GEOMETRIC CRITERIA FOR REALIZABILITY OF TENSEGRITIES IN HIGHER DIMENSIONS

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Abstract. In this paper we study a classical Maxwell question on the existence of self-stresses for frameworks, which are called tensegrities. We give a complete answer on geometric conditions of at most $(d+1)$-valent tensegrities in $\mathbb{R}^d$ both in terms of discrete multiplicative 1-forms and in terms of "meet" and "join" operations in the Grassmann-Cayley algebra.

Contents

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Introduction

In this paper we are dealing with a classical question on self-stresses of frameworks in arbitrary dimensions (that were later referred as tensegrities). Our main goal is to find geometric tensegrity existence characterizations on all (generic) k-valent graphs in $\mathbb{R}^d$ ($k \leq d+1$). We do this in two different geometric settings. The first one is based on discrete multiplicative 1-forms which belong to discrete differential geometry. For example, discrete multiplicative 1-forms have been used to characterize discrete Koenigs nets [3]. The second one is via geometric equations written in terms of "meet" and "join" operations of Grassmann-Cayley algebra.

Although the research on tensegrities was initiated already in 1864 by J.C. Maxwell [20], the term "tensegrity" itself appears much later. Tensegrity is a concatenation of the words "tension" and "integrity". This term was proposed by R. Buckminster Fuller who was inspired by the elegance of self-stressed constructions. Tensegrities form an essential part of modern architecture and in arts, they serve as a light structural support (like in a recent sculpture *TensegriTree* in the University of Kent). Tensegrities are traditionally used in the study of cells [11, 1, 2], viruses [5, 24], deployable mechanisms [26], etc.

In the second half of the 20th century the subject of tensegrities became popular in mathematics again: questions of rigidity and flexibility of structures were studied amongst others by R. Connelly, B. Roth, and W. Whiteley in [6, 7, 22, 31], etc. For a general modern overview of the subject we refer to the book [25].

Tensegrities were generalized to spherical and projective geometries (by F.V. Saliola and W. Whiteley [23]); to normed spaces (by D. Kitson and S.C. Power in [17] and by D. Kitson and B. Schulze in [18]); and to surfaces in $\mathbb{R}^3$ (by B. Jackson and A. Nixon in [12]); etc.

Realizability of tensegrities. If the amount of edges is not large enough, a generic realisation of a graph in $\mathbb{R}^d$ will not have a non-zero tensegrity. The non-zero tensegrities exist only for specific frameworks (that are actually semi-algebraic sets in the configuration spaces of tensegrities [9]). For instance, a framework for the $K_{3,3}$ graph admits a non-zero tensegrity if and only if all its six points are on a conic.

An algebraic description of realizability conditions for tensegrities was proposed by N.L. White and W. Whiteley in [29, 30]. It was given in terms of bracket rings for the determinants of extended rigidity matrices (see also [28]). This algebraic description provides us with large polynomial conditions which are very hard to observe, so a geometric approach was initiated. It is based on Grassmann-Cayley algebras for the affine lines and planes in $\mathbb{R}^d$. In their work M. de Guzmán and D. Orden [8] made first steps in the study of geometry of stresses by introducing atom decomposition techniques. In all the studied examples (see, e.g., [9, 30]) there is a simple geometric description for tensegrities in terms of the "meet-join" operations of Grassmann-Cayley algebra. This suggests such a description for all possible graphs. In this paper we develop techniques to write such conditions for the case of k-valent graphs ($k \leq d+1$) in an arbitrary dimension d.

A preliminary investigation of geometric conditions was made in [9]: the authors had introduced two surgeries that result in classification of all the geometric conditions for codimension one strata for graphs with 8 or less vertices. Topological properties of the configuration spaces of all tensegrities for graphs with 4 and 5 vertices were studied in [16]. A complete description of geometric conditions in the two-dimensional case was announced in [15]. Finally, a nice collection of problems on geometry and topology of stratification of tensegrities can be found in [14].

In the present paper we consider less than $d+1$ valent graphs in $\mathbb{R}^d$. We write geometric conditions both in terms of integrability of multiplicative 1-forms (Theorem 2.7) and in terms of Grassmann-Cayley algebra (Theorem 3.22).

Organization of the paper. We start in Section 1 with the definition of tensegrities and notions that we use throughout the paper. In Section 2 we discuss discrete multiplicative 1-forms and how exact 1-forms characterize frameworks admitting non-zero self-stresses. In Section 3 we work within the Grassmann-Cayley algebra to provide a recursive geometric characterization of tensegrities. Section 4 is devoted to point out a relation between tensegrities and harmonic maps. Finally in Section 5 we study as an example the case of octahedral tensegrities in $\mathbb{R}^3$.

1. Notions and definitions

In this section we give the necessary definitions of the setting around tensegrities. Additionally, we provide the notion of general position of the framework so that we can formulate our geometric conditions on frameworks admitting a tensegrity.

1.1. **Definition of tensegrities.** Let us first set the scene by recalling some basic notions before we come to the general definition of tensegrities.

Definition 1.1. Let G be an arbitrary graph without loops and multiple edges on n vertices.

— Let $V(G) = \{v_1, \dots, v_n\}$ and $E(G)$ denote the sets of vertices and edges for G, respectively. Denote by $(v_i; v_j)$ the edge joining v_i and v_j.

— Let $B(G)$ be the subset of all 1-valent vertices in $V(G)$, which we refer to as the *boundary of* G.

— Let $Z(G)$ be the subset of all vertices with valence greater than 1 in $V(G)$.

- A *framework* $G(P)$ is a map of the vertices $v_1, \dots, v_n$ of G onto a finite point configuration $P = (p_1, \dots, p_n)$ in $\mathbb{R}^d$, such that $G(P)(v_i) = p_i$ for $i = 1, \dots, n$. We say that there is an *edge* between p_i and p_j if $(v_i; v_j)$ is an edge of G and denote it by $(p_i; p_j)$. Note that the points $p_1, \dots, p_n$ are not necessarily distinct.
- A *stress* w on a framework is an assignment of real scalars $w_{i,j}$ (called *tensions*) to its edges $(v_i; v_j)$ with the property $w_{i,j} = w_{j,i}$. We also set $w_{i,j} = 0$ if there is no edge between the corresponding vertices.
- A stress w is called a *self-stress* if the following equilibrium condition is fulfilled at every vertex of valence greater than 1, i.e., at $v_i \in Z(G)$:

$$\sum_{\{j|j\neq i\}} w_{i,j}(p_i - p_j) = 0.$$

  By $p_i - p_j$ we denote the vector from the point p_j to the point p_i. Note that we do not consider equilibrium for the boundary points $B(G)$. There are the points where the tensegrity is attached to the exterior construction. Therefore, the corresponding forces are compensated by the forces of the exterior construction.
- A pair $(G(P), w)$ is called a *tensegrity* if w is a self-stress for the framework $G(P)$.
- A tensegrity $(G(P), w)$ (or stress w) is said to be *non-zero* if if there exists an edge $(v_i; v_j)$ of the framework that has non-vanishing tension $w_{i,j} \neq 0$.
- A tensegrity $(G(P), w)$ (or stress w) is said to be *everywhere non-zero* if each existing edge $(v_i; v_j)$ of the framework has non-vanishing tension $w_{i,j} \neq 0$.

Remark 1.2. If the set of boundary points $B(G)$ is empty, we have the classical case of tensegrities without boundary.

1.2. **Frameworks in various general positions.** To formulate our geometric conditions on frameworks admitting a non-zero self-stress via discrete multiplicative 1-forms, we need the vertices to lie in general position (Definition 1.3). A slightly stronger version of generality is needed to formulate our conditions within the setting of Grassmann-Cayley algebra.

Definition 1.3. A framework $G(P)$ is *linearly generic* if for every vertex (whose degree or valence we denote by k) the following two conditions hold:

- the k edges emanating from this vertex span a $(k-1)$-plane;
- every subset of $k-1$ edges emanating from this vertex spans this $(k-1)$-plane.

Remark 1.4. The valences of a linearly generic framework in $\mathbb{R}^d$ do not exceed $d+1$.

For the geometric characterization of tensegrities in terms of discrete multiplicative 1-forms (Section 2), the property on frameworks of being linearly generic is all we need. As for our characterization as formulated within Grassmann-Cayley algebra (Section 3), we have to include one further notion of general position.

Definition 1.5. A framework $G(P)$ is *in 3D-general position* if the following two conditions hold:

- $G(P)$ is linearly generic, and
- every 4-tuple of vertices in every cycle of $G(P)$ spans a 3-plane.

We conclude this chapter with the following general notion. Throughout the paper by $\mathrm{span}(s_1, \dots, s_k)$ we denote the *affine* or *projective span* of affine/projective spaces $s_1, \dots, s_k$ (and not the linear span as a vector space).

2. Characterizing k-valent tensegrities in terms of ratios

In this section we give a geometric characterization for linearly generic at most k-valent graphs in $\mathbb{R}^d$ ($k \leq d+1$) admitting a non-zero self-stress. It turns out to be practical to first provide a characterization for trivalent graphs before then generalizing it to k-valent graphs. Throughout this section all graphs G are connected. Our goal is to show that the product of certain ratios is one if and only if the framework admits a non-zero tensegrity (Theorem 2.7).

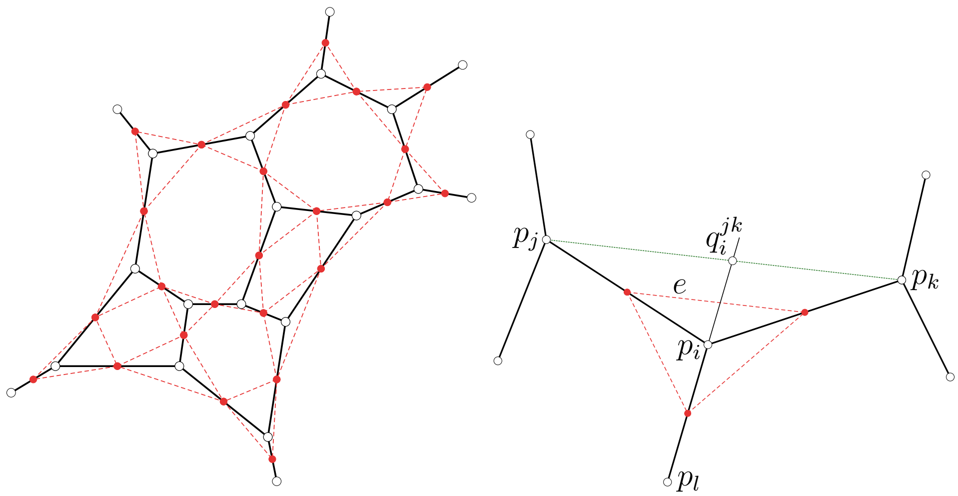

FIGURE 1. *Left*: A graph G (black lines; white vertices) and its mid-edge subdivision $M(G)$ (red dashed edges; red vertices). *Right*: A flat vertex star p_i, p_j, p_k, p_l. The edge e of the mid-edge subdivision $M(G)$ corresponds to the angle (v_j, v_i, v_k). The intersection point q_i^{jk} of the straight lines through p_jp_k and p_ip_l determines the value of the multiplicative 1-form q on that edge by $q(v_j, v_i, v_k) = (p_j - q_i^{jk}) : (q_i^{jk} - p_k)$.

2.1. **Tensegrities over trivalent graphs.** Our geometric characterization of a linearly generic framework to be a non-zero tensegrity is defined on the cycles of the underlying graph. The important notion here is the one of a discrete multiplicative 1-form which is well known in discrete differential geometry. We follow the definition in [3].

Definition 2.1. A real valued function $q : \vec{E}(G) \to \mathbb{R} \setminus \{0\}$ ($\vec{E}(G)$ denotes the set of oriented edges of the graph G) is called a *multiplicative 1-form*, if $q(-e) = 1/q(e)$ for every $e \in \vec{E}(G)$. It is called *exact* if for every cycle $e_1, \ldots, e_k$ of directed edges the values of the 1-form multiply to 1, i.e.,

$$q(e_1) \cdot \ldots \cdot q(e_k) = 1.$$

The following definition is about a particular subdivision of a trivalent graph.

Definition 2.2. The *mid-edge subdivision graph* $M(G)$ of a trivalent graph G has the following properties. Its vertices consist of the mid-points of the edges, and, its edges consist of all the triangle edges around each vertex of degree 3. See Figure 1 (left).

We now aim at constructing a multiplicative 1-form on the oriented edges of the mid-edge subdivision $M(G)$ of a trivalent graph G corresponding to a linearly generic framework $G(P)$.

Each edge e of $M(G)$ connects the midpoints of edges of the form $(v_j; v_i)$ and $(v_i; v_k)$, as illustrated in Figure 1 (right). We can therefore denote the oriented edges of $M(G)$ by triplets of the form $e = (v_j, v_i, v_k)$ with the property that the negatively oriented edge is $-e = (v_k, v_i, v_j)$.

Let us denote the third edge emanating from v_i by $(v_i; v_l)$. The framework being linearly generic implies that the corresponding vertices p_i, p_j, p_k, p_l lie in a common plane. Furthermore, the framework being linearly generic implies that the straight line connecting $p_i p_l$ intersects the line connecting $p_j p_k$ in a point q_i^{jk}. Consequently, this point gives rise to an affine ratio of the form

$$q(v_j, v_i, v_k) := \frac{p_j - q_i^{jk}}{q_i^{jk} - p_k}, \tag{1}$$

as ratio of parallel vectors. Clearly, $q(v_j, v_i, v_k) = 1/q(v_k, v_i, v_j)$ which implies that q is a multiplicative 1-form on the oriented edges of the mid-edge subdivision $M(G)$.

Theorem 2.3. *Let $G(P)$ be a linearly generic trivalent framework. Then there is a stress w on $G(P)$ such that the framework $(G(P), w)$ is a non-zero tensegrity if and only if the 1-form q given by Equation (1) on the mid-edge subdivision $M(G)$ is exact.*

Proof. Let us first assume that $(G(P), w)$ is a non-zero tensegrity. Since at every inner vertex p_i of a trivalent tensegrity the sum of forces adds up to zero we obtain

$$w_{i,j}(p_i - p_j) + w_{i,k}(p_i - p_k) + w_{i,l}(p_i - p_l) = 0. \tag{2}$$

The point q_i^{jk} lies on the straight line through $p_i p_l$ and can therefore be written in the form

$$q_i^{jk} = p_i + \lambda(p_i - p_l)$$

for some $\lambda \in \mathbb{R}$. Inserting Equation (2) yields

$$\begin{aligned} q_i^{jk} &= p_i + \lambda\Big(\frac{w_{i,j}}{w_{i,l}}(p_j - p_i) + \frac{w_{i,k}}{w_{i,l}}(p_k - p_i)\Big) \\ &= \Big(1 - \lambda\frac{w_{i,j}}{w_{i,l}} - \lambda\frac{w_{i,k}}{w_{i,l}}\Big)p_i + \lambda\frac{w_{i,j}}{w_{i,l}}p_j + \lambda\frac{w_{i,k}}{w_{i,l}}p_k. \end{aligned}$$

Since q_i^{jk} must lie on the line through $p_j p_k$ we obtain for $\lambda = \frac{w_{i,l}}{w_{i,j}+w_{i,k}}$ and therefore the affine combination

$$q_i^{jk} = \frac{w_{i,j}}{w_{i,j} + w_{i,k}}p_j + \frac{w_{i,k}}{w_{i,j} + w_{i,k}}p_k.$$

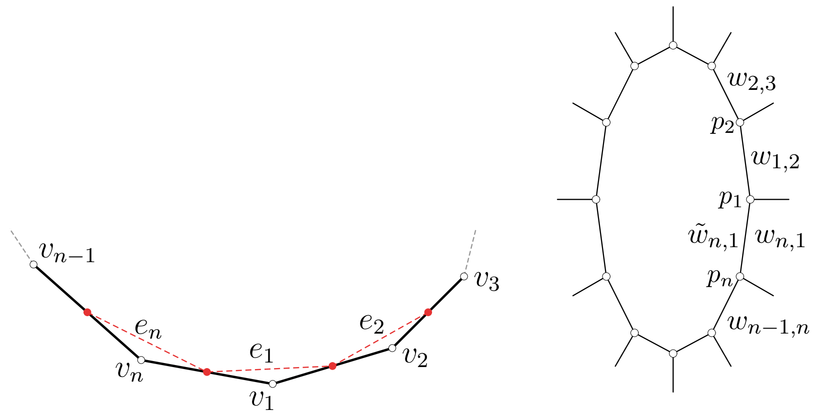

FIGURE 2. *Left*: Notations of edges in a cycle in the mid-edge subdivision $M(G)$. *Right*: A cycle in a trivalent graph. After prescribing a tension $w_{1,2}$ we can compute the tension at edge $(v_n; v_1)$ in two ways: First, by enforcing equilibrium at p_1 (and getting $\tilde{w}_{n,1}$), and second by transporting the tension along the cycle (resulting in $w_{n,1}$).

Consequently, for our q in Equation (1) we obtain

$$q(v_j, v_i, v_k) = \frac{w_{i,k}}{w_{i,j}}. \tag{3}$$

To show the exactness of q we have to show that the product of all values along any cycle multiply to 1. So let $(e_1, \dots, e_n)$ be a cycle of the mid-edge subdivision $M(G)$ where e_i are oriented edges. There is a corresponding cycle $(v_1, \dots, v_n)$ in G such that e_i corresponds to the angle (v_{i-1}, v_i, v_{i+1}), where we take the indices modulo n (see Figure 2 left). We compute the product of the corresponding values of q:

$$\prod_{i=1}^{n} q(e_i) = \prod_{i=1}^{n} q(v_{i-1}, v_i, v_{i+1}) = \prod_{i=1}^{n} \frac{w_{i,i+1}}{w_{i,i-1}} = \prod_{i=1}^{n} \frac{w_{i,i+1}}{w_{i-1,i}}$$
$$= \frac{w_{1,2}}{w_{n,1}} \cdot \frac{w_{2,3}}{w_{1,2}} \cdot \ldots \cdot \frac{w_{n-1,n}}{w_{n-2,n-1}} \cdot \frac{w_{n,1}}{w_{n-1,n}} = 1,$$

which shows the first direction of the statement.

As for the other direction, let us first note that prescribing one tension $w_{i,j}$ in a trivalent vertex of a linearly generic framework uniquely determines the other two tensions as well since Equation (2) is then a linear combination of two linearly independent vectors with coefficients $w_{i,k}$ and $w_{i,l}$. Consequently, after choosing one tension $w_{i,j}$ we can transport it to any other vertex along any connected path. This way we could define a stress w on the graph G, if this construction would be well-defined, i.e., if transporting the tension along different paths to the same edge would result in the same tensions. Or equivalently, if we transport the tension around any cycle we would have to get back to the same tension with which we started.

So let us take an arbitrary cycle $(v_1, \ldots, v_n)$. We choose a non-zero tension $w_{1,2}$ on the edge $(v_1; v_2)$ which immediately determines the tension $\tilde{w}_{n,1}$ on the edge $(v_n; v_1)$ due to the equilibrium condition shown in Equation (2). See also Figure 2 (right). The value of the multiplicative 1-form on the oriented edge (v_n, v_1, v_2) of $M(G)$ therefore has the value

$$q(v_n, v_1, v_2) = w_{1,2}/\tilde{w}_{n,1}.$$

On the other hand $w_{1,2}$ determines the tension $w_{2,3}$ as edge emanating from v_1. Repeating this propagation process we define all tensions in the cycle including the last one $w_{n,1}$. We have therefore defined the tension at $(v_n; v_1)$ twice: from the "left" and from the "right" as $\tilde{w}_{n,1}$ and $w_{n,1}$. Now the question is whether those tensions are the same.

Our assumption is that the multiplicative 1-form q is exact which implies

$$1 = \prod_{i=1}^{n} q(v_{i-1}, v_i, v_{i+1}) = \frac{w_{1,2}}{\tilde{w}_{n,1}} \cdot \frac{w_{2,3}}{w_{1,2}} \cdot \ldots \cdot \frac{w_{n-1,n}}{w_{n-2,n-1}} \cdot \frac{w_{n,1}}{w_{n-1,n}} = \frac{w_{n,1}}{\tilde{w}_{n,1}},$$

and therfore $w_{n,1} = \tilde{w}_{n,1}$. Consequently, we can consistently define a stress w (uniquely up to scaling) on $G(P)$ such that the framework $(G(P), w)$ is a non-zero tensegrity. ∎

The following corollary follows immediately from Theorem 2.3 and its proof, in particular from Equation (3).

Corollary 2.4. Let $G(P)$ be a linearly generic trivalent framework and let w be a non-zero stress on $G(P)$. Then the framework $(G(P), w)$ is a non-zero tensegrity if and only if the 1-form

$$\tilde{q}(v_j, v_i, v_k) := \frac{w_{i,k}}{w_{i,j}},$$

defined on the mid-edge subdivision $M(G)$ is exact. ∎

2.2. **Special cases of trivalent cycles.** In this section we will consider two special cases of cycles and briefly reflect on what Theorem 2.3 means in these cases.

$\underline{n = 3}$: In that case the cycle is a triangle and the points q_i^{jk} lie on the edges of the triangle opposite to p_i. Consequently, the exactness of the 1-form on that cycle is precisely the setting of the classical Ceva's theorem (see e.g., [21]). Therefore, the three lines $p_1 q_1^{2,3}$, $p_2 q_2^{3,1}$, and $p_3 q_3^{1,2}$ intersect in one point (cf. [15] and see Figure 3 left).

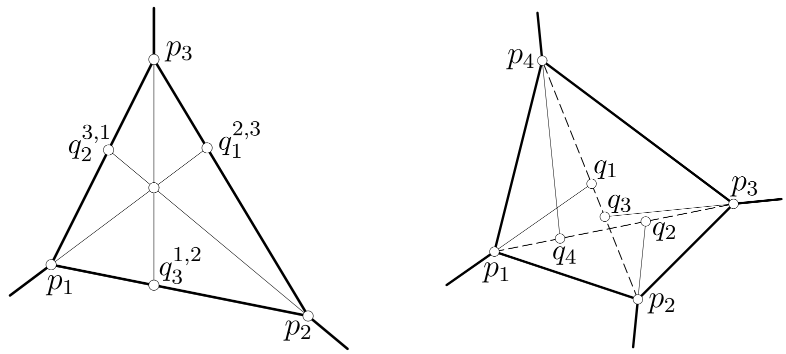

FIGURE 3. *Left*: The cycle is a triangle. The exactness of the 1-form on that triangle is equivalent to the three "outward" pointing edges intersecting in one point, i.e., Ceva's configuration. *Right*: The cycle is a quadrilateral. Then the three "outward" pointing edges intersect the respective diagonals in points q_i. We abbreviate $q_1^{2,4}$ simply by q_1 etc. The exactness of the 1-form is equivalent to $\mathrm{cr}(q_1, p_4, q_3, p_2) = \mathrm{cr}(q_2, p_3, q_4, p_1)$.

$\underline{n=4}$: In the case of a quadrilateral the points q_i^{jk} lie on the diagonals (see Figure 3 right). Exactness of the 1-form on that cycle is equivalent to

$$1 = q(v_4, v_1, v_2) \cdot q(v_1, v_2, v_3) \cdot q(v_2, v_3, v_4) \cdot q(v_3, v_4, v_1),$$

which is further equivalent to

$$q(v_4, v_1, v_2) \cdot q(v_2, v_3, v_4) = \frac{1}{q(v_4, v_1, v_2) \cdot q(v_2, v_3, v_4)},$$

and further to

$$\frac{p_4 - q_1^{4,2}}{q_1^{4,2} - p_2} \cdot \frac{p_2 - q_3^{2,4}}{q_3^{2,4} - p_4} = \frac{q_2^{1,3} - p_3}{p_1 - q_2^{1,3}} \cdot \frac{q_4^{3,1} - p_1}{p_3 - q_4^{3,1}}.$$

The last equation is an equation of cross-ratios, namely

$$\mathrm{cr}(q_1^{4,2}, p_4, q_3^{2,4}, p_2) = \mathrm{cr}(q_2^{1,3}, p_3, q_4^{3,1}, p_1). \tag{4}$$

Example 2.5. It is well known (see e.g., [9]) that the complete graph $K_{3,3}$, which is trivalent, with vertices in $\mathbb{R}^2$ is a tensegrity if and only if the vertices lie on a conic (see Figure 4). That property can also be shown easily within our setting of exact multiplicative 1-forms as follows. Assuming that the six points of $K_{3,3}$ form a tensegrity, we will show that the six points lie on a conic. To verify that we will show, according to Steiner's definition of conics, that the four lines p_5p_i and p_6p_i for $i = 1, \dots, 4$ are related by a projectivity (a projective map) or equivalently that their cross-ratio is the same. Let us consider the cycle (v_1, v_2, v_3, v_4) with four vertices. Then Equation (4) holds for this cycle which we will use in the following computation. Further, we have

$$\begin{aligned}&\operatorname{cr}(p_5p_2, p_5p_3, p_5p_4, p_5p_1) = \operatorname{cr}(q_2, p_3, q_4, p_1) \overset{(4)}{=} \operatorname{cr}(q_1, p_4, q_3, p_2)\\ =&\operatorname{cr}(p_6p_1, p_6p_4, p_6p_3, p_6p_2) = \operatorname{cr}(p_6p_2, p_6p_3, p_6p_4, p_6p_1),\end{aligned}$$

where the last equality holds because $\operatorname{cr}(a, b, c, d) = \operatorname{cr}(d, c, b, a)$.

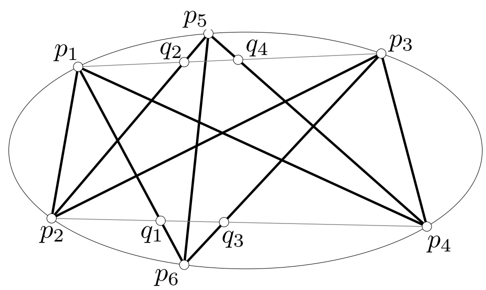

FIGURE 4. Six points $p_1, \dots, p_6$ in the plane $\mathbb{R}^2$ form a tensegrity if and only if the six points lie on a conic.

2.3. **Tensegrities over k-valent graphs.** Let us now generalize the geometric characterization of trivalent tensegrities (of Section 2.1) to linearly generic k-valent tensegrities in $\mathbb{R}^d$ with $k \leq d+1$.

Our first task here is to generalize the multiplicative 1-form defined on the linearly generic framework over the mid-edge graph $M(G)$ of trivalent graphs to the mid-edge graph of k-valent graphs.

Definition 2.6. The *mid-edge subdivision graph* $M(G)$ of a general graph G has the following properties. Its vertices consist of the midpoints of the edges, and, its edges consist of all the edges connecting two midpoints of two respective edges emanating from the same vertex. See Figure 5 (left).

Let us consider a k-valent vertex star with inner vertex v_i and adjacent vertices $v_1, \dots, v_k$. Again we can denote an oriented edge of the mid-edge subdivision $M(G)$ by (v_j, v_i, v_l) (with $1 \leq j \neq l \leq n$). See also Figure 5 (right). Since G is linearly generic, the subspaces $\operatorname{span}(p_i, p_j, p_l)$ and $\operatorname{span}(\bigcup_{m \neq j,l} p_m)$ intersect in a line L. Consequently, this line L intersects the line p_jp_l in a point q_i^{jl}. In the trivalent case, L is simply the line p_ip_l. Analogously to the trivalent case we define the discrete multiplicative 1-form as

$$q(v_j, v_i, v_l) := \frac{p_j - q_i^{jl}}{q_i^{jl} - p_l}. \tag{5}$$

Now the proof of Theorem 2.3 can be repeated basically word by word which implies the following theorem.

Theorem 2.7. *Let $G(P)$ be a linearly generic framework in $\mathbb{R}^d$ with vertices of valence at most $d+1$. Then there is a stress w on $G(P)$ such that the framework $(G(P), w)$ is a non-zero tensegrity if and only if the 1-form q given by Equation (5) on the mid-edge subdivision $M(G)$ is exact.* ■

3. Grassmann-Cayley conditions for frameworks in 3D-general position

In this section we construct Grassmann-Cayley conditions for frameworks whose all 4-tuples of vertices span a 3-plane. We start in Section 3.1 with the case of frameworks for so-called framed cycles. We introduce WU-surgeries on framed cycles that preserve the property to admit a non-zero tensegrity and that reduce the amount of vertices of framed cycles. These properties will lead to explicit formulae in terms of Grassmann-Cayley algebra. Further, in Section 3.2 we prove that a sufficiently generic framework admits a non-zero tensegrity if and only if all its associated framed cycle frameworks admit non-zero tensegrities (Theorem 3.13). Finally, in Section 3.3 we briefly recall basic notions

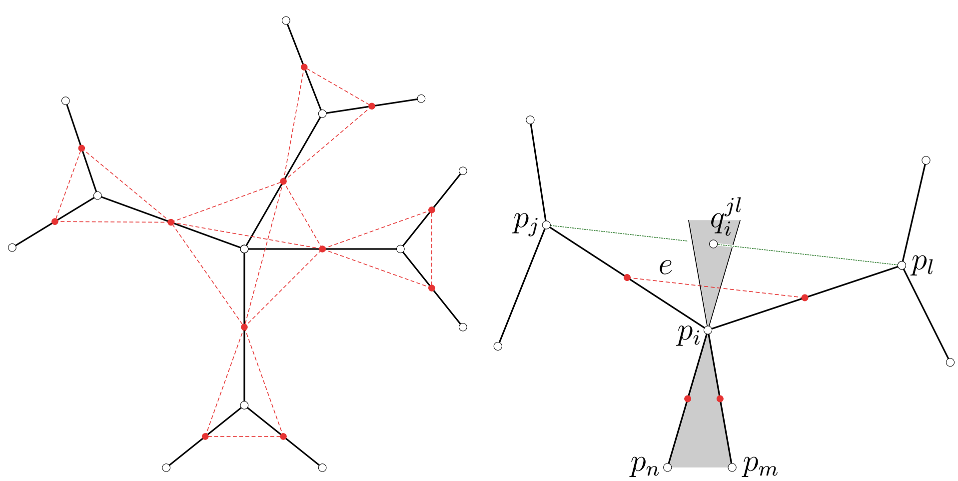

Figure 5. *Left*: The mid-edge subdivision graph $M(G)$ of a *general* graph G. The new edges (red dashed) connect midpoints of old adjacent edges. A vertex star of valence three generates three new edges, a vertex star of valence four generates six new edges. *Right*: We construct the discrete multiplicative 1-form on edges of the mid-edge subdivision graph by intersecting the line p_jp_l with the affine subspace $\text{span}(\bigcup_{m\neq j,l} p_m)$.

of Grassmann-Cayley algebra and construct Grassmann-Cayley conditions for the existence of tensegrities for given graphs that are not generically flexible (Theorem 3.22). All frameworks in this section are in $\mathbb{R}^d$ with $d \geq 3$.

3.1. **Framed cycles and their frameworks.** We start this section with basic definitions, some properties of framed cycles and their generic frameworks. Further, we introduce WU-surgeries that take frameworks in 3D-general position to generic frameworks of framed cycles. We show also that WU-surgeries preserve the property of admitting a non-zero tensegrity.

3.1.1. *General definitions.* We say that a graph is a *cycle* if it is homeomorphic to a circle.

Definition 3.1. Let $C = (c_1, \ldots, c_n)$ and $B = (b_1, \ldots, b_n)$ be two n-tuples of points. A *framed cycle* $C_B = (C, B)$ is the cycle $c_1, \ldots, c_n$ with attached edges $b_i c_i$ for $i = 1, \ldots, n$.

Definition 3.2. We say that a framework $C_B(P)$ of a framed cycle C_B is *in 3D-general flat position* if

- $C_B(P)$ is linearly generic (see Definition 1.3);
- there are no four points of $C(P)$ contained in a two-dimensional plane (only for the cycle C).

Remark 3.3. Notice that linear genericity in particular implies that all edges emanating from the same vertex of a framed cycle are contained in a 2-plane; and that $P(b_i) \neq P(c_i)$ for all admissible i.

3.1.2. *A preliminary observation.* Let us formulate a preliminary statement for the definition of WU-surgeries.

Proposition 3.4. Let a framed cycle framework $C_B(P)$ be in 3D-general flat position. Let also

$$P(b_i) = e_i, \quad P(c_i) = r_i, \quad i = 1, \ldots, n.$$

Then we have the following two statements:

- The line $e_{i-1}e_i$ is not contained in the plane $r_{i-2}r_{i-1}r_{i+1}$;

Denote by $\hat{e}_{i-1}$ the (projective) intersection point of the line $e_{i-1}e_i$ and the plane $r_{i-2}r_{i-1}r_{i+1}$ (see Figure 6). Then additionally we have:

- $\hat{e}_{i-1} \notin r_{i-2}r_{i-1}$;
- $\hat{e}_{i-1} \notin r_{i-1}r_{i+1}$.

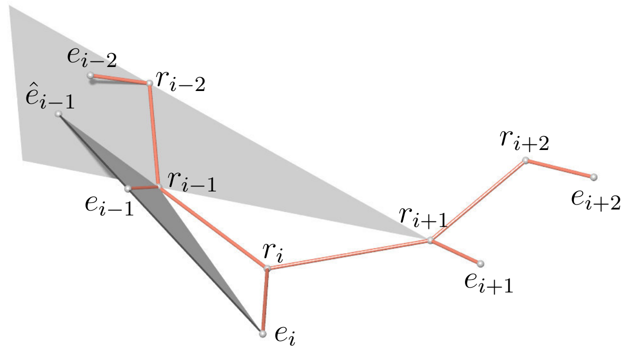

FIGURE 6. Definition of $\hat{e}_{i-1}$.

Remark 3.5. The theory of tensegrities (or equivalently the theory of infinitesimal rigidity) is projectively invariant. So we do not consider special cases of parallel objects. They are not parallel after an appropriate choice of an affine chart.

Proof of Proposition 3.4. First of all, the point e_{i-1} is not in the plane $r_{i-2}r_{i-1}r_{i+1}$ (cf. Figure 6), as otherwise

$$\operatorname{span}(r_{i-2}, r_{i-1}, r_i) = \operatorname{span}(r_{i-2}, r_{i-1}, e_{i-1}) = \operatorname{span}(r_{i-2}, r_{i-1}, r_{i+1}),$$

which would imply that the points $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ are contained in a 2-plane, and therefore $C(P)$ is not in a 3D-general flat position. Therefore, the line $e_{i-1}e_i$ is not in the plane $r_{i-2}r_{i-1}r_{i+1}$.

Secondly, if $\hat{e}_{i-1} \in r_{i-2}r_{i-1}$, then the points $e_i, e_{i-1}, r_{i-2}, r_{i-1}$ are in a 2-plane. Now the point r_i is in this plane as it is in the span of $e_{i-1}, r_{i-1}, r_{i-2}$; and additionally r_{i+1} is in this plane as it is in the span of e_i, r_{i-1}, r_i. Therefore, $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ are in this 2-plane, which contradicts to flat 3D-genericity of the cycle.

Finally, if $\hat{e}_{i-1} \in r_{i-1}r_{i+1}$, then the points $e_i, e_{i-1}, r_{i-1}, r_{i+1}$ are in a 2-plane. Now the point r_i is in this plane as it is in the span of e_i, r_{i-1}, r_{i+1}; and additionally r_{i-2} is in this plane as it is in the span of e_{i-1}, r_{i-1}, r_i. Therefore, $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ are in this 2-plane, which contradicts to flat 3D-genericity of the cycle.

This concludes the proof of all statements of the proposition. ∎

For the definition of WU-surgeries we need an index symmetric statement. The following corollary is just the index symmetric version of Proposition 3.4.

Corollary 3.6. Let a framed cycle framework $C_B(P) = (C(P), B(P))$ be in a 3D-general flat position. Let also

$$P(b_i) = e_i, \quad P(c_i) = r_i, \quad i = 1, \dots, n.$$

Then we have the following two statements:

- The line e_ie_{i+1} is not in the plane $r_{i-1}r_{i+1}r_{i+2}$; Denote by $\hat{e}_{i+1}$ the (projective) intersection point of the line e_ie_{i+1} and the plane $r_{i-1}r_{i+1}r_{i+2}$. Then additionally we have:
- $\hat{e}_{i+1} \notin r_{i+1}r_{i+2}$;
- $\hat{e}_{i+1} \notin r_{i-1}r_{i+1}$.

Proof. After swapping the indexes $i \to n-i$ for all i in C_B we arrive at the statement of Proposition 3.4 for $n-i$. ■

3.1.3. WU-*surgeries.* Let us continue with the definition of WU-surgeries.

Definition 3.7. Consider a framed cycle

$$C_B = \big((c_1, \dots, c_n), (b_1, \dots, b_n)\big),$$

and its framework

$$C_B(P) = \big((r_1, \dots, r_n), (e_1, \dots, e_n)\big)$$

in 3D-general flat position. Let $i \in 1, \dots, n$. The WU-*surgery* of the cycle C at node i is the cycle

$$\mathsf{WU}_i(C_B(P)) = \big((r_1, \dots, r_{i-2}, r_{i-1}, r_{i+1}, r_{i+2}, \dots, r_n),$$
$$(e_1, \dots, e_{i-2}, \hat{e}_{i-1}, \hat{e}_{i+1}, e_{i+2}, \dots, e_n)\big),$$

where

$$\hat{e}_{i-1} = e_ie_{i-1} \cap r_{i+1}r_{i-1}r_{i-2};$$
$$\hat{e}_{i+1} = e_ie_{i+1} \cap r_{i-1}r_{i+1}r_{i+2},$$

(see Figure 7).

Remark 3.8. Due to Proposition 3.4 and Corollary 3.6, the points $\hat{e}_{i-1}$ and $\hat{e}_{i+1}$ are uniquely defined.

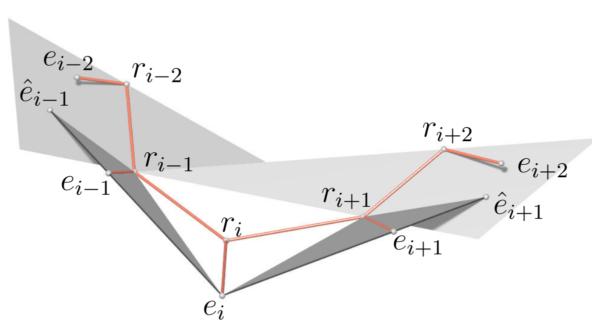

FIGURE 7. The construction of WU_i-surgery. Here we exclude the vertices e_i and r_i and replace e_{i-1} and e_{i+1} respectively by $\hat{e}_{i-1}$ and $\hat{e}_{i+1}$.

Corollary 3.9. A WU-surgery takes a framework of a framed cycle in 3D-general flat position to a framework of a framed cycle in 3D-general flat position.

Proof. The set of $C(P')$-vertices after the surgery is a subset of $C(P)$ therefore, there are no four points of $C(P')$ in a 2-plane.

By construction we have

$$\hat{e}_{i-1} \in \operatorname{span}(r_{i-2}, r_{i-1}, r_{i+1}) \quad \text{and} \quad \hat{e}_{i+1} \in \operatorname{span}(r_{i-1}, r_{i+1}, r_{i+2}).$$

Further, by Proposition 3.4 every two vectors from

$$\{r_{i-1} - \hat{e}_{i-1}, r_{i-1} - r_{i-2}, r_{i-1} - r_{i+1}\}$$

are not collinear.

Finally, by Corollary 3.6 every two vectors of

$$\{r_{i+1} - \hat{e}_{i+1}, r_{i+1} - r_{i+2}, r_{i+1} - r_{i-1}\}$$

are not collinear. Therefore, $\mathsf{WU}_i(C_B(P))$ is in 3D-general flat position. ■

3.1.4. *Static properties of* WU*-surgeries.* We continue with the following important property of WU-surgeries.

Proposition 3.10. Let C_B be a framed cycle of length m and $i \in \{1, \ldots, m\}$. A framework $C_B(P)$ in 3D-general flat position admits a non-zero tensegrity if and only if $\mathsf{WU}_i(C_B(P))$ admits a non-zero tensegrity for every admissible i.

Proof. Let $C_B(P)$ admit a tensegrity $G(C_B(P), w)$. First, we construct a framed cycle tensegrity $(C^3_{B,i}(P), \hat{w})$. Let

$$C^3_{B,i}(P) = \big((r_{i-1}, r_i, r_{i+1}), (e_i, e_i, e_i)\big)$$

(see Figure 8).

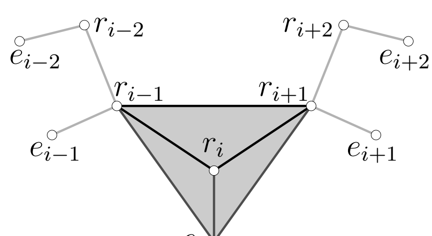

FIGURE 8. The framed cycle $C^3_{B,i}(P)$ is illustrated by the shaded area. Notice, that the boundary points for this cycle all coincide with e_i.

The stress $\hat{w}$ is defined from the following condition: at all edges adjacent to r_i the stress $\hat{w}$ coincides with the stress w for $(C_B(P), w)$.

It is clear that the tensions $\hat{w}$ on the remaining edges of $C^3_{B,i}(P)$ are defined in the unique way.

Let us now subtract $(C^3_{B,i}(P), \hat{w})$ from $(C_B(P), w)$. We have:

- zero stresses at all vertices adjacent to r_i.
- the sum of vectors of forces $\lambda r_{i-1}e_{i-1}$ and $\mu r_{i-1}e_i$ (for some non-zero λ and μ) should be in the plane spanned by $r_{i-2}, r_{i-1}, r_{i+1}$ and therefore it is in the line $r_{i-1}\hat{e}_{i-1}$.
- the sum of vectors of forces $\lambda r_{i+1}e_{i+1}$ and $\mu r_{i+1}e_i$ (for some non-zero λ and μ) should be in the plane spanned by $r_{i-1}, r_{i+1}, r_{i+2}$ and therefore it is in the line $r_{i+1}\hat{e}_{i+1}$.

Since the points $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ span a 3-plane, the planes $r_{i-1}e_{i-1}e_i$ and $r_{i-2}r_{i-1}r_{i+1}$ intersect by a line. Symmetrically, the planes $r_{i+1}e_{i+1}e_i$ and $r_{i-1}r_{i+1}r_{i+2}$ intersect by a line.

Therefore, the resulting tensegrity is a non-zero tensegrity on the framework $\mathsf{WU}_i(C_B(P))$.

Now let us assume that there is a non-zero tensegrity on $\mathsf{WU}_i(C_B(P))$. Then we consider a tensegrity $(C^3_{B,i}(P), \tilde{w})$, where $C^3_{B,i}(P)$ is the framed 3-cycle framework as above; the self-stress $\tilde{w}$ is defined by linearity starting from the fact that at edge $r_{i-1}r_{i+1}$ it coincides with the self-stress at $r_{i-1}r_{i+1}$ for $\mathsf{WU}_i(C_B(P))$.

Similarly, by subtracting $(C^3_{B,i}(P), \tilde{w})$ from $(\mathsf{WU}_i(C_B(P)), w)$ and summing the boundary force vectors at r_{i-1} and r_{i+1} we get a non-zero tensegrity for $C_B(P)$. ■

3.2. **On existence and uniqueness of tensegrities for frameworks in 3D-general position.** Recall that in this paper we work only with connected graphs. The uniqueness of tensegrities (up to a scalar) can be formulated as follows.

Proposition 3.11. All tensegrities on a linearly generic framework are proportional. In addition every non-zero tensegrity is everywhere non-zero.

Proof. The proof is straightforward as tensions at every vertex of a linearly generic framework are defined in the unique way up to a scalar. All stresses at this vertex are either all zero or all non-zero. ■

Before to formulate a criterium of existence of a tensegrity we give the following definition.

Definition 3.12. Consider a linearly generic framework $G(P)$ and a cycle C in G (without self-intersections). Furthermore, consider a framed cycle $C_B = (C, B)$. We say that a framed cycle framework

$$C_B(\tilde{P}) = \big((r_1, \ldots, r_n), (e_1, \ldots, e_n)\big)$$

is *associated* to $G(P)$ if

- $r_i = p_i$ at all corresponding points of C and G;
- for the boundary points we have:

$$e_i \in \operatorname{span}(p_i, p_{i-1}, p_{i+1}) \cap \operatorname{span}(p_i, p_{i,1}, \ldots, p_{i,k}),$$

  where $v_i v_{i,j}$ correspond to all edges adjacent to v_i except for the two edges $v_i v_{i-1}$ and $v_i v_{i+1}$, and $p_i p_{i,j}$ are their realizations in $G(P)$.
- In addition we require that $e_i \neq r_i$ for $i = 1, \ldots, n$.

The criterium of existence of a tensegrity can be formulated in the following way.

Theorem 3.13. *A linearly generic connected framework admits a non-zero tensegrity if and only if all its associated framed cycle frameworks admit a non-zero tensegrity.*

Proof. Assume that a framework admits a non-zero tensegrity. Then the associated framed cycle frameworks admit a non-zero tensegrity directly by Proposition 3.11.

Let now all associated framed cycle frameworks of $G(P)$ admit a non-zero tensegrity. Let us iteratively construct a non-zero tensegrity for $G(P)$.

We start with any vertex of degree greater than 1 and set the stress on one of its edges equal to 1. Therefore, the stresses for the other edges are defined in the unique way.

Assume now that we have constructed the stresses for the edges adjacent to all vertices of $V' \subset V$. In addition we assume that every pair of vertices in V' is connected by a path in G within V'.

Let us now consider some edge $v'v$ such that $v' \in V'$ and $v \in V \setminus V'$. If $v \in B$ then there is no equilibrium condition on stresses, we just add v to V'. Let now v' be k-valent ($k > 1$) with edges $vv_1, \ldots, vv_k$ adjacent to v. Consider the following two cases for these edges.
Case 1: $v_i \notin V'$. Then the stress at vv_i is not yet defined. Hence we define it from the equilibrium condition for v.
Case 2: $v_i \in V'$. Then there exists an associated framed cycle framework $C_B(\tilde{P})$ whose all non-boundary vertices correspond to vertices in $V' \cup \{v\}$ and that passes through v and v_i via edge vv_i. First of all, it has a non-zero self-stress by the theorem assumption. Secondly, this self-stress is proportional to the stresses defined on the edges adjacent to V' (since all the vertices but one are in V', and the equilibrium conditions in V' are fulfilled simultaneously for the self-stress on the cycle $C_B(\tilde{P})$ and the partially constructed stress). So the stress at vv_i

defined from v_i before coincides with the stress at vv_i defined by the equilibrium in v.

Now we add v to V' and continue to the next vertex of $V \setminus V'$. Note that after adding v to V' all the vertices of the new V' are connected by edge paths of G via vertices of V'.

At each step of iteration we add a new vertex and define the stresses on the edges adjacent to it (if they were not defined before) such that the equilibrium condition is fulfilled.

Since G is connected, the process terminates and we have a tensegrity on $G(P)$ at the end of the process. ■

3.3. **Grassmann-Cayley condition for existence of tensegrities.** Finally, we have all tools to formulate geometric conditions for the existence of non-zero tensegrities for frameworks in 3D-general position in terms of Grassmann-Cayley algebra. Let us first briefly recall important notions and definitions of Grassmann-Cayley algebra.

3.3.1. *Grassmann-Cayley algebra, operations and relations.* Let us briefly recall some notions of Grassmann-Cayley algebra. First of all the elements of Grassmann-Cayley algebra on $\mathbb{R}^d$ (or on $\mathbb{R}P^d$) are all the k-planes of all possible dimensions $k \leq d$.

There are two operations on Grassmann-Cayley algebra that are called *join* and *meet* operations and denoted by $\vee$ and $\wedge$, respectively.

Definition 3.14. Given the affine (or projective) planes $\pi_1, \ldots, \pi_n$ of arbitrary dimensions, the *join* and *meet* operations for these planes are respectively as follows:

$$\pi_1 \vee \ldots \vee \pi_n = \operatorname{span}(\pi_1, \ldots, \pi_n);$$
$$\pi_1 \wedge \ldots \wedge \pi_n = \bigcap_{i=1}^{n} \pi_i.$$

Finally, let us formulate relations on the elements of Grassmann-Cayley algebra.

Definition 3.15. Given the affine planes, for simplicity we consider their projectivisations $\pi_1, \ldots, \pi_n$. We say that

$$\pi_1 \wedge \ldots \wedge \pi_n = \text{true},$$

if there exist projective planes π_i' with $\dim \pi_i' = \dim \pi_i$ for $i = 1, \ldots, n$ such that

$$\dim(\pi_1 \wedge \ldots \wedge \pi_n) > \dim(\pi_1' \wedge \ldots \wedge \pi_n').$$

Otherwise we say that

$$\pi_1 \wedge \ldots \wedge \pi_n = \text{false}.$$

Here we consider the dimension of an empty set to be -1.

Example 3.16. Consider three lines ℓ_1, ℓ_2, ℓ_3 in the plane. Then

$$\ell_1 \wedge \ell_2 \wedge \ell_3 = \text{true},$$

if and only if these three lines have projectively at least one point in common. Note that we obtain the value "true" for three parallel lines as well.

For more information on Grassmann-Cayley algebra we refer to [10] and [19].

3.3.2. *Grassmann-Cayley condition for framed cycles.* Let us first start with a Grassmann-Cayley condition for a framed cycle on three vertices.

Definition 3.17. Let $C_B(P)$ be a framework of a framed cycle in 3D-general position

$$\big((r_1, r_2, r_3), (e_1, e_2, e_3)\big).$$

Then the *Grassmann-Cayley condition* for C is as follows

$$r_1e_1 \wedge r_2e_2 \wedge r_3e_3 = \text{true}.$$

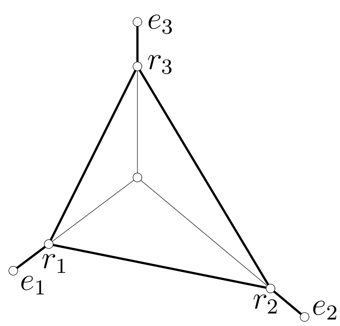

FIGURE 9. A framed cycle consisting of a triangle r_1, r_2, r_3 with external forces $w_i(e_i - r_i)$ is a tensegrity if and only if the three lines r_ie_i meet in a point.

Let us now expand the notion of Grassmann-Cayley condition to framed cycle frameworks of arbitrary length.

Definition 3.18. Let C_B be a framed cycle of length $n \geq 3$, and let $C_B(P)$ be its framework in 3D-general position. Then the *Grassmann-Cayley condition* for $C_B(P)$ is as follows

$$r_1e_1^{(n-3)} \wedge r_2e_2 \wedge r_3e_3^{(n-3)} = \text{true},$$

where $e_1^{(n-3)}$ is defined recursively by

$$e_1^{(0)} = e_1;$$
$$e_1^{(k)} = e_{n-k+1}e_1^{(k-1)} \wedge \big(r_{n-k} \vee r_1 \vee r_2\big),$$

and $e_3^{(n-3)}$ is defined recursively by

$$e_3^{(0)} = e_n;$$
$$e_3^{(k)} = e_{n-k}e_3^{(k-1)} \wedge \big(r_{n-k-1} \vee r_{n-k} \vee r_1\big).$$

Remark 3.19. For simplicity here and below we write uw instead of $u \vee v$.

Proposition 3.20. A framed cycle framework $C_B(P)$ in 3D-general flat position has a non-zero tensegrity if and only if $C_B(P)$ fulfills the Grassmann-Cayley condition.

Proof. The condition is written by iteratively application of WU-surgeries to the last vertex of C, reducing C_B to a triangular framed cycle in general flat position. Namely the resulting flat cycle is

$$\mathsf{WU}_4(\ldots \mathsf{WU}_n(C_B(P))\ldots).$$

The existence of a non-zero tensegrity is equivalent to the existence of a non-zero tensegrity after WU-surgeries by Proposition 3.10.

So the statement of proposition is reduced to triangular cycles. The statement for a triangular cycle (which has to be planar) is classical (see e.g., [15]). ■

Let us write explicitly the Grassmann-Cayley conditions for cycles on 3 and 4 vertices.

Example 3.21. If $n = 3$, then we have

$$r_1e_1 \wedge r_2e_2 \wedge r_3e_3 = \text{true}.$$

If $n = 4$, then we have (see Figure 10)

$$\big[r_1 \vee \big(e_4e_1 \wedge (r_3 \vee r_1 \vee r_2)\big)\big] \wedge r_2e_2 \wedge \big[r_3 \vee \big(e_3e_4 \wedge (r_2 \vee r_3 \vee r_1)\big)\big] = \text{true}.$$

3.3.3. *Grassmann-Cayley algebra criteria for tensegrities in 3D-general position.* The following theorem and its proof is the recipe to write the Grassmann-Cayley algebra criteria for tensegrities in 3D-general position.

Theorem 3.22. *The framework $G(P)$ in 3D-general position admits a non-trivial tensegrity, if and only if all the Grassmann-Cayley conditions for all its associated framed cycle frameworks are fulfilled.*

Proof. The Grassmann-Cayley conditions for $G(P)$ are written according to Definition 3.18. Due to Theorem 3.13 and Proposition 3.20 they are equivalent to the existence of a non-zero tensegrity on $G(P)$.

It remains to add the following detail to the above construction. In order to generate the boundary $\tilde{e}_i$ of an associated framed cycle

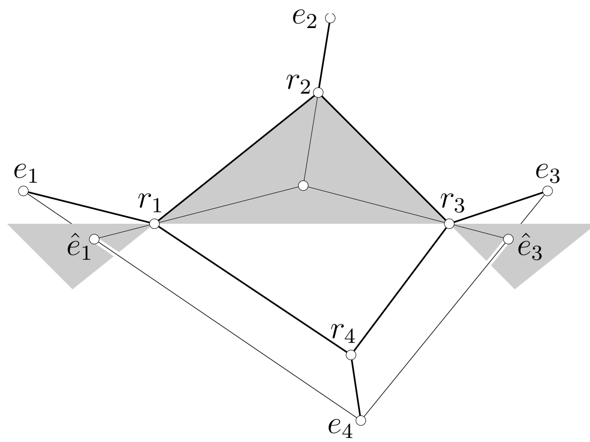

FIGURE 10. Illustration of Example 3.21 for $n = 4$.

framework $C_B(\tilde{P})$, one should take the intersection of the span of two edges in the cycle passing through $\tilde{r}_i = r_i$ (namely $r_{i-1}r_i$ and r_ir_{i+1}). and the span of all other edges adjacent to r_i, say $r_ir_{i,1}, \ldots, r_ir_{i,k}$. Let us denote the resulting line by ℓ. In terms of Grassmann-Cayley algebra ℓ is written as

$$\ell = \big(r_i \vee r_{i,1} \vee \ldots \vee r_{i,k}\big) \wedge \big(r_{i-1} \vee r_i \vee r_{i+1}\big).$$

Finally, we pick up a point $P(b_i)$ on ℓ distinct to r_i. For instance, set

$$\tilde{e}_i = \ell \wedge \big(r_{i,1} \vee \ldots \vee r_{i,k}\big). \qquad \blacksquare$$

Remark 3.23. In the previous theorem it is sufficient to consider only the non-intersecting framed cycles of $G(P)$ representing different classes in $H_1(G)$.

Remark 3.24. It is possible to write the conditions for arbitrary connected graphs (whose frameworks are not necessarily linearly generic) in terms of extended Grassmann-Cayley algebra. The two-dimensional case of non linearly generic frameworks was studied in [13] (see also [15]), and the higher dimensional case can be approached with similar techniques. We skip it here for simplicity.

4. APPLICATIONS TO DISCRETE HARMONIC MAPS

In this section we relate tensegrities to the notion of discrete harmonic functions, i.e., functions fulfilling a discrete Laplace equation $\Delta f = 0$. Tensegrities as well as harmonic functions with positive tensions or weights, respectively, can be found by minimizing a quadratic energy. We illustrate some results as numerical solutions of this minimization process.

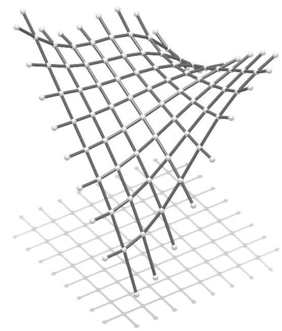

FIGURE 11. A discrete harmonic real valued function over the $\mathbb{Z}^2$ lattice $f : \mathbb{Z}^2 \to \mathbb{R}$.

Recall that the discrete Laplace operator acts on maps $f : G \to \mathbb{R}^d$ defined on arbitrary graphs G by

$$(\Delta f)(v_i) := \sum_{v_j \sim v_i} w_{i,j}(f(v_i) - f(v_j)),$$

where we sum over neighboring vertices v_j of v_i. This *discrete Laplace operator* has been used in several applications of geometry processing as well as in discrete complex analysis and discrete minimal surface theory (see e.g., [3, 4]). The weights $w_{i,j} \in \mathbb{R}$ are chosen depending on the application. Prominent examples are the cotangent-weights or the area of Voronoi cells around the vertex v_i. Furthermore, the choice of the weights implies which properties of the discrete Laplace operator "inherits" from its smooth counterpart [27].

As it follows from the definition of tensegrities (Definition 1.1) it can be seen as zeroes of the discrete Laplace operator (the graph Laplacian) for maps defined on the vertices of a graph. In this sense tensegrities are harmonic maps with respect to the discrete Laplace operator.

Definition 4.1. A function $f : G \to \mathbb{R}^d$ is called *discrete harmonic* if

$$(\Delta f)(v_i) = 0$$

for all vertices $v_i \in Z(G)$.

A real valued discrete harmonic function over some rectangular subgrid of the $\mathbb{Z}^2$ lattice is illustrated by Figure 11.

In the setting of tensegrities the function f describes the coordinates of the position of the vertices in space and the weight assignment $w_{i,j}$ represents the stress at each edge $(v_i; v_j)$. Consequently, we will allow positive and negative weights for tensile and compression forces.

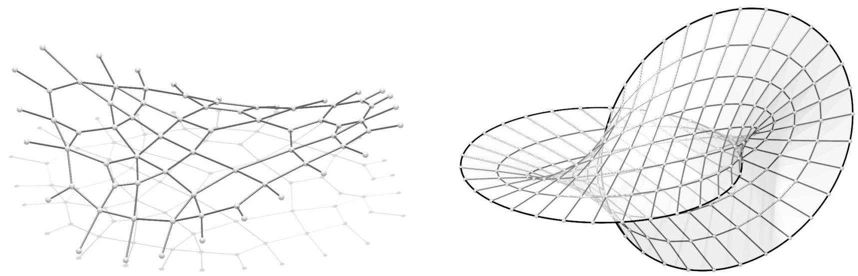

FIGURE 12. *Left*: A tensegrity with the combinatorics of an arbitrary cell decomposition of a disc. *Right*: Two circles are the boundaries of a tensegrity with regular quadrilateral combinatorics.

Tensegrities with arbitrary combinatorics and with only positive tensions can be easily constructed as a minimum of the energy

$$\sum_{v_i \in G} \sum_{v_j \sim v_i} w_{i,j} \|p_i - p_j\|^2, \tag{6}$$

viewing the coordinates of the vertices p_i as variables. Then at a critical point we obtain

$$\sum_{v_j \sim v_i} w_{i,j}(p_i - p_j) = 0,$$

for all i and therewith a tensegrity as critical point of an energy.

Example 4.2. Let us consider a function $f : U \subset \mathbb{Z}^2 \to \mathbb{R}$ where U is a rectangular patch and let us further fix the values of f on the boundary of U. To obtain a harmonic function we minimize

$$\sum_{v_i \in U} \sum_{v_j \sim v_i} w_{i,j} \|f_i - f_j\|^2,$$

where f_i denotes the value of f at v_i. We illustrate the graph $(v_i, f_i) \in \mathbb{R}^2 \times \mathbb{R}$ of that harmonic function f in Figure 11 above.

Example 4.3. Two tensegrities with everywhere unit tensions are illustrated by Figure 12 with the combinatorics of an arbitrary cell decomposition (left) and with regular quadrilateral combinatorics (right). We obtain these tensegrities by fixing the positions of the boundary vertices and minimizing the quadratic energy in Equation (6).

5. EXAMPLE: OCTAHEDRAL TENSEGRITIES IN $\mathbb{R}^3$

We conclude the paper with a brief description of a non-trivial three-dimensional example, an octahedral tensegrity (cf. [30]).

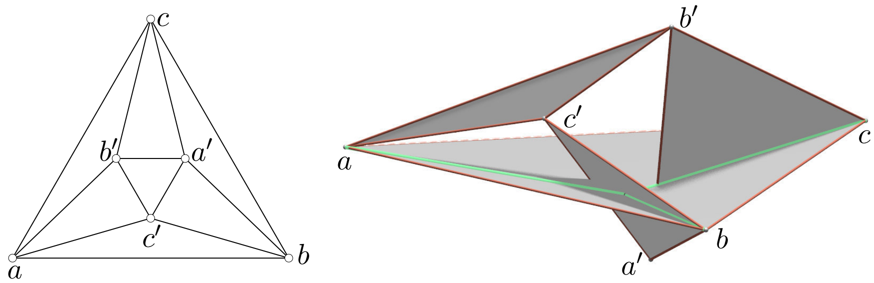

FIGURE 13. *Left*: The combinatorics of an octahedron. *Right*: An octahedron in $\mathbb{R}^3$. Its edges form a tensegrity if and only if any four alternate face planes, i.e., planes of the type abc, $ab'c'$, $a'bc'$, $a'b'c$, are concurrent in a point.

Proposition 5.1. An octahedral framework (a, b, c, a', b', c') in $\mathbb{R}^3$ is a tensegrity if and only if four alternate face planes abc, $ab'c'$, $a'bc'$, $a'b'c$ are concurrent in a point (see Figure 13).

Let us give two new proofs of this classical statement in terms of multiplicative 1-forms and in terms of meet and join operations.

Proof 1 (via multiplicative 1-forms). Let us consider the cycle with three vertices (a, b, c). The necessary condition for that cycle to be part of a tensegrity is that the product of the three values of the 1-form multiply to 1 which is equivalent to Ceva's theorem (see Section 2.2 for $n = 3$). Consequently, the three lines

$$\begin{gathered}\operatorname{span}(a, b', c') \cap \operatorname{span}(a, b, c),\\ \operatorname{span}(a', b, c') \cap \operatorname{span}(a, b, c),\\ \operatorname{span}(a', b', c) \cap \operatorname{span}(a, b, c),\end{gathered}$$

must intersect in one point and therefore all four planes intersect in one point. ■

Proof 2 (within Grassmann-Cayley algebra). Let

$$\begin{gathered}\ell_1 = ab'c' \wedge abc;\\ \ell_2 = a'bc' \wedge abc;\\ \ell_3 = a'b'c \wedge abc.\end{gathered}$$

Our condition for a triangle abc is

$$\ell_1 \wedge \ell_2 \wedge \ell_3 = \text{true}.$$

This is to say that $b'c'a$, $c'a'b$, $a'b'c$, and abc indeed meet in a point. ■

Remark 5.2. As one can notice, one can apply proofs 1 and 2 of Proposition 5.1 to any other triangle in the octahedron. In fact all these conditions would be equivalent.

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