int_{N_{min}(t)}^{N(t)} \Theta(i,j)A(j,t)dj \tag{C.8}$$

so any technology is subject to potential impact of those technologies present at t. The cross-technologies interactions are state-dependent and their intensity is given by the interactions operator Θ with $\Theta(i,j)$ measuring the impact of technology j on technology i. This operator then is a natural generalization of the commonly employed knowledge spillover, which acts as a main growth driver in the majority of endogenous growth literature (see Peretto and Smulders (2002) for discussion of knowledge spillovers).

However the general operator like (C.8) is difficult to analyze, since it might be the case that already outdated technologies would experience a revival due to new spillovers from emerging technologies. We thus impose the simplifying assumption on the structure of cross-technologies interactions to make sure this is impossible in the model:

Assumption C.1. *For any technology i the range of technologies influencing it is limited:*

$$\forall i \in [0,\infty): \ \Theta(i,j) = \begin{cases} \theta(i,j), \ \textit{if}\, N_{min}(\tau_0(i)) \leq j \leq N(\tau_{min}(i)), \\ 0, \ \textit{otherwise} \end{cases} \tag{C.9}$$

Here $N(\tau_{min}(i))$ denotes the newest invented technology at the time technology i becomes outdated ($\tau_{min}(i)$) and $N_{min}(\tau_0(i))$ denotes the oldest technology at the time technology invented ($\tau_0(i)$). Assumption C.9 thus requires that those technologies, which appear after i is outdated, cannot influence it and those, which are already outdated when

i is just invented cannot impact it too. In this way the introduced assumption seems to be not very much binding.

In fact this is the statement that spillovers are sufficiently *local*: there is a limit on the range of technologies which is influenced by any given technology.

This assumption makes the operator Θ time-invariant: the overall impact on any technology varies over time because of state-dependency, but the operator of these impacts is fixed for any i. We thus may proceed as in Bondarev and Krysiak (2017) with analysis of cross-technologies interactions[1]. The total evolution of productivity for every (invented) technology $i \in [N_{min}, N]$ is thus a sum of controlled individual investments and the externalities impact:

$$\dot{A}^T(i,t) = \dot{A}^P(i,t) + \int_{N_{min}(\tau_0(i))}^{N(\tau_{min}(i))} \theta(i,j)A(j,t)dj \tag{C.10}$$

where we use Assumption C.1 replacing $\dot{A}^{SP}(i,t)$ with a simpler bounded domain operator. Observe that once $j > N(t)$ the productivity for technology j is zero by the Lemma C.2, and thus in an effect the spillover operator takes into account only already invented technologies.

We next assume that in the decentralized economy individual R&D firms cannot track the individual impacts of different technologies on each other (non-atomicity assumption):

Assumption C.2. *For any R&D firm i the impact of cross-technologies interactions is a function of time only, independent of the state of any individual technology:*

$$\int_{N_{min}(\tau_0(i))}^{N(\tau_{min}(i))} \theta(i,j)A(j,t)dj \stackrel{def}{=} \Theta(i,t) : \frac{\partial \Theta(i,t)}{\partial A(j,t)} = 0 \tag{C.11}$$

Using this assumption we can set up the R&D firm problem as a standard optimal control problem in finite time (see Seierstad and Sydsaeter (1999) for example) and apply the Maximum Principle. As a result we obtain optimal investments and productivity evolution for each i as:

$$g^P(i) = \begin{cases} \frac{\partial p_A(i)/\partial A_i}{1+r}(1 - \mathrm{e}^{(1+r)(t-\tau_{min}(i))}), & \text{if } \tau_{min}(i) \geq t \geq \tau_0(i), \\ 0, & \text{otherwise} \end{cases} \tag{C.12}$$

$$\dot{A}^P(i,t) = \begin{cases} \frac{\partial p_A(i)/\partial A_i}{1+r}(1 - \mathrm{e}^{(1+r)(t-\tau_{min}(i))}) - A^P(i,t) + \Theta(i,t), & \text{if } \tau_{min}(i) \geq t \geq \tau_0(i), \\ -A^P(i,t) + \Theta(i,t), & \text{if } t > \tau_{min}(i), \\ 0, & \text{if } t < \tau_0(i) \end{cases} \tag{C.13}$$

with $\Theta(i,t)$ defined by (C.11) and $\partial p_A(i)/\partial A_i$ being the marginal return to the increase in productivity of i in terms of the patent price. Observe that this last is time-invariant (since the patent price itself is time-invariant) for each i, but varies across i. So for mathematical derivations this quantity may be treated as constant in time.

Given this observation the evolution of each technology, (C.13) is a linear non-autonomous differential equation within operational phase $\tau_{min}(i) \leq t \leq \tau_0(i)$. Once $t > \tau_{min}(i)$ productivity investments are zero, but the spillover part may still affect the productivity evolution. By Assumption C.1 the spillover affecting outdated technologies is limited: after $\tau_{min}(N(\tau_{min}(i)))$ (which is the time when the last invented during the development of i technology becomes outdated) it is strictly zero and the technology decays to zero as in the baseline model. During the period $t \in [\tau_{min}(i), \tau_{min}(N(\tau_{min}(i)))]$ the manufacturing firm may still use the technology as it may remain competitive even in the absence of R&D investments. This type of producers we refer to as *imitators* as their behavior resembles one in Acemoglu and Cao (2015).

So every technology potentially exhibits up to three phases of dynamics:

1. Normal R&D development till $\tau_{min}(i)$ when productivity is supported by the associated R&D firm receiving patent payments;

2. Free-rider development may start at any time after $\tau_{min}(i)$ and continues at max till $\tau_{min}(N(\tau_{min}(i)))$ but can stop earlier;

3. Irreversible decay of technology after $\tau_{min}(N(\tau_{min}(i)))$

The second phase is novel and appears due to the presence of cross-technologies interactions Θ in the model. Depending on the structure of these interactions this phase may have different duration and the development of technology may even exceed that during the normal phase.

These firms would stay operational for some positive time as soon as the accumulated profit during this spillover phase is non-negative. However, this will increase the competition at the labor market. We thus assume throughout the main part of the paper that spillovers intensity is bounded in some precise sense:

Definition C.1. *Technology i is normal, if intensity of spillovers it experiences is not high enough for potential free-riders to make positive profit:*

$$\int_{\tau_{min}(i)}^{\tau(N(\tau_{min}(i)))} \mathrm{e}^{r(t-\tau_{min}(i))}\Pi_i dt \leq 0 \tag{C.14}$$

Otherwise technology i is free-riding.

We thus assume

Assumption C.3. *Operator Θ is such that all technologies are normal in the sense of Definition C.1.*

It follows that once we adopt Assumption C.3, every technology has only one operational cycle which corresponds to the time when investments are positive[2].

D Government

As soon as there exist externalities across R&D, characterized by the operator Θ, there might be a need for government interventions. To account for this opportunity we include a government into the model, thus diverging from the benchmark case.

The government is caring only for market failures corrections, thus social planners' problem is to maximize welfare coming from R&D only:

$$J^G = \max_{g(\bullet),u(\bullet)} \int_0^{\infty} \mathrm{e}^{-\rho t} \left\{ \int_{N_{min}}^{N} \mathrm{e}^{-\rho(t-\tau_0(i))} \left\{ b(i) - \frac{1}{2} g^2(i,t) \right\} di - \frac{1}{2} u^2(t) \right\} dt \tag{D.1}$$

with $b(i)$ being the *social value of the technology i.* The focus of this paper is the market inefficiency stemming from cross-technologies interactions so I make a simplifying assumption concerning the social value of technologies, letting $\forall i \in [0,\infty) : \ b(i) = \pi^R(i)$. All the main results would hold for $b(i) \neq \pi^R(i)$ under suitable assumptions on this social welfare wedge (like boundedness) but will unnecessarily overcomplicate the exposition.

The objective (D.1) together with constraints (C.6), (C.10), (C.3) defines the optimization problem for the government which would yield a first-best schedule of technologies' development. Next, once this is different from the market solution, government would need subsidies (positive or negative) to different technologies. If this would be the case, we require government to run a balanced budget[3]:

$$\forall t \in [0, \infty): \; S(t) \overset{def}{=} \int_{N_{min}}^{N} \{s(i,t)di + s(N(t),t)\}\, dt = T(t)a(t) \tag{D.2}$$

where $S(t)$ is defined to be a total sum of subsidies (positive or negative), $T(t)$ is the income tax rate levied on households' assets[4]. Once $S(t) \neq 0$ the government optimization problem has to take (D.2) into account.

The social planner's solution differs from the decentralized one exactly by the impact of each technology on all others. Formally the social planner's problem is maximizing (D.1) subject to (C.3) and (C.10). This constitutes an infinite-dimensional infinite-horizon optimal control problem. As soon as $\tau_{0,min,max}(i)$ are taken as given the problem is equivalent to the one considered in Bondarev (2018). Using Maximum Principle as of Skritek et al. (2011), we may derive socially-optimal horizontal and vertical investments. The time of emergence and profitability of technologies are then obtained as inverse functions of variety expansion:

$$\begin{aligned}
&g^*(i,t) = \psi^*(i,t) = \psi^R(i,t) + \frac{1}{1+r}(1 - \mathrm{e}^{(1+r)(t-\tau^*_{min}(i))}) \int_{N_{min}}^{N_{max}} \Theta(j,i)\psi^*(j,t)dj, \\
&\dot{A}^*(i,t)) = \psi^*(i,t) - A^*(i,t) + \int_{N_{min}}^{N_{max}} \Theta(i,j)A^*(j,t)dj, \\
&u^*(t) = \lambda^*(t), \\
&\dot{N}^*(t) = \dot{N}^*_{min}(t) = \dot{N}^*_{max}(t) = \lambda^*(t), \\
&\tau^*_0(i) = (N^*(t)|_{N=i})^{-1}, \tau^*_{min}(i) = (N^*_{min}(t)|_{N=i})^{-1}, \tau^*_{max}(i) = (N^*_{max}(t)|_{N=i})^{-1} \qquad \text{(D.3)}
\end{aligned}$$

In (D.3) we used the fact that outdating and operational phase entering of technologies are proportional to the emergence of new ones (otherwise the economy would collapse in finite time) and once $b(i) = \pi^R(i)$ the variety expansion is still linear. Then we get

Lemma D.1 (Timing lemma)**.**

Under the assumption $b(i) = \pi^R(i)$ *and free entry the timing for all new technologies coincide under social planner's and market solutions:*

$$\forall i \in [N_0, \infty) : \; \tau_0^*(i) = \tau_0^P(i), \, \tau_{min}^*(i) = \tau_{min}^P(i), \, \tau_{max}^*(i) = \tau_{max}^P(i). \tag{D.4}$$

This provides consistency for government policy.

Proof. Once free entry holds, we get $\pi^R(i) = C$. Then under social planner's regime we still get constant returns to every technology. This grants linear expansion rate and thus equal timing for both economies. □

This lemma is necessary to implement policies, since otherwise the mass of technologies in operational phase/in existence would not coincide and any policy instrument targeted at more than a single technology would be ill-posed.

Once Lemma D.1 holds it is immediate to design the first-best subsidies schedule:

Lemma D.2. *The first-best subsidy to R&D is given by*

$$s(i,t) := \frac{1}{1+r}(1 - \mathrm{e}^{(1+r)(t-\tau_{min}^*(i))}) \int_{N_{min}}^{N_{max}} \Theta(j,i)\psi^*(j,t)dj \tag{D.5}$$

It is feasible as long as Θ *is a compact operator and* (D.2) *holds.*

Proof. Comparing decentralised and centralised solutions (C.13) and (D.3) we observe that investments differ solely by the term $\frac{1}{1+r}(1-\mathrm{e}^{(1+r)(t-\tau_{min}^*(i))}) \int_{N_{min}}^{N_{max}} \Theta(j,i)\psi^*(j,t)dj$. It thus suffices to assign a subsidy at this level to each technology to correct the decentralised solution to the first-best one.

This subsidy is well-defined for each i only if the associated integral equation (Fredholm equation) $\psi^*(i,t) = \frac{\partial p_A(i)/\partial A_i}{1+r}(1-\mathrm{e}^{(1+r)(t-\tau_{min}(i))})+\frac{1}{1+r}(1-\mathrm{e}^{(1+r)(t-\tau_{min}^*(i))}) \int_{N_{min}}^{N_{max}} \Theta(j,i)\psi^*(j,t)dj$

has a solution. This is the case as long as Θ is a compact operator and $\psi^*(i,t)$ is defined for each i.

At last, the budget constraint has to hold for the subsidy to be feasible, i. e. there are sufficient funds to cover all additional expenditures. □

Lemma D.2 states that first-best subsidy schedule exists, but the requirement of compactness is rather strong, since the operator changes its value every time new technology arrives. In the absence of structural change this is almost always the case, since Θ becomes time-invariant. However under structural change as it is understood in this paper, first.best subsidies are not always feasible, as the following analysis demonstrates.

To complete the description of the model we list market clearing conditions, which are exactly the same as in the benchmark model.

E Markets clearing

First, observe that $\dot{E} = 0$, since prices are moving in the opposite direction of productivities, total labor force is constant and labor income is a numeraire. Then the expenditures are a cosntant fraction of the labor income:

$$E = \frac{\epsilon}{\epsilon - 1} L \tag{E.1}$$

implying more or less that the labor income is consumed depending on ϵ value (for $\epsilon > 2$ some capital income is consumed, for $2 > \epsilon > 1$ fraction of labor income is saved).

Next, using $\dot{E} = 0$ and the Euler equation, we can derive the interest rate in equilibrium:

$$\frac{\dot{E}}{E} = r - T - \rho = 0 \to r = T + \rho. \tag{E.2}$$

For the real interest rate to be constant, taxes should be constant in time also, implying the requirement of government dynamic consistency.

Labour market clearing condition is given if the following holds:

$$\int\limits_{N_{min}}^{N} L^D(i,t)di = L = L \int\limits_{N_{min}}^{N} \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N} A_j^{-\alpha(1-\epsilon)}dj} di,$$

$$\int\limits_{N_{min}}^{N} \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N} A_j^{-\alpha(1-\epsilon)}dj} di = \frac{\int\limits_{N_{min}}^{N} A_i^{-\alpha(1-\epsilon)}di}{\int\limits_{N_{min}}^{N} A_j^{-\alpha(1-\epsilon)}dj} = 1. \tag{E.3}$$

But this last condition is automatically satisfied, hence the labour market is cleared.

At last assets are growing in the economy once the initial endowment is sufficiently high and taxes are low enough:

$$\dot{a} = (r-T)a - \frac{1}{\epsilon - 1}L, \tag{E.4}$$

which can be solved to obtain the assets as a function of time,

$$a(t) = \mathrm{e}^{(r-T)t}\left(a_0 - \frac{1}{(\epsilon-1)(r-T)}L\right) + \frac{1}{(r-T)(\epsilon-1)}L. \tag{E.5}$$

Assets accumulation is positive as long as the initial assets of households are sufficiently large:

$$a_0 > \frac{1}{\epsilon-1}\frac{1}{r}L. \tag{E.6}$$

As long as (E.6) holds, assets increase exponentially. Since horizontal investments are constant implying at most linear growth of assets demand from the horizontal R&D we have under assumption of horizontal R&D having the priority[5]

Lemma E.1. *As long as* (E.6) *holds, $T < r$, then*

$$\exists! \tau_{suff} : \forall t > \tau_{suff} : G(t) > 0, \tag{E.7}$$

Proof. Follows from the fact that $a(t)$ grows exponentially and $N(t)$ linearly. There exists at most one intersection point of two monotonically growing functions of such types. Denote it τ_{suff} and the result follows. □

F Definitions of spectral properties

It uses the operator spectral theory to some extent, so for exposition to be self-contained I list here some additional definitions.

Definition F.1. *The spectrum of Θ, denoted $\sigma(\Theta)$ is the set of λ such that for any x $\sigma(\Theta) = \{\lambda : \lambda x = \Theta x\}$.*

The spectral radius of operator Θ, denoted $\rho(\Theta)$ is the maximal absolute size of its spectrum

$$\rho(\Theta) \stackrel{def}{=} \max\{|\lambda|\}. \tag{F.1}$$

In particular for spillover operators defined over the Hilbert space we have $\sigma(\Theta) = \sigma_p \cup \sigma_c \cup \sigma_r$ where subscripts p, c, r denote pointwise, continuous and residual spectral components respectively and $\rho(\Theta) = ||\Theta||_{op}$, spectral radius equals the operator norm of Θ which is its maximal value (see e. g. Kolmogorov and Fomin (1999)).

Definition F.2. *The operator Θ is scalar, if $\forall i \neq j : \theta(i, j) = 0$ and $\forall i : \theta(i, i) = \theta > 0$, i. e. it is a scalar multiple of the identity operator.*

This definition simply resembles that of a scalar matrix but for possibly infinite-dimensional setting.

Definition F.3. *The operator Θ is of scalar type, if it admits the resolution of identity similar to the multiplication operator.*

We thus call Θ scalar-type if it resembles either the infinite-dimensional diagonal matrix, or by proper choice of the eigenbasis may be transformed into such a diagonal matrix (multiplication operator).

The next class limiting finite-dimensional case is the class of compact operators:

Definition F.4. *Operator $\Theta : X \mapsto Y$ is compact if it maps every bounded subset of X to relatively compact subset of Y*

In particular this means for any given technology i the total spillover size is bounded.

Next we use the notion of the nilpotent operator which is an extension of the nilpotent matrix notion:

Definition F.5. *The operator Θ is nilpotent if $\exists n \in \mathbb{N} : \Theta^n = 0$. It is topological nilpotent if $\sigma(\Theta) = 0$.*

Observe that these two notions coincide only for finite-dimensional operators. In general case the nilpotent operator does not need to dissipate. This is an important subclass called spectral class n operators.

At last we follow Dunford (1954) and define

Definition F.6. *The operator Θ is spectral, if it admits the canonical decomposition into the scalar-type and nilpotent parts.*

Observe that not every spectral operator is compact, but every compact operator is spectral.

For results of Section 4 we also need an infinite-dimensional version of Jordan-Chevalley decomposition (JCD) which is for completeness listed here.

Once we consider compact operators, they all have only point-wise spectrum (except zero) and thus we get full correspondence with the finite-dimensional Jordan-Chevalley decomposition (JCD, see Helgason (2001) for example).

Lemma F.1 (Infinite-dimensional JCD).
Any compact spectral operator over Hilbert space admits the canonical decomposition into the semi-simple part and the nilpotent part.

Proof. Follows from definitions of the semi-simple, spectral and nilpotent operators and from the canonical decomposition of the spectral operator in Dunford (1954). □

G Proof of Proposition 2

Proof. As long as Θ is scalar in the sense of Definition F.2 and its spectral radius is bounded by 1, it means it represents the scalar multiple $\theta \leq 1$ and the spillover qualitatively does not change the dynamics for the symmetric model. To see that just consider the symmetric case where $g^P(i,t) = \frac{G(t)}{N-N_{min}}$ following the baseline model and $\Theta(i,t) = \theta A(i,t)$. The long-run growth rate $\bar{g}_A$ is then independent of θ and equals r by direct computations. This implies BGP exists as defined by Definition 1 in the main text.

Once $\rho(\Theta) > 1$, every technology has increasing growth due to spillover. However the growth rate converges to the same value for any technology because of the turnpike property of the governing dynamical system in the same way as in the benchmark case (see e. g. Yano (1984) for details). Direct computation shows that $\bar{g}_A = \theta - 1$ in this case.

Once Θ is not a scalar, it has some heterogeneity and under assumption $\theta(i, j) > 0$ some technologies have competitive advantage and grow faster than others. Then either the economy collapses into one-sector either it experiences explosive growth as follows from Corollary 1. In both cases no BGP exists and sustained growth requires government interventions. □

H Proof of Proposition 3

Proof. As soon as cross-technologies interactions are such that the spillover operator is unitary similar to the multiplication one, it is always possible to redefine 'technologies' in such a way, that they become separated and the R&D system (D.3) admits solution as an infinite-dimensional ODE system. Since in the social optimum the shadow costs of investments for every technology take into account all possible interactions, the resulting solution is welfare- maximizing and as such admits the BGP.

On the other hand, once the operator is not necessarily of the scalar type, but is compact, its spectrum contains only pointwise eigenvalues (discrete spectrum) and the continuous spectrum is restricted to zero (see e. g. Kolmogorov and Fomin (1999)). Thus even if the spillover operator contains the non-zero nilpotent part, this last has at most countable non-zero entries which can be corrected for by appropriate subsidies/taxes (see Bondarev and Krysiak (2017) for details).

Once the operator is not compact, the nilpotent part may contain continuous spectrum components and usual regulation cannot correct for cascading cross-technologies spillovers. Thus the first-best solution will eventually be unstable and the BGP as of Definition 1 would not exist. □

I Proof of Proposition 4

Proof. For all t when $\Theta|_{t \in \mathcal{F}} = \mathbf{S}$, the BGP exists by Proposition 2. Since there are no interactions between technologies not accounted for by individual R&D firms, the welfare theorems grant the optimality of the decentralized solution.

As soon as $\Theta|_{t \in \mathcal{F}} \neq \mathbf{S}$, there exist at least some technologies with interactions not accounted for by market participants. This leads to the non-balanced development of the mass of technologies, whereas some are more efficient then others. Resources are thus concentrated by the market in the most efficient ones, while other technologies start to degrade. Even if at the next time instance it happens that $t \in \mathcal{F}$, the distortion caused on the previous step by the interactions is not smoothed out, since the competitive advantage builds up. Then the economy will slip off the decentralized BGP and cannot return there without government intervention. Thus as soon as $t = t_S$ as defined above, and operator's projection is no longer scalar(-type), the decentralized BGP is destroyed and cannot be recovered without government intervention. $\square$

J Proof of Proposition 5

Proof.

1. Follows from Proposition 3: once Θ remains compact, the socially optimal BGP exists and is feasible. Thus without further restrictions on subsidies it can be implemented.

2. If Θ is not compact, the social BGP does not exist by Proposition 3. The spectral decomposition implies that spectrum of Θ consists exactly of three parts: pointwise, continuous and residual components. By assertion the residual component is empty, thus only pointwise and continuous components have to be considered. Now observe

that (optimal) subsidy is proportional to the eigenvalues of Θ: indeed, these measure the difference between social and private values of the spillover as in Bondarev and Krysiak (2017). Still continuous spectrum components cannot be precisely defined and only approximate eigenvalues may exist for this part of the spectrum (and does not necessarily exist). Thus any government subsidy policy will differ from the potential optimal to the extent of the continuous spectrum. The sustained growth path which is approximated is exactly the one with all the spillovers being internalized and the proximity to it is then defined by the size of the continuous component of the spectrum.

3. If the residual component of the spectrum is not empty, operator becomes not dense in some part of its domain. This means there are sparse spillovers where the source cannot be even approximated. Thus the optimal SGP cannot be approximated either. Still by correcting for the overall effect of cross-technologies spillovers, the government may achieve some SGP (not close to the optimal one) which is an improvement over not following SGP at all (since positive welfare at infinite horizon is always better then the finite-time collapse of the economy).

□

K Proof of Proposition 6

Proof. 1. Under this assertion the operator can be made similar to the multiplication one (diagonal). Thus all the spillovers may be delineated through appropriate choice of the basis (eigenbasis). At the same time all spillovers are inessential in a sense that they do not grant crucial competitive advantage to the associated technology.

Thus the symmetry is fully restores by the proper rearrangement of property rights (basis change).

2. Under this assertion the operator can be made diagonal but some technologies experience high positive spillovers which make then more efficient than others. For each such a technology i_1 the associated eigenvalue measures the intensity of its impact and in the renormalized form of the spillover Θ this exactly corresponds to its competitive advantage. Duration of a subsidy is then given by the time technology stays operational.

3. If Θ contains non-zero nilpotent part, not all impacts may be reassigned through property rights (the full diagonalization is not feasible). Still if the complexity is finite, it corresponds to finite cascades of technologies subject to the spillover and thus given the ongoing structural change, the duration of the subsidy is still finite and given by the timeframe of the cascade of technologies subjected by the impact.

4. At last if simultaneously nilpotent part is nonempty, and its complexity is infinite, there are cascades of spillovers, which are not limited to certain range of technologies, but are persistent. In this case the regulation has to be permanent.

□

Notes

[1]it is of interest to account for interactions operators with unbounded dynamic range (non-local ones), that is, relaxing Assumption C.1. However this rises substantial technical difficulties and involve non-local operators theory, which is still under development, see e. g. Bernardis et al. (2016).

[2]again it would be of interest to allow for free-riding technologies, but this may lead to chaotic and even non-deterministic dynamics because of piecewise-smooth system resulting from (C.13), see e. g. Colombo and Jeffrey (2011) for details.

[3]we neglect the possibility of deficit-run budget for the sake of simplicity, but again this would not influence the main results of the paper

[4]it has been shown by Greiner and Bondarev (2015) that consumption tax actively discussed recently may cause instability of the taxed economy so we limit exposition to the income tax only

[5]if on the contrary, vertical R&D are prioritized, there is no structural change and the economy follows the setup of Peretto and Connolly (2007) with productivity growth only.

References

Acemoglu, D. and Cao, D. (2015) Innovation by entrants and incumbents. *Journal of Economic Theory*, 157(C): 255–294.

Bernardis, A., F. J. Martín-Reyes, P. R. Stinga, and J. L. Torrea (2016) Maximum principles, extension problem and inversion for nonlocal one-side equations. *Journal of Differential Equations*, 260(7): 6333–6362.

Bondarev, A. (2018) Does stronger intellectual property rights protection foster structural change? Effects of heterogeneity in innovations. *Structural Change and Economic Dynamics*, 46: 26–42.

Bondarev, A. (2016) Intensity of R&D competition and the generation ofg innovations in heterogeneous setting. *Journal of Evolutionary Economics*, 26(3): 621–653.

Bondarev, A. and F. Krysiak (2017) Dynamic heterogeneous r&d with cross-technologies interactions. WWZ Working paper 2017/13, University of Basel.

Colombo, A. and Jeffrey, M. R. (2011) Nondeterministic chaos, and the two-fold singularity in piecewise smooth flows *SIAM Journal on Applied Dynamical Systems*, 10(2): 423–451.

Dunford, N. (1954) Spectral operators. *Pacific Journal of Mathematics*, 4(3): 321–354.

Greiner, A. and Bondarev, A. (2015) Public debt and aggregate stability with endogenous growth and a state-dependent consumption tax. *Economics Bulletin*, 35(4): 2601–2609.

Griamud, A. and Rouge, L. (2004) Polluting non-renewable resources, tradeable permits and endogenous growth. *International Journal of Global Environmental Issues*, 4(1):38–57.

Helgason, S. (2001) *Differential Geometry, Lie Groups, and Symmetric Spaces.* Providence: American Mathematical Society.

Kolmogorov, A. and S. Fomin (1999) *Elements of the Theory of Functions and Functional Analysis.* Dover: Dover Publications; Dover Books on Mathematics edition.

Nordhaus, W. D. (1967) The optimal life of a patent. Cowles Foundation Discussion Papers 241, Cowles Foundation for Research in Economics, Yale University.

Peretto, P. and M. Connolly (2007) The manhattan metaphor. *Journal of Economic Growth*, 12 (4): 329–350.

Peretto, P. and S. Smulders (2002) Technological distance, growth and scale effects. *The Economic Journal*, 112(481): 603–624.

Romer, P. (1990) Endogenous technological change. *The Journal of Political Economy*, 98(5): S71-S102.

Seierstad, A. and K. Sydsaeter (1999) *Optimal Control Theory with Economic Applications.* Amsterdam: Elsevier.

Skritek, B., Stachev T., and V. Veliov (2014) Optimality conditions and the Hamiltonian for a distributed optimal control problem on controlled domain. *Applied Mathematical Optimization*, 70(1): 141–164.

Yano, M. (1984) The turnpike of dynamic general equilibrium paths and its insensitivity to initial conditions. *Journal of Mathematical Economics*, 13(3): 235–254.