Generalized Nash Equilibrium without Common Belief in Rationality*

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Abstract. We provide an existence result for the solution concept of generalized Nash equilibrium, which can be viewed as the direct incomplete information analogue of Nash equilibrium. Intuitively, a tuple consisting of a probability measure for every player on his choices and utility functions is a generalized Nash equilibrium, whenever some mutual optimality property is satisfied. This incomplete information solution concept is then epistemically characterized in a way that common belief in rationality is neither used nor implied. For the special case of complete information, an epistemic characterization of Nash equilibrium ensues as a corollary.

Keywords: common belief in rationality; complete information; epistemic characterization; epistemic game theory; existence; generalized Nash equilibrium; incomplete information; interactive epistemology; Nash equilibrium; solution concepts; static games.

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1 Introduction

In game theory Nash's (1950) and (1951) notion of equilibrium constitutes one of the most prevalent solution concepts for static games with complete information. Existence of this solution concept has been established by Nash (1950) based on

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Kakutani's generalized fixed point theorem (Kakutani, 1941, Theorem 1) for the class of finite static games with complete information. Besides, Nash (1951) gives a different proof of existence by only relying on Brouwer's original fixed point theorem (Brouwer, 1911, Satz 4).

In order to unveil the reasoning assumptions underlying Nash equilibrium, epistemic foundations have been provided for this classical solution concept by, for instance, Aumann and Brandenburger (1995), Perea (2007), Barelli (2009), as well as Bach and Tsakas (2014). In each of these epistemic foundations some correct beliefs assumption is needed to obtain Nash equilibrium. As correct beliefs seems to be a rather demanding requirement, Nash equilibrium does actually impose non-trivial conditions on the players' reasoning.

In static games with incomplete information, players face uncertainty about the opponents' utility functions. For this more general class of games the most widespread solution concept is Harsanyi's (1967-68) Bayesian equilibrium. In fact, Bayesian equilibrium does not generalize Nash equilibrium but correlated equilibrium to incomplete information (cf. Battigalli and Siniscalchi, 2003; Bach and Perea, 2017).

However, a direct incomplete information analogue to Nash equilibrium can be defined, by extending its mutual optimality property to payoff uncertainty. Accordingly, a tuple consisting of beliefs about each player's choice and utility function is called a generalized Nash equilibrium, whenever each belief only assigns positive probability to choice utility function pairs such that the choice is optimal for the utility function and the product measure of the beliefs on the opponents' choices. Coinciding with the mutual optimality property definition of Nash equilibrium in the case of complete information with mixed strategies interpreted as beliefs, the notion of generalized Nash equilibrium thus provides a direct generalization of Nash equilibrium to incomplete information.

As an illustration of the incomplete information solution concept of generalized Nash equilibrium, suppose a game between two players *Alice* and *Bob* who are both invited to a party. They need to – simultaneously and independently – choose the colour of their outfits to be black or pink, or alternatively, to stay at home. *Alice* prefers wearing the same colour as *Bob* to staying at home, but prefers staying at home to attending the party with a different colour than *Bob*. *Alice* is not sure about *Bob*'s preferences. She thinks that he either entertains the same preferences as she or that he prefers attending the party with a different colour than she to staying at home, but prefers staying at home to attending the party with the same colour as she. The utility functions for *Alice* and *Bob* are provided in Figure 1, and an interactive representation of the game is given in Figure 2.

Consider the two beliefs $(black, u_A)$ about *Alice*'s choice and utility function as well as $\frac{3}{4} \cdot (black, u_B) + \frac{1}{4} \cdot (pink, u'_B)\big)$ about *Bob*'s choice and utility function. Note that black is optimal for *Alice*'s utility function u_A, if she believes *Bob* to wear black with probability $\frac{3}{4}$ and pink with probability $\frac{1}{4}$. Also, black is optimal for *Bob*'s utility function u_B, if he believes *Alice* to wear black, and pink is optimal for *Bob*'s utility function u'_B, if he believes her to wear black.

u_A	*black*	*pink*	*stay*
black	3	0	0
pink	0	3	0
stay	2	2	2

u_B	*black*	*pink*	*stay*
black	3	0	0
pink	0	3	0
stay	2	2	2

u'_B	*black*	*pink*	*stay*
black	0	3	0
pink	3	0	0
stay	2	2	2

Fig. 1. Utility functions of *Alice* and *Bob*.

Alice \ *Bob*	*black*	*pink*	*stay*
black	$3, 3$	$0, 0$	$0, 2$
pink	$0, 0$	$3, 3$	$0, 2$
stay	$2, 0$	$2, 0$	$2, 2$

Alice \ *Bob*	*black*	*pink*	*stay*
black	$3, 0$	$0, 3$	$0, 2$
pink	$0, 3$	$3, 0$	$0, 2$
stay	$2, 0$	$2, 0$	$2, 2$

Fig. 2. Interactive representation of the two-player game with incomplete information and utility functions as specified in Figure 1.

The two beliefs $(black, u_A)$ and $\left(\frac{3}{4} \cdot (black, u_B) + \frac{1}{4} \cdot (pink, u'_B)\right)$ thus form a generalized Nash equilibrium.

This note first establishes the existence of generalized Nash equilibrium for the class of static games with incomplete information. Then, an epistemic characterization of this solution concept is provided. The epistemic conditions are intended to be as minimal as possible. In particular, it is shown that they actually do not imply common belief in rationality. Similarly to the special case of complete information with Nash equilibrium, a correct beliefs assumption also emerges as the decisive property for players to reason in line with generalized Nash equilibrium. Besides, for complete information games an epistemic characterization of Nash equilibrium ensues as a corollary.

2 Generalized Nash Equilibrium

A game with incomplete information is modelled as a tuple $\Gamma = \left(I, (C_i)_{i \in I}, (U_i)_{i \in I}\right)$, where I is a finite set of players, C_i denotes player i's finite choice set, and the finite set U_i contains player i's utility functions, where a utility function $u_i : \times_{j \in I} C_j \to \mathbb{R}$ from U_i assigns a real number $u_i(c)$ to every choice combination $c \in \times_{j \in I} C_j$. Complete information obtains as a special case, if the set U_i is a singleton for every player $i \in I$.

Before the solution concept of generalized Nash equilibrium for games with incomplete information is defined, attention is restricted to complete information and the classical solution concept of Nash equilibrium is recalled. For a given game $\Gamma = \left(I, (C_i)_{i \in I}, (\{u_i\})_{i \in I}\right)$ with complete information, a tuple $(\sigma_i)_{i \in I} \in \times_{i \in I} \Delta(C_i)$ of probability measures constitutes a *Nash equilibrium*, whenever for all $i \in I$ and for all $c_i \in C_i$, if $\sigma_i(c_i) > 0$, then $\sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i})\big) \geq$

$\sum_{c_{-i}\in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c'_i, c_{-i})\big)$ for all $c'_i \in C_i$.[1] A direct generalization of Nash equilibrium to incomplete information obtains as follows.

Definition 1. *Let Γ be a game with incomplete information, and $(\beta_i)_{i\in I} \in \times_{i\in I}\big(\Delta(C_i \times U_i)\big)$ be a tuple of probability measures. The tuple $(\beta_i)_{i\in I}$ constitutes a* generalized Nash equilibrium*, whenever for all $i \in I$ and for all $(c_i, u_i) \in C_i \times U_i$, if $\beta_i(c_i, u_i) > 0$, then*

$$\sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c'_i, c_{-i})$$

for all $c'_i \in C_i$.

Intuitively, the mutual optimality property of the players' supports required by the complete information solution concept of Nash equilibrium is extended to the augmented uncertainty space of choices and utility functions. In the specific case of complete information, i.e. $U_i = \{u_i\}$ for all $i \in I$, the notion of generalized Nash equilibrium formally indeed reduces to Nash equilibrium. In other words, generalized Nash equilibrium imposes the analogous condition on the – due to payoff uncertainty extended – space $\times_{i\in I}\big(\Delta(C_i \times U_i)\big)$ that Nash equilibrium imposes on the space $\times_{i\in I}\Delta(C_i)$. Note that for the game represented in Figure 2, the tuple $\big((black, u_A), \frac{3}{4} \cdot (black, u_B) + \frac{1}{4} \cdot (pink, u'_B)\big)$ indeed constitutes a generalized Nash equilibrium.

In order to characterize decision-making in line with generalized Nash equilibrium, the notion of optimal choice in a generalized Nash equilibrium is defined next.

Definition 2. *Let Γ be a game with incomplete information, $i \in I$ a player, and $u_i \in U_i$ some utility function of player i. A choice $c_i \in C_i$ of player i is* optimal for the utility function u_i in a generalized Nash equilibrium*, if there exists a generalized Nash equilibrium $(\beta_i)_{i\in I} \in \times_{i\in I}\big(\Delta(C_i \times U_i)\big)$ such that*

$$\sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c_i, c_{-i})\big) \geq \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c'_i, c_{-i})$$

for all $c'_i \in C_i$.

In fact, it can be shown that in terms of optimal choices generalized Nash equilibrium refines Harsanyi's (1967-68) solution concept of Bayesian equilibrium (cf. Bach and Perea, 2017).

Solution concepts are always defined relative to a class of games. An existence result ensures that a solution concept always generates a tuple of non-empty strategy sets – sometimes also called prediction – for any game within the respective class. In particular, existence excludes that a solution concept can

[1] Given collection $\{X_i : i \in I\}$ of sets and probability measures $p_i \in \Delta(X_i)$ for all $i \in I$, the set X_{-i} refers to the product set $\times_{j\in I\setminus\{i\}} X_j$ and the probability measure p_{-i} refers to the product measure $\Pi_{j\in I\setminus\{i\}} p_j \in \Delta(X_{-i})$ on X_{-i}.

only be applied to some strict subset of the intended class of games. For static games with complete information Nash (1950) provides an existence result for the solution concept of Nash equilibrium based on Kakutani's generalized fixed point theorem (Kakutani, 1941, Theorem 1). Also using Kakutani's generalized fixed point theorem the existence of generalized Nash equilibrium within the class of static games with incomplete information can be established as follows.

Theorem 1. *Let Γ be a game with incomplete information, and $\beta_i^U \in \Delta(U_i)$ a probability measure for every player $i \in I$. Then, there exists a generalized Nash equilibrium $(\beta_i)_{i\in I} \in \times_{i\in I}\big(\Delta(C_i \times U_i)\big)$ such that $\text{marg}_{U_i}\beta_i = \beta_i^U$ for all $i \in I$.*

Proof. For every player $i \in I$, and for every set $X_i \subseteq C_i \times U_i$ define a set $\Delta^{\beta_i^U}(X_i) := \{\beta_i \in \Delta(X_i) : \text{marg}_{U_i}\beta_i = \beta_i^U\}$, as well as a correspondence $f_i : \times_{j\in I}\big(\Delta^{\beta_j^U}(C_j \times U_j)\big) \twoheadrightarrow \Delta^{\beta_i^U}(C_i \times U_i)$ such that $f_i\big((\beta_j)_{j\in I}\big) := \Delta^{\beta_i^U}\big(\{(c_i,u_i) \in C_i \times U_i : \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i},u_{-i})\cdot u_i(c_i,c_{-i})\big) \geq \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i},u_{-i})\cdot u_i(c_i',c_{-i})$ for all $c_i' \in C_i\}\big)$. Consider the correspondece $f : \times_{j\in I}\big(\Delta^{\beta_j^U}(C_j \times U_j)\big) \twoheadrightarrow \times_{j\in I}\big(\Delta^{\beta_j^U}(C_j \times U_j)\big)$, where $f\big((\beta_j)_{j\in I}\big) := \times_{j\in I} f_j\big((\beta_k)_{k\in I}\big)$ for all $(\beta_j)_{j\in I} \in \times_{j\in I}\big(\Delta^{\beta_j^U}(C_j \times U_j)\big)$. Observe that the set $\times_{j\in I}\big(\Delta^{\beta_j^U}(C_j \times U_j)\big)$ as well as for all $(\beta_i)_{i\in I}$ the image set $f\big((\beta_i)_{i\in I}\big)$ are non-empty, compact, and convex. Let $\big((\beta_j)^n)_{j\in I}\big)_{n\in\mathbb{N}}$ be some converging sequence with limit $(\beta_j)_{j\in I}$, where $\beta_j^n \in \Delta(C_j \times U_j)$ for all $j \in I$ and for all $n \in \mathbb{N}$. Consider some player $i \in I$ and suppose that $\hat{\beta}_i^n \in f_i\big((\beta_j^n)_{j\in I}\big)$ for all $n \in \mathbb{N}$ as well as that the sequence $(\hat{\beta}_i^n)_{n\in\mathbb{N}}$ is converging with limit β_i. It is then the case that $\hat{\beta}_i \in f_i\big((\beta_j)_{j\in I}\big)$. Consequently, the function f is upper semi-continuous. By Kakutani (1941, Theorem 1) it follows that there exists a tuple $(\beta_i^*)_{i\in I} \in \times_{i\in I}\Delta^{\beta_i^U}(C_i \times U_i)$ such that $(\beta_i^*)_{i\in I} \in f\big((\beta_i^*)_{i\in I}\big)$. Therefore, $(\beta_i^*)_{i\in I}$ constitutes a generalized Nash equilibrium of Γ such that $\text{marg}_{U_i}\beta_i^* = \beta_i^U$ for all $i \in I$. ■

Accordingly, for every incomplete information game and for every tuple of probability measures about utility functions, it is possible to construct a generalized Nash equilibrium that matches these probability measures about utility functions. As an immediate corollary of Theorem 1 an existence result analogous to Nash (1951, Theorem 1) ensues: *every finite game with incomplete information has a generalized Nash equilibrium.*[2] However, Theorem 1 is stronger, since it requires generalized Nash equilibrium to satisfy additional conditions by fixing the

[2] If no specific probability measures on utility functions are imposed on generalized Nash equilibrium as additional conditions, then our solution concept can also be constructed in a more direct way based on Nash's existence theorem. For a given incomplete information game $\big(I, (C_i)_{i\in I}, (U_i)_{i\in I}\big)$, fix a utility function $u_i^* \in U_i$ for every player $i \in I$ and consider the complete information game $\big(I, (C_i)_{i\in I}, (\{u_i^*\})_{i\in I}\big)$. By Nash (1951, Theorem 1) a Nash equilibrium $(\sigma_i)_{i\in I}$ exists. Define for every player $i \in I$ a probability measure $\beta_i \in \Delta(C_i \times U_i)$ where

$$\beta_i(c_i,u_i) := \begin{cases} \sigma_i(c_i), & \text{if } u_i = u_i^*, \\ 0, & \text{otherwise,} \end{cases}$$

probability measures about utility functions. Intuitively, no matter what beliefs about payoffs agents may hold in a specific context of a complete information game, a corresponding generalized Nash equilibrium always exists. Besides, note that in a sense the formulation of Theorem 1 is similar to how Ely and Pęski (2006) as well as Dekel et al. (2007) define their incomplete information solution concepts of interim rationalizability by fixing the players' belief hierarchies on utility functions.

3 Common Belief in Rationality

From the perspective of a single player there exist two basic sources of uncertainty with respect to Γ. A player faces strategic uncertainty, i.e. what choices his opponents make, as well as payoff uncertainty, i.e. what utility functions represent the opponents' preferences. The notion of an epistemic model provides the framework to describe the players' reasoning about these two sources of uncertainty. Formally, an *epistemic model* of Γ is a tuple $\mathcal{M}^{\Gamma} = \big((T_i)_{i \in I}, (b_i)_{i \in I}\big)$, where for every player $i \in I$, the set T_i contains all of i's types and the function $b_i : T_i \rightarrow \Delta(C_{-i} \times T_{-i} \times U_{-i})$ assigns to every type $t_i \in T_i$ a probability measure $b_i[t_i]$ on the set of opponents' choice type utility function combinations. Given a game and an epistemic model of it, belief hierarchies, marginal beliefs, as well as marginal belief hierarchies can be derived from every type. For instance, every type $t_i \in T_i$ induces a belief on the opponents' choice combinations by marginalizing the probability measure $b_i[t_i]$ on the space C_{-i}. For simplicity sake, no additional notation is introduced for marginal beliefs. It should always be clear from the context which belief $b_i[t_i]$ refers to.

Some further notions are now introduced. For that purpose consider a game Γ, an epistemic model $\mathcal{M}^{\Gamma}$ of it, and fix two players $i, j \in I$ such that $i \neq j$. A type $t_i \in T_i$ of i is said to *deem possible* some choice type utility function combination $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$ of his opponents, if $b_i[t_i](c_{-i}, t_{-i}, u_{-i}) > 0$. Analogously, a type $t_i \in T_i$ deems possible some opponent j's type $t_j \in T_j$, if $b_i[t_i](t_j) > 0$. For each choice type utility function combination $(c_i, t_i, u_i) \in C_i \times T_i \times U_i$, the *expected utility* is given by

$$v_i(c_i, t_i, u_i) = \sum_{c_{-i} \in C_{-i}} \big(b_i[t_i](c_{-i}) \cdot u_i(c_i, c_{-i})\big)$$

for every player $i \in I$. Optimality can be viewed as a property of choices given a type utility function pair. Formally, given some utility function $u_i \in U_i$ and some type $t_i \in T_i$ of player i, a choice $c_i \in C_i$ is *optimal* for (t_i, u_i), if $v_i(c_i, t_i, u_i) \geq v_i(c_i', t_i, u_i)$ for all $c_i' \in C_i$. A player believes in his opponents' rationality, if he only deems possible choice type utility function triples – for each of his opponents – such that the choice is optimal for the type utility function pair, respectively. Formally, a type $t_i \in T_i$ *believes in the opponents' rationality*, if t_i only deems

for all $(c_i, u_i) \in C_i \times U_i$. It then follows that $(\beta_i)_{i \in I}$ constitutes a generalized Nash equilibrium.

possible choice type utility function combinations $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$ such that c_j is optimal for (t_j, u_j) for every opponent $j \in I \setminus \{i\}$.

Iterating belief in rationality gives rise to the interactive reasoning concept of common belief in rationality.

Definition 3. *Let Γ be a game with incomplete information, $\mathcal{M}^\Gamma$ an epistemic model of it, and $i \in I$ some player.*

- *A type $t_i \in T_i$ expresses* 1-fold belief in rationality*, if t_i believes in the opponents' rationality.*
- *A type $t_i \in T_i$ expresses* k-fold belief in rationality *for some $k > 1$, if t_i only deems possible types $t_j \in T_j$ for all $j \in I \setminus \{i\}$ such that t_j expresses $k-1$-fold belief in rationality.*
- *A type $t_i \in T_i$ expresses* common belief in rationality*, if t_i expresses k-fold belief in rationality for all $k \geq 1$.*

A player satisfying common belief in rationality entertains a belief hierarchy in which the rationality of all players is not questioned at any level. Observe that if an epistemic model contains for every player only types that believe in the opponents' rationality, then every type also expresses common belief in rationality. This fact is useful when constructing epistemic models with types expressing common belief in rationality.

4 Epistemic Characterization

Before the incomplete information solution concept of generalized Nash equilibrium can be characterized epistemically, some further epistemic notions need to be invoked. For this purpose, consider a game with incomplete information Γ, some epistemic model $\mathcal{M}^\Gamma$ of it, and fix some player $i \in I$.

A type $t_i \in T_i$ of player i is said to have *projective beliefs*, if for every opponent $j \in I \setminus \{i\}$ it is the case that $b_i[t_i](t_j) > 0$ implies that $b_i[t_i](c_k, u_k) = b_j[t_j](c_k, u_k)$ for all $(c_k, u_k) \in C_k \times U_k$ and for all $k \in I \setminus \{i, j\}$. Intuitively, a player with projective beliefs thinks that every opponent shares his belief on every other player's choice utility function combination.

Moreover, a type $t_i \in T_i$ of player i is said to have *independent beliefs*, if $b_i[t_i](c_{-i}, u_{-i}, t_{-i}) = \Pi_{j \in I \setminus \{i\}} b_i[t_i](c_j, u_j, t_j)$ for all $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$. Intuitively, a player with independent beliefs excludes the possibility that his opponents' choice utility function pairs could be correlated.

In addition, for every opponent $j \in I \setminus \{i\}$, a type $t_i \in T_i$ believes that j is *correct* about i's belief about the opponents' choice utility function combinations, if $b_i[t_i'](c_{-i}, u_{-i}) = b_i[t_i](c_{-i}, u_{-i})$ for all $t_i' \in \text{supp}(b_j[t_j])$, for all $t_j \in \text{supp}(b_i[t_i])$, and for all $(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}$.

Furthermore, a type $t_i \in T_i$ of player i is said to have *connected beliefs*, if for two opponents $j, k \in I \setminus \{i\}$ such that $j \neq k$, it is the case that $t_k \in \text{supp}(b_j[t_j])$ or $t_j \in \text{supp}(b_k[t_k])$ for all $t_j, t_k \in \text{supp}(b_i[t_i])$

Besides, for every opponent $j \in I \setminus \{i\}$, a type $t_i \in T_i$ of player i is said to believe that j expresses a certain property, if t_i only deems possible types $t_j \in T_j$ of player j that express the property.

Using these epistemic notions, the following epistemic characterization of generalized Nash equilibrium emerges.

Theorem 2. *Let Γ be a game with incomplete information, $i \in I$ some player, and $u_i^* \in U$ some utility function of player i. A choice $c_i^* \in C_i$ is optimal for u_i^* in a generalized Nash equilibrium, if and only if, there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ with a type $t_i \in T_i$ of player i such that c_i^* is optimal for (t_i, u_i^*) and t_i satisfies the following conditions:*

- (i) *t_i has projective beliefs,*
- (ii) *t_i believes that every opponent $j \in I \setminus \{i\}$ has projective beliefs,*
- (iii) *t_i has independent beliefs,*
- (iv) *t_i believes that every opponent $j \in I \setminus \{i\}$ has independent beliefs,*
- (v) *t_i believes in the opponents' rationality,*
- (vi) *t_i believes that every opponent $j \in I \setminus \{i\}$ believes in the opponents' rationality,*
- (vii) *t_i believes that every opponent $j \in I \setminus \{i\}$ deems possible t_i,*
- $(viii)$ *t_i believes that every opponent $j \in I \setminus \{i\}$ is correct about i's belief about the opponents' choice utility function combinations,*
- (ix) *t_i believes that every opponent $j \in I \setminus \{i\}$ believes that i is correct about j's belief about the opponents' choice utility function combinations.*
- (x) *t_i has connected beliefs.*

Proof. For the *only if* direction of the theorem, let c_i^* be optimal for u_i^* in a generalized Nash equilibrium $(\beta_j)_{j \in I}$. Construct an epistemic model $\mathcal{M}^\Gamma = \big((T_j)_{j \in I}, (b_j)_{j \in I}\big)$ of Γ, where $T_j := \{t_j\}$ and $b_j[t_j](c_{-j}, t_{-j}, u_{-j}) := \beta_{-j}(c_{-j}, u_{-j})$ for all $(c_{-j}, u_{-j}) \in C_{-j} \times U_{-j}$ and for all $j \in I$.

As

$$v_i(c_i^*, t_i, u_i^*) = \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i^*(c_i^*, c_{-i})$$

$$\geq \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i^*(c_i, c_{-i}) = v_i(c_i, t_i, u_i^*)$$

for all $c_i \in C_i$, it is the case that c_i^* is optimal for (t_i, u_i^*).

Observe that by definition of the marginal beliefs of $b_k[t_k]$ about the opponents' choice type utility function combinations to be the product measure $\Pi_{l \in I \setminus k} \beta_l$ for all $k \in I$, it directly holds that every type has projective and independent beliefs. It thus also directly follows that every type believes every opponent to have projective and independent beliefs.

Consider some opponent $j \in I \setminus \{i\}$ of player i and a choice type utility function tuple $(c_j, t_j, u_j) \in C_j \times \{t_j\} \times U_j$ of player j such that $b_i[t_i](c_j, t_j, u_j) > 0$. Then, $\beta_j(c_j, u_j) > 0$ and

$$v_j(c_j, t_j, u_j) = \sum_{(c_{-j}, u_{-j}) \in C_{-j} \times U_{-j}} \beta_{-j}(c_{-j}, u_{-j}) \cdot u_j(c_j, c_{-j})$$

$$\geq \sum_{(c_{-j},u_{-j})\in C_{-j}\times U_{-j}} \beta_{-j}(c_{-j},u_{-j})\cdot u_j(c'_j,c_{-j}) = v_j(c'_j,t_j,u_j)$$

for all $c'_j \in C_j$, by construction of $b_i[t_i]$ and by virtue of $(\beta_j)_{j\in I}$ being a generalized Nash equilibrium. Thus, c_j is optimal for (t_j,u_j). Therefore, t_i believes in the opponents' rationality. Analogeously, it can be shown that every type t_j of every player $j \in I\setminus\{i\}$ also believes in the opponents' rationality. As $b_i[t_i](t_j)=1$ for all $j\in I\setminus\{i\}$, it follows that t_i believes his opponents to believe in the opponents' rationality.

Note that it directly holds that t_i believes every opponent $j\in I\setminus\{i\}$ to deem possible his true type t_i, as there exists only this single type of i in the epistemic model $\mathcal{M}^{\Gamma}$.

Moreover, t_i's marginal belief on $C_{-i}\times U_{-i}$ coincides with $\Pi_{j\in I\setminus\{i\}}\beta_j$. Since $b_i[t_i](t_j)=1$ and $b_j[t_j](t_i)=1$ holds for every opponent $j\in I\setminus\{i\}$ of player i, type t_i believes that every opponent j believes that i's marginal belief on $C_{-i}\times U_{-i}$ is indeed given by $\Pi_{j\in I\setminus\{i\}}\beta_j$. Analogeously, it can be shown that the single type $t_j\in T_j$ for every player $j\in I\setminus\{i\}$ believes that every respective opponent $k\in I\setminus\{j\}$ is correct about j's marginal belief on $C_{-j}\times U_{-j}$. As for all $j\in I\setminus\{i\}$ it is the case that $b_i[t_i](t_j)=1$ and t_j believes that i is correct about j's marginal beliefs on $C_{-j}\times U_{-j}$, it follows that t_i believes every opponent j to believe that i is correct about j's marginal belief on $C_{-j}\times U_{-j}$.

Finally, as there exists only one type for each player, every type must have connected beliefs.

For the *if* direction of the theorem, consider an epistemic model $\mathcal{M}^{\Gamma}$ of Γ with a type $t_i\in T_i$ of player i that satisfies conditions $(i)-(x)$ and such that c_i^* is optimal for (t_i,u_i^*).

Construct a tuple $(\beta_j)_{j\in I}\in\Delta\big(\times_{j\in I}(C_j\times U_j)\big)$ of probability measures such that $\beta_j(c_j,u_j):=b_i[t_i](c_j,u_j)$ for all $(c_j,u_j)\in C_j\times U_j$ and for all $j\in I\setminus\{i\}$, and $\beta_i(c_i,u_i):=b_m[\hat{t}_m](c_i,u_i)$ for all $(c_i,u_i)\in C_i\times U_i$ and for some $m\in I\setminus\{i\}$ and for some $\hat{t}_m\in T_m$ with $b_i[t_i](\hat{t}_m)>0$.

We first show that for all players $j,k\in I\setminus\{i\}$, for every type $t_j\in T_j$ such that $b_i[t_i](t_j)>0$ and for every type $t_k\in T_k$ such that $b_i[t_i](t_k)>0$, it is the case that $b_j[t_j](c_i,u_i)=b_k[t_k](c_i,u_i)$ for all $(c_i,u_i)\in C_i\times U_i$. Fix some $(c_i,u_i)\in C_i\times U_i$. Suppose that $j=k$ and consider $t_j,t'_j\in T_j$ with $b_i[t_i](t_j)>0$ and $b_i[t_i](t'_j)>0$. Towards a contradiction assume that $b_j[t_j](c_i,u_i)\neq b_j[t'_j](c_i,u_i)$. By condition (vii), it is the case that $b_j[t_j](t_i)>0$. Hence, t_j deems it possible that i is not correct about j's belief about i's choice utility function combination, a contradiction with condition (ix). Now, suppose that $j\neq k$ and consider $t_j\in T_j$ as well as $t_k\in T_k$ with $b_i[t_i](t_j)>0$ and $b_i[t_i](t_k)>0$. By condition (x) and without loss of generality, it is the case that $b_j[t_j](t_k)>0$. By condition (ii), it follows that $b_j[t_j](c_i,u_i)=b_k[t_k](c_i,u_i)$.

Next, we show that $(\beta_j)_{j\in I}$ constitutes a generalized Nash equilibrium. Consider player i and suppose that $\beta_i(c_i,u_i)>0$. Then, $b_m[\hat{t}_m](c_i,u_i)>0$, and there thus exists a type $t'_i\in T_i$ of player i such that $b_m[\hat{t}_m](c_i,t'_i,u_i)>0$. By conditions $(viii)$ and (iii), it follows that $b_i[t'_i](c_{-i},u_{-i})=b_i[t_i](c_{-i},u_{-i})=\beta_{-i}(c_{-i},u_{-i})$. By condition (vi), c_i is optimal for (t'_i,u_i), and hence c_i is optimal for (t_i,u_i).

Therefore,

$$\sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i},u_{-i})\cdot u_i(c_i,c_{-i}) = v_i(c_i,t_i,u_i)$$

$$\geq v_i(c_i',t_i,u_i) = \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i},u_{-i})\cdot u_i(c_i',c_{-i})$$

for all $c_i' \in C_i$.

Now, consider some player $j \in I\setminus\{i\}$ and suppose that $\beta_j(c_j,u_j) > 0$ for some $(c_j,u_j) \in C_j \times U_j$. Then, $b_i[t_i](c_j,u_j) > 0$, and consequently $b_i[t_i](c_j,t_j,u_j) > 0$ for some type $t_j \in T_j$ of player j with $b_i[t_i](t_j) > 0$. By condition (i), it holds that $b_j[t_j](c_k,u_k) = b_i[t_i](c_k,u_k) = \beta_k(c_k,u_k)$ for all $(c_k,u_k) \in C_k \times U_k$ and for all $k \in I \setminus \{i,j\}$. Since $\beta_i(c_i,u_i) = b_m[\hat{t}_m](c_i,u_i)$ for all $(c_i,u_i) \in C_i \times U_i$, and as $b_i[t_i](t_j) > 0$, it follows from above that $b_j[t_j](c_i,u_i) = b_m[\hat{t}_m](c_i,u_i) = \beta_i(c_i,u_i)$ for all $(c_i,u_i) \in C_i \times U_i$. By condition (iv), it thus holds that $b_j[t_j](c_{-j},u_{-j}) = \beta_{-j}(c_{-j},u_{-j})$. Moreover, by condition (v), the choice c_j is optimal for (t_j,u_j), and thus

$$\sum_{(c_{-j},u_{-j})\in C_{-j}\times U_{-j}} \beta_{-j}(c_{-j},u_{-j})\cdot u_j(c_j,c_{-j}) = v_j(c_j,t_j,u_j)$$

$$\geq v_j(c_j',t_j,u_j) = \sum_{(c_{-j},u_{-j})\in C_{-j}\times U_{-j}} \beta_{-j}(c_{-j},u_{-j})\cdot u_j(c_j',c_{-j})$$

holds for all $c_j' \in C_j$. Consequently, $(\beta_j)_{j\in I}$ constitutes a generalized Nash equilibrium.

Since $b_i[t_i](c_{-i}) = \beta_{-i}(c_{-i})$ and c_i^* is optimal for (t_i,u_i^*), it is the case that

$$\sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i},u_{-i})\cdot u_i^*(c_i^*,c_{-i}) = v_i(c_i^*,t_i,u_i^*)$$

$$\geq v_i(c_i,t_i,u_i^*) = \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i},u_{-i})\cdot u_i^*(c_i,c_{-i})$$

for all $c_i \in C_i$. As $(\beta_j)_{j\in I}$ constitutes a generalized Nash equilibrium, c_i^* is optimal for u_i^* in a generalized Nash equilibrium. ■

The preceding theorem shows that correct beliefs conditions are inherently linked to the incomplete information solution concept of generalized Nash equilibrium. In fact, conditions $(vii)-(ix)$ together form the correct beliefs assumption that is needed. Intuitively, with the presence of incomplete information the correct beliefs assumption naturally does not only apply to strategic but also to payoff uncertainty.

However, only two layers of common belief in rationality are needed for the epistemic characterization of generalized Nash equilibrium. In fact, the epistemic conditions of Theorem 2 do not even imply common belief in rationality.

Remark 1. There exists a game Γ with incomplete information, an epistemic model $\mathcal{M}^{\Gamma}$ of Γ, $i \in I$ some player, and some type $t_i \in T_i$ of player i such that t_i satisfies conditions $(i) - (x)$ of Theorem 2, but t_i does not express common belief in rationality.

As complete information is a special case of incomplete information, the following example of a two person complete information game establishes Remark 1.

Example 1. Consider the two player game between Alice in Bob represented in Figure 3. Construct an epistemic model $\mathcal{M}^{\Gamma}$ of Γ given by $T_{Alice} = \{t_A, t'_A, t''_A\}$

		Bob	
		c	d
Alice	a	$0,0$	$0,0$
	b	$0,0$	$1,0$

Fig. 3. A two player game between Alice and Bob.

and $T_{Bob} = \{t_B, t'_B\}$ with $b_{Alice}[t_A] = (c, t_B)$, $b_{Alice}[t'_A] = (c, t'_B)$, and $b_{Alice}[t''_A] = (d, t_B)$, as well as $b_{Bob}[t_B] = 0.5 \cdot (a, t_A) + 0.5 \cdot (a, t'_A)$, and $b_{Bob}[t'_B] = (a, t''_A)$. Observe that t_A satisfies conditions $(i) - (x)$ of Theorem 2. However, t_A does not express common belief in rationality, as t_A believes that t_B deems possible that Alice is of type t'_A, which believes that Bob is of type t'_B, which in turn believes Alice to be of type t''_A and to choose a, i.e. which believes Alice to choose irrationally. ♣

Restricting attention to the specific class of complete information games, the epistemic characterization of generalized Nash equilibrium provides an epistemic characterization of the solution concept's complete information analogue i.e. Nash equilibrium. The result is a direct consequence of Theorem 2, if payoff uncertainty is eliminated.

Corollary 1. *Let Γ be a game with complete information, and $i \in I$ some player. A choice $c_i \in C_i$ is optimal in a Nash equilibrium, if and only if, there exists an epistemic model $\mathcal{M}^{\Gamma}$ of Γ with a type $t_i \in T_i$ of player i such that c_i is optimal for t_i and t_i satisfies the conditions $(i) - (x)$ of Theorem 2.*

With Corollary 1 a new epistemic characterization of Nash equilibrium is added to the analysis of static games with complete information.

5 Related Literature

The solution concept of Nash equilibrium for static games with incomplete information has been explored in terms of its underlying epistemic assumptions notably by Aumann and Brandenburger (1995), Perea (2007), Barelli (2009), as well as Bach and Tsakas (2014). The relation of our work to this previous literature is now discussed.

Most importantly, our epistemic characterization (Theorem 2) differs from the previous epistemic literature on Nash equilibrium by considering the more general framework of incomplete information. Also, the formulation of the solution concept of generalized Nash equilibrium does explicitly involve payoff uncertainty. From a classical game theoretic perspective, Theorem 1 can be viewed as an incomplete information analogue to Nash (1951, Theorem 1).

In contrast to Theorem 2, the epistemic characterizations by Aumann and Brandenburger (1995), Perea (2007), Barelli (2009), as well as Bach and Tsakas (2014) are all restricted to the special case of complete information. However, Corollary 1 provides an epistemic characterization of Nash equilibrium for static games with complete information and can thus be directly compared to the previous literature on Nash equilibrium.

First of all, for the case of more than two players, Aumann and Brandenburger (1995) use a common prior assumption in their model, which essentially states that the beliefs of all players are derived via Bayesian conditionalization from a single probability measure. Barelli's (2009) action consistency assumption weakens the common prior assumption. Accordingly, any belief about the expectation of any random variable – measurable with respect to the players' choices – must be equal to the expectation and coincide for all players. Bach and Tsakas (2014) further weaken Barelli's global assumption by only requiring action consistency between pairs of players on a biconnected graph. In a sense, both the common prior assumption as well as action consistency postulate that the players' beliefs are sufficiently aligned. In contrast to the epistemic characterizations of Nash equilibrium by Aumann and Brandenburger (1995), Barelli (2009), as well as Bach and Tsakas (2014), Corollary 1 does not use any form of common prior or action consistency.

The epistemic conditions for Nash equilibrium by Auman and Brandenburger (1995) imply common belief in rationality (cf. Polak, 1999). For Perea (2007) the same holds (this follows from some proofs in Perea, 2007). In comparison, Example 1 establishes that the epistemic conditions used by Corollary 1 do actually not imply common belief in rationality.

Furthermore, the approaches by Aumann and Brandeburger (1995), Barelli (2009), as well as Bach and Tsakas (2014) are state-based, whereas we employ a one-person perspective approach by modelling all epistemic conditions within the mind of the reasoner only. The elementary epistemic operator in Aumann and Brandenburger (1995) as well as in Barelli (2009) is knowledge, while we use the weaker epistemic notion of belief. In contrast to Perea's (2007) epistemic conditions for Nash equilibrium, Corollary 1 does not imply that a player believes his opponents to be correct about his full belief hierarchy: our conditions only imply that a player believes his opponents to be correct about his first-order belief, i.e. the first layer in his belief hierarchy. Unlike Bach and Tsakas (2014) we do not use any graph structure as additional modelling component.

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