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From Static to Cosmological Solutions of $\mathcal{N} = 2$ Supergravity

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Abstract

We obtain cosmological solutions with Kasner-like asymptotics in $\mathcal{N} = 2$ gauged and ungauged supergravity by maximal analytic continuation of planar versions of non-extremal black hole solutions. Initially, we construct static solutions with planar symmetry by solving the time-reduced field equations. Upon lifting back to four dimensions, the resulting static regions are incomplete and bounded by a curvature singularity on one side and a Killing horizon on the other. Analytic continuation reveals the existence of dynamic patches in the past and future, with Kasner-like asymptotics. For the ungauged STU-model, our solutions contain previously known solutions with the same conformal diagram as a subset. We find explicit lifts to five, six, ten and eleven dimensions which show that in the extremal limit, the underlying brane configuration is the same as for STU black holes. The extremal limit of the six-dimensional lift is shown to be BPS for special choices of the integration constants. We argue that there is a universal correspondence between spherically symmetric black hole solutions and planar cosmological solutions which can be illustrated using the Reissner-Nordström solution of Einstein-Maxwell theory.

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1 Introduction

Cosmological solutions of string theory are far less understood than stationary, let alone BPS solutions. It thus came as a surprise to the authors to obtain cosmological solutions by tweaking the horizon geometry of the well-studied class of STU black holes. The original aim of this paper was to continue previous work [1, 2, 3, 4] on the construction of non-extremal stationary solutions in theories of four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to gauged and ungauged supergravity, and to make contact in the extremal limit with the classification of supersymmetric near-horizon geometries. At a technical level, this paper extends the work of [3, 4] on solutions with planar horizons to solutions with more than one charge. The single charge solutions are 'Nernst branes', which are solutions with zero entropy in the extremal limit, where they reduce to the solutions of [5, 6].

In this paper, we construct non-extremal planar solutions with more than one charge, and we observe that the static region for solutions with three or four charges interpolates between a curvature singularity and a Killing horizon. By analytical continuation, we obtain time-dependent regions which are asymptotic to Kasner-like solutions in the infinite past and infinite future. These solutions are 'inside-out' compared to the causal structures of non-extremal black hole and black brane solutions, and should be interpreted as cosmological. Our family of solutions overlaps with previously found solutions of Einstein-Maxwell-Dilaton theories and truncations of supergravity theories, which in particular display the same conformal diagram [7, 8, 9]. The three-charge ansatz leads to solutions for a class of gauged vector multiplet theories, including the gauged STU-model, while the four-charge system is a solution to the ungauged STU model and thus admits lifts to ten and eleven-dimensional supergravity. In fact, from the ten- or eleven-dimensional point of view these solutions correspond to the same brane configurations as the STU-black hole; the only difference is that the planar rather than spherical horizon geometry leads to additional de-localisation along two non-compact directions. The cosmological character of our solutions crucially depends on them being non-extremal, and thus on our ability to construct non-extremal solutions. From the ten- and eleven-dimensional point of view, these solutions correspond to well-known BPS brane configurations. Besides non-extremality, the only ingredient for tweaking a black hole into a cosmological solution is to change the horizon geometry from spherical to planar. This is a robust feature, which can already be understood and demonstrated using the spherical and planar Reissner-Nordström solutions of Einstein-Maxwell theory, which are contained in our family of solutions as special cases.

On taking the extremal limit, defined by the Killing horizon having zero surface gravity, the distance between singularity and horizon becomes infinite and the cosmological patches disappear. With some additional fine-tuning of parameters, we identify our uplifted solution as a supersymmetric solution of six-dimensional supergravity. The most general classification of supersymmetric solutions in six-dimensional supergravity was constructed in [10, 11, 12] making use of spinorial geometry techniques [13], see also [14, 15] for classifications constructed via the Fierz identity/spinor bilinear method. The 6-dimensional supergravity which we consider is a simpler truncation of this, as it is coupled only to a single 3-form field strength and dilaton; the supersymmetric solutions of this theory were analysed in detail using the Fierz identity/spinor bilinears method in [16]. We determine how the uplifted solution is described by this classification, and show how the geometry is determined in terms of harmonic functions on a specific non-flat Gibbons-Hawking manifold. Such a construction is valuable in providing a solution generating technique which could be used to produce a potentially very large number of generalizations of our solutions. For example, it is known that many new black hole solutions, with regular horizons and nontrivial topology exterior to the horizon in the form of 2-cycles supported by magnetic flux, can be found by taking the standard BMPV black hole, which is written in terms of harmonic functions on the Gibbons-Hawking base space $\mathbb{R}^4$, and making appropriately chosen modifications to the harmonic functions appearing in the solution [17, 18]. Related methods were also previously employed to find examples of Black Saturn geometries [19, 20], as well as numerous examples of possible smooth horizonless black hole microstate geometries [21, 22]. By following similar reasoning, we expect to be able to construct large families of new solutions which are deformations of the planar brane geometries considered in this work and may exhibit novel topological structures.

The outline of this paper is as follows: in Section 2 we review the necessary background on four-dimensional vector multiplets and their dimensional reduction over time. We also explain which restrictions we impose on solutions in order to be able to integrate the reduced three-dimensional field equations explicitly. In Section 3 we solve the reduced field equations and obtain solutions with two, three and four charges. In Section 4 we lift these solutions back to four dimensions and reduce the number of independent integration constants by imposing that the solutions exhibit a regular Killing horizon. In Section 5 we show that the lifted, static solutions interpolate between a curvature singularity and the Killing horizon. By analytic continuation, we extend the solutions beyond the horizon and discover a dynamical patch with Kasner-like asymptotic behaviour at timelike infinity. In Section 6 we use Kruskal-like coordinates to obtain the full conformal diagram, which is 'Schwarzschild rotated by 90 degrees.' We observe that timelike geodesics are infinitely extendible. Using the Komar construction we define a 'position-dependent mass' in the static region which turns out to be negative, which is consistent with the behaviour of the timelike geodesics. We also compute the Brown-York mass, which while different from the Komar mass, is also negative. Finally, we study the proper acceleration of static test bodies, again finding that the singularities are repulsive. In Section 7 we provide explicit lifts of the four-charge solution, which is a solution to the ungauged STU model, to five and six, and as well to ten and eleven dimensions. By taking the extremal limit, we recover in ten and eleven dimensions well known BPS brane configurations which yield BPS black holes of the STU model. In Section 8 we study the extremal limit of the six-dimensional lift in detail and show that with additional fine-tuning, the extremal limit is BPS. Here we make contact with the classification of six-dimensional supersymmetric near horizon geometries. In Section 9 we discuss the physical interpretation of our solution and explain that their qualitative features already arise in the planar Reissner-Nordström solution.

Some technical material has been relegated to appendices. Appendix A discusses Kruskal-like coordinates in some more detail. Appendix B shows how the planar Reissner-Nordström solution arises as a special limit. In Appendix C we review non-extremal and extremal black hole solutions of the STU model and give their explicit lifts to five, six, ten and eleven dimensions for comparison with the planar case.

2 Planar Solutions with Multiple Gauge Fields

We follow the approach developed in [1, 2, 3, 4] to construct stationary solutions of $\mathcal{N} = 2$, $D = 4$ supergravity coupled to n_V vector multiplets, using

- the dimensional reduction over time to obtain an effective three-dimensional Euclidean theory.
- The real, rather than the more commonly used complex formulation of the special geometry of $\mathcal{N} = 2$ vector multiplets, which is based on a Hesse potential rather than a prepotential.
- A set of conditions which decouples the field equations and allows us to integrate them elementarily; this requires us to impose restrictions on the admissible prepotential/Hesse potential, and to consistently truncate the field content to a subset of 'purely imaginary' (PI) configurations.

Since the procedure follows with only small modifications as in the previous papers, in particular [3] where planar symmetry and a single charge were considered, we only summarise the essential steps to avoid unnecessary duplication of material.

2.1 Background

As in [3] our starting point is the general two-derivative Lagrangian for n_V $\mathcal{N} = 2$ vector multiplets coupled to Poincaré supergravity, including the most general (dyonic) FI-gauging. The bosonic Lagrangian is given by

$$\begin{aligned}\mathbf{e}_4^{-1}\mathcal{L} = &-\frac{1}{2}R_4 - g_{I\bar{J}}\partial_\mu X^I \partial^\mu \bar{X}^J + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu} \\ &+ V_4^{\text{dyonic}}(X, \bar{X}),\end{aligned} \tag{2.1}$$

where $\mu = 0, 1, 2, 3$ are the spacetime indices, $\mathbf{e}_4$ is the vierbein, R_4 is the Ricci scalar, $F^I_{\mu\nu}$ are the Abelian field strengths, $I, J = 0, 1, \ldots, n_V$. In our conventions the tilde represents the Hodge-dual

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \ , \tag{2.2}$$

and the Riemann tensor is

$$R^\mu{}_{\nu\rho\sigma} = -(\partial_\rho\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\rho} + \Gamma^\tau{}_{\nu\sigma}\Gamma^\mu{}_{\tau\rho} - \Gamma^\tau{}_{\nu\rho}\Gamma^\mu{}_{\tau\sigma}),$$

which introduces the minus sign in the first term of (2.1).

We are using the standard formulation of special Kähler geometry in terms of complex scalar fields X^I, which are subject to complex scale transformations $X^I \to \lambda X^I$, $\lambda \in \mathbb{C}^*$. Explicit expressions for the couplings $g_{I\bar{J}}$, $\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$, can be found in [1], but are not needed in the following. All data except the scalar potential V_4^{dyonic} are encoded in a *prepotential* $F = F(X^I)$, which is a holomorphic function, homogeneous of degree two in the complex scalar fields X^I. As already mentioned, the scalars X^I are not independent degrees of freedom. The Poincaré supergravity multiplet contains an Abelian vector field, called the graviphoton, so that a theory of n_V vector multiplets has $n_V + 1$ vector fields, but only n_V independent complex scalar fields z^A. Using the redundant parametrisation in terms of X^I is akin to using homogeneous coordinates on a projective complex space and has the advantage of formally balancing the number of scalar and vector fields.

The most important feature of $\mathcal{N}=2$ vector multiplets is that the field equations (though not the Lagrangian) are invariant under the action of the symplectic group $Sp(2n_V+2,\mathbb{R})$, which acts linearly on the field strength $(F^I_{\mu\nu}, G_{I|\mu\nu})^T$, where the dual field strengths are defined by $G_{I|\mu\nu} := \mathcal{R}_{IJ}F^J_{\mu\nu} - \mathcal{I}_{IJ}\tilde{F}^J_{\mu\nu}$. Symplectic transformations generalise the electric-magnetic duality of the source-free Maxwell equations, and contain stringy symmetries, such as T-duality and S-duality, if the theory under consideration arises as a low-energy effective theory from string theory. By supersymmetry, the full set of field equations is symplectically invariant, with the scalars described by the symplectic vectors $(X^I, F_I)^T$, where $F_I = \partial F/\partial X^I$. One important feature of our method is to preserve manifest symplectic covariance. The n_V physical scalars z^A can be parametrized as

$$z^A = \frac{X^A}{X^0}\ , \tag{2.3}$$

and we can extract them after solving the equations of motion.

One drawback of using the complex scalar fields X^I is that they do not form a symplectic vector by themselves. Similarly, the couplings $g_{I\bar{J}}$, $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ do not transform as tensors under the symplectic group, but have a more complicated behaviour. We will therefore switch to a formulation in terms of real scalar fields q^a, $a=1,\ldots,2n_V+2$, which are related to the complex scalars X^I by

$$(q^a) = \begin{pmatrix} x^I \\ y_J \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X^I \\ F_I \end{pmatrix},$$

and transform as a vector under symplectic transformations. In this formulation, all couplings are encoded in a real function $H = H(q^a)$, called the Hesse potential, which is homogeneous of degree two, and which up to a factor is the Legendre transform of the imaginary part of the prepotential F, see [1] for further details. It is helpful to think of the complex and real formulations of special geometry as being related to one another in the same way as the Lagrangian and Hamiltonian formulations of mechanics, see for example [23] for a detailed discussion.

To include a scalar potential we consider the most general (dyonic) FI gauging of the vector multiplet theory, which depends on $2n_V+2$ parameters $g^a = (g^I, g_I)$, transforming as a vector under symplectic transformations. The expression for the potential in terms of real scalar fields was worked out in appendix A of [3].

2.2 Dimensional Reduction

We will initially construct static solutions, and therefore decompose the four-dimensional spacetime metric as

$$ds_4^2 = -e^{\phi}dt^2 + e^{-\phi}ds_3^2\ , \tag{2.4}$$

where ϕ and all matter fields are independent of time t. The field ϕ is the Kaluza-Klein scalar. There is no Kaluza-Klein vector since we assume the field configuration to be static, that is stationary (time-independent) and hypersurface-orthogonal (no time-space cross-terms). We rearrange the fields using the same method as in [1] in the following way. The Kaluza-Klein scalar ϕ is absorbed into the scalar fields by introducing

$$Y^I := e^{\phi/2} X^I \ . \tag{2.5}$$

We do not introduce a new symbol for the corresponding real scalars q^a, which, are subject to the same rescaling, so that from now on

$$(q^a) = \begin{pmatrix} x^I \\ y_J \end{pmatrix} = \mathrm{Re} \begin{pmatrix} Y^I \\ F_I(Y) \end{pmatrix} .$$

The advantage of this field redefinition is that the Kaluza-Klein scalar is now on the same footing as the four-dimensional scalar fields. When needed, the Kaluza-Klein scalar can be extracted as $e^\phi = -2H$, where H is the Hesse potential considered as a function of the rescaled real scalars q^a. Upon dimensional reduction, the $n_V + 1$ four-dimensional vector fields split into scalars ζ^I and three-dimensional vector fields which can be dualised into a second set of scalars $\tilde{\zeta}_I$. These $2n_V + 2$ scalars can be combined into the symplectic vector $\hat{q}^a = \frac{1}{2}(\zeta^I, \tilde{\zeta}_I)^T$. We refrain from giving the explicit relations between the various fields at this point, and refer the interested reader to [1, 2, 3]. What matters is that all dynamical degrees of freedom are now encoded in the $4n_V + 4$ real scalars $q^a, \hat{q}^a$, which form two symplectic vectors.

2.3 Restricted Field Configurations

In order to obtain solutions of the field equations by decoupling and elementary integration, we make two further assumptions.

1. We will need to know the Hesse potential explicitly, but models are naturally defined in terms of their prepotential, *e.g.* in the context of Calabi-Yau compactifications of string theory. Since the Hesse potential is obtained by a Legendre transformation of the (imaginary part of the) prepotential, it cannot be computed in closed form for a generic prepotential. We will therefore restrict the form of the prepotential in such a way that we can obtain the Hesse potential explicitly.
2. We want the field equations to decouple. This is achieved by imposing a block structure, where the scalar fields within each block are proportional to each other, and where equations within a block do not couple to equations in other blocks. Such block structures appear if we consistently truncate out part of the scalar fields.

We remark that the two types of conditions we impose are not independent. We only need to know a Hesse potential for the subset of fields which are not truncated out consistently. The more fields we truncate out, the larger the class of prepotentials admissible. Conversely, when switching on more and more charges, more and more fields need to be kept in the effective three-dimensional theory, and the prepotentials we can admit become more and more restricted.

For the single charged Nernst brane solutions [3] of gauged supergravity it is sufficient to restrict the prepotential to be of the so-called very special form

$$F(Y) = \frac{f(X^1, \dots X^n)}{X^0}, \tag{2.6}$$

where f is a polynomial homogeneous of degree three. This condition is equivalent to imposing that the vector multiplet theory can be lifted to five dimensions. For solutions with 2, 3 and 4 charges we will show in the following sections that non-trivial solutions can be found when further restricting the prepotential to the forms

$$F_2(X) = \frac{X^1 f_2(X^i, \ldots, X^n)}{X^0}\ , \quad F_3(X) = \frac{X^1 X^2 f_3(X^i, \ldots, X^n)}{X^0}\ , \qquad F_4(X) = \frac{X^1 X^2 X^3}{X^0}\ , \tag{2.7}$$

where f_2 and f_3 are polynomials homogenous of degree 2 and 1 respectively. While $F_2(X)$ is still the generic form for a compactification of the heterotic string on $K3 \times T^2$ at string tree level, $F_4(X)$ is the well known STU model, which is also the minimal example for a prepotential of the form $F_3(X)$. While more general models can be defined and solved for by relaxing the condition that f_3 is a polynomial, we do not know how such models could be embedded into string theory and so restrict ourselves to the polynomial case.

Next we specify the consistent truncation of the scalar fields $q^a, \hat{q}^a$ that we impose to achieve decoupling. In [2] the truncated field configurations were called 'purely imaginary' (PI) because the corresponding four-dimensional physical scalars z^A are purely imaginary. This type of condition is also known as 'axion-free', as in our parametrization the real parts of z^A have an axion-like shift symmetry for prepotentials of the very special form. In terms of three-dimensional scalars the 'PI condition' takes the form

$$(q^a)\big|_{\mathrm{PI}} = (x^0, \ldots, 0; 0, y_1, y_2, \ldots, y_n), \tag{2.8}$$

This is extended to the scalars $\hat{q}^a$ which correspond to four-dimensional gauge fields by

$$(\partial_\mu \hat{q}^a)\big|_{\mathrm{PI}} = \frac{1}{2}(\partial_\mu \zeta^0, \ldots, 0; 0, \partial_\mu \zeta_1, \partial_\mu \zeta_2, \ldots, \partial_\mu \zeta_n).$$

In [3] it was shown that in the presence of FI gauging the analogous condition

$$(g^a)\big|_{\mathrm{PI}} = (g^0, \ldots, 0; 0, g_1, g_2, \ldots, g_n) \tag{2.9}$$

on the gauging parameters extends the factorization to the terms introduced by the scalar potential. While we will not directly use this here, we remark that the PI condition reflects the existence of a distinguished totally goedesic $(2n_V + 2)$-dimensional submanifold of the $(4n_V + 4)$-dimensional scalar manifold of the three-dimensional effective theory obtained by dimensional reduction.

For notational convenience, we adjust the assignment of indices $a, b, \ldots$ to the scalar fields $q^a, \hat{q}^a$ such that the non-constant scalars correspond to indices $a = 1, \ldots, n_V + 1$. We can further simplify the equations of motion through some simple manipulations. All terms in the three-dimensional Lagrangian and field equations which do not involve the scalar potential either involve the constant anti-symmetric matrix

$$\Omega_{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix},$$

or the Hesse potential H and its derivatives. It is convenient to introduce the auxiliary Hesse potential

$$\tilde{H} = -\frac{1}{2}\log(-2H),$$

and its derivatives $\tilde{H}_a, \tilde{H}_{ab}$. Moreover we replace the scalar fields q^a by their duals $q_a = \tilde{H}_a = -\tilde{H}_{ab} q^b$, where we used that $\tilde{H}_{ab}$ are homogeneous functions of degree

-2. While in general we cannot lower indices on $\hat{q}^a$ in the same way, we can lower the indices after differentiation: $\partial_\mu \hat{q}_a := \tilde{H}_{ab}\partial_\mu \hat{q}^b$ [2]. As the fields $\hat{q}^a$ are essentially the four-dimensional gauge potentials, which only enter into the field equations through their derivatives, this is sufficient for rewriting all field equations with indices a in the lower position.

3 Euclidean Instanton Solutions

We are now in the position to formulate the problem that we will solve. Starting from the Lagrangian (2.1), we impose that the four-dimensional metric (2.4) is static, the restrictions (2.7) on the prepotential, the PI truncation (2.8) – (2.9) and finally that the three-dimensional metric has planar symmetry

$$ds_3^2 = e^{4\psi}d\tau^2 + e^{2\psi}(dx^2 + dy^2). \tag{3.1}$$

All fields, including the unknown function ψ depend only on the overall transverse coordinate τ in this brane-like ansatz.

The resulting equations of motion, follow from the general three-dimensional equations for timelike dimensional reduction derived in [1] by imposing the above conditions:

$$\nabla^2 \hat{q}_a = 0, \tag{3.2}$$

$$\nabla^2 q_a + \frac{1}{2}\partial_a \tilde{H}^{bc}(\partial_\mu q_b \partial^\mu q_c - \partial_\mu \hat{q}_b \partial^\mu \hat{q}_c) - \frac{1}{2}\partial_a \tilde{H}_{bc} g^b g^c + 4\tilde{H}_{ab} g^b (g^c q_c) = 0, \tag{3.3}$$

$$-\frac{1}{2}R_{(3)\mu\nu} - \tilde{H}^{ab}(\partial_\mu q_a \partial_\nu q_b - \partial_\mu \hat{q}_a \partial_\nu \hat{q}_b) + g_{\mu\nu}\big(-\tilde{H}_{ab} g^a g^b + 4(g^a q_a)^2\big) = 0. \tag{3.4}$$

The first line are the equations for the scalars $\hat{q}^a$, which correspond to the four-dimensional vector field equations. The second line is the equations for the scalars q^a, which encode the four-dimensional scalars z^A and the Kaluza-Klein scalar ϕ. The last line are the three-dimensional Einstein's equations which determines the three-dimensional warp factor ψ.

To solve Einstein's equations we use that the non-zero part of the Ricci tensor is found to be

$$R_{\tau\tau} = 2\ddot{\psi} - 2\dot{\psi}^2, \quad R_{xx} = R_{yy} = e^{-2\psi}\ddot{\psi},$$

where we use a dot to denote differentiation by τ. The equations (3.4) then reduce to the following form for $\mu, \nu \neq \tau$

$$-\tilde{H}_{ab} g^a g^b + 4(q_a g^a)^2 - \tfrac{1}{2}e^{-4\psi}\ddot{\psi} = 0, \tag{3.5}$$

and for $\mu, \nu = \tau$

$$\tilde{H}^{ab}\big(\dot{q}_a \dot{q}_b - \dot{\hat{q}}_a \dot{\hat{q}}_b\big) = \dot{\psi}^2 - \tfrac{1}{2}\ddot{\psi}, \tag{3.6}$$

where we have substituted in (3.5) to reduce this condition. We see that (3.6) is the Hamiltonian constraint [2, 3].

3.1 Two-Charge Solution

We now turn our attention to generalising the single charge Nernst solution by starting with a solution carrying charge under two gauge fields. The prepotentials we admit take the factorised form

$$F_2(X) = \frac{X^1 f_2(X^2, \ldots, X^n)}{X^0},$$

where f_2 is homogenous of degree two. The corresponding Hesse potential [1] is

$$H = -\frac{1}{4}(-q_0 q_1 f_2(q_2, \dots, q_n))^{-\frac{1}{2}} \; . \tag{3.7}$$

The generality of the function $f_2(q)$ prevents us from obtaining each element of the metric $\tilde{H}^{ab}$, however as the fields q_0 and q_1 decouple we are able to calculate the components we actually need explicitly:

$$\tilde{H}^{00} = \frac{1}{4q_0^2} \; , \qquad \tilde{H}^{11} = \frac{1}{4q_1^2}.$$

We start by solving for $\hat{q}_a$. Since all fields are assumed to only depend on τ, equation (3.2) reduces to

$$\ddot{\hat{q}}_a = 0. \tag{3.8}$$

Integrating up we obtain

$$\dot{\hat{q}}_a = K_a \; . \tag{3.9}$$

The non-vanishing constants K_a are proportional to the electric charge Q_0 and magnetic charge P^1 of the two gauge fields in this solution[1]

$$\dot{\hat{q}}_0 = -Q_0, \qquad \dot{\hat{q}}_1 = P^1 \; . \tag{3.10}$$

We now turn our attention to (3.3) to solve for the scalar fields q_a. Due to the conditions we have imposed, the equations for q_0 and q_1 decouple from the rest and (3.3) reduces to:

$$\ddot{q}_0 - \frac{\dot{q}_0^2 - Q_0^2}{q_0} = 0, \qquad \ddot{q}_1 - \frac{\dot{q}_1^2 - (P^1)^2}{q_1} = 0, \tag{3.11}$$

where we used (3.10). These equations can be integrated up to obtain:

$$\begin{aligned} q_0(\tau) &= \mp \frac{Q_0}{B_0} \sinh\left(B_0 \tau + B_0 \frac{h_0}{Q_0}\right), \\ q_1(\tau) &= \pm \frac{P^1}{B_1} \sinh\left(B_1 \tau + B_1 \frac{h^1}{P^1}\right), \end{aligned} \tag{3.12}$$

where we have introduced the integration constants h_0, B_0, h^1, B_1, where without loss of generality we set $B_0, B_1 \geq 0$. To avoid curvature singularities associated with zeros of the fields q_0, q_1 we require that $\mathrm{sign}(h_0) = \mathrm{sign}(Q_0)$ and $\mathrm{sign}(h^1) = \mathrm{sign}(P^1)$. This ensures that there are no zeros for the domain $0 \leq \tau < \infty$. The remaining equations of motion corresponding to q_A for $A = 2, \dots, n$ are

$$e^{-4\psi}\ddot{q}_A + \tfrac{1}{2}e^{-4\psi}\partial_A \tilde{H}^{BC}\dot{q}_B\dot{q}_C - \tfrac{1}{2}\partial_A \tilde{H}_{BC} g_B g_C + 4\tilde{H}_{AB} g_B \left(g_C q_C\right)^2 = 0. \tag{3.13}$$

The corresponding components of the Einstein equations (3.5) are

$$-\tilde{H}_{AB} g_A g_B + 4\left(g_A q_A\right)^2 - \frac{1}{2}e^{-4\psi}\ddot{\psi} = 0. \tag{3.14}$$

Contracting (3.13) with q_A and substituting in (3.14) we obtain

$$q^A \ddot{q}_A + \tilde{H}^{AB}\dot{q}_A\dot{q}_B = \frac{1}{2}\ddot{\psi} = \frac{d}{d\tau}\left(q^A \dot{q}_A\right).$$

[1] The minus sign in front of Q_0 reflects that K_a transforms as a co-vector, and not as a vector, under symplectic transformations.

We have used here that the LHS can be written as a total derivative which upon integration yields

$$q^A \dot{q}_A = \frac{1}{2}\dot{\psi} - \frac{1}{4}a_0, \tag{3.15}$$

where a_0 is an integration constant and the pre-factor has been chosen for later convenience. This equation can be further rearranged using properties of the Hesse potential [1] and integrated a second time to obtain an expression for the function ψ

$$-2\psi + a_0\tau + b_0 = -2\log(-4H(-q_0q_1)^{\frac{1}{2}}) \ .$$

Substituting in the explicit form of the Hesse potential for this solution allows the realisation of the condition

$$\log(f_2(q_2,\ldots,q_n)) = -2\psi + a_0\tau + b_0. \tag{3.16}$$

Returning to the Hamiltonian constraint (3.6), substituting in the result from (3.12), we find

$$\tilde{H}^{AB}\dot{q}_A\dot{q}_B = \dot{\psi}^2 - \frac{1}{2}\ddot{\psi} - \frac{B_0^2+B_1^2}{4} \ . \tag{3.17}$$

To obtain an explicit expression for the remaining scalars q_A we need to make use of our final assumption that the scalars in the 'block', $q_2,\ldots,q_n$ are proportional to each other:

$$q_A(\tau) = \lambda_A Q(\tau).$$

This, together with the constraint (3.6) allows us to solve the field equations for general homogeneous f_2 by manipulating (3.13) into the form

$$\ddot{\psi} - \dot{\psi}a_0 - \dot{\psi}^2 + \frac{a_0}{4} + \frac{B_0^2+B_1^2}{2} = 0. \tag{3.18}$$

This is solved using the substitution

$$y \equiv \exp\left(-\psi - \frac{a_0\tau}{2}\right), \tag{3.19}$$

and thus the general solution is of the form

$$y = \frac{\alpha}{\omega}\sinh(\omega\tau + \omega\beta) = \exp\left(-\psi - \frac{a_0\tau}{2}\right), \tag{3.20}$$

with two new integration constants α and β, and $\omega^2 := \frac{1}{2}(a_0 + B_0^2 + B_1^2)$ where without loss of generality we set $\omega \geq 0$. This solution of y can then be back-substituted to obtain the form of the scalars

$$q_A(\tau) = \lambda_A e^{a_0}\sinh(\omega\tau + \omega\beta), \tag{3.21}$$

and the warp factor

$$e^{-4\psi} = e^{2a_0\tau}\left(\frac{\alpha}{\omega}\right)^4 \sinh^4(\omega\tau + \omega\beta). \tag{3.22}$$

The constants λ_A are determined by the gauging parameters through requiring consistency of (3.14)

$$\lambda_A = \pm\frac{\alpha^2}{2\omega m g_A} \ , \tag{3.23}$$

where $m = n - 1$ is the number of scalar fields belonging to the block $q_2, \dots, q_n$. From the homogeneity of f_2 we obtain an expression for the constant b_0 as a function of the gauging parameters

$$e^{b_0} = \frac{\alpha^2}{4m^2} f_2\left(\frac{1}{g_2}, \dots, \frac{1}{g_n}\right) . \tag{3.24}$$

In summary, we have obtained the following instanton solution of the reduced, three-dimensional Euclidean field equations, which depend on a single coordinate τ:

$$\begin{aligned}
q_0(\tau) &= \mp\frac{Q_0}{B_0}\sinh\left(B_0\tau + B_0\frac{h_0}{Q_0}\right),\\
q_1(\tau) &= \pm\frac{P^1}{B_1}\sinh\left(B_1\tau + B_1\frac{h^1}{P^1}\right),\\
q_A(\tau) &= \pm\frac{\alpha^2}{2\omega m g_A}e^{a_0\tau}\sinh(\omega\tau + \omega\beta),\\
e^{-4\psi} &= e^{2a_0\tau}\left(\frac{\alpha}{\omega}\right)^4\sinh^4(\omega\tau + \omega\beta).
\end{aligned}$$

We will later find that imposing regularity on the lifted four-dimensional solutions reduces the number of independent integration constants to five.

3.2 Three-Charge Solution

We can generate further solutions by following the previous method for a system supported by an additional gauge field. To do this we further restrict the form of the prepotential to

$$F(Y) = \frac{X^1X^2f_3(X)}{X^0}, \tag{3.25}$$

where $f_3(X)$ is a homogeneous polynomial of degree one. The corresponding Hesse potential is

$$H = -\frac{1}{4}(-q_0q_1q_2f_3(q_3, \dots, q_n))^{-\frac{1}{2}} .$$

The charges are arranged as

$$K_a = (-Q_0, 0, \dots, 0; 0, P^1, P^2, 0\dots, 0),$$

and we additionally switch off the gauging parameter for the $U(1)$ supporting the charge P^2,

$$(g^a)\big|_{\mathrm{PI}} = (0, \dots, 0; 0, 0, 0, g_3, \dots, g_n).$$

The solution of the equations of motion (3.2-3.4) follow the same steps as in the two-charge case with only minor changes. We begin by summarising the steps for this model which are identical to the two-charge solution.

There are now three non-trivial 'hatted' scalar fields

$$\dot{\hat{q}}_a = K_a. \tag{3.26}$$

The scalar fields q_0, q_1, q_2 have decoupled equations of motion which can be integrated as before:

$$\begin{aligned} q_0(\tau) &= \mp \frac{Q_0}{B_0} \sinh\left(B_0\tau + B_0\frac{h_0}{Q_0}\right), \\ q_1(\tau) &= \pm \frac{P^1}{B_1} \sinh\left(B_1\tau + B_1\frac{h^1}{P^1}\right), \\ q_2(\tau) &= \pm \frac{P^2}{B_2} \sinh\left(B_2\tau + B_2\frac{h^2}{P^2}\right). \end{aligned} \tag{3.27}$$

To keep f_3 general we assume that the scalar fields q_A for $A = 3, \ldots, n$ are proportional to each other, $q_A(\tau) = \lambda_A Q(\tau)$, for a set of constants λ_A.

The central difference between this and the two-charge case is caused by f_3 being of degree one rather than two. This changes the balance between the scalar and Hamiltonian conditions, and thus the form of the differential equation for the function ψ

$$\ddot{\psi} - 2\dot{\psi} a_0 + \frac{a_0^2 + B_0^2 + B_1^2 + B_2^2}{2} = 0.$$

This difference results in the disappearance of the $\dot{\psi}^2$ term. This missing term allows the differential equation to be solved using standard methods

$$\psi = \alpha + \beta e^{2a_0\tau} + X\tau, \tag{3.28}$$

where for simplicity we have collected integration constants together

$$X := \left(\frac{a_0^2 + B_0^2 + B_1^2 + B_2^2}{4a_0}\right).$$

Using this form of the warp factor ψ, the solution for the remaining scalars is found to be

$$q_a(\tau) = \lambda_A e^{b_0 - 2\alpha} e^{-2\beta e^{2a_0\tau}} e^{(a_0 - 2X)\tau} . \tag{3.29}$$

At this point we have fixed the functional form of all non-trivial fields:

$$\begin{aligned} q_a(\tau) &= \pm \frac{K_a}{B_a} \sinh\left(B_a\tau + B_a\frac{h^a}{K_a}\right), \\ q_A(\tau) &= \lambda_A e^{b_0 - 2\alpha} e^{-2\beta e^{2a_0\tau}} e^{(a_0 - 2X)\tau} , \\ e^{-4\psi} &= \exp\big(-4\big(\alpha + \beta e^{2a_0\tau} + X\tau\big)\big), \end{aligned}$$

for $a = 0, 1, 2$ and $A = 3, \ldots, n$.

We can reinsert these expressions into the scalar equation of motion as we did for the two-charge case, but unlike before where we were able to set λ_A in terms of the gauge parameters, we instead find that either all B_i must be zero or $\beta = 0$. As we will see later, the limit of $B_i \to 0$ is associated to the extremal limit of the $4D$ solution. To maintain a non-extremal solution we choose $\beta = 0$. From the previous condition $C = \lambda_A g_A$, the constant Λ_A is written explicitly as inversely proportional to the gauging parameters g_A

$$\lambda_A = \frac{C}{g_A}.$$

Finally using the same method as in the two-charge system we find that

$$e^{b_0} = C f_3 \left(\frac{1}{g_A}\right), \tag{3.30}$$

where α has been set to zero through a simple redefinition of the coordinate τ.

In summary, the three-dimensional solution is fully described by:

$$\begin{aligned} q_a(\tau) &= \pm\frac{K_a}{B_a}\sinh\left(B_a\tau + B_a\frac{h^a}{K_a}\right), \\ q_A(\tau) &= \frac{C^2}{g_A}f_3\left(\frac{1}{g_A}\right)e^{-2\beta e^{2a_0\tau}}e^{(a_0-2X)\tau}\ , \\ e^{-4\psi} &= e^{-4X\tau}. \end{aligned}$$

for $a = 0, 1, 2$ and $A = 3, \ldots, n$.

3.3 Four-Charge Solution

We now generate a final solution employing the same method by studying a brane supported by four gauge fields. This requires that the prepotential takes the form

$$F(X) = \frac{X^1X^2X^3}{X^0}, \tag{3.31}$$

which is the well known STU model. The charges are chosen such that

$$K_a = (-Q_0, 0, 0, 0; 0, P^1, P^2, P^3).$$

We necessarily turn off all gauging parameters, thus obtaining the ungauged STU model. From this, we will obtain a solution with planar symmetry. The STU prepotential gives us a simple Hesse potential

$$H = -\frac{1}{4}(-q_0q_1q_2q_3)^{-\frac{1}{2}}\ , \tag{3.32}$$

and we can now completely solve for the metric which is diagonal, with elements given by

$$\tilde{H}^{aa} = \frac{1}{4q_a^2}\ .$$

The scalar equations of motion are now simple to solve. For the hatted scalars we obtain the now familiar solution $\dot{\hat{q}}_a = K_a$. The scalar fields q_a completely decouple from each other, and we obtain the form of the scalars

$$q_a = \pm\frac{K_a}{B_a}\sinh\left(B_a\tau + B_a\frac{h_a}{|K_a|}\right), \tag{3.33}$$

where

$$B_a = (B_0, B_1, B_2, B_3), \quad h_a = (h_0, h^1, h^2, h^3).$$

Looking at the Hamiltonian constraint

$$\frac{1}{4q_a^2}\left(\dot{q}_a^2 - \dot{\hat{q}}_a\right) = \dot{\psi}^2 - \frac{1}{2}\ddot{\psi} = \frac{B_0^2 + B_1^2 + B_2^2 + B_3^2}{4}\ , \tag{3.34}$$

we find

$$-\frac{1}{2}e^{-4\psi}\ddot{\psi} = 0 \quad \Rightarrow \quad \ddot{\psi} = 0.$$

Returning to the Hamiltonian constraint we find

$$\dot{\psi} = \pm\frac{\sqrt{\sum_i B_i^2}}{2} \quad \Rightarrow \quad \psi = \pm\frac{\sqrt{\sum_i B_i^2}}{2}\tau + a_0.$$

This allows us to calculate

$$e^{-4\psi} = e^{-4a_0} e^{\pm 2\sqrt{\sum_i B_i^2}\tau} = e^{-4a_0} e^{\pm 2\sqrt{\sum_i B_i^2}\tau}\,. \tag{3.35}$$

In summary we have found the following planar solution to the time-reduced ungauged STU model:

$$\begin{aligned}\dot{\hat{q}}_a &= K_a,\\ q_a &= \pm\frac{K_a}{B_a}\sinh\left(B_a\tau + B_a\frac{h_a}{K_a}\right),\\ e^{-4\psi} &= e^{-4a_0} e^{\pm 2\sqrt{\sum_i B_i^2}\tau}\,.\end{aligned} \tag{3.36}$$

4 Four-Dimensional Planar Solutions

The three-dimensional Euclidean solutions found in the previous section can now be lifted to four dimensions using the dimensional reduction formulae found originally in [1, 2, 3]. In particular the four-dimensional metric is

$$ds_4^2 = -e^{\phi}dt^2 + e^{-\phi+4\psi}d\tau^2 + e^{-\phi+2\psi}(dx^2+dy^2). \tag{4.1}$$

The four-dimensional physical scalars are determined by the three-dimensional scalars through [2]

$$z^A = -i\left(-\frac{q_0 q_A^2}{f(q)}\right)^{\frac{1}{2}}.$$

Finally the four-dimensional gauge fields are calculated using $\hat{q}_a$ through the relation

$$\hat{q}^a = \frac{1}{2}\begin{pmatrix}\zeta^I\\ \tilde{\zeta}_I\end{pmatrix}, \tag{4.2}$$

where as displayed in (2.2), ζ^I are the components of the gauge fields along the reduction dimension and $\tilde{\zeta}_I$ are the Hodge duals of the three-dimensional vectors. Their relation to the four-dimensional gauge fields can be calculated from [2]

$$\partial_\mu\zeta^I := F^I_{\mu\eta}, \qquad \partial_\mu\tilde{\zeta}_I := G_{I|\mu\eta}, \tag{4.3}$$

where

$$G_{I|\mu\nu} := \mathcal{R}_{IJ}F^J_{\mu\nu} - \mathcal{I}_{IJ}\tilde{F}^J_{\mu\nu}, \tag{4.4}$$

is the dual field strength.

4.1 Two-Charge Solution

We first probe for the existence of a Killing horizon by looking for zeros of the norm of the Killing vector ($k^t = \partial_t$). As k^μ has only one non-zero component, its norm is given as $k^2 = g_{tt}$. In the limit $\tau \to \infty$, it takes the form

$$e^{\phi}\big|_{\tau\to\infty} \sim \exp\left(-\frac{B_0\tau}{2} - \frac{B_1\tau}{2} - a_0\tau - \omega\tau\right). \tag{4.5}$$

We see this always vanishes in the limit when $a_0 \geq 0$. If this restriction is lifted, the horizons position will change depending on the relative magnitudes of B_0, B_1 and a_0, so for now we choose to keep this restriction in place.

The area of the horizon is given by

$$A = \int dxdy e^{-\phi+2\psi}\Big|_{\tau\to\infty} . \tag{4.6}$$

As our x and y coordinates are not compact, this diverges, reflecting the planar symmetry of our ansatz. To obtain finite quantities we could identify x, y periodically, but we prefer to work with densities instead and take ratios relative to the coordinate volume $\int dxdy$. The area density of the horizon is

$$a = e^{-\phi+2\psi}\Big|_{\tau\to\infty} \sim \exp\left(\frac{B_0\tau}{2} + \frac{B_1\tau}{2} + a_0\tau + \omega\tau - a_0\tau - 2\omega\tau\right). \tag{4.7}$$

Imposing that the area density is finite and non-zero requires that

$$\omega = \frac{B_0 + B_1}{2} := B_a. \tag{4.8}$$

Recalling: $2\omega^2 = a_0 + B_0^2 + B_1^2$, we can write down a_0 in terms of B_0 and B_1, $a_0 = B_0B_1 - \frac{1}{2}(B_0^2 + B_1^2)$. We can condense this expression by including B_a

$$a_0 = 2(B_0B_1 - B_a^2),$$

and notice here that $a_0 = 0$ for $B_0 = B_1$. We also notice that in (3.22), the value for the constant β can be changed by a shift in the value of τ; we use this to shift τ such that β vanishes.

To constrain α we look at the limit where $B_0, B_1 \to 0$.[2] Then (3.22) reduces to

$$e^{-4\psi} = \left(\frac{\alpha}{\omega}\right)^4 (\omega\tau)^4,$$

which implies that $\alpha\tau = e^{-\psi}$. This allows us to scale τ such that $\alpha = 1$.

The integration constants can be further constrained by imposing regularity of the physical scalar fields z^1, z^A. According to [2]

$$Y^1 = -\frac{i}{2}e^{\phi}q_1, \quad Y^A = -\frac{i}{2}e^{\phi}q_A, \quad Y^0 = -\frac{1}{4q_0}, \tag{4.9}$$

where Y^I are rescaled scalar fields defined in (2.5). This yields:

$$z^1 = -i\left(\frac{-q_0q_1^2}{q_1 f_2(q)}\right)^{\frac{1}{2}}, \quad z^A = -i\left(\frac{-q_0q_A^2}{q_1 f_2(q)}\right)^{\frac{1}{2}}. \tag{4.10}$$

Next we impose that the physical fields take finite values on the horizon. Since

$$z^1\Big|_{\tau\to\infty} \sim e^{-a_0\tau}, \quad z^A\Big|_{\tau\to\infty} \sim (e^{B_0\tau}e^{-B_1\tau})^{\frac{1}{2}}, \tag{4.11}$$

this can be satisfied by setting $B_0 = B_1 = B \quad \Rightarrow \quad B_a = B \Rightarrow a_0 = 0$.

[2] We will see later that this is the extremal limit, that is, the limit where the surface gravity of the Killing horizon goes to zero.

Having imposed regularity we end up with the following solution:

$$
\begin{aligned}
q_0(\tau) &= \mp \frac{Q_0}{B} \sinh\left(B\tau + B\frac{h_0}{Q_0}\right), \\
q_1(\tau) &= \pm \frac{P^1}{B} \sinh\left(B\tau + B\frac{h^1}{P^1}\right), \\
q_A(\tau) &= \pm \frac{1}{2Bmg_A} \sinh(B\tau), \\
e^{\phi} &= \frac{1}{2}(-q_0 q_1 f_2(q_2, \ldots, q_n))^{-\frac{1}{2}} , \\
e^{-4\psi} &= \left(\frac{1}{B}\right)^4 \sinh^4(B\tau),
\end{aligned}
\tag{4.12}
$$

where only 5 out of the original 9 integration constants remain: h_0, h^1, Q_0, P^1, B.

4.2 Three-Charge Solution

We proceed as in the two-charge case. Since the computations are similar, we don't need to give many details. The condition for the existence of a Killing horizon at $\tau \to \infty$ is:

$$
k^2 = e^{\phi} = \exp\left(-\frac{B_0\tau}{2} - \frac{B_1\tau}{2} - \frac{B_2\tau}{2} + \beta e^{2a_0\tau} - \frac{(a_0 - 2X)}{2}\tau\right) \xrightarrow[\tau\to\infty]{} 0 \ ,
$$

implying

$$
B_0, B_1, B_2, a_0 - 2X > 0 \quad \text{and} \quad \beta \leq 0 \text{ or } a_0 < 0.
$$

Regularity of the physical scalar fields demands that the physical scalars

$$
z^a = -i\left(\frac{-q_0 q_a^2}{q_1 q_2 f_3(q_A)}\right)^{\frac{1}{2}}, \qquad \text{where } a = 1, \ldots, n,
$$

take finite values on the horizon. For $\tau \to \infty$ we find:

$$
\begin{aligned}
z^1\big|_{\tau\to\infty} &\sim \exp\left[\left(B_0 + B_1 - B_2 + \beta e^{2a_0\tau} - (a_0 - 2X)\right)\tau\right], \\
z^2\big|_{\tau\to\infty} &\sim \exp\left[\left(B_0 + B_2 - B_1 + \beta e^{2a_0\tau} - (a_0 - 2X)\right)\tau\right], \\
z^A\big|_{\tau\to\infty} &\sim \exp\left[\left(B_0 - \beta e^{2a_0\tau} + (a_0 - 2X) - B_1 - B_2\right)\tau\right].
\end{aligned}
$$

The exponential term is absent or decreasing if $\beta \leq 0$ or $a_0 < 0$. Imposing that the remaining terms cancel implies the following three simultaneous equations:

$$
\begin{aligned}
B_0 + B_1 - B_2 - (a_0 - 2X) &= 0, \\
B_0 + B_2 - B_1 - (a_0 - 2X) &= 0, \\
B_0 + (a_0 - 2X) - B_1 - B_2 &= 0.
\end{aligned}
$$

This implies $B_0 = B_1 = B_2 = a_0 - 2X$. We set $B := B_1 = B_2 = B_3$ and impose that the horizon area density is finite for $\tau \to \infty$, this is given by

$$
e^{-\phi+2\psi}\Big|_{\tau\to\infty} \sim \exp\left[\left(\frac{a_0^2 + 3B^2}{2a_0} + \frac{3B}{2} + \frac{a_0}{2} - \frac{a_0^2 + 3B^2}{4a_0}\right)\tau\right],
$$

giving us one last constraint

$$\frac{a_0^2 + 3B^2}{2a_0} + \frac{3B}{2} + \frac{a_0}{2} - \frac{a_0^2 + 3B^2}{4a_0} = 0,$$

and we find that

$$B = B_0 = B_1 = B_2 = -a_0 = -X. \tag{4.13}$$

Hence our solution takes the form

$$\begin{aligned}
q_0(\tau) &= \mp \frac{Q_0}{B} \sinh\left(B\tau + B\frac{h_0}{Q_0}\right), \\
q_1(\tau) &= \pm \frac{P^1}{B} \sinh\left(B\tau + B\frac{h^1}{P^1}\right), \\
q_2(\tau) &= \pm \frac{P^2}{B} \sinh\left(B\tau + B\frac{h^2}{P^2}\right), \\
q_A(\tau) &= \pm \frac{C}{g_A} \exp\left(B\tau\right),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
e^{\phi} &= \frac{1}{2}(-q_0 q_1 q_2 f_3(q_3, \ldots, q_n))^{-\frac{1}{2}}, \\
e^{-4\psi} &= e^{4B\tau},
\end{aligned} \tag{4.15}$$

we see that we have reduced the total number of integration constants to just 7.

4.3 Four-Charge Solution

The four-dimensional physical scalar fields take the form

$$z^A = -i\left(-\frac{q_0 q_A^2}{q_1 q_2 q_3}\right)^{\frac{1}{2}}. \tag{4.16}$$

Imposing that these fields are finite in the limit $\tau \to \infty$ implies:

$$\begin{aligned}
B_0 + B_1 - B_2 - B_3 &= 0, \\
B_0 + B_2 - B_1 - B_3 &= 0, \\
B_0 + B_3 - B_2 - B_1 &= 0,
\end{aligned}$$

which is satisfied when the integration constants obey $B_0 = B_1 = B_2 = B_3 = B$. The integration constant a_0 can be removed by a suitable shift in the τ coordinate which simplifies the warp factor to

$$e^{-4\psi} = e^{\pm 2\sqrt{\sum_i B_i^2}\tau}.$$

Then, regularity of the horizon area density further dictates that

$$e^{-\phi+2\psi}\Big|_{\tau\to\infty} \sim \exp\left(\frac{4B\tau}{2} \mp 2B\tau\right) \tag{4.17}$$

is finite, where we used $\sqrt{\sum_i B_i^2} = 2B > 0$. To cancel the terms inside the exponential we should therefore pick the (+)-sign in the solution (3.36) for ψ. The explicit solution takes the following form:

$$
\begin{aligned}
\dot{\hat{q}}_a &= K_a, \\
q_a &= \pm\frac{K_a}{B}\sinh\left(B\tau + B\frac{h_a}{|K_a|}\right), \\
e^{\phi} &= \frac{1}{2}(-q_0q_1q_2q_3)^{-\frac{1}{2}}\ , \\
e^{-4\psi} &= e^{4B\tau}.
\end{aligned}
\tag{4.18}
$$

Later, when oxidising the four-charge solution to ten and eleven dimensions we will need the explicit form of the gauge fields. As we assume all three-dimensional components depend only on the coordinate τ the non-zero components are found from (4.2-4.4)

$$
(\dot{A}^0)_\eta = 2\dot{\hat{q}}^0 = 2\tilde{H}^{00}\dot{\hat{q}}_0 = -\frac{Q_0}{2q_0^2(\tau)}, \qquad (\dot{\tilde{A}}_A)_\eta = \frac{P^A}{2q_A^2(\tau)}, \tag{4.19}
$$

where the dot references differentiation by the parameter τ.

5 Properties of Black Planar Solutions

In this section, we investigate the properties of the four-dimensional solutions obtained in the previous section. For the two-charge solution we find that we are able to follow the methodology of [3] to produce a meaningful discussion of the solution's geometry. For the three- and four-charged solutions we find that this is not the case. Their resulting four-dimensional spacetime is geodesically incomplete for both ends of the domain of the transverse coordinate. Through coordinate changes and analytic continuation, we show that these higher charged solutions have time-dependent asymptotics and timelike singularities.

5.1 Two-Charge Solution

Following previous experience [3] we introduce a new transverse coordinate ρ by setting

$$
e^{-2B\tau} = 1 - \frac{2B}{\rho} =: W(\rho). \tag{5.1}
$$

Applying the coordinate change to the scalars q_a in (4.12) we obtain

$$
q_0 = \mp\frac{\mathcal{H}_0}{W^{1/2}}, \quad q_1 = \pm\frac{\mathcal{H}_1}{W^{1/2}}, \quad q_A = \frac{1}{2mg_A}\frac{1}{\rho W^{1/2}}, \tag{5.2}
$$

where we have introduced the harmonic functions:

$$
\begin{aligned}
\mathcal{H}_0(\rho) &:= Q_0\left[\frac{1}{B}\sinh\left(\frac{Bh_0}{Q_0}\right) + \frac{e^{-\frac{Bh_0}{Q_0}}}{\rho}\right], \\
\mathcal{H}_1(\rho) &:= P^1\left[\frac{1}{B}\sinh\left(\frac{Bh^1}{P^1}\right) + \frac{e^{-\frac{Bh^1}{P^1}}}{\rho}\right].
\end{aligned}
$$

The physical scalars z^a as functions of ρ are given by

$$z^1 = -2im\rho\sqrt{\mathcal{H}_0\mathcal{H}_1}\, f_2^{-1/2}\left(\frac{1}{g}\right), \quad z^A = i\sqrt{\frac{\mathcal{H}_0}{\mathcal{H}_1}}\frac{1}{g_A} f_2^{-1/2}\left(\frac{1}{g}\right).$$

Their construction ensures regularity at the horizon. Their asymptotic behaviour in the limit $\rho \to \infty$ depends on h_0, h_1 and is summarised in the table below.

h_I	z^1	z^A
$h_0,\ h_1 \neq 0$	ρ	Const.
$h_0 = 0$	$\rho^{1/2}$	$\rho^{-1/2}$
$h_1 = 0$	$\rho^{1/2}$	$\rho^{1/2}$
$h_0 = h_1 = 0$	Const.	Const.

Next, we re-write the four-dimensional metric (4.1) in terms of our new transverse coordinate ρ. Applying (5.1) to the functions ψ and ϕ gives

$$e^{-4\psi} = \frac{1}{B^4}\sinh^4(B\tau), \qquad e^{\phi} = \frac{1}{2}\left[\frac{\mathcal{H}_0}{W^{1/2}}\frac{\mathcal{H}_1}{W^{1/2}}\frac{1}{4m^2\rho^2 W} f_2\left(\frac{1}{g}\right)\right]^{1/2}. \tag{5.3}$$

Combining this with (4.1) we obtain

$$ds^2 \;=\; -\frac{W\rho}{\mathcal{H}}dt^2 + \frac{\mathcal{H}}{W\rho}d\rho^2 + \mathcal{H}\rho(dx^2 + dy^2), \tag{5.4}$$

where it has been convenient to define a new function

$$\mathcal{H}(\rho) := \frac{\sqrt{\mathcal{H}_0\mathcal{H}_1}}{m} f_2^{1/2}\left(\frac{1}{g_2}, \dots, \frac{1}{g_n}\right).$$

The function $\mathcal{H}$ encodes the contributions of charges and gauging parameters, whereas $W(\rho)$ is a blackening factor which controls the deviation from extremality.

To study the near-horizon behaviour of the metric (5.4) we introduce a new transverse coordinate

$$r^2 \equiv \rho - 2B.$$

Then for $r \ll 1$

$$d\rho^2 = 4r^2 dr^2, \quad W \simeq \frac{r^2}{2B}.$$

The near horizon value for the harmonic functions are calculated

$$\mathcal{H}_0(2B) = \frac{Q_0}{2B}\exp\left(\frac{Bh_0}{Q_0}\right), \quad \mathcal{H}_1(2B) = \frac{P^1}{2B}\exp\left(\frac{Bh^1}{P^1}\right) \;\Rightarrow\; \mathcal{H}(2B) = \frac{Z\mathcal{E}}{B},$$

where we have defined

$$Z := \frac{\sqrt{Q_0 P^1}}{2m} f_2^{-1/2}\left(\frac{1}{g_2}, \dots, \frac{1}{g_n}\right), \qquad \mathcal{E} := \exp\left(\frac{B}{2}\left(\frac{h_0}{Q_0} + \frac{h^1}{P^1}\right)\right).$$

Using this compact form, the near horizon metric can be found by substitution into (5.4)

$$ds^2 \;=\; -\frac{Br^2}{Z\mathcal{E}}dt^2 + \frac{4Z\mathcal{E}}{B}dr^2 + 2Z\mathcal{E}(dx^2 + dy^2). \tag{5.5}$$

By Wick rotating the time component $t \to t_E = -it$ we can find the Hawking temperature of the Killing horizon[3]

$$4\pi T_H = \frac{B}{Z\mathcal{E}}. \tag{5.6}$$

This confirms the expected interpretation of B as a non-extremality parameter and of $B \to 0$ as an extremal limit. The solution can be interpreted as a black brane, with entropy density

$$s = 2Z\mathcal{E}. \tag{5.7}$$

We can then use (5.6) and (5.7) to obtain an expression for the integration constant

$$B = 2\pi T_H s,$$

which is interestingly the same as in [3]. We note that unlike the Nernst solution, this two-charge solution will have finite entropy density in the extremal limit.

Finally, we consider the behaviour in the limit $\rho \to \infty$. While $W \to 1$, the behaviour of the harmonic functions $\mathcal{H}_0, \mathcal{H}_1$ depends on whether the integration constants $h_0,\ h^1$ are finite or zero:

$$\lim_{\rho\to\infty} \mathcal{H}_a \sim \begin{cases} \text{Constant} & \text{for } h_a \neq 0, \\ \rho^{-1} & \text{for } h_a = 0. \end{cases} \tag{5.8}$$

where a runs over $0, 1$. This in turn gives the asymptotic form of $\mathcal{H}$. As the asymptotic behaviour of $\mathcal{H}$ is only sensitive to the number of finite integration constants, rather than the particular constant itself, there are three cases, namely both constants finite, one finite, or both zero

$$\mathcal{H}\big|^2_{\rho\to\infty} = C_2, \quad \mathcal{H}\big|^1_{\rho\to\infty} = \frac{C_1}{\rho^{1/2}}, \quad \mathcal{H}\big|^0_{\rho\to\infty} = \frac{C_0}{\rho}\ ,$$

where the superscript index labels the number of non-zero elements. The corresponding line elements are:

$$\begin{aligned} &h_0, h^1 \quad \neq 0 \qquad && ds^2 = -\frac{\rho}{C_2}dt^2 + C_2\frac{d\rho^2}{\rho} + C_2\rho(dx^2+dy^2), \\ &h_0 \text{ or } h^1 \quad \neq 0 \qquad && ds^2 = -\frac{\rho^{3/2}}{C_1}dt^2 + C_1\frac{d\rho^2}{\rho^{3/2}} + C_1\rho^{1/2}(dx^2+dy^2)\ , \\ &h_0, h^1 \quad = 0 \qquad && ds^2 = -\frac{\rho^2}{C_0}dt^2 + C_0\frac{d\rho^2}{\rho^2} + C_0(dx^2+dy^2). \end{aligned} \tag{5.9}$$

These geometries can be brought to standard forms using further coordinate transformations. For $h_0, h^1 \neq 0$, we define R by

$$\pm R = \log(\rho),$$

and the line element reduces to

$$ds^2 \ = \ e^{\pm R}\big(-dt^2 + dR^2 + dx^2 + dy^2\big),$$

showing that the metric is conformally flat.

[3] Computing the temperature via the surface gravity yields the same result.

When both h_0 and h^1 are zero we use the coordinate transformation $\rho = R^{-1}$ and obtain

$$ds^2 = \frac{1}{R^2}\Big(-dt^2 + dR^2\Big) + dx^2 + dy^2,$$

which decomposes as $\mathrm{AdS}_2 \times \mathbb{R}^2$.

Finally, when looking at the case when only one of the integration constants are finite we use the transformation $\rho = R^{-2}$ to obtain

$$ds^2 = \frac{\lambda}{R^3}\Big(-\frac{dt^2}{\lambda^2} + 4dR^2\Big) + \frac{\lambda}{R}\Big(dx^2 + dy^2\Big), \tag{5.10}$$

for some constant λ. The simplification achieved by the coordinate transformation is that the terms proportional to dt^2 and dR^2 have the same dependence on R.

5.2 Three-Charge Solution

As mentioned at the beginning of the section, this solution is fundamentally different from [3] and the two-charge solution. Previously, null geodesics were able to be extended to an infinite affine parameter in the limit of $\rho \to \infty$; this justified the definition of this as the asymptotic limit of the solution.[4]

As we will show below, for the three- and four-charge solutions, $\rho = \infty$ is reached by transverse null geodesics at finite affine parameter and therefore cannot be interpreted as the asymptotic region. Introducing a new transverse coordinate ζ, which is defined by $\rho = \zeta^{-1}$, we will arrive at the picture summarized in Figure 1: the locus $\rho \to \infty \Leftrightarrow \zeta \to 0$ is of no particular significance. The Killing horizon is located at $\zeta = (2B)^{-1}$ and the static region $\zeta < (2B)^{-1}$ terminates at some value $\zeta = \zeta_s$ with a timelike curvature singularity, which is reached by transverse null geodesics at finite affine parameter. By analytic continuation the solution can be extended to the region $(2B)^{-1} < \zeta < \infty$, where the coordinate ζ becomes timelike and where the limit $\zeta \to \infty$ is at infinite (timelike) distance. We will interpret region I, $\zeta_s < \zeta < (2B)^{-1}$ as the inside region, and region II, $(2B)^{-1} < \zeta < \infty$ as the outside region, because it has an asymptotic boundary at infinite distance.

We now give the details and make further comments on the properties of the three-charge solution. Beginning by applying the coordinate change (5.1) to the three-charge solution, the scalars q_0, q_1, q_2 are found to be:

$$q_0 = \mp\frac{\mathcal{H}_0}{W^{1/2}}, \quad q_1 = \pm\frac{\mathcal{H}_1}{W^{1/2}}, \quad q_2 = \pm\frac{\mathcal{H}_2}{W^{1/2}}, \tag{5.11}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined exactly as before and we have further defined

$$\mathcal{H}_2(\rho) := P^2\left[\frac{1}{B}\sinh\left(\frac{Bh^2}{P^2}\right) + \frac{e^{-\frac{Bh^2}{P^2}}}{\rho}\right].$$

The remaining scalars are given by

$$q_A = \pm\frac{C}{g_A}\frac{1}{W^{1/2}}.$$

[4] Commonly, an asymptotic limit is defined by the maximally symmetric geometry associated with the vacuum solution. For a four-dimensional solution with planar symmetry, this type of fall-off is not expected and so our best definition for an asymptotic region comes from the behaviour of the null geodesics of the solution.

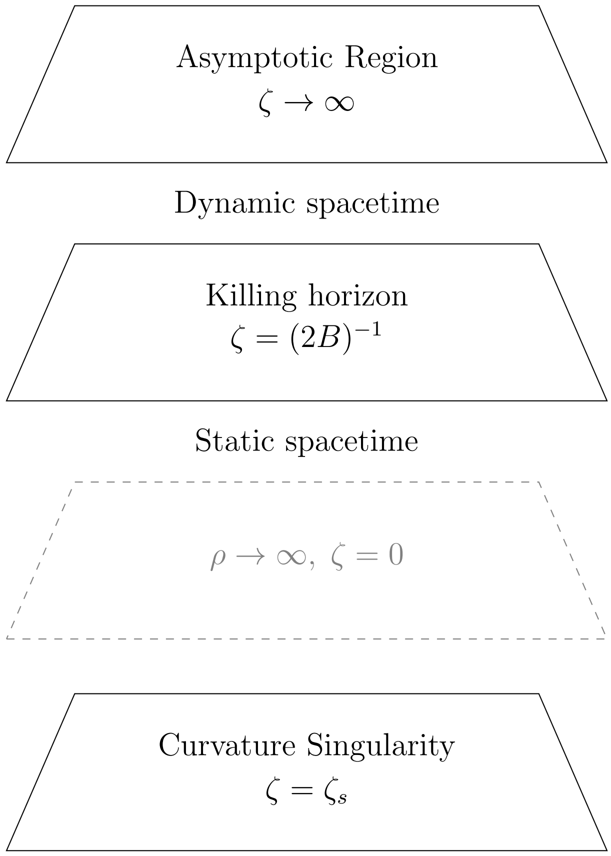

Figure 1: Diagram of the brane solution. When starting from the 3D solution, a static patch of the spacetime is found, parametrised by τ for $\zeta \in [0, (2B)^{-1}]$. We can extend this spacetime to a singularity where the Kretschmann invariant becomes infinite. Analytically continuing our parameter through the horizon to $\zeta > (2B)^{-1}$ we obtain a time-dependent geometry.

The metric degrees of freedom are

$$e^{-4\psi} = W(\rho)^{-2},$$

and

$$e^{\phi} = \frac{1}{2}\left(\frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2}{W^2} C f_3 \left(\frac{1}{g_A}\right)\right)^{-\frac{1}{2}} = \frac{W}{\mathcal{H}},$$

where we have defined the new function

$$\mathcal{H}(\rho) := 2\sqrt{C f_3 \mathcal{H}_0\mathcal{H}_1\mathcal{H}_2}\,.$$

This produces the line element

$$ds^2 = -\frac{W}{\mathcal{H}}dt^2 + \frac{\mathcal{H}}{W}\frac{d\rho^2}{\rho^4} + \mathcal{H}(dx^2 + dy^2). \tag{5.12}$$

The Lagrangian (energy functional) for transverse geodesics in the metric (5.12) is

$$\mathcal{L} = -\frac{W}{\mathcal{H}}\dot{t}^2 + \frac{\mathcal{H}}{W\rho^4}\dot{\rho}^2,$$

where the dot represents differentiation with respect to an affine parameter λ. Null geodesics satisfy $\mathcal{L} = 0$. The corresponding constant of motion

$$E = \frac{W\dot{t}}{\mathcal{H}},$$

is rearranged to give

$$\dot{\rho} = \pm\sqrt{\rho^4 E^2}, \qquad \lambda = \pm \int \frac{d\rho}{E\rho^2}. \tag{5.13}$$

This shows that light signals sent from $\rho > 2B$ reach $\rho = \infty$ at finite affine parameter, whereas $\rho \to 0$ is at infinite affine parameter. Therefore, $\rho \to 0$ should be interpreted as being at infinite distance and $\rho < 2B$ as the exterior region, while $\rho > 2B$ is the inside region.

Given this observation, we introduce the new transverse coordinate $\zeta = \rho^{-1}$ so that infinity is now at $\zeta \to \infty$. It is important to note that in order to reach the limit of $\zeta \to \infty$ we must cross the Killing horizon located at $\zeta = (2B)^{-1} =: \alpha^{-1}$ into a new, 'exterior' region. In the exterior ζ is a timelike coordinate, and since the line element depends explicitly on ζ, the exterior is non-stationary and as such the solution is interpreted as cosmological.

We also note that for causal information coming from an asymptotic distance the Killing horizon is a cosmological horizon which is located at a point in time $\zeta = \alpha^{-1}$ and so will be necessarily crossed for all causal geodesics.[5] Once the horizon has been crossed, the timelike singularity can be avoided and geodesics may leave the static patch into a second dynamic patch of spacetime. This statement is justified with calculations later in the paper.

Following our coordinate transformation the metric can be written in the following form

$$ds^2 = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2). \tag{5.14}$$

Note that we have relabelled the coordinate t, which is interpreted as time in the interior, as η. This is because t becomes a spacelike coordinate in the outside patch of spacetime. In the following we will use the neutral notation η, ζ instead of t, ρ. The metric functions are $W = 1 - \alpha\zeta$ and $\mathcal{H}(\zeta)$ and

$$\begin{aligned}
\mathcal{H}_0(\zeta) &= Q_0 \left[\frac{2}{\alpha}\sinh\left(\frac{\alpha h_0}{2Q_0}\right) + e^{-\frac{\alpha h_0}{2Q_0}}\zeta\right], \\
\mathcal{H}_1(\zeta) &= P^1 \left[\frac{2}{\alpha}\sinh\left(\frac{\alpha h^1}{2P^1}\right) + e^{-\frac{\alpha h^1}{2P^1}}\zeta\right], \\
\mathcal{H}_2(\zeta) &= P^2 \left[\frac{2}{\alpha}\sinh\left(\frac{\alpha h^2}{2P^2}\right) + e^{-\frac{\alpha h^2}{2P^2}}\zeta\right],
\end{aligned}$$

We will write this in a condensed format by redefining our integration constants such that

$$\mathcal{H}_a = (\beta_a + \gamma_a \zeta),$$

for $a = 0, 1, 2$.

[5]This mandatory crossing can be understood in the same way as all causal geodesics reaching the singularity for the Schwarzschild solution once the horizon has been crossed.

To demonstrate that the metric can be analytically continued to $\zeta > \alpha^{-1}$ despite the coordinate singularity at $\zeta = \alpha^{-1}$ we make an intermediate coordinate transformation to advanced Eddington-Finkelstein coordinates

$$v = \eta + \zeta^*, \qquad d\zeta^* = \frac{\mathcal{H}}{W} d\zeta,$$

where we have introduced the tortoise coordinate ζ^* such that the metric can be written in the form

$$ds^2 = -\frac{W}{\mathcal{H}} dv^2 + 2d\zeta dv + \mathcal{H}(dx^2 + dy^2),$$

This shows that the metric has no singularity at $\zeta = \alpha^{-1}$ and so we can analytically continue the coordinate ζ to $\zeta > \alpha^{-1}$, and then reverse the coordinate transformation to obtain the metric for the dynamic patch of the spacetime

$$ds^2 = -\frac{W(\zeta)}{\mathcal{H}(\zeta)} d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)} d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2)$$

for $\zeta > \alpha^{-1}$. Note that $W(\zeta)$ is an everywhere negative function within the domain of the dynamic patch of the spacetime. To have a clearer picture of our spacetime we define a new, always positive function within this domain $\mathcal{W}(\zeta) := \alpha\zeta - 1$.

Using this, we can write down the metric for $\zeta > \alpha^{-1}$ where it is immediately obvious that the coordinate ζ is timelike.

The exterior region (I) is the cosmological region where ζ is timelike and the metric is time-dependent. The inside region (II) where $\zeta < \alpha^{-1}$, η is timelike, and as we will see below spacetime ends at a timelike singularity located at ζ_s, where ζ_s is the first zero of $\mathcal{H}(\zeta)$. Their respective line elements are given by

$$\begin{aligned} ds_I^2 &= -\frac{\mathcal{H}(\zeta)}{\mathcal{W}(\zeta)} d\zeta^2 + \frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)} d\eta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2), \\ ds_{II}^2 &= -\frac{W(\zeta)}{\mathcal{H}(\zeta)} d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)} d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2). \end{aligned} \tag{5.15}$$

We postpone a discussion of Kruskal-like coordinates and of the Penrose-Carter diagram for after the discussion of the four-charge solution, as these two solutions lend themselves naturally to being discussed simultaneously.

Having found coordinates suitable for describing both regions of our solution, we can now start analysing its properties. We begin with the physical scalars

$$z_1 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_1 W^{1/2}}{\mathcal{H}_2 f_3(g)}\right)^{\frac{1}{2}}, \quad z_2 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_2 W^{1/2}}{\mathcal{H}_1 f_3(g)}\right)^{\frac{1}{2}}, \quad z_A = -i\left(\frac{\mathcal{H}_0 W^{-1/2}}{\mathcal{H}_1\mathcal{H}_2 f_3(g)}\right)^{\frac{1}{2}}.$$

The asymptotic behaviour in the limit $\zeta \to \infty$ depends on whether the integration constants h_0, h_1, h_2 are zero or non-zero, and is summarised in the table below.

Following standard calculations, we are able to find the curvature scalars corresponding to the metric (5.14). The Ricci scalar is found to be

$$R = \frac{W\left(\mathcal{H}'^2 - 2\mathcal{H}\mathcal{H}''\right)}{2\mathcal{H}^3}, \tag{5.16}$$

	z_1	z_2	z_A
All Finite	$\zeta^{3/4}$	$\zeta^{3/4}$	$\zeta^{3/4}$
$h_0 = 0$	$\zeta^{1/4}$	$\zeta^{1/4}$	$\zeta^{5/4}$
$h_1 = 0$	$\zeta^{1/4}$	$\zeta^{5/4}$	$\zeta^{1/4}$
$h_2 = 0$	$\zeta^{5/4}$	$\zeta^{1/4}$	$\zeta^{1/4}$
$h_0, h_1 = 0$	$\zeta^{1/4}$	$\zeta^{3/4}$	$\zeta^{3/4}$
$h_1, h_2 = 0$	$\zeta^{3/4}$	$\zeta^{3/4}$	$\zeta^{1/4}$
$h_0, h_2 = 0$	$\zeta^{3/4}$	$\zeta^{1/4}$	$\zeta^{3/4}$
All Zero	$\zeta^{1/4}$	$\zeta^{1/4}$	$\zeta^{1/4}$

and the Kretschmann scalar, $K = R^{abcd}R_{abcd}$, is given by

$$\begin{aligned} K = &\frac{3W^2\mathcal{H}''^2}{\mathcal{H}^4} - \frac{2W\mathcal{H}'\mathcal{H}''\left(4W\mathcal{H}' + 3\alpha\mathcal{H}\right)}{\mathcal{H}^5} \\ &+ \frac{\mathcal{H}'^2(27W^2\mathcal{H}'^2 + 44\alpha W\mathcal{H}\mathcal{H}' + 20\alpha^2\mathcal{H}^2)}{4\mathcal{H}^6}, \end{aligned} \tag{5.17}$$

where the prime denotes a derivative with respect to ζ. We find that both have singular behaviour for the limit of $\mathcal{H}(\zeta) \to 0$. As $\mathcal{H}(\zeta)$ is a polynomial of degree three which factorizes into three linear polynomials it will in general have three distinct zeros at $\zeta = \gamma_a\beta_a^{-1}$. The boundary of the spacetime domain $\zeta < (2B)^{-1}$ is given by the largest of these zeros, or 'first zero of $\mathcal{H}(\zeta)$', which we denote ζ_s. We remark that $\zeta_s \leq 0$ for all values of the integration constants.

We can study the near horizon geometry of this solution employing the same method as for the two-charge solution. The horizon values of the harmonic functions are:

$$\mathcal{H}_0(\alpha^{-1}) = \frac{Q_0}{\alpha}\exp\left(\frac{\alpha h_0}{2Q_0}\right), \quad \mathcal{H}_1(\alpha^{-1}) = \frac{P^1}{\alpha}\exp\left(\frac{\alpha h^1}{2P^1}\right),$$
$$\mathcal{H}_2(\alpha^{-1}) = \frac{P^2}{\alpha}\exp\left(\frac{\alpha h^2}{2P^2}\right).$$

Making a coordinate transformation

$$\chi^2 = \zeta - \alpha^{-1}, \qquad d\zeta^2 = 4\chi^2 d\chi^2, \tag{5.18}$$

we find

$$W \simeq \alpha\chi^2, \quad \mathcal{H}(\alpha^{-1}) = \frac{Z\mathcal{E}}{\alpha^{3/2}}, \tag{5.19}$$

where we have defined the constants

$$Z := 2\Lambda\sqrt{Q_0P^1P^2}, \quad \mathcal{E} := \exp\left(\frac{\alpha}{4}\left(\frac{h_0}{Q_0} + \frac{h^1}{P^1} + \frac{h^2}{P^2}\right)\right), \quad \Lambda = \sqrt{Cf_3}. \tag{5.20}$$

Substituting this in, we write down the near horizon line element

$$ds^2 = -\frac{\alpha^{5/2}\chi^2}{Z\mathcal{E}}d\eta^2 + \frac{4Z\mathcal{E}}{\alpha^{5/2}}d\chi^2 + \frac{Z\mathcal{E}}{\alpha^{3/2}}(dx^2 + dy^2). \tag{5.21}$$

To find the temperature of the Killing horizon we set

$$dR^2 = \left(\frac{4Z\mathcal{E}}{\alpha^{5/2}}\right) d\chi^2,$$

and Wick rotate $\eta \to -i\eta_E$ to obtain the Hawking temperature

$$2\pi T_H = \frac{\alpha^{5/2}}{2Z\mathcal{E}} \, . \tag{5.22}$$

This shows that α should still be interpreted as the non-extremality parameter, with extremal limit $\alpha \to 0$. We can read off the entropy density of the solution as

$$s = \frac{Z\mathcal{E}}{\alpha^{3/2}}, \tag{5.23}$$

and note that it diverges in the limit $\alpha \to 0$. Equating the above two equations, we can solve for the integration constant[6]

$$B = 2\pi s T_H,$$

which is the same relationship that we saw in the Nernst solution [3] and the two-charge solution.

In region I we can consider the asymptotic limit $\zeta \to \infty$, which corresponds to future timelike infinity. Taking the limit of the functions

$$\lim_{\zeta\to\infty} \frac{\mathcal{W}}{\mathcal{H}} = \frac{\alpha}{\Lambda\sqrt{\gamma_0\gamma_1\gamma_2}}\frac{1}{\sqrt{\zeta}}, \qquad \lim_{\zeta\to\infty} \mathcal{H} = \Lambda\sqrt{\gamma_0\gamma_1\gamma_2}\zeta^{\frac{3}{2}},$$

we find the asymptotic form of the metric

$$ds^2 = -\frac{\Lambda\sqrt{\gamma_0\gamma_1\gamma_2}}{\alpha}\sqrt{\zeta}d\zeta^2 + \frac{\alpha}{\Lambda\sqrt{\gamma_0\gamma_1\gamma_2}}\frac{1}{\sqrt{\zeta}}d\eta^2 + \Lambda\sqrt{\gamma_0\gamma_1\gamma_2}\zeta^{\frac{3}{2}}(dx^2+dy^2). \tag{5.24}$$

Taking the coordinate change $\zeta = t^2$ and absorbing constant factors, we can write this metric as

$$ds^2 = t^3(-dt^2 + dx^2 + dy^2) + \frac{1}{t}d\eta^2, \tag{5.25}$$

or taking

$$\zeta = \tau^{\frac{4}{5}} \qquad d\tau = \sqrt{\zeta}d\zeta^2,$$

to obtain a metric in the form

$$ds^2 = -d\tau^2 + \tau^{-\frac{2}{5}}d\eta^2 + \tau^{\frac{6}{5}}(dx^2+dy^2). \tag{5.26}$$

5.3 Four-Charge Solution

It will turn out that the qualitative behaviour of the four-charge solution is the same as that of the three-charge solution, and we proceed accordingly. We introduce the transverse coordinate ζ by

$$e^{-2B\tau} = 1 - \alpha\zeta := W(\zeta),$$

[6] Here we replace $\alpha = 2B$ to allow for better comparison to our previous results.

and the horizon is located at $\zeta = \alpha^{-1} = (2B)^{-1}$. Applying the same coordinate change to our scalars (4.16) we obtain

$$q_a = \pm\frac{\mathcal{H}_a}{W^{1/2}}, \tag{5.27}$$

where we have defined the harmonic functions

$$\mathcal{H}_a(\zeta) := |K_a|\left[\frac{2}{\alpha}\sinh\left(\frac{\alpha h_a}{2|K_a|}\right) + e^{-\frac{\alpha h_a}{2|K_a|}}\zeta\right], \quad a = 0, \ldots, 3.$$

Using (4.16) we obtain the following expressions for the physical scalars:

$$z^1 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_1}{\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{2}}, \quad z^2 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_2}{\mathcal{H}_1\mathcal{H}_3}\right)^{\frac{1}{2}}, \quad z^3 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_3}{\mathcal{H}_1\mathcal{H}_2}\right)^{\frac{1}{2}}.$$

Taking the limit $\zeta \to \infty$ we find that

$$\lim_{\zeta\to\infty}\mathcal{H}_a = K_a e^{-\frac{\alpha h_a}{2|K_a|}}\zeta,$$

and that the scalars all tend to a constant value as all $\mathcal{H}_a$ depend on ζ in the same manner.

We now wish to re-express the metric ansatz (4.1) with our new coordinates. We find that

$$e^{-4\psi} = e^{4B\tau} = \frac{1}{W^2}, \tag{5.28}$$

and

$$e^{\phi} = \frac{1}{2}(-q_0q_1q_2q_3)^{-\frac{1}{2}} = \frac{W}{\mathcal{H}}, \tag{5.29}$$

where we have defined:

$$\mathcal{H}(\zeta) := 2\sqrt{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}. \tag{5.30}$$

Simply substituting in these into our ansatz (4.1) we obtain the line element for the interior region of the solution as

$$ds^2_{II} = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2).$$

We notice here the same functional form of the four-charge solution as the three-charge solution (5.14), with the only difference coming from the polynomial order of $\mathcal{H}(\zeta)$. Analytically continuing to $\zeta > \alpha^{-1}$ we find that $W(\zeta)$ changes sign. This suggests we redfine $-W(\zeta) =: \mathcal{W}(\zeta) = \alpha\zeta - 1$ and now the metric between the asymptotic limit and the horizon is described by

$$ds^2_I = -\frac{\mathcal{H}(\zeta)}{\mathcal{W}(\zeta)}d\zeta^2 + \frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2), \tag{5.31}$$

where we notice that our metric depends only on the timelike coordinate ζ. As the metric is the same in form as (5.14) the curvature scalars will be given by (5.16, 5.17) and we see there is a curvature singularity when $\mathcal{H} = 0$, which happens whenever $\mathcal{H}_a(\zeta) = 0$. Without loss of generality, we assume that the first zero of $\mathcal{H}$ will be for $\mathcal{H}_0 = 0$, so that the singularity will occur at:

$$\zeta_s = \frac{1 - e^{\frac{\alpha h_0}{Q_0}}}{\alpha} = -\frac{\beta_0}{\gamma_0}. \tag{5.32}$$

From the relations (4.19) we can apply our coordinate transformation to write down the gauge fields in terms of the new ζ coordinate

$$F^0_{\zeta\eta} = -\frac{Q_0}{2(\beta_0 + \gamma_0\zeta)^2}\ , \qquad \tilde{F}_{A|\zeta\eta} = \frac{P^A}{2(\beta_A + \gamma_A\zeta)^2}\ . \tag{5.33}$$

We can again study the near-horizon geometry with the coordinate change (5.18) and probing for when $\zeta \simeq \alpha^{-1}$

$$d\zeta^2 = 4\chi^2 dr^2, \quad \mathcal{W} = \alpha\chi^2, \quad \mathcal{H} = \frac{2Z\mathcal{E}}{\alpha^2},$$

where we have defined

$$Z := \sqrt{Q_0 P^1 P^2 P^3}, \quad \mathcal{E} := \exp\left(\frac{\alpha}{4}\left(\frac{h_0}{Q_0} + \frac{h^1}{P^1} + \frac{h^2}{P^2} + \frac{h^3}{P^3}\right)\right).$$

We now substitute these expressions into our metric to obtain the near-horizon line element

$$ds^2 = -\frac{\alpha^3\chi^2}{2Z\mathcal{E}}d\eta^2 + \frac{8Z\mathcal{E}}{\alpha^3}d\chi^2 + \frac{2Z\mathcal{E}}{\alpha^2}(dx^2 + dy^2). \tag{5.34}$$

Following the method as before, we Wick rotate after the coordinate change to find the Hawking temperature associated to the brane

$$2\pi T_H = \frac{\alpha^3}{4Z\mathcal{E}},$$

and read off the entropy density

$$s = \frac{2Z\mathcal{E}}{\alpha^2}.$$

We notice that as for the three-charge solution, the entropy density diverges in the extremal limit of $\alpha \to 0$. We also find that the equation of state is again given by

$$B = 2\pi s T_H,$$

which thereby is established as a standard relation for the full set of solutions ranging from Nernst branes to the four-charge solution.

In the asymptotic limit, we take $\zeta \to \infty$ and we find that

$$\lim_{\zeta\to\infty} \mathcal{H}_a(\zeta) \simeq K_a e^{\frac{\alpha h_a}{2K_a}}\zeta, \qquad \lim_{\zeta\to\infty} \mathcal{H}(\zeta) \simeq 2Z\mathcal{E}\zeta^2, \qquad \lim_{\zeta\to\infty} \mathcal{W}(\zeta) \simeq \alpha\zeta.$$

We use this to write down the asymptotic metric

$$ds^2 = -\frac{2Z\mathcal{E}\zeta}{\alpha}d\zeta^2 + \frac{\alpha}{2Z\mathcal{E}\zeta}d\eta^2 + 2Z\mathcal{E}\zeta^2(d\bar{x}^2 + d\bar{y}^2),$$

and with a simple change of coordinates to absorb all of the constants we find the asymptotic metric is in the form

$$ds^2 = -\bar{\zeta}d\bar{\zeta}^2 + \frac{1}{\bar{\zeta}}d\bar{\eta}^2 + \bar{\zeta}^2(dx^2 + dy^2). \tag{5.35}$$

This can be identified with the planar Schwarzschild solution (AIII metric) [24] with the mass $M = \frac{1}{2}$. Through the coordinate transformation

$$\bar{\eta} = \left(\frac{3}{2}\right)^{\frac{1}{3}} z, \qquad \bar{\zeta} = \left(\frac{9}{4}\right)^{\frac{1}{3}} \tau^{\frac{2}{3}}, \qquad (x, y) = \left(\frac{4}{9}\right)^{\frac{1}{3}} (x, y),$$

we can rewrite the asymptotic metric in the form

$$ds^2 = -d\tau^2 + \tau^{2/3} dz^2 + \tau^{4/3}(dx^2 + dy^2),$$

which is the type D vacuum Kasner solution [25]. The Kasner solution is given generally by

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2,$$

where the constants (p_1, p_2, p_3) must satisfy the planar and spherical Kasner constraints

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1,$$

and we see that our asymptotic solution is the case for $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. The Penrose diagram for the vacuum Kasner type D solution is given by figure (2).

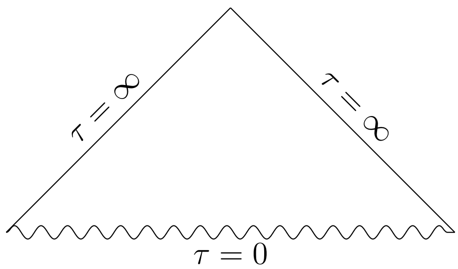

Figure 2: Penrose diagram for type D Kasner solution

6 Causal Structure of Cosmological Solutions

Of the three solutions that we have constructed, the higher charged solutions are more surprising. We have found that their static part is the interior of a solution that on the outside is time-dependent and thus can be interpreted as a cosmological solution. In this section, we further analyse the resulting spacetimes by studying the behaviour of causal geodesics and massive particles. We begin by outlining the coordinate transformations that lead to a Kruskal-type metric for the three- and four-charge solutions. This allows the construction of the Penrose-Carter diagram for these solutions and gives us an intuitive diagrammatic overview of the causal structure. This representation also allows us to realise that the solutions we study for the three- and four-charge systems have an intersection with a class of cosmological solutions studied in [7, 8] for the case of generalised Einstein-Maxwell-Dilaton and the orientifold constructions in [9]. We then study the static patch, and hence the singularity, in more detail by probing them with causal geodesics and the worldlines of stationary massive particles. We find that all timelike geodesics are repelled by the singularity and that stationary massive particles experience negative acceleration with respect to the singularity.

6.1 Kruskal Coordinates

While searching for an asymptotic region, we already found the Eddington-Finkelstein coordinates:

$$ds^2 \;=\; -\frac{W(\zeta)}{\mathcal{H}(\zeta)}dv^2 + 2d\zeta dv + \mathcal{H}(\zeta)(dx^2 + dy^2),$$

where for the three- and four-charge solutions

$$\mathcal{H}^{(3)}(\zeta) = 2\sqrt{Cf_3\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2}, \qquad \mathcal{H}^{(4)}(\zeta) = 2\sqrt{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}.$$

To find Kruskal coordinates we first make a second coordinate change into light-cone coordinates

$$u = \eta - \zeta^*, \qquad ds^2 = -\frac{W}{\mathcal{H}}dvd\zeta + \mathcal{H}(dx^2 + dy^2).$$

The key to finding Kruskal coordinates is integrating ζ^*

$$\zeta^* = \int d\zeta \frac{\mathcal{H}(\zeta)}{W(\zeta)},$$

and picking a suitable $\lambda(\zeta)$ for the transformation to Kruskal-like coordinates

$$U = -e^{-\lambda u}, \quad V = e^{\lambda v},$$

where $U \leq 0$ and $V \geq 0$, and λ is picked to remove the factor of W, and hence all zeros in the metric. Because of the form of the functions W and $\mathcal{H}$ this procedure can become algebraically involved. However finding the coordinate transformation explicitly is not needed to infer its existence and to draw the Penrose-Carter diagram.

A simple example, where finding the explicit form of the transformation is not too involved, is the case where all functions $\mathcal{H}_a$ are taken to be equal. This simplifies the calculation of ζ^* and hence allows for easy identification of λ. For those interested, the explicit expressions for this case are included in the appendix.

The final form to which the four-charge solution can be brought is

$$ds^2 = -\frac{1}{\lambda^2}\frac{e^{\xi(\zeta(U,V))}}{2(\beta + \gamma\zeta(U,V))^2}dUdV + 2(\beta + \gamma\zeta(U,V))^2(dx^2 + dy^2), \tag{6.1}$$

where ξ is a function defined from the integrand of ζ^* and packages together the results of the coordinate change into an everywhere non-zero, monotonically increasing function for the global domain of ζ, allowing to express ζ in terms of U, V. The constants β, γ are the constants coming from the simplification $\gamma = \gamma_0 = \gamma_1 = \gamma_2 = \gamma_3$ and $\beta = \beta_0 = \beta_1 = \beta_2 = \beta_3$ ensuring all $\mathcal{H}_a$ are equal.

If the global metric is known explicitly, it can then be used to construct the Penrose-Carter diagram. If no explicit expression is available, the causal diagram has to be constructed piece-wise, by patching together regions separated by regular horizons. The diagram for the general three- and four-charge solution is given in figure (3). It looks like the Penrose-Carter diagram of the maximally extended Schwarzschild spacetime, rotated by 90 degrees. Solutions with the same causal structure have been found previously in [7]. The regions I and III are time-dependent and asymptotic to Kasner solutions at late and early times, respectively. This cosmological solution is disturbed by the presence of two timelike singularities, which can be interpreted as brane-like sources, which create the static regions II and IV and are separated from the cosmological regions I and III by a bifurcate Killing horizon.

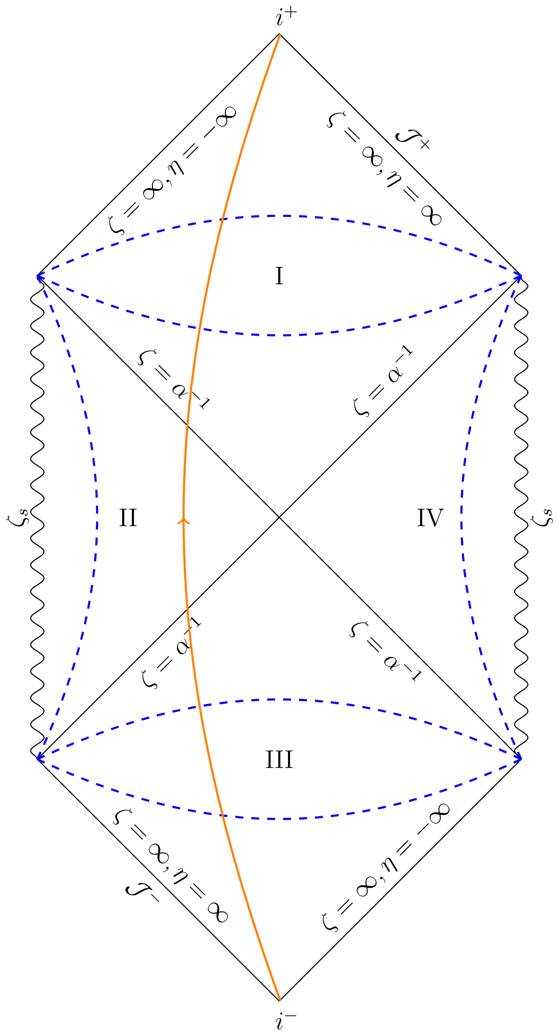

Figure 3: Penrose diagram for the planar cosmological solutions. Starting at $\zeta = \infty$ we have a cosmological spacetime (I) with a horizon located at a finite point in time; any observer must necessarily fall through the horizon. Passing through the horizon, the spacetime is static (II) with an avoidable (repulsive) naked singularity located at a point in space. Massive particles at rest experience negative acceleration and will leave the static region into a second dynamic spacetime. An example of a complete timelike geodesic is given in orange, spacelike hypersurfaces of constant time are given in blue.

6.2 Extremal Limit

For our solutions, the extremal limit is defined by the vanishing of the temperature T_H, which is equivalent to taking the limit of $\alpha \to 0$ and is represented in the metric functions and integration constants by:

$$\alpha \to 0 \quad \Rightarrow \quad \mathcal{W}(\zeta) \to -1, \qquad \beta_a \to h_a \zeta, \qquad \gamma_a \to K_a. \tag{6.2}$$

The resulting line element is given by

$$ds^2 = -\mathcal{H}^{-1}(\zeta)d\eta^2 + \mathcal{H}(\zeta)d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2),$$

where η, ζ are now everywhere timelike and spacelike respectively.

The extremal limit for the three- and four-charged solutions has a dramatic effect on the causal structure of the spacetime. As the function $\mathcal{W}$ becomes constant we find that the location of the horizon is set by $\mathcal{H}^{-1} \to 0$, which occurs when $\zeta \to \infty$. The horizon location is pushed from $\alpha^{-1} \to \infty$ and the resulting spacetime is everywhere static with a naked singularity. This change in the causal structure is a general feature of the planar symmetric solutions we consider, and the simplest example is found for the planar symmetric Riessner-Nordström solution. Further discussion of the relationship between the causal structure and the extremal limit is left for section 9 and the shifting of horizons under this limit is depicted in Figure 5.

6.3 Probing the Static Patch

In this section, we study the static regions II and IV. Obtaining a mass-like parameter for our solution is obstructed in two ways: firstly, the asymptotic region of the spacetime is time-dependent and so any conserved quantity associated to our Killing vector is going to be more physically related to momentum than mass.[7] Secondly, the solution is not asymptotically flat, and as such, there will be a complication as to how to properly normalise the norm of the Killing vector field and hence the conserved quantity associated to it.

To ameliorate these issues, we present two different methods of calculating a mass-like parameter. We first follow the work of [7] and perform a space-dependent calculation for the Komar mass. Secondly, we employ the Brown-York formalism [26] and calculate a quasi-local mass. Both techniques consistently give a mass-like parameter, which is negative and suggests that the singularity is repulsive to neutral massive particles. Following up on this observation, we show that there always exists a classical turning point for massive particles following geodesics. Further, we find that all massive particles at rest undergo negative acceleration. We conclude that the spacetime is 'timelike geodesically complete,' that is timelike geodesics can be extended to infinite proper time and that the singularity is repulsive.

For the following section, both the three- and four-charge solutions are analysed simultaneously; when differences between the solution arise, this is expressed explicitly for the metric functions. The difference for our integration constants is always left implicit.

6.3.1 Mass

Komar Mass

We begin our investigation of a local mass parameter within the static region by using the standard Komar integral

$$M_K = -\frac{1}{8\pi}\int_{\mathbb{R}^2} \star dk,$$

[7] Here we are interpreting our conserved charges as Noether charges. In static spacetimes, we associate time translation invariance with energy. In the dynamic region where our Killing vector is spacelike, invariance under spacelike translations is associated with conserved momentum.

for a timelike Killing vector k. In more conventional solutions this integral is evaluated while taking the asymptotic limit where the Komar mass matches with that of the ADM mass for solutions with the appropriate asymptotic fall off. For our solution, the domain of the static region is finite, and hence we leave k^2 unnormalised to obtain a mass-like parameter dependent on the spacelike coordinate ζ.

For a Killing vector: $k^\mu = (1, 0, 0, 0)$ and taking the Hodge dual, with orientation set by $\epsilon_{\eta\zeta xy} = 1$ we find

$$(\star dk)_{\eta\zeta} = -\mathcal{H}\partial_\zeta \left(\frac{W}{\mathcal{H}}\right).$$

The Komar integral is evaluated to

$$M_K = -\frac{1}{8\pi}\int_{\mathbb{R}^2} \left(\alpha + \frac{W}{\mathcal{H}}\partial_\zeta \mathcal{H}\right).$$

Due to the planar symmetry of our solution, this value is divergent when integrating over the plane, so we instead work with the mass density. The resulting position dependent mass density is

$$m_K = -\left(\frac{\alpha}{8\pi} + \frac{1}{8\pi}\frac{W g(\zeta)}{\mathcal{H}^2}\right), \tag{6.3}$$

where the function $g(\zeta)$ is related to the derivative of $\mathcal{H}$

$$\partial_\mu \mathcal{H} = \frac{g(\zeta)}{\mathcal{H}}.$$

For the three-charge solution

$$g(\zeta) = 2Cf_3(\gamma_0\mathcal{H}_1\mathcal{H}_2 + \gamma_1\mathcal{H}_0\mathcal{H}_2 + \gamma_2\mathcal{H}_0\mathcal{H}_1),$$

and for the four-charge solution

$$g(\zeta) = 2(\gamma_0\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3 + \gamma_1\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3 + \gamma_2\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3 + \gamma_3\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2).$$

Within the domain of the static region $\mathcal{H}, g, W > 0$. As α is always positive, the Komar mass will be everywhere negative within the static patch of the spacetime, regardless of the overall normalisation of k^2.

Brown-York Mass

Alternatively, we can calculate the Quasi-Local mass of the spacetime by using the Brown-York formalism [27]. The Brown-York quasi-local energy is found from[8]

$$E = -\frac{1}{8\pi}\int_{\mathbb{R}^2} \sqrt{\sigma} N(\mathtt{k} - \mathtt{k}_0), \tag{6.4}$$

In this formalism, we consider a physical spacetime M, which is topologically a hypersurface Σ, foliated over a real line interval. The boundary of Σ is B. Taking the product of B with the timelike worldlines orthogonal to Σ produces the

[8] We note here the inclusion of the lapse function N. For asymptotically flat spacetimes $\lim_{\zeta\to\infty} N = 1$ and so N is absent from many papers in the literature. The inclusion of N is talked about in more detail when considering non-asymptotically flat spacetimes, asymptotically AdS spaces being the most common example of this currently [28].

codimension-1 hypersurface ${}^3B$, a component of the 3-boundary of M. The full boundary of M includes the end points of timelike worldlines.

The spacetime M is equipped with the metric $g_{\mu\nu}$ and Levi-Civita connection ∇_μ. To calculate (6.4), we will need the geometric data of B in terms of the known data of (M, g). We take a future pointing unit vector u^μ, normal to the foliation Σ. A tensor T is said to be spatial when $T \cdot u = 0$. The metric $g_{\mu\nu}$ induces a metric on Σ, which, when regarded as a tensor $h_{\mu\nu}$ on M, is a spatial tensor. The induced covariant derivative $\mathcal{D}_\mu$ for spatial tensors is found through projection $\mathcal{D}_\mu = h^\nu_\mu \nabla_\nu$. The extrinsic curvature of Σ as an embedded submanifold of M is denoted $K_{\mu\nu}$. We use the notation h_{ij}, K_{ij}, where i, j run from one to the dimension of Σ, when regarding the metric and extrinsic curvature of Σ as tensors on Σ.

The ADM decomposition of the metric is given by

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt), \tag{6.5}$$

for a lapse function N and shift vector V^i.

We proceed in the same way with the 3-boundary ${}^3B$ by considering the outward pointing unit vector n^μ, normal to ${}^3B$. The metric induced by $g_{\mu\nu}$ is denoted γ_{mn} when regarded as a tensor on ${}^3B$ and $\gamma_{\mu\nu}$ when regarded as a horizontal tensor on M, *i.e.* as a tensor T on M satisfying $n \cdot T = 0$.

The boundary B, which is the intersection of Σ and ${}^3B$ has a metric $\sigma_{\mu\nu}$ which can be induced from either of the codimension-1 manifolds or the spacetime itself. The extrinsic curvature $\mathtt{k}_{\mu\nu}$ of B – the vital part needed to calculate (6.4) – is computed using the embedding of B in Σ:

$$\begin{aligned} \mathtt{k}_{\mu\nu} &= \sigma^\alpha_\mu \mathcal{D}_\alpha n_\nu \\ &= \gamma^\alpha_\mu h^\beta_\nu h^\rho_\alpha \nabla_\rho n_\beta \end{aligned} \tag{6.6}$$

We will also need the trace $\mathtt{k} = \sigma^{\mu\nu}\mathtt{k}_{\mu\nu}$ in our later calculations.

The quasi-local energy is evaluated by studying the Hamiltonian that generates a unit time translation orthogonal to Σ at the boundary B. This is related to the action together with a normalisation coming from a chosen background metric denoted with a subscript 0. For more details on the full derivation of (6.4) see [27].

Comparing our metric with the ADM decomposition (6.5) we identify

$$N^2 = \frac{W}{\mathcal{H}}, \quad V^i = 0, \quad \sigma_{xx} = \sigma_{yy} = 2\mathcal{H}.$$

The quasi-local energy is then simply calculated using these quantities together with the trace of the extrinsic curvature (6.6)

$$\mathtt{k} = \frac{1}{\mathcal{H}}\sqrt{\frac{W}{\mathcal{H}}}\partial_\zeta \mathcal{H}. \tag{6.7}$$

As there is no divergent contribution from the calculation there is no natural normalisation choice for $\mathtt{k}_0$, and so, for now, we set it to zero; simplifying our calculations

$$\begin{aligned} E_{BY} &= -\frac{1}{8\pi}\int d^2x \sqrt{\sigma} N \mathtt{k}, \\ &= -\frac{1}{4\pi}\int d^2x \frac{W g(\zeta)}{\mathcal{H}^2}. \end{aligned} \tag{6.8}$$

When the spacetime is static, the Brown-York energy is equivalent to the mass of the spacetime. Removing the divergent factor related to the planar symmetry and again looking at the mass density we find

$$m_{BY} = -\frac{1}{4\pi}\frac{Wg}{\mathcal{H}^2}. \tag{6.9}$$

This is negative definite in the static domain due to identical reasoning as for the Komar calculation.

We note here that despite both being everywhere negative $m_K \neq m_{BY}$. We could have handpicked $\mathtt{k}_0$ to match our result to the Komar calculation but without an asymptotic limit to properly normalise the Killing vector k^μ there is no reason pick either of our results as 'correct'. This undecidability of overall normalisation is not important for our current discussion as we focus on that the calculated mass-like parameter is everywhere negative throughout the static region rather than the precise value.

6.3.2 Geodesic Motion

We now turn our attention to studying the motion of causal geodesics within the static region of our spacetime. Using the metric, we can write down the Lagrangian of our system

$$s = \mathcal{L} = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}\dot{\eta}^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}\dot{\zeta}^2 + \mathcal{H}(\dot{x}^2 + \dot{y}^2), \tag{6.10}$$

where $s = 0, -1$ for null and timelike geodesics respectively. We calculate the constants of motion as

$$E = \frac{W}{\mathcal{H}}\dot{\eta}, \qquad a = \mathcal{H}\dot{x}, \qquad b = \mathcal{H}\dot{y},$$

allowing us to rewrite the Lagrangian as

$$s\frac{W}{\mathcal{H}} = -E^2 + \dot{\zeta}^2 + (a^2 + b^2)\frac{W}{\mathcal{H}^2}. \tag{6.11}$$

This can be rearranged into the familiar form

$$\dot{\zeta}^2 = E^2 - V(\zeta), \qquad V(\zeta) = \frac{W}{\mathcal{H}}\left(-s + \frac{(a^2 + b^2)}{\mathcal{H}}\right), \tag{6.12}$$

which can be interpreted as the equation of motion for a particle with mass $m = 2$. We rearrange this equation and package together the function $V(\zeta)$ to explicitly highlight that this piece can be interpreted as an effective 'potential' of the system. The domain of validity for the equation of motion is restricted by the inequality

$$V(\zeta) \leq E^2.$$

The point at which $V(\zeta_0) = E$ is interpreted as the classical truning point of the particle's trajectory. The domain is further restricted by the presence of the singularity such that $\zeta > \zeta_s$ and so the domain of ζ in region II is given by

$$\alpha^{-1} > \zeta > -\frac{\beta_0}{\gamma_0}. \tag{6.13}$$

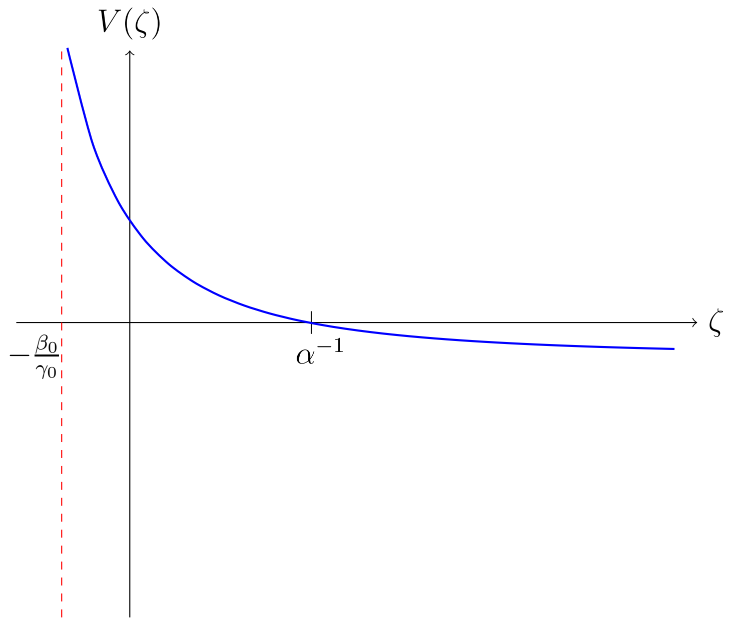

Figure 4: Behaviour of the effective 'potential' as a function of ζ for the set of casual geodesics excluding null transverse geodesics, for which $V = 0$.

Now, by studying the potential $V(\zeta)$ of our spacetime for the correct domain of ζ, we can look at the motion of causal information along geodesics.

In figure 4 we plot $V(\zeta)$ and see that for region II of our spacetime the potential is everywhere positive and therefore repulsive. When decreasing ζ from α^{-1} to ζ_s we see that the potential monotonically increases until it diverges in the limit of the singularity. As such we are guaranteed a unique solution for $V(\zeta_0) = E^2$, and hence the existence of a classical turning point.

There is one exception to this, the case when $s = a = b = 0$, specific to the motion along transverse null geodesics where the potential is everywhere zero. We see that our spacetime is not geodesically complete as transverse null rays can reach the singularity in a finite proper time.

We conclude that for non-zero potentials a particle will arrive from $\mathcal{J}^-$ and necessarily fall through the horizon at $\zeta = \alpha^{-1}$. The particle will then continue towards the singularity to a minimum distance ζ_0. At this point, it will then be reflected and continue off through the Killing horizon into a second dynamic spacetime towards $\mathcal{J}^+$. The only causal geodesics which do not follow these trajectories are those for which $V(\zeta) = 0$. These are precisely the transverse null geodesics which fall through the horizon from $\mathcal{J}^-$ and straight into the singularity.

6.3.3 Proper Acceleration

Seeing that all timelike geodesics are repelled by the singularity, it is interesting to also study the acceleration of massive particles at rest within the static patch of the spacetime.

Particles at rest follow orbits of the stationary Killing vector field k^μ with a proper velocity defined by

$$u^\mu = \frac{k^\mu}{\sqrt{-k^2}},$$

where the normalisation has been chosen such that $u^2 = -1$. From this, the proper acceleration can be found

$$A^\mu = u^\nu \nabla_\nu u^\mu = \frac{1}{2}\partial^\mu \log(-k^2). \tag{6.14}$$

The metric for the three- and four-charge solution is in the standard static form

$$ds^2 = -f(\zeta)d\eta^2 + f(\zeta)^{-1}d\zeta^2 + g(\zeta)(dx^2 + dy^2). \tag{6.15}$$

The Killing vector is given by

$$k^\mu = (1,0,0,0) \qquad \Rightarrow \qquad k^2 = -f(\zeta),$$

allowing us to calculate the proper acceleration of a massive particle at rest

$$A^\mu = \frac{1}{2}g^{\mu\nu}\partial_\nu \log\left(f(\zeta)\right).$$

For these symmetric, static solutions the metric depends only on one coordinate, ζ, and the only non zero component of the proper acceleration is

$$A^\zeta = \frac{1}{2}f(\zeta)\partial_\zeta \log\left(f(\zeta)\right) = \frac{1}{2}\partial_\zeta f(\zeta).$$

For the case of the three- and four-charge solutions we have

$$f(\zeta) = \frac{W(\zeta)}{\mathcal{H}(\zeta)},$$

and so the proper acceleration is found to be

$$A^\zeta = -\frac{\alpha\mathcal{H} + W\partial_\zeta\mathcal{H}}{\mathcal{H}^2}. \tag{6.16}$$

As the functions W, $\mathcal{H}$ and $\partial_\zeta\mathcal{H}$ are everywhere positive in the static region of spacetime we see that a particle at rest always experiences a force repelling it from the singularity. We remark that the qualitative behaviour of geodesics and Killing orbits is the same as for the interior region of the non-extremal Reissner-Nordström solution (usually called region III). Moreover in both cases, this static interior region is related by horizons both in the past and in the future to regions where the Killing vector field becomes spacelike. The main difference is that the Reissner-Nordstrom solution has a third type of region (usually called region I) which is static and asymptotically flat. We will come back to this comparison in Section 9.

7 Dimensional Lifting of the Cosmological STU Solution

To better understand the physical origin of the four-charge solution, we turn our attention to finding consistent higher dimensional embeddings. This is motivated by the success of [4] which offered a new understanding of the Nernst solution [3] by using a five-dimensional embedding. We are further motivated by the work of [9] and comments made by [7] which link cosmological solutions to higher dimensional theories reduced on orientifolds.

We remark that the cosmological solution of the STU model found in [29] can be shown to describe part of the space-time of our four-charge solution, using different coordinates.[9] Their solution covers part of our dynamic patch and does not include the Killing horizon. The higher-dimensional interpretation of their solution was through a lift from $4D$ to $10D$ on the orientifold $K3 \times T^2/\mathbb{Z}_2$. We will instead consider simpler, toroidal lifts, which will allow to relate our cosmological solution to black hole solutions of the STU model and to make contact with six-dimensional BPS solutions. We will follow the oxidation prescription of [30] to write down consistent truncated string/M-theory Lagrangians and their corresponding metric and gauge field content.

This section is organised as follows: first, we rewrite our Lagrangian (2.1) in a form that allows a direct comparison with [30]. We then uplift our non-extremal planar solutions of the STU model from $D = 4$ to $D = 5, 6, 10, 11$, expressing our solutions as embedded into truncations of string-/M-theory. Upon taking the extremal limit of the $4D$ solution, we make contact with well-known brane configurations in string/M-theory models. Additional fine-tuning of the $4D$ electric charges is shown to make its $6D$ uplift supersymmetric in addition to being extremal. This is an interesting result as we did not utilise Killing spinor equations and therefore the existence of supersymmetric limit was not guaranteed.

7.1 Rewriting the Lagrangian for Uplift

Our starting point is the Lagrangian (2.1), repeated here for reference

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^B + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}.$$

Explicit expressions for the gauge couplings are obtained from the prepotential $F(X)$ using standard special geometry formulae. We use the same conventions as [31]. When imposing the 'purely imaginary' conditions on the scalars the gauge couplings take the form:

$$\mathcal{R}_{IJ} = 0, \qquad \mathcal{I}_{IJ} = \text{diag}\left(-stu, -\frac{tu}{s}, -\frac{su}{t}, -\frac{st}{u}\right), \tag{7.1}$$

$$g_{A\bar{B}} = \text{diag}\left(\frac{1}{4s^2}, \frac{1}{4t^2}, \frac{1}{4u^2}\right), \tag{7.2}$$

where

$$s = -\text{Im}(z^1), \quad t = -\text{Im}(z^2), \quad u = -\text{Im}(z^3).$$

When evaluating the couplings $\mathcal{I}_{IJ}$ on our solution, we find, using the ζ-coordinate system (5.31)

$$\mathcal{I}_{00}^2 = \frac{\mathcal{H}_0^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}, \qquad \mathcal{I}_{11}^2 = \frac{\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3}{\mathcal{H}_1^3}, \qquad \mathcal{I}_{22}^2 = \frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3}{\mathcal{H}_2^3}, \qquad \mathcal{I}_{33}^2 = \frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2}{\mathcal{H}_3^3}.$$

[9] The explicit relation between their solution and ours is complicated, and we will not need it in the following.

After redefining our scalars

$$s = e^{-\phi_1}, \qquad t = e^{-\phi_2}, \qquad u = e^{-\phi_3},$$

the Lagrangian takes the following form

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{4}e^{-\phi_1-\phi_2-\phi_3}\left[(F^0)^2 + e^{2\phi_A}(F^A)^2\right], \tag{7.3}$$

where we sum over $i = 1, 2, 3$. Using the STU couplings (7.1) we can evaluate the scalars φ_i on our solution and thus express them as functions of ζ

$$e^{2\varphi_1} = \frac{\mathcal{I}_{11}}{\mathcal{I}_{00}} = \frac{\mathcal{H}_2\mathcal{H}_3}{\mathcal{H}_0\mathcal{H}_1}, \quad e^{2\varphi_2} = \frac{\mathcal{I}_{22}}{\mathcal{I}_{00}} = \frac{\mathcal{H}_1\mathcal{H}_3}{\mathcal{H}_0\mathcal{H}_2}, \quad e^{2\varphi_3} = \frac{\mathcal{I}_{33}}{\mathcal{I}_{00}} = \frac{\mathcal{H}_1\mathcal{H}_2}{\mathcal{H}_0\mathcal{H}_3}.$$

To embed our solution into higher dimensions, we will use various ansätze given in [30] as we obtain the results for 10D and 11D solutions via 6D and 5D solutions respectively. The relevant truncation of the $4D$ STU Lagrangian given in [30] is

$$e_4^{-1}\mathcal{L}_4 = -R - \frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - \frac{1}{4}e^{-\varphi_1-\varphi_2-\varphi_3}\left[(\mathbb{F}^4)^2 + e^{2\varphi_i}(\tilde{\mathbb{F}}_i)^2\right]. \tag{7.4}$$

This is related to our Lagrangian (7.3) by an overall factor of 2, together with the following rescaling of the gauge fields and the scalars

$$F^0 = \frac{1}{\sqrt{2}}\mathbb{F}^4, \qquad F^A = \frac{1}{\sqrt{2}}\tilde{\mathbb{F}}_i, \qquad \phi_i = \varphi_i.$$

We will need to keep track of these factors while oxidising and insert the exact values for the gauge fields into the ansatz of [32].

7.2 Oxidation to Five Dimensions

The STU model can be consistently embedded into five dimensions with the Lagrangian

$$\mathcal{L}_5 = -R \star 1 - \frac{1}{2}h_i^{-2}\left(\star dh_i \wedge dh_i + \star\tilde{\mathbb{F}}_i \wedge \tilde{\mathbb{F}}_i\right) + \tilde{\mathbb{F}}_1 \wedge \tilde{\mathbb{F}}_2 \wedge \tilde{\mathbb{A}}_3, \tag{7.5}$$

where the five-dimensional scalars h^i satisfy the constraint $h_1h_2h_3 = 1$. Using the Kaluza-Klein reduction ansatz

$$ds_5 = f^{-1}ds_4^2 + f^2(dz_5 - \mathbb{A}^4)^2, \qquad \tilde{\mathbb{A}}_{(5D)i} = \tilde{\mathbb{A}}_i, \tag{7.6}$$

we obtain the four-dimensional Lagrangian (7.4) when we make the choice $fh_i = e^{-\varphi_i}$. The vector field $\mathbb{A}^4$ is the KK vector field, while the vector fields $\mathbb{A}_i$ descend from the $5D$ vector fields.

Introducing new linear combinations σ, φ, λ for the three independent real four-dimensional scalars by

$$\varphi_1 = -\frac{2}{\sqrt{6}}\sigma + \frac{1}{\sqrt{3}}\lambda, \quad \varphi_2 = -\frac{1}{\sqrt{2}}\phi + \frac{1}{\sqrt{6}}\sigma + \frac{1}{\sqrt{3}}\lambda, \quad \varphi_3 = \frac{1}{\sqrt{2}}\phi + \frac{1}{\sqrt{6}}\sigma + \frac{1}{\sqrt{3}}\lambda,$$

the five-dimensional constrained scalars h^i can be expressed in terms of 2 independent fields

$$h_1 = e^{2\sigma/\sqrt{6}}, \qquad h_2 = e^{\phi/\sqrt{2}-\sigma/\sqrt{6}}, \qquad h_3 = e^{-\phi/\sqrt{2}-\sigma/\sqrt{6}}.$$

Combining these two relations we obtain

$$h_1 = \exp\left(-\frac{2\varphi_1}{3} + \frac{\varphi_2}{3} + \frac{\varphi_3}{3}\right) = \left(\frac{\mathcal{I}_{22}\mathcal{I}_{33}}{\mathcal{I}_{11}^2}\right)^{\frac{1}{6}},$$
$$h_2 = \exp\left(\frac{\varphi_1}{3} - \frac{2\varphi_2}{3} + \frac{\varphi_3}{3}\right) = \left(\frac{\mathcal{I}_{11}\mathcal{I}_{33}}{\mathcal{I}_{22}^2}\right)^{\frac{1}{6}},$$
$$h_3 = \exp\left(\frac{\varphi_1}{3} + \frac{\varphi_2}{3} - \frac{2\varphi_3}{3}\right) = \left(\frac{\mathcal{I}_{11}\mathcal{I}_{22}}{\mathcal{I}_{33}^2}\right)^{\frac{1}{6}},$$

and expressing the gauge couplings in terms of the harmonic functions $\mathcal{H}_i$,

$$h_i = \frac{\mathcal{H}_i}{(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{3}}}\,, \tag{7.7}$$

allows us to write down the Kaluza-Klein scalar f in terms of ζ

$$f = e^{-\varphi_i}h_i^{-1} = \left(\frac{\mathcal{I}_{00}^3}{\mathcal{I}_{11}\mathcal{I}_{22}\mathcal{I}_{33}}\right)^{\frac{1}{6}} = \left(\frac{\mathcal{H}_0^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{1/6}. \tag{7.8}$$

Five-Dimensional Metric

Using (7.6) together with (7.8) and $\mathbb{A}^4 = \sqrt{2}A^0$, as well as collecting common factors, we obtain the following five-dimensional metric for the uplift of our four-charge solution:

$$\begin{aligned} ds_5^2 = (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}}\Big[&\mathcal{H}_0 dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0}\left(\mathcal{W}\frac{\gamma_0^2}{Q_0^2}+1\right)d\eta^2 - \frac{2\mathcal{W}\gamma_0}{\sqrt{2}Q_0}d\eta dz_5 \\ &+ 2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right)\Big] \end{aligned} \tag{7.9}$$

Five-Dimensional Gauge Potential

To obtain expressions for the five-dimensional gauge potentials, it is necessary to express all the gauge fields in our solution in terms of electric components. This requires replacing the dual vector potentials $\tilde{A}_A$ by the 'standard' vector potentials A^A. The associated field strength $\tilde{F}_A$ and F^A are related by Hodge duality together with multiplication by inverse gauge coupling matrix

$$F^A = -\mathcal{I}^{AB} \star \tilde{F}_B.$$

Using the form of the gauge potential found in (5.33), standard calculations give their form

$$F^A = -\mathcal{I}^{AB} \star \tilde{F}_B = -\frac{P^A}{2\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}dx\wedge dy. \tag{7.10}$$

Integrating and relating to the gauge fields in the ansatz we obtain the form of the three $5D$ vector potentials

$$\tilde{\mathbb{A}}_i = \sqrt{2}A^A = \mathfrak{p}_a\,(ydx - xdy)\,, \qquad \mathfrak{p}_a = \frac{P^a}{2\sqrt{2\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{7.11}$$

We will return to these gauge fields before uplifting the solution to six dimensions; when it will be necessary to Hodge dualise in 5D to obtain a two-form potential.

Extremal Limit

We now investigate the effect of the $4D$ extremal limit defined in section 6.2 for the following higher-dimensional lifts.[10] Just as in the $4D$ case, the horizon for the $5D$ solution is pushed out to $\zeta \to \infty$ and the static region takes up the entirety of our spacetime; in other words, the extremal limit results in a solution containing a naked singularity. Simplifying the metric functions using the expressions from (6.2) we can write down the $5D$ line element in the form

$$ds_5^2 = (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}} \left[d\eta dz_5 + \mathcal{H}_0 dz_5^2 + \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3(d\zeta^2 + dx^2 + dy^2)\right].$$

We will consider the extremal limit for each of the following uplifts in term as we further oxidise the STU model.

7.3 Oxidation to Eleven Dimensions

To uplift our solution to eleven dimensions, we start with the bosonic part of the $11D$ supergravity Lagrangian

$$\mathcal{L}_{11} = -R \star 1 - \frac{1}{2} \star \mathcal{F} \wedge \mathcal{F} - \frac{1}{6}\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A},$$

where $\mathcal{A}$ is the three-form such that $\mathcal{F} = d\mathcal{A}$ is the four-form field strength. We can directly embed the $5D$ STU model into this theory through a Kaluza-Klein reduction on T^6 with the ansatz

$$ds_{11}^2 = ds_5^2 + h_1(dy_1^2 + dy_2^2) + h_2(dy_3^2 + dy_4^2) + h_3(dy_5^2 + dy_6^2),$$
$$\mathcal{A} = \tilde{\mathbb{A}}_1 \wedge dy^1 \wedge dy^2 + \tilde{\mathbb{A}}_2 \wedge dy^3 \wedge dy^4 + \tilde{\mathbb{A}}_3 \wedge dy^5 \wedge dy^6.$$

In a consistent truncation to five-dimensional minimal supergravity, the volume of the torus corresponds to a scalar in a hypermultiplet, while its shape is encoded by scalars in vector multiplets. This factorization imposes the condition $h_1h_2h_3 = 1$ on the scalars h_i. By restricting to backgrounds where hypermultiplets are trivial, we can consistently truncate out the hypermultiplets and remain with the five-dimensional STU model with two vector multiplets.

We now combine this 5D/11D lift with the previous 4D/5D lift. In our four-dimensional solution we can express the h_i as functions of ζ through the harmonic functions $\mathcal{H}_i$, see (7.7). The three-form gauge potential is found directly from the components of (7.11). Thus the full line element for the $11D$ lift of the non-extremal planar solution to the $4D$ STU model is

$$\begin{aligned} ds_{11}^2 = (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}} \Bigg[&\mathcal{H}_0 dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0}\left(\mathcal{W}\frac{\gamma_0^2}{Q_0^2} + 1\right) d\eta^2 - \frac{\mathcal{W}\gamma_0}{\sqrt{2}Q_0} d\eta dz_5 \\ &+ 2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right) \\ &+ \mathcal{H}_1(dy_1^2 + dy_2^2) + \mathcal{H}_2(dy_3^2 + dy_4^2) + \mathcal{H}_3(dy_5^2 + dy_6^2)\Bigg]. \end{aligned} \tag{7.12}$$

[10] We note here that we use the same symbols h_a for the integration constants in (6.2) and the constrained five-dimensional scalars, as this allows us to match notation with the literature on five-dimensional solutions. We trust that the reader will infer from context which quantity is meant in a particular expression.

Eleven-Dimensional Extremal Limit

By again substituting in the limit (6.2) we can write down (7.12) in the extremal limit to find

$$\begin{aligned} ds_{11}^2 =&(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}}\big[d\eta dz_5 + \mathcal{H}_0 dz_5^2 + \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3(d\zeta^2+dx^2+dy^2) \\ &+ \mathcal{H}_1(dy_1^2+dy_2^2) + \mathcal{H}_2(dy_3^2+dy_4^2) + \mathcal{H}_3(dy_5^2+dy_6^2)\big] \end{aligned} \tag{7.13}$$

This looks like a standard BPS solution of eleven-dimensional supergravity, a configuration of three stacks of M5-branes, encoded by $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ which triple intersect over a string, and with a gravitational wave, encoded by $\mathcal{H}_0$, superimposed along the string [33]. Compactification on $T^6 \times S^1$ gives rise to four-charged BPS black holes when the branes are delocalised along $y^1, \dots, y^6$ but localised in the remaining three spacelike directions [34]. In our solutions the M5-branes have in addition been delocalised in two of the non-compact directions, giving rise to planar rather than spherical symmetry.

7.4 Oxidation to Six Dimensions

Lifting the four-dimensional Lagrangian to six dimensions by extending the 4D/5D lift requires a tweak of the five-dimensional Lagrangian, namely to Hodge-dualise one of the three vector potentials into a two-form B. The reason is that the six-dimensional supergravity is chiral and both the supergravity multiplet and tensor multiplets contain self-dual or anti-self-dual tensor fields which do not admit a standard Lagrangian description. However, in supergravity coupled to one tensor multiplet (plus vector and hypermultiplets), one self-dual and one anti-self-dual tensor combine into an unconstrained tensor, which allows a standard Lagrangian description. String compactifications to six dimensions are of this type, with the tensor field descending from the ten-dimensional Kalb-Ramond field.

Matching our conventions with the work of [30] we define the three-form from the dualisation of the two-form field strength in five dimensions

$$\tilde{\mathbb{F}}_3 = d\tilde{\mathbb{A}}_3 = -h_1^{-2}h_2^{-2}\star_5 \mathbb{H}\,. \tag{7.14}$$

Making this transformation and substituting into the Lagrangian results in

$$\begin{aligned} \mathcal{L}_5 =& -R\star 1 - \frac{1}{2}h_i^{-2}\star dh_i\wedge dh_i + \frac{1}{2}h_1^{-2}\star\tilde{\mathbb{F}}_1\wedge\tilde{\mathbb{F}}_1 \\ &+ \frac{1}{2}h_2^{-2}\star\tilde{\mathbb{F}}_2\wedge\tilde{\mathbb{F}}_2 - \frac{1}{2}h_1^{-2}h_2^{-2}\star\mathbb{H}\wedge\mathbb{H}\,. \end{aligned} \tag{7.15}$$

We can now use the results of [32], and first work with the new three-form field strength in five dimensions in terms of ζ.

Dualization of the Five-Dimensional Gauge Field

Taking the Hodge dual of (7.14) again we find the three-form

$$\star_5\mathbb{H} = -h_1^2h_2^2\tilde{\mathbb{F}}_3\,, \qquad \star_5\star_5\mathbb{H} = -\star_5(h_1^2h_2^2\tilde{\mathbb{F}}_3)\,, \qquad \mathbb{H} = \star_5(h_1^2h_2^2\tilde{\mathbb{F}}_3)\,,$$

where we have used that for a k-form ω in n dimensions in the Lorentzian signature $\star\star\omega = (-1)^{k(n-k)+1}\omega$. Substituting in (7.10) together with:

$$\sqrt{-g_5} = 2(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{\frac{2}{3}}, \qquad \epsilon_{\eta\zeta xyz_5} = 1\,,$$

$$h_1^2 h_2^2 = h_3^{-2} = (\mathcal{H}_1\mathcal{H}_2)^{\frac{2}{3}}\mathcal{H}_3^{-\frac{4}{3}} , \qquad g^{xx} = g^{yy} = \frac{1}{2}(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{2}{3}} ,$$

we find that the three-form is

$$\mathbb{H} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2}\right) d\eta \wedge d\zeta \wedge dz_5 .$$

Lift to Six Dimensions

The six-dimensional Lagrangian is

$$\mathcal{L}_6 = -R \star 1 - \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{-\sqrt{2}\phi} \star H \wedge H,$$

where $H = dB$ is a three-form field strength. The reduction ansatz [30] which reproduces our five-dimensional Lagrangian (7.15) is

$$ds_6^2 = e^{\sigma/\sqrt{6}} ds_5^2 + e^{-3\sigma/\sqrt{6}}(dz_6 + \tilde{\mathbb{A}}_1)^2 , \qquad B_{(6D)} = B + \tilde{\mathbb{A}}_2 \wedge (dz_6 + \tilde{\mathbb{A}}_1),$$

with the field strengths decomposed as

$$H_{(6D)} = \mathbb{H} + \tilde{\mathbb{F}}_2 \wedge (dz_6 + \tilde{\mathbb{A}}_1) , \qquad \mathbb{H} = dB - \tilde{\mathbb{A}}_2 \wedge \tilde{\mathbb{F}}_1, \qquad \tilde{\mathbb{F}}_i = d\tilde{\mathbb{A}}_i .$$

We see that from our parameterisation of the h_i we can write the $6D$ Kaluza-Klein scalar as

$$e^{\sigma/\sqrt{6}} = \sqrt{h_1} = \left(\frac{\mathcal{H}_1^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{6}} .$$

We are now in the position to combine these results to write down the $6D$ metric for our embedded solution:

$$\begin{aligned} ds_6^2 = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}} \Bigg[& \mathcal{H}_0 dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0}\left(\mathcal{W}\frac{\gamma_0^2}{Q_0^2} + 1\right) d\eta^2 - \frac{2\mathcal{W}\gamma_0}{\sqrt{2}Q_0} d\eta dz_5 \\ & + 2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right)\Bigg] + \frac{(\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}}{\mathcal{H}_1}(dz_6 + \tilde{\mathbb{A}}_1)^2 , \end{aligned}$$

where the determinant of the metric is

$$\sqrt{-g_6} = 2\mathcal{H}_1\sqrt{\mathcal{H}_2\mathcal{H}_3} .$$

The piece containing the gauge field $\mathbb{A}_1$ can be expanded

$$\begin{aligned} (dz_6 + \tilde{\mathbb{A}}_1)^2 &= (dz_6 + (\tilde{\mathbb{A}}_1)_x dx + (\tilde{\mathbb{A}}_1)_y dy)^2 \\ &= (dz_6 + \mathfrak{p}_1(ydx - xdy))^2 . \end{aligned}$$

Six-Dimensional Gauge Fields

We now take the gauge fields and express them as a function of the $6D$ coordinates. We see that for the two remaining one-form potentials nothing has been changed compared to the lower dimensional solutions

$$\tilde{\mathbb{A}}_1 = \sqrt{2}A^1, \qquad \tilde{\mathbb{A}}_2 = \sqrt{2}A^2 .$$

The three-form H is found from two pieces

$$H_{(6D)} = \mathbb{H} + \tilde{\mathbb{F}}_2 \wedge (dz_6 + \tilde{\mathbb{A}}_1) .$$

This is simplified as the term

$$\tilde{\mathbb{F}}_2 \wedge \tilde{\mathbb{A}}_1 = 2\mathfrak{p}_2 dx \wedge dy \wedge \mathfrak{p}_1 \left(y dx - x dy \right) = 0,$$

is zero due to anti-symmetry. Using the work from the $5D$ calculations the $6D$ three-form field strength is therefore given by:

$$H_{(6D)} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2} \right) d\eta \wedge d\zeta \wedge dz_5 - (2\mathfrak{p}_2) dx \wedge dy \wedge dz_6 \ .$$

Six-Dimensional Extremal Limit

Taking the limit (6.2) the six-dimensional line element is given by

$$ds_6^2 = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}} \left[\mathcal{H}_2^{-1} \left(\mathcal{H}_0 dz_5^2 + dz_5 d\eta \right) + \mathcal{H}_3 \mathcal{H}_1 (d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_3 \mathcal{H}_1^{-1} (dz_6 + \tilde{\mathbb{A}}^1)^2 \right] . \tag{7.16}$$

The three-form in this limit is given by

$$H_{(6D)} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2} \right) d\eta \wedge d\zeta \wedge dz_5 - (2\mathfrak{p}_2) dx \wedge dy \wedge dz_6, \qquad \mathfrak{p}_a = \frac{P^a}{2\sqrt{2 Q_0 P^1 P^2 P^3}}.$$

7.5 Oxidation to Ten Dimensions

The six-dimensional STU model is a consistent truncation of the reduction of IIB supergravity on T^4. To lift our solution, we only need to include the overall volume of the T^4 as a modulus

$$ds_{10}^2 = ds_6^2 + e^{\phi/\sqrt{2}} (dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2), \qquad \Phi = \frac{\phi}{\sqrt{2}}, \qquad C \equiv B.$$

The expression for the six-dimensional dilaton ϕ in terms of ζ is

$$e^{\sqrt{2}\phi} = \frac{h_2}{h_3} = \left(\frac{\mathcal{I}_{33}}{\mathcal{I}_{22}} \right)^{\frac{1}{2}} \quad \Rightarrow \quad e^{\phi/\sqrt{2}} = \left(\frac{\mathcal{I}_{33}}{\mathcal{I}_{22}} \right)^{\frac{1}{4}} = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}} \ .$$

All other data follow straight from the six-dimensional solutions. The ten-dimensional dilaton is given by

$$\Phi = \frac{1}{2} \log \left(\frac{\mathcal{H}_2}{\mathcal{H}_3} \right) .$$

The ten-dimensional line element is given by

$$\begin{aligned} ds_{10}^2 = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}} \Bigg[& \mathcal{H}_0 \mathcal{H}_2^{-1} dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0 \mathcal{H}_2} \left(\mathcal{W} \frac{\gamma_0^2}{Q_0^2} + 1 \right) d\eta^2 - \frac{\mathcal{W}\gamma_0}{\sqrt{2} Q_0 \mathcal{H}_2} d\eta dz_5 \\ & + 2\mathcal{H}_1 \mathcal{H}_3 \left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2 \right) + \frac{\mathcal{H}_3}{\mathcal{H}_1} (dz_6 + \tilde{\mathbb{A}}_1)^2 \\ & + dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 \Bigg]. \end{aligned}$$

Ten-Dimensional Extremal Limit

Uplifting the extremal 6D solution using the same methods as (7.5) we find that the line element is

$$\begin{aligned} ds_{10}^2 =& \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}}\Big[\mathcal{H}_2^{-1}\left(\mathcal{H}_0 dz_5^2 + dz_5 d\eta\right) + \mathcal{H}_3\mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) \\ &+ \mathcal{H}_3\mathcal{H}_1^{-1}(dz_6^2 + \tilde{\mathbb{A}}^1)^2 + dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\Big]. \end{aligned} \tag{7.17}$$

which is the intersection of a D1 and D5 brane with momentum along the common direction and a Taub-NUT space.

8 Supersymmetry in Six Dimensions

Supersymmetric solutions of six-dimensional supergravity have been classified in detail. The first classification of supersymmetric solutions in the minimal ungauged six-dimensional theory, with a self-dual three-form, was constructed in [36]. Following on from this, the supersymmetric solutions of six-dimensional $U(1)$, and $SU(2)$ gauged supergravity were classified in [16]. This analysis was done using the spinor bilinears method. Supersymmetric solutions of more general theories coupled to arbitrary vector and tensor multiplets were classified using spinorial geometry methods in [10]; see also [11, 12, 14, 15]. These classifications have been used to find many new examples of solutions, and we shall show that in a certain limit, the 6D uplift solution we have constructed satisfies the necessary and sufficient conditions for supersymmetry.

8.1 Conditions Required for Supersymmetry

We now turn our attention to the $6D$ uplift of our solution and test to see whether there is a configuration of integration constants such that the solution is supersymmetric. In this particular case, the theory of interest is the $U(1)$ gauged supergravity whose supersymmetric solutions were classified in [16], in the special case for which the $U(1)$ gauge parameter is set to zero. The bosonic content of this theory is the metric g, a real three-form G, and a dilaton ϕ. The geometry of these solutions was also considered in [37]. Before considering the $6D$ uplift in detail, we first summarise the necessary and sufficient conditions on the bosonic fields in order for a generic solution of this theory to be supersymmetric.

The metric for the supersymmetic solutions is given by

$$ds_6^2 = -2H^{-1}(dv + \beta)(du + \omega + \frac{1}{2}\mathcal{F}(dv + \beta)) + H ds_4^2. \tag{8.1}$$

The metric for the four-dimensional base space $\mathcal{B}$ is written as

$$ds_4^2 = h_{mn} dx^m dx^n, \tag{8.2}$$

with $\beta = \beta_m dx^m$ and $\omega = \omega_m dx^m$ regarded as one-forms on $\mathcal{B}$. The vector $\frac{\partial}{\partial u}$ corresponds to a Killing spinor bilinear and the Killing spinor equations imply that this vector is an isometry, and moreover that the Lie derivative of the three-form G and the dilaton Φ with respect to $\frac{\partial}{\partial u}$ vanish. However, in general, the metric, the three-form and the dilaton may depend on the v and the x^m coordinates.

Analysis of the algebraic properties of the spinor bilinears by considering the Fierz identities implies that there are 3 anti-self-dual two-forms on the base $\mathcal{B}$: $J^{(A)}$, $A = 1, 2, 3$, which satisfy the algebra of the imaginary unit quaternions; $\mathcal{B}$ therefore admits an almost hyper-Kähler structure. In addition, the gravitino Killing spinor equations imply that

$$\tilde{d}J^{(A)} = \partial_v \Big(\beta \wedge J^{(A)} \Big), \tag{8.3}$$

where $\tilde{d}$ denotes the exterior derivative restricted to surfaces of constant u and v; and ∂_v denotes the Lie derivative with respect to $\frac{\partial}{\partial v}$. It is also useful to define the differential operator D by

$$D\chi = \tilde{d}\chi - \beta \wedge \partial_v \chi, \tag{8.4}$$

where χ is a u-independent differential form on $\mathcal{B}$. Then supersymmetry implies that

$$D\beta = \star_4 D\beta, \tag{8.5}$$

where $\star_4$ denotes the Hodge dual on $\mathcal{B}$. This exhausts the conditions on the geometry obtained from the gravitino Killing spinor equation. It remains to consider the conditions on the fluxes.

The Killing spinor equations determine the components of the three-form G as

$$\begin{aligned} e^{\sqrt{2}\Phi} G = {} & \tfrac{1}{2} \star_4 \Big(DH + H\partial_v \beta - \sqrt{2} H D\Phi \Big) \\ & - \tfrac{1}{2} e^+ \wedge e^- \wedge \Big(H^{-1} DH + \partial_v \beta + \sqrt{2} D\Phi \Big) \\ & - e^+ \wedge (-H\psi + \tfrac{1}{2}(D\omega)^- - K) + \tfrac{1}{2} H^{-1} e^- \wedge D\beta, \end{aligned} \tag{8.6}$$

where K is a self-dual form on the base $\mathcal{B}$, ψ is expressed as

$$\psi = \frac{1}{16} H \epsilon^{ABC} J^{(A)mn} (\partial_v J^{(B)})_{mn} J^{(C)}, \tag{8.7}$$

and we have adopted the null basis

$$e^+ = H^{-1}(dv + \beta), \qquad e^- = du + \omega + \frac{1}{2}\mathcal{F} H e^+, \qquad e^a = H^{\frac{1}{2}} \tilde{e}^a, \tag{8.8}$$

in which the metric is

$$ds_6^2 = -2e^+ e^- + \delta_{ab} e^a e^b, \tag{8.9}$$

and the basis $\tilde{e}^a = \tilde{e}^a{}_m dx^m$ is a basis for the base $\mathcal{B}$.

On imposing the Bianchi identity $dG = 0$, the following conditions are obtained

$$\begin{aligned} D\Big(H^{-1} e^{\sqrt{2}\Phi} \big(K - H\mathcal{G} - H\psi\big) \Big) + \frac{1}{2} \partial_v \star_4 \Big(D\big(He^{\sqrt{2}\Phi}\big) + He^{\sqrt{2}\Phi} \partial_v \beta \Big) & \\ - H^{-1} e^{\sqrt{2}\Phi} (\partial_v \beta) \wedge \big(K - H\mathcal{G} - H\psi\big) = 0, & \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} -D\Big(H^{-1} e^{-\sqrt{2}\Phi} \big(K + H\mathcal{G} + H\psi\big) \Big) + \frac{1}{2} \partial_v \star_4 \Big(D\big(He^{-\sqrt{2}\Phi}\big) + He^{-\sqrt{2}\Phi} \partial_v \beta \Big) & \\ + H^{-1} e^{-\sqrt{2}\Phi} (\partial_v \beta) \wedge \big(K + H\mathcal{G} + H\psi\big) = 0, & \end{aligned} \tag{8.11}$$

where

$$\mathcal{G} = \frac{1}{2H}\Big((D\omega)^+ + \frac{1}{2}\mathcal{F}D\beta\Big), \tag{8.12}$$

and $(D\omega)^\pm$ denote the self-dual and anti-self dual parts of $D\omega$.

The gauge field equations, $d(e^{2\sqrt{2}\Phi}\star_6 G) = 0$ also imply the following conditions

$$D\star_4\Big(D\big(He^{\sqrt{2}\Phi}\big) + He^{\sqrt{2}\Phi}\partial_v\beta\Big) = 2H^{-1}e^{\sqrt{2}\Phi}\big(K - H\mathcal{G}\big)\wedge D\beta, \tag{8.13}$$

and

$$D\star_4\Big(D\big(He^{-\sqrt{2}\Phi}\big) + He^{-\sqrt{2}\Phi}\partial_v\beta\Big) = -2H^{-1}e^{-\sqrt{2}\Phi}\big(K + H\mathcal{G}\big)\wedge D\beta, \tag{8.14}$$

As noted in [16], imposing these conditions implies that the dilaton field equation is automatically satisfied, and also all but one component of the Einstein field equations also hold. The remaining ++ component of the Einstein equations must be imposed as an additional condition. On defining

$$L = \partial_v\omega + \frac{1}{2}\mathcal{F}\partial_v\beta - \frac{1}{2}D\mathcal{F}, \tag{8.15}$$

this component of the Einstein equation is given by

$$\begin{aligned}\star_4 D\star_4 L &= \frac{1}{2}h^{mn}\partial_v^2\big(Hh_{mn}\big) + \frac{1}{2}\partial_v\big(Hh^{mn}\big)\partial_v\big(Hh_{mn}\big)\\ &- \frac{1}{2}H^{-2}\big(D\omega + \frac{1}{2}\mathcal{F}D\beta\big)^2 - 2L^m(\partial_v\beta)_m + 2H^2(\partial_v\Phi)^2\\ &+ 2H^{-2}\big(K - H\psi + \frac{1}{2}(D\omega)^-\big)^2. \end{aligned}\tag{8.16}$$

where we adopt the convention that if X is a two-form on $\mathcal{B}$ then $X^2 = \frac{1}{2}X_{mn}X^{mn}$.

8.2 Matching the Solutions

We begin by taking the $\alpha \to 0$ limit; in four dimensions we can think of this limit as taking the blackening factor to zero and thus, being associated with extremality.

In this limit the resulting metric was found to be (7.16) and for convenience, we repeat the full expression for the six-dimensional three-form

$$H_{(6D)} = -\frac{P^3}{2\mathcal{H}_3^2\sqrt{2Q_0P^1P^2P^3}}d\eta\wedge d\zeta\wedge dz_5 - \frac{P^2}{\sqrt{2Q_0P^1P^2P^3}}dx\wedge dy\wedge dz_6. \tag{8.17}$$

Comparing our metric with the metric (8.1) we extract a four-dimensional base space:

$$ds_6^2 = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}}dz_5(d\eta + \mathcal{H}_0 dz_5) + (\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}ds_4^2, \tag{8.18}$$

in the form

$$ds_4^2 = \mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_1^{-1}(dz_6 + \mathbb{A}^1)^2. \tag{8.19}$$

Direct comparison to (8.1) shows that we should make the following identifications:

$$\beta = \omega = 0, \qquad H = \sqrt{\mathcal{H}_2\mathcal{H}_3}, \qquad \mathcal{F} = \mathcal{H}_0, \qquad dv = dz_5, \qquad 2du = d\eta,$$

with all components of the metric and three-form independent of the v coordinate. The basis vectors are given as:

$$e^{+} = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}} dz_5, \qquad e^{-} = \frac{1}{2}d\eta + \frac{1}{2}\mathcal{H}_0 dz_5, \qquad e^{a} = (\mathcal{H}_1\mathcal{H}_2)^{\frac{1}{4}} \tilde{e}^{a}_{m} dx^{m}.$$

We begin by looking more closely at the base space (8.19)

$$ds_4^2 = \mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_1^{-1}(dz_6 + \mathbb{A}^1)^2,$$

which has a set of basis vectors:

$$\begin{aligned} e^1 &= \mathcal{H}_1^{\frac{1}{2}} d\zeta, & e^2 &= \mathcal{H}_1^{\frac{1}{2}} dx, \\ e^3 &= \mathcal{H}_1^{\frac{1}{2}} dy, & e^4 &= \mathcal{H}_1^{-\frac{1}{2}} (dz_6 + \mathbb{A}^1), \end{aligned}$$

with

$$\mathcal{H}_1 = h_1 + P^1\zeta, \qquad \mathbb{A}^1 = \frac{P^1}{2\sqrt{2Q_0 P^1 P^2 P^3}}\big(y dx - x dy\big). \tag{8.20}$$

As the solution is independent of the v coordinate, the condition (8.3) implies that the base is hyper-Kähler. In particular, we require that the Ricci scalar of the base must vanish, which imposes the following condition

$$2Q_0 P^1 P^2 P^3 = 1, \tag{8.21}$$

which we can interpret as a condition for the integration constant

$$Q_0 = \frac{1}{2P^1P^2P^3}, \tag{8.22}$$

and so we see that the supersymmetric limit occurs by fine-tuning of the $4D$ electric charge or alternatively the KK momentum in $5/6D$.

Given this fine tuning condition, the base metric is then given by

$$ds_4^2 = \big(h_1 + P^1\zeta\big)\Big(d\zeta^2 + dx^2 + dy^2\Big) + \big(h_1 + P^1\zeta\big)^{-1}\Big(dz^6 + \frac{1}{2}P^1\big(y dx - x dy\big)\Big)^2. \tag{8.23}$$

This metric is in the form of the Gibbons-Hawking instanton solution [38, 39]

$$ds^2_{GH} = U^{-1}(d\tau + \omega)^2 + U d\vec{x}\cdot d\vec{x},$$

where $\tau = z^6$ is the direction corresponding to the tri-holomorphic isometry $\frac{\partial}{\partial\tau}$ of the hyper-Kähler structure, and $U = h_1 + P^1\zeta$ is a linear harmonic function of the Cartesian coordinates $\{\zeta, x, y\}$ on $\mathbb{R}^3$, and the one-form $\omega = dz^6 + \frac{1}{2}P^1\big(y dx - x dy\big)$ is a $U(1)$ connection on $\mathbb{R}^3$ which satisfies

$$dU = \star_3 d\omega. \tag{8.24}$$

This base space corresponds to a constant density planar distribution of Taub-NUT instantons. Moreover, the conditions imposed on the three-form given in (8.6) are consistent with the three-form obtained from the uplift in (8.17), on setting $K = 0$ in (8.6), and also identifying

$$\Phi = -\frac{1}{2\sqrt{2}}\log\left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right). \tag{8.25}$$

We remark that the dilaton which appears in the classification of [16], which we have denoted by Φ, differs from the dilaton ϕ appearing in previous sections by a scaling

$$\Phi = -\frac{1}{2}\phi = -\frac{1}{2\sqrt{2}}\log\left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right) \quad \Rightarrow \quad e^{\sqrt{2}\Phi} = \left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right)^{-\frac{1}{2}}.$$

With these identifications, it is straightforward to match (8.6) with (8.17), on making use of the identities

$$d\zeta = \mathcal{H}_1^{-\frac{1}{2}} e^1, \qquad \star_4 d\zeta \ = -\mathcal{H}_1^{-\frac{1}{2}} e^2 \wedge e^3 \wedge e^4 = -dx \wedge dy \wedge dz_6. \tag{8.26}$$

and

$$e^{-\sqrt{2}\Phi} \star_4 \left(dH - \sqrt{2} H d\Phi\right) = -P^2 dx \wedge dy \wedge dz_6, \tag{8.27}$$

and

$$e^{-\sqrt{2}\Phi} e^+ \wedge e^- \wedge \left(H^{-1}dH + \sqrt{2}d\Phi\right) = \frac{P^3}{2\mathcal{H}_3^2} dz_5 \wedge d\eta \wedge d\zeta. \tag{8.28}$$

In addition, the Bianchi identities (8.10) and (8.11) hold with no further conditions imposed, as all terms are independent of v, and also $K = 0$, $\mathcal{G} = 0$ and $\psi = 0$. The condition $\psi = 0$ follows from (8.7), on using the fact that the hyper-complex structures are independent of v.

It remains to consider the gauge field equations (8.13) and (8.14). The RHS of these equations vanishes identically, as a consequence of the fact that $h = 0$. The remaining content of the gauge field equations is that $He^{\pm\sqrt{2}\Phi}$ be harmonic on the base space. This holds automatically as a consequence of the previously obtained conditions, because $He^{\sqrt{2}\Phi} = \mathcal{H}_3$ and $He^{-\sqrt{2}\Phi} = \mathcal{H}_2$, and ζ is harmonic on the base space as a consequence of (8.26). Similarly, the Einstein equation (8.16) holds automatically, because all terms on the RHS vanish individually, and also $L = -\frac{1}{2}Q_0 d\zeta$ which is co-closed on the base, again as a consequence of (8.26).

8.3 Analysis of the Spacetime

Now we have shown that by fine-tuning our integration constants we can obtain a supersymmetric solution, it is interesting to look at the geometric properties of this spacetime.

Our analysis is focused on the simplified metric

$$ds_6^2 = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}} dz_5(d\eta + \mathcal{H}_0 dz_5) + (\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}\left[\mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_1^{-1}(dz_6 + \mathbb{A}^1)^2\right].$$

In the limit of $\zeta \to \infty$ we find that the Riemann tensor falls off as ζ^{-n} for $n \geq 1$. The Ricci curvature of the spacetime is

$$R_{(6D)} = \frac{(h_3 P^2 - h_2 P^3)^2}{4\mathcal{H}_1(\mathcal{H}_2\mathcal{H}_3)^{\frac{5}{2}}},$$

and we notice here that we have the option to pick either a charge $P^{2/3}$ or $h_{2/3}$ value such that the spacetime is Ricci flat

$$h_3 = \frac{h_2 P^3}{P^2} \quad \Leftrightarrow \quad R_{(6D)} = 0.$$

We can understand this condition by looking back at the harmonic functions

$$\begin{aligned}\mathcal{H}_2\mathcal{H}_3 &= (h_2 + P^2\zeta)(h_3 + P^3\zeta) = \left(h_2 + P^2\zeta\right)\left(\frac{h_2 P^3}{P^2} + P^3\zeta\right) \\ &= P^2 P^3 \left(\zeta + \frac{h_2}{P^2}\right)^2 .\end{aligned}$$

and we see that picking the right integration constants we allow the zeros of both $\mathcal{H}_2$ and $\mathcal{H}_3$ to simultaneously occur.

9 Conclusions and Outlook

We have seen that, surprisingly, a method designed to produce static solutions has provided us with a class of cosmological solutions. In the four-charge case, we were able to lift these solutions to five and six, and then to ten and eleven dimensions; allowing an embedding into string theory and M-theory. In the extremal limit, we recover two of the best known intersecting brane solutions, which give rise to four-dimensional BPS black hole solutions upon dimensional reduction together with the usual delocalisation of the branes along the compact directions. The extremal limits of our planar solution are related to these configurations by the additional delocalisation along two of the non-compact spatial directions, which changes the symmetry from spherical to planar. The dimensional lift and embedding into string theory does not provide by itself any insight into why we obtain cosmological rather than black brane solutions since the additional dimensions are just spectators. Instead, we learn an interesting lesson about the importance of being able to make brane configurations non-extremal. The compactified BPS brane solutions used to obtain four-dimensional BPS black holes have the same causal structure as the extremal Reissner-Nordström solution, which is embedded as a 'double-extreme' limit [40], where all four-dimensional scalars are constant. The essential features of our cosmological solutions can be understood using the charged electro-vac solutions of Einstein-Maxwell theory.

We start with the spherically symmetric extremal Reissner Nordström solution, whose causal structure is shared by a large class of BPS solutions obtained by compactifying brane configurations. Its maximal analytical extension is a sequence of two types of regions, both static: one containing an asymptotically flat exterior, the other – the interior – containing a timelike singularity which is repulsive to massive neutral particles. In other words, timelike geodesics are infinitely extendable. For our purposes, we focus on just a single pair of such regions, see Figure (5) for illustration. If the solution is made non-extremal, a third type of region occurs, which is dynamical (non-stationary) and located between the two types of static patches. Let us now consider the effect of replacing spherical by planar symmetry, or, in brane language, of delocalisation of the constituent branes along two non-compact spatial directions. In this case, the solution cannot be asymptotically flat any more. For brane-type solutions, it is a well-known feature that asymptotic flatness requires more than two transverse dimensions: 'large branes' (those with two or less transverse dimensions, like the D7-brane in type-IIB) cannot be asymptotically flat. In terms of the causal structure, we loose the static, asymptotically flat patch and remain with a static patch containing the singularity, and a dynamical patch. More precisely, by maximal analytic extension, we end up with two patches

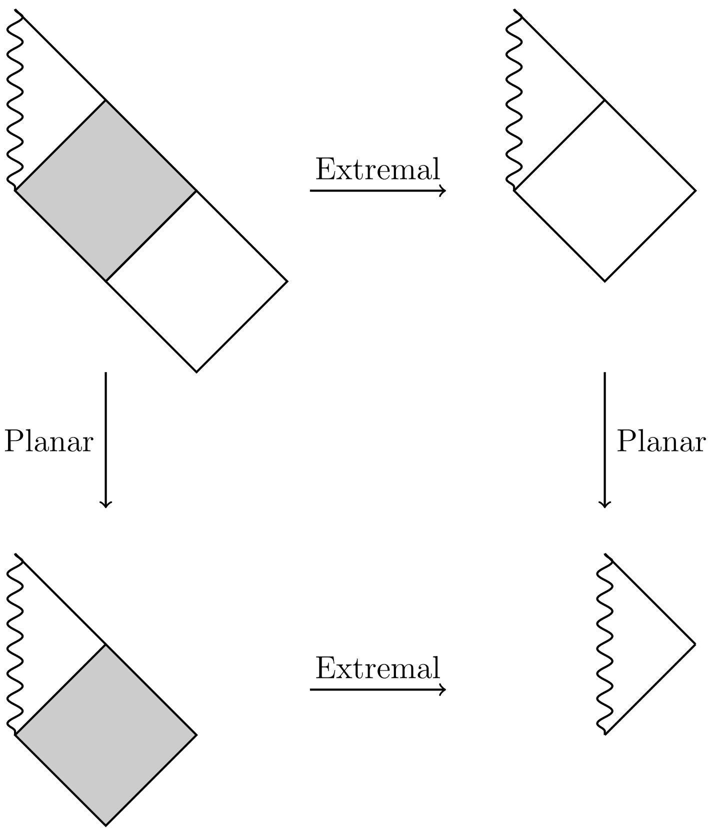

Figure 5: Comparison of the conformal diagrams for spherical and planar Reissner-Nordström-like spacetimes. We only display one copy of each type of region. Shaded regions are where the spacetime is dynamical (no timelike Killing vector).

of each type, resulting in a conformal diagram which is the same as Schwarzschild rotated by 90 degrees, see Figure (3). If we now perform an extremal limit, we also lose the dynamical patch and remain with a static patch containing a singularity. Comparing the four types of conformal diagrams, we see that going from spherical to planar symmetry removes the asymptotically flat region, while the existence of a dynamical patch depends on non-extremality. Viewed from this perspective, the presence of a cosmological patch in our solutions is completely natural, and results from physics already present in Einstein-Maxwell theory. These features are robust under dimensional lifting and persist for the three-charged solution, which is a solution of gauged supergravity and have the same conformal diagram. However, the Nernst brane solutions [3, 4, 5, 6] illustrate that these features do not persist if we modify essential features. Like the three- and four-charge solutions, the single-charged Nernst branes are planar and not asymptotically flat, but they do not share the 'inside-out' feature of a singularity at a finite distance inside the static patch. The essential difference is that Nernst branes require a non-constant scalar fields, and therefore there is no limit in which they become solutions of four-dimensional Einstein-Maxwell theory. Instead, as shown in [4], they lift to boosted AdS-Schwarzschild black brane solutions of five-dimensional AdS gravity.

The close relation of our cosmological solutions to the planar Reissner-Nordström solution also settles the question of whether we need to interpret it as being sourced by negative tension branes. We have found that the local Komar mass is negative in the static patch, which is consistent with the repulsive character of the singularity. However, this feature is also present in the spherical Reissner-Nordström solution, the only difference being that with planar symmetry we loose the asymptotically flat region, and hence the ability to define a 'proper' mass by evaluating the Komar expression at asymptotic infinity. This reflects the general insight, reviewed recently in [41], that the definition of global quantities through conservation laws a la Noether requires that general diffeomorphism invariance is 'broken naturally' by the presence of extra structure, such as boundary conditions. That we do not have a static asymptotic region does not provide a good reason to assign negative tension to the sources, because locally the situation is not different from Reissner-Nordström. Moreover, for the cases where we can lift to ten or eleven dimensions, the sources reveal themselves as conventional, positive tension branes.

The only caveat is that our four-dimensional solutions admit other embeddings into string theory, which might change their higher-dimensional interpretation. In particular, it can be shown that the solution found in [29] describes a region of our solution, although in different coordinates, where the existence of a Killing horizon is not obvious. The solution of [29] admits an uplift over the orientifold $K3\times\mathbb{T}^2/\mathbb{Z}_2$. This alternative embedding, which we have not analysed in detail in this paper, is interesting because it starts with a compactification which has less than maximal supersymmetry. In contrast, in our uplift we have used toroidal compactifications and start with maximally supersymmetric theories in ten and eleven dimensions. Therefore reduction to a four-dimensional $\mathcal{N}=2$ theory requires one to truncate the field content after compactification. In [29] the sources are orientifolds, rather than D-branes or M-branes. Some authors [9] have argued that in string theory, orientifolds naturally give rise to cosmological solutions. We leave the investigation of this alternative lift for future work.

Another aspect of our solutions which we have only noted in passing is the reduction of the number of integration constants resulting from imposing the presence of a regular Killing horizon. This is related to the question of whether there is an analogue or generalisation of the attractor mechanism for non-extremal solutions. The attractor mechanism [42, 43, 44, 45] forces scalar field to attain unique values, determined by the charges, at the event horizon of static extremal BPS black holes.[11] This mechanism reduces the number of integration constants in the second order scalar field equations by a factor of one half, since only the asymptotic values of the scalars at infinity remain integration constants that can be chosen arbitrarily. When constructing solutions using the Killing spinor equations, or more generally, by imposing that the scalar field equations reduce to first order gradient flow equations, this reduction is automatic. When solving the second order field equations directly, the reduction in the number of integration constants enters when imposing that the scalars should take regular values at the horizon, rather than displaying run-away behaviour. Interestingly, this link between regularity and the reduction of the number of integration constants also exists for non-extremal solutions, as we have seen in Section 4. While naively we could have expected to obtain solutions with two integration constants per scalar field, there is only one, corresponding to the fields value at infinity, and one additional constant, which corresponds to the

[11]For non-BPS extremal black holes the horizon values of some scalar fields may remain un-fixed, as long as the variation of these values does not change the black hole entropy [46].

non-extremality parameter. While the values of the scalars at the horizon are not exclusively determined by the charges, they are still fixed and completely determined by the other integration constants. Similar observations were made in [2, 47] for non-extremal five- and four-dimensional black holes, and in [3] for Nernst branes. In [47] this behaviour was dubbed 'dressed attractor mechanism', since the horizon values of the scalars are given by the same expressions as in the extremal limit, with the charges replaced by dressed charges which depend on the other integration constants. These observations are consistent with the idea of 'hot attractors', which was advocated in [48, 49, 50], and support the idea that the attractor mechanism is relevant, in a modified form, for non-extremal solutions.

Another loose end is the question of whether the Killing horizons of our solutions admit any thermodynamic interpretation. In an upcoming companion paper, we will show that despite our solutions being cosmological and despite the lack of any standard spacelike asymptotics (such as asymptotically flat or asymptotically AdS), a version of the first law of horizon mechanics can be established.

In the second part of this paper, we have focussed on the four-charge solution, since it admits various higher dimensional lifts and embeddings into string/M-theory. It would be interesting to do something similar for the three-charge solution, which however is a solution of a gauged supergravity, for which, to our knowledge, it is not known how to obtain it as a consistent truncation of a higher-dimensional theory.

One aspect which we have not investigated in this paper is the question of whether our solutions are stable. For this we refer to the discussion in [7, 8] which have addressed some aspects of the stability of the horizon. They found that the situation for the first horizon is the same as for the inner horizon of non-extremal Reissner-Nordstrom solution, while for the second horizon no indication for an instability was found.

While our analysis disfavours interpreting the sources of our solutions as negative tension branes, it has been argued that negative branes exist in string theory [51]. In [52] it was shown that when admitting timelike T-duality, the web of string/M-theories contains exotic theories with twisted supersymmetry and negative kinetic energy for some of the fields (type-II*). Moreover, there exists at least one version of type-II string theory for any possible space-time signature. According to [51], some of the branes of these exotic theories appear as 'negative branes' when viewed from the point of view of a dual theory. This could allow the construction of new, genuinely stringy cosmological solutions, and our formalism could easily be tweaked to study these solutions.

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A Kruskal Cordinates

We detail here the simplest case of calculating global coordinates for the four-charge solution using a Kruskal like coordinate change. Here we assume that the harmonic functions $\mathcal{H}_a$ are all equal; physically this is understood as trivialising the scalar fields within the spacetime, but it is a decision made to simplify the integral of ζ^* to highlight how to choose λ to obtain global coordinates.

As all $\mathcal{H}_a$ are equal, we rewrite the function

$$\mathcal{H} = 2(\beta + \gamma\zeta)^2,$$

and begin with the metric for $\zeta < \alpha^{-1}$

$$ds^2 = -\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}d\eta^2 + \frac{2(\beta+\gamma\zeta)^2}{1-\alpha\zeta}d\zeta^2 + 2(\beta+\gamma\zeta)^2(dx^2+dy^2).$$

We make the coordinate transformation into Eddington-Finkelstein coordinates by using the advanced coordinates

$$v = \eta + \zeta^*, \qquad d\zeta^* = \frac{2(\beta+\gamma z)^2}{1-\alpha\zeta}d\zeta,$$

where we have introduced the tortoise coordinate z^* such that the metric can be written in the form

$$ds^2 = -\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}dv^2 + 2d\zeta dv + 2(\beta+\gamma\zeta)^2(dx^2+dy^2),$$

and we can integrate up to find

$$\zeta^* = -\frac{2(\alpha\beta+\gamma)^2}{\alpha^3}\log(1-\alpha\zeta) - \frac{\gamma\zeta}{\alpha^2}(4\alpha\beta + 2\gamma + \alpha\gamma\zeta).$$

We can also define the advanced coordinate $u = t - \zeta^*$ to write the metric in lightcone coordinates

$$ds^2 = -\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}dvdu + 2(\beta+\gamma\zeta)^2(dx^2+dy^2).$$

We make the Kruskal-like change

$$U = -e^{-\lambda u}, \quad V = e^{\lambda v},$$

such that $U \leq 0$ and $V \geq 0$. Taking derivatives we find $dUdV = \lambda^2 UV dudv$, where

$$UV = (1-\alpha\zeta)^{-\frac{4\lambda}{\alpha^3}(\alpha\beta+\gamma)^2}\exp\left(-\frac{2\lambda\gamma\zeta}{\alpha^2}(4\alpha\beta+2\gamma+\alpha\gamma\zeta)\right).$$

To find the form of λ we substitute this all into the metric and pick λ to remove $(1-\alpha\zeta)$ from the metric to ensure that there globally are no zeros of the metric

$$\begin{aligned} ds^2 &= \frac{1}{\lambda^2 UV}\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}dUdV + 2(\beta+\gamma\zeta)^2(dx^2+dy^2) \\ &= -\frac{(1-\alpha\zeta)^{1+\frac{4\lambda}{\alpha^3}(\alpha\beta+\gamma)^2}}{2\lambda^2(\beta+\gamma\zeta)^2}\exp\left[\frac{2\lambda\gamma\zeta}{\alpha^2}(4\alpha\beta+2\gamma+\alpha\gamma\zeta)\right]dUdV + 2(\beta+\gamma\zeta)^2(dx^2+dy^2). \end{aligned}$$

Making the choice

$$\lambda = -\frac{\alpha^3}{4}(\alpha\beta + \gamma)^{-2},$$

we obtain the metric

$$ds^2 = -\frac{1}{\lambda^2}\frac{e^{\xi(\zeta(U,V))}}{2(\beta + \gamma\zeta(U,V))^2}dUdV + 2(\beta + \gamma\zeta(U,V))^2(dx^2 + dy^2), \tag{A.1}$$

where the new function $\xi(U,V)$ is an everywhere non-zero function in the global domain of ζ

$$\xi(\zeta(U,V)) = -\frac{\alpha\gamma\zeta(4\alpha\beta + 2\gamma + \alpha\gamma\zeta)}{2\alpha(\alpha\beta + \gamma)^2}.$$

B Planar Einstein-Maxwell Solution

If we take the limit of setting the physical scalars of the theory to be constant, the geometry of the four-charge solution becomes that of vacuum solution to the Einstein-Maxwell equations with planar symmetry. This behaviour is expected as the Reissner-Nordström solution is the resulting geometry for the spherically symmetric solution to the STU model with constant physical scalars, also known as the double-extremal limit [40].

The physical scalars are given by

$$z^A = -i\mathcal{H}_A\left(\frac{\mathcal{H}_0}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{2}},$$

and we see that they are everywhere constant under the restriction that $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$. This means the integration constants must be fine-tuned such that $Q_0 = P^1 = P^2 = P^3 = K$ and $h_0 = h^1 = h^2 = h^3 = h$.

Trivialising the constants in this way allows us to easily see the recovery of the Einstein-Maxwell system through studying the $4D$ Lagrangian in section (7.1). The kinetic term for the scalars will vanish; the choice that all integration constants are the same reduces the number of charge gauge fields from one to four, and the term in front of the gauge field can be simply removed through the redefinition of the remaining gauge potential.

This transition from the STU model to the Einstein-Maxwell system is also mirrored in our geometry. When we take the above limit for our integration constants, we recover the line element for the Einstein-Maxwell solution with planar symmetry. The metric for the static patch of the spacetime in the main body of the paper, repeated here

$$ds^2 = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2), \tag{B.1}$$

changes at the level of these functions, which are now given by

$$W(\zeta) = 1 - \alpha\zeta, \qquad \mathcal{H}(\zeta) = (\beta + \gamma\zeta)^2,$$

with constants simplified as

$$\alpha = 2B, \quad \beta = \frac{2K}{\alpha}\sinh\left(\frac{\alpha h}{2K}\right), \quad \gamma = \exp\left(-\frac{\alpha h}{2K}\right), \qquad \alpha, \beta, \gamma \in (0, \infty).$$

The metric written in terms of these new constants for $\zeta < \alpha^{-1}$ is given by

$$ds^2 = -\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}d\eta^2 + \frac{2(\beta+\gamma\zeta)^2}{(1-\alpha\zeta)}d\zeta^2 + 2(\beta+\gamma\zeta)^2(dx^2+dy^2). \tag{B.2}$$

The solution to Einstein-Maxwell's equations with planar symmetry is generally given in the form [24]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2+dy^2), \qquad f(r) = -\frac{2M}{r} + \frac{e^2}{r^2}. \tag{B.3}$$

We can show the equivalence of our solution (B.1) and (B.3) by making the following coordinate transformations

$$2(\beta+\gamma\zeta)^2 = \tilde{r}^2 \;\Rightarrow\; \tilde{r} = \sqrt{2}(\beta+\gamma\zeta), \qquad \zeta = \frac{1}{\gamma}\left(\frac{\tilde{r}}{\sqrt{2}} - \beta\right), \qquad d\zeta = \frac{d\tilde{r}}{\sqrt{2}\gamma},$$

we can then rewrite parts of the line element as

$$\begin{aligned}\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}d\eta^2 &= \left(-\frac{\alpha}{\sqrt{2}\gamma\tilde{r}} + \frac{\alpha\beta+\gamma}{\gamma\tilde{r}^2}\right)d\eta^2,\\ \frac{2(\beta+\gamma\zeta)^2}{(\alpha\zeta-1)}d\zeta^2 &= \left(-\frac{\alpha}{\sqrt{2}\gamma\tilde{r}} + \frac{\alpha\beta+\gamma}{\gamma\tilde{r}^2}\right)^{-1}\frac{d\tilde{r}^2}{2\gamma^2}.\end{aligned}$$

To ensure that the functions preceding the $d\eta^2$ and $d\tilde{r}^2$ are each other's multiplicative inverse we rescale $\tilde{r}$ such that

$$r = \frac{\tilde{r}}{\sqrt{2}\gamma}, \qquad dr = \frac{d\tilde{r}}{\sqrt{2}\gamma}, \qquad \tilde{r} = \gamma\sqrt{2}r.$$

Allowing us to write down the metric in the form

$$ds^2 = -f(r)d\eta^2 + \frac{dr^2}{f(r)} + 2\gamma^2 t^2(dx^2+dy^2),$$

where we have defined the function

$$f(r) := -\frac{\alpha}{2\gamma^2}\frac{1}{r} + \frac{\alpha\beta+\gamma}{2\gamma^3}\frac{1}{r^2}.$$

Finally we rescale the x and y coordinates to re-absorb the $2\gamma^2$ factor and rename η to t to arrive at the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2+dy^2). \tag{B.4}$$

Looking at the function $f(r)$ and comparing this to (B.3) we can relate the integration constants from our solution to the "mass"[12] and electric charge of the solution.

$$M = \frac{\alpha}{4\gamma^2}, \qquad e^2 = \frac{\alpha\beta+\gamma}{2\gamma^3}.$$

[12] M is much more loosely related to the mass for planar solutions as in the spherically symmetric case. As the solution is not asymptotically flat, we cannot identify the integration constant M with the Newtonian limit as is done for the Schwarzschild or Reissner-Nordström solutions. In contrast, the electric charge can still be set via Gauss' law and so is easier to pin down.

C Spherically Symmetric Solutions for STU Supergravity from the C-Map

While working on the oxidation of the planar STU model we noticed that the $4D$ metric with planar symmetry was only superficially different from the solution found in [2] where spherical symmetry was imposed on a class of prepotentials which included the STU model.

Due to the simplicity of the generalisation of the uplift and the popularity of spherically symmetric solutions for supergravity theories we have chosen to include here the uplift of the non-extremal STU model with spherical symmetry in ten and eleven dimensions. We hope that the line elements and gauge field content for these theories could be interesting for those looking at non-extremal STU models in the future.

In this appendix, we first show how the spherically symmetric solution is related to the four-charge solution solved in the main body of the paper. We then write down the metric and gauge field content of both string/M-theory embeddings. Finally, we show that taking the 4D extremal limit the resulting metrics are now in the form of the intersecting M5 branes [34] in $11D$ and the $D1$-$D5$-P-KK solutions in $10D$. Unlike the planar solution, the harmonic functions of the spherically symmetric solution diverge for $\rho = 0$ and so we obtain an interpretation for the position of the intersecting branes for each higher dimensional theory.

It is also interesting to note here how the two solutions differ; that when changing the geometric ansatz from spherical to planar the asymptotic region of the spacetime changes from static to dynamic.

We begin this section referring to [3] where the STU prepotential is picked out by setting $n = 3$ in (5.22) with the resulting four-dimensional metric

$$ds_4^2 = -\frac{W(\rho)dt^2}{\sqrt{-H_0 H^1 H^2 H^3}} + \sqrt{-H_0 H^1 H^2 H^3}\left(\frac{d\rho^2}{W(\rho)} + \rho^2 d\Omega_2^2\right). \tag{C.1}$$

The functions are:

$$W(\rho) = 1 - \frac{2c}{\rho},$$
$$H_0(\rho) = -\sqrt{2}Q_0\left[\frac{1}{c}\sinh\left(\frac{ch_0}{Q_0}\right) + e^{-\frac{ch_0}{Q_0}}\rho^{-1}\right],$$
$$H^A(\rho) = \sqrt{2}P^A\left[\frac{1}{c}\sinh\left(\frac{ch^A}{P^A}\right) + e^{-\frac{ch^A}{P^A}}\rho^{-1}\right],$$

for $A = 1, 2, 3$. These should remind the reader of the functions $W(\zeta)$ and $\mathcal{H}_a(\zeta)$ with $\zeta \to \rho^{-1}$ and $\alpha \leftrightarrow c$. Of course these two metrics are not related by $\zeta \to \rho^{-1}$ as this would affect $d\zeta^2 \to \frac{d\rho}{\rho^4}$.

The physical scalars are given by (5.24) in [2]

$$z^A = -iH_A\sqrt{\frac{-H_0}{H^1 H^2 H^3}},$$

and the gauge field strengths

$$F^0 = \frac{1}{2}\frac{Q_0}{q_0^2}dt\wedge d\tau, \qquad F^A = -\frac{1}{2}P^a \sin\theta d\theta\wedge d\phi, \qquad q_0^2 = \frac{H_0^2}{2W}.$$

The physical scalars for the spherically symmetric model are in an identical form but with $H_a \leftrightarrow \mathcal{H}_a$. In the asymptotic limit H_a tend to constants, where as $\mathcal{H}_a$ diverge $\mathcal{O}(\zeta)$ and although visually similar the physical behaviour of the scalars between the two solutions will be different. In particular, assuming that $h_a \neq 0$ all physical scalars are asymptotic to constant values.

Integrating up and applying boundary conditions gauge potential is found to be

$$A^0 = \frac{Wc}{Q_0}\left(e^{\frac{2ch_0}{Q_0}} - W\right)^{-1} d\eta, \qquad A^A = \frac{1}{2}P^A \cos\theta d\phi,$$

which can be manipulated into

$$A^0 = -\frac{W\gamma_0}{\sqrt{2}Q_0 H_0}d\eta, \qquad A^A = \frac{1}{2}P^A \cos\theta d\phi, \qquad \gamma_0 = Q_0 e^{-\frac{ch^A}{P^A}}.$$

We see that A^0 has the same form as A^0 from the planar solution and the A^A are now constants as before but over the two-sphere and not the two-plane.

As we are not required to take derivatives of the metric functions during the oxidation procedure, we find that the uplift of the spherically symmetric solution is unaffected by the difference in form of the harmonic functions H_a. This allows us to simply write down the higher dimensional uplifts of this solution straight from the work in the main body of the text.

Oxidation to Five Dimensions

$$\begin{aligned} ds_5^2 = (H_1H_2H_3)^{-\frac{1}{3}}\Bigg[&H_0 dz_5^2 + \frac{W}{2H_0}\left(W\frac{\gamma_0^2}{Q_0^2} - 1\right)dt^2 + \frac{2W\gamma_0}{\sqrt{2}Q_0}dtdz_5 \\ &+ 2H_1H_2H_3\left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2\right)\Bigg], \end{aligned} \tag{C.2}$$

where $d\Omega_2^2$ is the line element for the two sphere

$$\tilde{\mathbb{A}}_i = \sqrt{2}A^A = \mathfrak{p}_a \cos\theta d\phi, \quad \mathfrak{p}_a = \frac{P^A}{\sqrt{2}}. \tag{C.3}$$

Oxidation to Eleven Dimensions

Using an identical procedure, the uplift to $11D$ is trivial and given by

$$ds_{11}^2 = ds_5^2 + h_1(dy_1^2 + dy_2^2) + h_2(dy_3^2 + dy_4^2) + h_3(dy_5^2 + dy_6^2),$$

$$\mathcal{A} = \tilde{\mathbb{A}}_1 \wedge dy^1 \wedge dy^2 + \tilde{\mathbb{A}}_2 \wedge dy^3 \wedge dy^4 + \tilde{\mathbb{A}}_3 \wedge dy^5 \wedge dy^6.$$

This compactification is subject to the constraint that the torus has constant volume (which is equivalent to $h_1h_2h_3 = 1$). The explicit ζ dependence of h_i is

$$h_i = \frac{H_i}{(H_1H_2H_3)^{\frac{1}{3}}},$$

and the three-form gauge potential is found simply from the components of (7.11). Thus the full line element for the non-extremal planar STU model is

$$ds_5^2 = (H_1H_2H_3)^{-\frac{1}{3}}\Bigg[H_0dz_5^2 + \frac{W}{2H_0}\left(W\frac{\gamma_0^2}{Q_0^2}-1\right)dt^2 + \frac{2W\gamma_0}{\sqrt{2}Q_0}dtdz_5 \\ + 2H_1H_2H_3\left(\frac{d\rho^2}{W}+\rho^2d\Omega_2^2\right) \\ + H_1(dy_1^2+dy_2^2)+H_2(dy_3^2+dy_4^2)+H_3(dy_5^2+dy_6^2)\Bigg].$$

Oxidation to Six Dimensions

The appropriate reduction ansatz [30] to arrive back at the 5D Lagrangian is given by

$$ds_6^2 = e^{\sigma/\sqrt{6}}ds_5^2 + e^{-3\sigma/\sqrt{6}}(dz_6+\tilde{\mathbb{A}}_1)^2, \qquad B_{(6D)} = B + \tilde{\mathbb{A}}_2\wedge(dz_6+\tilde{\mathbb{A}}_1).$$

with the field strengths decomposed as

$$H_{(6D)} = \mathbb{H} + \tilde{\mathbb{F}}_2\wedge(dz_6+\tilde{\mathbb{A}}_1), \qquad \mathbb{H} = dB - \tilde{\mathbb{A}}_2\wedge\tilde{\mathbb{F}}_1, \qquad \tilde{\mathbb{F}}_i = d\tilde{\mathbb{A}}_i.$$

The metric is given in the same form as the main body of the text

$$ds_6^2 = (H_2H_3)^{-\frac{1}{2}}\Bigg[H_0dz_5^2 + \frac{W}{2H_0}\left(W\frac{\gamma_0^2}{Q_0^2}-1\right)d\eta^2 + \frac{2W\gamma_0}{\sqrt{2}Q_0}d\eta dz_5 \\ + 2H_1H_2H_3\left(\frac{d\rho^2}{W}+d\Omega_2^2\right)\Bigg] + \frac{(H_2H_3)^{\frac{1}{2}}}{H_1}(dz_6+\tilde{\mathbb{A}}_1)^2.$$

The gauge content is very similar. The gauge fields

$$\tilde{\mathbb{A}}_1 = \sqrt{2}A^1, \qquad \tilde{\mathbb{A}}_2 = \sqrt{2}A^2,$$

are identical to the $4D$ solutions. Using the work from the $5D$ calculations the $6D$ three-form field strength is given by:

$$H_{(6D)} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2}\right)d\eta\wedge d\rho\wedge dz_5 + (2\mathfrak{p}_2)\sin\theta d\theta\wedge d\phi\wedge dz_6.$$

Oxidation to Ten Dimensions

The reduction ansatz to uplift the solution to $10D$ is given by

$$ds_{10}^2 = ds_6^2 + e^{\phi/\sqrt{2}}(dy_1^2+dy_2^2+dy_3^2+dy_4^2), \qquad \Phi = \frac{\phi}{\sqrt{2}}, \qquad C\equiv B,$$

where ϕ is the dilaton from the $6D$ theory and can be found in terms of ρ by using

$$e^{\sqrt{2}\phi} = \frac{h_2}{h_3} = \left(\frac{\mathcal{I}_{33}}{\mathcal{I}_{22}}\right)^{\frac{1}{2}} \quad\Rightarrow\quad e^{\phi/\sqrt{2}} = \left(\frac{\mathcal{I}_{33}}{\mathcal{I}_{22}}\right)^{\frac{1}{4}} = \sqrt{\frac{H_2}{H_3}},$$

and everything else is found simply from the $6D$ analysis. The Dilaton is given by

$$\Phi = \frac{1}{2}\log\left(\frac{H_2}{H_3}\right).$$

The line element is given by

$$ds_{10}^2 = \sqrt{\frac{H_2}{H_3}}\Bigg[H_0 H_2^{-1} dz_5^2 + \frac{W}{2H_0H_2}\left(W\frac{\gamma_0^2}{Q_0^2} - 1\right) d\eta^2 + \frac{W\gamma_0}{\sqrt{2}Q_0H_2} d\eta dz_5$$
$$+ 2H_1H_3\left(\frac{d\rho^2}{W} + d\Omega_2^2\right) + \frac{H_3}{H_1}(dz_6 + \tilde{\mathbb{A}}_1)^2 + dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\Bigg].$$

Extremal Limit

We can again take the limit of $c \to 0$ for the $4D$ solution to obtain the extremal limit of the spherically symmetric solution.

Uplifting the extremal solution to $11D$ results in the following line element

$$\begin{aligned} ds_{11}^2 =& (H_1H_2H_3)^{-\frac{1}{3}}\big[d\eta dz_5 + H_0 dz_5^2 + H_1H_2H_3(d\rho^2 + d\Omega_2^2) \\ &+ H_1(dy_1^2 + dy_2^2) + H_2(dy_3^2 + dy_4^2) + H_3(dy_5^2 + dy_6^2)\big], \end{aligned} \tag{C.4}$$

which matches exactly with equation (4.1) in [34]. This allows us to identify this solution as the intersection of three $M5$ branes with momentum along the common intersection.

Uplifting the extremal solution to $10D$ we find that the line element is

$$\begin{aligned} ds_{10}^2 =& \sqrt{\frac{H_2}{H_3}}\Bigg[H_2^{-1}\left(H_0 dz_5^2 + dz_5 d\eta\right) + H_3H_1(d\rho^2 + d\Omega_2^2) \\ &+ H_3H_1^{-1}(dz_6^2 + \tilde{\mathbb{A}}^1)^2 + dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\Bigg], \end{aligned} \tag{C.5}$$

which is the intersection of a D1 and D5 brane with momentum along the common direction and a Taub-NUT space.

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