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Four-dimensional vector multiplets in arbitrary signature (I)

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We derive a necessary and sufficient condition for Poincaré Lie superalgebras in any dimension and signature to be isomorphic. This reduces the classification problem, up to certain discrete operations, to classifying the orbits of the Schur group on the vector space of superbrackets. We then classify four-dimensional $\mathcal{N} = 2$ supersymmetry algebras, which are found to be unique in Euclidean and in neutral signature, while in Lorentz signature there exist two algebras with R-symmetry groups U(2) and U(1, 1), respectively.

Keywords: Poincaré Lie superalgebras; extended supersymmetry; arbitrary signature

1. Introduction

Supersymmetry can be defined in any space-time signature. Besides Lorentzian signature, Euclidean signature has received a good deal of attention because of its relevance for the functional integral formalism, non-perturbative effects, and the construction of stationary solutions through dimensional reduction to an auxiliary Euclidean theory. Exotic signatures with more than one time-like dimension have been less studied, but seem to be mandatory in string theory, where space-time signature can be changed by combining time-like T-duality and S-duality [1,2,3]. On the mathematical side, $\mathcal{N}$-extended Poincaré Lie superalgebras in general signature (t, s) have been constructed and classified, in arbitrary dimension and for arbitrary $\mathcal{N}$, in [4]. This work was extended to a classification of polyvector charges

(BPS charges) in [5]. While this construction allows one to obtain all Poincaré Lie superalgebras, it does not immediately provide a classification up to isomorphism, for the following reason: the essential ingredients in extending a Poincaré Lie algebra $\mathfrak{g}_0 = \mathfrak{p}(V) = \mathfrak{so}(V) + V$, where $V \cong \mathbb{R}^{t,s}$, to a Poincaré Lie superalgebra are: (i) the specification of a spinorial module (spin 1/2 representation) S which serves as the odd part, $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 = (\mathfrak{so}(V) + V) + S$, and (ii), the specification of the superbracket on S. More precisely, as shown in [4], one needs to specify a real, symmetric, vector-valued, $\mathrm{Spin}_0(V)$-equivariant bilinear form $\Pi \ : S \times S \to V$, which defines the restriction of the superbracket to $S \times S$, $[s,t] := \Pi(s,t)$ for all $s, t \in S$. As shown in [4], a basis of the vector space of all such vector-valued bilinear forms can be constructed in terms of so-called *admissible* bilinear forms $\beta : S \times S \to \mathbb{R}$. While all possible Poincaré Lie superalgebras can be obtained this way, one still needs criteria which allow one to decide whether the algebras defined by any two given superbrackets are isomorphic, or not. This is the problem which we address and solve in this paper. Theorem 1 gives a necessary and sufficient condition for two Poincaré Lie superalgebras to be isomorphic, while subsequently Corollary 1 shows that the classification problem amounts to, essentially (see Remark 1), classifying the orbits of the so-called Schur group $\mathcal{C}^*(S)$ on the space of superbrackets. The Schur group is the subgroup of $\mathrm{GL}(S)$ the elements of which commute with the action of $\mathrm{Spin}_0(V)$. The stabilizer subgroup of the Schur group on a given orbit is the R-symmetry group of the corresponding supersymmetry algebra.

As an application of this general result we obtain the classification of four-dimensional $\mathcal{N} = 2$ supersymmetry algebras for all signatures $(0,4), \ldots,\ (4,0)$. Here $\mathcal{N} = 2$ supersymmetry refers to supersymmetry algebras whose odd part is the complex spinor module $\mathbb{S} \cong \mathbb{C}^4$, that is the representation by Dirac spinors. Note that for some signatures this is the minimal supersymmetry algebra. Since signatures (t,s) and (s,t) are physically equivalent, as they are related by going from a mostly plus to a mostly minus convention for the metric, or, for neutral signature, swapping of time-like against space-like dimensions, there are three cases to consider: Euclidean, Lorentzian and neutral signature. In all cases the space of $\mathcal{N} = 2$ superbrackets is four-dimensional, and different isomorphism classes of $\mathcal{N} = 2$ supersymmetry algebras are represented by elements in different open orbits of the Schur group. In cases where the $\mathcal{N} = 2$ supersymmetry algebra is non-minimal, $\mathcal{N} = 1$ supersymmetry algebras are related to lower-dimensional orbits. While in Euclidean and in neutral signature the $\mathcal{N} = 2$ supersymmetry algebra is shown to be unique up to isomorphism, we find that there are two Lorentzian $\mathcal{N} = 2$ supersymmetry algebras, distinguished by their R-symmetry groups, which are $\mathrm{U}(2)$ and $\mathrm{U}(1,1)$ respectively. The supersymmetry algebra with non-compact R-symmetry group is of the same type as the 'twisted' or 'type-*' supersymmetry algebras that occur when time-like T-duality is applied to 'conventional' theories, the prime example being the map between IIA/B and IIB*/IIA* string theory [1]. Explicit off-shell representations for four-dimensional vector multiplets in arbitrary signature will be presented in a companion paper [6].

2. Classification of Poincaré Lie superalgebras in arbitrary dimension, signature and number of supercharges

Consider the pseudo-Euclidean vector space $V = \mathbb{R}^{t,s} \cong \mathbb{R}^{t+s}$ with its standard scalar product $\langle v, w\rangle = -\sum_{i=1}^{t} v^i w^i + \sum_{i=t+1}^{t+s} v^i w^i$. We denote by S an arbitrary non-trivial module of the Clifford algebra $Cl(V)$, considered as a module of the Lie algebra $\mathfrak{so}(V) \cong \mathfrak{spin}(V)$, that is, an arbitrary sum of irreducible spinor modules. Then $\gamma : Cl(V) \to \mathrm{End}\, S$, $a \mapsto \gamma_a = \gamma(a)$, denotes the corresponding Clifford representation. Let $\mathfrak{g} = \mathfrak{so}(V) + V + S$ be the direct sum of the vector spaces $\mathfrak{so}(V)$, V, S. We endow $\mathfrak{g}$ with the $\mathbb{Z}_2$ grading $\mathfrak{g}_0 = \mathfrak{so}(V) + V$, $\mathfrak{g}_1 = S$.

We consider on $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ all possible Lie superbrackets $[\cdot,\cdot]$ of the following form:

$$[A, B] = AB - BA\ , [A, v] = Av\ , [v_1, v_2] = 0\ , [A, s] = A \cdot s := \rho_S(A)s\ ,$$

$$[s_1, s_2] = \Pi(s_1, s_2) \in V\ ,$$

for all $A, B \in \mathfrak{so}(V)$, $v, v_1, v_2 \in V$ and $s, s_1, s_2 \in S$, where ρ_S denotes the spinorial representation of $\mathfrak{so}(V)$ on S and where $\Pi \in (\mathrm{Sym}^2 S^* \otimes V)^{\mathrm{Spin}_0(V)}$, is a symmetric, Spin_0-equivariant vector-valued bilinear form on S.

Such Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot])$ are called *Poincaré Lie superalgebras*. All such brackets Π are linear combinations of brackets of the form Π_β, where β is a *super-admissible* bilinear form on S [4]. Π_β is defined as follows:

$$\langle \Pi_\beta(s_1, s_2)\ , v\rangle = \beta(vs_1, s_2)\ , \tag{2.1}$$

for all $s_1, s_2 \in S$, $v \in V$. The admissibility of the form β is defined by the existence of $\sigma, \tau \in \{\pm 1\}$, called the *symmetry*, and the *type* of β, respectively, such that

$$\begin{aligned} \beta(s_1, s_2) &= \sigma\beta(s_2, s_1)\ , \\ \beta(vs_1, s_2) &= \tau\beta(s_1, vs_2)\ , \end{aligned} \tag{2.2}$$

for all $s_1, s_2 \in S$, $v \in V$. An admissible form is called *super-admissible* if $\sigma\tau = 1$. All admissible bilinear forms were described in [4]. In particular, all the brackets Π defining Poincaré Lie superalgebras are known explicitly.

In general, the space of brackets is higher-dimensional and for a given pair $\Pi, \Pi' \in (\mathrm{Sym}^2 S^* \otimes V)^{\mathrm{Spin}_0(V)}$ one needs to decide whether the corresponding Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot] = [\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}, [\cdot,\cdot]' = [\cdot,\cdot]_{\Pi'})$ are isomorphic. This is the classification problem for Poincaré Lie superalgebras up to isomorphism. In this section we explain how this problem can be solved in general. In the next section we will apply the method in four dimensions for the case where the spinorial module S is the complex spinor module $\mathbb{S}$, that is the representation on Dirac spinors, regarded as a real representation.

Theorem 1. *Assume that the signature (t, s) of V is different from $(1, 1)$. Two Poincaré Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot])$ and $(\mathfrak{g}, [\cdot,\cdot]')$ are isomorphic if and only if there* *exists* $\psi = \psi' \cdot a \in \mathrm{Pin}(V) \cdot \mathcal{C}(S)^*$*, where* $\psi' \in \mathrm{Pin}(V)$ *and* $a \in \mathcal{C}(S)^*$*, such that*

$$\Pi'(\psi s_1, \psi s_2) = \varphi(\Pi(s_1, s_2)) \ , \tag{2.3}$$

or

$$\Pi'(\psi s_1, \psi s_2) = -\varphi(\Pi(s_1, s_2)) \ , \tag{2.4}$$

for all $s_1, s_2 \in S$*, where* φ *is the image of* ψ' *under the homomorphism* $\mathrm{Ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$ *induced by the adjoint representation of* $\mathrm{Pin}(V)$ *on* V*. Here* $\mathcal{C}(S)^* = Z_{\mathrm{GL}(S)}(\mathfrak{spin}(V))$ *denotes the group of invertible elements of the Schur algebra* $\mathcal{C}(S) = Z_{\mathrm{End}(S)}(\mathfrak{spin}(V))$*. The product* $\mathrm{Pin}(V) \cdot \mathcal{C}(S)^*$ *denotes the subgroup of* $\mathrm{GL}(S)$ *generated by* $\mathrm{Pin}(V)$ *and* $\mathcal{C}(S)^*$*. (Notice that* $\mathrm{Pin}(V)$ *normalizes* $\mathcal{C}(S)^*$*.)*

Proof: Every isomorphism $\phi : (\mathfrak{g}, [\cdot,\cdot]) \to (\mathfrak{g}, [\cdot,\cdot]')$ maps $\mathfrak{g}_i$ to $\mathfrak{g}_i$, $i = 0, 1$. It also maps V to V, since V is precisely the kernel of the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, which is induced by the adjoint representation of $\mathfrak{g}$ with either bracket. We define:

$$\varphi := \phi|_V \in \mathrm{GL}(V) \ , \quad \psi := \phi|_S \in \mathrm{GL}(S) \ .$$

It follows that ϕ induces an automorphism ξ of the quotient $\mathfrak{so}(V) = (\mathfrak{so}(V) + V)/V$. Even more is true. The subalgebra $\phi(\mathfrak{so}(V)) \subset \mathfrak{so}(V) + V$ is conjugate to $\mathfrak{so}(V)$ by a translation, as follows from $H^1(\mathfrak{so}(V), V) = 0$. Therefore, up to composition of ϕ with the inner automorphism of $(\mathfrak{g}, [\cdot,\cdot]')$ induced by the above translation, we can assume that $\phi(\mathfrak{so}(V)) = \mathfrak{so}(V)$. Now we can identify $\xi = \phi|_{\mathfrak{so}(V)} \in \mathrm{Aut}(\mathfrak{so}(V))$. Therefore ϕ is an isomorphism if and only if ξ, φ, ψ satisfy the following system of equations:

$$\xi(A)\varphi(v) = \varphi(Av) \ , \tag{2.5}$$

$$\xi(A)\psi(s) = \psi(As) \ , \tag{2.6}$$

and (2.3), for all $A \in \mathfrak{so}(V)$, $v \in V$ and $s_1, s_2 \in S$. Equation (2.5) determines $\xi \in \mathrm{Aut}(\mathfrak{so}(V))$ in terms of φ as $\xi = C_\varphi$, where $C_\varphi : A \mapsto \varphi \circ A \circ \varphi^{-1}$ denotes the conjugation by φ. Now (2.5) is a condition solely on φ:

$$\varphi \in N_{\mathrm{GL}(V)}(\mathfrak{so}(V)) = \{A \in \mathrm{GL}(V) \mid A^*\langle\cdot,\cdot\rangle = \pm\lambda\langle\cdot,\cdot\rangle, \quad \lambda > 0\} \ .$$

Here we have used that a linear transformation which normalizes the Lie algebra $\mathfrak{so}(V)$ (and therefore the group $\mathrm{SO}_0(V)$) preserves the scalar product up to a (possibly negative) factor, which is true for all signature (t, s) with the exception of $(t, s) = (1, 1)$. Note if $t \neq s$, the resulting group is precisely the linear conformal group

$$\mathrm{CO}(V) = \{A \in \mathrm{GL}(V) \mid A^*\langle\cdot,\cdot\rangle = \lambda\langle\cdot,\cdot\rangle, \quad \lambda > 0\} = \mathbb{R}^* \cdot \mathrm{O}(V),$$

since anti-isometries only exist if $t = s$. The next lemma shows that (2.6) implies $\varphi \in \mathrm{CO}(V)$ for all signatures $(t, s) \neq (1, 1)$.

Lemma 1. *Assume that* $t = s \geq 2$*, and let* ξ *be the automorphism of* $\mathfrak{so}(V)$ *induced by an anti-isometry* $\varphi \in \mathrm{GL}(V)$*. Then there is no* $\psi \in \mathrm{GL}(S)$ *normalizing the image of* $\mathfrak{spin}(V)$ *in* $\mathrm{End}\, S$ *and acting on* $\mathfrak{spin}(V) \cong \mathfrak{so}(V)$ *as* ξ*.*

Proof: Since the homomorphism $\mathrm{Ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$ is surjective we can assume without loss of generality that φ is given by $\varphi(e_i) = e'_i$, $\varphi(e'_i) = e_i$, where $(e_1, \dots, e_t, e'_1, \dots, e'_t)$ is an orthonormal basis with time-like vectors e_i. Then ξ interchanges e_ie_j with $-e'_ie'_j$ $(i \neq j)$ and $e_ie'_j$ with $-e'_ie_j = e_je'_i$ $(i, j$ arbitrary$)$.

We proceed by induction starting with the case $t = 2$ (since the claim is not true for $t = 1$). Without loss of generality we can assume that the Clifford module S is irreducible. Then we can realize S in signature $(2, 2)$ as $S = \mathbb{R}^2 \otimes \mathbb{R}^2$, where $\gamma_{e_1} = J \otimes I$, $\gamma_{e_2} = K \otimes I$, $\gamma_{e'_1} = \mathbb{1} \otimes J$, $\gamma_{e'_2} = \mathbb{1} \otimes K$, where $I, J, K = IJ$ are pairwise anti-commuting operators on $\mathbb{R}^2$ such that $J^2 = K^2 = \mathbb{1} = -I^2$. Then ξ preserves the elements $J \otimes K, K \otimes J$ and interchanges $1 \otimes I$ with $-I \otimes \mathbb{1}$ and $J \otimes J$ with $-K \otimes K$. In fact, these elements obtained by pairwise multiplying the above Clifford generators form a basis of $\mathfrak{spin}(V)$. Now we can write $\psi \in \mathrm{End}(S)$ in the form

$$\psi = \mathbb{1} \otimes A_0 + I \otimes A_1 + J \otimes A_2 + K \otimes A_3, \tag{2.7}$$

where $A_a \in \mathrm{End}(\mathbb{R}^2)$, $a = 0, \dots, 3$. Now one can easily solve the system of equations

$$\psi \circ (J \otimes K) = (J \otimes K) \circ \psi, \quad \psi \circ (K \otimes J) = (K \otimes J) \circ \psi,$$

$$\psi \circ (\mathbb{1} \otimes I) = -(I \otimes \mathbb{1}) \circ \psi, \quad \psi \circ (K \otimes K) = -(J \otimes J) \circ \psi,$$

which corresponds to (2.6). We find that the only solution is $\psi = 0$, showing that for $t = 2$ there is no $\psi \in \mathrm{GL}(S)$ with the desired properties.

To pass from t to $t + 1$ we write the irreducible Clifford module in signature $(t + 1, t + 1)$ as $S = \mathbb{R}^2 \otimes (\mathbb{R}^2)^{\otimes n}$, where $\gamma_{e_i} = J \otimes L_i$, $\gamma_{e'_i} = J \otimes L'_i$, $\gamma_{e_{n+1}} = I \otimes \mathbb{1}$, $\gamma'_{e_{n+1}} = K \otimes \mathbb{1}$ and L_i, L'_i are Clifford generators in signature (t, t). Then we write $\psi \in \mathrm{End}(S)$ as (2.7), where now $A_a \in \mathrm{End}((\mathbb{R}^2)^{\otimes n})$. The equation (2.6) is now a system of equations for the A_a, which contains the following equations:

$$A_aL_iL_j = -L'_iL'_jA_a \quad (i \neq j), \quad A_aL_iL'_j = L_jL'_iA_a \tag{2.8}$$

and also equations involving $\gamma_{e_{n+1}}$ and $\gamma'_{e_{n+1}}$. By induction, the equations (2.8) already imply $A_a = 0$. In fact, this system for a single A corresponds to the equation (2.6) in signature (t, t). □

Since a homothety with factor μ on S accompanied by μ^2 on V defines an automorphism of any Poincaré Lie superalgebra, we can assume that $\varphi \in \mathrm{O}(V)$. It is known that the homomorphism $\mathrm{Ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$ is surjective for $\dim V$ even, while the image is $\mathrm{SO}(V)$ if $\dim V$ is odd. Irrespective of the dimension of V, there either exists $\psi_1 \in \mathrm{Pin}(V)$, with $\mathrm{Ad}(\psi_1) = \varphi$, or there exists $\psi_2 \in \mathrm{Pin}(V)$ with $\mathrm{Ad}(\psi_2) = -\varphi$, or both. Any such ψ_i solves equation (2.6), and all solutions are of this type.

This shows that ψ coincides, up to an element of the Schur group $\mathcal{C}(S)^*$, either with a pre-image ψ_1 of φ or with a pre-image ψ_2 of $-\varphi$ under the map $\mathrm{Ad} : \mathrm{Pin}(V) \to O(V)$. In the former case (2.3) holds, whereas in the latter case the equation

$$\Pi'(\psi s_1, \psi s_2) = -\tilde{\varphi}(\Pi(s_1, s_2))$$

holds, where $\tilde{\varphi} = -\varphi$ is the image of ψ under $\mathrm{Ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$. Conversely, any solution (ψ, φ) of (2.3) or (2.4) defines an isomorphism from $(\mathfrak{g}, [\cdot,\cdot] = [\cdot,\cdot]_\Pi)$ to $(\mathfrak{g}, [\cdot,\cdot]' = [\cdot,\cdot]_{\Pi'})$ or from $(\mathfrak{g}, [\cdot,\cdot]_{-\Pi})$ to $(\mathfrak{g}, [\cdot,\cdot]' = [\cdot,\cdot]_{\Pi'})$, respectively. This proves the theorem since the Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}, [\cdot,\cdot]_{-\Pi})$ are isomorphic. An isomorphism is given by $(A, v, s) \mapsto (A, -v, s)$. □

The above theorem allows us to reduce the classification of Poincaré Lie superalgebras up to isomorphism to the classification of the orbits

$$\mathcal{O}_\Pi := \mathcal{C}(S)^* \cdot \mathrm{Pin}(V) \cdot \Pi \tag{2.9}$$

of the group $\frac{\mathcal{C}(S)^* \cdot \mathrm{Pin}(V)}{\mathrm{Spin}_0(V)}$ on $(\mathrm{Sym}^2 S^* \otimes V)^{\mathrm{Spin}_0(V)}$. Notice that the finite group $\mathrm{Pin}(V)/\mathrm{Spin}_0(V) \cong \mathrm{O}(V)/\mathrm{SO}_0(V)$ is isomorphic either to $\mathbb{Z}_2$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since we are ultimately interested in the four-dimensional case, we will now assume that $n = t + s = \dim V$ is even. If this case

$$\frac{\mathrm{Pin}(V)}{\mathrm{Spin}_0(V)} = \begin{cases} \{[1], [e_1], [\omega], [e_1\omega]\} , & \text{if V indefinite , } t, s \text{ odd ,} \\ \{[1], [e_1], [e_{t+s}], [e_1 e_{t+s}]\} , & \text{if V indefinite , } t, s \text{ even ,} \\ \{[1], [e_1]\} , & \text{if V definite ,} \end{cases}$$

where $(e_1, \ldots, e_n)$ is an orthonormal basis of V, and where $\omega = e_1 \cdots e_n$.

Since $\omega \in \gamma(\mathrm{Pin}(V)) \cap \mathcal{C}(S)^*$, we have

(1)
$$\begin{aligned} \mathcal{C}(S)^* \cdot \gamma(\mathrm{Pin}(V)) = \mathcal{C}(S)^* \cdot \gamma(\mathrm{Spin}_0(V)) \cup \mathcal{C}(S)^* \cdot \gamma(\mathrm{Spin}_0(V) e_1) \cup \\ \mathcal{C}(S)^* \cdot \gamma(\mathrm{Spin}_0(V) e_{t+s}) \cup \mathcal{C}(S)^* \cdot \gamma(\mathrm{Spin}_0(V) e_1 e_{t+s}) , \end{aligned}$$

if V is indefinite and t, s are both even.

(2)
$$\mathcal{C}(S)^* \cdot \gamma(\mathrm{Pin}(V)) = \mathcal{C}(S)^* \cdot \gamma(\mathrm{Spin}_0(V)) \cup \mathcal{C}(S)^* \cdot \gamma(\mathrm{Spin}_0(V) e_1) ,$$

if V is definite, or if V is indefinite and t, s are both odd.

This proves the following:

Proposition 1. *Assume that* $\dim V$ *is even. Then the orbit* $\mathcal{O}_\Pi$ *defined in (2.9) is given by*

$$\mathcal{O}_\Pi = \mathcal{C}(S)^* \cdot \Pi \cup \mathcal{C}(S)^* \cdot \gamma_{e_1} \cdot \Pi \cup \mathcal{C}(S)^* \cdot \gamma_{e_{t+s}} \cup \mathcal{C}(S)^* \cdot \gamma_{e_1 e_{t+s}}$$

if V *is indefinite and* t, s *are both even, and by*

$$\mathcal{O}_\Pi = \mathcal{C}(S)^* \cdot \Pi \cup \mathcal{C}(S)^* \cdot \gamma_{e_1} \cdot \Pi .$$

if V *is definite or if* V *is indefinite and* t, s *are both odd.*

Using Theorem 1 we obtain:

Corollary 1. *Assume that* $\dim V$ *is even, with* $V \ncong \mathbb{R}^{1,1}$.

(1) *V is definite, or V is indefinite and t, s are odd. Then two Poincaré Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot] = [\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}, [\cdot,\cdot]' = [\cdot,\cdot]_{\Pi'})$ are isomorphic if and only if Π, $-\Pi$, $\gamma_{e_1}\Pi$, or $-\gamma_{e_1}\Pi$ is related to Π' by an element of the Schur group $\mathcal{C}(S)^*$.*
(2) *V is indefinite and t, s are both even. Then two Poincaré Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot] = [\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}, [\cdot,\cdot]' = [\cdot,\cdot]_{\Pi'})$ are isomorphic if and only if Π, $-\Pi$, $\gamma_{e_1}\Pi$, $-\gamma_{e_1}\Pi$ $\gamma_{e_{t+s}}\Pi$, $-\gamma_{e_{t+s}}\Pi$, $\gamma_{e_1 e_{t+s}}\Pi$ or $-\gamma_{e_1 e_{t+s}}\Pi$, is related to Π' by an element of the Schur group $\mathcal{C}(S)^*$.*

Remark 1. We will find in Section 3 that in dimension four, and for $S = \mathbb{S}$ the complex spinor module, the elements γ_{e_1} and $\gamma_{e_{t+s}}$ in Proposition 1 and Corollary 1 are not needed, that is $\mathcal{O}_\Pi = \mathcal{C}(\mathbb{S})^* \cdot \Pi$ and two Poincaré Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot] = [\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}, [\cdot,\cdot]' = [\cdot,\cdot]_{\Pi'})$ are isomorphic if and only if Π or $-\Pi$ is related to Π' by an element of the Schur group $\mathcal{C}(S)^*$.

3. Classification of Poincaré Lie superalgebras based on four-dimensional Dirac spinors in arbitrary signature

3.1. *The general setting*

Now we apply the method in four dimensions for the case where the $\mathfrak{spin}(V)$ module S is the complex spinor module $\mathbb{S}$, regarded as a real module. According to Corollary 1, to classify the Poincaré Lie superalgebras in this case, we need to determine first the Schur group $\mathcal{C}(\mathbb{S})^*$ for all possible signatures (t, s), $t + s = 4$, and classify the orbits of the Schur group on $(\mathrm{Sym}^2\mathbb{S}^* \otimes V)^{\mathrm{Spin}_0(V)}$. Then we need to determine the orbits of the involution induced by γ_{e_1}, and for t, s both even also of γ_{e_4} and $\gamma_{e_1}\gamma_{e_4}$, on this set of orbits.

For reference, we will now list the Clifford algebras, spinor modules and Schur algebras that are relevant in four dimensions. We use a notation where $\mathbb{K}(N)$ denotes the algebra of $N \times N$ matrices over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and where $m\mathbb{K}(N) := \mathbb{K}(N) \oplus \cdots \oplus \mathbb{K}(N)$ is the m-fold direct sum of the algebras $\mathbb{K}(N)$. The algebra $m\mathbb{K}(N)$ has precisely m inequivalent irreducible representations, given by the natural action of one factor $\mathbb{K}(N)$ on $\mathbb{K}^N$, while the other factors act trivially. Recall that all real Clifford algebras $C_{t,s}$ are isomorphic to matrix algebras of the form $m\mathbb{K}(N)$, while all complex Clifford algebas $\mathbb{C}l_n$ are of the form $m\mathbb{C}(N)$, where $m \in \{1, 2\}$. The same is true for the even Clifford algebras $Cl^0_{t,s}$ and $\mathbb{C}l^0_n$. It follows that $Cl^0_{t,s}$ has either a unique irreducible module Σ (if $m = 1$), or precisely two irreducible modules $\Sigma_1 \not\cong \Sigma_2$ (if $m = 2$). The most general $Cl^0_{t,s}$ module is of the form $S = p\Sigma$ or $S = p_1\Sigma_1 \oplus p_2\Sigma_2$, and the corresponding Schur algebra is $\mathcal{C}(S) = \mathbb{K}(p)$ or $\mathcal{C}(S) = \mathbb{K}(p_1) \oplus \mathbb{K}(p_2)$. Similar results hold for $\mathbb{C}l^0_n$.

Now we specialize the discussion to four dimensions and the case where $S = \mathbb{S}$ is the complex spinor module. We start with the complex Clifford algebra $\mathbb{C}l_4$ and its even subalgebra $\mathbb{C}l^0_4$, which are listed in Table 1.

The complex spinor module $\mathbb{S}$, which is the $\mathrm{Spin}(\mathbb{C}^4)$-module obtained by restricting an irreducible $\mathbb{C}l_4$-module, decomposes in even dimensions into two in-

Table 1. The complex Clifford algebra $\mathbb{C}l_4$ together with its even part $\mathbb{C}l_4^0$, the spinor and semi-spinor modules, $\mathbb{S}, \mathbb{S}_\pm$, and their Schur algebras $\mathcal{C}(\mathbb{S}), \mathcal{C}(\mathbb{S}_\pm)$.

Complex case	$\mathbb{C}l_4$	$\mathbb{C}l_4^0$	$\mathcal{C}_\mathbb{C}(\mathbb{S})$	$\mathcal{C}_\mathbb{C}(\mathbb{S}_\pm)$	$\mathbb{S}$	$\mathbb{S}_\pm$
	$\mathbb{C}(4)$	$2\mathbb{C}(2)$	$2\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^4$	$\mathbb{C}^2$

equivalent irreducible complex semi-spinor modules $\mathbb{S}_\pm$. The complex Schur algebra of $\mathbb{S}$ is denoted $\mathcal{C}_\mathbb{C}(\mathbb{S}) := \mathrm{End}_{\mathbb{C}l_4^0}(\mathbb{S})$.

In Table 2 we list the real Clifford algebras, spinor modules and Schur algebras for all signatures that occur in four dimensions.

Table 2. The real Clifford algebras in four dimensions, together with their even subalgebras, the Schur algebras $\mathcal{C}(\mathbb{S})$ and $\mathcal{C}(S_\mathbb{R})$ of the complex and real spinor module, and the relations between the complex and real spinor modules $\mathbb{S}, S_\mathbb{R}$ and semi-spinor modules $\mathbb{S}_\pm, S_\mathbb{R}^\pm$.

Signature	$Cl_{t,s}$	$Cl_{t,s}^0$	$\mathcal{C}_{t,s}(\mathbb{S})$	$\mathcal{C}_{t,s}(S_\mathbb{R})$	$\mathbb{S}$	$\mathbb{S}_\pm$
$(0,4),(4,0)$	$\mathbb{H}(2)$	$2\mathbb{H}$	$2\mathbb{H}$	$2\mathbb{H}$	$S_\mathbb{R}$	$S_\mathbb{R}^\pm$
$(1,3)$	$\mathbb{R}(4)$	$\mathbb{C}(2)$	$\mathbb{C}(2)$	$\mathbb{C}$	$S_\mathbb{R}\otimes\mathbb{C}$	$S_\mathbb{R}$
$(2,2)$	$\mathbb{R}(4)$	$2\mathbb{R}(2)$	$2\mathbb{R}(2)$	$2\mathbb{R}$	$S_\mathbb{R}\otimes\mathbb{C}$	$S_\mathbb{R}^\pm\otimes\mathbb{C}$
$(3,1)$	$\mathbb{H}(2)$	$\mathbb{C}(2)$	$\mathbb{C}(2)$	$\mathbb{C}(2)$	$S_\mathbb{R} = S_\mathbb{R}^\pm\otimes\mathbb{C}$	$S_\mathbb{R}^\pm$

The real spinor module $S_\mathbb{R}$ is the $\mathrm{Spin}(t,s)$-module obtained by restricting an irreducible $Cl_{t,s}$-module. $S_\mathbb{R}$ is either irreducible or decomposes into two irreducible real semi-spinor modules $S_\mathbb{R}^\pm$, which may or may not be isomorphic to one another. The decide whether $S_\mathbb{R}$ is reducible, we need to compare $Cl_{t,s}$ to $Cl_{t,s}^0$. In four dimensions we find by inspection that the only signature where real spinors are irreducible is $(1,3)$. In the remaining cases real spinors decompose into real semi-spinors, $S_\mathbb{R} = S_\mathbb{R}^+ \oplus S_\mathbb{R}^-$. The real semi-spinor modules are isomorphic if and only if the algebra $Cl_{t,s}^0$ is simple. The relation between the complex spinor module $\mathbb{S}$ and the real spinor module $S_\mathbb{R}$, and the relation between the complex semi-spinor modules $\mathbb{S}_\pm$ and real semi-spinor modules $S_\mathbb{R}^\pm$ follow by dimensional reasoning. We have also listed the Schur algebras $\mathcal{C}_{t,s}(S_\mathbb{R}) = Z_{\mathrm{GL}(S_\mathbb{R})}(\mathfrak{spin}(t,s)) = \mathrm{End}_{Cl_{t,s}^0}(S_\mathbb{R})$ and $\mathcal{C}_{t,s}(\mathbb{S}) = Z_{\mathrm{GL}(\mathbb{S})}(\mathfrak{spin}(t,s)) = \mathrm{End}_{Cl_{t,s}^0}(\mathbb{S})$ of $S_\mathbb{R}$ and $\mathbb{S}$, where the latter is considered as a real module. While the Schur algebras $\mathcal{C}_{t,s}(\mathbb{S})$ are relevant for our classification problem, the Schur algebras $\mathcal{C}_{t,s}(S_\mathbb{R})$ are included for comparison with Table 1 of [5].

Elements $a \in \mathcal{C}_{t,s}(\mathbb{S})^*$ of the Schur group act on vector-valued bilinear forms $\Pi \in (\mathrm{Sym}^2\mathbb{S}^* \otimes \mathbb{R}^{t,s})^{\mathrm{Spin}_0(t,s)}$ by the contragradient (or dual) representation

$$(a,\Pi) \mapsto \Pi' = a\cdot\Pi = \Pi(a^{-1}\cdot\,, a^{-1}\cdot)\ .$$

By considering one-parameter subgroups $a(u) = \exp(uA)$, where $A \in \mathcal{C}_{t,s}(\mathbb{S})$ is an element of the Schur algebra regarded as a Lie algebra, we obtain the corresponding infinitesimal action

$$(A, \Pi) \mapsto \Pi' = A \cdot \Pi := -\Pi(A\cdot, \cdot) - \Pi(\cdot, A\cdot) .$$

Recall that if β is an admissible bilinear form on $\mathbb{S}$, as defined in (2.2), then the corresponding admissible vector-valued bilinear form Π_β is given by (2.1). If β is an admissible bilinear form, then an endomorphism $A \in \mathrm{End}(\mathbb{S})$ is called β-admissible if the following conditions hold:

(1) Clifford multiplication either commutes or anti-commutes with A. The type of A is $\tau(A) = 1$ in the first case and $\tau(A) = -1$ in the second.
(2) A is either β-symmetric or β-skew. The β-symmetry of A is $\sigma_\beta(A) = 1$ in the first case and $\sigma_\beta(A) = -1$ in the second.
(3) If $\mathbb{S}$ is reducible, $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$, then either $A\mathbb{S}_\pm \subset \mathbb{S}_\pm$ or $A\mathbb{S}_\pm \subset \mathbb{S}_\mp$. The isotropy of A is $\iota(A) = 1$ in the first case and $\iota(A) = -1$ in the second.

For reducible $\mathbb{S}$ we can also define the isotropy $\iota(\beta)$ of a bilinear form β to be $\iota(\beta) = 1$ if $\mathbb{S}_\pm$ are mutually β-orthogonal, $\beta(\mathbb{S}_\pm, \mathbb{S}_\mp) = 0$, and to be $\iota(\beta) = -1$ if $\mathbb{S}_\pm$ are mutually β-isotropic, $\beta(\mathbb{S}_\pm, \mathbb{S}_\pm) = 0$. A non-degenerate admissible bilinear form automatically has a well defined isotropy.

It was shown in [4] that if β is admissible and if A is β-admissible, then

$$\beta_A := \beta(A\cdot, \cdot)$$

is admissible. Moreover, the space of Spin_0-invariant bilinear forms admits a basis $(\beta_{A_1}, \ldots, \beta_{A_l})$, consisting of admissible forms β_{A_i}, where $A_i \in \mathcal{C}_{t,s}(\mathbb{S})$, $i = 1, \ldots, \dim \mathcal{C}_{t,s}(\mathbb{S})$ are the elements of a basis of the Schur algebra, and where β is a non-degenerate admissible bilinear form [4]. The vector-valued bilinear form Π_{β_A} associated to the admissible bilinear form β_A is symmetric, and hence defines a Poincaré Lie superalgebra, if and only if β_A is super-admissible, $\sigma(\beta_A)\tau(\beta_A) = 1$. Note that any basis of admissible forms will split into two disjoint subsets, one consisting of super-admissible forms, the other of admissible forms with $\sigma(\beta_A)\tau(\beta_A) = -1$.

The following short calculation shows that the infinitesimal action of the Schur group on vector-valued bilinear forms can be expressed as an action on the underlying bilinear forms:

$$\begin{aligned}
\langle \Pi_\beta(As_1, s_2) + \Pi_\beta(s_1, As_2)\ , v\rangle &= \beta(\gamma_v A s_1, s_2) + \beta(\gamma_v s_1, A s_2) \\
= (\tau(A) + \sigma_\beta(A))\beta(A\gamma_v s_1, s_2) &= (\tau(A) + \sigma_\beta(A))\beta_A(\gamma_v s_1, s_2) \\
= (\tau(A) + \sigma_\beta(A))\langle \Pi_{\beta_A}(s_1, s_2), v\rangle \ .
\end{aligned}$$

Therefore:

$$-A \cdot \Pi_\beta = (\tau(A) + \sigma_\beta(A))\Pi_{\beta_A} = \begin{cases} 2\tau(A)\Pi_{\beta_A}\ , & \text{if } \tau(A)\sigma_\beta(A) = 1\ , \\ 0\ , & \text{if } \tau(A)\sigma_\beta(A) = -1\ . \end{cases}$$

This shows that a β-admissible Schur algebra element only acts non-trivially on a super-admissible form if it maps it to another super-admissible form. The β-admissible Schur algebra elements $A \in \mathcal{C}_{t,s}(\mathbb{S})$ with $\sigma_\beta(A)\tau(A) = -1$ generate the connected component of the stabilizer (or isotropy group) of Π_β,

$$\mathrm{Stab}_{\mathcal{C}_{t,s}(\mathbb{S})^*}(\Pi_\beta) = \{a \in \mathcal{C}(\mathbb{S})^* | \beta(\gamma_v a\cdot, a\cdot) = \beta(\gamma_v\cdot, \cdot)\} \subset \mathcal{C}_{t,s}(\mathbb{S})^* \ .$$

Up to conjugation the stabilizer only depends on the $\mathcal{C}_{t,s}(\mathbb{S})^*$-orbit of Π_β, and is therefore isomorphic for all superbrackets which define isomorphic Poincaré Lie superalgebras. We define the *R-symmetry group* G_R of a Poincaré Lie superalgebra with bracket Π as $G_R = \mathrm{Stab}_{\mathcal{C}_{t,s}(\mathbb{S})^*}(\Pi)$.

3.2. *Minkowski signature*

Minkowski signature can be realised either with the mostly plus convention, $(t,s) = (1,3)$ or with the mostly minus convention $(t,s) = (3,1)$. While the Clifford algebras $Cl_{1,3} \cong \mathbb{R}(4)$ and $Cl_{3,1} \cong \mathbb{H}(2)$ are distinct, the even Clifford algebras $Cl^0_{1,3} \cong \mathbb{C}(2) \cong Cl^0_{3,1}$, and hence the resulting $\mathrm{Spin}_0(1,3)$- and $\mathrm{Spin}_0(3,1)$- representations are equivalent. Since also the Schur algebras $\mathcal{C}_{1,3}(\mathbb{S}) \cong \mathbb{C}(2) \cong \mathcal{C}_{3,1}(\mathbb{S})$ are the same, the classification of Schur group orbits, and hence of Poincaré Lie superalgebras will not depend on which convention we use for the signature. For definiteness we will work in the mostly plus convention, $(t,s) = (1,3)$. Our conventions for Clifford algebras and their representations are as follows: The real Clifford algebra $Cl_{t,s}$ is represented by matrices γ^μ, $\mu = 1,\ldots t+s = n$ satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1} \ , \quad (\eta^{\mu\nu}) = \mathrm{diag}(-1,\ldots,-1,1,\ldots 1) \ .$$

This is the same convention as in [7,8], which differs from [9] by a relative sign in the defining relation of the Clifford algebra, and a relative sign in the definition of $\eta^{\mu\nu}$. The net effect is that $Cl_{t,s}$ refers to the same real associative algebra.

A convenient model of $\mathbb{S}$ for signature $(1,3)$ can be constructed by taking tensor products of real factors $\mathbb{R}^2$, using that the real Clifford algebra can be realised as a product: $Cl_{1,3} \simeq Cl_{0,2} \otimes Cl_{1,1} \simeq \mathbb{R}(2) \otimes \mathbb{R}(2) \simeq \mathbb{R}(4)$. We define:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad K = IJ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

Note that I and J are two anticommuting involutions, so that their product K is a complex structure anticommuting with I, J. Combined with the 2×2 identity matrix $\mathbb{1} = \mathbb{1}_2$ they generate the real algebra $\mathbb{R}(2)$, which can be identified with the algebra $\mathbb{H}'$ of para-quaternions, see Appendix B of [8].

Clifford generators can be realised as follows:

$$\gamma_0 = K \otimes I \ , \quad \gamma_1 = I \otimes \mathbb{1} \ , \quad \gamma_2 = J \otimes \mathbb{1} \ , \quad \gamma_3 = K \otimes K \ .$$

These generators act on the real spinor module $S_\mathbb{R} \simeq \mathbb{R}^4 \simeq \mathbb{R}^2 \otimes \mathbb{R}^2$. The corresponding $\mathfrak{spin}(1,3)$ representation is real and corresponds to Majorana spinors. We could proceed to construct a Poincaré Lie superalgebra of the form $\mathfrak{g} = \mathfrak{so}(1,3)+\mathbb{R}^{1,3}+S_{\mathbb{R}}$, which in physics terminology is the $\mathcal{N}=1$ supersymmetry algebras based on Majorana spinors, and which is the minimal supersymmetry algebra in signature $(1,3)$. But our main interest is to classify Poincaré Lie superalgebras of the from $\mathfrak{g} = \mathfrak{so}(1,3) + \mathbb{R}^{1,3} + \mathbb{S}$, that is $\mathcal{N}=2$ supersymmetry algebra where the supercharges form a Dirac spinor. We will see later that in our description the $\mathcal{N}=1$ supersymmetry algebra corresponds to a special (higher co-dimension) orbit of the Schur group. We now proceed with the $\mathcal{N}=2$ case and therefore consider two copies of the real spinor module

$$S_{\mathbb{R}} \oplus S_{\mathbb{R}} \simeq S_{\mathbb{R}} \otimes \mathbb{R}^2$$

which we identify with the complex spinor module by equipping the additional factor $\mathbb{R}^2$ with the complex structure K:

$$\mathbb{S} \simeq S_{\mathbb{R}} \otimes \mathbb{C}\ , \quad \mathbb{C} \simeq (\mathbb{R}^2, K)\ .$$

Real bilinear forms on $\mathbb{S}$ can be constructed as tensor products of bilinear forms on the three factors $\mathbb{R}^2$. On each factor $\mathbb{R}^2$ we use the following basis of bilinear forms: $g = g_0$ is the standard positive definite symmetric bilinear form, with representing matrix the identity. Then we use I, J, K to define:

$$\begin{aligned} g_1 = \eta &= g(I\cdot,\cdot) = g(\cdot, I\cdot)\ ,\\ g_2 = \eta' &= g(J\cdot,\cdot) = g(\cdot, J\cdot)\ ,\\ g_3 = \epsilon &= g(K\cdot,\cdot) = -g(\cdot, K\cdot)\ . \end{aligned}$$

The symmetric bilinear forms g_1 and g_2 have split signature, while the antisymmetric bilinear form g_3 is the Kähler form associated to the metric g_0 and complex structure K.

For later use, we list the symmetry $\sigma_\beta(A)$ of the endomorphisms $A = \mathbb{1}, I, J, K$ with respect to the bilinear forms $\beta = g_0, g_1, g_2, g_3$ in Table 3.

Table 3. The symmetry of the endomorphisms $\mathbb{1}, I, J, K$ with respect to the bilinear forms g, η, η', ϵ.

	g_0	g_1	g_2	g_3
$\mathbb{1}$	$+$	$+$	$+$	$+$
I	$+$	$+$	$-$	$-$
J	$+$	$-$	$+$	$-$
K	$-$	$+$	$+$	$-$

On $S_{\mathbb{R}} \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ the even Clifford algebra is realized as

$$Cl^0_{1,3} = Cl_{0,3} = \langle \gamma_0\gamma_\alpha | \alpha = 1,2,3\rangle_{\text{algebra}} = \langle J\otimes I\ , I\otimes I\ , \mathbb{1}\otimes J\rangle_{\text{algebra}}\ .$$

By inspection, $K \otimes J$ and $\mathbb{1} \otimes \mathbb{1}$ form a basis for operators commuting with $Cl^0_{1,3}$. Since $(K \otimes J)^2 = -1$ the Schur algebra of the real spinor module is

$$\mathcal{C}(S_{\mathbb{R}}) = \langle \mathbb{1} \otimes \mathbb{1} \, , K \otimes J \rangle_{\text{algebra}} \simeq \mathbb{C} \, .$$

The action of the above Clifford and spin generators is trivially extended, by taking the tensor product with $\mathbb{1}$ acting on the third factor, to the complex spinor module $\mathbb{S} = \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$. Therefore, as in Table 2

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) \otimes \mathbb{R}(2) \cong \mathbb{C} \otimes \mathbb{R}(2) \simeq \mathbb{C}(2) \, .$$

The simple algebra $\mathbb{C}(2)$ contains both the quaternions $\mathbb{H}$ and the para-quaternions (aka split-quaternions) $\mathbb{H}' \simeq \mathbb{R}(2)$ as subalgebras, due to the following isomorphisms of real algebras:

$$\mathbb{C} \otimes \mathbb{H}' \simeq \mathbb{C}(2) \simeq \mathbb{C} \otimes \mathbb{H} \, .$$

A subalgebra of $\mathbb{C}(2)$ isomorphic to $\mathbb{H}'$ is

$$\langle 1 \otimes 1 \otimes I \, , \quad 1 \otimes 1 \otimes J \, , \quad 1 \otimes 1 \otimes K \rangle_{\text{algebra}} \, ,$$

and a subalgebra isomorphic to $\mathbb{H}$ is

$$\langle K \otimes J \otimes I \, , \quad K \otimes J \otimes J \, , \quad 1 \otimes 1 \otimes K \rangle_{\text{algebra}} \, .$$

These subalgebras do not commute, and they intersect in the subalgebra $\langle \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \mathbb{1} \otimes \mathbb{1} \otimes K \rangle \cong \mathbb{C}$.

We introduce

$$\gamma_* := i\gamma_0\gamma_1\gamma_2\gamma_3 \, , \quad \gamma_*^2 = 1 \, ,$$

which is, up to sign, the real volume element of $Cl_{1,3}$. In our model

$$\gamma_* = -K \otimes J \otimes K \, ,$$

where the last factor corresponds to multiplication by 'i' with our choice of complex structure on $\mathbb{S}$. The eigenspaces of γ_* are the complex semi-spinor modules $\mathbb{S}_\pm$, whose elements are the Weyl spinors.

To determine the super-admissible bilinear forms on $\mathbb{S} \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, we start by identifying those bilinear forms on $S_{\mathbb{R}}$ which have a definite type. Out of the sixteen basic forms, only the two listed in Table 4 qualify. Since the Spin group

Table 4. A basis for the admissible bilinear forms on $S_{\mathbb{R}}$, listing for each basis element its symmetry σ and type τ.

	σ	τ
$g_0 \otimes g_3$	$-$	$+$
$g_3 \otimes g_1$	$-$	$-$

does not act on the third factor $\mathbb{R}^2$, we obtain super-admissible forms by combining

$g_0 \otimes g_3$ with an antisymmetric form on the third factor and by combining $g_3 \otimes g_1$ with a symmetric form. This results in a basis of four super-admissible forms on $\mathbb{S}$, which are listed with their symmetry, type and isotropy in Table 5.

Table 5. A basis for the super-admissible real bilinear forms on $\mathbb{S}$, listing for each basis element its symmetry σ, type τ and isotropy ι.

β_i	σ	τ	ι
$\beta_0 := g_3 \otimes g_1 \otimes g_0$	$-$	$-$	$-$
$\beta_1 := g_3 \otimes g_1 \otimes g_1$	$-$	$-$	$+$
$\beta_2 := g_3 \otimes g_1 \otimes g_2$	$-$	$-$	$+$
$\beta_3 := g_0 \otimes g_3 \otimes g_3$	$+$	$+$	$-$

Now we can describe the action of the Schur algebra on the space of superbrackets explicitly. Since we know that Schur algebra elements A with $\tau(A)\sigma_{\beta_i}(A) = -1$ act trivially on β_i, we determine the type $\tau(A)$ and β_i-symmetry $\sigma_{\beta_i}(A)$ for the generators of the Schur algebra $\mathcal{C}(\mathbb{S})$ and list the results in Table 6.

Table 6. The type $\tau(A)$ and β_i-symmetry $\sigma_{\beta_i}(A)$ of the basis elements A of the Schur algebra $\mathcal{C}(\mathbb{S})$.

A	$\tau(A)$	$\sigma_{\beta_0}(A)$	$\sigma_{\beta_1}(A)$	$\sigma_{\beta_2}(A)$	$\sigma_{\beta_3}(A)$
$\mathrm{Id} = 1 \otimes 1 \otimes 1$	$+$	$+$	$+$	$+$	$+$
$E_1 := 1 \otimes 1 \otimes I$	$+$	$+$	$+$	$-$	$-$
$E_2 := 1 \otimes 1 \otimes J$	$+$	$+$	$-$	$+$	$-$
$E_3 := 1 \otimes 1 \otimes K$	$+$	$-$	$+$	$+$	$-$
$\mathcal{I} := K \otimes J \otimes 1$	$-$	$+$	$+$	$+$	$+$
$\mathcal{I}E_1 = K \otimes J \otimes I$	$-$	$+$	$+$	$-$	$-$
$\mathcal{I}E_2 = K \otimes J \otimes J$	$-$	$+$	$-$	$+$	$-$
$\mathcal{I}E_3 = K \otimes J \otimes K$	$-$	$-$	$+$	$+$	$-$

In Table 6 we have introduced the following notation for the Schur algebra generators. Id is the identity, and $\mathcal{I}$ a complex structure, $\mathcal{I}^2 = -\mathrm{Id}$. The endomorphisms E_a, $a = 1, 2, 3$ generate a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ while $(E_a, \mathcal{I}E_a)$ are a real basis for a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{R}) + i\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{gl}(2, \mathbb{C}) = \mathcal{C}(\mathbb{S})$.

From the Table 6 we obtain Table 7 that shows which Schur algebra generators act trivially, and which act non-trivially on the forms Π_{β_i}. The element Id generates

Table 7. Inserting the endomorphism A into one argument of a super-admissible form β_i creates a new superbracket $\Pi_{\beta_i A}$ if $\tau(A)\sigma_{\beta_i}(A) = +1$ and leaves the superbracket Π_{β_i} invariant if $\tau(A)\sigma_{\beta_i}(A) = -1$.

A	$\tau(A)\ \sigma_{\beta_0}(A)$	$\tau(A)\sigma_{\beta_1}(A)$	$\tau(A)\sigma_{\beta_2}(A)$	$\tau(A)\sigma_{\beta_3}(A)$
$\mathrm{Id} = 1 \otimes 1 \otimes 1$	$+$	$+$	$+$	$+$
$E_1 = 1 \otimes 1 \otimes I$	$+$	$+$	$-$	$-$
$E_2 = 1 \otimes 1 \otimes J$	$+$	$-$	$+$	$-$
$E_3 = 1 \otimes 1 \otimes K$	$-$	$+$	$+$	$-$
$\mathcal{I} = K \otimes J \otimes 1$	$-$	$-$	$-$	$-$
$\mathcal{I}E_1 = K \otimes J \otimes I$	$-$	$-$	$+$	$+$
$\mathcal{I}E_2 = K \otimes J \otimes J$	$-$	$+$	$-$	$+$
$\mathcal{I}E_3 = K \otimes J \otimes K$	$+$	$-$	$-$	$+$

a subgroup $\mathbb{R}^{>0}$ of the Schur group which acts by re-scalings. The element $\mathcal{I} = K \otimes J \otimes 1$ stabilizes all four super-admissible forms, which implies that the lower half of the table is obtained from the upper half by flipping signs. Together Id and $\mathcal{I}$ generate the center $\mathbb{C}^*$ of the Schur group $\mathcal{C}(\mathbb{S})^* = \mathrm{GL}(2, \mathbb{C})$.

Let us first study the action of the subgroup $\mathrm{SL}(2, \mathbb{C})$ which is the universal cover of the connected Lorentz group $\mathrm{SO}(1, 3)_0$. This has two (real-) inequivalent four-dimensional representations, the vector representation and the (Weyl) spinor representation. The latter has only one open orbit. To show that we have at least two open orbits, we compute the stabilizer groups of the forms Π_{β_i}, by reading off from the above tables which endomorphisms act trivially, see Table 8.

Table 8. The stabilizer Lie algebras of the four basic superbrackets.

Π_{β_i}	Stabilizer
Π_{β_0}	$\langle E_3, \mathcal{I}, \mathcal{I}E_1, \mathcal{I}E_2 \rangle \cong \mathfrak{u}(1) \oplus \mathfrak{su}(2)$
Π_{β_1}	$\langle E_2, \mathcal{I}, \mathcal{I}E_1, \mathcal{I}E_3 \rangle \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(1,1)$
Π_{β_2}	$\langle E_1, \mathcal{I}, \mathcal{I}E_2, \mathcal{I}E_3 \rangle \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(1,1)$
Π_{β_3}	$\langle E_1, E_2, E_3, \mathcal{I} \rangle \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(1,1)$

Since Π_{β_0} has a compact stabilizer, while Π_{β_a}, $a = 1, 2, 3$ have non-compact stabilizers, we have at least two open orbits, which implies that $\mathrm{SL}(2, \mathbb{C})$ operates in the vector representation.[a] In fact, the non-abelian factors are precisely the

[a]It is straightforward to work out the explicit matrix representation, which is indeed the vector

stabilizers $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ and $\mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1)$ of time-like and space-like vectors under the action of the Lorentz group. It follows that there are at least two non-isomorphic $\mathcal{N} = 2$ superalgebras with R-symmetry groups which are isomorphic to $\mathrm{U}(2) = \mathrm{U}(1) \cdot \mathrm{SU}(2)$ and $\mathrm{U}(1,1) = \mathrm{U}(1) \cdot \mathrm{SU}(1,1)$.

Since $\mathrm{U}(1) \subset \mathcal{C}(\mathbb{S})^*$ acts trivially, we see that $\mathcal{C}(\mathbb{S})^*$ acts as the linear conformal pseudo-orthogonal group $\mathrm{CSO}_0(1,3) := \mathbb{R}^{>0} \times \mathrm{SO}_0(1,3)$ on the space of superbrackets, which we can identify with four-dimensional Minkowski space $\mathbb{R}^{1,3}$ by choosing the spin-invariant scalar product for which the Π_{β_i} form an orthonormal basis. The Schur group $\mathcal{C}(\mathbb{S})^*$ acts with six orbits: the three open orbits of time-like future-directed, time-like past-directed and space-like vectors, the two three-dimensional orbits of non-zero null future or past-directed vectors, and the origin. Since the superbrackets Π_β and $\Pi_{-\beta}$ define isomorphic Poincaré Lie superalgebras, there are only four non-isomorphic Poincaré Lie superalgebra structures, distinguished by the isomorphism type of their stabilizers in the Schur group:

(1) The time-like orbits of $\Pi_{\pm\beta_0}$ define isomorphic supersymmetry algebras with non-degenerate superbrackets and R-symmetry group $\mathrm{U}(2)$. This is the standard $\mathcal{N} = 2$ superalgebra.
(2) The space-like orbit, which contains Π_{β_a}, $a = 1, 2, 3$, defines a supersymmetry algebra with non-degenerate superbracket and R-symmetry group $\mathrm{U}(1,1)$. This is a non-standard 'twisted' $\mathcal{N} = 2$ supersymmetry algebra similar to the twisted supersymmetry algebra of type-II* string theories described in [1].
(3) The orbits generated by null vectors correspond to isomorphic supersymmetry algebras with partially degenerate superbrackets. Without loss of generality, we can consider the bracket $\Pi_{\frac{1}{2}(\beta_0+\beta_1)}$. We note that $\frac{1}{2}(\beta_0+\beta_1) = \beta_0(\frac{1}{2}(1+E_1)\cdot,\cdot)$. Since $E_1^2 = \mathbb{1}$, $\Pi_\pm^{E_1} := \frac{1}{2}(1 \pm E_1)$ are projection operators onto the eigenspaces of E_1 with eigenvalues ± 1. The bilinear form $\Pi_{\frac{1}{2}(\beta_0+\beta_1)}$ has the four-dimensional kernel $\Pi_-^{E_1}\mathbb{S}$ and by restriction defines a Poincaré Lie superalgebra with spinor module $S_\mathbb{R} = \Pi_+^{E_1}\mathbb{S}$. The isotropy group of this bracket in the Schur group $\mathcal{C}^*(S_\mathbb{R}) = \mathbb{C}^*$ is the $\mathrm{U}(1)$ generated by $\mathcal{I}E_1$. Since in our classification there is no other non-trivial supersymmetry bracket with a non-trivial kernel, this supersymmetry algebra must be the standard $\mathcal{N} = 1$ supersymmetry algebra.
(4) The zero vector defines a completely degenerate superbracket corresponding to the trivial supersymmetry algebra.

3.3. *Neutral signature*

In signature $(2,2)$ the real Clifford algebra is $Cl_{2,2} \cong \mathbb{R}(4)$, and the real spinor module is $S_\mathbb{R} = \mathbb{R}^4$, which will allow us to use a real model similar to signature $(1,3)$. Since the even real Clifford algebra is $2\mathbb{R}(2)$, real spinors decompose into inequivalent real semi-spinors, $S_\mathbb{R} = S_\mathbb{R}^+ + S_\mathbb{R}^-$, $S_\mathbb{R}^+ \ncong S_\mathbb{R}^-$. The real and complex

representation of $\mathrm{SO}_0(1,3)$. In the following we will not need an explicit matrix representation.

spinor and semi-spinor modules are related by $\mathbb{S} = S_{\mathbb{R}} \otimes \mathbb{C}$ and $\mathbb{S}_{\pm} = S^{\pm}_{\mathbb{R}} \otimes \mathbb{C}$. The Schur algebras are

$$\mathcal{C}(S^{\pm}_{\mathbb{R}}) = \mathbb{R}\ , \quad \mathcal{C}(\mathbb{S}_{\pm}) = \mathbb{R}(2) = \mathbb{H}'\ , \quad \mathcal{C}(S_{\mathbb{R}}) = 2\mathbb{R}\ , \quad \mathcal{C}(\mathbb{S}) = 2\mathbb{R}(2) = 2\mathbb{H}'\ .$$

We used that $\mathbb{R}(2) = \mathbb{H}'$ are the para-quaternions, to emphasize that $\mathbb{S}$ carries two invariant real structures (which preserve chirality). The complex semi-spinor modules are the complexifications of the real semi-spinor modules, hence of real type, and self-conjugate as complex $Cl^0_{2,2}$ modules.

In physics terminology, elements of $S_{\mathbb{R}}$, $\mathbb{S}_{\pm}$ and $S^{\pm}_{\mathbb{R}}$ are Majorana spinors, Weyl spinors and Majorana-Weyl spinors respectively. Due to the absence of invariant quaternionic structures on $\mathbb{S}$, we cannot define symplectic Majorana spinors. The Majorana condition allows one to define an $\mathcal{N} = 1$ superalgebra, which we will recover when classifying the orbits of the Schur group. The existence of Majorana-Weyl spinors is consistent with the existence of an even smaller '$\mathcal{N} = 1/2$' superalgebra, which would be chiral in the sense of only involving superbrackets between supercharges of the same chirality. We will be able to decide later whether such a supersymmetry algebra exists.

As in signature $(1, 3)$ we take $S \cong \mathbb{R}^2 \otimes \mathbb{R}^2$ and $\mathbb{S} \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$. On $\mathbb{R}(2)$ we choose the following basis:[b]

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ , \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ , \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad K = IJ\ ,$$

where now I is a complex structure on $\mathbb{R}^2$, while J, K are involutions. Since I, J, K anti-commute they satisfy the para-quaternionic algebra, making manifest that $\mathbb{R}(2) \simeq \mathbb{H}'$ as associative algebras, where $\mathbb{H}'$ is the algebra of para-quaternions.

On $\mathbb{R}^2$ we choose the following basis of bilinear forms: $g_0 = g, g_1 = gI, g_2 = gJ, g_3 = gK$, where g is the standard symmetric positive definite bilinear form, and where $gI = g(I\cdot, \cdot)$, etc. The symmetry of these basic bilinear forms is listed in Table 9 together with the g_i-symmetry of the basic endomorphisms.

It is straightforward to verify that

$$\gamma_1 = J \otimes I\ , \quad \gamma_2 = K \otimes I\ , \quad \gamma_3 = 1 \otimes J\ , \quad \gamma_4 = 1 \otimes K$$

are generators of $Cl_{2,2}$ acting on $S = \mathbb{R}^2 \otimes \mathbb{R}^2$,

The resulting generators of $\mathfrak{spin}(2, 2)$ are

$$\gamma_1\gamma_2 = -I \otimes \mathbb{1}\ , \ \ \gamma_1\gamma_3 = J \otimes K\ , \ \ \gamma_1\gamma_4 = -J \otimes J\ ,$$
$$\gamma_2\gamma_3 = K \otimes K\ , \ \ \gamma_2\gamma_4 = -K \otimes J\ , \ \ \gamma_3\gamma_4 = -\mathbb{1} \otimes I\ .$$

By inspection, the only endomorphisms commuting with the spin generators are linear combinations of $1 \otimes 1$ and $I \otimes I$. The Schur algebra of the real spinor module

[b]Note that this basis is different from the one we used for signature $(1, 3)$ in Section 3.2.

Table 9. The symmetry of the four basic bilinear forms g_i, and the g_i-symmetry of the endomorphisms I, J, K.

	$\sigma(g_i)$	$\sigma_{g_i}(I)$	$\sigma_{g_i}(J)$	$\sigma_{g_i}(K)$
g_0	$+$	$-$	$+$	$+$
g_1	$-$	$-$	$-$	$-$
g_2	$+$	$+$	$+$	$-$
g_3	$+$	$+$	$-$	$+$

is

$$\mathcal{C}(S_{\mathbb{R}}) = \mathrm{End}_{Cl^0_{2,2}}(S_{\mathbb{R}}) = \langle 1 \otimes 1 \,, I \otimes I \rangle \cong \mathbb{R} \oplus \mathbb{R} \,.$$

Likewise by inspection, only two out of the sixteen bilinear forms $g_i \otimes g_j$, $i, j = 0, 1, 2, 3$ are admissible, namely those listed in Table 10.

Table 10. List of admissible forms on $S_{\mathbb{R}}$.

	σ	τ
$g_0 \otimes g_1$	$-$	$-$
$g_1 \otimes g_0$	$-$	$+$

We can realize the complex spinor module as $\mathbb{S} \cong S_{\mathbb{R}} \otimes \mathbb{R}^2 \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, where the complex structure of $\mathbb{S}$ is defined by $\mathbb{1} \otimes \mathbb{1} \otimes I$. The Clifford generators are extended trivially as $\gamma_\mu \otimes \mathbb{1}$. For notational simplicity we will write γ_μ instead of $\gamma_\mu \otimes \mathbb{1}$ in the following. Since the Clifford algebra does not act on the third factor $\mathbb{R}^2$, we obtain eight admissible bilinear forms on $\mathbb{S}$ by tensoring the two admissible forms on $S_{\mathbb{R}}$ with the four basic bilinear forms. Out of these, the four forms listed in Table 11 are super-admissible.

Table 11. List of super-admissible bilinear forms on $\mathbb{S}$.

	σ	τ
$\beta_1 = g_0 \otimes g_1 \otimes g_0$	$-$	$-$
$\beta_2 = g_0 \otimes g_1 \otimes g_2$	$-$	$-$
$\beta_3 = g_0 \otimes g_1 \otimes g_3$	$-$	$-$
$\beta_4 = g_1 \otimes g_0 \otimes g_1$	$+$	$+$

Generators of the Schur algebra $\mathcal{C}(\mathbb{S})$ are obtained by tensoring the two generators of $\mathcal{C}(S_{\mathbb{R}})$ with the four basic endomorphisms acting on the third factor $\mathbb{R}^2$. In other words we have the following direct decomposition of vector spaces:

$$\mathcal{C}(\mathbb{S}) = (1 \otimes 1 \otimes \mathbb{H}') \oplus (I \otimes I \otimes \mathbb{H}') ,$$

where $\mathbb{H}' = \langle 1, I, J, K \rangle$. To obtain a decomposition $\mathcal{C}(\mathbb{S}) = \mathcal{C}(\mathbb{S})_+ \oplus \mathcal{C}(\mathbb{S})_- \cong \mathbb{H}' \oplus \mathbb{H}'$ as an algebra it suffices to apply the projectors

$$P_\pm = \frac{1}{2} (1 \otimes 1 \otimes 1 \pm I \otimes I \otimes 1) \ .$$

The two $\mathbb{H}'$ factors $\mathcal{C}(\mathbb{S})_\pm$ are spanned by the operators

$$\begin{aligned} 1_\pm &= P_\pm(1 \otimes 1 \otimes 1) \ , \ I_\pm = P_\pm(1 \otimes 1 \otimes I) \ , \\ J_\pm &= P_\pm(1 \otimes 1 \otimes J) \ , \ K_\pm = P_\pm(1 \otimes 1 \otimes K) \ . \end{aligned}$$

We choose the basis $v_i = \Pi_{\beta_i}$, $i = 1, 2, 3, 4$ in the space of superbrackets. The infinitesimal action of the generators of the Schur algebra on superbrackets is summarized in table 12. It preserves the scalar product on the space of vector-valued bilinear forms for which the basis $(v_1, \ldots, v_4)$ is orthonormal, with v_1, v_4 time-like and v_2, v_3 space-like.

Table 12. Action of the generators of the Schur algebra on the basis of the space of superbrackets.

Generator	Action
$\mathrm{Id} = 1 \otimes 1 \otimes 1$	scaling
$1 \otimes 1 \otimes I$	rotation $2R_{23}$
$1 \otimes 1 \otimes J$	boost $-2B_{12}$
$1 \otimes 1 \otimes K$	boost $-2B_{13}$
$\gamma_1\gamma_2\gamma_3\gamma_4 = I \otimes I \otimes 1$	trivial
$I \otimes I \otimes I$	rotation $-2R_{14}$
$I \otimes I \otimes J$	boost $-2B_{34}$
$I \otimes I \otimes K$	boost $-2B_{24}$

In table 12 R_{ij} denotes the rotation by 90 degrees in the plane spanned by v_i, v_j, and B_{ij} the boost $v_i \mapsto v_j, v_j \mapsto v_i$. To determine the action of the full, non-connected Schur group

$$\mathcal{C}(\mathbb{S})^* = \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R}) = (\mathbb{R}^{>0} \times \mathrm{SL}^\pm(2, \mathbb{R})) \times (\mathbb{R}^{>0} \times \mathrm{SL}^\pm(2, \mathbb{R}))$$

where $\mathrm{SL}^\pm(2, \mathbb{R})$ is the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of matrices A with $|\det(A)| = 1$, it suffices to determine the action of the two group elements $P_- + J_+$ and $P_+ + J_-$ on the four-dimensional space of Lie superbrackets. In fact these two elements generate a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the Schur group which acts simply transitively on the four components of the Schur group. A straightforward calculation shows that $P_- + J_+$ interchanges v_1 and v_2 as well as v_3 and $-v_4$. Similarly $P_+ + J_-$ interchanges v_1 and v_2 as well as v_3 and v_4. This implies that the image of the Schur group under the representation on the four-dimensional space of superbrackets is precisely $\mathrm{CO}_0(2,2) \cup \xi \mathrm{CO}_0(2,2)$, where $\xi = P_+ + J_-$ is the involution which maps v_1 to v_2 and v_3 to v_4, and

$$\mathrm{CO}_0(2,2) = \mathbb{R}^{>0} \times \mathrm{SO}_0(2,2) = \mathbb{R}^{>0} \times \mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$$

is the connected component of the identity of the conformal linear group. Note that ξ is an anti-isometry and therefore interchanges space-like and time-like vectors.

The action of the connected group $\mathrm{CO}_0(2,2)$ has four orbits: the two open orbits of time-like and space-like vectors separated by the lightcone, the three-dimensional orbit of non-zero null vectors, and the origin. The two open orbits cannot be distinguished by the isomorphism type of their stabilizers, which are $\mathrm{CO}_0(2,1) \cong \mathrm{CO}_0(1,2) = \mathbb{R}^{>0} \times \mathrm{SO}_0(1,2)$. Under the full Schur group there are only three orbits since the orbits of time-like and space-like vectors are mapped to each other by ξ. The open orbit of the full Schur group corresponds to a unique $\mathcal{N} = 2$ supersymmetry algebra in signature $(2,2)$. The connected R-symmetry group is $\mathbb{R}^{>0} \times \mathrm{Spin}_0(1,2) \cong \mathbb{R}^{>0} \times \mathrm{SL}(2,\mathbb{R})$. Note that $\mathrm{Spin}_0(1,2) \subset \mathrm{Spin}_0(2,2) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is a diagonally embedded $\mathrm{SL}(2,\mathbb{R})$-subgroup of the maximally connected Schur group $\mathcal{C}(\mathbb{S})_0^* = \mathrm{GL}^+(2,\mathbb{R}) \times \mathrm{GL}^+(2,\mathbb{R})$.

Consider next the orbit of non-zero null vectors. Without restriction of generality, consider the bilinear form

$$\frac{1}{2}(\beta_1 + \beta_2) = \frac{1}{2}(g_0 \otimes g_1 \otimes (g_0 + g_2)) = g_0 \otimes g_1 \otimes g_0 \left(\frac{1}{2}(\mathbb{1} + J)\cdot, \cdot \right) .$$

Since $J^2 = 1$, the operators $\Pi^J_\pm = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \otimes (\mathbb{1} + J))$ are projection operators onto the eigenspaces $\Pi^J_\pm \mathbb{S}$ of $\mathbb{1} \otimes \mathbb{1} \otimes J$ with eigenvalues ± 1. Since $\mathbb{1} \otimes \mathbb{1} \otimes J$ commutes with the Clifford generators, the vector-valued bilinear form $\Pi_{\frac{1}{2}(\beta_1+\beta_2)}$ has a four-dimensional kernel $\Pi^J_- \mathbb{S}$ and defines a non-trivial Poincaré Lie superalgebra with spinor module $\mathbb{S}_\mathbb{R} = \Pi^J_+ \mathbb{S}$. Therefore there is a unique $\mathcal{N} = 1$ supersymmetry algebra in signature $(2,2)$. Its connected R-symmetry group, that is the stabilizer of $\Pi_{\frac{1}{2}(\beta_1+\beta_2)}$ in the identity component of the Schur group $\mathcal{C}(S_\mathbb{R})^*$, is the group $\mathrm{SO}_0(1,1)$ generated by $I \otimes I \otimes J$.

The volume element ω of the Clifford algebra is[c]

$$\omega = -\gamma_* = \gamma_1\gamma_2\gamma_3\gamma_4 = I \otimes I \ .$$

All four super-admissible bilinear forms β_i have isotropy $\iota(\beta_i) = 1$, that is $\beta_i(\mathbb{S}_\pm, \mathbb{S}_\mp) = 0$. Since ω anti-commutes with the Clifford generators, the corresponding vector valued bilinear forms are isotropic, $\Pi_{\beta_i}(\mathbb{S}_\pm, \mathbb{S}_\pm) = 0$. This implies that one cannot define a non-trivial '$\mathcal{N} = \frac{1}{2}$' supersymmetry algebra where the

[c]The definition of γ_* includes a minus sign, which is needed for consistency with our conventions in the companion paper [6].

independent supercharges form a single Majorana-Weyl spinor. This also follows from our classification of orbits. The absence of chiral supersymmetry algebras in four dimensions reflects a general principle. As follows from the commutation relations between the charge conjugation matrix C and the chirality matrix γ_* [10], super-brackets are isotropic in dimensions $d = 4n$, $n = 1, 2, 3, \ldots$, and orthogonal ($\Pi_{\beta_i}(\mathbb{S}_\pm, \mathbb{S}_\mp) = 0$) in dimensions $d = 4n + 2$, irrespective of signature. This implies that chiral supersymmetry algebras can only exist in dimensions $d = 2, 6, 10, \ldots$. A chiral supersymmetry algebra requires in addition that $S^+_{\mathbb{R}} \not\cong S^-_{\mathbb{R}}$, which is a signature dependent condition. Signatures with chiral supersymmetry algebras have a larger number of inequivalent supersymmetry algebras and associated physical theories than other signatures.

3.4. *Euclidean signature*

In signature $(0, 4)$ the real Clifford algebra is $Cl_{0,4} = \mathbb{H}(2)$ and the real spinor module is $S_{\mathbb{R}} = \mathbb{H}^2 \cong \mathbb{C}^4$. This shows that $S_{\mathbb{R}}$ carries a quaternionic, and therefore a complex structure, and is equal to the complex spinor module, $S_{\mathbb{R}} = \mathbb{S}$. Since the even Clifford algebra is $Cl^0_{0,4} = 2\mathbb{H}$, the real spinor module decomposes into two inequivalent real semi-spinor modules, $S_{\mathbb{R}} = S^+_{\mathbb{R}} + S^-_{\mathbb{R}}$, $S^+_{\mathbb{R}} \not\cong S^-_{\mathbb{R}}$, which coincide with the complex semi-spinor modules, $S^\pm_{\mathbb{R}} = \mathbb{S}_\pm$. The semi-spinor modules carry a quaternionic structure, and therefore are self-conjugate as complex modules, $\overline{\mathbb{S}}_\pm \cong \mathbb{S}_\pm$. The complex spinor module is also self-conjugate, $\overline{\mathbb{S}} \cong \mathbb{S}$. Since the semi-spinor modules are not equivalent the Schur algebra of $\mathbb{S} = S_{\mathbb{R}}$ is

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) = 2\mathbb{H} \; .$$

Due to the absence of an invariant real structure, there are no Majorana spinors. The existence of an invariant quaternionic structure allows us to rewrite a Dirac spinor as a pair of symplectic Majorana spinors, and since the quaternionic structure preserves chirality (maps $\mathbb{S}_\pm$ to $\mathbb{S}_\pm$), Weyl spinors can be rewritten as pairs of symplectic Majorana-Weyl spinors. Since $Cl_{0,4} \cong Cl_{4,0}$, we do not need to consider signature $(4, 0)$ explicitly.

Since $Cl_{0,4}$ is a quaternionic matrix algebra, we will use a different type of model than for the other signatures. We define the following operators on $\mathbb{H}^2$:

$$(I_a)_{a=0,1,2,3} = (\mathrm{Id}, R_i, R_j, R_k) \; , \quad (I'_a)_{a=0,1,2,3} = (\mathrm{Id}, L_i, L_j, L_k) \; ,$$

where R_q, L_q, with $q \in \mathbb{H}$ denotes right and left multiplication by quaternions, respectively. We also introduce the following matrix operators which act on $\mathbb{H}^2$ from the left:

$$D = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \; , \quad E = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \; .$$

We note that I_a, I'_a span quaternionic algebras which commute with each other and with D, E. The operators D and E are two anti-commuting involutions,

$$D^2 = \mathrm{Id} \; , \quad E^2 = \mathrm{Id} \quad \text{and} \quad \{D, E\} = 0 \; ,$$

and therefore their product is a complex structure, $(DE)^2 = -\mathrm{Id}$, which anticommutes with D and E. Hence they generate an algebra isomorphic to the para-quaternionic algebra $\mathbb{H}' \cong \mathbb{R}(2)$.

It is straightforward to verify that

$$\gamma^\alpha = IDI'_\alpha \ , \alpha = 1,2,3 \ , \quad \gamma^4 = IDE \ ,$$

where $I = I_1$, satisfy the relations of generators for $Cl_{0,4}$. The generators

$$\gamma^1\gamma^2 = -L_k \ , \gamma^1\gamma^3 = L_j \ , \gamma^1\gamma^4 = -L_iE \ , \gamma^2\gamma^3 = -L_i \ ,$$
$$\gamma^2\gamma^4 = -L_jE \ , \gamma^3\gamma^4 = -L_kE$$

of $\mathfrak{spin}(4)$ act diagonally on $\mathbb{H}^2$. We also note that the $Cl_{0,4}$ volume element

$$\gamma_* = \gamma^1\gamma^2\gamma^3\gamma^4 = -E \tag{3.1}$$

is proportional to the identity on the factors of $\mathbb{S} = S_{\mathbb{R}} = \mathbb{H}+\mathbb{H}$, which are therefore the semi-spinor modules $\mathbb{S}^\pm_{\mathbb{R}} = S^\pm_{\mathbb{R}} = \mathbb{H}$.

We remark that by adding $\gamma^0 = IE$, we obtain a set of generators for the five-dimensional Clifford algebra $Cl_{1,4}$, which is associated to a theory in signature $(1,4)$. By dimensional reduction over time one can then obtain a theory in signature $(0,4)$ [7]. The model used in this paper differs from the one used in [7] by exchanging D and E. The representation used in the present paper is a 'Weyl' representation where the volume element acts diagonally on $S_{\mathbb{R}} = S^+_{\mathbb{R}} + S^-_{\mathbb{R}}$.

We now turn to the construction of admissible bilinear forms. On $S_{\mathbb{R}} = \mathbb{H}+\mathbb{H}$ we obtain a non-degenerate $\mathfrak{spin}(4)$-invariant positive definite scalar product g by taking the direct sum of the standard scalar products on the factors. The group $\mathrm{Spin}_0(4) \cong \mathrm{SU}(2)\times\mathrm{SU}(2)$ acts isometrically on $\mathbb{H}^2$ by left multiplication, while the Schur algebra

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) = \langle I_a, I_aE | a = 0,1,2,3\rangle \cong 2\mathbb{H}$$

acts by multiplication from the right. On each factor $\mathbb{S}_\pm \cong \mathbb{H}$, L_q and R_q with $q = i,j,k$ are isometries of the standard scalar product, and therefore leave the scalar product g on $\mathbb{H}^2$ invariant. Since $L_q^2 = -1$, these operators are g-skew. D and E are isometries of g, but since they are involutions, they are g-symmetric. The Clifford generators act isometrically with respect to g, and since they are involutions, $(\gamma^\alpha)^2 = \mathrm{Id} = (\gamma^4)^2$, they are g-symmetric. Hence g is super-admissible: $\sigma_g = \tau_g = 1$. To obtain a basis of admissible forms for the space of $\mathrm{Spin}(4)$-invariant real bilinear forms, we take $g_A := g(A\cdot,\cdot)$, where A runs over a basis of the Schur algebra which consists of admissible endomorphisms. To show that we can choose $\{I_a, I_aE | a = 0,1,2,3\}$ as such a basis, we compute the g-symmetry and type of these endomorphisms. Obviously the complex structures I_α are g-skew, wheras D and E are g-symmetric. Since I_α and E commute, $\sigma_g(I_\alpha E) = -1$. With regard to the type we note that $I = I_1$ commutes with $\gamma^\alpha = IDI'_\alpha$ and $\gamma^4 = IDE$, while $I_{2,3}$ anticommute: $\tau(I_1) = 1$, $\tau(I_{2,3}) = -1$. Since E anticommutes with D it

Table 13. The g-symmetry and type of the generators of the Schur algebra, where g is the standard positive definite bilinear form. If $\sigma_g(A)\tau(A) = 1$, then $g_A = g(A\cdot, \cdot)$ is super-admissible and defines a superbracket.

A	I_0	I_1	$I_{2,3}$	E	EI_1	$EI_{2,3}$
$\sigma_g(A)$	$+$	$-$	$-$	$+$	$-$	$-$
$\tau(A)$	$+$	$+$	$-$	$-$	$-$	$+$
$\sigma_g(A)\tau(A)$	$+$	$-$	$+$	$-$	$+$	$-$

anticommutes with γ^α and γ^4: $\tau(E) = -1$, and $\tau(I_1E) = -1$, $\tau(I_{2,3}E) = 1$. See Table 13 for a summary.

Using that with $\sigma_g = \tau_g = 1$ we have $\sigma(g_A) = \sigma_g\sigma_g(A) = \sigma_g(A)$ and $\tau(g_A) = \tau_g\tau(A) = \tau(A)$, it follows from the table that all eight forms are admissible, and that four of them, namely

$$\{\beta_i | i = 1, 2, 3, 4\} = \{g\ , g(I_2\cdot, \cdot)\ , g(I_3\cdot, \cdot)\ , g(EI_1\cdot, \cdot)\}$$

are super-admissible. Therefore Π_{β_i} form a basis for the space of symmetric Spin(4)-equivariant bilinear forms on $\mathbb{S}$ with values in the vector representation, and therefore for the space of Poincaŕe Lie superalgebra structures. To make explicit the action of the Schur algebra on this space, we need the symmetry of all eight Schur generators with respect to the four super-admissible forms. This follows from the previous data upon using that

$$\sigma_{g_B}(A) = \begin{cases} +\sigma_g(A) \text{ if } [A, B] = 0\ , \\ -\sigma_g(A) \text{ if } \{A, B\} = 0\ . \end{cases}$$

The relevant information has been collected in table 14.

Table 14. This table lists, for all Schur algebra generators, their type and their symmetry with respect to the super-admissible forms.

A	$\tau(A)$	$\sigma_g(A)$	$\sigma_{g_{I_2}}(A)$	$\sigma_{g_{I_3}}(A)$	$\sigma_{g_{EI_1}}(A)$
I_0	$+$	$+$	$+$	$+$	$+$
I_1	$+$	$-$	$+$	$+$	$-$
I_2	$-$	$-$	$-$	$+$	$+$
I_3	$-$	$-$	$+$	$-$	$+$
E	$-$	$+$	$+$	$+$	$+$
EI_1	$-$	$-$	$+$	$+$	$-$
EI_2	$+$	$-$	$-$	$+$	$+$
EI_3	$+$	$-$	$+$	$-$	$+$

Table 15. Entries + in this table indicate that the Schur algebra generator A displayed in the first column acts non-trivially on the bilinear form g_B indicated by the first row. Entries $-$ indicate that A leaves the corresponding bilinear form g_B invariant; such A generate the R-symmetry group of the corresponding superbracket.

A	$\tau(A)\sigma_g(A)$	$\tau(A)\sigma_{g_{I_2}}(A)$	$\tau(A)\sigma_{g_{I_3}}(A)$	$\tau(A)\sigma_{g_{EI_1}}(A)$
I_0	$+$	$+$	$+$	$+$
I_1	$-$	$+$	$+$	$-$
I_2	$+$	$+$	$-$	$-$
I_3	$+$	$-$	$+$	$-$
E	$-$	$-$	$-$	$-$
EI_1	$+$	$-$	$-$	$+$
EI_2	$-$	$-$	$+$	$+$
EI_3	$-$	$+$	$-$	$+$

To see how the Schur algebra acts on the four super-admissible forms it is convenient to convert Table 14 into 15. I_0 acts by an overall rescaling on all forms, while E generates the one-dimensional kernel of the representation. The stabilizers of all forms are four-dimensional with Lie algebra $\mathbb{R} + \mathfrak{su}(2)$. By factorizing the one-dimensional kernel of the representation, we obtain the seven-dimensional Lie algebra

$$\langle \mathrm{Id}, I_\alpha, EI_\alpha \rangle \cong \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \ .$$

The group $\mathrm{SO}(4) \cong \mathrm{SU}(2) \cdot \mathrm{SU}(2)$ generated by $\mathfrak{su}(2) + \mathfrak{su}(2)$ acts in a four-dimensional irreducible representation. Since both factors $\mathfrak{su}(2)$ act non-trivially, this is the vector representation, and we see that the Schur group acts as the linear conformal orthogonal group

$$\mathrm{CSO}(4) := \mathbb{R}^{>0} \times \mathrm{SO}(4)$$

on the four-dimensional space of superbrackets. This action is transitive once we remove the origin. Therefore there are two orbits: the open orbit of non-zero vectors and the origin. There is one non-zero superbracket up to isomorphism, corresponding to a unique Euclidean $\mathcal{N} = 2$ supersymmetry algebra. Its R-symmetry group is $\mathbb{R}^{>0} \times \mathrm{Spin}(3) \cong \mathbb{R}^{>0} \times \mathrm{SU}(2)$, where $\mathrm{Spin}(3) \subset \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ is a diagonally embedded $\mathrm{SU}(2)$-subgroup of the Schur group $\mathcal{C}(\mathbb{S})^* = \mathbb{H}^* \times \mathbb{H}^* = \mathbb{R}^{>0} \times \mathrm{SU}(2) \times \mathbb{R}^{>0} \times \mathrm{SU}(2)$.

We close this section by showing explicitly how each of the brackets Π_{β_i}, $i = 1, 2, 3$ can be obtained from $\Pi_{\beta_0} = \Pi_g$. This amounts to finding $A \in \mathcal{C}(\mathbb{S})^*$ such that

$$A^{-1} \cdot \Pi_g = \Pi_g(A\cdot, A\cdot) = \Pi_{g_{f(A)}}$$

with $f(A) = I_2, I_3, EI_1$. We consider invertible elements of the Schur algebra of the form

$$A = aId + bI_1E + cI_2 + dI_3 \ , \quad a^2 + b^2 + c^2 + d^2 \neq 0 \ .$$

We compute

$$\begin{aligned} g(\gamma_v As, At) = (a^2 - b^2 - c^2 - d^2) g(\gamma_v s, t) - 2abg(I_1 E\gamma_v s, t) \\ -2acg(I_2\gamma_v s, t) - 2adg(I_3\gamma_v s, t) \ , \end{aligned}$$

using the symmetry and type of the various automorphisms. This determines:

$$\begin{aligned} f: \quad & A = aId + bI_1E + cI_2 + dI_3 \\ & \mapsto f(A) = (a^2 - b^2 - c^2 - d^2)Id - 2abI_1E - 2acI_2 - 2adI_3 \ . \end{aligned}$$

Now we can read off how to obtain the basis Π_{β_i} by action with elements of the Schur group on Π_g, see Table 16. Note that the overall sign of A is free, since we insert it twice into the bilinear form.

Table 16. This table shows how the four basic bilinear forms Π_{β_i} can be obtained from Π_g by the action of the Schur group.

Form	Coefficients	Schur group element
Π_g	$a = 1, b = c = d = 0$	$\pm A = \mathrm{Id}$
$\Pi_{g_{I_1E}}$	$c = d = 0 \ , \ \ a = -b = \frac{1}{\sqrt{2}}$	$\pm A = \frac{1}{\sqrt{2}}(Id - I_1E)$
$\Pi_{g_{I_2}}$	$b = d = 0 \ , \ \ a = -c = \frac{1}{\sqrt{2}}$	$\pm A = \frac{1}{\sqrt{2}}(Id - I_2)$
$\Pi_{g_{I_3}}$	$b = c = 0 \ , \ \ a = -d = \frac{1}{\sqrt{2}}$	$\pm A = \frac{1}{\sqrt{2}}(Id - I_3)$

We remark that the semi-spinor modules are g-orthogonal, $g(\mathbb{S}_\pm, \mathbb{S}_\mp) = 0$. Since the operators EI_1, I_2, I_3 commute with the volume element $\gamma^1\gamma^2\gamma^3\gamma^4 = -E$, all superbrackets vanish on $\mathbb{S}_+ \otimes \mathbb{S}_+ + \mathbb{S}_- \otimes \mathbb{S}_-$.

4. Outlook

In this paper we have derived the necessary and sufficient condition for Poincaré Lie superalgebras to be isomorphic, and we have obtained a complete classification of isomorphism classes of supersymmetry algebras whose odd part is the complex spinor module $\mathbb{S}$ for all possible space-time signatures in four dimensions. In a companion paper [6] we will present physical theories which realize all these algebras as symmetries. The fields of these theories will belong to $\mathcal{N} = 2$ vector multiplets, which can be viewed as an extensions of Maxwell theory by fermions and scalars. Based on earlier work on five-dimensional vector multiplets with arbitrary signature [8] we will obtain representations of the four-dimensional supersymmetry algebras on fields which close 'off-shell', that is without imposing equations of motions. In addition, we will also present the corresponding supersymmetry invariant Lagrangians. We will also show how one can explicitly construct field redefinitions which relate the supersymmetry transformations and Lagrangians representing isomorphic supersymmetry algebras to one another.

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