CONTACT, WITH APPLICATIONS TO SUBMANIFOLDS OF $\mathbb{R}^n$

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CONTENTS

INTRODUCTION

Given two plane curves X and Y passing through the origin, there are two classical ways of measuring the contact between them at that point. The first is to consider one of them, say X, as immersed by $g : R \hookrightarrow R^2$, $X = g(R)$, and the other as the zero-set of a submersion $f : R^2 \longrightarrow R$, $Y = f^{-1}(0)$. The contact is then measured by the order of the zero of the 'contact map' $f \circ g$, that is the largest value of k such that the k-jet of f g vanishes (of course k may be infinite). The other method is to consider the ideals of X and Y,

$$I(X) = \{h \in \mathcal{E}_2 : h|_X = 0\},$$

and similarly for Y, where $\mathcal{E}_2$ is the ring of germs at 0 of functions on R^2, and put $I(X, Y) = I(X) + I(Y)$. The multiplicity of the contact can then be defined as the dimension of the local algebra $Q(X, Y) = \mathcal{E}_2/I(X, Y)$. These two methods are of course equivalent, and multiplicity = 1 + order. The geometrical intuition of the multiplicity is if 2 curves have multiplicity μ they can be perturbed by an arbitrarily small amount to 2 curves with μ points of intersection.

This all carries over word for word to R^n, where X is a curve and Y is any submanifold. However, for X and Y arbitrary submanifolds of R^n the situation is not so clear. Chapter 1 is dedicated to extending these notions to this general case starting from a purely geometrical definition of contact: Two pairs of submanifolds of R^n have the same contact type if there is a (local) diffeomorphism of R^n taking one pair to the other. We consider the contact map, formed in the same way as for curves above, and find that its singularity type (more precisely its $\mathcal{K}$-class) is a contact invariant, and indeed the $\mathcal{K}$-class determines the contact type. This result is central to the remainder of the thesis which studies the geometry of submanifolds of R^n in terms of their contact with each of a family of 'model submanifolds' (in particular, n = 3,4, and the model submanifolds are spheres of codimension 1 and 2). We also see in Chapter 1 that the local algebra Q(X, Y) is a contact invariant which in fact nearly always determines the contact type. Moreover, we see that for surfaces the dimension of Q(X, Y) has the same geometric interpretation, the multiplicity

of intersection, as for curves.

Chapter 2 proves two genericity theorems (similar to Looijenga's theorem, [L4]) which enable us to apply standard singularity theory - deformations, codimension etc. - to our study of contact. The first of these theorems is for the general case of measuring the contact of a given submanifold with each of any family of model submanifolds. The second is for the particular case where the model submanifolds are spheres of codimension p (Looijenga's theorem was only for p = 1), but allowing the family to contain degenerate spheres (i.e. points) - the motivation behind this extension being that it gives some new information on surfaces in 3-space, in particular near umbilics.

Chapter 3, on classical differential geometry, has two purposes. Firstly to present the standard ideas of curvature, umbilics, semiumbilics, etc., for use in subsequent chapters, and secondly to commit to paper some more recent results, mostly due to Porteous (see [P2] and [P3]), the proofs and precise statements of which have not been written down previously. There is also some new material generalizing the classical notions of the parabolic curve and asymptotic lines of a surface in R^3 to curves of constant principal curvature and curves of constant normal curvature, and a discussion of the behaviour of these near an umbilic.

Chapters 4 and 5 apply the ideas of the first two chapters to surfaces in R^3 and R^4 respectively, giving many new results on the higher order local geometry of surfaces. For example, through any umbilic there pass 3 curves at each point of which there is a circle with 6-point contact with the surface, and the behaviour of these circles as we approach the umbilic is governed by the type of umbilic (more precisely by its index). Chapter 4 is completed with a generalization of a theorem of Banchoff, Gaffney and McCrory which appears in [B-G-M]. Their theorem is about ridge points where the principal curvature is zero, the generalization is to ridge points with any value of the principal curvature. In fact this is the philosophy behind much of the geometry in this thesis: that there is nothing special about flatness as opposed to any non-zero curvature, for example planes are just a special case of spheres.

Because of the central rôle of singularity theory in this thesis, Appendix 1 is devoted to a discussion of the relevant aspects of this, in particular to $\mathcal{K}$-equivalence. Appendix 2 is a brief synopsis of standard results on binary cubic forms which are used a great deal in chapters 3, 4 and 5 for describing the geometry of surfaces near umbilics and semiumbilics.

Finally I would like to thank my supervisor, Dr. I.R. Porteous, for his help, encouragement and stimulation. I would also like to thank the rest of the Mathematics department of the University of Liverpool, and my postgraduate colleagues, for useful discussions and a generally sympathetic atmosphere. Lastly I thank Alex Flegmann for producing figures 3(v) and 3(vi) on the University computer for me.

CHAPTER 1 CONTACT

In this chapter we first give a definition of the contact between two submanifolds of R^n at a point, and proceed by giving two useful (i.e. calculable) invariants of the contact involving the singularity type of a particular map. We then go on to discuss the multiplicity associated to a contact type, and finish off with a brief discussion on contact with curves. As in later chapters, we are very lax about distinguishing between map-germs and their representatives, however to begin with we do refer to germs and it is always clear what is meant, for example, by a diffeomorphism-germ taking one manifold to another (it means, of course, that there is a representative of that germ and a neighbourhood of its source-point in each of these submanifolds such that it carries one neighbourhood into the other!).

Definition of Contact.

Throughout this chapter, X_i and Y_i, $i = 1,2$, will be submanifolds of R^n, with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$.

Definition 1.1 We say that the contact of X_1 and Y_1 at y_1 is of the same type as the contact of X_2 and Y_2 at y_2 if there is a diffeomorphism-germ

$$H \ : \ R^n, y_1 \longrightarrow R^n, y_2$$

such that $H(X_1) = X_2$, and $H(Y_1) = Y_2$ (see the comment in the introduction). In this case we write,

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2).$$

Remarks 1.2 If $y \notin X \cap Y$ then we say that the contact is <u>empty</u>. Also, it is clear that in the definition R^n could be replaced by any manifold Z, or indeed by two equi-dimensional manifolds Z_1 and Z_2.

Our definition of contact is slightly more general than that given by Golubitsky and Guillemin in [G-G], where they require all the submanifolds to have the same dimension. This generalisation becomes critical later in the chapter, in particular in the proof of the main theorem 1.4. The reason that Golubitsky and Guillemin stick to the equi-dimensional case is that they are using it to define $\mathcal{K}$-equivalence.

<u>Contact & $\mathcal{K}$-equivalence.</u>

Let $g : X \hookrightarrow R^n$ be an immersion, and $f : R^n \longrightarrow R^p$ be a submersion at each point of $Y = f^{-1}(0)$, we say such a map <u>cuts out</u> Y. We will see that the contact type of X and Y at y is measured by the $\mathcal{K}$-class of the composite map $f \circ g$ at $x = g^{-1}(y)$, and we will call this map a <u>contact map</u> for X and Y. But first we prove that this $\mathcal{K}$-class depends only on the submanifolds X and Y, and not on the actual choice of immersion g and submersion f. (See Appendix 1 for a detailed discussion of the singularity theory.)

Proposition 1.3 Let X, Y, g, f be as above, and let $g' : X \hookrightarrow R^n$ be another immersion with the same image as g, and let $f' : R^n \rightarrow R^p$ be another map which cuts out Y, then $f \circ g$ and $f' \circ g'$ are $\mathcal{K}$-equivalent.

Proof: We show that the ideals of $f \circ g$ and $f' \circ g'$ are induced isomorphic. As g and g' are immersions with the same image, there is a diffeomorphism h of X on to itself with $g' = g \circ h$. By Hadamard's lemma (see appendix 1), we can write

$$f'(y) = \sum_{i=1}^{p} f_i(y).a_i(y),$$

for some functions a_i, where $f(x) = (f_1(x),\ldots,f_p(x))$. Consequently,

$$f' \circ g'(x) = \sum_{i=1}^{p} f_i \circ g'(x).a_i \circ g'(x).$$

So, $I(f' \circ g') \subset I(f \circ g')$, and similarly $I(f \circ g') \subset I(f' \circ g')$, thus $I(f' \circ g') = I(f \circ g') = I(f \circ g \circ h) = h^* I(f \circ g)$.

Example: This example shows that it is necessary to consider $\mathcal{K}$-equivalence, and that $\mathcal{A}$-(left-right-) equivalence is too restrictive: Let $f, f' : R^3 \rightarrow R^2$ be given by (y,z) and $(y, z - xy)$ - both of which cut out the x-axis. Now consider their contact with the curve $t \mapsto (t,t^2,t^3)$, the contact maps become, $t \mapsto (t^2,t^3)$, and $t \mapsto (t^2,0)$, respectively. These two maps are certainly $\mathcal{K}$-equivalent, but that they are not $\mathcal{A}$-equivalent follows from the fact that their images are not diffeomorphic.

The main importance of $\mathcal{K}$-equivalence in studying contact is contained in the following theorem:

Theorem 1.4 For i = 1,2, let $g_i : X_i, x_i \hookrightarrow R^n, y_i$ be immersion-germs, and $f_i : R^n, y_i \rightarrow R^p, o$ be maps which cut out Y_i, with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$. Then,

$$K(X_1,Y_1;y_1) = K(X_2,Y_2,y_2) \iff \mathcal{K}(f_1 \circ g_1) = \mathcal{K}(f_2 \circ g_2).$$

To prove this theorem we first prove two lemmas.

Lemma 1.5 (Invariance of contact under suspension.) Let a be any positive integer, and let $X_i' = X_i \times R^a$, $Y_i' = Y_i \times \{0\}$, and $y_i' = (y_i, 0)$, all in $R^n \times R^a$, then

$$K(X_1,Y_1;y_1) = K(X_2,Y_2;y_2) \iff K(X_1',Y_1';y_1') = K(X_2',Y_2';y_2')$$

Proof: (of lemma 3.5)
(i) "$\Rightarrow$": This part is immediate, as the suspension of the diffeomorphism taking X_1 and Y_1 to X_2 and Y_2 respectively, itself takes

X_1' and Y_1' to X_2' and Y_2' respectively.

(ii) " $\Leftarrow$ ": Let H' be the diffeomorphism required by the second statement, so $H'(X_1') = X_2'$, and $H'(Y_1') = Y_2'$, and write $H' = (H_1,H_2)$, where $H_1: R^{n+a},y_1' \to R^n,y_1$, and H_2 has image R^a,o. It follows that each of H_1 and H_2 are submersions.

Suppose that we can find a map-germ $u : R^n,y_1 \to R^a,o$ satisfying, (a) $u|_Y = 0$, and (b) the map $H : x \mapsto H_1(x,u(x))$ is a diffeomorphism, then the map H will be the required map, for:

$$x\in X_1 \Rightarrow (x,u(x))\in X_1' \Rightarrow H'(x,u(x))\in X_2' \Rightarrow H(x)= H_1(x,u(x)) \in X_2,$$
$$y\in Y_1 \Rightarrow (y,u(y)) = (y,0)\in Y_1' \Rightarrow H'(y,0)\in Y_2' \Rightarrow H(y)\in Y_2.$$

There remains to show that the map u does indeed exist. For (b) it is enough that the differential $dH : \hat{x} \mapsto (\hat{x}, du\hat{x})$ be injective. Let $dH_1 = (A,B)$, with $A\in L(R^n,R^n)$, $B\in L(R^a,R^n)$, and $U = du$, then (A,B) has rank n, and we require U so that A + BU has rank n, and for condition (a) we require U to be zero on TY_1. It is straightforward to show that such a U exists (since A restricted to TY_1 is injective).

Lemma 1.6 (Invariance of $\mathcal{K}$-class of $f\circ g$ under suspension.) Let X_i, Y_i, f_i and g_i be as above, $i = 1,2$.
(i) Let $X_i' = X_i \times R^a$ and $Y_i' = Y_i \times \{0\}$ all in $R^n \times R^a$ and let g_i' and f_i' be immersions with images X_i' and subersions with zero-set Y_i' respectively, then $f_1'\circ g_1'$ and $f_2'\circ g_2'$ are $\mathcal{K}$-equivalent if and only if $f_1\circ g_1$ and $f_2\circ g_2$ are.
(ii) Let $X_i'' = X_i \times \{0\} \subset R^n \times R^a$, and $Y_i'' = Y_i \times R^a$, and let g_i'' and f_i'' be immersions with images $X_i \times \{0\}$, and submersions with zero-sets $Y_i \times R^a$ respectively, then $f_1''\circ g_1''$ and $f_2''\circ g_2''$ are $\mathcal{K}$-equivalent if and only if $f_1\circ g_1$ and $f_2\circ g_2$ are.

Proof: By proposition 1.3 we can choose g_i', f_i', g_i'', f_i'' without affecting the $\mathcal{K}$-classes of the composite maps. We therefore make the obvious choices:

$$g_i' : (x,u) \mapsto (g_i(x), u),$$
$$f_i' : (y,u) \mapsto (f_i(y), u),$$

$$g_i'' : x \mapsto (g_i(x), 0),$$
$$f_i'' : (y,u) \mapsto f_i(y).$$

Then $f_i'\circ g_i'(x,u) = (f_i\circ g_i(x), u)$, and $f_i''\circ g_i''(x) = f\circ g(x)$, and the result follows.

Proof: (Of theorem 1.4)
(i) $K(X_1,Y_1;y_1) = K(X_2,Y_2;y_2) \Rightarrow f_1\circ g_1$ and $f_2\circ g_2$ are $\mathcal{K}$-equivalent:
Let H be the diffeomorphism of R^n taking X_1 and Y_1 to X_2 and Y_2 respectively. Now, $H|_{X_1} : X_1 \to X_2$ is a diffeomorphism, thus there exists a diffeomorphism $h : X_1 \to X_2$, such that $H\circ g_1 = g_2\circ h$. We also have that $(f_2\circ H)^{-1}(0) = f_1^{-1}(0)$, and so as in the proof of proposition 1.3 we can write,

(1.1) $$f_2\circ H(y) = \sum_{i=1}^{p} f_{1i}(y).a_i(y),$$

where $f_1 = (f_{11}, \dots, f_{1p})$, and for each i, $a_i(y)\in R^p$. $f_2\circ H$ is a submersion, and therefore so is the right-hand side of (1.1), which implies, since $f_1(y_1) = 0$, that the p x p matrix $[a_1(y),\dots a_p(y)]$ is invertible.

Define $\theta : R^n \times R^p \to R^p$ by

$$\theta(y,z) = \sum_{i=1}^{p} z_i a_i(y),$$

and

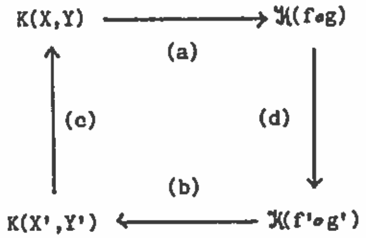
$$\theta' : X \times R^p \to R^p : (x,z) \mapsto \theta(g_1(x),z).$$

Then, $\theta'(x, f_1\circ g_1(x)) = f_2\circ g_2\circ h(x)$, and it is easy to show that θ' has the required properties to ensure that $f_1\circ g_1$ and $f_2\circ g_2$ are $\mathcal{K}$-equivalent.

(ii) $f_1\circ g_1$ and $f_2\circ g_2$ $\mathcal{K}$-equivalent $\Rightarrow K(X_1,Y_1;y_1) = K(X_2,Y_2;y_2)$:
It is this part of the proof that requires lemmas 1.5 and 1.6, for it does not lend itself to a direct proof unless dim X_i = dim Y_i. First we treat this equidimensional case, and then use the lemmas to extend this to the general result.

So, suppose dim X_i = dim Y_i = k. We wish to express each X_i as the graph of some map $\phi_i : R^k \to R^p$, and the Y_i as the graph of the zero map from R^k to R^p. To do this we first choose coordinates on R^n so that $f_1(x_1,\ldots,x_n) = (x_{k+1},\ldots,x_n)$, thus $Y_1 = R^k \times \{0\}$. Secondly, choose a p-dimensional subspace V_1 transverse to both X_1 and Y_1, and write $R^n = Y_1 \times V_1$. Let $\pi : R^n \to Y_1$ be the projection on to the first factor, then $\pi|_{X_1} : X_1 \to Y_1$ is a diffeomorphism which induces a coordinate system on X_1. With respect to these coordinates Y_1 is the graph of the zero map, while X_1 is the graph of the map $f_1 \circ g_1$ (thinking of f_1 as the projection : $Y_1 \times V_1 \to V_1$). A similar construction can be done for X_2 and Y_2. Then any diffeomorphism $H: R^k \times R^p \to R^k \times R^p$ preserving the graph of the zero map and taking the graph of $f_1 \circ g_1$ to the graph of $f_2 \circ g_2$ is then a diffeomorphism taking X_1 to X_2 and Y_1 to Y_2, so concluding the equidimensional case.

In the case where $\dim X_i \neq \dim Y_i$, we can suspend whichever is of the lower dimension with R^a, a being the difference in the dimensions, to give X_i' and Y_i' in $R^n \times R^a$, and define the appropriate maps g_i' and f_i'. We then have the following correspondences:



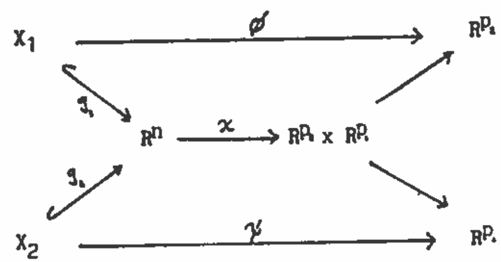
with (a) from part (i) of this proof, (b) from part (ii), (c) from lemma 1.5 and (d) from lemma 1.6 (1.6(i) for $\dim Y > \dim X$, and 1.6(ii) for $\dim Y < \dim X$).

In the theorem above we chose to immerse X and cut out Y, and we considered the $\mathcal{K}$-class of the resulting composite map. What is not clear is whether we would obtain the same information had we chosen to immerse Y and cut out X. The following lemma ensures that we would. I am grateful to Prof. C.T.C. Wall for the idea of using unfoldings in the proof of this

lemma.

Lemma 1.7 For i = 1,2, let X_i be submanifolds of R^n. Let $g_i : X_i \hookrightarrow R^n$ be immersions of X_i, and let $f_i : R^n \to R^{p_i}$ cut out X_i (i.e. they are submersions at each point of $X_i = f_i^{-1}(0)$), and suppose, w.l.o.g., that $p_2 > p_1$ - so $\dim X_1 > \dim X_2$. Then $f_2 \circ g_1$ is $\mathcal{K}$-equivalent to a suspension of $f_1 \circ g_2$.

Proof: Consider the following commutative diagram:



where $\chi(x) = (f_2(x), f_1(x))$, and the two maps from $R^{p_2} \times R^{p_1}$ to R^{p_2} and R^{p_1} respectively are the obvious projections. Now, if we express R^n as $X_1 \times R^{p_1}$ then it is easy to see that χ is an unfolding of ϕ, and similarly χ can be seen to be an unfolding of ψ. Since an unfolding of any map is $\mathcal{K}$-equivalent to a suspension of that map (see Appendix 1, lemma A1.10), it follows that χ is $\mathcal{K}$-equivalent to a suspension of both ϕ and ψ. It is now straightforward to show that ϕ is $\mathcal{K}$-equivalent to a suspension of ψ as required.

We now introduce another, though not unrelated, invariant of the contact of submanifolds which has the advantage of being more obviously symmetric, and as we will see its dimension has an important geometric meaning. The disadvantage is that it is not clear whether this invariant always distinguishes between different contact types.

Definition 1.8 Given two submanifolds X and Y of R^n, we define the <u>local algebra of contact</u> of X and Y at y R^n to be,

$$Q(X, Y; y) = \mathcal{E}_n / (I(X) + I(Y)) ,$$

where $\mathcal{E}_n$ is the ring of function-germs on R^n at y, I(X) is the ideal in $\mathcal{E}_n$ of germs vanishing on X, and similarly I(Y). If $y \notin X \cap Y$, then $I(X) + I(Y) = \mathcal{E}_n$, so $Q(X,Y;y) = \{0\}$.

Theorem 1.9 Let X_i, Y_i, y_i be as usual, then
(i) $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2) \Rightarrow Q(X_1, Y_1; y_1)$ and $Q(X_2, Y_2; y_2)$ are induced isomorphic (see Appendix 1).
(ii) If the contact is of <u>finite type</u> (which will mean that the associated contact maps $f_i \circ g_i$ are of finite $\mathcal{K}$-codimension) then the converse of (i) is also true.

Proof: (i) Suppose H is the diffeomorphism of R^n taking X_1 and Y_1 to X_2 and Y_2 respectively, then $H^* : \mathcal{E}_n \to \mathcal{E}_n$ is an induced isomorphism, with $I(X_1) = H^*I(X_2)$ and $I(Y_1) = H^*I(Y_2)$, and the result follows.
(ii) It is straightforward to show that Q(X,Y) and $Q(f \circ g)$ are isomorphic (choose coordinates on R^n and X so that $g(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$). It then follows that if $Q(X_1,Y_1)$ and $Q(X_2,Y_2)$ are isomorphic so are $Q(f_1 \circ g_1)$ and $Q(f_2 \circ g_2)$. In [M4], Mather shows that if two stable map-germs have isomorphic local algebras, then they are $\mathcal{K}$-equivalent, and it is easy to extend this to the case of finite codimension. The result then follows from theorem 1.4.

Remark 1.10 The correspondence between the contact types of submanifolds and the singularity types of maps proved in theorem 1.4 enables us to borrow the well established names of these singularity types for describing the contact types. Thus, for example, we will say that two plane curves have A_1 contact at a point x if they are tangent there but have different

curvatures, since the contact map $f \circ g$ will have an A_1 singularity at x.

<u>Multiplicity</u>

In this section we introduce the idea of the multiplicity of a contact type, and use some results in singularity theory to give estimates for this. Suppose X is an immersed submanifold, and $g : X \hookrightarrow R^n$, then an <u>a-parameter perturbation</u> of X is a smooth map $G : X \times R^a \to R^n$ with $G(, 0) = g$, such that for each $u \in R^a$ $G(, u)$ is an immersion of X into R^n. Similarly, if Y is cut out by a map f, then a smooth map $F : R^n \times R^a \to R^p$ is an a-parameter perturbation of Y. It is clear that perturbing the submanifolds X and Y of R^n induces a perturbation of the contact map $f \circ g$. What is not so obvious is that we can so induce <u>any</u> perturbation of $f \circ g$:

Lemma 1.11 With X, Y, f, g and $\phi = f \circ g$ as usual, and given any a-parameter deformation $\tilde{\phi}: X \times R^a \to R^p$ of ϕ, we can obtain $\tilde{\phi}$ from an a-parameter perturbation of X and Y. In particular, if ϕ has finite $\mathcal{K}$-codimension then there is a perturbation of X and Y which induces a versal deformation of ϕ - we will call such a perturbation of X and Y a <u>versal perturbation</u>.

Proof: Choose coordinates on R^n so that,

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_p) .$$

Then considering R^n as the product $R^p \times R^{n-p}$ (so f is projection on the first factor), we write the immersion g as (g_1,g_2), so $\phi = g_1$. Let $\tilde{\phi}: X \times R^a \to R^p$ be the given deformation of ϕ, and define the map $G : X \times R^a \to R^n$ by,

$$G(x, u) = (\tilde{\phi}(x, u), g_2(x)),$$

then $G(x, 0) = g(x)$ and consequently, for sufficiently small u, G_u is an immersion ($Imm(X, R^n)$ being an open subset of $C^\infty(X, R^n)$), and its image is a perturbation of X, and clearly $\not{G} = f \circ G$.

Under perturbation of X and Y, the point y in the intersection may split up into several (possibly infinitely many) distinct points, the maximum number of which will depend only on the contact type of X and Y. We therefore make the following definitions (a more precise definition of multiplicity being given below):

Definition 1.12 (i): The multiplicity of contact, $\mu = \mu(X, Y; y)$, of X and Y at y is the largest number of points that y can split up (bifurcate) into under perturbation of X and Y. If no such number exists then $\mu = \infty$.
(ii) The dimension of contact, $\delta = \delta(X, Y; y)$, is the dimension of the local algebra $Q(X, Y; y)$.

Remarks 1.13 (i): Both of these integers depend only on the contact type: for the multiplicity this follows from the definition of contact type, while for the dimension it follows from theorem 1.9(i).
(ii) Note that since $y \in X \cap Y$, $\mu \geqslant 1$, and also $I(X) + I(Y) \subset \mathcal{E}_n$, consequently $\delta \geqslant \dim(\mathcal{E}_n/m_n) = 1$. If $y \notin X \cap Y$, we can put $\mu = \delta = 0$.
(iii) It is of course possible for δ and μ to be infinite, as, for example, in the case where X and Y are the x-y-plane and the paraboloid $z = x^2 + y^2$ in R^3 respectively. By shifting the plane to $z > 0$, the intersection is an entire circle, so $\mu = \infty$, and the local algebra $Q(X,Y,0)$ is $\mathcal{E}_3/ < z, x^2 + y^2 >$ which has infinite dimension. Indeed this will always be the case when $\dim X + \dim Y > n$.

There are some results, in particular due to Damon and Galligo, see [D-G], which we can use to relate δ and μ in certain cases. (Note that δ has the advantage of being fairly easy to calculate.) In order to use these results we first make a short digression.

Let $f : R^n,o \to R^p,o$ be a map-germ (or its representative), then we define its real multiplicity $m(f)$ to be the maximum $m \in Z$ such that for

every neighbourhood U of O in R^p there is a y in U such that $|f^{-1}(y)| = m$. If no such m exists, then $m(f) = \infty$. Also let $\delta(f) = \dim Q(f)$, then there are the following results concerning $m(f)$ and $\delta(f)$, none of which we prove, but proofs can be found where indicated.

Proposition 1.14 Let $\delta(f) < \infty$, then $m(f) \leqslant \delta(f)$. (Proof in [G-G])

Proposition 1.15 Let f be stable. Then either of the following conditions will ensure that $m(f) = \delta(f)$:
(i) The kernel rank of f at O is at most 2;
(ii) f has discrete algebra type at O (or ker rk (f) = i, and there are at most finitely many Σ^i - types nearby f).

Most of the paper [D-G] is devoted to proving this fact.

We now return to our situation to see how these results apply.

Suppose we are given X and Y as usual, with $\phi = f \circ g$ of finite $\mathcal{K}$-codimension (this is a necessary restriction in the discussion that follows). Let $\Phi : X \times R^a \to R^p$ be a versal deformation of ϕ. By lemma 1.10 there exist $F : R^n \times R^a \to R^p$, and $G : X \times R^a \to R^n$ such that $\Phi_u = F_u \circ G_u$, F and G then defining a versal perturbation of X and Y. Writing $X_u = \text{image}(G_u)$, and $Y_u = F_u^{-1}(0)$, we see that $g(x) \in X_u \cap Y_u$ iff $\Phi(x,u) = 0$.

Associated to the map Φ is the map $\pi : \Phi^{-1}(0) \to R^a$. One of the standard properties of π (see Appendix 1) is that it is stable if and only if Φ is a versal deformation of ϕ. Also, $y \in \pi^{-1}(u)$ iff $\Phi(u,y) = 0$, indeed, $\pi^{-1}(u) = X_u \cap Y_u$. Thus $\mu(X,Y) = m(\pi)$, and we have the following theorem:

Theorem 1.16 Given X, Y and y as usual in R^n, and let μ be the multiplicity of contact and δ be the dimension of contact, then

(i) If $\delta < \infty$, then $\mu \leq \delta$;
(ii) If the contact map $f \circ g$ has finite $\mathcal{K}$-codimension, and either has kernel rank at most 2, or is of discrete algebra type, then $\mu = \delta$.

Proof: This follows from propositions 1.14 and 1.15 together with the discussion above.

Remark 1.17 (i) Geometrically, the kernel of $d\phi$ at x corresponds to the intersection of the tangent spaces to X and Y at $y = g(x)$. Thus the above result holds whenever that intersection has dimension at most 2, and in particular when one of the submanifolds has dimension at most 2, as is the case in our applications in chapters 4 and 5.
(ii) In this chapter we have been dealing with submanifolds of R^n. Had we instead been considering complex submanifolds of C^n, we would obtain a map $\phi : C^k \to C^p$ (replacing X by C^k). In the case where X and Y are of complementary dimension, so $k = p$, the contact map would be between spaces of equal dimension, and it is well known (see, e.g. [M10]) that in that case $m(f) = \delta(f)$, thus we would have $\mu = \delta$. (I think this is probably true for $k < p$ as well.)

Contact with curves.

Let X be a curve and Y any submanifold in R^n, with g immersing X, and $f : R^n \to R^p$ cutting out Y as usual. The resulting contact map maps R to R^p, and the $\mathcal{K}$-classification of such maps is very simple: Any map from R to R^p (of finite $\mathcal{K}$-codimension) is $\mathcal{K}$-equivalent to the map :

$$t \mapsto (t^{k+1}, 0, 0, \ldots, 0)$$

for some k, and such a map is said to have an A_k singularity. The local algebra for an A_k is isomorphic to $R\{1, t, \ldots, t^k\}$ which has dimension $(k + 1)$. Thus by theorem 1.16, the multiplicity of contact of a curve and a submanifold with A_k contact is $(k + 1)$, which coincides with the classical intuition of $(k + 1)$-point contact. We will be using this term whenever one

of the submanifolds involved is a curve, and we see from this discussion that the integer k in the phrase 'k-point contact' (provided it is finite, in which case the contact map is of finite codimension) distinguishes between different contact types.

Final Remarks

Recall the classical Lagrange multipliers theorem: Let $X = f^{-1}(0)$ be a submanifold of R^n, and h a function on R^n, then $h|_X$ has a critical point at x if and only if $dh = q.df$ for some covector q ($q_1, \ldots, q_p$ being the Lagrange multipliers), or equivalently iff the matrix [df] augmented with [dh] has kernel rank 1. The theory we have developed in this chapter throws more light on this result: we see that the local algebra of $h|_X$ is isomorphic to $Q(X,Y)$, where $Y = h^{-1}(0)$, or to $\mathcal{E}_n/\langle f,h\rangle$; thus we can generalize the theorem to tell us not only where $h|_X$ is singular, but also the singularity that occurs - at least up to $\mathcal{K}$-equivalence.

It would be possible to define a multi-local contact type between two submanifolds of R^n if they intersect in at most finitely many points, however this does not appear to be more than just listing the individual contact types that occur at each of the points of intersection. On the other hand, a global picture using global diffeomorphisms in definition 1.1 would of course be far more complicated.

CHAPTER 2 GENERICITY THEOREMS

In the remainder of this thesis we will be applying the ideas of the first chapter in the following manner: For X an immersed submanifold of R^n we look at the contact between X and each of a smooth family of "model submanifolds", which in particular will be circles or spheres. What we wish to know, for any given situation, is which contact types we can expect to occur generically (i.e. for most immersions of X, in a sense to be made precise at the end of this chapter). The results of this chapter enable us to answer this question, and also to use unfolding, or deformation, theory to predict which contact types will occur near a given one and what their structure will be.

Setting up the problem.

Let our family of model submanifolds of R^n be smoothly parametrized by $m \in M$ (M being a manifold). To measure the contact between X and $m \in M$, we can either immerse m, cut out X and take the composite map, or immerse X, cut out m and take that composite map. We know from corollary 1.9 that we will get the same $\mathcal{K}$-class in either case, and thus the same information on the contact types. However there is some benefit to be gained from the latter approach as we can extend the family of maps cutting out the m to include some whose zero-set is singular, and it turns out that in one case in particular - that of the contact of surfaces in R^3 with circles - this extension will give some new information.

Thus for each m M we have a map $f_m: R^n \to R^p$, with $f_m^{-1}(0)$ being the model submanifold m of codimension p in R^n. We require that the map

$$F : R^n \times M \to R^p : (y,m) \mapsto f_m(y)$$

be smooth.

Let $g : X \hookrightarrow R^n$ be an immersion. Then by theorem 1.4 the contact between X, or rather g(X), and the model submanifold m at g(x) is

given by the $\mathcal{K}$-class of the composite $f_m \circ g$ at x.

We denote by $\bar{\phi}_g$ and $\phi_{g,m}$ the maps

$$(2.1) \qquad \bar{\phi}_g : X \times M \to R^p : (x, m) \mapsto \phi_{g,m}(x) = f_m \circ g(x)$$

though when it causes no confusion we drop the subscript 'g'.

Transversality theorems.

Given two manifolds X and Y, the set $C^\infty(X,Y)$ of smooth maps from X to Y will be endowed with the Whitney C^∞ topology. A base of open sets consists of the sets

$$\mathfrak{J}(U) = \{ f \in C^\infty(X,Y) : j^r f(X) \subset U \}$$

with U open in $J^r(X,Y)$ for some r. Recall that a <u>residual</u> set is a countable intersection of open dense sets, which, in $C^\infty(X,Y)$, is also dense as this is a Baire space. The subset Imm(X,Y) of immersions of X into Y is open in $C^\infty(X,Y)$, and if dim Y $\geq$ 2.dim X, it is also dense (Whitney Immersion Theorem, see [G-G]).

Central to the proofs of this section is the following variant of Thom's transversality theorem, see [T], the proof of which follows that of Thom's theorem almost verbatim, see for example [W2]. The reason for using this variant, rather than the original statement which both Looijenga in [L4] and Wall in [W2] use to prove Looijenga's genericity theorem, is that I feel it makes the proof of theorem 2.2 more straightforward.

Proposition 2.1 Let M, X and Y be any (smooth) manifolds, and let Z be a submanifold of $J^r(X,Y) \times M$, then $T_Z = \{ g : (j^r g, 1) \pitchfork Z \}$ is residual in $C^\infty(X,Y)$, where $(j^r g, 1) : X \times M \longrightarrow J^r(X,Y) \times M$ is the product map.

[In fact the only change from the proof in [W2] is as follows: On p.740 Wall defines a map $\phi: N \times B_\epsilon \longrightarrow P$ (his N and P correspond to our X and Y) and the crux of the proof lies in showing that the map $j^r\phi \;:\; N \times B_\epsilon \longrightarrow J^r(N,P)$ is a submersion, he then applies the arguments on the preceeding page. For the proof of our version we would need that $(j^r\phi, 1) : X \times B_\epsilon \times M \longrightarrow J^r(X,Y) \times M$ is a submersion, the argument then continuing in the same manner. But this is true if and only if $j^r\phi$ is a submersion, which is proved by Wall.]

Our first genericity theorem is for the general setting described above, where for each $m \in M$, the map $f_m : R^n \longrightarrow R^p$ cuts out its zero-set. The later theorems will extend this result to a particular case where this condition may fail.

Denote by $J^r_y(X,R^p)$ the subset of jets with target y, and note (see Remark A1.2(iii)) that any $\mathcal{K}$-invariant submanifold of $J^r(X,R^p)$ is either all of the complement of $J^r_0(X,R^p)$, or a submanifold of $J^r_0(X,R^p)$.

Theorem 2.2 Let X, M, f_m, g and ϕ_g be as above. Let W be either a submanifold of $J^r_0(X,R^p)$ or all of its complement, then the set

$$R_W = \{\, g \in \mathrm{Imm}(X,R^n) : j^r_1\phi_g \pitchfork W \,\}$$

is residual in $\mathrm{Imm}(X,R^n)$, where j^r_1 is the jet with respect to the first variable, so $j^r_1\phi$ maps $X \times M$ to $J^r(X, R^p)$.

Proof: The central idea of the proof is as follows: define the map

(2.2)
$$\Gamma^r : J^r(X,R^n) \times M \longrightarrow J^r(X,R^p)$$
$$: (\, j^r g(x)\;,\; m\,) \longmapsto j^r(f_m \circ g)(x) = j^r\phi_{g,m}(x).$$

Now, if $\Gamma^r \pitchfork W$, then $Z = (\Gamma^r)^{-1}(W) \subset J^r(X,R^n) \times M$ is a submanifold, and the theorem follows from proposition 2.1 as $R_W = T_Z \cap \mathrm{Imm}(X,R^n)$. Note that as $\mathrm{Imm}(X,R^n)$ is open in $C^\infty(X,R^n)$, then the intersection of the former with any residual subset of the latter is itself residual in the former.

If W is the complement of $J^r_0(X,R^p)$ then it is open and the transversality condition $\Gamma^r \pitchfork W$ is empty. We are therefore left with showing that $\Gamma^r \pitchfork W$ whenever W is a submanifold of $J^r_0(X,RP)$. We show that for any $m \in M$, Γ^r_m is a submersion at x if $\Gamma^r_m(x) \in J^r_0(X,R^p)$. Now,the tangent space to the fibre over x in X of $J^r(X,R^n)$ at $j^r g(x)$ can be identified with $\theta(g)/\mathcal{M}_k^{r+1}$, where $\theta(g)$ is the set of germs at x of vector fields along g (see appendix 1). Let $\phi = \Gamma_m(g)$, where Γ is the map:

(2.3)
$$\Gamma : X \times C^\infty(X,R^n) \times M \longrightarrow X \times C^\infty(X,R^p)$$
$$:(x\;,\; g\;,\; m) \longmapsto (x\;,\; f_m \circ g\,),$$

from which Γ^r can be induced. Γ_m then induces a map

(2.4)
$$\Gamma_{m*} : \theta(g) \longmapsto \theta(\phi)$$
$$: \xi_x \longmapsto df_m(\xi_x),$$

where the differential is taken at g(x), and

(2.5)
$$\Gamma^r_{m*} : \theta(g)/\mathcal{M}_k^{r+1} \longrightarrow \theta(\phi)/\mathcal{M}_k^{r+1}$$

is the differential of Γ^r_m restricted to the fibre over x. We claim that Γ_{m*} is surjective if f_m is a submersion at g(x). Given that, then it follows that Γ_m is a submersion, and consequently so is Γ^r_m as required.

Suppose now that f_m is a submersion at $y = g(x)$, we can choose coordinates in R^n and R^p so that f_m takes the form

$$f_m(y_1,\dots,y_n) = (y_1,\dots,y_p).$$

Let $\xi_x = [h_1(x),\dots,h_n(x)]$, then

$$\Gamma_{m*}(\xi_x) = df_m(\xi_x) = [h_1(x),\dots,h_p(x)],$$

so Γ_{m*} is clearly surjective, and the proof is complete.

The next theorem is for the particular situations that we will be interested in - namely the contact of immersed submanifolds of R^n with spheres of codimension p. Denote by M the set of (n-p)-spheres in R^n, each such sphere being cut out by an appropriate map $f_m : R^n \rightarrow R^p$. If we were merely interested in this geometrical situation then theorem 2.2 would be enough. However, as we indicated earlier, some extra information can be found by extending the family of maps f_m, and so the set M, to include some maps which do not cut out $f_m^{-1}(0)$.

We now make our choice of maps f. Any (n-p)-sphere in R^n can be given by the intersection of p hyperspheres. Let these p hyperspheres have centres $c_1,\dots,c_p$, and radii $r_1,\dots,r_p$, then the (n-p)-sphere is the zero-set of the map:

$$y \longmapsto (\,|c_1 - y|^2 - r_1^2\,,\,\dots\,,\,|c_p - y|^2 - r_p^2\,),$$

where $c_i \neq c_j$ when $i \neq j$, and the r_i are such that the zero-set is indeed a genuine (n-p)-sphere. However we also wish to include planes as a special case of spheres, so we add the points at infinity in R^n and deal with points $[c:s] \in RP^n$, and $[c:s:\rho] \in RP^{n+1}$, and we define the two sets,

$$M_e = \{([c_1:s_1:\rho_1],\dots,[c_p:s_p:\rho_p]) : [c_i:s_i] \neq [c_j:s_j] \text{ for } i \neq j, \text{ and } (c_i,s_i) \neq 0 \text{ for all } i\,\},$$

and,

$$M = \{\, m \in M_e : f_m^{-1}(0) \text{ is a submanifold of } R^n \text{ of codimension } p\,\},$$

where

(2.6) $$f_m(y) = (c_1.y - 1/2\, s_1|y|^2 + \rho_1,\ \dots\ ,\ c_p.y - 1/2\, s_p|y|^2 + \rho_p).$$

Thus if $s_i \neq 0$ the i^{th} component of f_m defines the hypersphere

$$|y - c_i/s_i|^2 - \rho_i/s_i - |c_i|^2/s_i^2 = 0,$$

while if $s_i = 0$, it defines the hyperplane $c_i.y + \rho_i = 0$. Note that these sets are independent of the choice of representative of $[c_i:s_i:\rho_i]$, though the map itself does depend on this choice. However the $\mathcal{K}$-class of $f_m \circ g$ is unaffected and as that is all we are interested in we can ignore this ambiguity (when necessary, though, we can limit ourselves to the condition $|c_i|^2 + s_i^2 + \rho_i^2 = 1$). Note also that there is a lot of redundancy in our parametrization of the set of (n-p)-spheres in R^n as there are infinitely many hyperspheres passing through any given (n-p)-sphere, and our choice of p of them is almost arbitrary. We will return to this point for our applications, but for the moment it is unimportant.

Before stating and proving the theorem we ask, when does f_m fail to be a submersion? Now, with f_m as in (2.6),

(2.7) $$df_m = (\,c_1 - s_1 y\,,\ \dots\ ,\ c_p - s_p y\,),$$

which fails to be surjective at y if and only if $c_i = s_i y$ for some i. (Note that as $[c_i:s_i] \neq [c_j:s_j]$ for $i \neq j$, the rank of df_m can drop by at most 1.) Suppose, without loss of generality, that $c_p = s_p y_o$, for some $y_o \in R^n$, then $c_i \neq s_i y_o$ for $i < p$, and, as $(c_p,s_p) \neq 0$, $c_p = s_p y_0 \Rightarrow s_p \neq 0$. Thus the second derivative of the p^{th} component of f_m at y_o is non-degenerate, and it follows that we can choose coordinates about y_o in R^n such that

(2.8) $$f(y_1,\dots,y_n) = (y_1,\dots,y_{p-1}, \sum_{i=p}^{n} \epsilon_i y_i^2),$$

where $\epsilon_i = \pm 1$, $i = 1,\dots,n$.

Theorem 2.3 In the situation described above, with Γ^r as in (2.2), if W is a $\mathcal{K}$-invariant submanifold of $J^r(X,\mathbb{R}^p)$, then $\Gamma^r \pitchfork W$, and consequently (by proposition 2.1) the set R_W is residual in $\mathrm{Imm}(X,\mathbb{R}^n)$, where

$$R_W = \{\, g \in \mathrm{Imm}(X,\mathbb{R}^n) \;:\; j_1^r \tilde{\phi}_g \pitchfork W \,\}.$$

Proof : If W is the complement of $J_0^r(X,\mathbb{R}^p)$, or if f_m is a submersion at $g(x) \in f_m^{-1}(0)$, then the proof of theorem 2.2 ensures that $\Gamma_m^r \pitchfork W$ in the first case, and that Γ_m^r is a submersion at $j^r g(x)$ in the second. There remains the case where f_m fails to be a submersion at $g(x) \in f_m^{-1}(0)$.

As in theorem 2.2, we have

$$\Gamma^r_* : T_z(J^r(X,\mathbb{R}^n)_x \times M_e) \longrightarrow T_{z'}J^r(X,\mathbb{R}^p)_x \;,$$

where $z = j^r g(x)$, $z' = j^r \phi_{g,m}(x)$, and the map

$$\text{(2.9)} \qquad \Gamma_* : \theta(g) \times T_m(M_e) \longrightarrow \theta(\phi_{g,m}) = \mathbb{R}^p + m_k\,\theta(\phi_{g,m})$$

from which Γ^r_* can be induced, and we define Γ_{m*} to be Γ_* restricted to $\theta(g) \times \{0\}$. Thus

$$\Gamma_*(\xi_x,\hat{m}) = \Gamma_{m*}(\xi_x) + d\tilde{\phi}_g(o,\hat{m}).$$

Now, $m = ([c_1:s_1:\rho_1], \ldots, [c_p:s_p:\rho_p])$, so we choose $\hat{m} = ([0:0:\hat{\rho}_1], \ldots, [0:0:\hat{\rho}_p])$, and then

$$d\tilde{\phi}_g(0,\hat{m}) = (\hat{\rho}_1, \ldots, \hat{\rho}_p).$$

Thus, restricting Γ_* to $\{0\} \times T_m(M_e)$, we can span $\mathbb{R}^p \times \{0\}$ in (2.9), so there remains to show that $m_k\theta(\phi_{g,m}) \subset \mathrm{Image}(\Gamma_{m*}) + T_e\mathcal{K}(\phi_{g,m})$, for if $j^r\phi_{g,m}(x) \in W$, then $T_{z'}W = T_e\mathcal{K}(\phi_{g,m})/m_k^{r+1}$, and so

$$\mathrm{Image}(\Gamma^r_*) + T_{z'}(W) = \theta(\phi_{g,m}).$$

We now choose coordinates in $\mathbb{R}^n$ and $\mathbb{R}^p$ so that f_m takes the form (4.6), then at $g(x)$,

$$df_m = \begin{bmatrix} I_{p-1} & 0 & \ldots & 0 \\ 0 & \epsilon_p g_p(x) & \ldots & \epsilon_n g_n \end{bmatrix} ,$$

where $g_1(x), \ldots, g_n(x)$ are the component functions of $g(x)$. Let $\xi_x = [h_1(x), \ldots, h_n(x)] \in \theta(g)$, then

$$\Gamma_{m*}(\xi_x) = df_m(\xi_x) = [\, h_1(x), \ldots, h_{p-1}(x), \sum_{i=p}^{n} \epsilon_i g_i(x) h_i(x) \,]$$

and

$$\mathrm{Image}(\Gamma_{m*}) = \mathcal{E}_k + \mathcal{E}_k + \ldots + \mathcal{E}_k + \langle g_p, \ldots, g_n \rangle,$$

($p-1$ copies of $\mathcal{E}_k$). With this choice of coordinates,

$$\phi_{g,m}(x) = (g_1(x), \ldots, g_{p-1}(x), \sum_{i=p}^{n} \epsilon_i g_i(x)^2),$$

so, $\langle g_1, \ldots, g_{p-1} \rangle . \theta(\phi_{g,m}) \subset T\mathcal{C}(\phi_{g,m}) \subset T_e\mathcal{K}(\phi_{g,m})$,

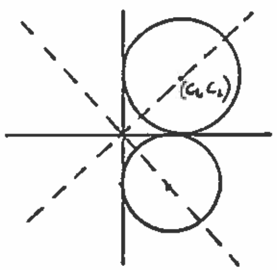
and so,

$$\mathrm{Image}(\Gamma_{m*}) + T_e\mathcal{K}(\phi_{g,m}) \supset \mathcal{E}_k + \ldots + \mathcal{E}_k + m_k \supset m_k.\theta(\phi_{g,m}).$$

In the case $p = 1$, this result is essentially a monojet version of Looijenga's theorem [L4], or [W2]. The ρ that occurs in our parametrization of M_e enables us to omit the condition "W invariant under addition of constants" that appears in Looijenga's theorem.

Remarks on multijets.

It is possible to prove a multi-jet version of theorem 2.2, in fact the proof is exactly the same, provided some care is taken in defining the class of 'interesting' submanifolds W of ${}_sJ^r(X,R^p)$ (if $(z_1,..,z_i,..,z_s)$ is in W, and z_i has non-zero target, then we would require that $(z_1,..,\hat{z}_i,..,z_s)$ be in W for all $\hat{z}_i$ with non-zero target). For theorem 2.3 the situation is more complex. Looijenga's theorem is for multi-jets, but only holds for the set Emb(X,R^n) of embeddings of X into R^n, and the same restriction would allow a multi-jet version of theorem 2.3 to be proved. That this restriction is necessary can be seen from the following example.



Consider the immersion bi-germ

$$g : \{ s \mapsto (s,0) \ ; \ t \mapsto (0,t) \}$$

of an immersed curve in R^2. Let M be the collection of circles in the plane, parametrized by centre and radius, extended as in theorem 2.3 to include the singular circles (we do not bother with the compactification here as it is not relevant). Thus,

$$f_m(x,y) = (c_1 - x)^2 + (c_2 - y)^2 - \rho \ .$$

The resulting bi-germ $f_m \circ g$ would be

$$\{ s \mapsto (c_1 - s)^2 + c_2{}^2 - \rho \ ; \ t \mapsto (c_2 - t)^2 + c_1{}^2 - \rho \}.$$

Consider the set $A \subset R^2 \times M$ where the bi-germ has two A_1 singularities: we require both maps to have zero target and to be singular, so

$$A = \{ (s,s,s,s,s^2) \ \ R^2 \times M : s \in R \} \cup \{ (t,-t,t,-t,t^2) \ \ R^2 \times M : t \in R \},$$

and is therefore singular at (0,0,0,0,0). It is clear that this situation is stable under small perturbations of the immersion.

Now, in ${}_2J(R,R^2)$ the set of double A_1's is a submanifold (${}_2J^r(R,R^2)$ is an open subset of the Cartesian product of two copies of $J^r(R,R^2)$, and the A_1-subset of $J^r(R,R^2)$ is non-singular), and thus in the example above, and in any sufficiently nearby example, the bi-jet extension map ${}_2j^r(f_m \circ g)$ cannot be transverse to the double A_1 set.

Genericity

We now discuss what is meant by a 'generic' immersion for any given setting - that is, given X, n, M and consequently p. First note that in [M6] Mather calculates the codimension $\sigma(k,p)$ of the set of modal singularities in $J^r(k,p)$ - the fibre of $J^r(X, R^p)$ over a point (x, 0) in $X \times R^p$ - for sufficiently large r. Because of the preferred rôle of the zero target, the codimension of the set of all modal singularities in $J^r(X, R^p)$ is $\sigma(k,p) + p$. Note also that for given g, the dimension of the image of the associated jet-extension map $j_1^r \phi_g$ is $k + d$, where $d = \dim M$.

In the cases where $k + d < \sigma(k,p) + p$ then we let $\{W_1, . . ,W_s\}$ be the finite set of $\mathcal{K}$-orbits in $J^r(X, R^p)$ of codimension less than $k + d$, and $\{W_{s+1} , . . . ,W_t\}$ a finite stratification of the complement of $W_1 \ \ .. \ \ W_s$. Then for the general situation described in theorem 2.2 and the particular extension in theorem 2.3, let R be the intersection of the R_W , which will itself be a residual subset of Imm(X, R^n). For $g \in R$, the associated jet-extension map will then miss the W_i for $i > s$, and be transverse to the W_i for $i \leqslant s$, and such a mapping will be termed generic for the given situation. In our applications in chapters 4 and 5, the hypothesis $k + d < \sigma(k,p) + p$ will indeed be fulfilled, and for those chapters the term generic will have the meaning described in this paragraph.

If, on the other hand, $k + d \geqslant \sigma(k,p) + p$, then the meaning the word generic should have is not so straightforward. However in practice it seems that one can always take the strata to be the union of the $\mathcal{K}$-orbits as the moduli (or modal parameters) vary, (usually excluding some exceptional values).

When the dimensions are such that the modal singularities are not encountered for a generic immersion, the transversality to the stratification ensures that all the singularities are presented transversely, (i.e. they are versally deformed). If, on the other hand, the modal singularities do occur, then the singularities cannot all be presented transversely. Thus in our applications, all the singularities that arise for a generic immersion will be presented transversely.

We now ask the question: For which situations are the modal singularities avoided by generic immersions? In [M6], Mather calculates the codimension of the algebraic variety consisting of all the modal singularities in $J^r(X, R^p)$. If dim X = k, then this number is $\sigma(k,p)$ - to be precise it is $\sigma_r(k,p)$, which is a decreasing function of r, and $\sigma(k,p)$ is defined to be $\inf_r \sigma_r(k,p)$. $\sigma(k,p)$ is given by the following formula:

Case I, $k \leqslant p$.

$$\sigma(k,p) = \begin{cases} 6(p-k)+8 & \text{if } p-k \geqslant 4, \text{ and } k \geqslant 4 \\ 6(p-k)+9 & \text{if } 3 > p-k > 0, \text{ and } k > 4 \\ & \text{or if } k = 3 \\ 7(p-k)+10 & \text{if } k = 2 \\ \infty & \text{if } k = 1. \end{cases}$$

Case II, $k > p$

$$\sigma(k,p) = \begin{cases} 9 & \text{if } k-p = 1 \\ 8 & \text{if } k-p = 2 \\ k-p+7 & \text{if } k-p > 3. \end{cases}$$

In the present context of measuring the contact of a given submanifold X of R^n with each of a family of model submanifolds $m \in M$, p is the codimension of the m. If d = dim M, then the image of the jet extension

map $j_1^r \Phi$ has dimension k + r, so to avoid the modal singularities we need $k + d < \sigma(k,p) + p$. (Notice that the two cases above become dim X + dim m $\leqslant$ n and dim X + dim m > n respectively).

For example, if the model submanifolds are spheres of codimension p then $d = p(n - p + 2)$, and we require $k + p(n - p + 2) < \sigma(k,p) + p$. We find that this inequality first breaks down in the following cases:

$X^2 \hookrightarrow R^7$, p = 1;
$X^2 \hookrightarrow R^5$, p = 2;
$X^2 \hookrightarrow R^7$, p = 3.

$X^3 \hookrightarrow R^5$, p = 1;
$X^3 \hookrightarrow R^4$, p = 1,2,3.

$X^4 \hookrightarrow R^6$, p = 1;
$X^4 \hookrightarrow R^5$, p = 2,3,4.

$X^5 \hookrightarrow R^6$, p = 1,2,3,4,5 (all p).

For p = n - 1, contact with circles, modal singularities never occur for contact with surfaces, while for 3-folds the only possibility is $X^3 \hookrightarrow R^4$.

It should be pointed out that in these examples there may be other (geometric) factors which prevent modal singularities from occuring. For example for $X^3 \hookrightarrow R^4$ and contact with circles, the singularity can only be modal if the circle is singular, for otherwise the contact map is a suspension of a map $R \to R$ (by lemma 1.7), which cannot be modal. The only case I have checked is $X^2 \hookrightarrow R^5$ and p = 2, and in this case the modal singularities occur at isolated points which appear to be interesting for other reasons as well.

CHAPTER 3 CLASSICAL GEOMETRY

Introduction

The primary purpose of this chapter is to present some of the classical local differential geometry required for later chapters. It is divided into three sections, the first on the geometry of space curves, the second on the geometry of surfaces in 3-space, and the last on surfaces in 4-space. For the first and third sections the adjective 'classical' in the title of the chapter is accurate, while for the second section this is less true as I have included some material which is too recent for that epithet. Of this, the material on curves of constant principal curvature appears to be original, while that on the third order geometry, i.e. ridges, is mostly due to Porteous (see [P2] and [P3]), though some to Markatis [M1]. There are two reasons for including this, firstly for its use in chapters 4 and 5, and secondly because many of the proofs have not previously been committed to paper, and indeed some of the results themselves have not been made explicit.

SPACE-CURVES

Let $r : R \hookrightarrow R^3$ be an immersed curve in R^3. In this thesis we only study local properties of curves and surfaces, so the language of germs ought to be adopted. However we shall not do so, except in discussions of singularity theory, and we will leave implicit the possibility that the domain of r is not all of R. We denote the successive derivatives of r at t by $r_1(t)$, $r_2(t)$,... etc., thus $r_1(t)$ can (and will) be thought of as a tangent vector to the curve r at $r(t)$.

Given a curve in R^3 there are two types of parametrization that will be useful. The first, parametrization by unit length, is global, and has the property that $|r_1(t)| = 1$ for all t. The second type is a local parametrization which has the property that at the point p on the curve, the second and higher derivatives are all perpendicular to the first, and that the tangent vector r_1 is of unit length at p. One way of achieving this is as follows: for simplicity let p be at the origin in R^3, and let the x-axis be tangent to the curve at 0, and the y- and z-axes be chosen to give an orthogonal coordinate system on R^3. The projection of the curve on to the x-axis is then non-singular and so induces a non-singular parametrization of the curve near 0, given by

$$r(t) = (t, \bar{r}(t), \hat{r}(t)), \tag{3.1}$$

where $\bar{r}$ and $\hat{r}$ are singular functions of t. The immersion expressed in this form is said to be in Monge form (the term "Monge form" is usually applied to immersions of surfaces, but as the principle is the same, we borrow the term for this case as well).

The geometry of curves.

The curvature vector, $k(t)$, of a curve r at $r(t)$ is defined to be the component of $r_2(t)/|r_1(t)|^2$ normal to the curve at $r(t)$. In both of the special parametrizations introduced above, (3.2) $$k(t) = r_2(t).$$

The curvature of the curve $\kappa(t)$ is the magnitude of the curvature vector, and can be given by

(3.3) $$\kappa = |r_1 \times r_2| \;/\; |r_1|^3.$$

The possibility of zero curvature is often ignored in treatments of space curves as any curve can be perturbed by an arbitrarily small amount to a curve with nowhere vanishing curvature. However we will need to keep this possibility in mind for our applications. Flatness is often reflected in phenomena at infinity, so for the following definitions we include the points at infinity by considering the projective space RP^3, whose points we express as $[c:s]$, $c \in R^3$, $s \in R$, and as usual $[c:s] = [d:t]$ if (c,s) and (d,t) are parallel as vectors in R^4. We recover R^3 from the subset of RP^3 with $s \neq 0$ by the map : $[c:s] \mapsto c/s$, while the plane at infinity corresponds to the subset of RP^3 with $s = 0$.

The normal space to the curve r at r(t) is

(3.4) $$N(t) = \{\, [x:s] \in RP^3 : (c - sr(t)).r_1(t) = 0 \,\}.$$

The parametrized union of these N(t) is the normal bundle Nr. Contained in N(t) is the focal line F(t) (sometimes called the polar axis),

(3.5) $$F(t) = \{\, [c:s] \in N(t) : (c - sr(t)).r_2(t) - s|r_1|^2 = 0 \,\}$$

The focal line has the property that if $[c:s] \in F(t)$ then the sphere centre $[c:s]$ through r(t) has 3-point contact with the curve at r(t). Such spheres will be called focal spheres. Note that if $k(t) = 0$, then F(t) is the line at infinity in N(t).

The focal set is the union of the focal lines, and can be viewed either as a subset of the normal bundle or of the ambient space RP^3. As a subset of RP^3 the focal set is the envelope of the normal planes, or, in classical language, where nearby normal planes meet.

There are two points of interest in F(t), namely the centres of the osculating sphere and of the osculating circle. The osculating sphere is the (unique, unless $r_3 = 3r_2.r_1k$) sphere with 4-point contact with the curve at r(t), its centre, the centre of spherical curvature is the point $[c:s] \in F(t)$ satisfying

(3.6) $$(c - sr(t)).r_3(t) = 3sr_1(t).r_2(t).$$

The osculating circle is the (always unique) circle with 3-point contact with the curve at r(t). When $k(t) \neq 0$ its centre is the inverse of k(t) with respect to the unit circle in N(t), that is the point [c:s] on F(t) satisfying,

(3.7) $$\kappa(t)^2(c - sr(t)) - sk(t) = 0,$$

while if $k(t) = 0$ it is not determined. However, in this case the osculating circle itself is: it is the tangent line to the curve at r(t). Thus the osculating circle to the curve at r(t) is always well-defined, and as can be seen from the diagram below, is contained in all focal spheres at r(t).

Figure 3 (i) shows the normal space to the curve r at r(t), F being the focal line. The large circle shown represents an arbitrary focal sphere with centre b, and the point a is the centre of the osculating circle, which is also shown.

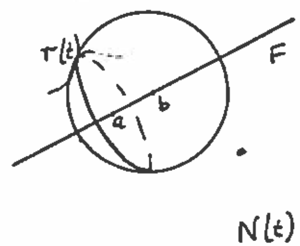


Figure 3 (i)

SURFACES IN 3-SPACE

Let X be a 2-dimensional manifold, and $g : X \hookrightarrow R^3$ an immersion, the image of g will also be called X. X inherits, via its immersion, a Riemannian metric, called the first fundamental form and denoted I. Another symmetric bilinear form, the second fundamental form II, can be defined on X, and all of the local differential geometry of the surface can be described in terms of these two forms: the fundamental theorem of surfaces, due to Bonnet, says that these two forms determine a surface uniquely, up to rigid Euclidean motion. Indeed, given any two quadratic forms on X, satisfying certain integrability conditions - the Gauss-Codazzi equations - then there exists an immersion of X into R^3 with these as first and second fundamental forms respectively, see Struik [S]. Because of the importance of these forms, we will describe many of our results in terms of these two forms and their derivatives.

Let u and v be tangent vectors to X at x, and n a unit vector normal to the surface, then the two fundamental forms at x are given by,

$$I_x(u,v) = dg_x u . dg_x v \tag{3.8}$$

$$II_x(u,v) = n . d^2 g_x(u,v).$$

The second fundamental form is subject to an arbitrary choice of sign, from the choice of n. However, if X is oriented we can make a global choice, while if it is not, we can still make a local choice (which is all we need as we are only interested in the local geometry).

As for curves, there is a special class of local parametrizations of surfaces in 3-space which will be particularly useful. Let $x \in X$, then choose coordinates in R^3 so that $g(x) = 0$, where g is the immersion, and the x-y plane is tangent to X at 0. The orthogonal projection of R^3 on to that plane then induces a coordinate system on X, via the immersion g. The resulting parametrization can then be written as

$$g(x,y) = (x,y,h(x,y)) \tag{3.9}$$

where h is a real valued function on R^2 with $h(0) = 0$, $dh(0) = 0$. An immersion expressed in this way is said to be in Monge form.

Principal directions, lines of curvature and the focal set.

At each point of the surface we have an eigenvalue equation:

$$II_x(u,\ ) = \kappa I_x(u,\ ). \tag{3.10}$$

The eigenvalues of (3.10) are called the principal curvatures at x, and the eigenvectors are called principal directions (with an abuse of nomenclature that we use throughout!), and provided I_x and II_x are not linearly dependent, there are two orthogonal principal directions, orthogonality being defined with respect to I. A point x where the two quadratic forms are linearly dependent is called an umbilic, and in general umbilics are isolated, a point we will return to later.

Thus, away from the set of (isolated) umbilics, there are two distinct line-fields which can be integrated to to obtain two families of curves on the surface. These curves are called lines of curvature.

As we have already stated, an umbilic is a point on the surface where the two principal curvatures are equal and every direction is then principal. Much of the geometry at and near the umbilic is governed by the cubic form

$$C = n.d^3g - 3\kappa dg.d^2g = dII - \kappa dI \tag{3.11}$$

where κ is the unique principal curvature at the umbilic x, and n is the unit normal we chose to define II. We call C the intrinsic cubic at the umbilic, and say an umbilic is elliptic, parabolic or hyperbolic accordingly as C is. See appendix 2 for a brief discussion of cubic forms, in particular in the presence of a positive definite quadratic form.

In [D], Darboux discusses the behaviour of lines of curvature near umbilics, relating it to the cubic form C, or more precisely to the Jacobian of C and I. A more recent study of the stability of the global structure of lines of curvature has been made by Gutiérrez and Sotomayor, see [G-S].

The focal points are the two centres of curvature corresponding to the principal directions. They can either be thought of as sitting in the ambient space or in the normal bundle. When a principal curvature is zero the associated focal point is at infinity, so we replace $c \in R^3$ by $[c:s] \in RP^3$ $(s \in R)$, we can choose c and s so that $|c|^2 + s^2 = 1$. Define the distance-squared function V on $X \times RP^3$ by

$$V(x,[c:s]) = c.g(x) - 1/2\, s|g(x)|^2. \tag{3.12}$$

The reason for the name being that for $s \neq 0$, $V(\ ,[c:s])$ has the same level sets as the function $|c - sg(x)|^2$, but our variant is used as it is non-singular when $s = 0$, and in this case the level sets are planes. Denote by $V_1, V_2, V_3, \ldots$ the successive derivatives of V with respect to x. Thus $V_1(x,[c:s]) = (c - sg(x)).dg_x$. Note that any ambiguity in the definition of V can be locally eliminated by restricting (c,s) to lie on the unit sphere in R^4. We now define the compactified normal bundle,

$$N = \{ (x,[c:s]) \in X \times RP^3 : V_1(x,[c:s]) = 0 \}.$$

Let $\tilde{N} = N \oplus S^1X$, where S^1X is the unit tangent bundle of X. We then define the focal set F as follows: Let

$$\tilde{F} = \{ (x:[c:s],u) \in \tilde{N} : V_2(x,[c:s])u = 0 \},$$

and then F is the image of $\tilde{F}$ under the obvious projection : $\tilde{N} \to N$ (i.e. ignoring the vector u). Thus $(x,[c:s]) \in N$ is a focal point if and only if there is a non-zero tangent vector u satisfying $(c - sg(x)).d^2gu - sdgu.dgu = 0$, which is equivalent to (3.10) with

$$\kappa(c - sg(x)) = sn \, . \tag{3.13}$$

Recall that a cubic form C is said to be orthogonal if there is a pair of non-zero orthogonal vectors u and v such that $Cuv = 0$.

Proposition 3.1 $\tilde{F}$ is a (two dimensional) submanifold of $\tilde{N}$ if at each umbilic of X the intrinsic cubic is not orthogonal. (We will see in chapter 4 that this is indeed the case for a generic immersion.)

Proof: Define the map

$$\psi : TX \times RP^3 \to R^2 \times R^2 \times R$$

$$: (x,[c:s],u) \mapsto (V_1(x,[c:s]),\ V_2(x,[c:s])u,\ (dgu.dgu - 1)/2),$$

then $\tilde{F} = \psi^{-1}(0)$. Now,

$$d\psi = \begin{bmatrix} V_2 & .dg & -g(x).dg & 0 \\ V_3u & .d^2gu & -g(x).d^2gu - dgu.dg & V_2 \\ d^2gu.dgu & 0 & 0 & dgu.dg \end{bmatrix}$$

and F is a submanifold if ψ is a submersion at $\psi^{-1}(0)$, so if $[d\psi]$ is surjective, or equivalently if $[d\psi]^t$ (transpose of $[d\psi]$) is injective. Suppose, therefore, that $[d\psi]^t(a,b,t) = 0$ $(a,b \in R^2,\ t \in R)$ and take the immersion to be in Monge form at x, then

(i) $V_2a + V_3ub = 0$

(ii) $dga + d^2gub = 0$

(iii) $dgu.dgb = 0$

(iv) $V_2b + tdgu.dg = 0$

as $g(x) = 0$ in Monge form. Then, (ii) $\Rightarrow a = 0$. Letting (iv) act on u we get that $t = 0$, and so $V_2 b = 0$. If x is not an umbilic, then $V_2 b = 0 \Rightarrow b$ is a multiple of u, but then (iii) $\Rightarrow b = 0$, and $[d\psi]$ is surjective. If x is an umbilic, then $V_2 = 0$ and we are left with $V_3 ub = 0$ and $dgu.dgb = 0$, so either $b = 0$ or V_3 is orthogonal. However, V_3 is a scalar multiple of C and so by hypothesis is not orthogonal, so $b = 0$ and again $[d\psi]$ is surjective.

Remarks (i) There is an alternative way of defining the focal set that we use for surfaces in 4-space and could equally well have been used here. The idea is to define a map $\mu : N \to Q$, the bundle of quadratic forms on TX. Then F is the inverse image of the cone of parabolic forms. For more details see the section on surfaces in 4-space in this chapter.
(ii) The focal set, considered as a subset of the ambient space, is the envelope of the normal lines, and is thus the generalisation to surfaces of the evolute of a plane curve.

At an umbilic, the focal set F is singular (conical) although F is not (unless the intrinsic cubic is orthogonal). Let $\pi : \tilde{F} \to X$ be the obvious projection, then if x is an umbilic $\pi^{-1}(x)$ is the unit circle in the tangent space to X at x. We now prove a useful lemma.

Lemma 3.2 Let $(x,[c:s],u) \in \tilde{F}$, and $\pi : \tilde{F} \to X$ be as above, then
(i) If x is not an umbilic, the map π is an immersion at $(x,[c:s],u)$. Consequently, away from umbilics it is a local diffeomorphism.
(ii) If x is an umbilic, then the image under this projection of any non-singular curve in F through $(x,[c:s],u)$ transverse to $\pi^{-1}(x)$ is itself a non-singular curve in X.

Proof (of both parts): By differentiating the map ψ (from proposition 3.1) we see that $(\hat{x},(\hat{c},\hat{s}),\hat{u})$ is tangent to F if and only if,

$$\begin{aligned} V_2\hat{x} \quad + \quad (\hat{c} - \hat{s}g(x)).dg \quad &= \quad 0, \\ V_3u\hat{x} \; + \; (\hat{c} - \hat{s}g(x)).d^2gu \; - \; \hat{s}dgu.dg \; + V_2\hat{u} \; &= \; 0, \\ d^2gu\hat{x}.dgu \; + \; dgu.dg\hat{u} \quad &= \; 0\,, \\ c.\hat{c} + s.\hat{s} \; &= \; 0. \end{aligned} \tag{3.14}$$

Now, π_* has a non-zero kernel vector iff a non-zero vector of the form $(0,(\hat{c},\hat{s}),\hat{u})$ is tangent to F. Let A be the linear map,

$$A : R^3 \times R \times R^2 \longrightarrow R^2 \times R^2 \times R \times R$$

$$: (\hat{c},\hat{s},\hat{u}) \longmapsto ((\hat{c} - \hat{s}g(x)).dg \,,\, (\hat{c} - \hat{s}g(x)).d^2gu^2 - \hat{s}dgu.dgu \,,$$

$$d^2gu\hat{x}.dgu + dgu.dg\hat{u}, \;\; c.\hat{c} + s\hat{s}).$$

If x is not an umbilic, then it is easy to show that A is invertible (by putting the immersion in Monge form and showing that A^t is injective). Consequently, $\hat{x} = 0 \Rightarrow A(\hat{c},\hat{s}) = 0 \Rightarrow (\hat{c},\hat{s}) = 0$, and π_* is an immersion. If, on the other hand, x is an umbilic, then $V_2 = 0$ and A has corank 1. However $(0,0,v) \in \ker A$, for v orthogonal to u, but by hypothesis the curve in (ii) is transverse to $\pi^{-1}(x)$, so $(\hat{x},(\hat{c},\hat{s}),\hat{u})) \neq (0,(0,0),v)$. •

Remark 3.3 The projection $\pi : \tilde{F} \to X$ factors through the focal set F, and away from umbilics the map $\tilde{F} \to F$ is a global diffeomorphism. Let x be an umbilic and $(x,[c:s]) \in F$, then $\pi^{-1}(x) = (x,[c:s]) \times S^1 \subset \tilde{F}$ and the map $\tilde{F} \to F$ is a blow-up of the umbilical centre $(x,[c:s])$, as in the diagram below. Now, any non-singular curve passing through $(x,[c:s],u)$ transverse to $\pi^{-1}(x)$ passes from one component of $\tilde{F} \setminus \pi^{-1}(x)$ to the other, and consequently its projection on to F passes from one sheet to the other of F, and its image on X would be non-singular. In contrast we would expect the projection of any curve passing through $\pi^{-1}(x)$ but contained in one component of $F \setminus \pi^{-1}(x)$ to be singular.

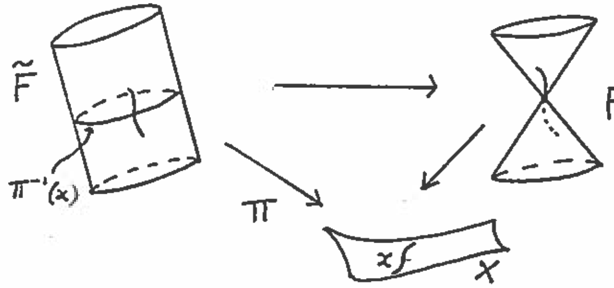


Figure 3(ii)

Also of importance are the <u>harmonic directions</u> at an umbilic. They are the root directions of the harmonic representative of the intrinsic cubic C with respect to I (see appendix 2). Unless this representative is zero (which we will see is non-generic), there are always three such directions at 120° to one another. The relevance of these directions will be seen in chapter 4.

<u>Curves of Constant Principal Curvature</u>

The principal curvature function κ can be defined intrinsically on $\tilde{F}$ by (3.13) (note that $(x,[c:s]) \in F \Rightarrow c \neq sg(x)$), so

$$\kappa : \tilde{F} \longrightarrow R$$

is a smooth function. Away from umbilics we have seen that the projection : $\tilde{F} \to X$ is a double covering, and thus induces a pair of principal curvature functions on X which are smooth away from umbilics, see (3.10). At an

umbilic the principal curvature functions take on a (unique) well-defined value, so giving two globally continuous functions which are, however, not differentiable at umbilics. Note that we will freely use the term principal curvature function both for the function defined on $\hat{F}$, and for those on X.

We now present two propositions on singularities of the principal curvature functions, the first for non-umbilical points, and the second for umbilics.

Proposition 3.4 Let $x \in X$ not be an umbilic, and let κ_1 be one of the principal curvature functions with principal direction u at x. Then κ_1 is singular at x if and only if both the following hold:

$$(3.15)\qquad \begin{aligned} n.d^3gu^3 - 3\,\kappa_1 dgu.d^2gu^2 &= 0 \\ n.d^3gu^2v - 2\,\kappa_1 dgu.d^2guv - \kappa_2 dgv.d^2gu^2 &= 0, \end{aligned}$$

where κ_2 is the other principal curvature at x, and v is a unit tangent vector orthogonal to u. This condition can easily be shown to be equivalent to

$$(dII - \kappa_1 dI)u^2 = 0.$$

Proof: To simplify this proof we express the immersion in Monge form, as in (3.9), and then condition (3.15) reduces to

$$n.d^3gu^2 = 0,$$

as $dg.d^2g = 0$. It ought to be shown that the expression in (3.15) is intrinsic for the proposition to then follow, but this is just a standard, though tedious, excersise which we do not reproduce here.

Let $\phi : R \hookrightarrow \tilde{F}$ be a smooth curve, and let $\phi_1(0) = (\hat{x},\hat{c},\hat{s},\hat{u})$ with $\phi(0) = (x,[c:s],u)$, so $(\hat{x},\hat{c},\hat{s},\hat{u})$ satisfies (3.14). Denote by $\hat{\kappa}$ the differential of κ along ϕ at 0. Then differentiating (3.13), or rather the equivalent equation $\kappa\, n.(c - sg(x)) - s = 0$, we get

(3.16) $\hat{\kappa}\, n.(c - sg(x)) + \kappa \hat{n}.(c - sg(x)) + \kappa n.(\hat{c} - \hat{s}g(x) - sdg\hat{x}) - \hat{s} = 0.$

Note that, since n is always a unit vector and normal to the surface, so $n.n = 1$ and $n.dg = 0$, we get

(3.17) $$n.\hat{n} = 0, \quad \text{and} \quad \hat{n}.dg + n.d^2g\hat{x} = 0.$$

Thus (3.16) simplifies, with $g(x) = 0$, to

$$\hat{\kappa}\, n.c + \kappa n.\hat{c} - \hat{s} = 0,$$

and so $d\kappa(\hat{x},\hat{c},\hat{s},\hat{u}) = 0$ iff $\hat{s} = \kappa n.\hat{c}$. If $\kappa = 0$ then this becomes $\hat{s} = 0$, while if $\kappa \neq 0$ then $s \neq 0$ so $s\hat{s} = \kappa sn.\hat{c} = \kappa^2 c.\hat{c}$, and using the last equation of (3.14), $s\hat{s}(1 + \kappa^2) = 0$, so again the condition is $\hat{s} = 0$. Thus $d\kappa(\phi_1) = 0$ iff $\hat{s} = 0$. Putting this into (3.14) we get,

$$V_3u\hat{x} + \hat{c}.d^2gu + V_2\hat{u} = 0,$$

and equivalently,

$$V_3u^2\hat{x} + \hat{c}.d^2gu^2 = 0.$$

However, since $\hat{c}.c = -s\hat{s}$, $\hat{s} = 0 \Rightarrow c.\hat{c} = 0 \Rightarrow \hat{c}.d^2g = 0$, so $d\kappa(\phi_1) = 0$ iff $V_3u^2\hat{x} = 0$ for the immersion in Monge form. Thus κ is singular at $(x,[c:s],u)$ iff $V_3u^2 = 0$, so iff $n.d^3gu^2 = 0$ as required.

Near an umbilic the situation is more interesting as we shall see shortly, but first we introduce some concepts. The level sets on X of the principal curvature functions will be called <u>curves of constant principal curvature</u>. These curves are a generalisation of the classical parabolic curve on a surface which is a curve of constant principal curvature zero. (We do not use the term parabolic in this context as it can cause confusion with other uses of the term.) Away from umbilics the curves of constant principal curvature will be smooth with possible singularities at the points described in proposition 3.4 above. We will see in chapter 4 that for a generic surface every umbilic is either elliptic or hyperbolic (so $Cu^2 \neq 0$

for all $u \neq 0$). Also if $Cuz = 0$ then u and z are said to be <u>Hessian directions</u> at the umbilic: for a hyperbolic umbilic there are two such directions, while for an elliptic one there are none.

Proposition 3.5 Let x be an umbilic whose intrinsic cubic form C is not orthogonal, then:
(i) if x is elliptic there are no curves of constant principal curvature passing through x, and nearby curves of constant principal curvature are closed curves around x;
(ii) if x is hyperbolic there are two non-singular curves of constant principal curvature through x with tangents in the Hessian directions at x. (See also remark 3.6 below.)

Proof: Let $w = (x,[c:s],u) \in \tilde{F}$. The tangent vectors $\hat{w} = (\hat{x},\hat{c},\hat{s},\hat{u})$ to $\tilde{F}$ at w satisfy (3.14) with $V_2 = 0$ (i.e. $II - \kappa I = 0$). The second equation implies $V_3uv\hat{x} = 0$, i.e. $Cuv\hat{x} = 0$, where v is orthogonal to u (recall that C not orthogonal $\Rightarrow Cuv \neq 0$), and note that if u is a Hessian direction then $V_3uv\hat{x} = 0 \Rightarrow V_3u\hat{x} = 0$, so $\hat{x}$ would be the other Hessian direction, and it follows from (3.14) that $\hat{c} = 0$, $\hat{s} = 0$. Thus u is a Hessian direction iff the tangent space at w is

$$\{ (\hat{x},0,0,\hat{u}) : V_3uv\hat{x} = 0,\ dgu.dg\hat{u} = 0 \}.$$

Now, in the proof of proposition 3.4 we saw that $d\kappa(\phi_1) = 0$ iff $\hat{s} = 0$. Thus κ is singular at w iff $\hat{s} = 0$ for all tangent vectors to $\tilde{F}$ at w, and so iff u is a Hessian direction. Thus for case (i), where the umbilic is elliptic, we have that κ is non-singular at all points of $\pi^{-1}(x)$, and since it is constant on $\pi^{-1}(x)$ there are no other curves of constant κ passing through any point of $\pi^{-1}(x)$, and so there are no curves of constant principal curvature on the surface X passing through x.

For case (ii) we have two points in $\pi^{-1}(x)$ where κ is singular. If we show that $d^2\kappa$ is non-degenerate at these points, then we can deduce that there is a curve of constant κ passing through each of them transverse to $\pi^{-1}(x)$ (by the Morse lemma), so we have part (ii) of the proposition, using proposition 3.2(ii), since $V_3 u\hat{x} = 0$.

We now show that $d^2\kappa$ is indeed non-degenerate. Differentiating (3.16) again, with $\hat{\kappa} = 0$, and using, from a further differentiation of (3.17), $\hat{n}.\hat{n} + n.\hat{\hat{n}} = 0$, $\hat{\hat{n}}.dg + 2\hat{n}.d^2g\hat{x} + n.d^3g\hat{x}^2 = 0$, we get, putting $g(x) = 0$ and $\not{p}_2 = (\hat{x},\hat{c},\hat{s},\hat{u})$,

$$n.c\hat{\hat{\kappa}} + \kappa\hat{\hat{n}}.c - 2s\kappa\hat{n}.dg\hat{x} + \kappa n.(\hat{\hat{c}} - sd^2g\hat{x}^2) - \hat{\hat{s}} = 0.$$

Thus,

$$n.c\hat{\hat{\kappa}} = s(1 - \kappa^2)dg\hat{x}.dg\hat{x} + (\kappa n.\hat{\hat{c}} - \hat{\hat{s}}).$$

Let a be a unit vector satisfying $V_3ua = 0$, and let $\hat{x} = \lambda a$, then $\sigma = dga.dgu \neq 0$ ($\sigma = 0 \Rightarrow V_3$ is orthogonal). Differentiating the second equation in (3.14) we get,

$$V_4u\hat{x}^2 + 2V_3\hat{u}\hat{x} + \hat{\hat{c}}.d^2gu - \hat{\hat{s}}dgu.dg = 0.$$

Consequently, with $\hat{u} = \mu v$, and g in Monge form,

$$\lambda^2V_4ua^3 + 2\lambda\mu V_3va^2 + V_3u\hat{\hat{x}}a + \hat{\hat{c}}.d^2gua - \hat{\hat{s}}dgu.dga = 0$$

$$\therefore \quad \lambda^2V_4ua^3 + 2\lambda\mu V_3va^2 + (\kappa n.\hat{\hat{c}} - \hat{\hat{s}})dgu.dga = 0.$$

Eliminating $\kappa n.\hat{\hat{c}} - \hat{\hat{s}}$ between this equation and

$$n.c\hat{\hat{\kappa}} = s(1 - \kappa^2)\lambda^2 + (\kappa n.\hat{\hat{c}} - \hat{\hat{s}}),$$

we get

$$(dgu.dga)n.c\hat{\hat{\kappa}} = [s(dgu.dga)(1 - \kappa^2) - V_4ua^3]\lambda^2 - 2\lambda\mu V_3va^2.$$

Thus $\hat{\hat{\kappa}} = 0$, $\lambda \neq 0$, can be solved uniqly for μ unless $V_3va^2 = 0$, in which case $V_3a^2 = 0$ (since $V_3ua = 0$) and V_3 is parabolic. Thus $d^2\kappa$ is non-degenerate - it has one root in the v-direction and the other in the $(\lambda a, \mu v)$-direction (which are distinct). Thus (ii) is proved.

Remark 3.6 On the surface X one usually considers there being two principal curvature functions, one corresponding to each sheet of the focal set. However, the non-singular curves of constant κ in the proposition above pass from one sheet to the other, and so their projections on X will not be the curves of constant principal curvature, while these latter will be singular at an umbilic. See the diagram below of the curves of constant principal curvature for the two types of umbilic, with continuous lines for one sheet and broken lines for the other.

Hyperbolic umbilic

elliptic umbilic

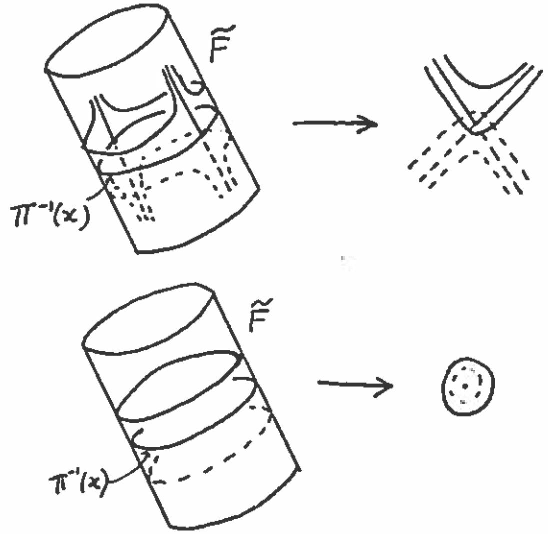


Figure 3(iii)

We have seen that the curves of constant principal curvature are a generalization of the classical parabolic curve. In a similar manner we can generalize the notion of asymptotic curves as follows: Fix $\kappa \in R$ and consider the equation,

$$II(u,u) = \kappa I(u,u).$$

At each point of the surface, this is a quadratic equation for u, which on an open subset of X will have two distinct real solutions. The boundary of this set, where the roots coincide, will be the curve of constant principal curvature κ. The integral curves of this equation will have, at each point, normal curvature κ (any curve on the surface with unit tangent vector u at x, its curvature vector k at x will have normal component $n.d^2gu^2$), so we call these integral curves, curves of constant normal curvature. If $\kappa = 0$, these are the classical asymptotic curves.

In [B-G-M], they discuss briefly the behaviour of asymptotic curves near the parabolic curve, using some results of D. Lak on multiform differential equations, see [L1], and that they have interesting properties near flat ridge points (ridge points where the associated principal curvature is zero), which are generally first order ridge points. The same sort of investigation could be done for curves of constant normal curvature κ, and the results of [B-G-M] and [L1] would carry through for most values of κ . However, if κ is a principal curvature at a higher order ridge point, or at an umbilic, then more work would need to be done on classifying multiform differential equations. I conjecture that the behaviour of these curves of constant normal curvature near umbilics, is as in the following diagrams.

Hyperbolic Umbilic Elliptic Umbilic

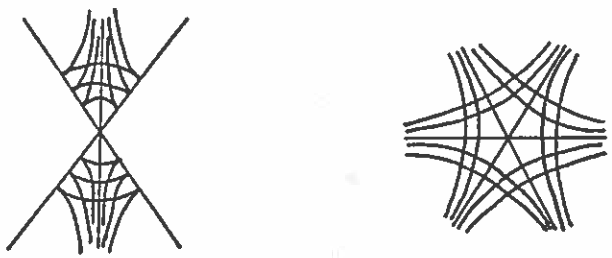


Figure 3(iv)

Ridges

Ridges on a surface were first introduced as a mathematical concept by Porteous in [P2] where he also calls them base ribs. In [P3] they are called rib lines, while Bruce in [B2] calls them ridges. I use the term ridge as I feel it is a closer intuition to the definition that I give below. I will use the term 'rib' for the appropriate subset of the focal

set which projects down on to the ridge on the surface (this, at least, is consistent with [P2] and [P3]!).

Let $\gamma : R \hookrightarrow X$ be a non-singular parametrization of a line of curvature, with $\gamma(0)$ not an umbilic, thus for each t, $\gamma_1(t)$ is a principal direction at $\gamma(t)$. This determines a curve $t \mapsto (\gamma(t), \tilde{\gamma}(t))$ in the focal set, with $\tilde{\gamma}(t)$ in RP^3 (the ambient space). Let κ be the associated principal curvature function, i.e. that associated to the principal direction $\gamma_1(t)$.

Definition 3.7 We say a point $x = \gamma(0) \in X$ is a k-th order ridge-point if one of the principal curvature functions measured along the associated line of curvature is stationary to order k at x, i.e. $(\kappa \circ \gamma)_1 = \ldots = (\kappa \circ \gamma)_k = 0$, $(\kappa \circ \gamma)_{k+1} \neq 0$. Generally we say x is a ridge point if $(\kappa \circ \gamma)_1 = 0$. Thus associated to any ridge point x is a principal direction u at that point, and a focal point (x,[c:s]) which is called a rib point.

Let $\phi: R \hookrightarrow X$ be a parametrization of a curve of constant principal curvature, and suppose that if the image of ϕ is non-singular then so is its parametrization (thus ϕ can only be singular where $d\kappa(x) = 0$). Since each principal curvature function is associated (locally) to a definite sheet of the focal set, ϕ lifts to a curve $(\phi(t), \tilde{\phi}(t))$ in the focal set. Suppose $(\phi(0), \tilde{\phi}(0)) = (\gamma(0), \tilde{\gamma}(0)) = (x,[c:s])$ then it is clear (see chapter 1) that for $d\kappa(x) \neq 0$, x is a k-th order ridge point iff ϕ and γ have (k + 1)-point contact. We now turn to some alternative definitions of a ridge point.

Theorem 3.8 With x, ϕ, γ , ... etc., as above, the following are equivalent:

(i) x is a ridge-point, i.e. $(\kappa \circ \gamma)_1 = 0$,

(ii) either $d\kappa(x) = 0$, or ϕ and γ are tangent at x,

(iii) $\tilde{\gamma}_1(0) = 0$,

(iv) the projection of the focal set into the compactified ambient space

RP^3 is singular at (x,[c:s]), (this was the original definition of a rib point, see [P2]),

(v) either $d\kappa(x) = 0$, or $\tilde{\phi}_1(0) = 0$,

(vi) $V_3(x,[c:s])u^2 + V_2v = 0$ for some v (equivalently $V_3u^3 = 0$). (This condition is interpreted in theorem 4.2.)

Proof: We show (iv)⟺(iii)⟺(i)⟺(ii)⟺(v), (iii)⟺(vi), though not in that order.

(i)⟺(ii): This is immediate from the definition - see the observation immediately preceeding this theorem.

(i)⟺(iii): We saw in the proof of proposition 3.4 that d $(\hat{x}) = 0$ iff $\hat{s} = 0$, where $(\hat{x},\hat{c},\hat{s})$ is tangent to F. Since $\hat{x} = u = \gamma_1$, $V_2\hat{x} = 0$ so the first equation in (3.14) becomes $\hat{c}.dg = 0$, and assuming $\hat{s} = 0$, the last becomes $c.\hat{c} = 0$, thus $\hat{s} = 0$ and $\hat{x} = u$ iff $\hat{c} = 0$, i.e. $\tilde{\gamma}_1 = 0$.

(iii)⟺(iv): Under the projection : $F \to RP^3$, $(\gamma_1, \tilde{\gamma}_1) \mapsto \tilde{\gamma}_1$, and since $\gamma_1 \neq 0$, $\tilde{\gamma}_1 = 0$ the projection is singular. Conversely, suppose $(\hat{x},0,0)$ is tangent to F, then from the first of (3.14) we conclude that $\gamma_1 = a\hat{x}$, for some real a, and so $\tilde{\gamma}_1 = a(0,0) = 0$.

(ii)⟺(v): Suppose $d\kappa(x) \neq 0$. Then $\phi_1 = a\gamma_1 \Rightarrow \tilde{\phi}_1 = a\tilde{\gamma}_1$ (the projection $F \to X$ being a local diffeomorphism), and since (ii)⟺(iv) it follows that $\tilde{\phi}_1 = 0$. Conversely, $d\kappa(x) \neq 0 \to (\phi_1, \tilde{\phi}_1) \neq 0$, and the projection $F \to RP^3$ maps this to $\tilde{\phi}_1 = 0$, and we have (iv) which is in turn equivalent to (ii).

(iii)⟺(vi): Now, $(\gamma_1, \tilde{\gamma}_1)$ is tangent to F, so from (3.14) with $(\gamma_1, \tilde{\gamma}_1) = (u,\hat{c},\hat{s})$, we get

$$(\hat{c} - \hat{s}g(x)).dg = 0,$$

and

$$V_3u^3 + (\hat{c} - \hat{s}g(x)).d^2gu^2 - \hat{s}dgu.dgu = 0.$$

Thus, using $c.\hat{c} + s\hat{s} = 0$, we see that $V_3u^3 = 0 \Leftrightarrow (\hat{c},\hat{s}) = 0$.

We will see in chapter 4 that there are two types of first order ridge point: hyperbolic and elliptic, with higher order ridge points being

in some sense parabolic. We will also see (theorem 4.3) that for a generic surface, each ridge is a non-singular curve on the surface.

The following analogous theorem for higher order ridge points can be proved in the same way, differentiating twice, rather than once, and of course, using the results of theorem 3.8.

Theorem 3.9 Suppose x is a ridge point, and ϕ, γ, etc. are as above, and $d\kappa(x) \neq 0$, then the following are equivalent:

(i) x is a higher order ridge point (so $(\kappa\cdot\gamma)_1 = (\kappa\cdot\gamma)_2 = 0$),

(ii) ϕ and γ have at least 3-point contact at x,

(iii) $\tilde{\gamma}_2 = 0$,

(iv) $\tilde{\phi}_2 = 0$,

(v) for v as in theorem 3.8(vi), $V_4u^4 + 3V_3u^2v = 0$,

and if (as is generically the case) the ridge through x is a non-singular curve then these are all equivalent to :

(vi) the projection of the rib in to RP^3 is singular, and

(vii) the ridge through x is tangent to γ (and so to ϕ) at x.

SURFACES IN 4-SPACE

In this final section of the chapter we introduce some of the classical concepts for surfaces in 4-space needed in Chapter 5. A lot of material on surfaces in 4-space, particularly on the fundamental forms and the curvature ellipse, can be found in Little's article [L3]. The set I have called the centre set has not been defined before, and the reason for introducing it now is that it will be useful in Chapter 5 for describing the contact with 2-spheres, it turns out that if a 2-sphere has higher order contact then its centre lies on this set. One classical reference is [P1].

Let $g : X \hookrightarrow R^4$ be an immersion of a surface. As usual we write X for g(X). The <u>first fundamental form</u> at $x \in X$ is, as always, a quadratic form on the tangent space to X at x:

$$I_x(u,v) = dg_x u.dg_x v . \tag{3.18}$$

However, the <u>second fundamental form</u> is no longer a quadratic form, but is best viewed as a symmetric bilinear map from the tangent space at x to the normal space at x defined by

$$\begin{aligned} II_x : T_xX \times T_xX &\longrightarrow N_xX \\ : (u, v) &\longmapsto P_x(d^2g_x uv) \end{aligned} \tag{3.19}$$

where P_x denotes orthogonal projection of R^4 on to the normal space to X at g(x). Sometimes it will be useful to think of II as a pair of quadratic forms, and so n.II, as n varies in the normal plane, defines a pencil of quadratic forms (see the map μ below).

Since the second fundamental form is no longer a quadratic form we do not have an eigenvalue problem as we did for surfaces in 3-space, so there is no natural generalization of principal directions and lines of curvature to surfaces in 4-space. There are, though, other ways of producing orthogonal pairs of tangent directions at most points on the surface, see [L3] and [J], but none of these has yet been studied in much

depth (except for what is done in [L3]).

We now introduce three subsets of the normal space at x, each of which reflect the second order geometry of the surface at x: the curvature ellipse E, the focal set F (sometimes known as Kommerell's conic), and the centre set C. We then describe the relationships between these sets, introducing the normal sections and their focal lines en route. The different intuitions of the sets E, F and C will prove useful in describing the contacts with spheres in Chapter 5.

Let $k_x : T_xX \longrightarrow N_xX$ be the quadratic map associated to II_x, so k(u) = II(u,u) (as usual we drop any reference to x when there is no room for ambiguity). This map could be called the normal curvature map: any curve on the suface with unit tangent vector u at x has normal curvature k(u). Now, let {u,v} be an orthonormal basis for T_xX (orthonormality being defined by the first fundamental form), and put $\alpha = k(u)$, $\beta = II(u,v)$ and $\gamma = k(v)$, then

$$\begin{aligned} k(u.\cos\theta + v.\sin\theta) &= \alpha\cos^2\theta + 2\beta\cos\theta\sin\theta + \gamma\sin^2\theta \\ &= 1/2(\alpha+\gamma) + 1/2(\alpha-\gamma)\cos2\theta + \beta\sin2\theta . \end{aligned} \tag{3.20}$$

Thus the image under k of the unit circle in T_xX is a doubly covered ellipse, the <u>curvature ellipse</u>, denoted E, with centre $1/2(\alpha+\gamma)$ and the property that the images of orthogonal (respectively antipodal) points on the unit circle are diametrically opposed (respectively equal) on the ellipse, moreover the vector II(u,v) will be tangent to E at k(u). The second order geometry of the immersion is governed by the ellipse and its position relative to the origin in the normal space. It should be pointed out that this ellipse may degenerate to a line segment, or even to a point (though this last possibility is highly non-generic). Classically, a point on the surface for which E degenerates is called a <u>semiumbilic</u>. There are two curvature functions that can be defined on a surface in 4-space: the Gauss curvature and the normal curvature, the latter can be shown to be proportional to the area of the curvature ellipse. Thus the ellipse degenerating to a line segment (or to a point) corresponds to the normal curvature vanishing, see [L3] and [B-M].

The curvature ellipse is a bounded subset of the normal space so we did not need the compactified normal bundle, however for F and C we do:

$$(3.21) \qquad N = \{ (x, [c:s]) \in X \times RP^4 \ : \ (c - sg(x)).dg_x = 0 \}$$

Note that with this definition, the normal planes are affine planes in R^4. However when we need their linear structure we consider the point g(x) as the origin in the normal plane N_xX.

The <u>focal set</u> can be defined as it was for surfaces in 3-space, so

$$(3.22) \qquad F = \{ (x,[c:s]) \in N : \exists u \neq 0, (c - sg(x)).d^2g_x u - sdg_x u.dg_x = 0 \}.$$

But another approach can be used, from which some of its properties become clear. The two definitions are easily seen to be equivalent. Let Q_x be the space of (projective) quadratic forms on T_xX , and define the affine linear map at x

$$\mu : N_xX \longrightarrow Q_x$$

$$: [c:s] \longmapsto (c - sg(x)).k - sI,$$

where k and I are as above. The focal set is then the inverse image under μ of the cone in Q_x of parabolic forms, and is therefore a conic section (μ being affine linear). Note that $\mu([g(x):1]) = -I$ is an elliptic form, so any linear subspace of N_xX meets the focal set in two, possibly coincident, points ([g(x):1] being the origin in N_xX). This observation limits the possible positions of the conic with respect to the origin in the normal space: the origin must be 'inside' the conic. When x is a semiumbilic, the focal set degenerates to a pair of lines intersecting at the <u>semiumbilical centre</u>, and the two branches of F are focal lines F_u for the directions u for which k(u) is a singular point of E (these two directions are orthogonal as their images under k are antipodal). See below for further comments.

To define the centre set we use a construction that Euler introduced for surfaces in 3-space. Fix $x \in X$, then for each unit tangent vector u at x we have the 3-dimensional subspace V_u of R^4 passing through g(x) and spanned by $dg_x u$ and the normal plane to X at g(x). Since V_u contains the normal plane at g(x) it is transverse to the surface at g(x), and so they intersect in a non-singular curve r_u, which has tangent vector $dg_x u$ and normal plane N_xX at x. Being contained in V_u, r_u is a space curve, so we can draw on the standard ideas presented in the first section of this chapter. In particular we have the focal line F_u and the centre of the osculating circle $[c:s]_u$, both of which lie in the normal plane to the surface.

We are now in a position to define the <u>centre set</u> C at x. Essentially it is the collection of the osculating centres $[c:s]_u$, but that is not quite a full picture, firstly because if k(u) = 0 then $[c:s]_u$ is undefined, and secondly because when x is a semiumbilic we will wish to include some other points. As was pointed out in the first section of this chapter, the osculating centre of a curve is the inverse of the curvature vector with respect to the unit circle in the normal space. Thus $[c:s]_u$ is the inverse of k(u), and when k(u) ≠ 0 and x is not a semiumbilic, C is the inverse of E. Now consider the case where k(u) = 0 for some u. Firstly, and most pathologically, if k(u) = 0 for all u we can take C to be the line at infinity in N_xX. Secondly suppose k(u) = 0 for some isolated u, then E approaches the origin in a definite direction, so its inverse can defined to include the appropriate point at infinity (we are essentially blowing up the origin - replacing it with a projective line - and then the inversion map can be made bijective). Finally consider the case where x is a semiumbilic. In this case E is a line segment so its inverse is an arc of a circle passing through the origin, and we take C to be that entire circle; if we need to distinguish between the two arcs, then the inverse of E will be the <u>real part</u> of C (which is a closed arc), and the remainder will be the <u>imaginary part</u> of C - note that since the map k is bounded, the origin always lies on the imaginary part of C. In the non-degenerate case, where x is not a semiumbilic, C is in fact the pedal curve of F. See the diagram at the end of this chapter for clarification.

Thus for $k(u) \neq 0$, the real part of C is the set

(3.23) $\{ (x, [c:s]) \in N : |k(u)|^2(c - sg(x)) - sk(u) = 0 \}$.

While if $k(u) = 0$ (but k is not identically 0), then $[c:s]_u = [II(u,v):0]$.

We now turn to the relationships between these 3 subsets of the normal space. We have seen that essentially C is the inverse of E. Now, the focal line F_u of r_u is the set,

$$F_u = \{ [c:s] \in N_xX : \mu([c:s])(u) = 0 \} ,$$

remembering that $\mu([c:s])(u) = (c - sg(x)).d^2g_xu^2 - sdg_xu.dg_xu$. Differentiating this condition with respect to u, we find that the focal set at x is the envelope of the lines F_u as u varies round the unit tangent circle at x. It follows that E and F are polar conjugates with respect to the unit circle in the normal space - recall that the polar conjugate of a point z with respect to the unit circle is the line $\{ y : z.y = 1 \}$, and the polar conjugate of a curve is then the envelope of these lines as z varies along the curve.

Figure 3(v) (page 55, drawn by Alex Flegmann): These show the situations for a general hyperbolic point (see below), and for a semiumbilic. At a hyperbolic point the two points at infinity in F are [c:0], $c \neq 0$, satisfying $c.d^2gu = 0$ for some u. Now II(u,) is a linear map from R^2 to itself, so this condition ensures that it has rank 1, and consequently k(u) is one of the points on E where lines from the origin tangent to E meet E - these tangents are shown as dotted lines on the diagram. Such directions u in the tangent space are called asymptotic directions. (Note that the tangencies between C and F on the diagram are not coincidental - given any non-singular plane curve, then its inverse and polar conjugate are tangent at the nearest and furthest points on the inverse from the origin. An example of discovery by computer!).

Figure 3(vi) (p.55, also by A. Flegmann) A semiumbilic was defined as a point x for which the curvature ellipse degenerates. Suppose k(u) is one of the singular points (end points) of E, then again II(u,) is degenerate (II(u,v) is a tangent vector to E at k(u) for v orthogonal to u), and it follows that $F_u \subset F$. Since F is a conic, it consists only of these two focal lines F_u and F_v, and the semiumbilical centre $[c:s] \in F_u \cap F_v$ satisfies,

(3.24) $(c - sg(x)).d^2g_x - sdg_x.dg_x = 0.$

In the figure, d is the semiumbilical centre, and a and b are the end points of the real part of C.

Table 3.1 gives the correspondences between E and F . A point x on the surface is sometimes said to be elliptic, hyperbolic or parabolic accordingly as F is an ellipse, a hyperbola or a parabola. However, it should be pointed out that in contrast to the case for surfaces in 3-space, this description does not coincide with the sign of the Gauss curvature K (which is defined via I), except for the relationship $K > 0 \Rightarrow$ hyperbolic.

CURVATURE ELLIPSE E	FOCAL SET F
0 inside E	ellipse
0 outside E	hyperbola
0 on E	parabola
non-radial line-segment	pair of non-parallel lines
radial line-segment	pair of parallel lines
single point ($\neq 0$)	double line
single point at 0	double line at infinity

Table 3.1 Correspondences between E and F .

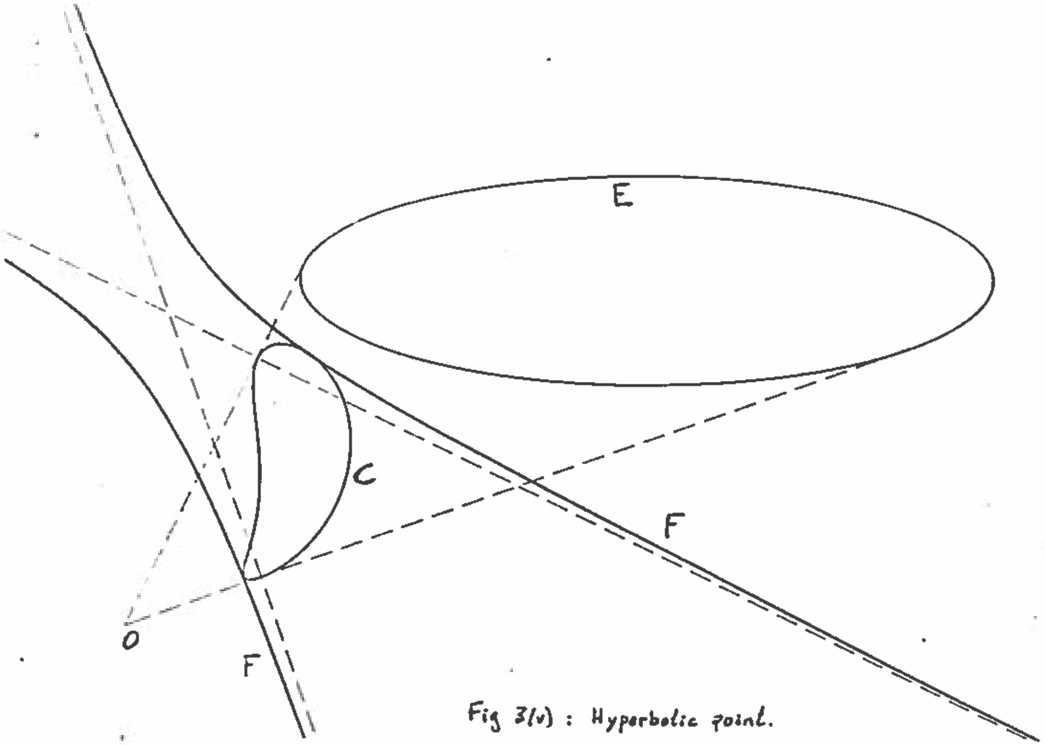


Fig 3(v) : Hyperbolic point.

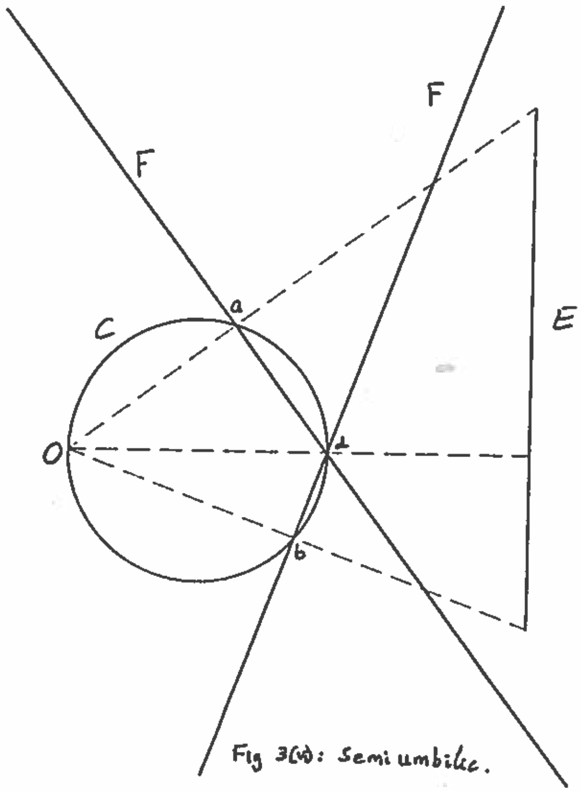


Fig 3(vi): Semi umbilic.

CHAPTER 4 GENERIC SURFACES IN 3-SPACE

This chapter is divided into two sections, the first on the contact of generic surfaces with spheres, and the second on their contact with circles - the precise meaning of generic was discussed in Chapter 2. Many of the results of the first section - though not all - are well-known, but have not been written down in detail. These are mostly due to Porteous, see [P2] and [P3], see also Markatis [M1]. The second section, on the other hand, was largely motivated by some conjectures of Porteous in [P3], and is original material (though one result is a consolidation of Meusnier's theorem which dates back to 1785).

CONTACT WITH SPHERES

Let M be the collection of spheres in R^3, and let M_e extend M as in Chapter 2. Thus,

(4.1) $$M_e = \{ [c:s:\rho] \in RP^4 : s, \rho \in R, (c,s) \neq 0 \}$$

Let $m = [c:s:\rho] \in M_e$, then $f_m : R^3 \to R$ given by

(4.2) $$f_m : y \mapsto c.y - 1/2\, s|y|^2 - \rho .$$

is a map with zero-set m. For $m \in M$, $m = f_m^{-1}(0)$ is non-singular, and f_m cuts out m (i.e. it is a submersion at each point of m). If $s \neq 0$, then m is the sphere centre c/s, radius $[|c/s|^2 + \rho/s]^{1/2}$, while if s = 0, m is the plane $c.y = \rho$. As in Chapter 2 we also have the map

(4.3) $$\Phi : X \times M_e \to R$$
$$: (x, m) \mapsto \phi_m(x) = f_m \circ g(x),$$

and from Chapter 3 the distance-squared function V on $X \times RP^3$:

(4.4) $$V(x, [c:s]) = c.g(x) - 1/2\, s|g(x)|^2.$$

We denote by V_i the i^{th} derivative of V with respect to x. It should be pointed out that the maps f_m, and hence $f_m \circ g$, $\tilde{\Phi}$ etc., are not well-defined, as their values depend on the choice of representative of [c:s]. Most of this ambiguity can be eliminated by restricting to the sphere $|c|^2 + s^2 = 1$ (the maps then being determined up to a sign), but in any case, the zero-sets - or level sets in the case of V - are unaffected by the choice of representative, as are their $\mathcal{K}$ -classes.

By theorem 1.4, the contact of the sphere m and the surface X at g(x) is given by the $\mathcal{K}$-class of the map $f_m \circ g$ at x. Now, suppose that $(x,m) \in \tilde{\Phi}^{-1}(0)$, with $m = [c:s:\rho]$, then it follows that $\rho = V(x, [c:s])$, and we see there is a diffeomorphism between $\tilde{\Phi}^{-1}(0)$ and $X \times RP^3$. Moreover, ϕ_m has a particular probe structure (see Appendix 1) if and only if V(,[c:s]) has that probe structure. In this section we therefore consider the probe structure of the function V, ignoring the constant term ρ , and if this corresponds to a singularity type S at x, it follows that the corresponding sphere m has contact type S with X at g(x).

The first proposition is on the second order properties of the surface, while later ones will be on higher order phenomena, such as ridge points.

Proposition 4.1 V(· ,[c:s]) has an A_1, an A_2 or a D_4 singularity (or worse) at x if and only if (x, [c:s]) is in the normal bundle to X, the focal set of X, or is an umbilical centre (i.e. x is an umbilic with centre [c:s]) respectively. Moreover the kernel vector for A_2 contact is the principal direction associated to the focal point.

Proof: This is all shown in Chapter 3, the conditions for the A_1, A_2 and D_4 singularities (or worse) being : $V_1 = 0$, $V_2u = 0$ for some $u \neq 0$, and $V_2 = 0$, respectively.

Spheres with A_2 contact with the surface will be called focal

spheres since under the correspondence $\tilde{\Phi}^{-1}(0) \longrightarrow X \times RP^3$ described above, the focal spheres are taken to points on the focal set F , and the set $\tilde{F}$ corresponds to $\{ (m,x,u) \in M \times S^1X : d\phi_m = d^2\phi_m u = 0 \}$. Thus at each non-umbilic there are two focal spheres, while at umbilics they coincide, and in that case they will be called umbilical spheres. Note that if $c = sg(x)$, so [c:s] is at the origin in the normal plane, then $m \in M_e \setminus M$, and in this case the singularity involved is an $A_1^\dagger$. From now on we will only be dealing with focal spheres so we can assume that $c \neq sg(x)$ and $m \in M$.

We mentioned in Chapter 3 that for a generic immersion umbilics are isolated. The reason for this is that the $\mathcal{K}$-codimension of the D_4 singularities is 4, and so the codimension of the D_4-orbit in $J^r(X,R)$ is 4 + 2 = 6 (see the last part of Chapter 2 for a discussion of codimensions of orbits), and dim $(X \times M_e) = 6$, so since, for a generic immersion, the jet-extension map $j_1^r\tilde{\Phi}$ is transverse to the D_4-orbit, they must intersect in a discrete set of points.

The next proposition gives yet another alternative definition for a ridge point, interpreting the last conditions in theorems 3.8 and 3.9.

Proposition 4.2 Let $x \in X$ not be an umbilic, then
(i) x is a ridge point iff one of the focal spheres at x has at least A_3 contact with the surface;
(ii) x is a higher order ridge point iff one of the focal spheres at x has at least A_4 contact with the surface.

Proof: Let [c:s] be a focal point at x, with principal direction u, then $V_1 = V_2u = 0$ (V = V(,[c:s])). That x is not an umbilic ensures that $V_2(x, [c:s]) \neq 0$. The A_3 and A_4 conditions for V are, cumulatively,
A_3 : $V_3u^2 + V_2v = 0$ for some v,
A_4 : $V_4u^3 + 3V_3uv + V_2w = 0$ for some w.
The first is equivalent to $V_3u^3 = 0$, and the second to $V_4u^4 + 3V_3u^2v = 0$. That these two equations are equivalent to x being a ridge point or

a higher order ridge point respectively, was shown in theorems 3.8 and 3.9.

Remark 4.3 There are two distinct types of A_3 singularities of functions on R^2, namely hyperbolic and elliptic, or A_3^- and A_3^+ respectively, and we say that a first order ridge point is hyperbolic or elliptic accordingly. These two types of ridge are also said to be fertile and sterile respectively, we will see why when we consider umbilics. Whether an A_3 is hyperbolic or elliptic corresponds to the type of quadratic in v given by

$$V_4u^4 + 6V_3u^2v + 3V_2v^2.$$

In the parabolic case, when the two roots coincide, this reduces to the A_4 condition above. The normal forms for the $A_3^{\pm}$ are $x^2 \pm y^4$, and we see that their zero-sets are two curves tangent at O (but with different curvatures), and an isolated point, respectively. The zero-set of an A_4 is a ramphoidal cusp, normal form $x^2 - y^5$. See the table at the end of this section.

Before stating theorem 3.9 we mentioned the fact that for a generic immersion the ridges form a collection of smooth curves on the surface, indeed part of the statement of that theorem depended on this fact, so we now prove it.

Theorem 4.4 The set of rib points in the focal set of a generic immersed surface is, away from umbilical centres, a one-dimensional manifold.

Proof: This theorem can be proved directly, but we will use the singularity theory that we have already set up. Now, a rib point is represented by a point $(x,m) \in X \times M$, where ϕ_m has an A_3 singularity or worse at x, but by hypothesis not a D_4 or worse. We therefore only expect A_3 and A_4 singularities - the set of A_k singularities, $k > 4$, forming a subset of $J^r(X,R)$ of codimension 7 (for large enough r) which will be avoided altogether by the genericity hypothesis. It follows from standard unfolding theory that the union of the A_3 and A_4

orbits is a submanifold of the jet space (the standard versal deformation of an A_4 is $(x,s,t,u,v) \longmapsto x^5 + sx^3 + tx^2 + ux + v$), and since the jet-extension map is transverse to each of the orbits it must be transverse to their union. Thus the appropriate subset of $X \times M$ is a submanifold, and because of the correspondence between $\bar{\phi}^{-1}(0)$ and $X \times RP^3$ described above so is the set of rib points.

Remark It follows from this and the fact that away from umbilics the projection of the focal set on to the surface is a local diffeomorphism (proposition 3.2), that the set of ridge points on the surface, away from umbilics, is a union of non-singular curves. If two of these curves meet at any non-umbilical point then for a generic immersion they necessarily come from different sheets of the focal set, and in [B2] Bruce shows that for an open dense set of immersions these crossings are transverse. For a non-generic surface, however, ridge lines from the same focal sheet can cross, and at such a point the relevant focal sphere has A_4-contact, see [P3]. We will see shortly that ridge curves are also non-singular at umbilics.

Umbilics

We have seen in proposition 4.1 that x is an umbilic if and only if for some $m = [c:s:\rho] \in M$, the contact map ϕ_m has a D_4 singularity at x (or worse : D_5, E_6, etc., but for generic immersions these do not occur as their codimension is too high). For a generic immersion this singularity will be presented transversely, and we now find the condition on the immersion at the umbilic that ensures this.

Proposition 4.5 Let x be an umbilic of the immersion g, then the D_4 singularity of ϕ_m (m as above) is presented transversely if and only if the intrinsic cubic C at the umbilic (see (3.11)) is not orthogonal with respect to the first fundamental form, I.

Proof: We write the immersion in Monge form,

$$g(x, y) = (x, y, h(x, y)),$$

with $h \in \mathcal{M}_2^2$. We can put $h(x, y) = \kappa/2\,(x^2 + y^2) + 1/6\, C(x, y)^3 + H(x,y)$, with $H \in \mathcal{M}_2^4$, κ is then the curvature at x, and C is the intrinsic cubic. Then with $c = (c_1, c_2, c_3)$, ϕ_m becomes,

$$\phi_m(x, y) = c_1x + c_2y + c_3h(x, y) - 1/2\, s(x^2 + y^2 + h(x, y)^2) - \rho.$$

For $d\phi_m = 0$ we need $c_1 = c_2 = 0$, and for $d^2\phi_m = 0$ we need $s = \kappa c_3$. The umbilical centre is therefore at $m = [c:s:\rho] = [(0,0,1): \kappa: 0]$. Then the $\mathcal{K}_e$-tangent space at ϕ_m is

$$T\mathcal{K}_e(\phi_m) = \langle C(x, y)^3 + H(x, y),\ C(1,0)(x, y)^2 + H_1(x, y),\ C(0,1)(x, y)^2 + H_2(x,y) \rangle,$$

where the $H_i \in \mathcal{M}_2^3$. Now, since the singularity cannot be worse than a D_4 - from considerations of codimension - it follows that C is not parabolic, and it is then easy to show that, $T\mathcal{K}_e(\phi_m) \supset \mathcal{M}_2^3$.

With the notation used in Appendix 1, we have that

$$T\mathcal{K}_e(\phi_m) + R\{\dot{\Phi}_i\} = T\mathcal{K}_e(\phi_m) + R\{\dot{\phi}_i\} + \mathcal{M}_2^3$$

$$= R\{ C(1,0)(x, y)^2, C(0,1)(x, y)^2, x^2 + y^2, x, y, 1\} + \mathcal{M}_2^3.$$

which is all of $\mathcal{E}_2$ provided the three quadratic terms are independant, which is the case precisely when C is not orthogonal with respect to $(x^2 + y^2)$ - the first fundamental form.

The relationship between this proposition and proposition 3.1 is that the set $\{ (m,x,u)\ M \times S^1X : d\phi_m = d^2\phi_m u = 0 \}$ corresponding to $\tilde{F}$ is a submanifold of $M \times S^1X$ near an umbilic if the D_4 singularity is presented transversely. Let us now look more closely at the D_4 singularity. There are two types of D_4: hyperbolic or D_4^+, with normal form $x(x^2 + y^2)$; and

elliptic or D_4^-, with normal form $x(x^2 - y^2)$. Since the first two derivatives of a D_4 vanish the third derivative is intrinsic, and the D_4 is hyperbolic or elliptic accordingly as its third derivative is. At an umbilic on a surface, this third intrinsic derivative is (up to a scalar multiple) the intrinsic cubic at the umbilic (see (3.11)). Any text on catastrophe theory details some of the properties of the versal deformation (or unfolding) of a D_4 singularity, we will be interested in two of these in particular. Let $f(x, y) = x(x^2 + y^2)$, then

$$F(x, y, s, t, u, v) = f(x, y) + sx^2 + tx + uy + v.$$

is a versal deformation of f. It is straightforward to check that for a D_4^- there are three non-singular curves passing through the origin in $R^2 \times R^3$ - namely $(x, 4x, 6x^2, 2x^2, -12x^3)$, $(x, -x, 4x, 6x^2, -2x^2, -12x^3)$, and $(0,0,u, 0, 0)$ - at each point of which F has an A_3 singularity, while for a D_4^+ there is only one such curve - the last one of the three above. All the A_3's that occur are hyperbolic, i.e. A_3^-. These curves lift to curves in the desingularization of the A_2-set (corresponding to the set given at the beginning of this paragraph), and in the elliptic case the lifted curves do not intersect, that is they have different limiting kernel vectors, and these limits are the roots of the intrinsic cubic.

Another interesting feature of the unfolding of the D_4^-, but not the D_4^+, is that through the origin in the parametrization space, there passes a curve at each point of which (excepting the origin) the perturbed function has 3 A_1 singularities. In the deformation given above, the curve is $(s, t, u, v) = (2a, a^2, 0, 0)$ and the singularities are at $(x, y) = (0, a)$, $(0, -a)$ and $(-a, 0)$.

Remark It should be checked that the above properties hold for any versal deformation of a D_4 singularity. In fact this follows from theorem A1.8 in Appendix 1.

We now return to the geometry of a generic surface near an umbilic, using the properties of deformations we discussed above.

Theorem 4.6 Let x be an elliptic (resp. hyperbolic) umbilic of the immersion g. Then passing through x there are 3 (resp. 1) ridges, each being a non-singular curve. Moreover in a sufficiently small neighbourhood of the umbilic all the ridge points are fertile (or hyperbolic - see remark 4.3). At each ridge point there is an associated principal direction, and the root directions of the intrinsic cubic at the umbilic are the limiting values of the principal direction as we approach the umbilic along each of the ridges. Moreover if u is one of these root directions and C is the intrinsic cubic, then the tangent vector to the associated ridge is $\hat{x}$ satisfying $Cuv\hat{x} = 0$, where v is orthogonal to u.

Proof: The existance of the 3 (resp. 1) ridges, that they are fertile in a neighbourhood of the umbilic, and that the root directions of C are the limiting principal directions all follows immediately from the deformation we discussed above. The deformation tells us that the rib-lines in X x M are non-singular, but we need proposition 3.2 to infer that their projections on to the surface - the ridges - are also non-singular. This follows because the A_3 curves lifted to the desingularized A_2 set are transverse to the unit circle corresponding to the umbilic. The final statement, that $\hat{x}$ satisfies $Cuv\hat{x} = 0$, follows from equation (3.14), and can be found in the proof of proposition 3.5.

It is because of this last result that the adjectives 'fertile' and 'sterile' are applied to ridge points. When considering one-parameter families of surfaces pairs of umbilics may be born (similarly to singularities of vector fields), but from the theorem this can only happen at fertile ridge points.

Using the other property of versal deformations of a D_4^- singularity that we discussed above we get the following theorem about elliptic umbilics.

Theorem 4.7 Let x be an elliptic umbilic of the generic immersion g, with m the umbilical sphere. Then there is a curve in M through m such that each point of the curve, other than m, represents a tritangent sphere (i.e. a sphere tangent to the surface X at three distinct points). As the sphere approaches m along this curve, so the three points of tangency approach x.

Proof: This follows immediately from the versal deformation of a D_4^- singularity. It corresponds to the perturbation  (the first being the zero-set of a D_4^-, and the double points represent tangencies, or A_1^- contact).

This result could be rephrased as follows: In any neighbourhood of an elliptic umbilic there are three points with a common tangent sphere.

We conclude this section with a list of the various contact types expected for a generic immersion, what properties they are associated with, and the intersection (up to diffeomorphism) of the given sphere with the surface.

Contact type	Situation	Intersection (up to diffeo.)
A_1^+	Tangent, but not	• Isolated point
A_1^-	focal sphere.	× 2 transverse curves
A_2	Focal sphere (not ridge or umbilic)	 Simple cusp
A_3^+	Sterile ridge	• Isolated point
A_3^-	Fertile ridge	 Two tangent curves
A_4	Higher order ridge	 Ramphoid cusp
D_4^+	Hyperbolic umbilic	 Curve - plus singular point
D_4^-	Elliptic umbilic	 3 concurrent curves, pairwise transverse

CONTACT WITH CIRCLES

In this section our model submanifolds of R^3 will be circles (the word 'circle' will always include straight lines). There are several ways of representing circles, though not all will include lines, but here we choose to represent a circle as the intersection of a pair of spheres. Some care is needed with this as firstly not all pairs of spheres intersect, but more importantly because for a given circle there is a one parameter family of spheres on which it lies. Thus initially there will be a two dimensional redundancy in our representation (one for each of the two spheres), but we will be making certain choices later which will simplify the presentation.

For this section we put

$$(4.5) \qquad M_e = \{ ([c_1:s_1:\rho_1],\ [c_2:s_2:\rho_2]) \in RP^4 \times RP^4 : (c_1,s_1),\ (c_2,s_2) \neq 0,\ [c_1:s_1] \neq [c_2:s_2] \}$$

then for $m \in M_e$, the map f_m is defined by

$$(4.6) \qquad f_m(y) = (c_1.y - 1/2\, s|y|^2 - \rho_1,\ c_2.y - 1/2\, s|y|^2 - \rho_2).$$

Note that $f_m^{-1}(0)$ is straight line only if $s_1 = s_2 = 0$. We then define M to be the subset of M_e for which the set $m = f_m^{-1}(0)$ is a genuine circle (so M excludes the m for which $f_m^{-1}(0)$ is empty or singular). As usual, given an immersion g, we define the map

$$(4.7) \qquad \begin{aligned} \Phi &: X \times M_e \longrightarrow R^2 \\ &: (x, m) \longmapsto \phi_m(x) = f_m \circ g(x), \end{aligned}$$

and the contact of X with m at x is determined by the $\mathcal{K}$-class of the contact map $\phi_m : X \longrightarrow R^2$ at x. We refer freely to Appendix 1 for the $\mathcal{K}$- classes of maps from R^2 to R^2 and how to distinguish them. Note that if $m \in M$, the map ϕ_m will be $\mathcal{K}$-equivalent to a suspension of a map from R to R (by the symmetry lemma 1.7), and so the only contact types we expect to occur for $m \in M$ are A_k's, and since the codimension of an A_k singularity

is (k+1) we can expect genuine circles to have up to A_6 (or 7-point) contact. However we will find that interesting features occur for $m \in M_e \setminus M$ where Σ^2 singularities will arise. We return to this point later.

Now, $\phi_m(x) = 0$ if and only if $g(x) \in f_m^{-1}(0)$, which if $m \in M$ means that $g(x)$ lies on the circle m, and ϕ_m is singular if and only if m is tangent to X at $g(x)$ – at least for $m \in M$, for $m \notin M$ this is meaningless as m consists solely of the isolated point $g(x)$. For $m = ([c_1:s_1:\rho_1],[c_2:s_2:\rho_2])$

$$(4.8) \qquad d\phi_m = ((c_1 - s_1 g(x)).dg\ ,\ (c_2 - s_2 g(x)).dg),$$

and this has an A_1 singularity or worse at x if and only if there is some non-zero cokernel vector (a,b) satisfying

$$((ac_1 + bc_2) - (as_1 + bs_2)).dg = 0.$$

Now, $[ac_1 + bc_2 : as_1 + bs_2]$ spans the axis of the circle as (a,b) varies, so we see that the circle m is tangent to X at $g(x)$ if and only if its axis intersects the normal to the surface. Now we make our choice of spheres for a given tangent circle: we can choose any two points on the axis of the circle as the centres of the spheres. From now on, for a circle with at least A_1 contact with X at $g(x)$, we take $[c_1:s_1]$ to be the point on the normal to the surface at $g(x)$, and we take $[c_2:s_2]$ to be the point on the tangent plane to X at $g(x)$, either or both of these points may be at infinity. We denote by V the usual distance squared function from $[c_1:s_1]$, and by W that from $[c_2:s_2]$, see (3.12). Then $\phi_m = (V - \rho_1,\ W - \rho_2)$, and the conditions for ϕ_m to have an A_k singularity, $k = 1,2,3,4$, at x are, cumulatively:

$$(4.9) \qquad \begin{cases} A_1: & W_1 u = 0, \text{ for some } u \neq 0 \quad (V_1 = 0 \text{ by definition}); \\ A_2: & V_2 u^2 = 0,\quad W_2 u^2 + W_1 v = 0 \quad \text{for some } v; \\ A_3: & V_3 u^3 + 3V_2 uv = 0,\quad W_3 u^3 + 3W_2 uv + W_1 w = 0, \quad \text{for some } w; \\ A_4: & V_4 u^4 + 6V_3 u^2 v + 3V_2 v^2 + 4V_2 uw = 0, \\ & \qquad W_4 u^4 + 6W_3 u^2 v + 3W_2 v^2 + 4W_2 uw + W_1 z = 0, \text{ for some } z. \end{cases}$$

We now turn to higher order contact, but first a definition.

Definition 4.8 Let u be a unit tangent vector at x. The <u>Meusnier sphere</u> for the direction u at x, denoted M_u, is defined to be the sphere tangent to the surface at g(x) with curvature II(u,u). (In the case where u is an asymptotic direction, so II(u,u) = 0, M_u is the tangent plane at g(x).)

Remark We name M_u the Meusnier sphere after Meusnier's theorem, which states: For any plane section of the surface at g(x) in the direction u, the circle of curvature of the resulting plane curve lies on M_u. This extends to: The osculating circle at g(x) of any curve with tangent vector u lies on M_u. I have not found this extension elsewhere, though it is easy enough to prove. Note that Meusnier's theorem does not usually include asymptotic directions, though this extension is also immediate.

The following result is a consolidation of Meusnier's theorem.

Proposition 4.9 A circle with tangent vector $dg_x u$ at x has at least A_2 contact with the surface at g(x) if and only if it lies on the Meusnier sphere M_u.

Proof: From (4.9) the circle in question has A_2 contact iff $V_2u^2 = 0$ and $W_2u^2 + W_1v = 0$ for some v. The second equation can be solved for either v or $[c_2:s_2]$. (Note that $c_2 \neq sg(x)$, otherwise the circle would be singular.) The first equation,

$$V_2u^2 = (c_1 - s_1g(x)).d^2gu^2 - s_1dgu.dgu = 0,$$

tells us that the axis of the circle must pass through the centre of M_u. Since the circle already has one point on the sphere, namely g(x), it follows that it must lie entirely on the sphere.

Remarks (i): This result, or at least the 'if' part, could have been derived from Meusnier's theorem : If a given circle has 3-point contact with a curve then under perturbation of that circle we can produce 3 points of intersection, but this perturbed circle will therefore also have 3 points of intersection with the surface, thus the circle has 3-point contact with the surface (see the discussion on contact with curves in Chapter 1).

(ii) There is a useful intuition for the contact of these circles on the Meusnier sphere. If u is not a principal direction, then $V_2u \neq 0$, $V_2u^2 = 0$ so M_u has $A_{\bar{1}}$ contact with the surface at g(x). Thus they intersect in two transverse curves - see the end of the last section - one of which has tangent u. Any circle lying on the sphere with tangent u has 2-point contact with one curve, and 1-point contact with the other one, giving a total of 3-point contact with the surface. We will be developing this intuition further, after each of the subsequent propositions.

Proposition 4.10 If u is not a principal direction at x, then there is a unique circle on M_u, with tangent vector dgu, with at least 4-point (A_3) contact with the surface.

Proof: From (4.9), the A_3 condition is

$$V_3u^3 + 3V_2uv = 0, \quad W_3u^3 + 3W_2uv + W_1w = 0,$$

where v satisfies $W_2u^2 + W_1v = 0$. Now if u is not principal then $V_2u \neq 0$, and the first equation can be solved uniquly for v (modulo u). The third equation can then be solved uniquly (despite the ambiguity in v) for $[c_2:s_2]$, which determines the circle uniquely (the second equation being solved for w).

To continue the geometrical intuition in the preceeding remark, the circle in this proposition is the osculating circle of one of the branches of the intersection of M_u and X, giving 3-point contact with that branch and 1-point contact with the other.

We now turn to the case where u is a principal direction, and in this case the Meusnier sphere will be a focal sphere.

Proposition 4.11 Let u be a principal direction at a non-umbilic x, then:
(i) if u is not associated to a ridge point, there is no circle on M_u, with tangent vector u, that has at least A_3 contact with the surface;
(ii) if u is the principal direction associated to a ridge point, every circle on M_u with tangent vector u has at least A_3 contact with the surface, and moreover, there are 2, 0 or 1 circles with at least A_4 contact accordingly as the ridge point is fertile, sterile or higher order (hyperbolic, elliptic or parabolic, respectively).

Proof: (i) Since u is principal, M_u is a focal sphere and $V_2u = 0$. The A_3 condition - see (4.9) - then implies $V_3u^3 = 0$, which is precisely the condition for x to be a ridge point.
(ii) As x is a ridge point, $V_3u^3 = 0$, and given $[c_2:s_2]$ the equations involving W can be solved for v and w, so leaving the axis of the circle indetermined. The A_4 condition now reduces to,

$$V_4u^4 + 6V_3u^2v + 3V_2v^2 = 0.$$

This is a quadratic equation for v which is the same as that which determines the type of A_3: hyperbolic, elliptic or parabolic (see remark 4.3). The solutions v of this equation then determine the point $[c_2:s_2]$ from the A_2 condition, the remaining equations being solvable for w and z.

The geometrical intuition discussed above illuminates this proposition as well. If M_u is a focal sphere, not associated to a ridge or an umbilic, then it intersects the surface in a simple cusp which has infinite curvature at the point g(x), and the osculating circle of the intersection collapses to a point. Moreover as we approach the cusp, the

plane spanning the circle tends to the tangent plane to X at g(x). We call this phenomenon 'curling up and dying', and it is something we return to when considering umbilics. In part (ii), where x is a ridge point, the sphere M_u intersects the surface in one of three ways: two tangent curves (fertile ridge); an isolated point (sterile ridge); and a ramphoid cusp (second order ridge). For the sterile ridge, our geometrical intuition is not very helpful (from the intersection, we might not expect any circles of A_2 contact either!), however for the fertile ridge we see that every circle on M_u with tangent u has 2-point contact with each branch of the intersection, giving 4-point - A_3 - contact. The two osculating circles of the branches then have 5-point contact with the surface. For the ramphoid cusp the curve has a well-defined limiting centre of curvature, and it can be shown that the circle of curvature at the cusp has 5-point contact with the curve at the cusp.

We now prove a similar proposition for umbilical points. Note that since all directions at an umbilic have the same curvature, there is a unique Meusnier sphere there - namely the umbilical sphere which has D_4 contact with the surface at the umbilic. Recall that the intrinsic cubic C at an umbilic is a scalar multiple of V_3 (V measured from the umbilical centre), and that the root directions at an umbilic are the roots of C.

Proposition 4.12 If x is an umbilic on a generic surface, then a circle has at least A_3 contact with the surface at g(x) if and only if its tangent vector at g(x) is in a root direction. If u is a root direction then every circle with tangent vector u has at least A_3 contact, and exactly one of these circles has at least A_4 contact with the surface at the umbilic. Thus for a hyperbolic umbilic there are three circles with at least A_4 contact, while for an elliptic umbilic there is just one.

Proof: This runs similarly to previous proofs except that $V_2 = 0$ in this case, so the A_4 condition becomes $V_4u^4 + 6V_3u^2v = 0$, which is now linear, giving a unique circle with at least A_4 (5-point) contact.

Our running geometrical interpretation of these results also works for the elliptic umbilic, but not entirely for the hyperbolic umbilic (it will fail whenever the intersection does not accurately reflect the contact type), this is left as an exercise for the reader!

We now turn to the situation at a general point on the surface. Looking back over the last few propositions we see that at a given point x and for most tangent directions u, we have a unique circle with at least A_3 contact, and for finitely many directions (usually none!) there is a one-dimensional family of such circles. Thus over each point there is a one-dimensional family of circles with A_3 contact with the surface, so we expect there to be finitely many circles with at least A_4 contact with the surface. Question: How many? We saw in proposition 4.12 above that at an umbilic there are at most 3 such circles (or 1 if it is an elliptic umbilic), and we now put an upper bound on this for the general point.

Theorem 4.13 For a generic immersion $g : X \hookrightarrow \mathbb{R}^3$, and any point x on X, there will be at most 10 circles with at least A_4 contact with the surface at g(x). For an umbilic we have already seen this reduces to 3.

Proof: To simplify this proof we express the immersion in Monge form at the point x, so

$$g(x, y) = (x, y, h(x, y)),$$

with

$$h(x, y) = 1/2\ B(x, y)^2 + 1/6\ C(x, y)^3 + 1/24\ D(x, y)^4 + H(x, y),$$

where $H(0) = dH(0) = \ldots = d^4H(0) = 0$, and we can choose the coordinate directions to be the principal directions at x, so $B(x,y)^2 = ax^2 + by^2$. As usual we take $[c_1:s_1]$ to be on the normal and $[c_2:s_2]$ to be on the tangent plane, thus $[c_1:s_1] = [(0,0,r):s_1]$, and $[c_2:s_2] = [(p,q,0):s_2]$ for some p,q,r. The first equation in (4.9), with u = (x,y), becomes $(p,q).(x,y) = 0$, so we can put $(p,q) = t(-y,x)$ for some t (note that $(x,y) \neq 0$). The other equations become, with $v = e(-y,x)$ for some e

as we can always choose v, w, z,.. to be perpendicular to u,

A_2: $r(ax^2 + by^2) - s_1(x^2 + y^2) = 0;\quad s_2 = et;$

A_3: $rC(x, y)^3 + 3rt(b - a)xy = 0;\quad w = 0;$

A_4: $rD(x,y)^4 - 3s_1(ax^2 + by^2)3 + 6rtC(x,y)^2(-y,x) + 3rt^2(b^2 + ay^2) - 3s_1t^2(x^2 + y^2) = 0;$

and the second A_4 equation can always be solved for z so is not of interest. The second A_2 equation can be solved for s_2, so it too can be ignored, and we are thus left with 3 equations, A_2, A_3 and A_4. We can eliminate $[r:s_1]$ between the A_2 and A_4 equations, to obtain the 2 equations (note that r = 0 is not of interest):

$$C(x, y)^3 + 3(b - a)txy = 0,$$

(4.10)

$$D(x,y)^4(x^2 + y^2) - 3(ax^2 + by^2)^3 + 6t(x^2 + y^2)C(x,y)^2(-y,x) + 3t(b - a)(x^4 - y^4) = 0.$$

These equations are homogeneous in x, y, t, of degrees 3 and 6 respectively, and therefore represent 2 algebraic curves in CP^2, of degrees 3 and 6, so by Bezout's theorem (see any standard text on algebraic curves, e.g. [W1]) they either have a common component or else have 6 x 3 = 18 points of intersection, counting multiplicity. The possibility of having a common component is of too high codimension to occur generically (though it seems that this is a slightly more restrictive use of the word than usual, in that the definition of generic in chapter 2 does not appear to exclude this possibility). Thus there are 18 points in the intersection of the two curves, but the intersection at [x:y:t] = [0:0:1] does not correspond to a circle. We therefore need to know the multiplicity of the intersection at that point, and this is shown to be 8 in the lemma below. We are therefore left with 10 (= 18 - 8) solutions (counting multiplicity) with $(x,y) \neq 0$, and the result is proved.

Lemma The two curves in CP^2 given by equations (4.11) have an intersection of multiplicity 8 at [0:0:1], provided the point on the surface is not an umbilic.

Proof: Choose local coordinates in CP^2 about [0,0,1] by putting t = 1. Then by standard algebraic geometry, the multiplicity of the point of intersection is equal to the codimension in $\mathcal{E}_2$ of the ideal

$$< C(x,y)^3 + 3(b - a)xy, \quad D(x,y)^4(x^2 + y^2) - 3(ax^2 + by^2)^3 + (x^2 + y^2)C(x,y)^2(-y,x) + 3(b - a)(x^4 - y^4)>.$$

Routine calculations show that if $b \neq a$ (so x is not an umbilic) then this ideal has codimension 8.

It would be worth knowing whether or not it is possible to realize this number (i.e 10) of circles with A_4 contact at a point on the surface, or whether some of the solutions of (4.10) are forced to be complex. The answer is not clear. That it is possible to have precisely six circles with A_4 contact at a point follows from an article by Blum [B1]. He produces an example of a cyclide with the 6-circle property, i.e. through every point on the surface there pass precisely 6 distinct real circles all lying on the surface. This cyclide is clearly non-generic, but as the set of generic immersions is dense in $Imm(X,R^3)$, we can make an arbitrarily small perturbation to Blum's cyclide to produce a generic surface. Now, any circle actually lying in a surface has infinite order contact with the surface at each point, so will certainly satisfy the A_4 condition expressed by (4.10). Since the 6 real solutions of these equations at each point of Blum's cyclide are distinct, any sufficiently small perturbation of the cyclide will also have A_4 equations with 6 distinct real solutions at each point, and so 6 distinct circles at each point with 5-point contact. It also follows, from proposition 4.12, that the perturbed surface has no umbilics, and so must have the topology of a torus, and thus so does Blum's cyclide. Indeed we can use this argument to show that any surface with the

n-circle property, $4 \leqslant n \leqslant 10$, must have the topology of a torus, and that there is no surface with the n-circle property for $10 < n < \infty$, so giving a partial answer to Blum's conjecture: " there are no surfaces with the n-circle property with $6 < n < \infty$ ".

Many of these circles with at least 5-point contact will in fact have 6-point contact. It is not easy to say very much about these, though we can obtain some information near umbilics using deformation theory. Before that, we need to study how the Σ^2 singularities that we referred to earlier in this chapter can arise. See [L2] for details on maps from R^2 to itself, and their versal deformations.

It is clear from (4.8) that if $d\phi_m(x) = 0$, then $[c_1:s_1]$ and $[c_2:s_2]$ both lie on the normal to the surface at g(x), thus the set m is just {g(x)}, and $m \in M_e \setminus M$. As before there is a redundancy, and we can make choices of these centres, so we choose $[c_2:s_2]$ to be the point g(x) (i.e. $c_2 = s_2 g(x)$), and we let $[c_1:s_1]$ be any other point on the normal at g(x). Now in this case we can choose $g(x) = 0$ and $s_2 = -1$, so $\phi_m(x) = (V(x, [c_1:s_1]) - \rho_1, \ 1/2 \ |g(x)|^2)$. Then,

$$d^2\phi_m = (V_2 , \ dg.dg).$$

If x is not an umbilic then the two quadratic forms that appear are linearly independent with one being positive definite, and it follows that the Σ^2 singularity that occurs is a $I_{2,2}$ in Mather's notation, which has normal form $(x^2 + y^2, xy)$. If, on the other hand, x is an umbilic then the two quadratic forms are linearly dependent and the singularity is of a higher type. Since in this case the one quadratic form that does occur is (positive) definite, it follows that the singularity will be a IV_k for some $k \geqslant 3$, which has normal form $(x^2 + y^2, x^k)$. Since the codimension of such a singularity is 2k, the only possibility for a generic surface is a IV_3 singularity. For this umbilical case we can choose $[c_1:s_1]$ to be the umbilical centre, so $d^2\phi_m = (0, dg.dg)$, so now,

$$d^3\phi_m = (V_3 , \ dg.d^2g).$$

From this we see that ϕ_m has a IV_3 singularity at the umbilic if and only if the cubic form V_3 is not a multiple of the quadratic form dg.dg, that is, we need C to have a non-zero harmonic part. We will see the geomerical significance of this harmonic part in a moment.

Let $f : R^2 \to R^2$ have a IV_3 singularity at 0, and let F be any 6-parameter versal deformation : $R^2 \times R^6 \to R^2$ $(\mathrm{codim}(IV_3) = 6)$. In [L2] Lander shows that there are three non-singular curves in $R^2 \times R^6$ through the origin such that at each point of the curves (apart from the origin) the deformed map has an A_5 singularity. In our application, these A_5 singularities will occur at points in $X \times M_e$ representing genuine circles, i.e. in X x M, as for $m \in M_e \setminus M$, f_m is either non-zero at x or has a Σ^2 singularity there. Thus through any umbilic there are three curves each point of which represents a circle with 6-point contact with the surface. As we approach the umbilic along any of these curves the radius of the circle shrinks to zero, and the plane spanning the circle tends to the tangent plane at the umbilic - or, more graphically, it curls up and dies.

To study this phenomenon of curling up and dying more closely we need to study in the IV_3 singularity in greater depth. Let $f : R^2,o \to R^2,o$ be any map-germ with a IV_3 singularity at 0. Then $df(0) = 0$, and $d^2f(0)$ as a quadratic map : $R^2 \to R^2$ has rank 1. Let $q \in \mathrm{coker}(d^2f)$ be non-zero, so $q.d^2f = 0$, and let $a \notin \mathrm{coker}(d^2f)$ and $B = a.d^2f$.

Definition 4.14 With f etc. as above, the harmonic part of f, denoted H_f, is defined to be the harmonic representative of $q.d^3f$ with respect to B (see appendix 2). Note that this is well-defined despite the choice in a because the harmonic representative of C with respect to tB is independent of t (see proposition A2.1).

Proposition 4.15 Let $f : R^2,o \to R^2,o$ be a map-germ with a IV_3 singularity at 0, then the harmonic part of f is intrinsic: it is a $\mathcal{K}$-invariant cubic form on the tangent space at 0.

Proof: Suppose $g: R^2,o \to R^2,o$ is $\mathcal{K}$-equivalent to f, then $\exists$ h, a diffeomorphism of R^2,o , and $\theta : R^2 \times R^2,o \to R^2,o$ satisfying the usual conditions for $\mathcal{K}$-equivalence, in particular

(4.11) $$g \circ h(x) = \theta(x, f(x)).$$

We want to show that $H_g.dh^3 = H_f$. Differentiating (4.11) twice and thrice, and letting subscripts denote successive differentiation, and A the first derivative of θ with respect to its second argument (so $A : R^2 \to R^2$ is invertible), we get

(4.12) $$g_2.h_1^2 = A.f_2,$$

(4.13) $$g_3.h_1^3 + 3g_2h_1h_2 = 3\theta_2(1,0)(0,f_2) + A.f_3$$

Now f_2 and g_2 are quadratic maps with rank 1, so let $p \in \mathrm{coker}(g_2)$ be non-zero, then from (4.12) $q = p.A \in \mathrm{coker}(f_2)$ and is non-zero. Letting p act on the left on (4.13), we get

(4.14) $$p.g_3.h_1^3 = 3p.\theta_2(1,0)(0,f_2) + q.f_3.$$

Let $B' = a.g_2$ for some $a \notin \mathrm{coker}(g_2)$, and $B = a.A.f_2$, so $B = h_1^*B'$, then the first term on the right hand side of (4.14) is a product of a linear form and B, and so $p.g_3.h_1^3$ and $q.f_3$ have the same harmonic representative (h. rep.) with respect to B.

We now have to show that the h. rep. of $p.g_3.h_1^3$ w.r.t. B is the pull-back under h_1 of the h. rep. of $p.g_3$ w.r.t. B. Consider the linear form $L = -1/8\, \Delta_{B'}(p.g_3)$ (see appendix 2), then define

$$H_g = p.g_3 + LB',$$

which is the h. rep. of $p.g_3$ w.r.t. B', and so the h. part of f. Let $H = H_g.h_1^3 = h_1^*H_g$, then

$$\Delta_B(H) = \Delta_{h\,B'}(h_1^*H_g)$$

$$= h_1^{*}\Delta_{B'}(H_g) = 0$$

from the properties of Δ_B (see appendix 2). So $H = H_f$, the harmonic part of f.

We now examine the standard IV_3 singularity and a versal deformation of it:

$$f(x, y) = (x^2 + y^2, x^3),$$

(4.15)

$$F(x, y, u, v, w, s, l, m) = (x^2 + y^2 + l, x^3 + ux^2 + vxy + wx + sy + m)$$

Then the 3 A_5-curves referred to above are parametrized by:

$$t \mapsto (t, 0, -3t, 0, 3t^2, 0, -t^2, -t^3),$$

$$t \mapsto (t, 3t, -3t, 3\ 3t, -6t^2, -15\ 3/2\ t^2, -4t^2, 43/2\ t^3),$$

$$t \mapsto (t, -\ 3t, -3t, -3\ 3t, -6t^2, 15\ 3/2\ t^2, -4t^2, 43/2\ t^3).$$

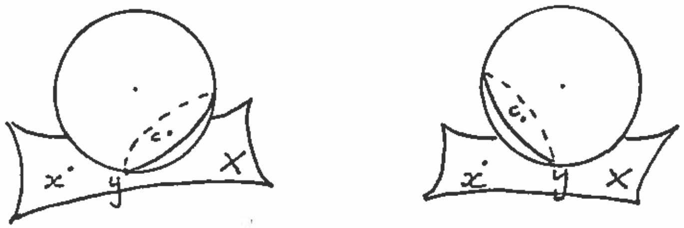
The limiting kernel directions of dF_u as we approach the origin along each of these curves are, (0,1), $(\sqrt{3}, -1)$, and $(\sqrt{3}, 1)$ respectively. Now, the harmonic part of f is $x^3 - 3xy^2$ which has these three directions as root directions. It is not difficult to show that any versal deformation of any IV_3 singularity has this property, namely that the limiting kernel directions of the A_5 singularities coincide with the root directions of the harmonic part of the IV_3.

From this discussion we can deduce the following theorem about the behaviour of circles of A_5 contact with the surface, which we will call <u>A_5-circles</u>, near umbilics. Recall from chapter 3 that the harmonic directions at an umbilic are the root directions of the harmonic representative of the intrinsic cubic C with respect to the first fundamental form I.

Theorem 4.17 Through any generic umbilic, there are 3 curves at each point of which there is an A_5-circle, and as we approach the umbilic along any of these curves, so the tangent directions to the circles approach the harmonic directions at the umbilic. Moreover, the radius of the A_5-circles will tend to zero, and the plane spanning them tends to the tangent plane to the surface at the umbilic.

Proof: We have already seen how a IV_3 singularity will arise for the contact map at an umbilic when the circle is singular. The theorem then follows from the discussion above on the IV_3-singularity. For the last statement it must be remembered that each of these circles lies on its appropriate Meusnier sphere, so the only possible way that the radius can shrink to zero is the one described.

There are two distinct ways in which a circle can 'curl up and die' as it approaches the umbilic x. Let y be a point on one of the A_5 curves near x, and consider the projection of the A_5-circle at y and its tangent line at y on to the tangent plane at x, for y sufficiently close to x this projection will be non-singular. For type (i), this projection of the circle and the point x lie on the opposite sides of the projection of the tangent line, while for type (ii) they lie on the same side, see figure 4(i) below. Equivalently, if c is the centre of the circle, then for types (i) and (ii), (c - y).(x - y) is respectively negative and positive. More graphically these two types could be termed 'slipping down' and 'falling over' respectively.



Type (i) Type(ii)

Figure 4(i) : Curling up and dying.

Theorem 4.18 For a generic immersion, as we approach the umbilic along any of the A_5 curves, the way the circle curls up and dies corresponds to the index of the intrinsic cubic at the umbilic as follows:

Type(i) : Index = 1/2

Type(ii) : Index = -1/2.

Proof: I have only been able to prove this theorem by reversing the rôles of the surface and the circle in expressing the contact, thus each circle will be given by an immersion, while the suface will be cut out by a function, so each contact map will be a map from R to itself. From Lemma 1.7 this will not alter the problem. Let the umbilic be at the origin in R^3, and the surface be given by

$$f(x, y, z) = h(x, y) - z = 0,$$

with h as in the proof of theorem 4.13. We will still write g(x) = (x, h(x)). The A_5-circle at g(x) can be immersed by

$$r_x(t) = g(x) + (a,a_3)\sin(t) + (b,b_3)(1 - \cos(t)),$$

where $a,b \in R^2$, $(b,b_3) = c - g(x)$, and (a,a_3) is the vector product of (b,b_3) with the unit normal to X at g(x). This circle has centre c and

tangent vector (a,a_3) at g(x) (we do not complicate matters further by considering c as a point in projective space since for this theorem c cannot be at infinity). The contact map is then

$$f \circ r_x(t) = h(\, a.\sin(t) + b(1 - \cos(t))\,) - a_3\sin(t) - b_3(1 - \cos(t)).$$

The A_5 condition is that the first 5 derivatives vanish at t = 0, so

A_1: $dha - a_3 = 0,$

A_2: $d^2ha^2 + dhb - b_3 = 0,$

A_3: $d^3ha^3 + 3d^2hab = 0,$

A_4: $d^4ha^4 + 6d^3a^2b + 3d^2hb^2 - 3d^2ha^2 = 0,$

A_5: $d^5ha^5 + 10d^4ha^3b + 15d^3hab^2 - 10d^3ha^3 - 15d^2hab = 0,$

where all derivatives are at x.

Now, as the circle curls up and dies, so c, and hence (a,a_3) and (b,b_3), tend to 0. Let u and v be the limiting directions of (a,a_3), (b,b_3) respectively, then from the A_1 and A_2 conditions and the fact that by definition (a,a_3) and (b,b_3) are always perpendicular, we obtain that u and v are tangent to the surface at 0, that u.v = 0 and that $|u| = |v|$, and we can put them = 1. It is not difficult to show that differentiating the A_3, A_4 and A_5 conditions three times each and evaluating at 0, we get

$$(4.16) \quad \begin{cases} Cu^3 + 3Cuv\hat{x} = 0, \\ 2Cu^2v + Cv^2\hat{x} - Cu^2\hat{x} = 0, \\ 3Cuv^2 - 2Cu^3 - 3Cuv\hat{x} = 0. \end{cases}$$

Eliminating $Cuv\hat{x}$ between the first and third of these we get,

$$Cu^3 - 3Cuv^2 = 0,$$

which can be shown to be precisely the condition that u be a harmonic direction at the umbilic, providing an alternative to part of the proof of theorem 4.17.

For this theorem we are interested in the sign of $\hat{x}.v$ since this determines the limiting sign of $(c - y).(x - y)$ as in the definition: for type (i) $\hat{x}.v > 0$, for type (ii) $\hat{x}.v < 0$. Refer to the end of appendix 2 for the definition of the index of a cubic form. To use this definition we need to express C in the form of (A2.3), and choose coordinates in the plane so that $\alpha = 1$ (we are assuming $\alpha \neq 0$, as otherwise C has zero harmonic rep. w.r.t. I), and let $\beta = s + i\,t$. Then, for $u = (x, y)$ and $v = (-y, x)$,

$$Cu^3 = (1 + 3s)x^3 + 3tx^2y + (s - 1)xy^2 + 3ty^3,$$

and

$$Cu^3 - 3Cuv^2 = 4x(x^2 - 3y^2)$$

which is zero for $u = (0, 1), (-1, -\sqrt{3}), (-1, \sqrt{3})$. Without loss of generality we only look at $u = (0, 1)$ (the others follow either by similar arguments or by rotating the x-y plane so that they are each (0, 1) in turn). So $v = (1, 0)$, and in this case equations (4.16), with $\hat{x} = (x, y)$, become

$$tx + (s - 1)y = t,$$

$$(s + 1)x - ty = s - 1.$$

Solving these we get $\hat{x}.v = x = [t^2 + (s - 1)^2]/(s^2 + t^2 - 1)$, thus for C not orthogonal, so $s^2 + t^2 \neq 1$, we get $\hat{x}.v > 0$ for $s^2 + t^2 > 1$ (so index = 1/2) and $\hat{x}.v < 0$ for $s^2 + t^2 < 1$ (so index = -1/2).

A Generalization of a theorem of Banchoff Gaffney & McCrory

To conclude this chapter we present a generalization of the central theorem in the book, 'Cusps of Gauss Mappings' by Banchoff, Gaffney and McCrory, [B-G-M]. The philosophy behind this generalization is as follows, the Gauss map gives information on the 'flatness' of the surface, for example it is singular at points on the classical parabolic curve, while as we have seen in chapter 3, many of the properties of 'flatness', or zero curvature, are just special cases of phenomena at points with arbitrary curvature.

Let $g : X \hookrightarrow R^3$ be an immersion of a surface X, then the Gauss Map of g is the map $G : X \longrightarrow S^2$ - the unit sphere in R^3 - associating to each point x on X the outward normal to the surface at g(x) (if X is orientable this can be defined globally, otherwise only locally, or else with RP^2 replacing S^2). The Gauss map is singular at points where one of the principal curvatures is zero - the classical parabolic curve. Denote by $g_t : X \longrightarrow R^3$ the parallel map, $g_t(x) = g(x) + tG(x)$, its image being the parallel surface to X at distance t, and it is well known that g_t is singular at x if and only if $g_t(x)$ is a focal point at x.

First we state the original theorem, using our notation, and then our generalization.

Theorem (Banchoff, Gaffney and McCrory.)

If the Gauss map G of g has a cusp at $x \in X$, then the following statements (a)-(h) are true. Conversely, if G is stable and any of the following conditions hold, then G has a cusp at x.

(a) A zero principal curvature direction of g is tangent to the parabolic curve at x.

(b) For each $\epsilon > 0$ there exist 3 points $y_1, y_2, y_3 \in X$ such that for i = 1,2,3 $|y_i - x| < \epsilon$ and the tangent planes to X at $g(y_i)$, i = 1,2,3, are parallel.

(c) For each $\epsilon > 0$ there exist 2 points $y_1, y_2 \in X$ such that for i = 1, 2, $|y_i - x| < \epsilon$, and the tangent planes to X at $g(y_1)$ and $g(y_2)$ are equal.

(d) x is a ridge point of g as well as a parabolic point, and the principal curvature associated to the ridge is zero at x.

(e) Given $D, \varepsilon > 0$, there exists a $d > D$ and a point $y \in X$ such that $|y - x| < \varepsilon$ and y is a swallowtail point of the parallel map g_d.

(f) For any point $z \in R^3$ which is not on the tangent plane to X at g(x), the point x is a swallowtail point of the pedal surface of X from z.

(g) x is a parabolic point of g and a line in R^3 has at least 3-point contact with X at x.

(h) x is a parabolic point of g, and given any $\varepsilon > 0$ $\exists\, y \in X$ such that $|y - x| < \varepsilon$ and y is an inflexion point of an asymptotic curve of g.

If the image of the parabolic curve of g in X has non-zero-curvature at x, then condition (i) can be included in the theorem, and if the asymptotic direction map along the parabolic curve of g is regular at x then so can condition (j), where

(i) The osculating plane at g(x) of the image of the parabolic curve of g is the tangent plane of X at g(x).

(j) x is an inflexion point of the asymptotic map of g.

Note that in [B-W] Bleeker and Wilson show that for an open dense subset of Imm(X, R^3) the Gauss map is indeed stable.

The generalization does not make use of the Gauss map as such, though one could construct, at least locally, a sort of 'section map' which would be a finite analogue of the Gauss map. For our purposes we extend the parallel maps to include that at infinity, so we let $t = [a:b] \in RP^1$, and put $g_t(x) = [ag(x) + bG(x): a]$, so for $t = [0:1]$ this is just the Gauss map. I have been unable to find the right generalization of parts (f), (h) and (j), though it would be particularly interesting to find the generalizations of (h) and (j). Much of the proof is fairly standard, so we only give an outline.

Theorem 4.19 Let g be a generic immersion. Let $x \in X$ not be an umbilic, and suppose that κ is one of the principal curvatures, with associated principal direction u and focal point [c:s], then the following statements are equivalent.

(a) u is tangent to the curve of constant principal curvature κ through x, or $d\kappa(x) = 0$.

(b) For each $\varepsilon > 0$ there exist 3 points $y_1, y_2, y_3 \in X$ such that for i = 1,2,3 $|y_i - x| < \varepsilon$, and there are 3 concentric spheres tangent to X at $g(y_i)$, i = 1,2,3, respectively.

(c) For each $\varepsilon > 0$ there exist 2 points y_1, y_2 X such that $|y_i - x| < \varepsilon$, i, = 1,2, and there is a sphere tangent at both points.

(d) x is a ridge point of g and the principal curvature associated to the ridge is κ, with principal direction u.

(e) Let t be such that $g_t(x) = [c:s]$, then x is a swallowtail point of g_t.

(g) There is a circle in R^3 with tangent vector dgu at x which has at least 4-point contact with X at g(x).

(i) For a curve on the surface with tangent vector dgu at g(x) the osculating sphere is tangent to X at g(x) (and so must be M_u, the only focal sphere tangent to X). If this is true for one such curve it follows that it must hold for all such curves.

Proof: We show that each statement is equivalent to (d).

(a): Proved in theorem 3.8.

(b): We saw in proposition 4.2 that (d) is equivalent to V(,[c:s]) having an A_3 singularity (or worse) at x, and the result then follows from standard catastrophe theory, as in the original theorem.

(c): Let $m = [c:s:\rho]$ for the appropriate value of ρ. Then from proposition 4.2 (d) is equivalent to ϕ_m having an A_3 singularity (or worse) at x. Now the standard $A_3^{\pm}$ singularity is $(x,y) \mapsto x^2 \pm y^4$. Now the A_3^- singularity can be deformed to the function

$$x^2 - y^4 + 2ty^2 - t^2 = (x - y^2 + t)(x + y^2 - t)$$

which, for each $t > 0$, has as zero-set 2 curves which intersect at $(x, y) = (0, \pm\sqrt{t})$, giving 2 A_1^- singularities which represent tangencies with the nearby spheres. Similarly, the A_3^+ singularity can be deformed to

$$x^2 + y^4 - 2ty^2 + t^2 = x^2 + (y^2 - t)^2$$

which, for each $t > 0$, has zero-set consisting of 2 isolated points, giving 2 A_1^+ singularities also representing bitangent spheres. If instead the contact type were A_4 the same result would hold, since there would be A_3's arbitrarily nearby. For the converse it is enough to observe that in versal deformations of contact types of lower codimension, these phenomena do not occur.

(e): The condition for g_t to have a swallowtail singularity at x is similar to the usual A_3 condition, that is, for t = [a:b],

$$dg_t u = d^2 g_t u^2 + dg_t v = 0.$$

Writing this out in terms of g and G, and using the fact that $dG\hat{x} = \hat{n}$ and the properties of $\hat{n}$ (cf. proposition 3.4), the first equation tells us that u is indeed the principal direction, and the second becomes $V_3u^2 + V_2v = 0$, which is the A_3 condition for contact, so is equivalent to (d).

(g): This follows from proposition 4.11, for if u is a principal direction and x is not a ridge point, then there are no circles with 4-point contact, while if x is a ridge point then part (ii) ensures that all the circles with tangent vector u lying on M_u have 4-point contact with X at g(x).

(i): From proposition 4.9 we see that the point [c:s] is on the focal line of every curve on X through g(x) with tangent vector u. The condition that [c:s] be the centre of the osculating centre (the centre of spherical curvature) is then (cf. (3.6)) $V_3u^3 + 3V_2uv + V_1w = 0$, but $V_1 = V_2u = 0$ (since [c:s] is a focal point) and the condition reduces to $V_3u^3 = 0$, which is precisely the A_3 condition for [c:s].

CHAPTER 5 GENERIC SURFACES IN 4-SPACE

As with chapter 4, this chapter is divided into 2 sections, the first on contact of surfaces with hyperspheres (spheres of dimension 3), and the second on contact with 2-spheres, and as always, a surface is a two-dimensional submanifold. In [P2] Porteous presents some results for surfaces in 4-space and the distance-squared function (which is equivalent to contact with hyperspheres), and the object of the first section is to consolidate these existing results as well as taking the study further. The second section is, on the other hand, entirely new and gives a nice picture of how the $\mathcal{K}$-classes of maps from R^2 to itself fit together, particularly the Σ^2 maps. (It is also interesting to see how the two sections inter-relate, and I feel that much of this relationship could be explained at the singularity theory level, though except in one instance, I do not do this).

CONTACT WITH HYPERSPHERES

The problem for this section is set up in exactly the same way as that for surfaces in R^3 and their contact with spheres. Thus we refer to the first part of chapter 4 for the definitions of M_e, M, f_m, $\mathscr{F}$, V, with RP^4 and RP^5 replacing RP^3 and RP^4 respectively, and the same remark applies in this case that the probe-structure of ϕ_m at x is the same as that of V(, [c:s]), with m = [c:s:ρ] and ρ = V(x, [c:s]).

We start with the second order proderties of the contact, we will refer to the last section of chapter 3 for an account of the terms involved. Let $m \in M_e$ and ρ = V(x, [c:s]), and let $g : X \hookrightarrow R^4$ be a generic immersion.

Proposition 5.1 The map ϕ_m has an A_1, A_2, or D_4 (or worse) singularity at x if and only if [c:s] is in the normal plane, the focal set, or x is a semiumbilic with centre [c:s], respectively.

Proof: The conditions for A_1, A_2 and D_4 (or worse) are precisely those appearing in the definitions of N, F and the semiumbilical centre in

(3.21), (3.22), and (3.24) respectively.

Let us first consider the case where x is a semiumbilic, with centre [c:s]. The only possibilities for the singularity type of ϕ_m are then D_4 and D_5, anything worse (D_6, E_6, etc.) being of too high codimension. In the first part of chapter 4 we discussed briefly the D_4^+ and D_4^- singularities, where the intrinsic third derivative is hyperbolic and elliptic respectively. The transitional - parabolic - case corresponds to the D_5 singularity, which has normal form $f(x, y) = x(x^3 + y^2)$, and its zero-set is of the form  , the union of a simple cusp and a non-singular curve transverse to the limiting tangent direction at the cusp. A versal deformation of f can be given by,

$$F(x, y, s, t, u, v, w) = f(x, y) + sx^2 + ty^2 + ux + vy + w.$$

The D_4 set is then the smooth curve $r \mapsto (r, 0, -6r^2, -r, 8r^3, 0, -3r^4)$, with D_4^+ for $r > 0$, and D_4^- for $r < 0$ ($r = 0$ being the D_5 point). This is a non-singular curve through the origin, and it follows that for any versal deformation of any D_5 singularity there is a non-singular curve through the D_5 point along which the singularity is a D_4.

The following theorem is proved in [B-M], though their open dense set of immersions does not necessarily coincide with our set of generic immersions. I feel it is worth proving here with the techniques we have developed, though to complete this proof we need to anticipate a result of the next section (see remark 5.8), namely that for a generic immersion E does not degenerate to a point.

Theorem 5.2 For a generic surface in R^4, the set of semiumbilics is a one-dimensional submanifold of X.

Proof: At each semiumbilical centre (x, [c:s]) in the normal bundle, V(, [c:s]) will have either a D_4 or a D_5 singularity at x (nothing of higher codimension can occur for a generic immersion). The genericity

hypothesis ensures that these singularities are presented transversely, and it follows from the standard deformation of a D_5, given above, that the semiumbilical centres occur along a one-dimensional submanifold of the normal bundle. There now remains to show that the projection of this curve down on to the surface is non-singular.

At each point of this curve (x, [c:s]) we have,

$$V_1 = (c - sg(x)).dg = 0, \tag{5.1}$$
$$V_2 = (c - sg(x)).d^2g - sdg.dg = 0.$$

The tangent vector $(\hat{x}, (\hat{c},\hat{s}))$ then satisfies

$$(\hat{c} - \hat{s}g(x)).dg = 0,$$

$$V_3\hat{x} + (\hat{c} - \hat{s}g(x)).d^2g - \hat{s}dg.dg = 0.$$

The projection on to the surface is then singular if $\hat{x} = 0$, $(\hat{c},\hat{s}) \neq 0$. It then follows that $[\hat{c}:\hat{s}]$ is another semiumbilical centre at x (different from [c:s] as it is tangent to RP^4 at [c:s]), and this can only occur if the focal set is a double line, so the curvature ellipse is an isolated point. We will see in remark 5.8 that this cannot occur for a generic immersion.

As for surfaces in R^3 (in chapter 3), let $\tilde{N} = N \oplus S^1X$, and $\tilde{F}$ be the 'blow up' of the focal set, so

$$\tilde{F} = \{ (x, [c:s], u) \in \tilde{N} : V_2(x, [c:s])u = 0 \}.$$

Recall that in proposition 3.1 we showed that for surfaces in 3-space the set F is a submanifold of N if the intrinsic cubics at the umbilics are not orthogonal. We now prove the analogous result for surfaces in 4-space.

Proposition 5.3 Let $\tilde{N}$ and $\tilde{F}$ be as above. Then $\tilde{F}$ is a submanifold if the

Hessian directions of the intrinsic cubic at each semiumbilical centre do not coincide with the asymptotic directions at x. (Note that at a semiumbilic the asymptotic directions are orthogonal, so this condition is not unrelated to the one for surfaces in 3-space.)

Proof: The proof follows that of proposition 3.1 obtaining the same equations (i), (ii), (iii), and (iv). If (x, [c:s]) in F is not a semiumbilical centre then the proof is completed as in proposition 3.1. If it is a semiumbilical centre, then we still obtain a = 0 (having put the immersion in Monge form), and the first three equations become

(i) $V_3ub = 0$,

(ii) $II(u,b) = 0$,

(iii) $dgu.dgb = 0$.

If $b \neq 0$, then the first equation requires that u and b are the Hessian directions, and the second that k(u) and k(b) are the end points of E (consequently u and b are orthogonal, and equation (iii) is redundant).

Remarks: (i) If we have a D_5 singularity at (x, [c:s]) then $V_3u^2 = 0$, and the Hessian directions coincide. Thus condition (i) in the proof implies that b is a multiple of u (unless $V_3u = 0$, which is the E_6 condition, and is therefore non-generic), which with (ii) implies that b = 0. Consequently at a D_5, F cannot be singular.
(ii) If the singularity at the semiumbilical centre is a D_4 and not a D_5, then the condition for $\tilde{F}$ to be a submanifold of $\tilde{N}$ given in the proposition is the same as the condition for the singularity to be presented transversely (cf. proposition 4.5). However, when the singularity is a D_5, the condition for transverse presentation involves the 4-jet of the immersion and so cannot be the same as the condition above (which in this case is empty, as pointed out in (i) above).

We now turn to the third order properties of the contact. Recall

from the last part of chapter 3 the definition of the space curve r_u for a given tangent vector u at x, and its focal line F_u, and note equation (3.6) for its centre of spherical curvature at g(x).

Proposition 5.4 Let [c:s] be a focal point at x, but not a semiumbilical centre. Then the corresponding hypersphere, centre [c:s] through g(x), has at least A_3 contact with the surface at g(x) if and only if [c:s] is the centre of spherical curvature of r_u, for the appropriate u.

Proof: The A_3 condition is $V_1 = V_2u = 0$, $V_3u^3 = 0$ for some u, but $V_2 \neq 0$. Now r_u is a curve on the surface X, so $r_u = g\circ\gamma$, for some curve γ with tangent vector u at x. Then (omitting the subscript u) $r_1 = dgu$, $r_2 = d^2gu^2 + dg\,\gamma_2$, and $r_3 = d^3gu^3 + 3\,d^2gu\,\gamma_2 + dg\,\gamma_3$. From (3.6), [c:s] is the centre of spherical curvature of r_u if and only if,

$$(c - sg(x)).(d^3gu^3 + 3d^2guv + dgw) - 3sdgu.(d^2gu^2 + dgv) = 0,$$

where $v = \gamma_2$, and $w = \gamma_3$. But since $V_1 = V_2u = 0$, this becomes,

$$(c - sg(x)).d^3gu^3 - 3sdgu.d^2gu^2 = 0,$$

which is precisely the A_3 condition $V_3u^3 = 0$.

Since for any point on the surface there is a one dimensional family of 3-spheres with A_2-contact with the surface at that point, we expect there to be finitely many spheres with A_3-contact. How many?

Theorem 5.5 Let $x \in X$, and $g : X \hookrightarrow R^4$ be a generic immersion. Then there are at most five 3-spheres with at least A_3 contact with X at g(x).

Proof: Express the immersion in Monge form at x:

$$g(x, y) = (x, y, h_1(x, y), h_2(x, y)),$$

with $h_i(x, y) = 1/2\, B_i(x, y)^2 + 1/6\, C_i(x, y)^3 + H_i(x, y)$, where the H_i have vanishing 3-jet at 0. For the A_1 condition to be satisfied we need [c:s] to be in the normal plane, so let [c:s] = [(p,q):s]. Then the A_2 condition becomes,

$$pB_1u + qB_2u - su = 0.$$

Let this act on u and v respectively, with v orthogonal to u. We get

(i) $$pB_1u^2 + qB_2u^2 - su.u = 0,$$

(ii) $$pB_1uv + qB_2uv = 0,$$

and for the A_3 condition,

(iii) $$pC_1u^3 + qC_2u^3 = 0.$$

Given p, q and u, (i) can then be solved for s. Now eliminating p and q between (ii) and (iii) we get

(5.2) $$B_1uvC_2u^3 - B_2uvC_1u^3 = 0$$

as the A_3 condition. This is a quintic equation in u, and the condition that this is identically zero is of codimension at least 3 so will not occur for generic surfaces (as in theorem 4.13 this seems to be a slightly more special meaning of the term generic).

Remarks 5.6 (i) It is easy to see that equation (5.2) can be any quintic equation, so in particular 5 real roots is possible.
(ii) If we denote $P_x(d^3g)$ by III (cf. definition of II in (3.19), though it should be pointed out that this is not standard notation, and III usually has a different meaning), then (5.2) becomes,

$$II(u,v) \times IIIu^3 = 0,$$

where x is the usual vector product (or determinant) in the plane. At a semiumbilic there are 2 asymptotic directions - corresponding to the end-points of E - and these directions u satisfy II(u,v) = 0, thus the asymptotic directions satisfy (5.2) and so give rise to spheres with A_3 contact. In fact this could have been predicted from proposition 5.4 since for the asymptotic directions the entire focal line F_u is contained in the focal set F. We can also see from proposition 5.4 that if u is not asymptotic then the only candidate for A_3 contact is the 'semiumbilical' sphere, i.e. that with D_4 contact or worse, which we do not usually class as having A_3 contact. There is one possibility that has been omitted, namely what happens when the centre of spherical curvature of r_u for u asymptotic coincides with the semiumbilical centre? Well, as far as contact with 3-spheres is concerned, it means that the sphere with A_3 contact coincides with that with D_4 contact (or worse), and so the number of A_3 contact spheres drops by 1. We return to this point at the end of the chapter and find that this phenomenon gives rise to a special type of contact with 2-spheres.

CONTACT WITH 2-SPHERES

The problem of contact of surfaces in R^4 with 2-spheres is set up in much the same way as that for surfaces in R^3 and their contact with circles. However, the presentation is simplified if we represent each 2-sphere as the intersection of a 3-sphere and a hyperplane through its centre, rather than as the intersection between two 3-spheres. We will be dealing almost exclusively with Σ^2 contact in this part of the chapter, and we refer to appendix 1 for the classification of the necessary maps. On page 104, at the end of the chapter, figures 5(i) and 5(ii) show the relationship between points in the normal space and the contact type of the associated 2-spheres.

For this section we put,

$$M_e = \{([c:s:\rho], (n,h)) \in RP^5 \times S^2 \times R : (c,s) \neq 0, n.c = sh\},$$

then for $m \in M_e$, the map $f_m: R^4 \longrightarrow R^2$ is defined by

$$f_m(y) = (c.y - 1/2\, s|y|^2 - \rho \;,\; n.y - h).$$

As in chapter 4 the set M is the subset of M_e for which $f_m^{-1}(0)$ is a genuine 2-sphere in R^4, and as usual 'sphere' includes 'plane'.

Given an immersion $g : X \hookrightarrow R^4$ we have the usual maps $\bar{\phi}$ and ϕ_m, and the contact between X and m at g(x) is determined by the $\mathcal{K}$-class of ϕ_m, and we continue to refer freely to appendix 1 for the classification of maps from R^2 to itself.

Now,

$$\phi_m(x) = (c.g(x) - 1/2\, s|g(x)|^2 - \rho \;,\; n.g(x) - h),$$

so

$$(5.3) \quad \begin{cases} d\phi_m(x) = ((c - sg(x)).dg_x \;,\; n.dg_x), \\ d^2\phi_m(x) = ((c - sg(x)).d^2g_x - sdg_x.dg_x \;,\; n.d^2g_x). \end{cases}$$

To find the contact types of various 2-spheres we first treat the Boardman symbol of the map ϕ_m at a point $x \in \phi_m^{-1}(0)$. We have pointed out before that the kernel of $d\phi_m$ represents the intersection of the tangent spaces, so ϕ_m has a Σ^2 singularity if and only if X and m have the same tangent plane at g(x). Let us now consider the set of 2-spheres which have Σ^2 contact at g(x). First note that for $d\phi_m(x) = 0$, both [n:0] and [c:s] are in the normal plane at g(x), and secondly that $\phi_m(x) = 0 \Rightarrow$ c is in the plane $n.y = h$, so $n.(c - sg(x)) = 0$, so in the normal plane the vectors n and [c:s] are orthogonal and if [c:s] is fixed then n is determined up to a sign. We can therefore consider the set of 2-spheres with Σ^2 contact at g(x) to be parametrized by $[c:s] \in N_xX$.

We now turn to the second order Boardman symbol. Recall that a map $\phi: R^2 \to R^2$ has a $\Sigma^{2,i}$ singularity if $d\phi = 0$ and $d^2\phi$ as a map from R^2 to itself has kernel rank i.

Proposition 5.7 Suppose ϕ_m has a Σ^2 singularity at x, then
(i) ϕ_m has a $\Sigma^{2,2}$ singularity at x if and only if the curvature ellipse E at x degenerates to a point, and the point [c:s] is the inverse of that point (i.e. is on the centre set C)
(ii) ϕ_m has a $\Sigma^{2,1}$ singularity at x if and only if the curvature ellipse E at x is a line segment and the point [c:s] is the inverse of one of the end-points of E (i.e. is one of the end-points of the real part of C - see the end of chapter 3).

Proof: (i) For simplicity let us take $g(x) = 0$, then

$$\phi_m \text{ has a } \Sigma^{2,2} \Leftrightarrow d^2\phi_m = 0 \Leftrightarrow c.d^2g - sdg.dg = n.d^2g = 0.$$

Now $n.d^2g = 0 \Rightarrow \text{rank}(k) < 2$ (since $n \neq 0$) which is equivalent to E being contained in a line through the origin in the normal plane. However, as $c.n = 0$, c is on that same line, and so $c.k(u) = s$ for all unit vectors u can be satisfied if and only if E is a single point and [c:s] is the inverse of that point.

(ii) Again, take $g(x) = 0$. Then,

$$\phi_m \text{ has a } \Sigma^{2,1} \Leftrightarrow d^2\phi_m u = 0 \text{ for some unique (up to sign) unit tangent vector } u$$

$$\Leftrightarrow n.d^2gu = c.d^2gu - sdgu.dg = 0.$$

This is equivalent to $n.II(u,v) = c.II(u,v) = 0$, and $n.k(u) = 0$, $c.k(u) = s$. Since n and c are linearly independent (from the last equality, $c \neq 0$) these are in turn equivalent to $II(u,v) = 0$ and [c:s] is the inverse of k(u).

Remarks 5.8 (i) The codimension of the $\Sigma^{2,2}$ set in $J^r(X, R^2)$ is 12, while $\dim(X \times M_e) = 10$, so for a generic immersion g the family of maps ϕ_m will not exhibit a $\Sigma^{2,2}$ singularity, and consequently from the proposition for a generic immersion the curvature ellipse never degenerates to a point, a result we anticipated in theorem 5.2.

(ii) We see from part (ii) of the proposition that the $\Sigma^{2,1}$ locus in X x M_e is a double cover of the curve of semiumbilics. The two points in the normal space at a semiumbilic are always distinct - for a generic immersion - since otherwise the two end-points of E would have to coincide, giving a point. See figure 5(ii) on p.104.

For all pairs ([n:0], [c:s]) in the normal space with $n.(c - sg(x)) = 0$ other than those at the end-points of the real part of C, ϕ_m has a $\Sigma^{2,0}$ singularity at x. In [M6] Mather classifies such maps dividing them into 5 infinite families. Two of the families ($III_{a,b}$ and V_a) only occur for maps from R^n to R^p with $n < p$, while the other 3 ($I_{a,b}$, $II_{a,b}$ and IV_a) can occur for $n = p$ (= 2 in our case). See appendix 1 for details on these maps.

The first $\Sigma^{2,0}$ maps to occur are $I_{2,2}$ and $II_{2,2}$, collectively referred to as $B_{2,2}$ (from the complex classification). For a generic immersion the set $\{(x, m) : \phi_m \text{ is } \Sigma^2 \text{ but not a } B_{2,2} \text{ at } x\}$ is of codimension 1 in the Σ^2 set. Recall that the 2-spheres with Σ^2 contact at x are parametrized by [c:s], n then being determined up to a sign.

Proposition 5.9 Suppose ϕ_m has a Σ^2 singularity at $x \in X$, then it is a $B_{2,2}$ if and only if the appropriate point [c:s] is not on the centre set C.

Proof: By hypothesis, $d\phi_m = 0$ (at x), so [c:s] and [n:0] are in the normal plane (at x). Now, ϕ_m is not $B_{2,2}$ if and only if $\exists u \neq 0$ (possibly complex) such that $d^2\phi_m u^2 = 0$, which with $g(x) = 0$ becomes,

$$c.d^2gu^2 - sdgu.dgu = 0,$$

(5.4)

$$n.d^2gu = 0.$$

First suppose u is real. Since $n.c = 0$, we see that $c = stk(u)$ for some $t \in R$ (provided $k(u) \neq 0$), and it follows that for $|u| = 1$, $t = |k(u)|^{-2}$. If $k(u) = 0$, then $s = 0$ and [c:0] is then not determined, and we can choose [c:0] to be the point at infinity in C. In this case m is the tangent plane to X at g(x). If, on the other hand, u is not real but k(u) is, then E must be degenerate and the image of this formal extension of k will be the affine line in the normal bundle containing E. The point [c:s] will then lie on the inverse of the point k(u), that is on C.

To distinguish between the two cases $I_{2,2}$ and $II_{2,2}$ note that for $c = sg(x)$ (i.e. [c:s] at the origin in N_xX),

$$d^2\phi_m = (-sdg.dg, \ n.d^2g).$$

If E is non-degenerate the resulting pencil of quadratic forms then contains an elliptic form (namely dg.dg), so the singularity of ϕ_m is of type $I_{2,2}$. It follows that if [c:s] is in the same connected component of the complement of C in N_xX as the origin then ϕ_m has a $I_{2,2}$ singularity, since the only way for a $I_{2,2}$ to change to a $II_{2,2}$ is to go through a worse singularity, for example a $I_{2,3}$. We will see that if E is non-degenerate then for [c:s] in the other component ϕ_m has a $II_{2,2}$. If E is degenerate - so x is a semiumbilic - every point not on C gives rise to a $I_{2,2}$, as we will see when we consider IV_3 singularities. See figures 5(i) and 5(ii) on p.104.

There is a nice geometrical result which ties together the 2-sphere m associated to a point [c:s] in C and the space-curve r_u and its focal line F_u. Note that any 2-sphere in R^4 has an axis, as for a circle in R^3.

Proposition 5.10 Let $x \in X$, and [c:s] be the inverse of k(u) for some unit tangent vector u at x (so if x is a semiumbilic, [c:s] is on the real part of C), and let V_u, r_u and F_u be as usual (see chapter 3), then
(i) F_u is the axis of m in R^4, and
(ii) the osculating circle of r_u at x is $m \cap V_u$.

Proof: (i) Let [n:0] be perpendicular to [c:s] as usual, then [c:s] and [n:0] satisfy (5.4). The set [tc + (1 - t)n: ts] as t varies is therefore the axis of m, and is also F_u.
(ii) Let S denote the osculating circle of of r_u. Now, F_u is the axis of S in V_u, and since n.c = 0 (taking g(x) = 0) m and S have the same centre, namely [c:s]. Further, both m and S lie in the subspace orthogonalto n, and m is the intersection of the 3-sphere centre [c:s] with this subspace, consequently S is contained in m. Since m and X are tangent at g(x) $V_u \bar{\pitchfork} X \Rightarrow V_u \bar{\pitchfork} m$ so V_u intersects m in a circle which therefore must be S.

We have seen that for m tangent to X at g(x), m has contact type $B_{2,2}$ ($I_{2,2}$ or $II_{2,2}$) if and only if [c:s] $\notin$ C, the centre set. We now turn to higher contact types and we will take, from now on , [c:s]$\in$ C. Broadly we divide the $B_{a,b}$, $b \geqslant a \geqslant 2$, $b > 2$, into 2 families, a = 2 and a > 2.

Proposition 5.11 The contact of m and X at g(x) is of type $B_{a,b}$ with a > 2 if and only if x is a semiumbilic and [c:s] $\in$ C is not an end-point of the real part of C (if it were then the contact would be $E_{3,3}$ or worse, see

proposition 5.7).

Proof: ϕ_m has a $B_{a,b}$ with a > 2 (or worse) iff the pencil of quadratic forms is singular, i.e. $d^2\phi_m$ has rank 1. Let $(q_1,q_2) \neq 0$ be a cokernel vector of $d^2\phi_m$, so from (5.3) with g(x) = 0,

$$(q_1c + q_2n).d^2g - q_1sdg.dg = 0.$$

It follows that $[q_1c + q_2n : q_1s]$ is the semiumbilical centre, see (5.1), so x is a semiumbilic.

Remark 5.12 The first $B_{a,b}$, a > 2, to appear (i.e. of least codimension) are the $I_{3,3}$ and the IV_3 (both of codimension 6). In these cases the single quadratic form in $d^2\phi_m$ is respectively hyperbolic and elliptic - if it is parabolic the singularity would be an $E_{3,3}$ (codimension 7) or worse. Consider the $I_{3,3}$ case. Let u and u' be the 2 distinct roots of the hyperbolic quadratic form, so

$$d^2\phi_m u^2 = d^2\phi_m u'^2 = 0.$$

It follows that [c:s] is the inverse of both k(u) and k(u'), and so is on the real part of C. The IV_3 singularity then occurs for the [c:s] on the imaginary part of C.

In versal deformations of a IV_3 singularity, the only Σ^2 singularities to occur are $I_{2,2}$'s (never $II_{2,2}$'s), consequently for a semiumbilic, [c:s] $\notin$ C $\Rightarrow$ ϕ_m has a $I_{2,2}$ singularity, as anticipated in the discussion following proposition 5.9.

By exclusion, the only possibility for $B_{2,b}$, b > 2, is for [c:s] $\in$ C, but x not a semiumbilic. The first $B_{2,b}$, b > 2, to occur is the $I_{2,3}$ (of codimension 5) which in the versal deformation separates the $I_{2,2}$ and the $II_{2,2}$ singularities. It follows that for x not a semiumbilic, one of the connected components of $N_xX \setminus C$ gives $I_{2,2}$ contact while the other gives $II_{2,2}$ contact. We saw in the discussion following proposition 5.9 that the component containing the origin gives $I_{2,2}$.

We now have a complete picture of the second-order properties of the contact, so we turn to the higher-order geometry and contact of type $B_{a,b}$ with $b > 3$. Recall that on the focal line F_u of r_u there lies the centre of spherical curvature of r_u. This next result should be compared to proposition 5.4.

Proposition 5.13 (i) Suppose x is not a semiumbilic, and $[c:s] \in C$ is the inverse of $k(u)$. Then the associated 2-sphere m has at least $B_{2,4}$ contact if and only if the centre of spherical curvature of r_u lies on F.
(ii) If x is a semiumbilic, and $[c:s] \in C$ is the inverse of $k(u)$ and $k(u')$ $(u \neq u')$ - so it is not an end-point of the real part of C - then m has at least $B_{3,4}$ (resp. $B_{4,4}$) contact with X at g(x) iff one of (resp. both of) the centres of spherical curvature of r_u and $r_{u'}$ lies (lie) at the semiumbilical centre.

Proof: The conditions for a $B_{2,4}$ singularity for ϕ are: $\exists u \neq 0$, v such that $d^2\phi u^2 = 0$, $d^3\phi u^3 + 3d^2\phi uv = 0$. For a $B_{3,4}$ we also require that $\exists u' \neq 0$ and $d^2\phi u'^2 = 0$, and for a $B_{4,4}$ in addition $\exists v'$ such that $d^3\phi u'^3 + 3d^2\phi u'v' = 0$.

Consider $d^2\phi u \in L(R^2, R^2)$. Since $d^2\phi u^2 = 0$, $d^2\phi u$ has rank 1. Let $q = (q_1, q_2)$ be a non-zero cokernel vector, so $q.d^2\phi u = 0$, and the condition for a $B_{2,4}$ is equivalent to $q.d^3\phi u^3 = 0$. First we identify the vector q. Now, $q.d^2\phi u = 0$ iff, with $g(x) = 0$,

$$(q_1c + q_2n).d^2gu - q_1sdgu.dg = 0,$$

so q is the point for which $[q_1c + q_2n: q_1s] \in F$. Then $q.d^3\phi u^3 = 0$ is precisely the A_3 condition on the distance-squared function at the point $[q_1c + q_2n: q_1s]$ which from proposition 5.4 is equivalent to this point being the centre of spherical curvature of r_u. So part (i) is proved, and the same argument proves part (ii) as well.

This proposition has essentially used the following correspondence between singularities of functions and maps. Let $f: R^2 \to R^2$, then f has a

$B_{2,4}$ singularity, or worse, at 0 if and only if $\exists q \in R^2$ such that $q.f: R^2 \to R$ has an A_3 singularity, or worse. It might be expected that this correspondence carries on, with f having a $B_{2,5}$ iff q.f has an A_4, but this is not the case as the following examples show.

Let,

$$f(x,y) = (xy,\ x^2 + y^4 + axy^2),$$

which has a $I_{2,4}$ for all a. Then u = (0,1) and q = (0,1), so

$$q.f(x, y) = x^2 + y^4 + axy^2$$

which is only an A_4 for $a^2 = 2$. For $a^2 < 2$ this is an A_3^+, while for $a^2 > 2$ it is an A_3^-.

Conversely, let

$$g(x, y) = (xy,\ x^2 + y^5 + bxy^2),$$

which has a $I_{2,5}$ for all b. Again u = (0,1), q = (0,1), so

$$q.g(x, y) = x^2 + y^5 + bxy^2$$

which only has an A_5 singularity for b = 0, otherwise it is an A_3^-.

So we see that while $B_{2,b}$ and A_{b-1} do correspond for b = 2,3 and 4, they do not for b = 5. However there is a limited correlation between the two as follows:

Lemma 5.14 Let $f: R^2 \to R^2$ have a $II_{2,4}$ singularity at 0, then $\exists q \in R^2$ such that q.f has a singularity of type A_3^-.

Proof: Let $g(x,y) = (xy,\ x^2 - y^4)$ - the standard $II_{2,4}$ map. First we show that the lemma is true for g, and then show that it is also true for any f $\mathcal{K}$-equivalent to g. Now, $d^2gu^2 = 0$ for u = (0,1), and $q.d^2gu = 0$ for $q = (0,1)$. Then,

$$q.g(x, y) = x^2 - y^4,$$

which is the standard A_3^- singularity.

Any $\mathcal{K}$-equivalence can be decomposed into a $\mathcal{C}$-equivalence and an $\mathcal{R}$-equivalence. The case of $\mathcal{R}$-(right-)equivalence is obvious, so let f be $\mathcal{C}$-equivalent to g. Then,

$$f(x) = \theta(x).g(x),$$

where $\theta(x) \in L(R^2, R^2)$ is invertible for all x near 0. Letting subscripts denote differentiation with respect to x at 0 and $\theta = \theta(0)$, we get, putting $g(0) = 0$, $g_1(0) = 0$,

$$\begin{aligned} f_1 &= 0, \\ f_2 &= \theta.g_2, \\ f_3 &= \theta.g_3 + 3\theta_1.g_2, \\ f_4 &= \theta.g_4 + 4\theta_1.g_3 + 6\theta_2.g_2. \end{aligned}$$

Let $p = q.\theta^{-1}$, then $p.f_2u = q.g_2u$, so p is a cokernel vector of f_2u (where $f_2u^2 = g_2u^2 = 0$). Now, $(p.f)_i = p.f_i$, so the condition for p.f to have an A_3^- singularity is,

$$p.f_1 = 0, \quad p.f_2u = 0, \quad p.f_3u^2 + 3p.f_2v = 0,$$

for some v, and the quadratic equation for w,

$$p.f_4u^4 + 6p.f_3u^2w + p.f_2w^2 = 0,$$

has 2 distinct real roots. We have seen that $p.f_1 = 0$, and $p.f_2u = 0$. Now,

$$\begin{aligned} p.f_3u^2 + p.f_2v &= q.g_3u^2 + p\theta_1g_2u^2 + q.g_2v \\ &= q.g_3u^2 + q.g_2v, \end{aligned}$$

since $g_2u^2 = 0$, and $q.g_3u^3 + q.g_2v = 0$ can be solved for v as q.g has an A_3, therefore so does $p.f_3$. Finally,

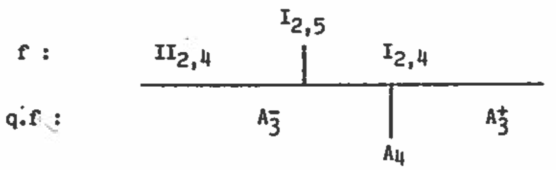
$$p.f_4u^4 + 6p.f_3u^2w + 3p.f_2w^2 = 0$$

$$\begin{aligned} \Leftrightarrow \quad & [q.g_4u^4 + 4p.\theta_1g_3u^3 + 6\,\theta_2(u^2)g_2u^2] \\ & + 6[q.g_3u^2w + 2p.\theta_1(u)g_2uw + p.\theta_1(w)g_2u^2] \\ & + 3q.g_2w^2 = 0 \end{aligned}$$

$$\Leftrightarrow \quad q.g_4u^4r + 2p.\theta_1(u).g_2uw + 3q.g_2w^2 = 0$$

as $g_3 = 0$, $g_2u^2 = 0$ and $q.g_2u = 0$. We can show that independently of θ_1 this equation for w has 2 distinct roots (cf. for a quadratic equation $ax^2 + bx + c = 0$, then $ac < 0 \Rightarrow b^2 - 4ac > 0$ for all b)

This lemma tells us that the correspondences are:



though it is possible for A_4 and $I_{2,5}$ to correspond, as in $(xy, x^2 + y^5)$

Remarks (i) We have seen how the cokernel vector q gives the correspondence between 2-spheres with centre on C and 3-spheres with centre on F, and proposition 5.13 tells us that the 2-sphere has at least $B_{2,4}$ contact if and only if the 3-sphere has at least A_3 contact. However the lemma shows that $B_{2,5}$ and A_4 contact do not correspond, and it would be interesting to understand why not, and what each does correspond to geometrically.

(ii) In the case of $B_{3,b}$ the correspondence breaks down sooner. Consider the map f given by

$$f(x, y) = (xy, x^3 + y^b + axy^2),$$

which is a $B_{3,b}$ ($I_{3,b}$) for all a. Then q.f is an A_1 unless q = (0, 1), in which case it is a D_4 (for a ≠0) irrespective of the value of b > 3.

To conclude this section we return to the $\Sigma^{2,1}$ case, and to a situation described at the end of the last section. Recall that the only $\Sigma^{2,1}$ singularities of codimension ⩽ 8 are the $E_{3,3}$ and the $F_{3,2}$, see appendix 1. The probe method does not distinguish between these two types, so in order to do so we use a cokernel method.

Lemma 5.15 Let f have a $\Sigma^{2,1}$ singularity, $f : R^2,o \longrightarrow R^2,o$, and let q and u satisfy $q.d^2f = 0$, $d^2fu = 0$. Then f has an $F_{3,2}$ singularity or worse if and only if $q.d^3fu^3 = 0$.

Proof: This follows the proof of lemma 5.14 exactly, with $f(x, y) = (x^2 + y^3, xy^2)$.

Returning to the geometry, recall that ϕ_m has a $\Sigma^{2,1}$ singularity at x if and only if x is a semiumbilic and [c:s] is an end-point of the real part of C. Let u satisfy $d^2\phi_m u = 0$, then F_u is one of the branches of the focal set, so the centre of spherical curvature of r_u is automatically contained in F

Proposition 5.16 ϕm has contact of type $F_{3,2}$ with X at g(x) if and only if it has $\Sigma^{2,1}$ contact and the centre of spherical curvature of r_u , u as above, lies at the semiumbilical centre.

Proof: We proceed as in proposition 5.14. First we identify the cokernel vector (q_1, q_2), putting g(x) = 0,

$$(q_1c + q_2n).d^2g - q_1sdg.dg \;=\; 0.$$

Thus $[q_1c + q_2n: q_1s]$ is the semiumbilical centre, and the proof then follows that of proposition 5.14, but using lemma 5.15 above.

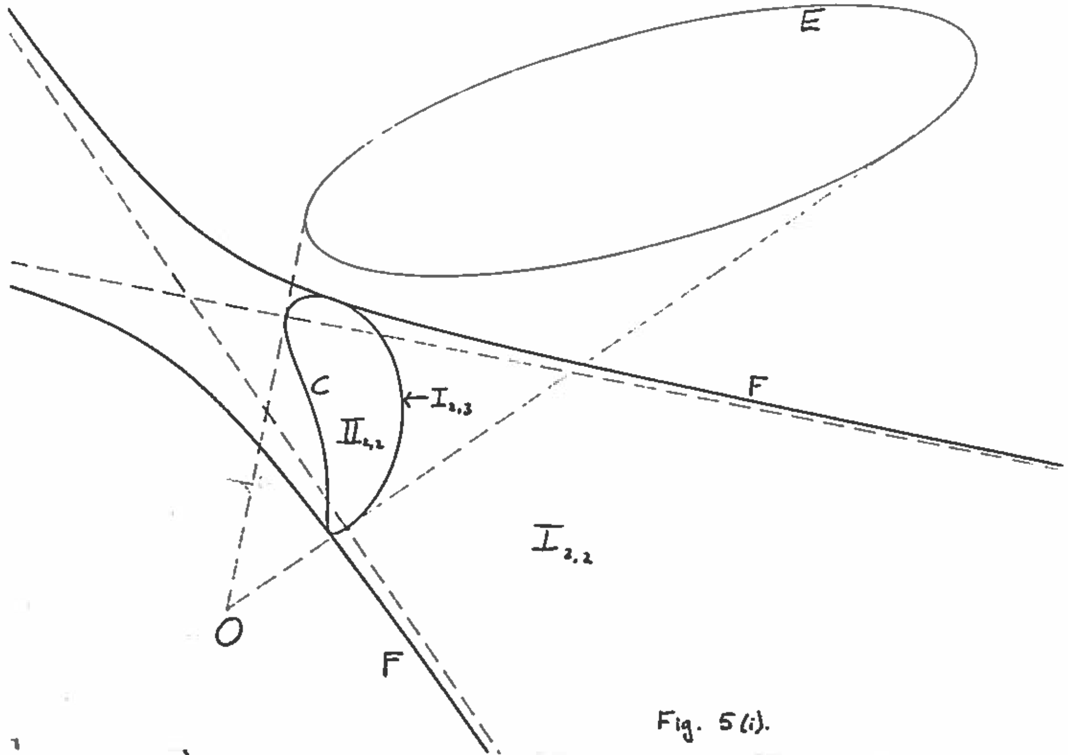


Fig. 5(i).

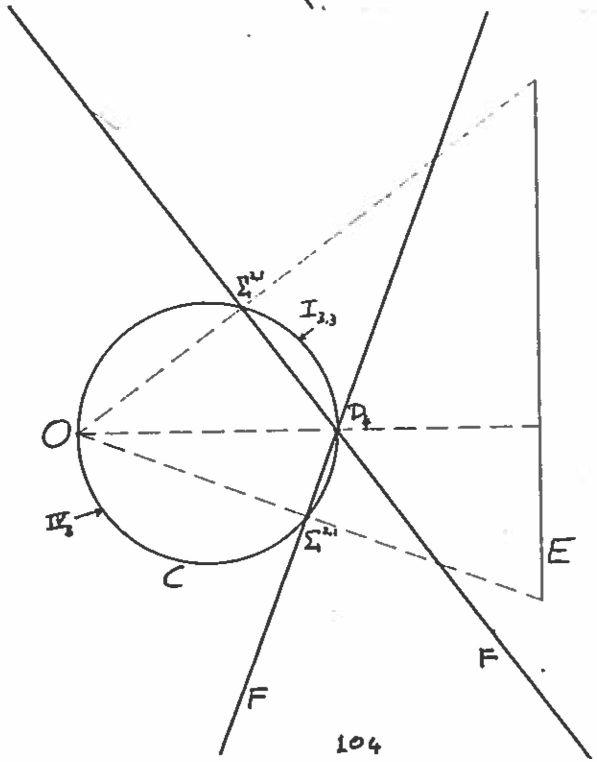


Fig 5(ii).

APPENDIX 1 SINGULARITY THEORY

In this appendix we present some standard concepts and results from singularity theory which are needed throughout this thesis. Familiarity with elementary notation is assumed.

Definition A1.1 Two map-germs $f,g : R^n,o \longrightarrow R^p,o$ are said to be $\mathcal{K}$-equivalent, denoted $\mathcal{K}(f) = \mathcal{K}(g)$, if there exist two diffeomorphism-germs

$$H : R^n \times R^p,o \longrightarrow R^n \times R^p,o$$

$$h : R^n,o \longrightarrow R^n,o ,$$

such that

$$H(x, 0) = (h(x), 0)$$

and

$$H(x, f(x)) = (h(x), g \circ h(x)).$$

Remarks A1.2 (i) We see from this definition that f and g are $\mathcal{K}$-equivalent if and only if there is a diffeomorphism H of $R^n \times R^p,o$ to itself, taking the graph of f to the graph of g, and leaving the graph of the constant zero-map invariant. This is often expressed by saying that the two graphs have isomorphic contact (or the same contact type) with the graph of the zero-map at 0, as we see in chapter 1. Consequently $\mathcal{K}$-equivalence is often called <u>contact-equivalence</u>.
(ii) In the definition we could write $H(x,y) = (h(x), \theta(x,y))$, where $\theta : R^n \times R^p,o \longrightarrow R^p,o$, where we would need the p x p matrix $[d_y\theta(0)]$ to be invertible, and $\theta(x,0) = 0$ for all $x \in R^n$. Thus by Hadamard's lemma, see below, we can put

$$\theta(x, y) = \sum y_i \, \theta_i(x, y), \tag{A1.1}$$

where for each i, $\theta_i : R^n \times R^p \rightarrow R^p$, and $[d_y\theta(0)] = [\theta_i(0)]_{i=1,\dots,p}$. This last expression (A1.1) is made use of in the body of the thesis.
(iii) It is not necessary that the two map-germs have the same source-point for them to be $\mathcal{K}$-equivalent. However the question is not so cut and

dried for the target point and it depends on what use the $\mathcal{K}$-equivalence is being put to. In his series of papers on the stability of C^∞ mappings, see [M4] - [M6], Mather uses $\mathcal{K}$-equivalence as a means towards classifying map-germs up to $\mathcal{A}$ - (left-right) equivalence, and consequently the targets may well be distinct. In our application there is a preferred target, namely 0, as we are interested in submanifolds given as $f^{-1}(0)$. In this latter context we will consider all maps with non-zero target as being $\mathcal{K}$-equivalent, while if one map has target 0, and another does not then they cannot be equivalent.

Lemma A1.3 (Hadamard's Lemma) Let $f: R^n,o \longrightarrow R^p,o$ be the germ of a submersion. Suppose $h : R^n \rightarrow R$ is zero on $f^{-1}(0)$, then there are p functions $a_1, \ldots, a_p$ such that

$$h(x) = \sum a_i(x) f_i(x)$$

where the f_i are the components of f.

The proof of this lemma for f just the projection π appears in [M7]. The extension to arbitrary submersions f can be obtained by choosing a diffeomorphism ϕ of R^n so that $f = \pi \circ \phi$.

<u>$\mathcal{K}$-invariants</u>

There are two $\mathcal{K}$-invariants that we will be interested in (a $\mathcal{K}$-invariant being some object associated to a map that depends only on its $\mathcal{K}$-class). The first is algebraic, the local algebra of a map, which is not only invariant, but also determines the $\mathcal{K}$-class of the map. The second, due to I.R.Porteous [P4], is what I shall call the probe-structure of a map. This second is more geometrical in nature, though it has the disadvantage that it does not distinguish between all the different $\mathcal{K}$-classes, as we see below. First some notation. Let $\mathcal{E}_n$ denote the ring of function-germs on R^n at 0, and $\mathcal{M}_n$ the maximal ideal in $\mathcal{E}_n$ of germs vanishing at 0.

Definition A1.4 (i) Let $f : R^n,o \to R^p,o$ be a map-germ, then we define the <u>ideal</u> of f and the <u>local algebra</u> of f by,

$$I(f) = \mathcal{E}_n . f^* \mathcal{M}_p,$$

and

$$Q(f) = \mathcal{E}_n / I(f),$$

where $f^* : \mathcal{E}_p \to \mathcal{E}_n : h \mapsto h \circ f$ is the ring- (or algebra-) homomorphism induced by f.
(ii) Two ideals I_1 and I_2 are said to be <u>induced isomorphic</u> if there is a diffeomorphism ϕ of R^n such that $I_1 = \phi^* I_2$, and in that case we also say that the algebras Q_1 and Q_2 are induced isomorphic, where $Q_i = \mathcal{E}_n / I_i$.

Theorem A1.5 (Mather) Let $f, g : R^n,o \to R^p,o$ be two map-germs, then f and g are $\mathcal{K}$-equivalent if and only if Q(f) and Q(g) are induced isomorphic (which is true if and only if I(f) and I(g) are).

For a proof see [G]. Mather also proved it but only for the finite jet case. Indeed if f and g are finitely $\mathcal{K}$-determined then the qualifier 'induced' can be dropped.

We now turn to the probe-structure of a map $f : R^n,o \to R^p,o$. A <u>probe</u> is an immersed curve $\gamma : R,o \hookrightarrow R^n$. The <u>probe structure</u> of f is the set of 'singularity types' of $f \circ \gamma$, $(df) \circ \gamma$, $(d^2 f) \circ \gamma$, . . . as γ varies through the set of all probes. More precisely to any probe γ, we associate a sequence of non-negative integers $(\nu_k, \ldots, \nu_1, \nu_0)$, where ν_i is the degree of the zero of $(d^i f) \circ \gamma$, that is the ν_i-jet of $(d^i f) \circ \gamma$ is zero, while the $(\nu_i + 1)$-jet is not, (if the ∞-jet is zero, then $\nu_i = \infty$), and k is such that $j^k f = 0$, $j^{k+1} f \neq 0$. An example is discussed below.

Theorem A1.6 (Porteous) The probe-structure of a map is a $\mathcal{K}$-invariant.

The proof of this appears in [M1].

Remark A1.7 The probe-structure of a map does not determine the $\mathcal{K}$-class of the map, as can be seen by the following examples. Let $f, g : R^2 \to R^2$ be

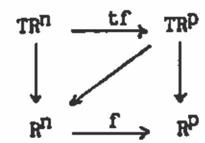
$$f(x, y) = (x^2, y^3)$$

$$g(x, y) = (x^2 + y^3, y^4).$$

Then $f_1 = g_1 = 0$ (as usual subscripts denote successive differentials at 0). $f_2(u, v)^2 = (2u^2, 0)$, $f_3(x, y)^3 = (0, 6v^3)$. So if $\gamma_1 \neq (0,1)$ then the probe sequence is (0, 1) as $f_1 \gamma_1 = 0$ but $f_2 \gamma_1^2 + f_1 \gamma_2 \neq 0$ for any γ_2, and $f_1 = 0$, $f_2 \gamma_1 \neq 0$. If $\gamma_1 = (0,1)$, then $f_2 \gamma_1^2 + f_1 \gamma_2 = 0$ for $\gamma_2 = 0$, and indeed $f_2 \gamma_1 = 0$, but in no case can $(f \circ \gamma)_3 = 0$, so we get (1,2). The same argument applies to g. However the two maps are not $\mathcal{K}$-equivalent as their local algebras are not isomorphic, indeed they have dimensions 6 and 8 respectively.

<u>Orbits and Tangent Spaces</u>

Let $f : R^n \to R^p$ be a map-germ. We denote by $\theta(f)$ the <u>set of vector fields along</u> f, that is germs of sections of the pull-back of the tangent bundle of R^p, TR^p, under f, and by tf we denote the tangent mapping of f, $tf(x, \xi_x) = (x, df_x \xi_x)$ for a vector field ξ_x on R^n.



$\theta(f)$ is isomorphic to $\mathcal{E}_{n,p}$, the p-fold Cartesian product of $\mathcal{E}_n$, and is thus an $\mathcal{E}_n$-module. The submodule $\mathcal{M}_n \theta(f)$ is the set of germs of vector fields along f which vanish at 0.

Now $\theta(f)$ can be identified in a natural way with the tangent space at f of maps $R^n,o \to R^p$, so we expect the tangent space to the $\mathcal{K}$-orbit of f, among germs with source zero, $T\mathcal{K}f$ to be a subspace of $\theta(f)$. (If we allow the source point to vary, the tangent space is $T_oR^n \times T\mathcal{K}f$ in $T_oR^n \times \theta(f)$.) Indeed, because of the preferred rôle of the zero target, the tangent space will be in $m_n\theta(f)$. In [M2] Martinet shows that

$$T\mathcal{K}f = tf(m_n\theta(n)) + f^*m_p.\theta(f) \subset m_n\theta(f),$$

where $\theta(n)$ is the set of germs of vector fields on R^n, so $T\mathcal{K}f$ is an $\mathcal{E}_n$-module. The <u>$\mathcal{K}$-codimension</u> of f, d(f), is defined to be the codimension of $T\mathcal{K}f$ as a vector subspace of $m_n\theta(f)$. Also useful is the extended tangent space, $T\mathcal{K}_ef$, given by

$$T\mathcal{K}_ef = tf(\theta(n)) + f^*m_p.\theta(f) \subset \theta(f).$$

The <u>$\mathcal{K}_e$-codimension</u> $d_e(f)$ of f is the codimension of $T\mathcal{K}_e(f)$ as a vector subspace of $\theta(f)$. Martinet shows that

$$T\mathcal{K}_ef = T\mathcal{K}f + R\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\},$$

(in fact he only shows this for maps with rank 0 at 0, but it is straightforward to show that it is true for submersions, and any map is equivalent to a direct product of a map of rank 0 and a submersion). Now, if f is not a submersion, $R\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ has dimension n and it follows that

$$d(f) = d_e(f) + n - p,$$

while if f is a submersion, then $d(f) = d_e(f) = 0$.

<u>Versal Deformations</u>

Let f be as usual, and $F : R^n \times R^a \to R^p$, with

$$F(x, u) = F_u(x), \quad F_o = f,$$

then we say F is an <u>a-parameter deformation</u> of f. (Note that R^a could be replaced by any a-dimensional manifold.)

Let F and G be two a-parameter deformations of f (or indeed of two $\mathcal{K}$-equivalent maps f and g), then we say that F and G are <u>equivalent</u> deformations if there is a diffeomorphism h of R^a,o such that $G_{h(u)}$ and F_u are $\mathcal{K}$-equivalent for all u in a neighbourhood of the origin.

Suppose $h : R^b,o \to R^a,o$ is any smooth map, then h <u>induces</u> from f a b-parameter deformation h^*F defined by,

$$(h^*F)_{h(u)} = F_u.$$

A deformation is said to be <u>versal</u> if every other deformation of f is equivalent to one induced from it. Associated to any a-parameter deformation F of f is the map $\pi_F : F^{-1}(0) \to R^a$, $\pi_F(x, u) = u$. We now give two theorems due to Martinet, the proofs of which are to be found in [M3].

Theorem A1.8 (Martinet)
(i) F is a versal deformation of F if and only if,

$$T\mathcal{K}_ef + R\{\dot{F}_i\} = \theta(f),$$

where $\dot{F}_i$ denotes the map : $x \mapsto \partial F/\partial u_i(x,0)$.
(ii) Suppose F and G are a-parameter deformations of f. They are equivalent if and only if there are diffeomorphisms ϕ: $F^{-1}(0) \to G^{-1}(0)$, and h of R^a,o such that $\pi_G \circ \phi = h \circ \pi_F$, i.e. π_F and π_G are $\mathcal{A}$-equivalent.
(iii) F is a versal deformation if and only if π_F is stable (i.e. any perturbation of π_F is $\mathcal{A}$-equivalent to π_F).

A map f is said to be k-determined with respect to $\mathcal{K}$ if every other map g with $j^k g = j^k f$ is $\mathcal{K}$-equivalent to f. There is the following theorem on finite determinacy:

Theorem A1.9 If $d_e(f) = a$, then f is $(a+1)$-determined w.r.t. $\mathcal{K}$.

See [W3] for further discussion on finite determinacy and references.

Associated to any a-parameter deformation F of f is an a-parameter unfolding $F : R^n \times R^a \to R^p \times R^a : (x, u) \mapsto (F_u(x), u)$. We now present a little lemma that is used in chapter 1.

Lemma A1.10 If F is any a-parameter unfolding of f as above, then F is $\mathcal{K}$-equivalent to the suspension $\tilde{f}$ of f, where $\tilde{f}(x, u) = (f(x), u)$.

Proof: Let H be the diffeomorphism of $(R^n \times R^a,o) \times (R^p \times R^a,o)$ defined by,

$$H((x, u), (y, v)) = ((x, u), (y + f(x) - f_v(x), v)).$$

This is the diffeomorphism required in the definition of $\mathcal{K}$-equivalence, with h the identity on $R^n \times R^a$.

Jet Spaces

For the purposes of this section, denote by J^r the jet bundle $J^r(R^n, R^p)$, which can be considered as a bundle over R^n with fibre J^r_x, over R^p with fibre J^r_y (and in particular the fibre with target 0 is J^r_0 - this will not be used for source 0!), or over $R^n \times R^p$ with fibre $J^r_{x,y}$. The jet map is $j^r : R^n \times C^\infty(R^n, R^p) \to J^r : (x,f) \mapsto j^r f(x)$, and its tangent

mapping at (x,f), which we also denote j^r, is

$$j^r : \theta(f) \to T_z(J^r_x)$$

$$: \xi_x \mapsto j^r \xi_x,$$

where $z = j^r(x)$, and T_z means 'tangent space at z'. So $T_z(J^r_x)$ can be identified with $\theta(f)/\mathcal{M}_n^{r+1}$, and $T_z(J^r_{x,o})$ with $\mathcal{M}_n\theta(f)/\mathcal{M}_n^{r+1}$.

Given f with $j^r f(x) = z$, then we can define the linear projection

$$\pi_f : T_z(J^r) \to T_z(J^r_x)$$

with the properties that π_f is the identity on $T_z(J^r_x)$, and has kernel $T_z(j^r f)$, the tangent space to the section induced from f (see [M5]). If we let $\mathcal{K}^r$ be the 'subgroup' of $\mathcal{K}$ consisting of r-jets of diffeomorphisms, then the tangent space to the $\mathcal{K}^r$-orbit through z is $TR^n + T\mathcal{K}f/\mathcal{M}_n^{r+1}$, and the image of this under π_f is precisely $T\mathcal{K}_e f/\mathcal{M}_n^{r+1}$ (giving an understanding of the space $T\mathcal{K}_e f$). The use of this is that $j^r f$ is transverse to a submanifold W iff $\pi_f(W) = \theta(f)/\mathcal{M}_n^{r+1}$, thus it is transverse to $\mathcal{K}^r(f)$ iff $T\mathcal{K}_e f/\mathcal{M}_n^{r+1} = \theta(f)/\mathcal{M}_n^{r+1}$, or for a deformation F of f, $j^r_1 F \pitchfork \mathcal{K}^r(f)$ iff $(T\mathcal{K}_e(f) + R\{\dot{F}_i\})/\mathcal{M}_n^{r+1} = \theta(f)/\mathcal{M}_n^{r+1}$.

Now, if $d_e(f) = a$, then we saw in theorem A1.10 that f is $(a+1)$-determined, so if we let $r > a + 1$ the $\mathcal{M}_n^{r+1}$ can be dropped from the two conditions above for transversality, for in that case the $\mathcal{K}$-orbit of f is the inverse image under j^r of the $\mathcal{K}^r$-orbit of $j^r f$. This argument enables us to use jet-spaces to provide genericity results in chapter 2.

Classifications

We give 3 partial classifications here, of maps $R \to R^p$, $R^2 \to R$, and $R^2 \to R^2$, together with their normal forms, their Boardman symbol, the dimension δ of their local algebra, their $\mathcal{K}_e$-codimension and their probe structure.

$R \longrightarrow R^p$

Every such map of finite codimension is $\mathcal{K}$-equivalent to the map $x \mapsto (x^{k+1}, 0, 0, \ldots, 0)$ for some k:

A_k: $x \mapsto (x^{k+1}, 0, \ldots, 0)$, Σ^{1_k}, $\delta = k + 1$, $d_e = k$

The probe-structure is just given by the order of the zero of the function at 0, so for an A_k, $f(0) = df(0) = \ldots = d^k f(0) = 0$, $d^{k+1}f(0) \neq 0$, and $(\nu_k, \ldots, \nu_o) = (0, 1, 2, \ldots, k)$.

$R^2 \longrightarrow R$

Such maps, when singular, are either Σ^2 or Σ^1, and all have $\delta = \infty$. See [A] for a fuller classification.

A_k: $(x, y) \mapsto x^2 + y^{k+1}$, $\Sigma^{2,1_{k-1}}$, $d_e = k$
for k odd we can distinguish between A_k^+ and A_k^- : $x^2 + y^{k+1}$ and $x^2 - y^{k+1}$.
The probe structure of an A_k is given by the existance of a single probe for which $(df)\circ\gamma$ has vanishing k-jet, so $(\nu_1, \nu_o) = (k, k-1)$.

D_k, $k \geq 4$, $(x, y) \mapsto x^2y + y^{k-1}$, $\Sigma^{2,2,0}$, $d_e = k$.
The only D_k we meet in the thesis are for k = 4,5, and for these the probe structures are: for D_4 $df = d^2f = 0$, d^3f is an elliptic or a hyperbolic cubic form, distinguishing between D_4^- and D_4^+ respectively, and the D_5 has d^3f a parabolic cubic form. Thus for D_4^- we get 3 $(1,2,\infty)$'s, for D_4^+ only one $(1,2,\infty)$, while for D_5 we get a (1,3,4) and a $(1,2,\infty)$.

$R^2 \longrightarrow R^2$

These maps are either Σ^1 or Σ^2. We treat those of codimension up to 8.

A_k: $(x, y) \mapsto (x, y^{k+1})$, Σ^{1_k}, $\delta = k + 1$, $d_e = k$.
Probe: $\exists\gamma$ such that $f\circ\gamma$: $R \to R^2$ has an A_k.

$\Sigma^{2,0}$:
$I_{a,b}$, $b \geq a \geq 2$: $(x, y) \mapsto (xy, x^a + y^b)$, $\delta = a + b$, $d_e = a + b$.
$II_{a,b}$, $b \geq a \geq 2$, both even, $(x, y) \mapsto (xy, x^a - y^b)$, $\delta = a + b$, $d_e = a + b$.
IV_a, $a \geq 3$, $(x, y) \mapsto (x^2 + y^2, x^a)$, $\delta = 2a$, $d_e = 2a$.

$\Sigma^{2,1,0}$
$E_{3,3}$ $(x, y) \mapsto (x^2, y^3)$, $\delta = 6$, $d_e = 7$.
$F_{3,2}$ $(x, y) \mapsto (x^2 + y^3, xy^2)$, $\delta = 7$, $d_e = 8$.

Probes:
$\Sigma^{2,1}$: $df = 0$, $d^2fu = 0$, so $(\nu_1, \nu_o) = (2,3)$
$\Sigma^{2,0}$: $df = 0$, $d^2fu \neq 0$, for all $u \neq 0$,
$I_{2,2}$ and $II_{2,2}$: $d^2fu^2 \neq 0$, so (1,2)
$I_{2,b}$ and $II_{2,b}$ for b > 2: $\exists\gamma$ such that $(f\circ\gamma)$ has zero (b - 1)-jet.
$I_{a,b}$ and $II_{a,b}$ for b ≥ a > 2: there are 2 probes γ and ϕ (with distinct tangents u and u') such that $(f\circ\gamma)$ and $(f\circ\phi)$ have zero (a - 1)- and (b - 1)-jets respectively.

APPENDIX 2 BINARY CUBIC FORMS

The geometry of a surface at and near an umbilic is largely governed by a cubic form on the tangent space, the intrinsic cubic, together with a positive definite quadratic form, the first fundamental form. We therefore dedicate a short appendix to a brief discussion of cubic forms, in particular in the presence of a positive definite quadratic form. We will think of these cubic forms as symmetric trilinear forms, and our notation will reflect this.

Let x, y be two real variables, then an arbitrary cubic form C can be written,

(A2.1) $$C(x, y)^3 = ax^3 + 3bx^2y + 3cxy^2 + dy^3.$$

As we are only interestd in the geometry of C - i.e. its roots and associated structure - we consider $[a:b:c:d] \in RP^3$. Two subsets are of immediate interest. First, C is a perfect cube if and only if $[a:b:c:d] = [s^3:s^2t:st^2:t^3]$ for some s,t, in which case $C(x, y)^3 = (sx + ty)^3$, this subset is a twisted cubic in RP^3. Second, C has a double root if [a:b:c:d] lies on the tangent developable of the twisted cubic, and so is of the form $[s^3 + 2ps^2: s^2t + 1/3 (qs^2 + 2pst): st^2 + 1/3 (2qst + pt^2): t^3 + qt^2]$, for some p and q, and in this case the double root is (x, y) = (-t, s).

Associated to any cubic is its Hessian quadratic,

$$H(x, y)^2 = \det[C(x, y)],$$

where again this is only defined up to a scalar multiple. For C in (A2.1),

(A2.2) $$H(x, y)^2 = (ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2.$$

The roots of H are called the Hessian directions of C. If $Hu^2 = 0$, u = (x, y), then Cuv = 0 for some v, in which case v is also a Hessian direction. The Hessian directions therefore satisfy Cuv = 0. It follows that C is a perfect cube if and only if its Hessian is zero, and C has a

repeated root if and only if H does: $Hu = 0 \Leftrightarrow Cu^2 = 0$ (for $H \neq 0$). The other correspondences are:

C has 3 distinct real roots iff H is elliptic ;

C has 1 real and 2 complex roots iff H is hyperbolic.

C is therefore said to be elliptic, hyperbolic or parabolic accordingly as H is, and sometimes a perfect cube is said to be 'symbolic' (a term we do not use here). By a linear change of coordinates, an elliptic, hyperbolic, parabolic cubic form can be reduced to $x(x^2 - y^2)$, $x(x^2 + y^2)$, x^2y respectively, and a perfect cube to x^3.

Suppose now that a positive definite cubic form Q is also present, then arbitrary linear changes of coordinates are no longer allowed, and C cannot in general be reduced to one of these four 'normal forms': if $Q(x,y)^2 = (x^2 + y^2)$ then the only coordinate changes allowed are rotations and reflections. The quadratic form defines an orthogonality relation: u and v are orthogonal if Quv = 0. A cubic form is said to be orthogonal if its Hessian directions are orthogonal w.r.t. Q. It can be seen in the body of the thesis that this a fairly important notion. We can also form the Jacobian of C w.r.t. Q, this is the cubic form,

$$J(x, y)^3 = \det \begin{vmatrix} C(x, y)^2 \\ Q(x, y) \end{vmatrix} .$$

The Jacobian is relevant to studying the lines of curvature in the vicinity of an umbilic, though we do not make any use of it here. It is not difficult to see that C is orthogonal if and only if 2 of the roots of the Jacobian are orthogonal.

Another important, but less well known, notion is that of the harmonic representative of a cubic form. We treat this in a little more detail. Given C and Q, we consider the family of cubic forms {C + LQ} as L varies through the set of linear forms. Suppose

$$Q(x, y) = px^2 + qxy + ry^2,$$

then define the second order linear differential operator Δ_Q by

$$\Delta_Q = r\partial^2/\partial x^2 - q\partial^2/\partial x\partial y + p\partial^2/\partial y^2.$$

Δ_Q has the following properties which are straightforward to check,
$\Delta_Q(Q(x, y)^2) = 4$,
$\Delta_Q(L(x,y)Q(x,y)^2) = 8L(x, y)$,
$\Delta_Q(C(x,y)^3) = (g^*)^{-1}\Delta_I((g^*C)(x, y)^3)$, where $I = (x^2 + y^2)$, g is a linear change of coordinates so that $I = g^*Q$.
It follows that $\Delta_Q(C + LQ) = \Delta_Q(C) + 8L$, so there is a unique linear form $L = -1/8\,\Delta_Q(C)$ such that $\Delta_Q(C + LQ) = 0$, the resulting cubic form $C + LQ$ being the harmonic representative of C with respect to Q. The harmonic representative has several properties, which we collect together in a proposition.

Proposition A2.1 (i) The harmonic representative of C w.r.t. tQ is independent of the number t;
(ii) The harmonic representative of C w.r.t. Q is zero if and only if $C = LQ$ for some L
(iii) If C is harmonic w.r.t. Q, then Q is (a multiple of) the Hessian of C;
(iv) If C is harmonic w.r.t. Q, then C is elliptic (has 3 distinct real roots).

Proof: (i) $\Delta_{tQ}(C + LtQ) = 0 \Leftrightarrow \Delta_Q(C + L'Q) = 0$, for $L' = tL$;
(ii) This is immediate;
(iii) This can be checked laboriously, comparing the condition on Q that $\Delta_Q(C) = 0$ with the Hessian in (A2.2);
(iv) This follows immediately from (iii).

There is an alternative representation of cubic forms which is sometimes useful, in particular in the presence of a positive definite quadratic form. Let x and y be coordinates such that $Q(x, y)^2 = x^2 + y^2$, and let $z = x + iy$ be a complex variable, then C can be written

(A2.3) $$C(x, y)^3 = \alpha z^3 + 3\bar{\beta} z^2\bar{z} + 3\beta z\bar{z}^2 + \bar{\alpha}\bar{z}^3$$

and $Q(x, y)^2 = |z|^2$. The coefficients in (A2.1) and (A2.3) are related by,

$$8\alpha = (a - 3c) + i(d - 3b), \quad 8\beta = (a + c) + i(b + d).$$

In this representation, the orthogonal cubic forms satisfy $|\alpha| = |\beta|$. The harmonic representative of C with respect to Q is just the same form but with $\beta = 0$ (i.e. $\alpha z^3 + \bar{\alpha}\bar{z}^3$). Thus the cubic form has zero harmonic representative if and only if $\alpha = 0$ (cf. part (ii) in the proposition). If $\alpha \neq 0$, then by a rotation and dilatation $z \mapsto \alpha^{-1/3} z$, we can take $\alpha = 1$, and the cubic forms are then represented by the point ξ on the complex-plane. See the figure below. The perfect cubes are the points for which $\beta^3 = 1$, the parabolic cubics lie on the tricuspidal deltoid parametrized by $t \mapsto 1/3(2e^{it} + e^{-it})$, the elliptic ones lie inside the deltoid while the hyperbolic ones lie outside. The orthogonal cubics lie on the unit circle.

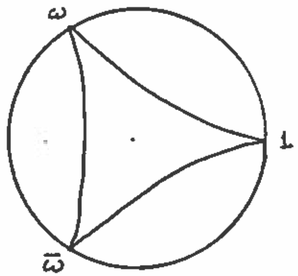


Figure A2(i)

An index can be associated to a cubic form which has an importance in studying the configurations of the lines of curvature near an umbilic. For C not orthogonal, the index of C is $+1/2$ if $|\beta| > |\alpha|$.
$-1/2$ if $|\beta| < |\alpha|$

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