# Modal decomposition method for response spectrum based analysis of nonlinear and non-classically damped systems


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**Abstract**
A novel inelastic modal decomposition method for random vibration analysis in alignment with contemporary aseismic code provisions (e.g., Eurocode 8) considering non-classically damped and nonlinear multi degree-of-freedom (MDOF) systems is developed. Relying on statistical linearization and state-variable formulation the complex eigenvalue problem considering inelastic MDOF structural systems subject to a vector of stochastic seismic processes is addressed. The involved seismic processes are characterized by power spectra compatible in a stochastic sense with an assigned elastic response uniform hazard spectrum (UHS) of specified modal damping ratio. Equivalent modal properties (EMPs) of the linearized MDOF system, namely equivalent pseudo-undamped natural frequencies and equivalent modal damping ratios are provided. To this aim, each mode of vibration is assigned with a different stationary random process compatible with the excitation response spectrum adjusted to the corresponding equivalent modal damping ratio property. Next, an efficient iterative scheme is devised achieving convergence of the equivalent modal damping ratios and the damping premises of the excitation response spectrum corresponding to each mode of the system. Subsequently, the stochastically derived forced vibrational modal properties of the structure are utilized together with the appropriate mean response elastic UHS for determining peak nonlinear responses in modal coordinates. The modal participation factors are determined for the complex-valued mode shapes and generalized square-root-of-sums-squared (SRSS) is employed as the modal combination rule for determining the peak total responses of the system in physical space. The pertinency and applicability of the proposed framework is numerically illustrated using a three-storey bilinear hysteretic frame structure exposed to the Eurocode 8 elastic response spectrum. Nonlinear response time-history analysis (RHA) involving a large ensemble of stationary Eurocode 8 spectrum compatible accelerograms is conducted to assess the accuracy of the proposed methodology in a Monte Carlo-based context.

**Keywords:** nonlinear stochastic dynamics, complex modal analysis, bilinear MDOF hysteretic systems, statistical linearization, forced vibrational characteristics, stochastic processes

## 1 INTRODUCTION

Modal decomposition method has been developed over the last five decades for conducting response analysis of dynamically excited linear systems, and the available tools today are mature and accessible by engineers of practice (e.g. [1,2]). However, when it is expected that the vibratory system will be excited in the nonlinear range, the linearity framework should be abandoned in favor of a nonlinear modal analysis. In conceptual relation with its linear counterpart, nonlinear modal analysis could be formulated either in time-history or in the more convenient response-spectrum variant. In this regard, the concept of determining nonlinear normal modes (NNMs) which was regarded as a theoretical curiosity until the beginning of the 1990s comes to the fore (e.g. [3]). Clearly, NNMs offer a solid theoretical and mathematical basis for interpreting the underlying structural dynamics. However, the concept of NNMs which phenomenologically forms an excellent basis for conducting modal analysis as well as system identification has found limited use in structural dynamics due to a number of reasons. Among them, one can find the computation of NNMs which necessitates considerable implementation effort, rendering it almost impractical [4]. Further, an important limitation compared to their linear counterparts (i.e. normal modes) should be considered the fact that the general motion in physical coordinates of a nonlinear system cannot be expressed as a superposition of individual NNMs. The principle of superposition which is the cornerstone of linear theory does not apply to nonlinear systems. However, in the approximate sense an interesting nonlinear version of superposition, inspired by the theory of invariant manifolds, can be found in the work of Shaw and Pierre [5]. Clearly, addressing nonlinearity and determining alternative routes for computing efficiently nonlinear system properties for the forced response case in realistic vibratory systems is identified as one of the greatest challenges in structural engineering. In this context, the need for developing efficient and broadly applicable nonlinear analysis methodologies, exploiting time-frequency analysis techniques (e.g. [6]) that can track the temporal evolution of the frequency of oscillation (frequency-energy dependence) has been highlighted as a potential research path in the work of Kerschen et al. [7].

In the dynamic analysis of structural systems subjected to seismic excitations, the classical damping assumption is commonly employed. In this setting, the linear vector-matrix differential equation of motion (see Eq.(1)) can be decoupled into a set of independent modal equations using the eigenvalues and the associated real-valued eigenvectors of the undamped system. However, in the majority of systems of engineering interest, where the modal damping matrix is in general non-diagonal, the modal equations are anticipated to remain coupled; these systems are generally defined as non-classically damped. For non-classically damped systems the mode shapes are complex-valued and the employed modal decomposition method should consider it carefully (e.g., [8]).

It is vital to bear in mind that in real structural and mechanical systems nonlinearities arise in various forms, and usually become progressively more significant as the amplitude of vibration increases. Specifically, in the engineering discipline of earthquake resistant structures such issues fairly emerge (e.g., [9]). This fact brings to the fore the need for a more pertinent representation of the system model by considering thoroughly the real mechanisms which determine to a great extent the overall system behaviour. In this setting, a suitable stochastic representation of seismic excitation in conjuction with nonlinear and non-classically damped system modelling provides a solid basis for formulating a realistic analysis and design procedure (e.g. [10]). Clearly, persistent nonlinear stochastic structural dynamics problems faced by engineers in their daily practice are amenable to efficient and comprehensive solutions, harnessing the potential of random vibration theory. Under these circumstances, the need for development of an efficient inelastic joint time-frequency system response analysis technique with respect to contemporary seismic code provisions is recognised as a topic of considerable importance. A well-documented review of current approaches for code-compliant inelastic seismic demand estimation can be found in [11].

This paper proposes a novel inelastic modal decomposition method for random vibration analysis in allingnment with contemporary aseismic code provisions (e.g., Eurocode 8) considering non-classically damped and nonlinear multi degree-of-freedom (MDOF) systems. Relying on statistical linearization and state-variable formulation the complex eigenvalue problem considering MDOF systems subject to a vector of stochastic seismic processes characterized by power spectra compatible in the median sense with a given elastic response UHS of specified modal damping ratio is addressed. The stochastically derived forced vibrational modal properties which capture well the trend of the inelastic behavior are utilized together with the appropriate mean response elastic UHS for determining peak nonlinear responses in modal coordinates. The modal participation factors are then determined for the complex-valued mode shapes and a generalized variant of SRSS rule is employed for determining the peak total responses of the system.

Comparing to the state of the art schemes available in the literature, the proposed stochastic dynamics technique exhibits a number of noteworthy attributes such as: (i) it accounts for *nonlinear* and *MDOF* structural systems, following aseismic code-prescribed criteria which dictate a ductile behavior under the design seismic action, (ii) it considers for *non-classically damped* systems which represent the majority of systems of engineering interest (e.g., [2,13]), (iii) it is tailored to facilitate code-compliant seismic demand estimation using linear UHS along with the well-established concept of damping modification factors to specify the seismic input action and aims to relax heuristic approximating assumptions made by current code-prescriptive simplified methods, (iv) it is considerably less computationally demanding compared to nonlinear RHA for UHS compatible ground motion records, (v) it furnishes with forced vibrational modal properties which offer a solid basis for interpreting the underlying structural dynamics. The efficient identification of the dynamic character of the system cannot be determined following nonlinear RHA. Specifically, the novel meaningful modal decoupling iterative scheme forms an excellent basis for conducting system identification, since it seamlessly provides with equivalent modal properties dependent on the degree of the exhibited nonlinearity. Lastly, (vi) it addresses cases of structures well entered into the inelastic range.

In the remainder of this paper Section 2 reviews briefly basic aspects of modal analysis, Sections 3.1-3.5 review the mathematical background supporting the proposed framework, Section 3.6 furnishes pertinent comments on the assumptions and practical usage of the implementation algorithm, Section 4, presents a numerical application of the framework to a yielding building frame exposed to the Eurocode 8 UHS [12] and assesses its accuracy against nonlinear RHA data, and Section 5 summarizes the main conclusions.

## 2 STATEMENT OF THE PROBLEM

The dynamic response of a *n*-DOF structure excited by a base motion, the acceleration of which is $\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t})$, is governed by the system of differential equations of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = -\mathbf{M}\boldsymbol{\gamma}\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t}) \tag{1}$$

where $\mathbf{x}(t)$, $\dot{\mathbf{x}}(t)$, and $\ddot{\mathbf{x}}(t)$ are the response displacement, velocity, and acceleration vectors of the nodes relative to the base motion, respectively. The dot superscript denotes differentiation with respect to time; $\boldsymbol{\gamma}$ is a unit $(\mathrm{n} \times 1)$ column vector. Further, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ denote the $(\mathrm{n} \times \mathrm{n})$ real-valued mass, damping, and stiffness matrices, respectively. The objective is to elucidate the analysis process for the linear system response where no particular restrictions are imposed on the form of the damping matrix (e.g. Rayleigh damping). In this setting, a commonly employed approximate modal analysis procedure is the decoupled analysis or classical modal analysis (e.g. [14]). The natural frequencies $\omega_r^o$, and mode shapes $\boldsymbol{\theta}_r$, are first determined for the undamped structure by admitting a harmonic response solution of the form

$$\mathbf{x}(t) = \boldsymbol{\theta}\mathrm{e}^{i\omega t} \tag{2}$$

to the system of Eq.(1) in free vibration. The mode shapes are normalized with respect to mass matrix so that $\boldsymbol{\theta}_r^T \boldsymbol{M} \boldsymbol{\theta}_r = 1$. Subsequently, the damping ratios are provided by the diagonal elements of the modal damping matrix $\zeta_r^o = \boldsymbol{\theta}_r^T \boldsymbol{C} \boldsymbol{\theta}_r / (2\omega_r^o)$. Note that the free vibrational characteristics provide an insight into the dynamic character of the system. In the work of Warburton and Soni [15], it has been shown that decoupled analysis yields reliable estimates provided that the modal coupling parameter $\rho_{r,q} = \boldsymbol{\theta}_r^T \boldsymbol{C} \boldsymbol{\theta}_q \omega_r^o / \left| {\omega_r^o}^2 - {\omega_q^o}^2 \right|$ is small relative to unity for all pairs of modes, $r \neq q$. However, a more pertinent addressing of the non-classical damping feature would necessitate resorting to complex-valued modes stemming from the treatment of the damped eigenvalue problem. In this regard, a completely decoupled set of equations is possible to obtain [16]. Further, a more realistic and representative modelling of real engineering systems subjected to severe excitations should thoroughly consider for the presence of nonlinear mechanisms. In this setting, the development of an efficient analysis method with respect to the above needs while retaining the broad applicability and attractiveness of its linear counterpart among structural engineering practice is discussed.

## 3 MATHEMATICAL BACKGROUND

This section reviews the mathematical details involved in undertaking the steps of (i) defining a power spectrum compatible in a stochastic sense with an assigned seismic response spectrum, of (ii) applying statistical linearization to a nonlinear *n*-DOF structural system under a state variable formulation, of (iii) decoupling the equivalent linear MDOF system by conducting complex modal analysis to derive the forced vibrational characteristics corresponding to each mode of vibration, of (iv) establishing an iterative algorithm based on the above three steps, and of (v) determining the modal combination rule and subsequently the modal participation factors for the complex-valued mode shapes. Particular attention has been given on elucidating the various simplifications and assumptions made in support of numerical efficiency.

### 3.1 *Consistent discrete power spectra and peak factor estimation*

An efficient numerical scheme is employed to statistically fit a stationary Gaussian acceleration process $\ddot{x}_g(t)$ of finite duration $T_s$, to an assigned elastic pseudo-acceleration response spectrum, $S_a(\omega, \zeta)$ defined along the axis of natural frequencies. In this section the most important elements of a computationally efficient approach [17] for the derivation of response spectrum compatible stationary power spectra are included for completeness. In this regard, the following nonlinear equation consists the basis for relating a pseudo-acceleration response spectrum $S_a(\omega_i, \zeta_o)$ to an one-sided power spectrum corresponding to a Gaussian stationary stochastic process $X_i(t)$; that is,

$$S_a(\omega_i, \zeta_o) = \eta_{X_i} \omega_i^2 \sqrt{\lambda_{0,X_i}} \tag{3}$$

where $\eta_{X_i}$ and $\lambda_{0,X_i}$ stand for the peak factor and the variance of the stationary stochastic response process $X_i(t)$ of an elastic oscillator of natural frequency $\omega_i$ and damping ratio $\zeta_o$. Further, the spectral moment of zeroth order of the stationary response process that appears in Eq.(3), reads for the general case of $n$th order

$$\lambda_{n,X_i} = \int_0^\infty \omega^n \frac{1}{(\omega_i^2 - \omega^2)^2 + (2\zeta_o\omega_i\omega)^2} G_{X_i}^{\zeta_o}(\omega) d\omega. \tag{4}$$

Note that the pseudo-acceleration response spectrum $S_a(\omega_i, \zeta_o)$ appearing in Eq.(3) can be estimated in a straightforward manner, once the power spectrum $G_{X_i}^{\zeta_o}(\omega)$ and the duration $T_s$ of the input are provided. However, the evaluation of the stochastically compatible power spectrum $G_{X_i}^{\zeta_o}(\omega)$, which does not appear explicitly in Eq.(3), necessitates a careful handling of the inverse stochastic dynamics problem.

The determination of the peak factor $\eta_{X_i}$ is related with the first-passage problem (e.g. [18,19]). Following the hypothesis of a barrier outcrossing in clumps [18], the peak factor is expressed as

$$\eta_{X_i}(T_s, p) = \sqrt{2\ \ln\{2\ v_{X_i}[1 - \exp[-\delta_{X_i}^{1.2}\sqrt{\pi\ \ln(2\ v_{X_i})}]]\}} \tag{5}$$

where the mean zero crossing rate $v_{X_i}$ and the spread factor $\delta_{X_i}$ of the stochastic response process $X_i(t)$ are defined as

$$v_{X_i} = \frac{T_s}{2\pi}\sqrt{\frac{\lambda_{2,X_i}}{\lambda_{0,X_i}}}(-\ln p)^{-1} \tag{6}$$

and

$$\delta_{X_i} = \sqrt{1 - \frac{\lambda_{1,X_i}^2}{\lambda_{0,X_i}\lambda_{2,X_i}}}. \tag{7}$$

respectively. The peak factor $\eta_{X_i}$ consists the critical factor by which the standard deviation of the considered elastic oscillator response should be multiplied to predict a level $S_a$ below which the peak response will remain, with probability $p$ (see Eq.(6)). Utilizing Vanmarcke's [20] approximate formula for obtaining a reliable estimation of the variance of the response process $X_i(t)$ of an oscillator of natural frequency $\omega_i$ and damping ratio $\zeta_o$, the following direct scheme for the evaluation of the stochastically compatible power spectrum $G_{X_i}^{\zeta_o}(\omega)$ is derived

$$G_{X_i}^{\zeta_o}(\omega_i) = \begin{cases} \dfrac{4\zeta_o}{\omega_i\pi - 4\zeta_o\omega_{i-1}}\left(\dfrac{S_\alpha^2(\omega_i,\zeta_o)}{\eta_{X_i}^2} - \Delta\omega\displaystyle\sum_{q=1}^{i-1} G_{X_i}^{\zeta_o}(\omega_q)\right), & \omega_i > \omega_b^l \\ 0, & \omega_i \le \omega_b^l \end{cases} \tag{8}$$

where the discretization scheme $\omega_i = \omega_b^l + (i - 0.5)\Delta\omega$ is employed. The value of $\omega_b^l$ is related with the lowest bound of the frequency domain of Eq.(5); see also [21]. Obviously, a preselection of an input power spectrum shape has to be preceded for deriving a stochastically compatible spectrum, according to the numerical scheme of Eq.(8). Further, it is appropriate to remark that due to the fact that the crossing rate $v_{X_i}$ and the spread factor $\delta_{X_i}$ are not very sensitive to the input power spectrum shape, given that the input is broad-band and the oscillator natural frequency belongs to the band of frequencies over which the input power spectrum has significant values, a satisfactory estimation by utilizing even a white-noise input is feasible.

Note in passing that the time-limited stationary power spectrum (and underlying stochastic process) is only used as a first numerical step to represent the seismic input action, defined in terms of a pseudo-acceleration response spectrum. The stochastic dynamics technique discussed in the remainder of this section is independent from the herein presented approach which is only one of the numerous proposed in the literature that can be used (e.g. [22]).

### 3.2 *Statistical Linearization for non-classically damped nonlinear MDOF structures under stationary seismic excitation*

Consider a non-classically damped, nonlinear structural system with *n* number of DOFs base-excited by the Gaussian stationary acceleration stochastic process $\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t})$, characterized in the frequency domain by the power spectrum $G_{X_i}^{\zeta_o}(\omega)$. The dynamic response of the structure is governed by the system of differential equations written in vector-matrix form as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{g}\big(\mathbf{x}(t), \dot{\mathbf{x}}(t)\big) = \mathbf{F}(\mathrm{t}) = -\mathbf{M}\boldsymbol{\gamma}\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t}) \tag{9}$$

where $\mathbf{g}\big(\mathbf{x}(t), \dot{\mathbf{x}}(t)\big)$ is a nonlinear $\mathrm{n} \times 1$ vector function of the variables $\mathbf{x}(\mathrm{t})$ and $\dot{\mathbf{x}}(\mathrm{t})$, used to model the inelastic response of seismically excited yielding structures. For non-classically damped systems which do not satisfy Caughey and O'Kelly identity [23] which states that

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \tag{10}$$

the eigenvalues as well as the modal shapes are expected to be complex-valued. The exponent in Eq.(10) denotes the inverse of the matrix. Note that in case the damping matrix of a system satisfies the above identity the natural modes are real-valued and equal to those of the associated undamped system. The eigenvalues of a classically damped system appear in complex conjugate pairs and the modulus of its pair is equal with the natural frequency $\omega_j^o$ of the associated undamped system. The Rayleigh form of damping where the damping matrix is defined as a linear combination of the mass and stiffness matrices is a sub-case of Caughey and O'Kelly's identity.

Further, $\mathbf{F}(\mathrm{t})$ can be expressed in the frequency domain by the power spectrum matrix as

$$\mathbf{S}_{\mathbf{FF}}(\omega) = G_{X_i}^{\zeta_o}(\omega)\mathbf{M}\boldsymbol{\gamma}\boldsymbol{\gamma}^{\boldsymbol{T}}\mathbf{M}. \tag{11}$$

Relying on the standard assumption that the response processes are Gaussian, the standard spectral matrix solution procedure of the classical statistical linearization [24] is employed to estimate the response power spectrum matrix of the nonlinear and non-classically damped MDOF structure. In this setting, a linearized version of Eq. (9) is considered

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \big(\mathbf{C} + \mathbf{C}_{\mathbf{eq}}\big)\dot{\mathbf{x}}(t) + \big(\mathbf{K} + \mathbf{K}_{\mathbf{eq}}\big)\mathbf{x}(t) = \mathbf{F}(t) = -\mathbf{M}\boldsymbol{\gamma}\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t}), \tag{12}$$

Further, it is possible to write the linearized equations in the following standard matrix form

$$\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ m_2 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ m_n & m_n & m_n & m_n \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} c_{1e} & -c_{2e} & \cdots & 0 \\ 0 & c_{2e} & -c_{3e} & 0 \\ 0 & 0 & \ddots & -c_{ne} \\ 0 & 0 & 0 & c_{ne} \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{bmatrix} + \begin{bmatrix} k_{1e} & -k_{2e} & \cdots & 0 \\ 0 & k_{2e} & -k_{3e} & 0 \\ 0 & 0 & \ddots & -k_{ne} \\ 0 & 0 & 0 & k_{ne} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = - \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & m_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \ddot{x}_g(t) \quad (13)$$

where $y_j(t)$ is the inter-story drift $y_j(t) = x_j - x_{j-1}$, $x_j$, $j = 0,1,2,\ldots,n$ is the lateral floor displacement relative to the ground displacement with $x_0$=0; alternatively, it can be written as $y_j(t) = \boldsymbol{w}^T \boldsymbol{x}$, where the $1 \times n$ transformation vector $\boldsymbol{w}^T$ for the case of the top floor relative displacement takes the values $[1 \; -1 \; 0 \ldots 0]$. Next, $k_{je} = k_j + k_{jj}^{eq}$ and $c_{je} = c_j + c_{jj}^{eq}$. The $(d,l)^{th}$ element of the equivalent linear matrices $\mathbf{C_{eq}}$ and $\mathbf{K_{eq}}$ are given by the expressions

$$c_{d,l}^{eq} = E\left[\frac{\partial g_d}{\partial \dot{y}_l}\right], \quad (14)$$

and

$$k_{d,l}^{eq} = E\left[\frac{\partial g_d}{\partial y_l}\right], \quad (15)$$

in which E[·] is the mathematical expectation operator. Subsequently, the Fourier transform of the response cross-correlations matrix defined by convoluting the impulse response function matrix with the vector of the applied stochastic loads leads for the general case of a linear *n*-DOF system in the celebrated frequency domain relation

$$\mathbf{S_{yy}}(\omega) = \mathbf{H_y}(i\omega)\mathbf{S_{FF}}(\omega)\mathbf{H_y}^{T*}(i\omega), \quad (16)$$

where the superscript (*) denotes Hermitian transposition and the frequency response function (FRF) matrix is defined as

$$\mathbf{H_y}(i\omega) = \left[\left[\left(\mathbf{K} + \mathbf{K_{eq}}\right) + \mathbf{M}(i\omega)^2\right] + i\omega\left(\mathbf{C} + \mathbf{C_{eq}}\right)\right]^{-1}, \quad (17)$$

where *i* is the imaginary unit. Furthermore, the cross–variance of the response due to a vector of stochastic excitation processes characterized by power spectra of the form $G_{X_i}^{\zeta_o}(\omega)$ can be evaluated by the expression

$$E[y_d(t)y_l(t)] = \int_{-\infty}^{\infty} S_{y_d y_l}(\omega)\, d\omega \quad (18)$$

where $S_{y_d y_l}(\omega)$ is the $(d,l)^{th}$ element of the response power spectrum matrix $\mathbf{S_{yy}}(\omega)$. It can be readily seen that Eqs.(13-18) constitute a coupled nonlinear system of algebraic equations to be solved iteratively for the system response covariance matrix. Actually, a simple iterative while-loop is sufficient to simultaneously satisfy Eqs. (13-18) until convergence of the elements of $\mathbf{C_{eq}}$ and $\mathbf{K_{eq}}$ matrices is achieved within a pre-specified tolerance (e.g. [19,25]). The iterations are initialized by neglecting the $\mathbf{C_{eq}}$ and $\mathbf{K_{eq}}$ matrices in Eq. (17) in determining the covariance matrix terms, which are then used to compute the updated values of the elements of $\mathbf{C_{eq}}$ and $\mathbf{K_{eq}}$ via Eqs. (14) and (15).

Further, the coupled linearized equations can be written in the following normalized form on dividing every single equation throughout by the corresponding $m_j$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} 2\zeta_{1e}\omega_{1e} & -2\zeta_{2e}\omega_{2e}\mu_{21} & & \cdots & 0 \\ 0 & 2\zeta_{2e}\omega_{2e} & & -2\zeta_{3e}\omega_{3e}\mu_{32} & 0 \\ 0 & 0 & \ddots & & -2\zeta_{ne}\omega_{ne}m_n\mu_{nn-1} \\ 0 & 0 & 0 & & 2\zeta_{ne}\omega_{ne}m_n \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{bmatrix}$$
$$+ \begin{bmatrix} \omega_{1e}^2 & -\omega_{2e}^2\mu_{21} & \cdots & 0 \\ 0 & \omega_{2e}^2 & -\omega_{3e}^2\mu_{32} & 0 \\ 0 & 0 & \ddots & -\omega_{ne}^2\mu_{nn-1} \\ 0 & 0 & 0 & \omega_{ne}^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = - \begin{bmatrix} \ddot{x}_g(t) \\ \ddot{x}_g(t) \\ \vdots \\ \ddot{x}_g(t) \end{bmatrix} \tag{19}$$

where $\omega_{je}^2 = \frac{k_{je}}{m_j}$, $\zeta_{je} = \frac{c_{je}}{2\sqrt{k_{je}m_j}}$ and $\mu_{j+1j} = \frac{m_{j+1}}{m_j}$.

In the engineering discipline of aseismic design the bilinear hysteretic force-deformation law, shown in Fig. 2(b), consists a commonly employed model to capture the hysteretic behavior of structural members and systems under seismic excitation (e.g. [26,27]). The governing equation of motion for a bilinear hysteretic oscillator can be expressed with the aid of an auxiliary state $z_j(t)$ (e.g. [16,28])

$$g_j\left(x_j(t), \dot{x}_j(t)\right) = \alpha_j y_j(t) + (1 - \alpha_j) z_j(t), \tag{20}$$

with

$$\dot{z}_j(t) = \dot{y}_j \left\{1 - \Phi\left(\dot{y}_j(t)\right) \Phi\left(z_j(t) - x_y\right) - \Phi\left(-\dot{y}_j(t)\right) \Phi\left(-z_j(t) - x_y\right)\right\}, \tag{21}$$

where $\Phi(\cdot)$ denotes the Heaviside step function, namely, $\Phi(m) = 1$ for $m \geq 0$, and $\Phi(m) = 0$ for $m < 0$, $x_y$ is the yielding deformation and $\alpha_j$ is the post-yield to pre-yield stiffness ratio. Through a consideration of a free-body diagram for the j-th story, it is evident that the equation of motion, in general, can be written as

$$m_j \sum_{i=1}^{j} \ddot{y}_i(t) + c_j \dot{y}_j(t) - c_{j+1}\dot{y}_{j+1}(t) + k_j g_j\left(x_j(t), \dot{x}_j(t)\right) - k_{j+1} g_{j+1}\left(x_j(t), \dot{x}_j(t)\right) = -m_j \ddot{x}_g(t) \tag{22}$$

Adopting the assumptions that the response of a viscously damped bilinear hysteretic SDOF oscillator is contained within a narrow frequency band and that the probability density function (PDF) of its response amplitude is a Rayleigh distribution, the following equivalent linear parameters are determined [29,30]

$$\omega_{je}^2 = \omega_j^2 \left\{1 - \frac{8(1-\alpha_j)}{\pi} \int_1^\infty \left[u^{-3} + \left(v_j u\right)^{-1}\right] (u-1)^{1/2} e^{-u^2/v_j} du\right\}, \tag{23}$$

and

$$\zeta_{je} = \zeta_j \frac{\omega_j}{\omega_e} + \left(\frac{\omega_j}{\omega_e}\right)^2 (1 - \alpha_j)(\pi v_j)^{-1/2} erfc(v_j^{-1/2}), \tag{24}$$

where

$$v_j = \frac{2E\left[\mathrm{y_j}^2(t)\right]}{\mathrm{x}_y^2}. \tag{25}$$

Since the equivalent linear parameters (ELPs) are defined for a bilinear hysteretic SDOF oscillator, the method is shown to be appropriate for being applied to the coupled linearized set of equations; see Eq.(19). The cross-correlation terms in the determination of the expressions for the ELPs are neglected due to their relatively low contribution. An alternative possible route accompanied with higher computational cost could include the determination of the joint PDFs between the system response amplitudes; see [31]. It can be readily seen that Eqs.(16-19) and Eqs.(23-25) constitute a coupled nonlinear system of algebraic equations to be solved iteratively for the system ELPs determination. The decoupling step reviewed in the next section utilizes the ELPs in Eqs. (23) and (24) in conjunction with a state space formulation to define equivalent linear uncoupled oscillators in modal coordinates with effective damping and natural frequency properties corresponding to the $r$-th mode of vibration.

### 3.3 *Complex modal decomposition of nonlinear structural dynamics*

For the linearized *n*-DOF system of Eq.(19) excited by a response spectrum compatible power spectrum, the Caughey and O'Kelly identity reads

$$\left(\mathbf{C} + \mathbf{C_{eq}}\right)\mathbf{M^{-1}}\left(\mathbf{K} + \mathbf{K_{eq}}\right) \neq \left(\mathbf{K} + \mathbf{K_{eq}}\right)\mathbf{M^{-1}}\left(\mathbf{C} + \mathbf{C_{eq}}\right) \tag{26}$$

It is readily conceived that the equivalent linear damping matrix does not obey to any particular restrictions, thus the identity would not be satisfied in the general case. Evidently, an appropriate treatment of the non-classically damped character of the equivalent linear MDOF system is needed. In this setting, the addressing of the following complex eigenvalue problem

$$\left[\boldsymbol{\lambda}^2\mathbf{M} + \boldsymbol{\lambda}\left(\mathbf{C} + \mathbf{C_{eq}}\right) + \left(\mathbf{K} + \mathbf{K_{eq}}\right)\right]\boldsymbol{\psi} = \mathbf{0} \tag{27}$$

is deemed necessary. Rather than through the solution of the system of equations (Eq.(27)), the eigenvalues $\boldsymbol{\lambda}$ and the associated eigenvectors $\boldsymbol{\psi}$ may be determined more conveniently by first reducing the system of *n* second order differential equations to a system of 2*n* first order differential equations. The general linearized equations of motion for a *n*-DOF system as expressed by Eq.(12), can be recast into the state variable form by defining a 2*n* state vector, $\mathbf{q}(t)$, as follows

$$\mathbf{q}(t) = \begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix} \tag{28}$$

A first-order matrix equation of motion may then be written as

$$\dot{\mathbf{q}}(t) = \boldsymbol{G}\mathbf{q}(t) + \mathbf{f}(t) \tag{29}$$

where

$$\boldsymbol{G} = \begin{bmatrix}\mathbf{0} & \boldsymbol{I}\\ -\mathbf{M^{-1}}\left(\mathbf{K} + \mathbf{K_{eq}}\right) & -\mathbf{M^{-1}}\left(\mathbf{C} + \mathbf{C_{eq}}\right)\end{bmatrix} \tag{30}$$

and

$$\mathbf{f}(t) = \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{F}(t)\end{bmatrix} \tag{31}$$

Next, the eigenvalues $\lambda_1, \lambda_2, \dots \lambda_{2n}$ of the 2n × 2n matrix $\boldsymbol{G}$ are computed by solving

$$|\boldsymbol{G} - \boldsymbol{\lambda I}| = \boldsymbol{0} \tag{32}$$

Provided the amount of damping in the system is not very high, the eigenvalues occur in complex conjugate pairs with negative real parts.

$$\begin{matrix} \lambda_r \\ \bar{\lambda}_r \end{matrix} = \begin{Bmatrix} -\zeta_{\mathrm{req}}\omega_{\mathrm{req}} + i\omega_{\mathrm{reqD}} \\ -\zeta_{\mathrm{req}}\omega_{\mathrm{req}} - i\omega_{\mathrm{reqD}} \end{Bmatrix}, \qquad r = 0,1,2, \dots, \mathrm{n} \tag{33}$$

The equivalent modal properties (EMPs), namely the equivalent pseudo-undamped natural circular frequency $\omega_{\mathrm{req}}$ and the equivalent modal damping ratio $\zeta_{\mathrm{req}}$, which correspond to the uncoupled equations of motion in modal coordinates are related with the eigenvalues as follows

$$\omega_{\mathrm{req}} = |\lambda_r|, \qquad \zeta_{\mathrm{req}} = -\frac{Re(\lambda_r)}{|\lambda_r|}. \tag{34}$$

The usage of the prefix ‘‘pseudo’’ intends to denote that for non-classically damped systems, $\omega_{\mathrm{r}}$ is a function of the amount of system damping and, hence, differs from the corresponding frequency of the associated undamped system $\omega_r^o$. Further, the corresponding frequency with damping $\omega_{\mathrm{reqD}}$ is given

$$\omega_{\mathrm{reqD}} = \omega_{\mathrm{req}}\left(1 - \zeta_{\mathrm{req}}^2\right)^{1/2} \tag{35}$$

Determining the EMPs could be especially important for a number of reasons such as tracking and avoiding moving resonance phenomena or developing efficient approximate techniques for defining nonlinear system survival probabilities and first-passage PDFs (e.g. [32,33]). For a *n*-DOF system, there are *n* pairs of eigenvalues, and to each such pair corresponds a complex conjugate pair of eigenvectors

$$\begin{matrix} \boldsymbol{\psi}_r \\ \bar{\boldsymbol{\psi}}_r \end{matrix} = \begin{Bmatrix} \boldsymbol{\varphi}_r + i\boldsymbol{y}_r \\ \boldsymbol{\varphi}_r - i\boldsymbol{y}_r \end{Bmatrix} \tag{36}$$

The columns of the 2n × 2n complex modal matrix $\boldsymbol{T_G}$ formed from the eigenvectors or complex modes

$$\boldsymbol{T_G} = [\boldsymbol{\psi}_1, \bar{\boldsymbol{\psi}}_1, \boldsymbol{\psi}_2, \bar{\boldsymbol{\psi}}_2, \dots \boldsymbol{\psi}_n, \bar{\boldsymbol{\psi}}_n] \tag{37}$$

can be used as an appropriate transformation matrix for introducing in matrix form the power spectrum of modal forces

$$\mathbf{S_{QQ}}(\omega) = \boldsymbol{T_G^{-1}}\mathbf{S_{FF}}(\omega)\, \boldsymbol{T_G^{-1}}^{*\mathrm{T}}. \tag{38}$$

The next section details the pertinent algorithm which utilizes the above three steps in a unified framework for conducting stochastic complex modal analysis of nonlinear MDOF systems without any restrictions on the nature of the damping matrix, in an iterative base.

### 3.4 *Identification of the forced vibrational modal properties based on an iterative scheme*

The proposed methodology incorporates an efficient iterative scheme which includes successive solution of an inverse stochastic dynamics problem for the determination of stochastically compatible power spectra with an assigned elastic design/response UHS of specified damping ratio. In the herein study, pseudo-acceleration design spectra prescribed by the European aseismic code provisions (EC8) are utilized for the determination of compatible design spectrum power spectra. At this point, it is deemed appropriate to note that the choice of EC8 is not binding and that the proposed methodology can readily be modified to account for provisions defined by various aseismic codes.

Relying on statistical linearization and utilizing the equivalent complex modal decomposition method, delineated in sections 3.2 and 3.3 respectively, the nonlinear *n*-DOF system is decoupled and cast into (*n*) uncoupled oscillators in modal coordinates with equivalent modal properties (EMPs), namely equivalent pseudo-undamped natural circular frequency $\omega_{\mathrm{req}}$ and equivalent modal damping ratio $\zeta_{\mathrm{req}}$. Next, the stochastically derived equivalent modal damping ratios $\zeta_{\mathrm{req}}$ redefine the damping premises of the updated input elastic response UHS which in turn define stochastically compatible design spectrum power spectra. The aforementioned procedure, establishes a cyclic relationship between the stochastically equivalent modal damping coefficients of the uncoupled oscillators $\zeta_{\mathrm{req}}$ in modal space and the damping ratios of the input elastic response UHS. In this setting, the elastic response UHS is included in the iterative process; it is considered as a variable of the optimization problem rather than a constant parameter. Graphically, a flowchart of the proposed methodology can be seen on Fig.(1). The idea of iteratively updating the nominal damping ratio of the input response spectrum corresponding to each mode has a twofold meaning; (i) it secures compliance with the basic definition of the response/design spectrum-based analysis which requires the considered decoupled linear/linearized oscillators and the imposed elastic response UHS to share the same damping premises. This well-detected critical point only recently received the appropriate attention in the literature (Mitseas et al. 2018). (ii) performs an efficient time-domain identification of the force-dependent vibrational modal parameters. Specifically, the proposed iterative scheme identifies the presence of various modal damping ratio values in the system, and adjusts appropriately the characteristics of the imposed seismic excitation corresponding to each mode of vibration, in order to render the generated EMPs function of the intensity of the imposed excitation. This is achieved by enforcing equality, within some allowance, between the stochastically equivalent damping coefficients and the damping ratio of the input UHS corresponding to each and every mode of the system. Lastly, once convergence between $\zeta_{\mathrm{req}}^{(k)}$ and $\zeta_{\mathrm{req}}^{(k-1)}$ is achieved after k iterations for a *r* mode, the forced vibrational modal properties, $\omega_{\mathrm{req}}^{(k_{end})}$ and $\zeta_{\mathrm{req}}^{(k_{end})}$ are defined.

The herein discussed modal decoupling iterative algorithm provides with forced vibrational modal properties which determine the dynamic character of the system and are amenable to a clear physical interpretation. Actually, they appear to capture the inelastic response of any MDOF system depending on the excitation intensity by taking on values in alignment with engineering intuition. Specifically, stronger nonlinear response due to higher excitation intensity leads to heavier damped modal oscillators shifted towards lower frequencies.

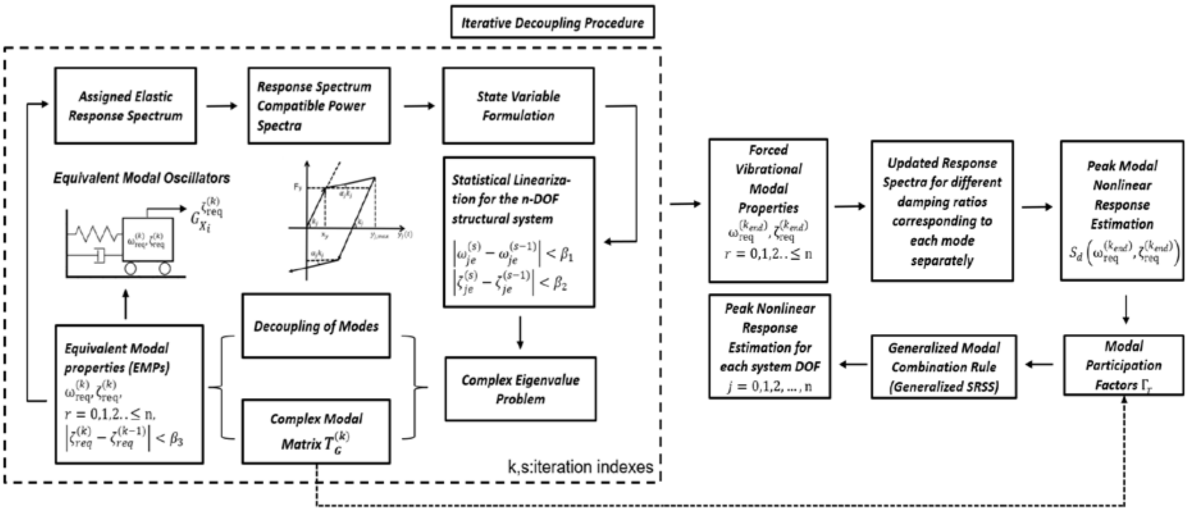


Figure 1. Flowchart of the proposed methodology for inelastic design spectrum analysis.

In this work, the above discussed attributes of the forced vibrational modal properties motivate their use to estimate the peak response of a given nonlinear MDOF structure exposed to a linear response spectrum $S_a(\omega_i, \zeta_o)$. However, these spectral ordinates correspond to peak structural responses associated with each mode and therefore an appropriate mode combination method which considers the non-classically damped character of the equivalent system is required.

### 3.5 Modal combination rule for non-classically damped systems. The case of generalized SRSS method

The method uses the relative displacement response spectra and real-valued participation factors $\Gamma_r$ which have been determined from the complex-valued mode shapes $\boldsymbol{\theta}_r$. The mode shapes, $\boldsymbol{\theta}_r$, are given by the upper half of the eigenvector $\boldsymbol{\psi}_r$. As in the classical modal combination analysis, a subset of the total number of modes can selectively be used (i.e. $r = 0,1,2, \dots, \leq$ n). The $j$-th response displacement is given [8] by

$$y_j(t) = \sum_{r=1}^{n} \left[ a_r h_r(t) + c_r \dot{h}_r(t) \right] \tag{39}$$

where $h_r(t)$ is the modal relative displacement response of an oscillator with natural frequency $\omega_r$ and damping ratio $\zeta_r$. The real-valued coefficients $a_r$ and $c_r$ are defined as $a_r = -2Re(\eta_r \bar{\lambda}_r)$ and $c_r = 2Re(\eta_r)$, where

$$\eta_r = (\boldsymbol{w}^T \boldsymbol{\theta}_r)(\boldsymbol{\theta}_r^T \boldsymbol{M} \boldsymbol{\gamma})(-\lambda_r \boldsymbol{\theta}_r^T \boldsymbol{M} \boldsymbol{\theta}_r + \lambda_r^{-1} \boldsymbol{\theta}_r^T \mathbf{K} \boldsymbol{\theta}_r)^{-\mathbf{1}}. \tag{40}$$

The displacement response spectrum is defined as $S_d(\omega_r, \zeta_r) = max|h_r(t)|$. The peak of each modal response over time can be provided in the following approximate form

$$max\left|a_r h_r(t) + c_r \dot{h}_r(t)\right| \approx \sqrt{a_r^2 max|h_r(t)|^2 + c_r^2 max\left|\dot{h}_r(t)\right|^2} \tag{41}$$

The stationarity assumption of the input which is preserved in the response processes allows for $h_r(t)$ and $\dot{h}_r(t)$ to be uncorrelated. Following the simplification that $max\left|\dot{h}_r(t)\right| \approx$

$\omega_r max|h_r(t)| = \omega_r S_d(\omega_r, \zeta_r)$ which is commonly employed to approximate pseudo-velocity response spectra, and considering Eqs.(39-41) the peak total responses in physical coordinates can be determined. The modal responses can be combined by the Square-Root-of-Sums-Squared (SRSS) method to obtain

$$max|y_j(t)| = \sqrt{\sum_{r=1}^{n} \Gamma_r^2 \, S_d^2(\omega_r, \zeta_r)} \qquad (42)$$

where the real-valued modal participation factors are defined as

$$\Gamma_r = \sqrt{(a_r^2 + \omega_r^2 c_r^2)} \qquad (43)$$

The SRSS rule is known to be adequate in cases where the modal coupling $\rho_{r,q}$ is relatively low or alternatively in cases where the modal frequencies are well separated. In an attempt to enhance the proposed method applicability, the utilization of a combination rule (e.g. [33]) which considers also the correlation between the modal responses (e.g. generalized Complete-Quadratic-Combination (CQC) method) has also been considered, and relevant remarks are provided on the successive section.

### 3.6 Discussion

A discussion on a number of important attributes which concerns advantages, limitations as well as potential practical applications of the proposed framework is herein presented.

Firstly, the proposed stochastic dynamics framework is in alignment with contemporary aseismic code provisions, since the seismic excitation is represented by a linear (pseudo-acceleration) response spectrum, $S_a$. Note in passing that $S_a$ is treated as the median response spectrum in the derivation of response spectrum statistically consistent power spectra.

Secondly, pertinent remarks should be given regarding the expected level of accuracy since the proposed method encompass a number of techniques which bear plausible limitations. The accuracy of the peak inelastic response estimates obtained by the proposed approach depends on the accuracy of the damping adjustment factors (e.g. [35,36]) used to define heavily damped spectra. Contemporary seismic design codes address this practical need by including relevant empirical formulae (e.g., Eq. (A.2) in the Appendix). Further, the well-reported in the literature accuracy of statistical linearization (e.g. [16]) may render the proposed method not sufficiently accurate for cases of particularly low-performing structures.

Another important issue which necessitates appropriate commenting is related with the selection of the modal combination method. The utilization of CQC combination rule in the proposed methodology has found to be unnecessary since it does not lead to any remarkable changes in terms of accuracy. Actually, in the iterative algorithm (see Fig. 1), and specifically in the redetermination of the diagonal excitation response spectrum in modal coordinates the contribution of the cross-spectral densities of the modal forces is omitted. The underlying simplification is based on the fact that in time-domain the response due to a modal force $Q_r(t)$ (see Eq.(38)) is almost statistically independent of the response due to $Q_q(t)$. In this regard, the modal forces cross-correlation terms are almost zero, with the only nonzero terms arising for $r = q$. Passing to the frequency-domain, this takes the form of low-valued cross spectral densities which justifies from a computational point of view their omission. The fully populated applied forces power spectrum needed for the iterative algorithm, can then be computed as

$$\mathbf{S_{FF}}(\omega) = \boldsymbol{T_G} \mathbf{S_{QQ}}(\omega)\, \boldsymbol{T_G}^{*\mathrm{T}}. \tag{44}$$

The proposed methodology can be applied to a time-variant nonlinear/hysteretic MDOF system, whether its linear part is classically damped or not (see Eq.(9)). Besides, in view of the statistical linearization implementation, the equivalent linearized system is inevitably accompanied by a damping matrix which does not follow any particular form comparing with its linear counterpart. Meanwhile, no restrictions are imposed on the excitation, with the only exception being the Gaussian assumption. Further, the proposed method retains the particular advantageous property of the standard modal decomposition method which allows for obtaining a reliable approximate estimation of the response considering only the first few modes (primary contributors) that capture the majority of the system energy. A particularly useful attribute for studying large-scale engineering structures such as bridges or high-rise buildings that may need thousands of DOFs to be modelled.

Code-compliant seismic design permits a structure to experience inelastic deformation under the imposed loads of the design earthquake. Relying upon the inherent ductility of conscientiously detailed buildings, a certain level of structural and nonstructural damage is being accepted. Towards achieving resilience in infrastructure, the structural protection concept of the dynamic vibration absorber is among the first strategies for passive vibration control of dynamically excited systems. Much of the early development has been devoted to the study of tuned mass-damper (TMD), which currently consists the most widely used dynamic vibration absorber in the practical engineering field. The objective of incorporating a TMD into a structural system, which in its simplest form, consists of a mass-spring-dashpot system attached to the primary structure, is basically to reduce the energy dissipation demand on the primary structure under the action of the induced forces. The reduction is achieved by transferring some of the structural vibrational energy to the TMD which has the limitation of being tuned to a single structural frequency; usually controlling the fundamental mode. It is thus expected that its effectiveness is the highest around this predominant structural mode of vibration. However, in cases of ordinary code-compliant structural systems where a nonlinear behaviour under the design loads is permitted, the implementation of a TMD device will face the known problem of ‘‘detuning’’ which is related with the softer behaviour of the yielding system and is accompanied by a significant decrease in the TMD vibration suppression effectiveness (e.g., [9]). The problem actually stems from the fact that the yielding structure would not retain the linear modal properties which formed the basis for tuning the TMD device in order to control its fundamental mode shape. The meaningful modal decoupling iterative algorithm which lies in the core of the proposed methodology can be well-exploited in resilience-based improvement strategies and particularly in the design and implementation of control systems. In this setting, the modal characteristics of the TMD (e.g., [37]) can be even dictated in real-time considering the induced seismic excitation based on the exhibited degree of nonlinearity rather than the design earthquake supplemented by the linearity assumption. The study of these potential applications warrants further research left for future work.

## 4 ILLUSTRATIVE APPLICATION

In this section the proposed stochastic dynamics technique (see Figure 1) is numerically exemplified by considering a yielding multi-storey frame structure subject to the Eurocode 8 elastic response spectrum [12] provided in the Appendix. The degree of accuracy of the predicted peak mean inter-storey drifts is quantified by comparison with pertinent results derived from nonlinear RHA for a large ensemble of time-realizations compatible with the considered Eurocode 8 response spectrum.

### 4.1 Non-classically damped elastoplastic MDOF frame structure

The three-storey non-classically damped inelastic shear frame shown in Figure 2 is considered to illustrate the proposed approach. The lumped masses $m_j$, the stiffness and damping coefficients of the $j$-th story, $k_j$ and $c_j$, respectively, are provided as $m_1 = m_2 = m_3 = 50\ ton$, $k_1 = 7.25\ MNm^{-1}$, $k_2 = 4\ MNm^{-1}$, $k_3 = 2\ MNm^{-1}$, $c_1 = 30\ kNsm^{-1}$, $c_2 = 20\ kNsm^{-1}$, and $c_3 = 10\ kNsm^{-1}$. The elastoplastic behavior of the shear frame is governed by the hysteretic relationship between the resisting story shearing forces and the corresponding inter-story drifts shown in Eqs.(20-22). The same relationship is assumed for all the three stories, whereas the yielding displacement $x_y$ is considered to be $5\ cm$. In an attempt to consider a range of inelastic behaviors various values of $\alpha_j$ (post-yield to pre-yield stiffness ratio) are assumed, specifically $\alpha_1 = 0.5$, $\alpha_2 = 0.6$ and $\alpha_3 = 0.7$.

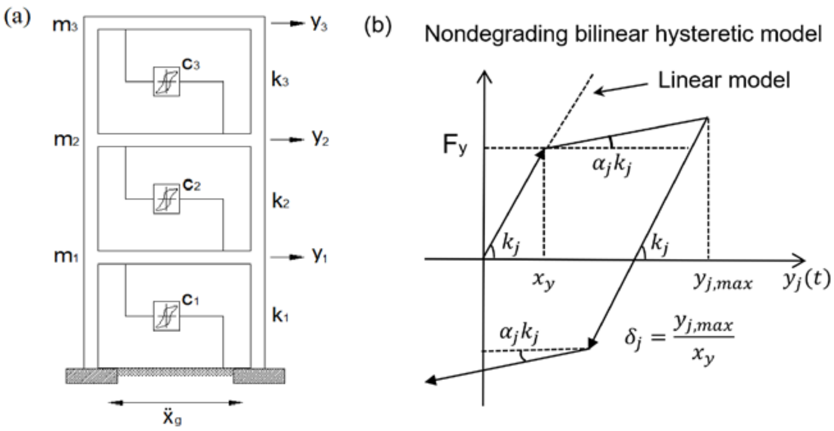


Figure 2. (a) The three-storey non-classically damped elastoplastic shear frame, and (b) the governing nonlinear restoring force-deformation law and definition of ductility ratio $\delta_j$

For excitations that will deform the structure beyond the nominal $x_y$ it is expected that the bilinear hysteretic model under the implementation of statistical linearization, will provide equivalent modal damping ratios with higher values comparing to their linear counterparts, namely $\zeta_{\mathrm{req}} > \zeta_r^o$.

### 4.2 Derivation of Eurocode 8 compatible power spectra

The Eurocode 8 elastic response spectrum for soil conditions B, critical damping ratio $\zeta_o = 5\%$, and peak ground acceleration (PGA) equal to $0.36g$ is initially considered for exciting the structure in Figure 2. The employed spectrum is provided in the Appendix and plotted in Figure 3 (black continuous curve) against the natural period $T$=2π/$\omega$. Next, the duration $T_s$ is taken equal to $20\ s$, whereas the discretization step $\Delta\omega$ in Eq.(8) is set equal to $0.1\ rad/s$. A preselection of an input power spectrum shape has to be preceded for deriving a stochastically compatible spectrum, according to the numerical scheme presented in section 3.1. In the ensuing analysis, the Clough and Penzien (CP) spectrum is considered, i.e.,

$$D_{X_i}(\omega_i) = \frac{(\omega_i/\omega_f)^4}{\left(1-(\omega_i/\omega_f)^2\right)^2 + 4\xi_f^2(\omega_i/\omega_f)^2} \frac{\omega_g^4 + 4\xi_g^2\omega_g^2\omega_i^2}{(\omega_g^2-\omega_i^2)^2 + 4\xi_g^2\omega_g^2\omega_i^2} \tag{45}$$

where the requisite parameters are $\xi_g = 0.78,\ \omega_g = 10.78\,rad/s,\ \ \xi_f = 0.92\ \ and\ \ \omega_f = 2.28\,rad/s$. The parameters $\omega_g$ and $\xi_g$ describe the filtering effects of the geological formations on the propagation of the seismically induced waves, whereas $\omega_f$ and $\xi_f$ control the incorporated CP high-pass filter to suppress the low frequencies. Note that a selection of a white-noise (WN) input spectrum shape could lead to reliable approximations of the power spectrum for the most of the practical applications of engineering interest.

The achieved level of compatibility between the derived power spectrum $G_{X_i}^{\zeta_o}(\omega)$ and the response spectrum $S_a$ is presented in Figure 3 by comparing the assigned $S_a$ with the response spectrum computed by Eq. (3) (broken line). A further comparison is shown in terms of a pertinent Monte Carlo based analysis which involves a large ensemble of 5000 stationary signals of 20s duration each compatible with the $G_{X_i}^{\zeta_o}(\omega)$ spectrum. The median response spectrum of these signals is plotted (dotted line) in Figure 3; a more detailed documentation accompanied by the appropriate commentary can be found in Mitseas et al., 2018.

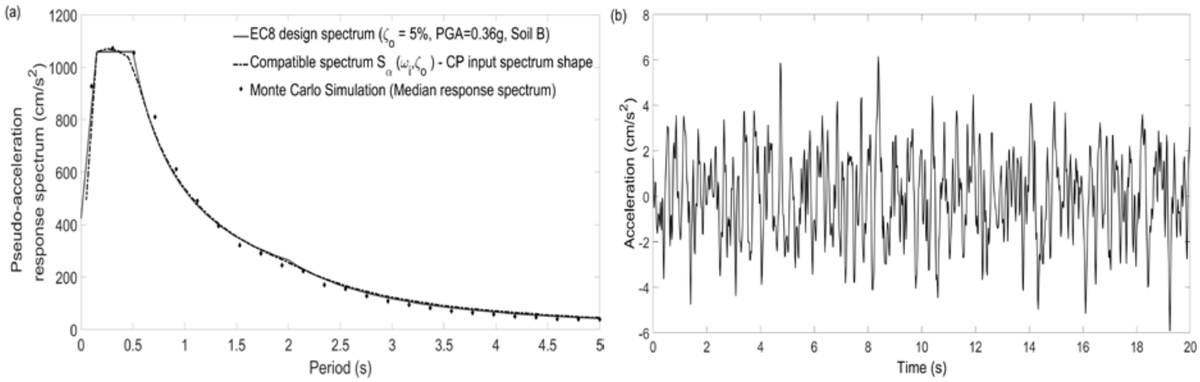


Figure 3. (a) Eurocode 8 response spectrum for 5% damping ratio and compatibility assessment with the power spectrum using Eq.(3) and nonlinear RHA. (b) Time-history of an arbitrarily chosen artificial accelerogram from the ensemble of 5000 signals compatible with the assigned Eurocode 8 spectrum.

### 4.3 System decoupling and derivation of the forced vibrational modal properties

Following the efficient modal decoupling iterative algorithm delineated in sections 3.1-3.4, the nonlinear *n*-DOF system is decoupled and cast into (*n*) uncoupled modal oscillators. To this aim, a number of successive iterations is deemed necessary. The employed thresholds concerning the convergence checks are set to $\beta_1 = \beta_2 = \beta_3 = 10^{-4}$. Sets of three $\omega_{\mathrm{req}}^{(k)}$ and $\zeta_{\mathrm{req}}^{(k)}$ EMPs $r = 0,1,2,3$ corresponding to the three mode shapes of the system are derived as by-products of the iterative algorithm as a function of the iteration index k. The iteratively repeated convergence process concerning each mode separately is terminated when successive values of the corresponding equivalent modal damping ratio $\zeta_{\mathrm{req}}^{(k)}$ display difference lower than the very small threshold $\beta_3$ (see also Figure 1). The values of the stochastically derived sets of

EMPs attained at the last iteration correspond to the forced vibrational modal properties of the system. To illustrate the convergence rate, the derived EMPs $\omega_{\mathrm{req}}^{(k)}$ and $\zeta_{\mathrm{req}}^{(k)}$ are plotted in Figures 4(a) and (b), respectively, as a function of the iteration index k. It is readily seen that convergence is achieved after a small number of iterations for all the modes.

It is vital to bear in mind that deriving in general effective/equivalent linear properties (either stochastically or deterministically), and using them directly on response/design spectra defined for different damping ratios without updating the excitation response spectrum itself, violates the basic definition of the response spectrum-based analysis which requires the considered linear/linearized oscillators and the imposed elastic response UHS to share the same damping premises. In this regard, the herein defined forced vibrational modal properties can be used directly in conjunction with the corresponding appropriately updated response UHS to estimate the peak inelastic modal responses of the system. Subsequently, the generalized modal combination method, outlined in section 3.5, is employed for nonlinear response estimates in physical coordinates.

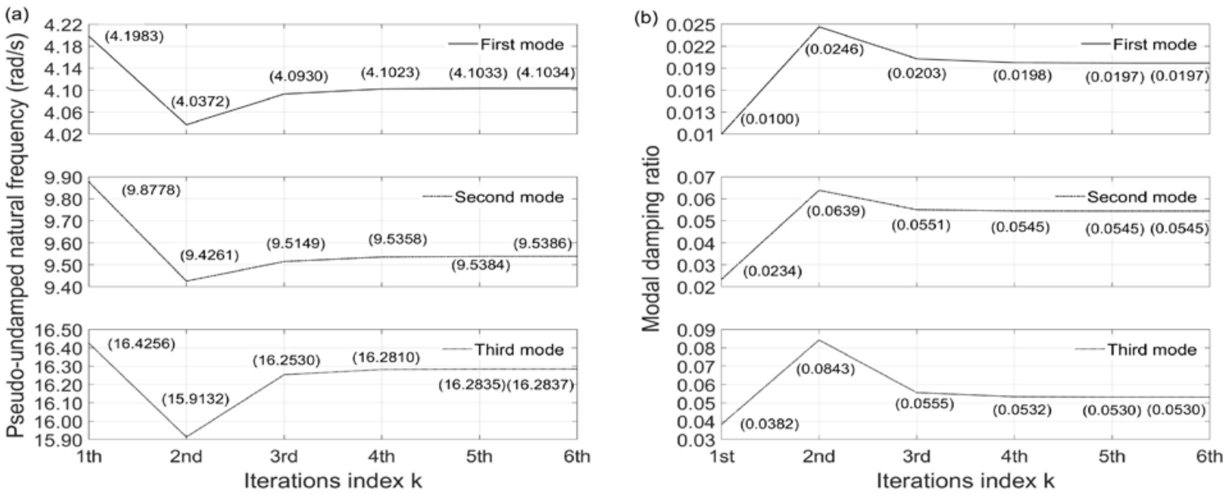


Figure 4. Equivalent pseudo-undamped natural frequency and modal damping ratio coefficients from successive iterations. The forced vibrational modal properties correspond to the EMPs values of the last iteration.

It is noteworthy that the EMPs show considerable versatility and efficacy by fluctuating significantly their values in an attempt to imprint and follow in the highest possible detail the impact of the stimulation of the system in the nonlinear range. In this setting, the forced vibrational modal properties are not only able to capture the trend of the inelastic behavior but they are also clearly amenable to a physical meaning, offering a solid basis for interpreting the underlying structural dynamics. Note, however, that as the degree of nonlinearity increases, the rate of convergence for the three involved checks (see Figure 1) will tend to slow down.

To illustrate the effect of nonlinearity, the equivalent modal impulse response functions are plotted in Figures 5 (a) and (b) for the non-classically damped elastoplastic shear frame shown in Figure 2(a), for the cases of the design earthquake delineated in section 4.2 as well as for an earthquake of reduced intensity not adequate enough to force the structure into the nonlinear range.

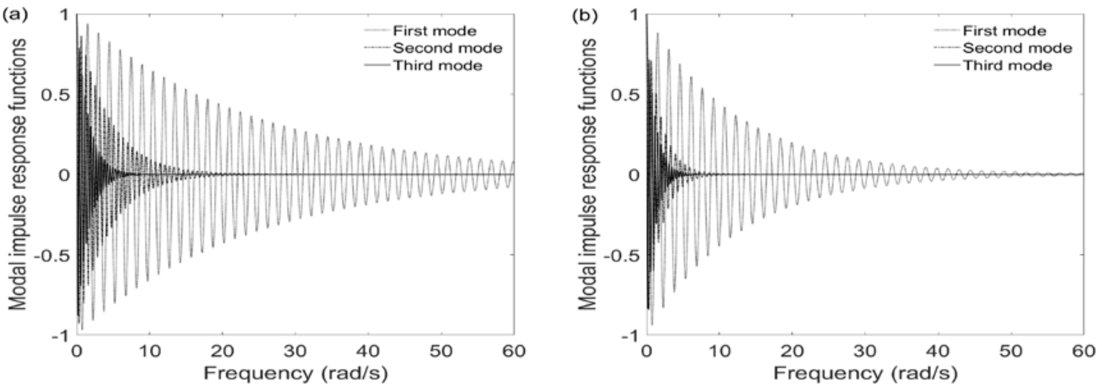


Figure 5. (a) Modal impulse response functions (elastic behavior), and (b) Equivalent modal impulse response functions (elastoplastic behavior).

It can be readily seen that for the case of the yielding system, softer and heavier damped equivalent modal oscillators are determined. Note that the higher values of the real parts of the eigenvalues, lead to higher $\zeta_{\mathrm{req}}$ reflecting the increased energy dissipation mechanism through the severe nonlinear/hysteretic behaviour. Clearly, the decaying trend of the modal vibrations has been amplified, whereas a closer look reveals also higher $\mathrm{T}_{\mathrm{req}}$ values comparing to the corresponding pre-yield $\mathrm{T}_r^o$ periods; see cases (b) and (c) in Table 1.

In the special case of an undamped linear system all the eigenvalues occur in entirely imaginary complex pairs (see Table 1), while the associated eigenvectors $\boldsymbol{\psi}_r$ are real-valued, namely $\boldsymbol{y}_r = \mathbf{0}$, $\boldsymbol{\psi}_r = \bar{\boldsymbol{\psi}}_r = \boldsymbol{\varphi}_r$ (see Eq.(36)), and $\omega_r = \omega_r^o$. In the herein study the eigenvectors are normalized such that the real part of the displacement of the first floor is unity whereas the corresponding imaginary is zero. Further, it is shown that the natural frequency of the highest mode of a damped system is always less than or equal to the corresponding undamped frequency, no matter whether damping is classical or nonclassical. The pseudo-undamped natural frequency $\omega_r$ of the lowest mode may be equal or even higher than the corresponding undamped frequency $\omega_r^o$ of a linear MDOF system. These observations concerning systems that have not entered into the inelastic range yet are in perfect alignment with pertinent numerical results reported in the literature (e.g. [2,8]).

Table 1. Free and forced vibrational modal characteristics of the system shown in Figure 2(a)

| Parameter | Floor level | First mode | Second mode | Third mode |
|---|---|---|---|---|
| | | (a) Undamped linear system | | |
| $\omega_r^o$ | | 4.1983 | 9.8777 | 16.4257 |
| $\lambda_r$ | | 4.1983i | 9.8777i | 16.4257i |
| $\boldsymbol{\theta}_r$ | 1 | 1.0000 | 1.0000 | 1.0000 |
| | 2 | 2.5922 | 1.5929 | -0.5601 |
| | 3 | 4.6343 | -1.1068 | 0.0975 |
| $\zeta_r^o$ | | 0.0000 | 0.0000 | 0.0000 |
| | | (b) Non-classically damped pre-yield system (linear range) | | |
| $\omega_r$ | | 4.1983 | 9.8778 | 16.4256 |
| $\lambda_r$ | | -0.0419 + 4.1981i | -0.2308 + 9.8751i | -0.6273 +16.4136i |
| $\boldsymbol{\theta}_r$ | 1 | 1.0000 + 0.0000i | 1.0000 + 0.0000i | 1.0000 + 0.0000i |
| | 2 | 2.5921 + 0.0063i | 1.5926 + 0.0122i | -0.5603 + 0.0063i |
| | 3 | 4.6341 + 0.0150i | -1.1067 + 0.0035i | 0.0975 + 0.0017i |
| $\zeta_r$ | | 0.0100 | 0.0234 | 0.0382 |
| | | (c) Non-classically damped yielding system (nonlinear range) | | |

| | | | | |
|---|---|---|---|---|
| $\omega_{req}$ | | 4.1034 | 9.5386 | 16.2837 |
| $\lambda_r$ | | -0.0808 + 4.1026i | -0.5194 + 9.5245i | -0.8635 +16.2608i |
| $\boldsymbol{\theta}_r$ | 1 | 1.0000 + 0.0000i | 1.0000 + 0.0000i | 1.0000 + 0.0000i |
| | 2 | 2.6359 + 0.0284i | 1.6875 + 0.1036i | -0.5374 + 0.0421i |
| | 3 | 4.9710 + 0.1640i | -1.1001 + 0.0065i | 0.0831 + 0.0211i |
| $\zeta_{req}$ | | 0.0197 | 0.0545 | 0.0530 |

The real $\boldsymbol{\varphi}_r$ and imaginary $\boldsymbol{y}_r$ elements of the complex-valued eigenvectors for the considered MDOF system are plotted in Figure 6. The imaginary component $\boldsymbol{y}_r$ seems to be particularly sensitive to the amount of damping as well as to the implementation of statistical linearization itself. On the contrary, the real component is almost solely affected from the degree of the exhibited nonlinearity, rather than from the amount of the total damping present. In the undamped linear system case, the mode shapes are purely real-valued. Note that the imaginary element $\boldsymbol{y}_r$ may be quite substantial for high damping values of the vibrating modes.

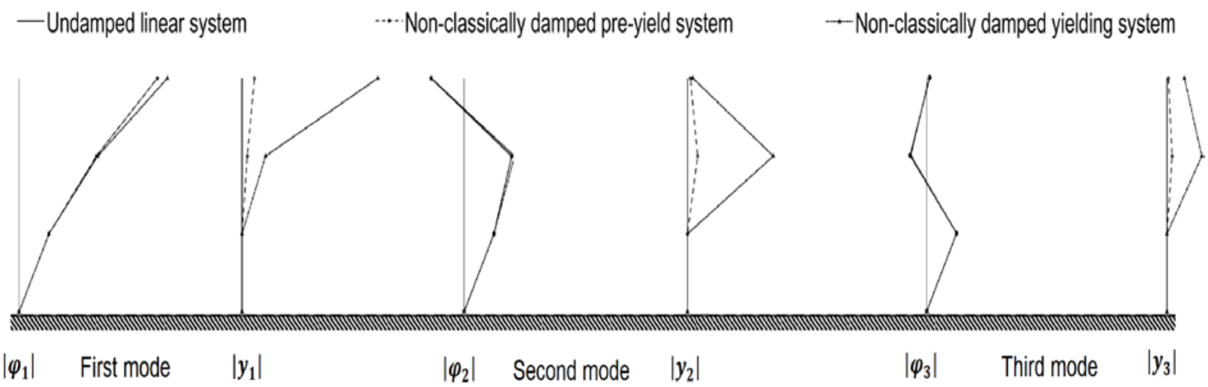

Figure 6. Effect of damping and nonlinearity on the natural modes of the system.

It should be recalled that the arrangement of the pseudo-undamped natural frequencies is in an ascending order. The forced vibrational modal properties can now be used in conjunction with the appropriately defined linear response spectra $S_a(\omega_i, \zeta_o)$ for spectral ordinates reading.

### 4.4 Modal peak inelastic responses utilizing the forced vibrational modal properties and assessment via nonlinear RHA

The convergence rate of the proposed iterative scheme is reasonably fast, and the stabilization of the estimates is achieved after a small number of iterations, as it is depicted in Figures 4 (a) and (b). Upon convergence of the equivalent modal damping ratios (i.e., $\left|\zeta_{\text{req}}^{(k)} - \zeta_{\text{req}}^{(k-1)}\right| < \beta_3$), the obtained pairs of the forced vibrational modal properties $\omega_{\text{req}}^{(k_{end})}$ and $\zeta_{\text{req}}^{(k_{end})}$ can be used directly in conjunction with the assigned Eurocode 8 elastic response spectrum for estimating the peak modal inter-storey drifts of the shearing frame shown in Figure 2(a). Note that the equivalent linear modal oscillator natural period, is computed as $\text{T}_{\text{req}} = 2\pi/\omega_{\text{req}}$. In Figure 7 the Eurocode 8 spectrum is plotted against the natural period for the different values of damping ratio $\zeta_{\text{req}}$ found, in terms of spectral acceleration, $S_\alpha$, and spectral displacement, $S_{\text{d}}$.

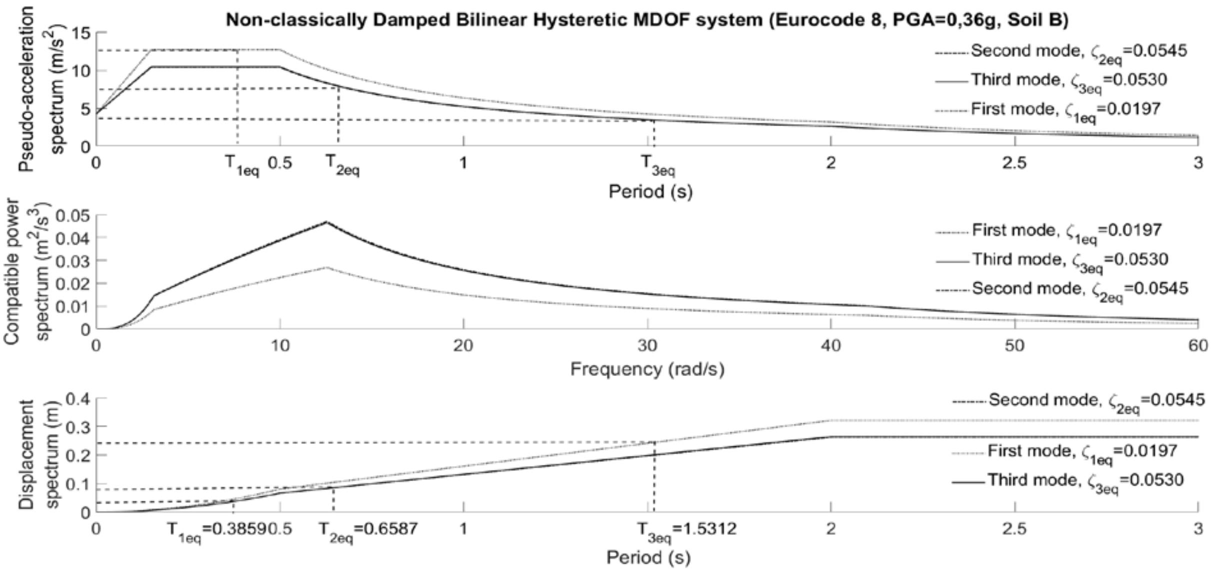


Figure 7. Peak modal inelastic responses using the forced vibrational modal properties in conjunction with Eurocode 8 response spectrum.

Subsequently, the complex-valued mode shapes associated with the obtained forced vibrational modal properties define the participation factors $\Gamma_r$ which in turn participate through Eq.(42) in the determination of the peak inter-storey drifts in physical coordinates.

Proposed methodology-based data are compared with nonlinear RHA within a Monte-Carlo simulation context utilizing an ensemble of 5000 artificial acceleration time-histories compatible in the mean sense with the Eurocode 8 elastic response spectrum for 5% damping. The considered time-histories are stationary in amplitude: an arbitrarily chosen time-history is plotted in Figure 3(b). Pictorially, the achieved degree of compatibility between the median response spectrum and the Eurocode 8 is shown in Figure 3(a). These time-histories are derived by generating 5000 stationary realizations compatible with the Clough-Penzien spectrum using the spectral representation technique of Shinozuka and Deodatis [38] applied in a number of studies (e.g. [39]). Lastly, the nonlinear differential equations of motion in Eq.(9) are numerically integrated via a standard fourth order Runge-Kutta scheme, and finally, system response statistics are obtained based on the ensemble of the response realizations. Further, the average ductility ratio which is defined as $\delta_j = |y_{j,max}(t)|/x_y$ where $j = 0,1,2, \dots,$ n, can serve as an excellent indicator of the degree of the exhibiting nonlinearity (e.g. [40]). For comparison reasons, results pertained to the derived EMPs after the first iteration of the proposed framework are included, hence, the corresponding estimates for the peak inter-storey drifts (in cm) are 6.79, 10.32, and 15.30, whereas the associated estimation error is gauged as 7.74%, 5.49%, and 11.51%. Harnessing the potential of the forced vibrational modal properties $\omega_{\mathrm{req}}^{(k_{end})}$ and $\zeta_{\mathrm{req}}^{(k_{end})}$ of the system, comprehensive numerical data are presented in Table 2.

Table 2. Average ductility ratios and comparison of peak inter-storey drift estimates

| Floor level | $\delta_j$ | Peak inter-storey drifts (cm) | Nonlinear RHA (cm) | Percentage error |
|---|---|---|---|---|
| 1 | 1.47 | 7.04 | 7.36 | 4.30% |
| 2 | 2.18 | 10.80 | 10.92 | 1.06% |
| 3 | 3.46 | 16.52 | 17.29 | 4.43% |

Based on the presented results, the proposed methodology evinces that the iterative scheme contributes substantially in the enhancement of the accuracy through identifying efficiently equivalent forced vibrational modal properties based on the degree of the exhibiting nonlinearity. Note that the proposed method leads to substantial reduction of computational effort as compared with nonlinear RHA within a MCS framework. In this setting, to provide with an indicative order of magnitude for the computational cost involved, utilizing a laptop computer with standard configurations, the proposed technique requires 1-2 min, whereas the MCS based system peak response estimation (5000 time histories) requires 5–6 h. It is further important to note that a range of ductility demands has been considered herein, reflecting nonlinear behaviors of various ratings, from a mild ($\delta_1$=1.47) to a much stronger ($\delta_3$=3.46) one. Specifically, this observation confirms that the proposed forced vibrational modal properties imprint in the highest possible detail the impact of the stimulation of the system in the nonlinear range, thus, they turned out to be particular useful for system identification purposes as well as for conducting efficiently the nonlinear counterpart of modal analysis in the response-spectrum variant. Lastly, it is worth-mentioning that the modal nature of the proposed stochastic dynamics method allows for physical insights into the underlying structural dynamics that classic computationally demanding time-history methods cannot provide.

## 5 CONCLUDING REMARKS

A novel inelastic modal decomposition method has been proposed for conducting dynamic response analysis of non-classically damped bilinear hysteretic MDOF structural systems excited by an elastic response UHS (e.g., Eurocode 8), circumventing the need of undertaking computationally demanding nonlinear RHA. Specifically, the proposed methodology is provided in the rather advantageous response-spectrum variant rather than in a time-history version, in an attempt to increase its attractiveness among the engineers of practice who are mainly accustomed with this idea. This asset hopefully qualifies the herein proposed approach as a potent analysis tool for preliminary seismic design of yielding structures, without any restrictions on the nature of the damping matrices (e.g. such problems fairly arise in equipment-structure-type systems). It is noteworthy that the seismic demands are imposed by an assigned response spectrum representing the seismic hazard, thus, the proposed approach can readily handle specifications prescribed by various aseismic code provisions.

The developed framework initiates by solving a series of inverse stochastic dynamics problems for the determination of input power spectra representing time-limited stationary processes compatible in a stochastic sense with an assigned response spectrum for a nominal damping ratio. Relying on statistical linearization and state-variable formulation the complex eigenvalue problem considering the MDOF system subject to a vector of stochastic seismic processes characterized by the derived power spectra is addressed. Then, EMPs, namely equivalent pseudo-undamped natural frequencies and equivalent modal damping ratios are assigned to each mode of oscillation. The determination of EMPs is repeated iteratively for each monitored mode of vibration upon updating the damping ratio of the corresponding excitation response spectrum with the equivalent modal damping ratio. Upon convergence of the damping ratio of the response spectrum with the equivalent modal damping ratio (i.e., equality within some tolerance), the forced vibrational modal properties of each mode of vibration are used together with the excitation response spectrum for different damping ratios to obtain peak modal nonlinear response estimates. Lastly, the real-valued modal participation factors are determined for the complex-valued mode shapes corresponding to the displacement coordinates and generalized SRSS is employed as the modal combination rule for determining the peak total nonlinear response estimates of the MDOF structure in physical coordinates.

Particular attention has been given to identify and elucidate the physical significance of the forced-dependent vibrational modal properties and to simplify their implementation for the evaluation of the MDOF system dynamic response and interpretation of the underlying structural dynamics. The modal nature of the proposed method allows for physical insights that classic time-history methods cannot provide. It has been shown that the displacements of a non-classically damped, nonlinear MDOF system may be expressed as a linear combination of the displacements of a number of excited modal oscillators by response spectra appropriately adjusted to the forced vibrational modal characteristics corresponding to each mode of vibration. Still, it is recognized that the exhibited accuracy of the approach is inevitably constraint by the well-reported in the literature accuracy of statistical linearization and, therefore, it may not provide sufficiently accurate peak response estimates for low-performing structures.

The concepts involved have been numerically illustrated using a three-storey bilinear hysteretic frame structure exposed to a Eurocode 8 elastic response spectrum. Moreover, nonlinear RHA involving a large ensemble of stationary accelerograms whose median response spectrum matches closely to the considered Eurocode 8 spectrum has been conducted to assess the accuracy of the proposed framework.

## APPENDIX: EUROCODE 8 ELASTIC RESPONSE SPECTRUM

The elastic pseudo-acceleration response spectrum for linear oscillators with critical damping ratio $\zeta$ and natural period $T = 2\pi/\omega$ is defined in the European aseismic code [12] by the following expressions:

$$S_\alpha(T,\zeta) = \alpha \times \begin{cases} S\left[1 + \dfrac{T}{T_B}(2.5\eta - 1)\right], & 0 \leq T \leq T_B \\ 2.5S\eta, & T_B \leq T \leq T_c \\ 2.5S\eta\dfrac{T_C}{T}, & T_C \leq T \leq T_D \\ 2.5S\eta\dfrac{T_C T_D}{T^2}, & T_D \leq T \leq T_E \\ S\dfrac{T_C T_D}{T^2}\left[2.5\eta + \dfrac{T - T_E}{T_F - T_E}(1 - 2.5\eta)\right], & T_E \leq T \leq T_F \\ S\dfrac{T_C T_D}{T^2}, & T_F \leq T \end{cases} \qquad (A.1)$$

with

$$\eta = \sqrt{\frac{10}{5 + \zeta}} \geq 0.55 \qquad (A.2)$$

where α is the peak ground acceleration (PGA), $S$ is a soil-dependent amplification factor, and $T_B$, $T_C$, $T_D$, $T_E$ and $T_F$ are soil-dependent corner periods. For soil type B: $S = 1.20, T_B = 0.15, T_C = 0.5, T_D = 2.0, T_E = 5.0, T_F = 10$.

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