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Generalized Perron Identity for broken lines

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Abstract. In this paper we generalize the Perron Identity for Markov minima. We express the values of binary quadratic forms with positive discriminant in terms of continued fractions associated to broken lines passing through the points where the values are computed.

Introduction

Consider a binary quadratic form f with positive discriminant $\Delta(f)$. In this paper we give a geometric interpretation and generalization of the *Perron Identity* relating the minimal value of $|f|$ at integer points except the origin and their corresponding continued fractions:

$$\inf_{\mathbb{Z}^2\setminus\{(0,0)\}} |f| = \inf_{i\in\mathbb{Z}} \left(\frac{\sqrt{\Delta(f)}}{a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots]} \right). \tag{0.1}$$

Here $[a_0; a_1 : \ldots]$ and $[0; a_{-1} : a_{-2} : \ldots]$ are regular continued fractions of the slopes of linear factors of corresponding reduced linear forms. Recall that a continued fraction is regular if all its elements are non negative. We discuss this in more detail further in Section 1.

The Perron Identity was shown by A. Markov in his paper on minima of binary quadratic forms and the Markov spectrum in the open interval $(-\infty, 3)$ in [14]. The statement holds for the entire Markov spectrum (see, e.g., the books by O. Perron [15], and T. Cusick and M. Flahive [1]). Recently Markov numbers were used in relation to Federer-Gromovs stable norm, ([5, 16]). There is not much known about a higher dimensional analogue of Markov spectrum. It is believed to be discrete (note that the existence of an accumulation point in the higher dimensional Markov spectrum will give a counterexample of the Oppenheim conjecture on best approximations, see in Chapter 18 of [10]). Various values of three-dimensional Markov spectrum were constructed by H. Davenport in [2, 3, 4].

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In this paper we show the geometric interpretation of the Perron Identity in terms of sails of the form (Remark 3.5) and generalise this expression in the spirit of integer geometry. This establishes a relationship between non-regular continued fractions and the values of the corresponding binary quadratic form at any point on the plane (Theorem 2.1 and Corollary 3.4). The result of this paper is based on recent results of the first author in geometric theory of continued fractions for arbitrary broken lines, see [9, 6, 7, 10].

Organization of the paper. We start in Section 1 with necessary definitions and background. We discuss reduced forms, LLS sequences, and formulate the classical Perron Identity. In Section 2 we formulate and prove the Generalized Perron Identity for finite broken lines. Finally in Section 3 we prove the Generalized Perron Identity for infinite broken lines, and discuss the relation with the classical the Perron Identity.

1. Basic notions and definitions

In this section we give necessary notions and definitions. We start in Subsection 1.1 with classical definitions of Markov minima and the Markov spectrum. Further in Subsection 1.2 we discuss reduced forms of quadratic binary forms with positive discriminant. In Subsection 1.3 we discuss the classical Perron Identity. Finally in Subsection 1.4 we introduce LLS sequences for broken lines, which is the central notion in the formulation of the main results.

1.1. Markov minima and the Markov spectrum. Let f be a binary quadratic form with positive discriminant. Recall that in this case f is decomposable into two real factors, namely

$$f(x, y) = (ax - by)(cx - dy),$$

for some real numbers a, b, c, and d. The discriminant of this form is

$$\Delta(f) = (ad - bc)^2.$$

The *Markov minimum* of the form f is the following number:

$$m(f) = \min_{\mathbb{Z}^2 \setminus \{(0,0)\}} |f|.$$

The set of all possible values of $\Delta(f)/m(f)$ is called *Markov Spectrum*. (Note that $\Delta(f)/m(f)$ is invariant under multiplication of the form f by a non-zero scalar.) The spectrum points of the open interval $(-\infty, 3)$ correspond to special forms with integer coefficients, we refer an interested reader to an excellent book [1] by T. Cusick and M. Flahive on the Markov spectrum and related subjects.

1.2. Reduced forms, and LLS-sequences. It is clear that $m(f)$ is invariant under the action of the group of $\mathrm{SL}(2,\mathbb{Z})$. Therefore in order to study the Markov spectrum one can restrict to so called *reduced forms* which are simple to describe. There are several ways to pick reduced forms, although the algorithmic part is rather similar to all of them, it is a subject of a Gauss reduction theory (see, e.g., [12], [13], [11], and [8]).

We consider the following family of *reduced forms.* For every $\alpha \geq 1$ and $1 > \beta \geq 0$ set

$$f_{\alpha,\beta} = (y - \alpha x)(y + \beta x).$$

Every form is multiple to some reduced form in an appropriate basis of the integer lattice $\mathbb{Z}^2$. However such a representation is not unique. The following notion provides a complete invariant distinguishing different classes of reduced forms.

Definition 1.1. Let $\alpha \geq 1$, $1 > \beta \geq 0$ and let

$$\alpha = [a_0; a_1 : \ldots] \quad \text{and} \quad \beta = [0; a_{-1} : a_{-2} : \ldots]$$

be the regular continued fractions for α and β. Then the sequence

$$(\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

is called the *LLS sequence* of the form $f_{\alpha,\beta}$.

This sequence can be either finite or infinite from one or both sides. The name for the LLS sequence (Lattice Length-Sine sequence) is due its lattice trigonometric properties, e.g., see in [6] and [7].

Proposition 1.2. *Two reduced forms are equivalent (i.e., multiple to each other after an* $\mathrm{SL}(2,\mathbb{Z})$*-change of coordinates) if and only if they have the same LLS sequence up to shifts of the sequence by* k*-elements for some integer* k *and a reversing of the order of a sequence.*

Remark 1.3. This statement follows directly from geometric properties of continued fractions. As we do not use this statement in the proof of the results of this paper we skip the proof here. We refer an interested reader to [10].

Due to Proposition 1.2 we can extend the notion of LLS-sequences to any binary quadratic form with positive discriminant.

Definition 1.4. Let f be a binary quadratic form with positive discriminant. The *LLS sequence* for f is the *LLS sequence* for any reduced form $f_{\alpha,\beta}$ equivalent to f. We denote it by $\mathrm{LLS}(f)$.

1.3. Classical Perron Identity. We are coming to one of the most mysterious statements in the theory of Markov minima. It is known as the *Perron Identity.*

Let f be a binary quadratic form with positive discriminant $\Delta(f)$. Let also

$$\mathrm{LLS}(f) = (\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

Then we have the following result by A. Markov in [14]:

$$\frac{m(f)}{\sqrt{\Delta(f)}} = \inf_{i\in\mathbb{Z}} \left(\frac{1}{a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots]} \right).$$

This result is based on the following observation. Let $\alpha \geq 1$, $1 > \beta \geq 0$ and let

$$\alpha = [a_0; a_1 : \ldots] \quad \text{and} \quad \beta = [0; a_{-1} : a_{-2} : \ldots]$$

be the regular continued fractions for α and β. Then

$$\frac{f_{\alpha,\beta}(0,1)}{\alpha+\beta} = \frac{1}{a_0 + [0; a_1 : a_2 : \ldots] + [0; a_{-1} : a_{-2} : \ldots]}.$$

Our goal is to investigate the lattice geometry behind this expression. It will lead us to a more general rule relating continued fractions whose elements are arbitrary non zero real numbers, and the values of the corresponding binary form at any point on the plane (see Theorem 2.1, Corollary 3.4 and Remark 3.5).

1.4. LLS sequences for broken lines. We start with the following general definition. Here and below we denote the origin $(0,0)$ by O.

Definition 1.5. Consider a quadratic binary form f with positive discriminant. A broken line $A_0 \ldots A_n$ is an f*-broken line* if the following conditions hold:

- $A_0, A_n \neq O$ belong to the two distinct loci of linear factors of f;
- all edges of the broken line are of positive length;
- for every $k = 1, \ldots, n$ the line $A_{k-1}A_k$ does not pass through the origin.

Recall the definition of oriented Euclidean area for parallelograms.

Definition 1.6. Consider three points A, B, C in the plane. Then the determinant for the matrix of vectors AB and AC is called the *the oriented Euclidean area* for the parallelogram spanned by AB and AC and denoted by

$$\det(AB, AC).$$

Definition 1.7. Let $\mathcal{A} = A_0A_1 \ldots A_n$ be a broken line with vertices A_0 A_n distinct from the origin O. Then the sign function of the determinant $\det(OA_0, OA_n)$ is called the *signature* of $\mathcal{A}$ with respect to the origin and denoted by $\mathrm{sign}(\mathcal{A})$.

We conclude this section with the following important definition.

Definition 1.8. Given an f-broken line $\mathcal{A} = A_0 \ldots A_n$ define

$$a_{2k} = \det(OA_k, OA_{k+1}), \quad k = 0, \ldots, n-1;$$

$$a_{2k-1} = \frac{\det(A_kA_{k-1}, A_kA_{k+1})}{a_{2k-2}a_{2k}}, \quad k = 1, \ldots, n-1.$$

The sequence $(a_0, \ldots, a_{2n-2})$ is called the *LLS sequence* for the broken line and denoted by $\mathrm{LLS}(\mathcal{A})$.

The expression $[a_0; \ldots : a_{2n-2}]$ is said to be the *continued fraction for the broken line* $A_0 \ldots A_n$. Note that the values $a_i \neq 0$ may be negative.

The LLS sequence encodes the integer angles and integer lengths of the broken line (see [10] for further details).

Remark 1.9. Note that the vertices of f-broken lines are not necessarily lattice points, and the elements of the LLS sequences for them can be arbitrary numbers (not necessarily integers). In some sense the definition of LLS sequences for broken lines (Definition 1.8) generalizes the definition of LLS sequences for forms (Definition 1.1). See Remark 3.5 for further discussions.

2. Generalized Perron Identity for finite broken lines

Now we are in position to formulate and prove the main result of this paper.

Theorem 2.1. (Generalized Perron Identity: case of finite broken lines.) *Consider a binary quadratic form with positive discriminant f. Let $\mathcal{A} = A_0 \ldots A_{n+m}$ be an f-broken line (here n and m are arbitrary positive integers), and let*

$$\mathrm{LLS}(\mathcal{A}) = (a_0, a_1, \ldots, a_{2n+2m-2}).$$

Then

$$f(A_n) = \frac{\mathrm{sign}(\mathcal{A}) \cdot \sqrt{\Delta(f)}}{a_{2n-1} + [0; a_{2n-2} : \ldots : a_0] + [0; a_{2n} : \ldots : a_{2n+2m-2}]}. \tag{2.1}$$

Let us first consider the following example.

Example 2.2. Consider the following binary quadratic form

$$f(x, y) = (x + y)(x - 2y).$$

Let $\mathcal{A} = A_0 \ldots A_7$ be the broken line with vertices

$$\begin{array}{llll} A_0 = (2, -2), & A_1 = (4, -1), & A_2 = (3, -2), & A_3 = (2, 0), \\ A_4 = (3, 1), & A_5 = (4, 0), & A_6 = (3, -1), & A_7 = (4, 2), \end{array}$$

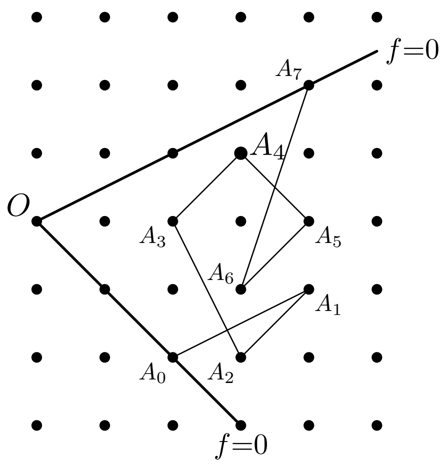

FIGURE 1. The kernel of f and the f-broken line $\mathcal{A}$.

see Figure 1. Let us check Theorem 2.1 for the broken line $\mathcal{A}$ at point $A_4 = (3, 1)$. We leave the computations of LLS-sequences to a reader as an exercise, the result is as follows:

$$\mathrm{LLS}(\mathcal{A}) = \left(6, -\frac{1}{30}, -5, -\frac{3}{20}, 4, \frac{3}{8}, 2, -\frac{1}{4}, -4, \frac{1}{8}, -4, -\frac{1}{20}, 10\right)$$

(here we denote the elements of $\mathrm{LLS}(\mathcal{A})$ by $a_0, \ldots, a_{12}$). Finally we have $\Delta(f) = 9$ and $\mathrm{sign}(\mathcal{A}) = 1$.

According to Theorem 2.1 we expect the following.

$$\begin{aligned} f(A_4) &= \frac{\mathrm{sign}(\mathcal{A}) \cdot \sqrt{\Delta(f)}}{a_7 + [0; a_6 : \ldots : a_0] + [0; a_8 : \ldots : a_{12}]} \\ &= \frac{1 \cdot 3}{-\frac{1}{4} + \left[0; 2 : \frac{3}{8} : 4 : -\frac{3}{20} : -5 : -\frac{1}{30} : 6\right] + \left[0; -4 : \frac{1}{8} : -4 : -\frac{1}{20} : 10\right]} \\ &= 4. \end{aligned}$$

Indeed, direct computation shows that

$$f(A_4) = (3 + 1)(3 - 2 \cdot 1) = 4.$$

We start the proof with three lemmas.

Lemma 2.3. *Consider a binary quadratic form with positive discriminant f. Let $P \neq O$ and $Q \neq O$ annulate distinct linear factors of f. Then for every point A it holds*

$$f(A) = \mathrm{sign}(POQ) \cdot \frac{\det(OP, OA) \cdot \det(OA, OQ)}{\det(OP, OQ)} \cdot \sqrt{\Delta(f)}.$$

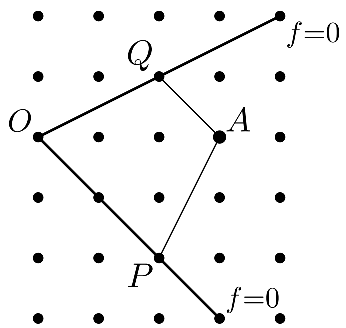

FIGURE 2. The kernel of f and the f-broken line PAQ.

Example 2.4. Consider the following binary quadratic form

$$f(x,y) = (x+y)(x-2y).$$

Let PAQ be an f-broken line, with $P=(2,-2)$, $A=(3,0)$, and $Q=(2,1)$, see Figure 2. Direct calculations show that

$$\begin{array}{lll} \det(OP,OA)=6, & \det(OA,OQ)=3, & \det(OP,OQ)=6, \\ \operatorname{sign}(POQ)=1, & f(A)=9, & \Delta(f)=9. \end{array}$$

Therefore, we have

$$\begin{aligned} \operatorname{sign}(POQ)\cdot\frac{\det(OP,OA)\cdot\det(OA,OQ)}{\det(OP,OQ)}\cdot\sqrt{\Delta(f)} &= 1\cdot\frac{6\cdot 3}{6}\cdot\sqrt{9}=9 \\ &= f(A). \end{aligned}$$

Proof of Lemma 2.3. The statement is straightforward for the form

$$f_\alpha(x,y)=\alpha xy.$$

Assume that $P=(p,0)$, $Q=(0,q)$, and $A=(x,y)$. Then we have

$$f_\alpha(A)=\alpha xy=\frac{py\cdot qx}{pq}\cdot\alpha=\frac{\det(OP,OA)\cdot\det(OA,OQ)}{\det(OP,OQ)}\cdot\sqrt{\Delta(f)}.$$

For $P=(0,p)$ and $Q=(q,0)$ we have

$$\begin{aligned} f_\alpha(A)=\alpha xy &= \frac{(-px)\cdot(-qx)}{-pq}\cdot\alpha \\ &= -\frac{\det(OP,OA)\cdot\det(OA,OQ)}{\det(OP,OQ)}\cdot\sqrt{\Delta(f)}. \end{aligned}$$

This conclude the proof for the case of f_α.

The general case follows from the invariance of the expressions of the equality of the lemma under the group of linear area preserving transformations (i.e., whose determinants equal 1) of the plane. □

Now we prove a particular case of Theorem 2.1.

Lemma 2.5. *Let f be a binary quadratic form with positive discriminant. Consider an oriented f-broken line $\mathcal{B} = B_0B_1B_2$ with* $\mathrm{LLS}(\mathcal{B}) = (b_0, b_1, b_2)$. *Then*

$$f(B_1) = \frac{\mathrm{sign}(\mathcal{B}) \cdot \sqrt{\Delta(f)}}{b_1 + [0; b_0] + [0; b_2]}.$$

Proof. Set $B_i = (x_i, y_i)$ for $i = 0, 1, 2$. Then Definition 1.8 implies

$$b_0 = \det(OB_0, OB_1) = x_0y_1 - x_1y_0,$$

$$b_2 = \det(OB_1, OB_2) = x_1y_2 - y_1x_2,$$

$$b_1 = \frac{\det(B_1B_0, B_1B_2)}{b_0b_2} = \frac{x_0y_2 - x_2y_0 - x_0y_1 + x_1y_0 - x_1y_2 + y_1x_2}{b_0b_2}.$$

After a substitution and simplification we get

$$\begin{aligned} \frac{1}{b_1 + [0; b_0] + [0; b_2]} &= \frac{(x_0y_1 - x_1y_0)(x_1y_2 - y_1x_2)}{x_0y_2 - x_2y_0} \\ &= \frac{\det(OB_0, OB_1) \cdot \det(OB_1, OB_2)}{\det(OB_1, OB_2)}. \end{aligned}$$

Finally recall that

$$\mathrm{sign}(\mathcal{B}) = \mathrm{sign}(B_0B_1B_2).$$

Now Lemma 2.5 follows directly from Lemma 2.3. □

For the proof of general case we need the following important result.

Lemma 2.6. **(**[10, Corollary 11.11, p. 144]**.)** *Consider a broken line $A_0 \ldots A_n$ that has the LLS sequence $(a_0, \ldots, a_{2n-2})$, with $A_0 = (1, 0)$, $A_1 = (1, a_0)$, and $A_n = (x, y)$. Let*

$$\alpha = [a_0; a_1 : \ldots : a_{2n-2}]$$

be the corresponding continued fraction for this broken line. Then

$$\frac{y}{x} = \alpha.$$

For the case of an infinite value for $\alpha = [a_0; a_1 : \ldots : a_{2n-2}]$,

$$\frac{x}{y} = 0.$$

For a proof of Lemma 2.6 we refer to [10]. As a consequence of Lemma 2.6 we have the following statement.

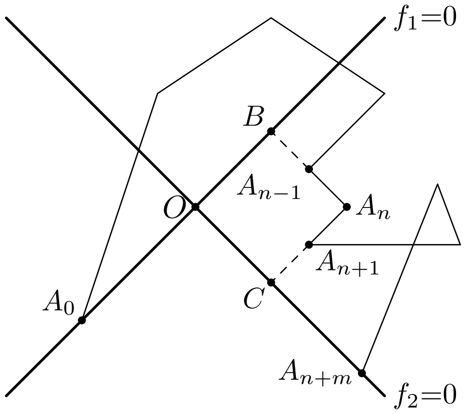

FIGURE 3. The original f-broken line $\mathcal{A}$ and the resulting f-broken line BA_nC.

Corollary 2.7. *Consider two broken lines $A_0 \ldots A_n$ and $B_0 \ldots B_m$ that have the LLS sequences $(a_0, \ldots, a_{2n-2})$, and $(b_0, \ldots, b_{2m-2})$ respectively. Suppose that the following hold:*

- *$A_0 = B_0$;*
- *the points A_n, B_m, and O are in a line;*
- *the points $A_0 = B_0$, A_1, and B_1 are in a line.*

Then

$$[a_0; a_1 : \ldots : a_{2n-2}] = [b_0; b_1 : \ldots : b_{2m-2}].$$

Proof. In coordinates of the basis

$$e_1 = OA_0, \qquad e_2 = \frac{A_0A_1}{|A_0A_1||OA_0|}$$

the coincidence of continued fractions follows from Lemma 2.6. □

Proof of Theorem 2.1. Let f be a binary quadratic form with positive discriminant. Denote the linear factors of f by f_1 and f_2. Consider an f-broken line $\mathcal{A} = A_0 \ldots A_{n+m}$. Without loss of generality we assume that A_0 and A_{n+m} annulate f_1 and f_2 respectively.

Let us first prove the statement of the theorem for the cases when $f(A_n) \neq 0$, A_nA_{n+1} is not parallel to one of the lines $f_i = 0$, and $A_{n-1}A_n$ is not parallel to one of the lines $f_i = 0$.

Denote by B the intersection of the line A_nA_{n-1} with the line $f_1 = 0$. Denote by C the intersection of the line A_nA_{n+1} with the line $f_2 = 0$. (See Figure 3.) Then the continued fraction for the broken line BA_nC is $[b_0 : a_{2n-1} : b_2]$ for some real numbers b_0 and b_2.

By Corollary 2.7 we have

$$[b_0] = [a_{2n-2}; \ldots : a_0];$$
$$[b_2] = [a_{2n}; \ldots : a_{2n+2m-2}].$$

By construction

$$\operatorname{sign}(BA_nC) = \operatorname{sign}(\mathcal{A}).$$

Therefore by Lemma 2.5 we have

$$\begin{aligned} f(A_n) &= \frac{\operatorname{sign}(BA_nC) \cdot \sqrt{\Delta(f)}}{a_{2n-1} + [0; b_0] + [0; b_2]} \\ &= \frac{\operatorname{sign}(\mathcal{A}) \cdot \sqrt{\Delta(f)}}{a_{2n-1} + [0; a_{2n-2} : \ldots : a_0] + [0; a_{2n} : \ldots : a_{2n+2m-2}]}. \end{aligned}$$

This concludes the proof of Theorem 2.1 for generic case.

Finally let us study the case when $f(A_n) = 0$, A_nA_{n+1} parallel to one of the lines $f_i = 0$, or $A_{n-1}A_n$ parallel to one of the lines $f_i = 0$.

Denote by $\Omega_{f,k}$ the space of all broken lines $A_0 \ldots A_k$ such that $f_1(A_0) = 0$ and $f_2(A_k) = 0$. We endow $\Omega_{f,k}$ with the induced topology of the Euclidean linear subspace of $\mathbb{R}^{2k+2}$ of codimension 2. In this topology the set of all f-broken lines is a dense open subset in $\Omega_{f,k}$.

Consider an f-broken line $\mathcal{A}_0 \in \Omega_{f,n+m}$. From the above there exists a neighbourhood of $\mathcal{A}_0$ whose elements are all f-broken lines (denote it by U). From Definition 1.8 the elements a_i smoothly depend on a variation of $\mathcal{A}_0$ in U. Hence both the right hand side and the left hand side of Equation 2.1 smoothly depend on a variation of $\mathcal{A}_0$ in U. As we have already proved, these expressions coincide in the general case (i.e., when $f(A_n) \neq 0$, A_nA_{n+1} is not parallel to one of the lines $f_i = 0$, and $A_{n-1}A_n$ is not parallel to one of the lines $f_i = 0$) which is an everywhere dense subset in U. Together with smoothness this implies that Equation 2.1 holds for all points of U. Therefore, Equation 2.1 holds for all f-broken lines. $\square$

3. Generalized Perron identity for asymptotic infinite broken lines

In this section we extend the Generalized Perron Identity (of Theorem 2.1) to the case of certain infinite broken lines and discuss the relation to the classical Perron Identity.

We start with the following definition.

Definition 3.1. Consider a binary quadratic form f with positive discriminant. An infinite in both sides broken line $\ldots A_{-2}A_{-1}A_0A_1A_2 \ldots$ is an *asymptotic f-broken line* if the following conditions hold (here we assume that $A_k = (x_k, y_k)$ for every integer k):

- the two side infinite sequence $\left(\frac{y_n}{x_n}\right)$ converges to different slopes of the linear factors in the kernel of f as n increases and decreases respectively;
- all edges of the broken line are of positive length;
- for every admissible k the line $A_{k-1}A_k$ does not pass through the origin.

Remark 3.2. Here and below one can consider one side infinite broken lines. All the proofs are similar, so we leave this case as an exercise.

The signature of an asymptotic f-broken line is defined as a determinant for two vectors in the kernel of f, the first with the starting limit direction and the second with the end limit direction.

Finally we have a definition of LLS-sequences similar to Definition 1.8.

Definition 3.3. Given an asymptotic f-broken line

$$\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$$

define

$$a_{2k} = \det(OA_k, OA_{k+1}), \quad k \in \mathbb{Z};$$

$$a_{2k-1} = \frac{\det(A_kA_{k-1}, A_kA_{k+1})}{a_{2k-2}a_{2k}}, \quad k \in \mathbb{Z}.$$

The sequence $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2\ldots)$ is called the *LLS sequence* for the broken line and denoted by $\mathrm{LLS}(\mathcal{A})$.

Let us extend the Generalized Perron Identity (of Theorem 2.1) to the case of asymptotic f-broken line.

Corollary 3.4. (Generalized Perron Identity: case of infinite broken lines.) *Consider a binary quadratic form with positive discriminant f. Let*

$$\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$$

be an asymptotic f-broken line, and let

$$\mathrm{LLS}(\mathcal{A}) = (\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots).$$

Assume also that both continued fractions

$$[0; a_{-1} : a_{-2} : \ldots] \quad \textit{and} \quad [0; a_1 : a_2 : \ldots]$$

converge. Then

$$f(A_0) = \frac{\mathrm{sign}(\mathcal{A}) \cdot \sqrt{\Delta(f)}}{a_0 + [0; a_{-1} : a_{-2} : \ldots] + [0; a_1 : a_2 : \ldots]}. \tag{3.1}$$

Proof. Without loss of generality we consider the form

$$f = \lambda f_{\alpha,\beta} = \lambda(y - \alpha x)(y + \beta x),$$

for some nonzero λ and arbitrary $\alpha \neq \beta$.

Let $\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$ be an asymptotic f-broken line, where $A_k = (x_k, y_k)$ for all integer k. Also we assume that $x_k \neq 0$ for all k (otherwise, switch to another coordinate system, where the last condition holds).

Set

$$\mathcal{A}_n = A_{-n}\ldots A_{-2}A_{-1}A_0A_1A_2\ldots A_n;$$
$$\alpha_n = \frac{y_{-n}}{x_{-n}}; \qquad \beta = \frac{y_n}{x_n}.$$

First of all, by definition $\mathrm{LLS}(\mathcal{A}_n)$ coincides with $\mathrm{LLS}(\mathcal{A})$ for all admissible entries.

Secondly, we immediately have that

$$\lim_{n\to\infty} \lambda f_{\alpha_n,\beta_n}(A_0) = \lambda f_{\alpha,\beta}(A_0).$$

Thirdly, the sequence of signatures stabilizes as n tends to infinity. In other words

$$\lim_{n\to\infty} \mathrm{sign}(\mathcal{A}_n) = \mathrm{sign}(\mathcal{A}).$$

Fourthly,

$$\lim_{n\to\infty} \Delta(\lambda f_{\alpha_n,\beta_n}) = \Delta(\lambda f_{\alpha,\beta}).$$

Finally since both continued fractions

$$[0; a_{-1} : a_{-2} : \ldots], \quad \text{and} \quad [0; a_1 : a_2 : \ldots]$$

converge and by the above four observations we have

$$\begin{aligned} f(A_0) &= \lim_{n\to\infty} \lambda f_{\alpha_n,\beta_n}(A_0) \\ &= \lim_{n\to\infty} \frac{\mathrm{sign}(\mathcal{A}_n)\cdot\sqrt{\Delta(\lambda f_{\alpha_n,\beta_n})}}{a_0 + [0; a_{-1} : a_{-2} : \ldots : a_{2-2n}] + [0; a_1 : a_2 : \ldots : a_{2n-2}]}. \\ &= \frac{\mathrm{sign}(\mathcal{A})\cdot\sqrt{\Delta(f)}}{a_0 + [0; a_{-1} : a_{-2} : \ldots] + [0; a_1 : a_2 : \ldots]}. \end{aligned}$$

The second equality holds as it holds for the elements in the limits for every positive integer n by Theorem 2.1.

This concludes the proof of the corollary. □

We conclude this paper with the following important remark.

Remark 3.5. (Lattice geometry of the Perron Identity.) Let f be a binary quadratic form with positive discriminant. Consider an angle in the complement to the kernel of f. The *sail* of this angle is the boundary of the convex hull of all integer points inside the angle except the origin.

Note that each form f has four angles in the complement, and, therefore, it has four sails.

It is important that the sail of any angle in the complement to the set $f = 0$ is an asymptotic f-broken line, so the Corollary 3.4 holds for each of four sails of f. From the general theory of geometric continued fractions, the Markov minimum is an accumulation point of the values at vertices of all sails.

For every vertex V_i of a sail there exists a reduced form f_{α_i,β_i} with $\alpha_i \geq 1$ and $1 \geq \beta_i > 1$ such that V_i corresponds to $(0,1)$. In particular we have

$$f(V_i) = f_{\alpha_i,\beta_i}(0,1).$$

The point $(0,1)$ is a vertex of the sail for f_{α_i,β_i}. Then from the general theory of continued fractions (see Part 1 of [10]) the sequence $\mathrm{LLS}(f_{\alpha,\beta})$ coincides with the LLS sequence for the sail containing $(0,1)$.

Hence the expressions in the Perron Identity (0.1) for which the minimum is computed, i.e.,

$$\frac{\sqrt{\Delta(f)}}{a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots]}$$

for $i = \ldots, -2, -1, 0, 1, 2, \ldots$ correspond to the formula of Corollary 3.4 for vertices V_i of all four sails. We consider the vertex V_i, with the sail containing it, as an f-broken line.

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