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Playing Muller Games in a Hurry*†

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This work considers a finite-duration variant of Muller games, and their connection to infinite-duration Muller games. In particular, it studies the question of how long a finite-duration Muller game must be played before the winner of the finite-duration game is guaranteed to be able to win the corresponding infinite-duration game. Previous work by McNaughton has shown that this must occur after $\prod_{j=1}^{n}(j!+1)$ moves, and the reduction from Muller games to parity games gives a bound of $n \cdot n! + 1$ moves. We improve upon both of these results, by giving a bound of 3^n moves.

Keywords: Muller Games; Zielonka's Algorithm; Winning Strategies.

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1. Introduction

In an infinite game, two players move a token through a finite graph, thereby constructing an infinite path. The winner is determined by a winning condition, which partitions the infinite paths of the graph into winning paths for Player 0 and winning paths for Player 1. Many winning conditions depend on the vertices that are visited infinitely often, i.e., the winner of a play cannot be determined after a finite number of steps. We study the following question: is it possible to give a criterion to define a finite duration variant of an infinite game? Such a criterion has to stop

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a play after a finite number of steps and then declare a winner based on the finite play constructed thus far. It is sound if Player 0 has a winning strategy for the infinite duration game if and only if Player 0 has a winning strategy for the finite duration game.

McNaughton considered the problem of playing infinite games in finite time from a different perspective. His motivation was to make infinite games suitable for "casual living room recreation" [8]. As human players cannot play infinitely long, he envisions a referee that stops a play at a certain time and declares a winner. The justification for declaring a winner is that "if the play were to continue with each [player] playing forever as he has so far, then the player declared to be the winner would be the winner of the infinite play of the game" [8].

Besides this recreational aspect of infinite games there are several interesting theoretical questions that motivate this problem. A sound criterion to stop a play after at most n steps yields a simple algorithm to determine the winner of the infinite game: the finite duration game can be seen as a reachability game on a finite tree of depth at most n that is won by the same player that wins the infinite duration game. There exist simple and efficient algorithms to determine the winner in reachability games on trees and thus also to determine the winner of the infinite duration game. Furthermore, if winning strategies for the reachability game can be turned into (small) finite-state winning strategies for the infinite duration game, then this may yield strategies with memory bounds that are better than those obtained through game reductions. This is because the bounds obtained from game reductions ignore the structure of the arena. Therefore, we may be able to improve upon these results in the average case, although the worst case bounds given by Dziembowski, Jurdziński, and Walukiewicz [3] will continue to hold.

Consider the following criterion: the players move the token through the arena until a vertex is visited twice. An infinite play can then be obtained by assuming that the players continue to play the loop that they have constructed, and the winner of the finite play is declared to be the winner of this infinite continuation. If the game is determined with positional strategies for both players, then this criterion is sound: if a player has a positional winning strategy for the infinite game, then this strategy can be used to win the finite version of the game and vice versa.

Therefore, McNaughton considered games that are not positionally determined. Here, the first loop does not determine an entire infinite play, as memory allows a player to make different decisions when a vertex is seen again. Therefore, the players have to play longer before the play can be stopped and analyzed.

McNaughton considered Muller games, which have the form $(G, \mathcal{F}_0, \mathcal{F}_1)$, where G is a finite arena and $(\mathcal{F}_0, \mathcal{F}_1)$ is a partition of the powerset of the vertices. Player i wins a play if the set of vertices visited infinitely often is in $\mathcal{F}_i$. Muller winning conditions allow us to express all other winning conditions that depend only on the infinity set of a play (e.g., Büchi, co-Büchi, parity, Rabin, and Streett conditions).

To give a sound criterion for Muller games, McNaughton defined for every set of vertices F a scoring function Sc_F that keeps track of the number of times the set F

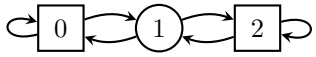

Fig. 1. An arena G.

was visited entirely since the last visit of a vertex that is not in F. In an infinite play, the set of vertices seen infinitely often is the unique set F such that Sc_F tends to infinity after being reset to 0 only a finite number of times.

Let G be the arena in Figure 1 (Player 0's vertices are shown as circles and Player 1's vertices are shown as squares) and consider the Muller game $\mathcal{G} = (G, \mathcal{F}_0, \mathcal{F}_1)$ with $\mathcal{F}_0 = \{\{0,1,2\},\{0\},\{2\}\}$. In the play 100122121 the score for the set $\{1,2\}$ is 3, as it was seen thrice (i.e., with the infixes 12, 21, and 21). Note that the order of the visits to the elements of F is irrelevant and that it is not required to close a loop in the arena. The following winning strategy for Player 0 bounds the scores of Player 1 by 2: arriving from 0 at 1 move to 2 and vice versa. However, Player 0 cannot avoid a score of 2 for Player 1, as either the play prefix 1001 or 1221 is consistent with every winning strategy.

McNaughton proved the following criterion to be sound [8]: stop a play after a score of $|F|! + 1$ for some set F is reached for the first time, and declare the winner to be the Player i such that $F \in \mathcal{F}_i$. However it can take a large number of steps for a play to reach a score of $|F|! + 1$, as scores may increase slowly or be reset to 0. It can be shown that a play must be stopped by this criterion after at most $\prod_{j=1}^{|G|}(j!+1)$ steps. Furthermore, there are examples in which it takes at least $\frac{1}{2}\prod_{j=1}^{|G|}(j!+1)$ steps before the criterion declares a winner.

The reduction from Muller games to parity games [5, 7] provides another sound criterion. The reduction constructs a parity game of size $|G| \cdot |G|!$, and since parity games are positionally determined, a winner can be declared after the players construct a loop in the parity game. This gives a sound criterion that stops a play after at most $|G| \cdot |G|! + 1$ steps.

Our contribution. Our goal is to improve the stopping criterion given by McNaughton. While he showed that stopping the play after a score of $|F|! + 1$ has been reached for some set F yields a sound criterion, we will show that stopping the play after a score of 3 has been reached for some set still yields a sound criterion. This is somewhat surprising, since the threshold is independent of the size of the arena and the complexity of the partition $(\mathcal{F}_0, \mathcal{F}_1)$. This result is obtained by using the internal structure of the winning regions computed by Zielonka's algorithm [10] to carefully define a winning strategy that bounds the scores of the opponent by 2. This suffices, since the score of some set must be unbounded in every infinite play.

In the example above, we have shown that Player 0 cannot avoid a score of 2 for some set in $\mathcal{F}_1$. However, this does not rule out that using the threshold of 2 still yields a sound criterion. This is because in the example there is always some set in $\mathcal{F}_0$ that reaches a score of 2 before a set in $\mathcal{F}_1$ does. In contrast to this, we will provide an example upon which the threshold of 2 does not yield a sound criterion.

Hence, the threshold of 3 in our main theorem is optimal.

We complement this by proving that a score of 3 must be reached after at most $3^{|G|}$ steps. Hence, we obtain a better bound than $|G| \cdot |G|! + 1$ steps and $\prod_{j=1}^{|G|}(j! + 1)$ steps, which were derived from waiting for a repetition of memory states or McNaughton's criterion, respectively.

Related work. Usually, the quality of a strategy is measured in terms of memory needed to implement it. However, there are other quality measures of winning strategies. Chatterjee, Henzinger, and Horn have studied a strengthening of parity objectives, where a bound between the occurrences of even colors is required [2]. Another quality measure appears in work on request-response games [6, 11], where waiting times between requests and their responses are used to define the value of a play. There it is shown that time-optimal winning strategies can be computed effectively. The maximal score achieved by the opponent is a quality measure for winning strategies in a Muller game. Player 0 prefers plays with small scores for Player 1, which corresponds to not spending a long time in a set of the opponent.

Bernet, Janin, and Walukiewicz used a reduction from parity games to safety games in order to compute the most permissive multi-strategy in a parity game [1]. Such a strategy encompasses the behaviors of all positional winning strategies. Furthermore, the reduction also allows us to compute the winning regions in the parity game by computing the winning regions in the safety game.

This paper is structured as follows. Section 2 contains basic definitions and fixes our notation. In Section 3, we introduce the scoring functions, prove some properties about scoring, and define finite-time Muller games. In Section 4, we present Zielonka's algorithm which is used in Section 5 to prove the main result. Section 6 ends the paper with a conclusion and some pointers to further research.

2. Definitions

The power set of a set S is denoted by 2^S and $\mathbb{N}$ denotes the non-negative integers. The prefix relation on words is denoted by $\sqsubseteq$, its strict version by $\sqsubset$. Given a word $w = xy$, define $x^{-1}w = y$ and $wy^{-1} = x$.

An arena $G = (V, V_0, V_1, E)$ consists of a finite, directed graph (V, E) and a partition (V_0, V_1) of V denoting the positions of Player 0 (drawn as circles) and Player 1 (drawn as squares). We require that every vertex has at least one outgoing edge. A set $X \subseteq V$ induces the subarena $G[X] = (V \cap X, V_0 \cap X, V_1 \cap X, E \cap (X \times X))$, if every vertex in X has at least one successor in X. A Muller game $\mathcal{G} = (G, \mathcal{F}_0, \mathcal{F}_1)$ consists of an arena G and a partition $(\mathcal{F}_0, \mathcal{F}_1)$ of 2^V.

A play in G starting in $v \in V$ is an infinite sequence $\rho = \rho_0\rho_1\rho_2\ldots$ such that $\rho_0 = v$ and $(\rho_n, \rho_{n+1}) \in E$ for all $n \in \mathbb{N}$. The occurrence set $\mathrm{Occ}(\rho)$ and infinity set $\mathrm{Inf}(\rho)$ of ρ are given by $\mathrm{Occ}(\rho) = \{v \in V \mid \exists n \in \mathbb{N} \text{ such that } \rho_n = v\}$ and $\mathrm{Inf}(\rho) = \{v \in V \mid \exists^\omega n \in \mathbb{N} \text{ such that } \rho_n = v\}$. We also use the occurrence set of a finite play w. A play ρ in a Muller game is winning for Player i if $\mathrm{Inf}(\rho) \in \mathcal{F}_i$.

A strategy for Player i is a function $\sigma\colon V^*V_i \to V$ satisfying $(v, \sigma(wv)) \in E$ for

all $w \in V^*$ and all $v \in V_i$. The play ρ is consistent with σ if $\rho_{n+1} = \sigma(\rho_0 \dots \rho_n)$ for every $n \in \mathbb{N}$ with $\rho_n \in V_i$. The set of strategies for Player i is denoted by Π_i. The unique play starting at $v \in V$ that is consistent with $\sigma \in \Pi_i$ and $\tau \in \Pi_{1-i}$ is denoted by $\mathrm{Play}(v, \sigma, \tau)$. A strategy σ for Player i is positional, if $\sigma(wv) = \sigma(v)$ for every $w \in V^*$ and every $v \in V_i$. Hence, we denote a such a strategy by $\sigma \colon V_i \to V$.

A strategy σ for Player i is a winning strategy from a vertex $v \in V$, if every play that starts in v and is consistent with σ is won by Player i. The strategy σ is a winning strategy for a set of vertices $W \subseteq V$, if every play that starts in some $v \in W$ and is consistent with σ is won by Player i. The winning region W_i contains all vertices from which Player i has a winning strategy. A game is determined if W_0 and W_1 form a partition of V.

Theorem 1 ([5]) *Muller games are determined.*

Let $G = (V, V_0, V_1, E)$ be an arena and let $X \subseteq V$ be a set that induces a subarena. The attractor for Player i of a set $F \subseteq V$ in X is $\mathrm{Attr}_i^X(F) = \bigcup_{n=0}^{|V|} A_n$ where $A_0 = F \cap X$ and

$$\begin{aligned} A_{n+1} = A_n &\cup \{v \in V_i \cap X \mid \exists v' \in A_n \text{ such that } (v, v') \in E\} \\ &\cup \{v \in V_{1-i} \cap X \mid \forall v' \in X \text{ with } (v, v') \in E : v' \in A_n\} \ . \end{aligned}$$

A set $X \subseteq V$ is a trap for Player i, if all outgoing edges of the vertices in $V_i \cap X$ lead to X and at least one successor of every vertex in $V_{1-i} \cap X$ is in X.

Lemma 2. *Let G be an arena with vertex set V and $F, X \subseteq V$ such that X induces a subarena.*

(1) Player i has a positional strategy to bring the play from every $v \in \mathrm{Attr}_i^X(F)$ into F.
(2) The set $V \setminus \mathrm{Attr}_i^X(F)$ induces a subarena and is a trap for Player i in G.

A strategy as in (1) is called attractor strategy.

3. The Scoring Functions and Finite-time Muller Games

This section introduces the notions that are required to formally define finite-time Muller games. In his study of these games, McNaughton introduced the concept of a score. For every set of vertices F the score of a finite play w is the number of times that F has been visited entirely since w last visited a vertex in $V \setminus F$.

Definition 3 (Score) *For every $F \subseteq V$ we define $\mathrm{Sc}_F \colon V^+ \to \mathbb{N}$ as*

$$\begin{aligned} \mathrm{Sc}_F(w) = \max\{k \in \mathbb{N} \mid &\exists x_1, \dots, x_k \in V^+ \textit{ such that} \\ &\mathrm{Occ}(x_i) = F \textit{ for all } i \textit{ and } x_1 \cdots x_k \textit{ is a suffix of } w\} \ . \end{aligned}$$

We extend this notion by introducing the concept of an accumulator. For every set F, the accumulator measures the progress that has been made towards the next score increase of F.

Definition 4 (Accumulator) *For every $F \subseteq V$ we define* $\mathrm{Acc}_F \colon V^+ \to 2^F$ *by* $\mathrm{Acc}_F(w) = \mathrm{Occ}(x)$*, where x is the longest suffix of w such that* $\mathrm{Sc}_F(w) = \mathrm{Sc}_F(wy^{-1})$ *for every suffix y of x, and* $\mathrm{Occ}(x) \subseteq F$*.*

A simple consequence of these definitions is that sets with non-zero score and the accumulators of all sets are all pairwise comparable.

Lemma 5 (cf. Theorem 4.2 of [8]) *Let $w \in V^+$. The sets F with* $\mathrm{Sc}_F(w) \geq 1$ *together with the sets* $\mathrm{Acc}_F(w)$ *for some F form a chain in the subset relation.*

Proof. It suffices to show that all such sets are pairwise comparable: let F and F' be two sets such that either $\mathrm{Sc}_F(w) \geq 1$ or $F = \mathrm{Acc}_H(w)$ for some $H \subseteq V$ and either $\mathrm{Sc}_{F'}(w) \geq 1$ or $F' = \mathrm{Acc}_{H'}(w)$ for some $H' \subseteq V$. Then, there exist two decompositions $w = w_0 w_1$ and $w = w_0' w_1'$ with $\mathrm{Occ}(w_1) = F$ and $\mathrm{Occ}(w_1') = F'$. Now, either w_1 is a suffix of w_1' or vice versa. In the first case, we have $F \subseteq F'$ and in the second case $F' \subseteq F$. □

Note that Lemma 5 implies that there can be at most $|V|$ sets that have a non-zero score at the same time.

Finally, we define the maximum score function. This function maps a subset $\mathcal{F} \subseteq 2^V$ and a play ρ to the highest score that is reached during ρ for a set in $\mathcal{F}$.

Definition 6 (MaxScore) *For every $\mathcal{F} \subseteq 2^V$ we define* $\mathrm{MaxSc}_{\mathcal{F}} \colon V^+ \cup V^\omega \to \mathbb{N} \cup \{\infty\}$ *by* $\mathrm{MaxSc}_{\mathcal{F}}(\rho) = \max_{F \in \mathcal{F}} \max_{w \sqsubseteq \rho} \mathrm{Sc}_F(w)$*.*

To illustrate these definitions, consider the play $w = 12210122$ in the arena G shown in Figure 1, and the set $F = \{1, 2\}$. We have that $\mathrm{Sc}_F(w) = 1$, because 122 is the longest suffix of w that is contained in F, and the entire set $\{1, 2\}$ is seen once during this suffix. We have $\mathrm{Acc}_F(w) = \{2\}$, because only vertex 2 has been seen since the score of F increased to 1. On the other hand, we have $\mathrm{MaxSc}_{\{F\}}(w) = 2$ because the prefix $w' = 1221$ of w has $\mathrm{Sc}_F(w') = 2$.

McNaughton proposed that scores should be used to decide the winner in a finite-time Muller game. As soon as a threshold score of k for some set F is reached, the play is stopped and if $F \in \mathcal{F}_i$ then Player i is declared the winner. The next lemma shows that this is sufficient to ensure that the game always terminates.

Lemma 7. *Let $k \in \mathbb{N}$. Every $w \in V^*$ with $|w| \geq k^{|V|}$ satisfies* $\mathrm{MaxSc}_{2^V}(w) \geq k$*.*

Proof. We show by induction over $|V|$ that every word $w \in V^*$ with $|w| \geq k^{|V|}$ contains an infix x that can be decomposed as $x = x_1 \cdots x_k$ where every x_i is a non-empty word with $\mathrm{Occ}(x_i) = \mathrm{Occ}(x)$. This implies $\mathrm{MaxSc}_{2^V}(w) \geq k$.

The claim holds trivially for $|V| = 1$ by choosing x to be the prefix of w of length k and $x_i = s$ for the single vertex $s \in V$. For the induction step, consider a set V with $n + 1$ vertices. If w contains an infix x of length k^n which contains at most n distinct vertices, then we can apply the inductive hypothesis and obtain a

decomposition of an infix of v with the desired properties. Otherwise, every infix x of w of length k^n contains every vertex of V at least once. Let x be the prefix of length k^{n+1} of w and let $x = x_1 \cdots x_k$ be the decomposition of x such that each x_i is of length k^n. Then, we have $\mathrm{Occ}(x_i) = \mathrm{Occ}(x) = V$ for all i. Therefore, the decomposition has the desired properties. □

Lemma 7 implies that a finite-time Muller game with threshold k must end after at most $k^{|V|}$ steps. We show that this bound is tight. For every $k > 0$ we inductively define a word over the alphabet $\Sigma_n = \{1, \ldots, n\}$ by $w_{(k,1)} = 1^{k-1}$ and $w_{(k,n)} = (w_{(k,n-1)}n)^{k-1}w_{(k,n-1)}$. The word $w_{(k,n)}$ has length $k^n - 1$, and it can also be shown that $\mathrm{MaxSc}_{2^{\Sigma_n}}(w_{(k,n)}) = k - 1$. This can easily be turned into a game where Player 1 loses, but can produce $w_{(k,n)}$ to avoid losing for as long as possible.

Finally, to declare a unique winner in every play of a finite-time Muller game we must exclude the case where two sets hit score k at the same time. McNaughton observed that this cannot happen.

Lemma 8 ([8]) *Let $k, l \geq 2$, let $F, F' \subseteq V$, let $w \in V^*$ and $v \in V$ such that $\mathrm{Sc}_F(w) < k$ and $\mathrm{Sc}_{F'}(w) < l$. If $\mathrm{Sc}_F(wv) = k$ and $\mathrm{Sc}_{F'}(wv) = l$, then $F = F'$.*

We can now define a finite-time Muller game. Such a game $\mathcal{G} = (G, \mathcal{F}_0, \mathcal{F}_1, k)$ consists of an arena $G = (V, V_0, V_1, E)$, a partition $(\mathcal{F}_0, \mathcal{F}_1)$ of 2^V, and a threshold $k \geq 2$. By Lemma 7 we have that every infinite play must reach score k for some set F after a bounded number of steps. Therefore, we define a play for the finite-time Muller game to be a finite path $w = w_0 \cdots w_n$ with $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_n) = k$, but $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_{n-1}) < k$. Due to Lemma 8, there is a unique $F \subseteq V$ such that $\mathrm{Sc}_F(w) = k$. Player 0 wins the play w if $F \in \mathcal{F}_0$ and Player 1 wins otherwise. The notions of strategies and winning regions can all be redefined for finite games. Applying a result of Zermelo to finite-time Muller games yields the following lemma.

Lemma 9 ([9]) *Finite-time Muller games are determined.*

In fact, McNaughton considered a slightly different definition of a finite-time Muller game. Rather than stopping the play when the score of a set reaches the global threshold k, in his version the play is stopped when the score of a set F reaches $|F|! + 1$. He obtained the following result.

Theorem 10 ([8]) *If W_i is the winning region of Player i in a Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$, and W'_i is the winning region of Player i in McNaughton's finite-time Muller game, then $W_i = W'_i$.*

Adapting the proof of Lemma 7 one can show that a play in this version is stopped after at most $\prod_{j=1}^{|G|}(j! + 1)$ steps. Furthermore, adapting the construction of the lower bounds $w_{(k,n)}$ above, one can also show that there are words $w_n \in \Sigma_n^*$ such that $|w_n| \geq \frac{1}{2}\prod_{j=1}^{|G|}(j! + 1)$ and $\mathrm{MaxSc}_{\{F\}}(w_n) < |F|! + 1$ for every $F \subseteq \Sigma_n$.

The threshold in McNaughton's game grows factorially in the size of the arena. Our goal is to find the smallest value of k for which a Muller game and the corre-

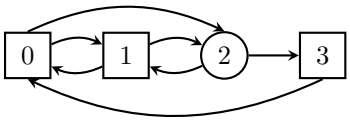

Fig. 2. The arena G.

sponding finite-time Muller game with threshold k have the same winning regions. As the singleton set $\{v\}$ has a score of 1 as soon as a play starts in v, the threshold 1 is obviously too small. We finish this section by proving that 3 is the smallest possible threshold for which this equivalence can hold. The rest of this paper is dedicated to showing that it does indeed hold for threshold 3.

Theorem 11. *There is a Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$ with winning region W_0 and corresponding finite-time Muller game $(G, \mathcal{F}_0, \mathcal{F}_1, 2)$ with winning region W_0' such that $W_0 \neq W_0'$.*

Proof. Consider the arena G in Figure 2 with $\mathcal{F}_1 = \{\{0, 1, 2\}, \{0, 2, 3\}\}$. The following strategy σ is winning for Player 0 from every vertex: at vertex 2 alternate between moving to 1 and to 3. Every play ρ consistent with σ either ends up in the loop between 0 and 1 or visits every vertex infinitely often. In both cases, ρ is won by Player 0.

On the other hand, Player 1 has a winning strategy from vertex 3 in $(G, \mathcal{F}_0, \mathcal{F}_1, 2)$: starting at 3, Player 1 moves to 0 and then 2. Now, if Player 0 moves to 3, Player 1 answers by moving to 0 and 2. The resulting play 302302 is won by Player 1, as the set $\{0, 2, 3\} \in \mathcal{F}_1$ has reached a score of 2 and no set of Player 0 has reached a score of 2. If Player 0 moves to 1, then Player 1 answers by moving to 0, 1, and then to 2, which gives the play 3021012 that is also won by Player 1. □

4. Zielonka's Algorithm For Muller Games

This section presents Zielonka's algorithm for Muller games [10], a reinterpretation of an earlier algorithm due to McNaughton [7]. Our notation mostly follows [3, 4]. The internal structure of the winning regions computed by the algorithm is used in Section 5 to define a strategy that bounds the scores of the losing player by 2.

As we consider uncolored arenas, we have to deal with Muller games where $(\mathcal{F}_0, \mathcal{F}_1)$ is a partition of $2^{V'}$ for some finite set $V' \supseteq V$, as the algorithm makes recursive calls for such games. This does not change the semantics of Muller games, as we have $\mathrm{Inf}(\rho) \subseteq V$ for every infinite play ρ.

We begin by introducing Zielonka trees, a representation of winning conditions $(\mathcal{F}_0, \mathcal{F}_1)$. Given a family of sets $\mathcal{F} \subseteq 2^{V'}$ and $X \subseteq V'$, we define $\mathcal{F} \restriction X = \{F \in \mathcal{F} \mid F \subseteq X\}$. Given a partition $(\mathcal{F}_0, \mathcal{F}_1)$ of $2^{V'}$, we define $(\mathcal{F}_0, \mathcal{F}_1) \restriction X = (\mathcal{F}_0 \restriction X, \mathcal{F}_1 \restriction X)$. Note that $\mathcal{F} \restriction X \subseteq \mathcal{F}$.

Definition 12 (Zielonka tree [3]) *For a winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ defined over a set V', its Zielonka tree $\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}$ is defined as follows: suppose that $V' \in \mathcal{F}_i$ and let $V'_0, V'_1, \ldots, V'_{k-1}$ be the $\subseteq$-maximal sets in $\mathcal{F}_{1-i}$. The tree $\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}$ consists of a root vertex labelled by V' with k children which are defined by the trees $\mathcal{Z}_{(\mathcal{F}_0,\mathcal{F}_1)\restriction V'_0}, \ldots, \mathcal{Z}_{(\mathcal{F}_0,\mathcal{F}_1)\restriction V'_{k-1}}$.*

For every Zielonka tree T, we define $\mathrm{RtLbl}(T)$ to be the label of the root in T, we define $\mathrm{BrnchFctr}(T)$ to be the number of children of the root, and we define $\mathrm{Chld}(T, j)$ for $0 \leq j < \mathrm{BrnchFctr}(T)$ to be the j-th child of the root. Here, we assume that the children of every vertex are ordered by some fixed linear order.

The input of Zielonka's algorithm (see Algorithm 1) is a finite arena G with vertex set V and the Zielonka tree of a partition $(\mathcal{F}_0, \mathcal{F}_1)$ of $2^{V'}$ for some finite set $V' \supseteq V$. For the sake of exposition, we assume that $\mathrm{RtLbl}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}) \in \mathcal{F}_1$ in the subsequent paragraphs, which implies that Zielonka's algorithm chooses i to be 1. If this is not the case then the roles of the two players can be swapped. The same assumption is made in Section 5. The algorithm computes the winning regions of the players by successively removing parts of Player 0's winning region (the sets $U_0, U_1, U_2, \ldots$). By doing this, the algorithm computes an internal structure of the winning regions that is crucial to proving our results in the next section.

Algorithm 1 Zielonka$(G, \mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1})$.

```
i := The index j such that RtLbl(Z_{F0,F1}) ∈ F_j
k := BrnchFctr(Z_{F0,F1})
if The root of Z_{F0,F1} has no children then
   W_i = V; W_{1-i} = ∅
   return(W_0, W_1)
end if
U_0 := ∅; n := 0
repeat
   n := n + 1
   A_n := Attr^V_{1-i}(U_{n-1})
   X_n := V \ A_n
   T_n := Chld(Z_{F0,F1}, n mod k)
   Y_n := X_n \ Attr^{X_n}_i(V \ RtLbl(T_n))
   (W^n_0, W^n_1) := Zielonka(G[Y_n], T_n)
   U_n := A_n ∪ W^n_{1-i}
until U_n = U_{n-1} = ··· = U_{n-k}
W_i = V \ U_n; W_{1-i} = U_n
return (W_0, W_1)
```

Figure 3 depicts the situation in the n-th iteration of the algorithm. The vertices in U_{n-1} have already been removed and belong to W_0. Thus, all vertices in the 0-

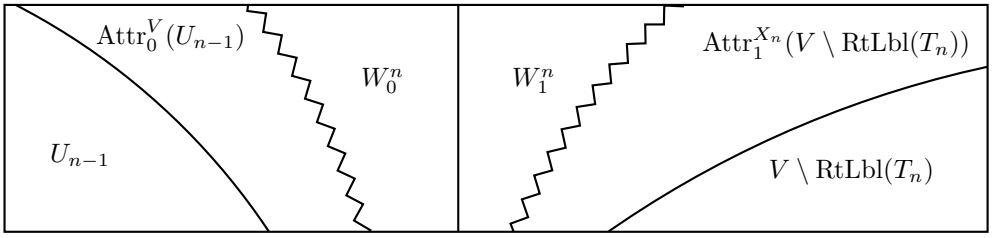

Fig. 3. The sets computed by Zielonka's algorithm.

attractor of U_{n-1} also belong to W_0. After removing these vertices from the arena, the algorithm also removes the vertices in the 1-attractor of $V \setminus \mathrm{RtLbl}(T_n)$. The remaining vertices form a subarena whose vertex set is a subset of $\mathrm{RtLbl}(T_n)$. Hence, the algorithm can recursively compute the winning regions W_i^n in this subarena with Zielonka tree T_n. By construction, the winning region W_0^n is also a subset of the winning region W_0, and so the algorithm can move into the next iteration with $U_n = A_n \cup W_0^n$. The algorithm only terminates when the size of the set U_n does not increase for $k = \mathrm{BrnchFctr}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1})$ consecutive iterations.

The execution of Zielonka's algorithm gives us a structure for W_0 and W_1 that we use in Section 5. The set W_0 is partitioned into the attractors given by the sets $A_n \setminus U_{n-1}$, and the recursively computed winning regions given by the sets W_0^n. On the other hand, the structure of W_1 is given by the final k iterations of the algorithm. In each of these iterations, the algorithm computes an attractor $\mathrm{Attr}_1^{X_n}(V \setminus \mathrm{RtLbl}(T_n))$, where $X_n = W_1$, and it recursively computes a winning region W_1^n. The attractor and the winning region are a partition of the set W_1. Since we have $T_n = \mathrm{Chld}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}, n \bmod k)$, the final k iterations of the algorithm give k distinct partitions, one for each child of the root of the Zielonka tree.

Theorem 13 ([10]) *Algorithm 1 terminates with a partition* (W_0, W_1)*, where Player* 0 *has a winning strategy for* W_0 *and Player* 1 *has a winning strategy for* W_1*.*

Zielonka's winning strategies are defined inductively: Player 0 plays the attractor strategy to U_{n-1} on each set $A_n \setminus U_{n-1}$, and the recursively computed winning strategy on each set W_0^n. Every play consistent with this strategy must eventually be contained within one of the sets W_0^n, hence the strategy is winning for Player 0.

Player 1 plays using a cyclic counter c ranging over $0, \ldots, k-1$: suppose $c = j$ and let n be the index at which the algorithm terminated. In W_1^{n-j}, the strategy plays according to the recursively computed winning strategy. If Player 0 chooses to leave W_1^{n-j}, then the strategy starts playing an attractor strategy to reach $V \setminus \mathrm{RtLbl}(T_{n-j})$. Once this set has been reached, the counter c is incremented modulo k, and the strategy begins again. There are two possibilities for a play consistent with this strategy: if it stays from some point onwards in some W_1^{n-j}, then it is winning by the inductive hypothesis. Otherwise, it visits infinitely many vertices in $V \setminus \mathrm{RtLbl}(\mathrm{Chld}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}, j))$ for every j in the range $0 \leq j < \mathrm{BrnchFctr}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1})$, which

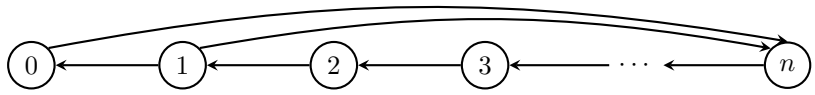

Fig. 4. The arena G_n for Lemma 14.

implies that the infinity set of the play is not a subset of any $\mathrm{RtLbl}(\mathrm{Chld}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}, j))$. Hence, it is in $\mathcal{F}_1$ and the play is indeed winning for Player 1.

We continue by showing that these winning strategies do not bound the score of the opponent by a constant.

Lemma 14. *There exists a family of Muller games $\mathcal{G}_n = (G_n, \mathcal{F}_0^n, \mathcal{F}_1^n)$ with $|G_n| = n+1$ and $|\mathcal{F}_0^n| = 1$ such that $W_0 = V$, but $\mathrm{MaxSc}_{\mathcal{F}_1^n}(\mathrm{Play}(v, \sigma, \tau)) = n$, where σ is Zielonka's strategy, $v \in V$, and $\tau \in \Pi_1$.*

Proof. Let $G_n = (V_n, V_n, \emptyset, E_n)$ with $V_n = \{0, \ldots, n\}$, $E_n = \{(i+1, i) \mid i < n\} \cup \{(0, n), (1, n)\}$ (see Figure 4), and $\mathcal{F}_0^n = \{V_n\}$. The Zielonka tree for the winning condition $(\mathcal{F}_0^n, \mathcal{F}_1^n)$ has a root labeled by V_n and $n+1$ children that are leaves and are labeled by $V_n \setminus \{i\}$ for every $i \in V_n$. Assume the children are ordered as follows: $V_n \setminus \{0\} < \cdots < V_n \setminus \{n\}$. Zielonka's strategy for $\mathcal{G}_n$, which depends on the ordering of the children, can be described as follows. Initialize a counter $c := 0$ and repeat:

(1) Use an attractor strategy to move to vertex c.
(2) Increment c modulo $n+1$.
(3) Go to 1.

This strategy is winning from every vertex. Now assume a play consistent with this strategy has just visited 0. Then, it visits all vertices $1, \ldots, n$ in this order by cycling through the loop $n, \ldots, 1$ exactly n times. Hence, the score for the set $\{1, \ldots, n\} \in \mathcal{F}_1$ is infinitely often n. □

By contrast, Player 0 has a positional winning strategy for $\mathcal{G}_n$ that bounds the opponents scores by 2 (and even 1). The reason the strategy described above fails to do this is that it ignores the fact that all other vertices are visited while moving to the vertex 0. In the next section we construct a strategy that recognizes such visits, and it turns out that this is sufficient to bound the opponent's scores by 2.

5. Bounding the Scores in a Muller Game

In this section, we prove our main result: the finite-time Muller game with threshold 3 is equivalent to the corresponding Muller game.

Theorem 15. *If W_i is the winning region of Player i in a Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$, and W_i' is the winning region of Player i in the finite-time Muller game $(G, \mathcal{F}_0, \mathcal{F}_1, 3)$, then $W_i = W_i'$.*

To prove Theorem 15, we show that if a player has a winning strategy for the Muller game, then this player also has a winning strategy for the Muller game that bounds the scores of the opponent by 2. Since the player could use this strategy in order to win the finite Muller game with threshold 3, this implies that for $i \in \{0, 1\}$ we have $W_i \subseteq W'_i$. Since W_0 and W_1 partition the set of vertices, this fact is sufficient to prove Theorem 15. Note that this actually proves a stronger statement: for every threshold $k \geq 3$ the finite-time Muller game $(G, \mathcal{F}_0, \mathcal{F}_1, k)$ is equivalent to the Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$.

The rest of this section is dedicated to proving the following lemma.

Lemma 16. *Player i has a winning strategy σ for her winning region W_i in a Muller game $\mathcal{G} = (G, \mathcal{F}_0, \mathcal{F}_1)$ such that* $\mathrm{MaxSc}_{\mathcal{F}_{1-i}}(\mathrm{Play}(v, \sigma, \tau)) \leq 2$ *for every vertex $v \in W_i$ and every $\tau \in \Pi_{1-i}$.*

In Lemma 14 we saw that the strategies computed by Zielonka's algorithm do not necessarily satisfy the property required by Lemma 16. Our task is to produce strategies that do bound the opponent's scores by 2. Our strategies are similar in structure to those that are produced by Zielonka's algorithm, but we must take much more care to ensure that the properties required by Lemma 16 are satisfied.

The winning strategies produced by Zielonka's algorithm have a recursive structure, which means that a winning strategy σ for a set of vertices W often proceeds by playing a recursively computed winning strategy σ' for a set of vertices $W' \subset W$. For example, the two players could construct a path $v_0 \dots v_n$, where $v_n \in W'$, and then σ could start executing σ' with the starting vertex v_n. However, the vertex v_n may not be the first point at which the play entered the set W', and there could be a suffix $v_m v_{m+1} \dots v_n$ of the play such that each vertex in the suffix is contained in W'. The strategies produced by Zielonka's algorithm ignore this suffix, because it is not relevant when we only want to construct a winning strategy.

By contrast, when we want to construct a winning strategy that satisfies the properties given by Lemma 16, this suffix turns out to be vitally important. We now give some definitions that allow us to work with such suffixes. Firstly, we redefine the notion of a play. Previously we had that a play begins at a starting vertex, but now we allow a play to begin with a finite initial path over which the players have no control. This new definition is useful, because it allows strategies to base their decisions on the properties of the finite initial path.

Definition 17 (Play) *For a non-empty finite path $w = w_0 \cdots w_m$ and strategies $\sigma \in \Pi_i$, $\tau \in \Pi_{1-i}$, we define the infinite play* $\mathrm{Play}(w, \sigma, \tau) = \rho_0 \rho_1 \rho_2 \cdots$ *inductively by $\rho_n = w_n$ for $0 \leq n \leq m$ and for $n > m$ by*

$$\rho_n = \begin{cases} \sigma(\rho_0 \cdots \rho_{n-1}) & \text{if } \rho_{n-1} \in V_i, \\ \tau(\rho_0 \cdots \rho_{n-1}) & \text{if } \rho_{n-1} \in V_{1-i}. \end{cases}$$

In fact, the finite paths that are passed to our strategies are not totally arbitrary. As described previously, these paths arise out of decisions made before the strategy

was recursively applied. Therefore, we have some control over the form that these paths take. We construct our strategy so that every path passed to a recursive strategy has the following property.

Definition 18 (Burden) *Let $\mathcal{F} \subseteq 2^{V'}$. A finite path w is an $\mathcal{F}$-burden if* $\mathrm{MaxSc}_{\mathcal{F}}(w) \leq 2$ *and for every $F \in \mathcal{F}$ either* $\mathrm{Sc}_F(w) = 0$ *or* $\mathrm{Sc}_F(w) = 1$ *and* $\mathrm{Acc}_F(w) = \emptyset$.

A path w satisfies the criteria of a burden if it has the following two properties. Firstly, the requirement that $\mathrm{MaxSc}_{\mathcal{F}}(w) \leq 2$ means that the score of every set $F \in \mathcal{F}$ must be bounded by 2 at *every* point along the path w. Secondly, the score of each set $F \in \mathcal{F}$ *at the end* of the path must either be 0 or 1. Additionally, if the score is 1, then the accumulator of this set must be empty. In other words, while the scores are allowed to reach 2 during the path, we insist that they satisfy a more restricted condition at the end of the path.

Before we begin proving Lemma 16, we state a useful property of burdens that is applied when we pass burdens to recursively computed strategies.

Remark 19. *Let $\mathcal{F}' \subseteq \mathcal{F}$. Every suffix of an $\mathcal{F}$-burden is an $\mathcal{F}'$-burden.*

We are now ready to prove Lemma 16. We assume that $\mathrm{RtLbl}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}) \in \mathcal{F}_1$. If this is not the case then the roles of the two players can be swapped. The proof is an induction over the structure of the Zielonka tree. The inductive hypothesis is that, if Zielonka's algorithm computes the partition into winning regions as (W_0, W_1), then Player i has a winning strategy for the set W_i that bounds the scores of every set in $\mathcal{F}_{1-i}$ by 2, even if the play starts with an $\mathcal{F}_{1-i}$- burden.

We begin with the base case of the induction, which occurs when the Zielonka tree is a leaf. Since we assume $\mathrm{RtLbl}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}) \in \mathcal{F}_1$, we must have that $W_1 = V$. Therefore, Player 0 can be ignored in this proof.

Lemma 20. *Let $(G, \mathcal{F}_0, \mathcal{F}_1)$ be a Muller game with vertex set V such that $\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}$ is a leaf. Then, Player 1 has a strategy τ such that* $\mathrm{MaxSc}_{\mathcal{F}_0}(\mathrm{Play}(wv, \sigma, \tau)) \leq 2$ *for every strategy $\sigma \in \Pi_0$ and every $\mathcal{F}_0$-burden wv with $v \in V$.*

Proof. As $\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}$ is a leaf and $\mathrm{RtLbl}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}) \in \mathcal{F}_1$ by assumption, we have $\mathcal{F}_0 = \emptyset$. Hence, any strategy τ for Player 1 guarantees $\mathrm{MaxSc}_{\mathcal{F}_0}(\mathrm{Play}(wv, \sigma, \tau)) \leq 2$. □

For the inductive step, we give two proofs: one for the set W_0, and the other for the set W_1. We begin with the proof for the set W_0. The structure of W_0, as computed by Zielonka's algorithm, is shown in Figure 5. Recall that the set W_0 consists of a number of sets W_0^n, which are winning subregions of W_0 that have been recursively computed by the algorithm. We denote the recursively computed winning strategy for W_0^n as σ_n^{R}. This strategy satisfies the inductive hypothesis, so we know that $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0^n}(\mathrm{Play}(wv, \sigma_n^{\mathrm{R}}, \tau)) \leq 2$ for every strategy τ of Player 1 in $G[W_0^n]$ and every $\mathcal{F}_1 \restriction W_0^n$-burden wv with $v \in W_0^n$. The sets $A_n \setminus U_{n-1}$ are attractors, and for each set $A_n \setminus U_{n-1}$ we denote the attractor strategy as σ_n^{A}.

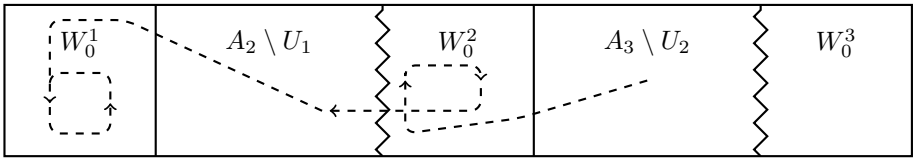

Fig. 5. The structure of W_0. The dashed line shows an example play according to σ^*.

We can now construct our proposed winning strategy. This strategy is similar to the one that is constructed by Zielonka's algorithm, but our strategy is careful to pass the appropriate finite path to the recursively computed strategy σ_n^{R}. For every path w and every vertex v, we define:

$$\sigma^*(wv) = \begin{cases} \sigma_n^{\mathrm{R}}(w'v) & \text{if } v \in W_0^n \text{ and } w' \text{ is the longest suffix of } w \text{ with} \\ & \qquad \mathrm{Occ}(w') \subseteq W_0^n, \\ \sigma_n^{\mathrm{A}}(v) & \text{if } v \in A_n \setminus U_{n-1}. \end{cases}$$

Note that σ^* passes the complete suffix of wv that is contained in W_0^n to σ_n^{R}. Applying Remark 19 yields, that if wv is an $\mathcal{F}_1 \restriction W_0^n$-burden, then $w'v$ is also an $\mathcal{F}_1 \restriction W_0^n$-burden. This allows us to apply the inductive hypothesis for σ_n^{R} in the following proof, which shows that σ^* has the property required by Lemma 16. Therefore, the next lemma proves the part of the inductive step that deals with W_0.

Lemma 21. *For every $\mathcal{F}_1 \restriction W_0$-burden wv with $v \in W_0$ and every strategy $\tau \in \Pi_1$ we have* $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(\mathrm{Play}(wv, \sigma^*, \tau)) \leq 2$.

Proof. The sets $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n$ form a sequence of hierarchical traps for Player 1. This means that once $\mathrm{Play}(wv, \sigma^*, \tau)$ enters a set U_j, it may never again visit a vertex in $V \setminus U_j$. Therefore, we can represent $\mathrm{Play}(wv, \sigma^*, \tau)$ as $wa_nw_na_{n-1}w_{n-1} \ldots a_kw_k$, where w is the burden without its last vertex, a_j is the portion of the play after w that is contained in $A_j \setminus U_{j-1}$, and w_j is the portion of the play after w that is contained in W_0^j. One or both of these infixes could be empty, and the portion w_k contains the infinite suffix of the play. We prove the claim by induction over this decomposition. The base case follows from the fact that wv is an $\mathcal{F}_1 \restriction W_0$ burden, and therefore $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(w) \leq 2$.

We have two cases to consider. Firstly we must prove that if we have $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(wa_nw_n \ldots a_j) \leq 2$, then we have $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(wa_nw_n \ldots a_jw_j) \leq 2$. Here we assume that w_j is nonempty, as the claim trivially holds if $w_j = \varepsilon$. Let s be the first vertex of w_j and let $F \in \mathcal{F}_1 \restriction W_0$. If F contains at least one vertex in $W_0 \setminus W_0^j$, then the score of F can increase by at most one during the portion w_j, because the play is confined to the set W_0^j. Since wv is a burden, we must have $\mathrm{Sc}_F(wv) \leq 1$. Since $a_nw_n \ldots a_j$ does not visit the set W_0^j, and since $\mathrm{Acc}_F(wv) = \emptyset$ in case $\mathrm{Sc}_F(wv) = 1$, we must therefore have $\mathrm{Sc}_F(wa_nw_n \ldots a_j) \leq 1$. Thus, even if the score of F is increased by 1 during w_j, it cannot increase to more than 2 throughout

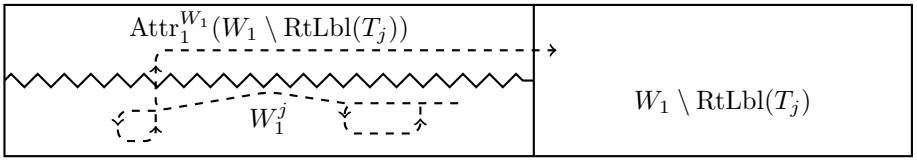

Fig. 6. The structure of W_1 with respect to T_j. The dashed line indicates a part of a play according to τ^* between two change points.

w_j. Finally, we consider the sets $F \subseteq W_0^j$. In this case the claim follows from the inductive hypothesis given by Lemma 16 for the recursively computed strategy σ_j^{R}. However, to invoke the inductive hypothesis, we must have that $wa_nw_n \dots a_js$ is an $\mathcal{F}_1 \restriction W_0^j$-burden. If $a_nw_n \dots a_j$ is non-empty, then this holds, because then we have $\mathrm{Sc}_F(wa_nw_n \dots a_j) = 0$ for every set $F \subseteq W_0^j$. This implies that $wa_nw_n \dots a_js$ is indeed an $\mathcal{F}_1 \restriction W_0^j$-burden. On the other hand, if $a_nw_n \dots a_j$ is empty, then we have $s = v$. Thus, as $wa_nw_n \dots a_js = wv$ is an $\mathcal{F}_1 \restriction W_0$-burden by assumption, it is also an $\mathcal{F}_1 \restriction W_0^j$-burden.

Secondly, we must prove that if $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(wa_nw_n \dots w_{j+1}) \leq 2$, then $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(wa_nw_n \dots w_{j+1}a_j) \leq 2$. Let $F \in \mathcal{F}_1 \restriction W_0$. If F contains a vertex in $W_0 \setminus (A_j \setminus U_{j-1})$, then the score of F must remain below 2 for exactly the same reasons as in the previous case. Otherwise, if $F \subseteq A_j \setminus U_{j-1}$, then we claim that the score of F can rise to at most 2 during the portion a_j. By construction of the decomposition we have that the score of F is at most 1 at the start of the portion a_j. It is easy to show that if an attractor strategy is played, then every vertex in the attractor can be seen at most once. This implies that the score of F can increase to at most 2 during a_j. □

We now turn our attention to the set W_1. Let $k = \mathrm{BrnchFctr}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1})$. The last k iterations of Zielonka's algorithm produce for each child $T_j = \mathrm{Chld}(\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}, j)$ with $0 \leq j \leq k-1$ an instance of the situation depicted in Figure 6. The set $\mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j))$ has an associated attractor strategy τ_j^{A}, and the set W_1^j has a recursively computed winning strategy τ_j^{R}. This strategy satisfies the inductive hypothesis, so we know that $\mathrm{MaxSc}_{\mathcal{F}_0 \restriction W_1^j}(\mathrm{Play}(wv, \sigma, \tau_j^{\mathrm{R}})) \leq 2$ for every strategy σ of Player 0 in $G[W_1^j]$ and every $\mathcal{F}_0 \restriction W_1^j$-burden wv with $v \in W_1^j$.

Figure 6 shows the outcome when Player 1 plays τ_j^{R} and τ_j^{A}. The play remains in the set W_1^j until Player 0 chooses to leave, at which point the play is forced to visit some vertex in $W_1 \setminus \mathrm{RtLbl}(T_j)$. Once the play enters $W_1 \setminus \mathrm{RtLbl}(T_j)$, a new index $j' \neq j$ is selected, and $\tau_{j'}^{\mathrm{R}}$ and $\tau_{j'}^{\mathrm{A}}$ is played. The strategy produced by Zielonka's algorithm chooses j' to be $j+1 \bmod k$, and Lemma 14 shows that this method does not bound the scores of the losing Player by 2. Our goal is to provide a method for choosing a new index that does bound the scores of the opponent by 2.

Recall that Lemma 5 implies that the sets that have non-zero score and the non-empty accumulators form a chain with respect to the subset relation. Note

that this property still holds even if we restrict ourselves to sets in $\mathcal{F}_0$. We define the indicator function of a play to be the function that selects the maximal element of this chain, when it is restricted to sets in $\mathcal{F}_0$. For every play w we define:

$$\mathrm{Ind}(w) = \bigcup_{\substack{F \in \mathcal{F}_0 : \\ \mathrm{Sc}_F(w) > 0}} F \cup \bigcup_{F \in \mathcal{F}_0} \mathrm{Acc}_F(w) \ .$$

The next lemma gives an important property that is used in our index selection method: there is always some child whose label contains the indicator.

Lemma 22. *For every play w, there is some j in the range $0 \leq j \leq k-1$ such that* $\mathrm{Ind}(w) \subseteq \mathrm{RtLbl}(T_j)$.

Proof. Lemma 5 implies that there is a maximal set C such that $\mathrm{Ind}(w) = C$, with either $\mathrm{Sc}_C(w) > 0$ or $\mathrm{Acc}_F(w) = C$ for some $F \in \mathcal{F}_0$ with $C \subseteq F$. Hence, $\mathrm{Ind}(w) \subseteq F$ for some $F \in \mathcal{F}_0$, and, by definition of $\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}$, there is some child of the root labeled by $\mathrm{RtLbl}(T_j)$ such that $F \subseteq \mathrm{RtLbl}(T_j)$. □

When a new child must be chosen, our strategy chooses one whose label contains the value of the indicator function for the play up to that point. Lemma 22 implies that such a child must always exist. It is also critically important that this condition is used when picking the child in the first step, which is why we had to introduce the concept of a burden.

We can now formally define this strategy. The strategy uses an auxiliary function $c : W_1^* \to \{0, 1, \ldots, k-1, \bot\}$ that specifies which child the strategy is currently considering. For each play w, if $c(w) = j$ then the strategy follows τ_j^{A} and τ_j^{R}. If $c(w) = \bot$ then the strategy moves arbitrarily.

We begin by defining the function c. This definition encompasses the idea that the strategy should always choose a child that contains the indicator. Therefore, we define $c(\varepsilon) = \bot$, and for every play w and every vertex v we define:

$$c(wv) = \begin{cases} c(w) & \text{if } v \in \mathrm{RtLbl}(T_{c(w)}), \\ j & \text{if } v \notin \mathrm{RtLbl}(T_{c(w)}),\ \mathrm{Ind}(wv) \neq \emptyset \text{ and } j \text{ minimal with} \\ & \qquad \mathrm{Ind}(wv) \subseteq \mathrm{RtLbl}(T_j), \\ \bot & \text{if } v \notin \bigcup_{0 \leq j \leq k-1} \mathrm{RtLbl}(T_j). \end{cases}$$

Note that c is defined for every wv, as $\mathrm{Ind}(wv) = \emptyset$ implies $v \notin \bigcup_{0 \leq j \leq k-1} \mathrm{RtLbl}(T_j)$. We can now define τ^* for W_1 as:

$$\tau^*(wv) = \begin{cases} \tau_j^{\mathrm{R}}(w'v) & \text{if } c(wv) = j, v \in W_1^j \text{ and } w' \text{ is the longest suffix of } w \text{ with} \\ & \qquad \mathrm{Occ}(w') \subseteq W_1^j, \\ \tau_j^{\mathrm{A}}(v) & \text{if } c(wv) = j, v \in \mathrm{RtLbl}(T_j) \setminus W_1^j, \\ x & \text{if } c(wv) = \bot \text{ where } x \in W_1 \text{ with } (v, x) \in E. \end{cases}$$

Note that τ^* passes the complete suffix of wv that is contained in W_1^j to τ_j^{R}. Applying Remark 19 yields, that if wv is an $\mathcal{F}_0 \restriction W_1$-burden, then $w'v$ is an $\mathcal{F}_0 \restriction$

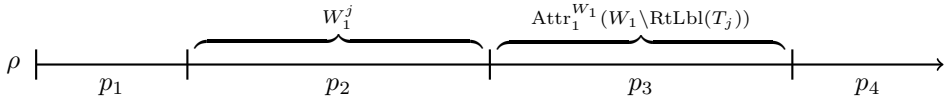

Fig. 7. The decomposition of a play for Lemma 24. The first vertex of p_4 is not in $\mathrm{RtLbl}(T_j)$.

W_1^j-burden. This allows us to apply the inductive hypothesis for τ_j^{R} in the part of the inductive step that deals with the set W_1.

We now prove that τ^* has the required properties. Our proof uses change points, which are positions in a play where the c function changes its value.

Definition 23 (Change Point) *Let $\rho_0\rho_1\rho_2\ldots$ be a play. We say that $n \in \mathbb{N}$ is a change point in ρ if $c(\rho_0\rho_1\ldots\rho_{n-1}) \neq c(\rho_0\rho_1\ldots\rho_{n-1}\rho_n)$.*

In the next Lemma, we prove that if Player 1 plays according to τ^* starting from a burden, then the play up to the next change point n is also a burden. Our intention is to use this as part of an inductive proof that every play bounds the scores of the opponent's sets by 2.

Lemma 24. *Let $\rho = \rho_0\rho_1\rho_2\ldots$ be a play, and let $\rho_0\ldots\rho_m$ be an $\mathcal{F}_0 \restriction W_1$-burden such that ρ is consistent with τ^* from at least m onwards. If n is the smallest change point in ρ satisfying $m < n$, then $\rho_0\ldots\rho_n$ is an $\mathcal{F}_0 \restriction W_1$-burden.*

Proof. Let $j = c(\rho_0\ldots\rho_m)$ be the index of the child that is chosen at the point ρ_m. We first provide a proof for the case where $j = \perp$. By definition this implies that $\rho_{n'} \notin \mathrm{RtLbl}(T_l)$ for all n' in the range $m \leq n' < n$ and all l in the range $0 \leq l \leq k-1$. Therefore, for every $F \in \mathcal{F}_0$ we must have $\mathrm{Sc}_F(\rho_0\ldots\rho_{n'}) = 0$ and $\mathrm{Acc}_F(\rho_0\ldots\rho_{n'}) = \emptyset$ for all n' in the range $m \leq n' < n$. From this, it is easy to see that $\rho_0\ldots\rho_n$ is an $\mathcal{F}_0 \restriction W_1$-burden.

For the case $j \neq \perp$ we split the play ρ into four pieces, as depicted in Figure 7. The piece p_1 contains the portion of ρ up to and including the point ρ_m and the piece p_4 contains the portion of ρ after and including the change point ρ_n. The piece p_2 contains the portion of ρ between the points ρ_m and ρ_n that is contained in the set W_1^j, and the piece p_3 contains the portion of ρ between the points ρ_m and ρ_n that is contained in the set $\mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j))$. Clearly, we have $\rho = p_1p_2p_3p_4$.

We now prove that $\mathrm{MaxSc}_{\mathcal{F}_0}(p_1p_2p_3) \leq 2$. The scores at position ρ_n will be considered later. For the portion p_1 the scores are bounded by 2 by assumption. Now, consider a set $F \in \mathcal{F}_0$. During the portion p_2, we know that τ_j^{R} is being played, and therefore the inductive hypothesis given by Lemma 16 is sufficient to prove the claim for the case where $F \subseteq W_1^j$. On the other hand, if there is a vertex $s \in F$ such that $s \notin W_1^j$, then s cannot be visited during the portion p_2. This implies that the score of F can increase by at most 1 during p_2. Since p_1 is a burden, we have that $\mathrm{Sc}_F(p_1) \leq 1$, which implies that $\mathrm{MaxSc}_{\{F\}}(p_1p_2) \leq 2$.

During the portion p_3 we know that the attractor strategy τ_j^{A} is being played, which implies that each vertex in $\mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j))$ can be seen at most once during this portion. Consider a set $F \in \mathcal{F}_0$. If $F \cap \mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j)) = \emptyset$ then the score of F is 0 during the portion p_3. Therefore, we only need to consider the case where $F \cap \mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j)) \neq \emptyset$. The assumption that p_1 is a burden implies that $\mathrm{Sc}_F(p_1) \leq 1$. If $F \cap W_1^j = \emptyset$ then the score of F cannot increase during p_2, and since p_3 never sees the same vertex twice, we have that the score of F can increase by at most 1 during p_3.

If $F \cap W_1^j \neq \emptyset$, then we consider two cases. If $\mathrm{Sc}_F(p_1) = 0$, then the score of F can increase only once during p_2, as the vertex in $\mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j))$ cannot be visited in p_2. Similarly, the score of F can increase only once during p_3, as the vertex in W_1^j cannot be visited in p_3. Hence, it can only increase to 2 during p_3. Otherwise, if $\mathrm{Sc}_F(p_1) = 1$ and $\mathrm{Acc}_F(p_1) = \emptyset$, then the score of F cannot increase during p_2, as the vertex in $\mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j))$ cannot be visited. Furthermore, the score can only be increased once during p_3, as no vertex in $\mathrm{Attr}_1^{W_1}(W_1 \setminus \mathrm{RtLbl}(T_j))$ is visited twice by p_3. Therefore, we have shown that $\mathrm{MaxSc}_{\mathcal{F}_0}(p_1p_2p_3) \leq 2$.

To complete the proof, we must show that for every set $F \in \mathcal{F}_0$, either we have $\mathrm{Sc}_F(p_1p_2p_3\rho_n) = 0$, or we have $\mathrm{Sc}_F(p_1p_2p_3\rho_n) = 1$ and $\mathrm{Acc}_F(p_1p_2p_3\rho_n) = \emptyset$. We split this proof into two cases. Firstly, we consider sets $F \in \mathcal{F}_0$ such that $\mathrm{Sc}_F(p_1) = 1$ and $\mathrm{Acc}_F(p_1) = \emptyset$. By definition of c we have $F \subseteq \mathrm{Ind}(p_1)$, and therefore by definition of our strategy, we must have $F \subseteq \mathrm{RtLbl}(T_j)$. Since $\rho_n \in W_1 \setminus \mathrm{RtLbl}(T_j)$, we must have $\rho_n \notin F$. This implies that $\mathrm{Sc}_F(p_1p_2p_3\rho_n) = 0$.

We now consider the case where $\mathrm{Sc}_F(p_1) = 0$. If $\rho_n \in F$, then $\rho_n \notin \mathrm{Acc}_F(p_1)$, as we have $\mathrm{Acc}_F(p_1) \subseteq \mathrm{RtLbl}(T_j)$ and $\rho_n \notin \mathrm{RtLbl}(T_j)$. Hence, we must have $\mathrm{Sc}_F(p_1p_2p_3) = 0$, as p_2p_3 is confined to $\mathrm{RtLbl}(T_j)$. Therefore, if $\mathrm{Sc}_F(p_1p_2p_3\rho_n) = 1$ then we must have $\mathrm{Acc}_F(p_1p_2p_3\rho_n) = \emptyset$. On the other hand, if $\rho_n \notin F$ then we must have $\mathrm{Sc}_F(p_1p_2p_3\rho_n) = 0$. □

Lemma 24 explains why burdens must be passed between recursive strategies. We use Lemma 24 inductively to show that the strategy τ^* bounds the scores of Player 0 by 2. However, for the base case of this inductive proof to hold, the finite path that was passed to the strategy must satisfy the burden property. The next lemma shows that τ^* satisfies the properties required by Lemma 16.

Lemma 25. *We have* $\mathrm{MaxSc}_{\mathcal{F}_0 \restriction W_1}(\mathrm{Play}(wv, \sigma, \tau^*)) \leq 2$ *for every strategy* $\sigma \in \Pi_0$ *and every* $\mathcal{F}_0 \restriction W_1$*-burden* wv *with* $v \in W_1$.

Proof. Let $\rho = \mathrm{Play}(wv, \sigma, \tau^*)$. Since wv is a burden, we can use Lemma 24 inductively to show that, if $n \geq |wv|$ is a change point in ρ, then $\rho_0\rho_1 \ldots \rho_n$ is a burden. If ρ contains infinitely many change points, then the proof is complete. This is because if the play up to every change point is a burden and there is an infinite number of change points, then $\mathrm{MaxSc}_{\mathcal{F}_0 \restriction W_1}(\rho) \leq 2$.

On the other hand, if there is only a finite number of change points, then let n be the final change point in ρ. Since $\rho_0 \dots \rho_n$ is a burden, we have that $\mathrm{MaxSc}_{\mathcal{F}_0 \restriction W_1}(\rho_0 \cdots \rho_n) \leq 2$. If $c(\rho_0 \cdots \rho_n) = j$ for some j in the range $0 \leq j \leq k-1$, then we must have $\rho_m \in W_1^j$ for every $m \geq n$. This implies that τ^* follows σ_j^{R} from the point n onwards. Since $\rho_0 \dots \rho_n$ is also an $\mathcal{F}_1 \restriction W_1^j$-burden, we can apply the inductive hypothesis given by Lemma 16 to obtain $\mathrm{MaxSc}_{\mathcal{F}_0 \restriction W_1}(\rho) \leq 2$.

If $c(\rho_0 \cdots \rho_n) = \bot$, then also $c(\rho_0 \cdots \rho_m) = \bot$ for every $m > n$. This implies $\rho_m \notin \mathrm{RtLbl}(T_j)$ for every j in the range $0 \leq j \leq k-1$, and hence $\mathrm{Sc}_F(\rho_0 \cdots \rho_m) = 0$ for every $m > n$ and every $F \in \mathcal{F}_0$. Therefore, $\mathrm{MaxSc}_{\mathcal{F}_0 \restriction W_1}(\rho) \leq 2$. □

Finally, we can prove Lemma 16, which also completes the proof of Theorem 15.

Proof. Theorem 13 yields that Algorithm 1 is correct, which means that the sets W_i returned are indeed the winning regions of the players. We prove the following stronger statement by induction over the height of $\mathcal{Z}_{\mathcal{F}_0,\mathcal{F}_1}$: *Player i has a winning strategy σ for her winning region W_i such that* $\mathrm{MaxSc}_{\mathcal{F}_{1-i}}(wv, \sigma, \tau) \leq 2$ *for every strategy $\tau \in \Pi_{1-i}$ and every $\mathcal{F}_{1-i} \restriction W_i$-burden wv with $v \in W_i$.* This implies Lemma 16, as the finite play v for every $v \in W_i$ is an $\mathcal{F}_{1-i} \restriction W_i$-burden.

For the induction start, apply Lemma 20. In the induction step, use the strategies obtained from the inductive hypothesis to define σ^* and τ^* as above. Lemma 21 guarantees $\mathrm{MaxSc}_{\mathcal{F}_1 \restriction W_0}(\mathrm{Play}(wv, \sigma^*, \tau)) \leq 2$ for every $\tau \in \Pi_1$ and every $\mathcal{F}_1 \restriction W_0$-burden wv with $v \in W_0$. As $\mathrm{Play}(wv, \sigma^*, \tau)$ is confined to W_0, we also have $\mathrm{MaxSc}_{\mathcal{F}_1}(\mathrm{Play}(wv, \sigma^*, \tau)) \leq 2$ for every $\tau \in \Pi_1$ and every $\mathcal{F}_1 \restriction W_0$-burden wv with $v \in W_0$. The reasoning for W_1 is analogous and applies Lemma 25. Both σ^* and τ^* are winning, as they bound the scores of the opponent by 2. □

6. Conclusion

We have presented a criterion to stop plays in a Muller game after a finite amount of time that preserves winning regions. Our bound $3^{|G|}$ on the length of a play improves the bound $|G| \cdot |G|! + 1$ obtained by a reduction to parity games. Furthermore, our techniques show that the winning player can bound the scores of the opponent by 2 and that this bound is tight.

A finite-time Muller game with threshold k can be viewed as a reachability game defined over the unraveling of the original arena up to depth at most $k^{|G|}$, which is of doubly-exponential size in $|G|$. Simple algorithms can be applied to solve this game. Our results also allow us to reduce Muller games to safety games: for each Muller game we can produce a safety game in which Player i wins if and only if Player i is able to avoid a score value of 3 for all sets of the opponent.

Another interesting direction is to find a construction which turns a winning strategy for a finite-time Muller game with threshold 3 into a finite-state strategy for the original Muller game. It is conceivable that such a construction would yield memory structures that are optimized for a given arena, something which does not hold for the LAR respectively Zielonka tree structures.

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