TENSEGRITIES ON THE SPACE OF GENERIC FUNCTIONS

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Introduction

In this small note we introduce a notion of self-stresses on the set functions in two variables with generic critical points. The notion naturally comes from a rather exotic representation of classical Maxwell frameworks in terms of differential forms.

For the sake of clarity we work in the two-dimensional case only. However all the definitions for the higher dimensional case are straightforward.

1. Preliminaries

1.1. **Classical definition of tensegrity.** For completeness of the story we start with the classical approach introduced in [7] by J. C. Maxwell in 1864. We refer any interested in rigidity and flexibility questions to [1, 8].

We use the following slightly modified definition of tensegrity from [4].

Definition 1.1. Let $G = (V, E)$ be an arbitrary graph on n vertices.

- A *framework* $G(P)$ in the plane is a map $f = (f_v, f_e)$:
$$f_v : V \to \mathbb{R}^2, \qquad f_e : E \to S^1,$$
such that for every edge v_iv_j the vector $f_v(v_i)f_v(v_j)$ is a multiple of $f_e(v_iv_j)$.
- A *stress* w on a framework is an assignment of real scalars $w_{i,j} = w_{j,i}$ (called *tensions*) to its edges.
- A stress w is called a *self-stress* if, in addition, the following equilibrium condition is fulfilled at every vertex p_i:
$$\sum_{\{j|j\neq i\}} w_{i,j}e_{ij} = 0.$$
- A pair $(G(P), w)$ is a *tensegrity* if w is a self-stress for the framework $G(P)$.

1.2. **Tensegrities and exterior forms.** In this section we recall a rather exotic interpretation of two-dimensional tensegrities as a collection of 2-forms in $\mathbb{R}^3$ with certain relations. It is rather in common with projective approach discussed by I. Izmestiev in [5].

Consider a tensegrity $(G(P), w)$ with $P = (P_1, \ldots, P_n)$. For every point $P_i = (x_i, y_i)$ we associate a 1-form in $\mathbb{R}^3$:
$$dP_i := x_idx + y_idy + dz.$$

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(one can say that dz is a normalization factor to extract tensions.)

For every edge E_iE_j we consider a 2-form:

$$dP_i \wedge dP_j.$$

It turns out that self-stressability conditions is precisely equivalent to

$$\sum_{\{j|j\neq i\}} w_{i,j} dP_j \wedge dP_i = 0.$$

So any framework in tensegrity can be defined simply by a collection of decomposable 2-forms in $\mathbb{R}^3$ (which we denote as $G(dP)$) and a self-stress w as before, We denote it by $(G(dP), w)$.

Remark 1.2. This definition perfectly suits the "meet" and "join" Cayley algebra expressions arising with the description of existence conditions of tensegrities (see, e.g., in [9, 2, 6]). It also rather straightforwardly provides projective invariance of tensegrity existence.

2. Case of generic functions

One of the mysterious questions related the notion of $(G(dP), w)$ is as follows: *what is a natural generalizations of the tensegrity to the case of decomposable differentiable 2-forms (not-necessarily with constant coefficients)?*

The aim of this section is to give a partial answer to this question for differential forms whose factors are of type

$$df + dz,$$

where $f = f(x, y)$ is a function of two variables.

Let us first give a definition of tensegrity. Secondly we show a geometric interpretation and link it to the classical case.

2.1. **Main definitions.** Let $F = (f_1, \ldots, f_n)$. Denote by dF the collection of forms

$$dF_1 = df_1 + dz, \quad dF_2 = df_2 + dz, \quad \ldots, \quad dF_n = df_n + dz.$$

Definition 2.1. Let F be a collection of functions $F = (f_1, \ldots, f_n)$ with finitely many critical points. A *tensegrity* $(G(dF), w)$ is a triple: a graph (G, F, w), where functions f_i are associated with vertices of a graph and edges are associated with stresses $w_{i,j}$.

For a function f denote the set of its critical points by $\mathrm{Cr}(f)$; the index of a critical point P is denoted by $\mathrm{ind}(P)$.

Definition 2.2. A self-stress condition on $(G(dF), w)$ at a function F_i

$$\sum_{P_{i,k}\in \mathrm{Cr}(f_i)} (-1)^{\mathrm{ind}(P_{i,k})} \Bigg(\sum_{\{j|j\neq i\}} w_{i,j} dF_j(P_{i,k}) \wedge dF_i(P_{i,k}) \Bigg) = 0.$$

Remark 2.3. It might be also useful to consider critical points separately (say if this increases durability for certain overconstrained system).

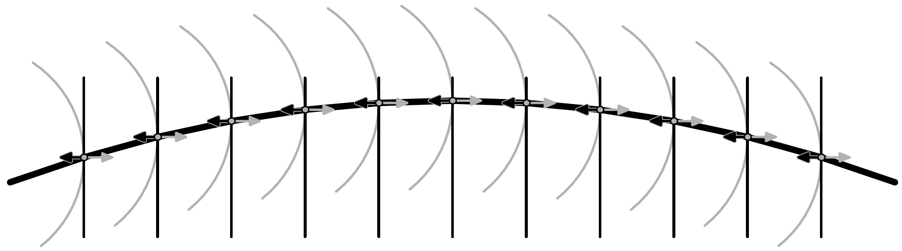


FIGURE 1. Level sets of two functions (gray and black), and the line where their gradients have the same direction.

Remark 2.4. Recall that at a critical point of F_i

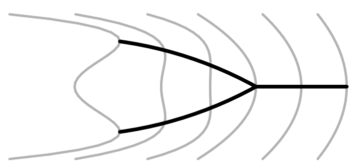
$$dF_i \wedge dF_j = dz \wedge df_j.$$

So one can replace every 2-form $dF_i \wedge dF_j$ in the equilibrium condition simply by df_j.

2.2. **Geometric discussions.** *Lines of forces* are precisely the points when the total force $dF_i \wedge dF_j = 0$ (see Figure 1). Lines of forces are defined by the equation

$$df_i \wedge df_j = 0.$$

It is clear that lines of forces are connecting critical points of F_i (including critical points at infinity) to critical points F_j usually by a graph rather than by a line. A possible picture for a splitting of a force line is as follows:



Remark 2.5. One might consider the classical theory of tensegrities as follows: For each point $P_i = (x_i, y_i)$ consider a function functions

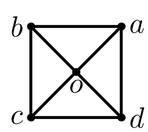
$$f_i = (x - x_i)^2 + (y - y_i)^2.$$

Then (possible after a simple rescaling of stresses) one has a classical tensegrity.

This works also for case of point-hyperplane frameworks introduced recently in [3], where hyperplanes are defined as linear functions.

In some sense the proposed techniques can be considered as a deformation of a classical tensegrity.

Example 2.6. First we start with a graph G on 5 vertices:



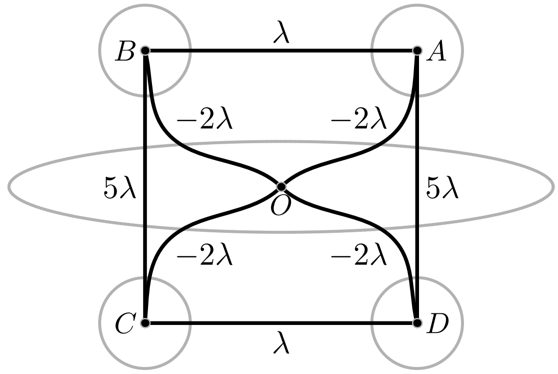


Figure 2. Level sets, compact lines of forces, and stresses.

Let us consider the following 5 functions corresponding to vertices of graphs:

$$\begin{array}{ll} a: & f_1=(x-3)^2+(y-3)^2; \\ b: & f_2=(x+3)^2+(y-3)^2; \\ c: & f_3=(x+3)^2+(y+3)^2; \\ d: & f_4=(x-3)^2+(y+3)^2; \\ o: & f_5=x^2+5y^2. \end{array}$$

Then the line of forces and the corresponding stresses are as on Figure 2. They are defined up to a choice of a real parameter λ. Here grey curves are level sets; black curves are compact lines of forces between critical points; the numbers indicate the stresses on edges. The critical points of functions are marked by the corresponding capital letters.

Remark 2.7. Finally we would like to admit that the situation in three and higher dimensional cases repeats the two-dimensional case discussed above.

Remark 2.8. In three dimensional case consider the following functions (potentials):

$$f_{a,b,c}=\frac{k_e}{(x-a)^2+(y-b)^2+(z-c)^2}$$

(here k_e is the Coulomb constant) and take the unit stresses. Then we arrive to classical Coulomb situation for points with unit charges in three-space.

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