Does the ratio of Laplace transforms of powers of a function identify the function?

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Abstract

In auction theory, one is interested in identifying the distribution of bids based on the distribution of the highest ones. We study this problem as a special case of the following question. Let m, n be two distinct nonnegative integers and f a nonzero measurable function on $[0,\infty)$ of at most exponential order. Let $H_{n,m} := \widehat{f^n}/\widehat{f^m}$ be the ratio of the Laplace transforms of f^n and f^m. Does knowledge of the function $H_{n,m}$ uniquely specify the function f? This is a generalization of Lerch's theorem (Laplace transform specifies the function). Under some rather strong assumptions on f we show that the answer is affirmative.

1 Introduction

There are N bidders for a single item. Bidder i bids X_i units of money. We assume that $X_1,\dots,X_N$ are random variables. They cannot be independent because there is a tacit common understanding about the value of the item. A simple model (see [9]) is thus

$$X_i = X^* + \varepsilon_i, \quad i = 1,\dots,N,$$

where X^* is a random variable representing the common understanding of the item value. In auction theory, X^* is called "unobserved heterogeneity". The random variable ε_i is the additional value of the item as perceived by bidder i. It is called the "idiosyncratic part" of the bid. Since the bidders act independently, it is reasonable to assume that $\varepsilon_1,\dots,\varepsilon_N$ are independent random variables. We also assume that they are independent of X^*. Moreover, we assume that bidders behave identically which means that the idiosyncratic parts have a common distribution denoted by

$$F(x) = \mathbb{P}(\varepsilon \le x).$$

An identification problem appearing in practice [9] is this: Given the distributions of the two highest bids can we find the distribution of ε? In other words, if $X_{(1)} \le \cdots \le X_{(N)}$ is the ordered version of $(X_1,\dots,X_N)$, and if we know the distributions of $X_{(N-1)}$ and $X_{(N)}$ can we find F? Quite clearly, knowledge of the distribution of X_N (which is the same as the distribution of X_{N-1}) does not imply

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knowledge of F. The catch here is that we have information about the highest and second highest bid, rather than two arbitrary bids; and this is what can possibly lead to an affirmative answer. To take a concrete case, suppose that we use a parametric model, for example, suppose ε is exponential with unknown rate. In this case,

$$\varepsilon_{(N)} \overset{(\mathrm{d})}{=} \varepsilon_{(N-1)} + \eta,$$

where η is an independent copy of ε and so we can find the distribution of η since we know its Laplace transform:

$$\mathbb{E}e^{-\lambda\varepsilon} = \mathbb{E}e^{-\lambda\eta} = \frac{\mathbb{E}e^{-\lambda X_{(N)}}}{\mathbb{E}e^{-\lambda X_{(N-1)}}}.$$

But, in general, the problem is not as trivial. In fact, we do not even know whether, indeed, we can identify the law of ε. For more information on the identification problem in auction theory, we refer to, among others, [7, 8, 6, 1, 3, 10, 4].

It will be seen (Section 3) that this question can be answered by means of the main result of this paper. We present this result next. We say that a nonzero measurable function $f : [0,\infty) \to \mathbb{R}$ is of exponential order if there are positive numbers C and c such that

$$|f(x)| \le Ce^{cx}, \quad x \ge 0.$$

Then the Laplace transform

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda x} f(x)dx$$

exists for $\lambda > c$. If n is a positive integer then f^n is also of exponential order and $\widehat{f^n}$ denotes its Laplace transform. Let m, n be nonnegative integers. Define

$$H_{n,m}(f,\lambda) := \frac{\widehat{f^n}(\lambda)}{\widehat{f^m}(\lambda)}.$$

The question of interest here is the following:

> **Uniqueness question:** For given distinct nonnegative integers n and m, does knowledge of the function $H_{n,m}(f,\cdot)$ uniquely specify f?

For $m > 0$, both $\widehat{f^n}(\lambda)$ and $\widehat{f^m}(\lambda)$ are analytic when λ ranges on the complex plane and the real part of λ is large enough, see, e.g., [2, Theorem 6.1]. So $H_{n,m}(f,\cdot)$ is a well-defined meromorphic function.

Clearly, if $m = 0$ then, by the classical theorem of Laplace transform inversion [11], we know f^n and so we know f if n is odd. But if n and m are distinct positive integers, the problem seems to be hard. We aim at giving an answer when we restrict f to a certain class of functions. Having in mind the probabilistic problem arising in auctions, where f plays the role of a distribution function, it is not unreasonable to assume that f is piecewise smooth. (By this we mean a function which is analytic except finitely many jump discontinuities.) This corresponds, e.g., to the case where ε has piecewise smooth distribution function.

It is easy to see that uniqueness, in strict sense, is impossible because translations do not affect $H_{n,m}(f,\cdot)$. Suppose that, for some $c > 0$, the function f is identically 0 on an interval $[0,c)$ and let

$$\theta_{-c}f(x) := f(x+c).$$

Then

$$\widehat{\theta_{-c}f}(\lambda) = e^{\lambda c}\widehat{f}(\lambda).$$

Clearly then,

$$H_{n,m}(f,\cdot) = H_{n,m}(\theta_{-c}f,\cdot).$$

So $H_{n,m}(f,\cdot)$ specifies f up to a translation. Hence, to obtain uniqueness, it is necessary to assume

$$\inf\{x : f(x) \ne 0\} = 0. \tag{1}$$

Even under this condition, we cannot answer the problem in general, i.e. under the sole assumption that the Laplace transform of f exists.

The case where f is a polynomial is of independent interest:

Theorem 1. *Let m, n be distinct positive integers and f, g polynomials such that*

$$H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot).$$

If $n-m$ is odd, then f is identical to g. If $n-m$ is even, then either f is identical to g or f is identical to $-g$.

For the general case, we shall restrict ourselves to functions that are a bit more general than piecewise smooth. We consider functions f on $[0,\infty)$ that are right-continuous and with left limits at each point (the so called càdlàg functions) and impose smoothness on the right. We say that f is right analytic on a set A if it is right analytic at any point $a \in A$, which is defined as follows: there exists $h > 0$ such that $[a, a+h) \subset A$ and f has right derivatives at a of all orders, denoted by $f^{(i)}(a+), i \geq 0$, and for all $a \leq x < a+h$

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(a+)\frac{(x-a)^i}{i!}.$$

The series on the right also converges on $a-h < x < a+h$ (see, e.g., [5, Prop. 1.1.1]). By [5, Cor. 1.2.3], the function $g(x) := \sum_{i=0}^{\infty} f^{(i)}(a+)\frac{(x-a)^i}{i!}$ is real analytic on $(a-h, a+h)$. So f is right analytic at a if and only if there exists a function g which is real analytic at a and there exists $h > 0$ such that $f(x) = g(x)$ for any $x \in [a, a+h)$. The right analyticity only imposes smoothness on the right of a point. A càdlàg and right analytic function f on $[0,\infty)$ can have countably many discontinuous point on a compact interval. For example, take

$$f(x) = \frac{1}{2^n}, \text{ if } x \in \left[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right), n = 0, 1, 2, \ldots; \quad f(x) = 2, \text{ if } x \geq 1.$$

The f defined above is càdlàg and right analytic on $[0,\infty)$ with discontinuities at points $\frac{1}{2^n}$, $n = 0, 1, 2, \ldots$, and at 1.

Theorem 2. *Let m, n be distinct positive integers. Suppose that f, g are nonnegative nondecreasing càdlàg functions, right analytic at every point $a \geq 0$, of exponential order and such that $f(x), g(x) > 0$ for all $x > 0$. If*

$$H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot).$$

then $f = g$.

The paper is organized as follows. Theorems 1 and 2 are proved in Section 2. Their relation to the auction theory case discussed above is presented in Section 3.

2 The uniqueness question

We start with a preliminary observation. For a function f that has sufficiently many derivatives at 0 let

$$I(f) := \min\{k \geq 0 : f^{(k)}(0) \neq 0\}.$$

We use the phrase "sufficiently many derivatives at 0" as equivalent to the phrase "at least as many derivatives as required for the definition of $I(f)$". So, if $f(0) \neq 0$ then f is allowed to have no derivative at 0. But if $f(0) = 0$ then we assume that f is at least once differentiable; if $f'(0) \neq 0$ then $I(f) = 1$ and f does not need to be twice differentiable. The observation is that if f and g have finite $I(f)$ and $I(g)$ then $H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot)$ implies that $I(f) = I(g)$. We explain this in the following lemma.

Lemma 1. *Suppose that f and g are of exponential order, have sufficiently many derivatives at 0, and $I(f) < \infty, I(g) < \infty$. Let m, n be distinct positive integers. Assume $H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot)$. Then $I(f) = I(g)$. Let $k = I(f) = I(g)$. If $n - m$ is odd then $f^{(k)}(0) = g^{(k)}(0)$. If $n - m$ is even then $|f^{(k)}(0)| = |g^{(k)}(0)|$.*

Proof. The assumption $H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot)$ is equivalent to

$$\widehat{f^n}(\lambda)\widehat{g^m}(\lambda) = \widehat{g^n}(\lambda)\widehat{f^m}(\lambda) \text{ for sufficiently large } \lambda$$

which is further equivalent to

$$f^n * g^m = f^m * g^n, \tag{2}$$

where $*$ denotes convolution. Write the left-hand side as

$$(f^n * g^m)(t) = \int_0^t f(s)^n g(t-s)^m ds = t\int_0^1 f(tu)^n g(t(1-u))^m du. \tag{3}$$

Define

$$k := I(f), \quad \ell := I(g), \quad a := f^{(k)}(0), \quad b := f^{(\ell)}(0).$$

Divide both sides of (3) by $t^{kn+\ell m+1}$. Then, as $t \to 0$,

$$\begin{aligned}\frac{(f^n * g^m)(t)}{t^{kn+\ell m+1}} &= \int_0^1 \left(\frac{f(tu)}{t^k}\right)^n \left(\frac{g(t(1-u))}{t^\ell}\right)^m du \\ &\to \int_0^1 \left(\frac{au^k}{k!}\right)^n \left(\frac{b(1-u)^\ell}{\ell!}\right)^m du = \frac{a^n b^m}{k!^n \ell!^m} \mathrm{B}(kn+1, \ell m+1),\end{aligned} \tag{4}$$

where B is the beta function. To obtain this, we used the assumption that the first nonzero derivative of f at zero is the derivative of order k, so that $f(tu)/t^k \to f^{(k)}(0)u^k/k!$ and, similarly, $g(t(1-u))/t^\ell \to g^{(\ell)}(0)(1-u)^\ell/\ell!$. Reversing the roles of n and m, we obtain

$$\frac{(f^m * g^n)(t)}{t^{km+\ell n+1}} \to \frac{a^m b^n}{k!^m \ell!^n} \mathrm{B}(km+1, \ell n+1), \tag{5}$$

as $t \to 0$. Comparing (4) and (5), and in view of (2), we are forced to conclude that

$$p_1 := kn + \ell m \;=\; km + \ell n =: p_2.$$

Indeed, by (2),we have $f^n * g^m = f^m * g^n = h$. The function h satisfies $t^{-p_1}h(t) \to C_1$ and $t^{-p_2}h(t) \to C_2$, as $t \to 0$, where C_1, C_2 are the constants appearing on the right-hand sides of (4) and (5), respectively. These constants are nonzero. If $p_1 > p_2$ we obtain $t^{-p_1}h(t) = t^{p_1-p_2}t^{-p_1}h(t) \to 0 \cdot C_2 = 0$. Hence $C_1 = 0$, which is impossible. Similarly, $p_1 < p_2$ is impossible, and thus $p_1 = p_2$. Thus, $k(n-m) = \ell(n-m)$ and so

$$k = \ell.$$

But then C_1 and C_2 are equal and this entails $a^m b^n = a^n b^m$, or

$$(a/b)^{n-m} = 1.$$

If $n - m$ is odd we have $a = b$. If $n - m$ is even we can only deduce that $|a| = |b|$. □

Lemma 2. *Suppose that f and g are of exponential order and have sufficiently many derivatives at 0. Assume that $I(f) = I(g) = k < \infty$ and $f^{(k)}(0) = g^{(k)}(0)$. Let m, n be distinct positive integers. If $H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot)$ then $f^{(\ell)}(0) = g^{(\ell)}(0)$ for all $\ell \geq k$ for which the two derivatives exist.*

Proof. Assume that, for some $\ell > k$, we have

$$f^{(j)}(0) = g^{(j)}(0), \quad k \le j \le \ell - 1.$$

We will show that $f^{(\ell)}(0) = g^{(\ell)}(0)$. With

$$c_j := f^{(j)}(0)/j!,\ k \le j < \ell, \quad a := f^{(\ell)}(0)/\ell!, \quad b := g^{(\ell)}(0)/\ell!,$$

we have

$$f(x) = \sum_{i=k}^{\ell-1} c_i x^i + a x^\ell + f_1(x), \quad g(x) = \sum_{i=k}^{\ell-1} c_i x^i + b x^\ell + g_1(x),$$

where $f_1(x) = o(x^\ell)$ and $g_1(x) = o(x^\ell)$ as $x \to 0$. We will show that $a = b$. We have

$$\begin{aligned} \frac{f^n * g^m(t)}{t} &= \int_0^1 f(tu)^n g(t(1-u))^m du \\ &= \int_0^1 \left(\sum_{i=k}^{\ell-1} c_i u^i t^i + \alpha u^\ell t^\ell + f_1(ut) \right)^n \left(\sum_{i=k}^{\ell-1} c_i (1-u)^i t^i + \beta (1-u)^\ell t^\ell + g_1((1-u)t) \right)^m du. \quad (6) \end{aligned}$$

Note th integrand in the last integral of (6) is a product of $n + m$ terms. Let[1]

$$d = \ell + k(n-1) + km.$$

After multiplication and integration, we shall keep track of the monomial terms of degree at most d and combine everything else into terms of order $o(t^d)$. Notice that if f_1 or g_1 is involved in the multiplication and integration, the resulting term must be of order $o(t^d)$. That means if we keep track of the monomial terms of degree at most d, f_1 and g_1 are not involved. So we can write

$$\frac{f^n * g^m(t)}{t} = P_{n,m}(t) + o(t^d).$$

Note that $P_{n,m}(t)$ can be obtained if we set f_1 and g_1 to zero in the last integral of (6) and integrate so that we obtain a polynomial in t of degree $n\ell + m\ell$, and keep only the monomials up to power t^d. We now split $P_{n,m}(t)$ into a polynomial $Q_{n,m}(t)$ of degree at most $d - 1$ and a monomial of degree d whose coefficient is split into two parts:

$$P_{n,m}(t) = Q_{n,m}(t) + (C_{n,m}(a,b) + D_{n,m}) t^d.$$

The first coefficient $C_{n,m}(a,b)$ contains all terms that depend on a or b. Explicitly,

$$\begin{aligned} C_{n,m}(a,b) t^d = &\int_0^1 a u^\ell t^\ell \binom{n}{1} (c_k u^k t^k)^{n-1} (c_k (1-u)^k t^k)^m du \\ &+ \int_0^1 b (1-u)^\ell t^\ell \binom{m}{1} (c_k (1-u)^k t^k)^{m-1} (c_k u^k t^k)^n du \\ &= \frac{t^{k(n+m-1)+l}}{l!(k!)^{n+m-1}} \int_0^1 \left(a n u^{k(n-1)+l} (1-n)^{km} + b m (1-u)^{k(m-1)+l} u^{kn} \right) du \\ &= \frac{t^{k(n+m-1)+l}}{l!(k!)^{n+m-1}} \big(a n \,\mathrm{B}(k(n-1)+l+1, km+1) + b m \,\mathrm{B}(k(m-1)+l+1, kn+1) \big). \quad (7) \end{aligned}$$

The coefficient $D_{n,m}$ is obtained as the coefficient in t^d when we set a and b to zero. In other words, $D_{n,m}$ is the coefficient of t^d in the following polynomial (in t)

$$\int_0^1 \left(\sum_{i=k}^{\ell-1} c_i u^i t^i \right)^n \left(\sum_{i=k}^{\ell-1} c_i (1-u)^i t^i \right)^m du.$$

[1]Ignoring for the moment the terms f_1 and g_1, so that the integrand is a polynomial, we can easily see that the term t^d of this polynomial has a coefficient that depends on α or β, whereas all smaller degree terms do not.

Notice that $Q_{n,m}(t)$ does not involve a or b neither, because when a or b is involved in the multiplication and integration, the resulting term must be at least of order t^d. So $D_{n,m}$ is the coefficient of t^{d-1} in the above polynomial. By symmetry, $D_{n,m} = D_{m,n}, Q_{n,m} = Q_{m,n}$. Reversing the roles of m and n we obtain

$$\frac{f^m * g^n(t)}{t} = P_{m,n}(t) + o(t^d) = Q_{m,n}(t) + (C_{m,n}(\alpha,\beta) + D_{m,n})t^d + o(t^d),$$

as $t \to 0$. The assumptions imply that $f^n * g^m = f^m * g^n$. We thus have

$$Q_{n,m}(t) + (C_{n,m}(\alpha,\beta) + D_{n,m})t^d + o(t^d) = Q_{m,n}(t) + (C_{m,n}(\alpha,\beta) + D_{m,n})t^d + o(t^d),$$

in a neighbourhood of 0. Since $D_{n,m} = D_{m,n}, Q_{n,m} = Q_{m,n}$,

$$C_{n,m}(a,b) = C_{m,n}(a,b).$$

Looking at the expression for $C_{n,m}$ from equation (7) we obtain

$$(a-b)\big[n\,\mathrm{B}(k(n-1)+\ell+1, km+1) - m\,\mathrm{B}(k(m-1)+\ell+1, kn+1)\big] = 0.$$

To conclude that $a = b$ we only have to show that the coefficient in the bracket is nonzero. To see this, recall that $\ell > k$, assume that $n > m \geq 1$, and use the notation $(p)_q := p(p-1)\cdots(p-q+1)$ to obtain that

$$\frac{n\,\mathrm{B}(k(n-1)+\ell+1, km+1)}{m\,\mathrm{B}(k(m-1)+\ell+1, kn+1)} = \frac{n}{m}\frac{(km)!}{(kn)!}\frac{(kn+\ell-k)!}{(km+\ell-k)!} = \frac{n}{m}\frac{(kn+\ell-k)_{(k(n-m))}}{(kn)_{k(n-m)}}$$

is the product of $1 + k(n-m)$ integers all strictly bigger than 1. Similarly, the ratio is strictly smaller than 1 if $n < m$. □

Corollary 1. *Suppose that f and g are of exponential order and that they have sufficiently many derivatives at 0. Let m, n be distinct positive integers. Suppose $H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot)$. Assume $k = I(f) = I(g) < \infty$. If $f^{(k)}(0) = g^{(k)}(0)$, then $f^{(j)}(0) = g^{(j)}(0)$ for all $j \geq 0$ for which the two derivatives exist. If $f^{(k)}(0) = -g^{(k)}(0)$, then $f^{(j)}(0) = -g^{(j)}(0)$ for all $j \geq 0$ for which the two derivatives exist.*

Proof. If $f^{(k)}(0) = g^{(k)}(0)$, by Lemma 2, $f^{(j)}(0) = g^{(j)}(0)$ for all $j \geq k$ and hence for all $j \geq 0$ for which the derivatives exist. If $f^{(k)}(0) = -g^{(k)}(0)$, by Lemma 1, $n - m$ must be even. Then $H_{n,m}(f,\cdot) = H_{n,m}(-g,\cdot)$. Using $f^{(k)}(0) = (-g)^{(k)}(0)$ and Lemma 2, $f^{(j)}(0) = (-g)^{(j)}(0)$ for any $j \geq 0$ for which the derivatives exist. □

Proof of Theorem 1. Since f, g are polynomials they are infinitely differentiable and are of exponential order. Moreover, $I(f) < \infty, I(g) < \infty$. By Lemma 1, $I(f) = I(g) =: k$, say. Moreover, we have $f^{(k)}(0) = g^{(k)}(0)$, if $n - m$ is odd; $|f^{(k)}(0)| = |g^{(k)}(0)|$, if $n - m$ is even. Suppose first that $n - m$ is odd. By Corollary 1, $f^{(j)}(0) = g^{(j)}(0)$ for all $j \geq 0$. Since polynomials are determined by their derivatives of all orders at zero, we have f identical to g. Suppose next that $n - m$ is even. We have two possibilities, i.e., either $f^{(k)}(0) = g^{(k)}(0)$ or $f^{(k)}(0) = -g^{(k)}(0)$. Consequently, we have either $f^{(j)}(0) = g^{(j)}(0)$ for all $j \geq 0$, or $f^{(j)}(0) = -g^{(j)}(0)$ for all $j \geq 0$. Hence f is identical to g or identical to $-g$. □

We now aim at proving Theorem 2. We need the preliminary result of Lemma 3 below. This lemma is inspired by the approach taken in [9].

Lemma 3. *Suppose that f and g are of exponential order, càdlàg and nondecreasing with $f(x) > 0, g(x) > 0$ for any $x > 0$. Assume that $H_{n,m}(f,\cdot) = H_{n,m}(g,\cdot)$. Assume further that there exists $a > 0$ such that $f(x) = g(x)$ for any $x \in [0,a)$. Then $f^{(i)}(a+) = g^{(i)}(a+)$ for any $i \geq 0$ if they exist.*

Proof. We argue by contradiction. Assume there exists $i \geq 0$ such that $f^{(j)}(a+), g^{(j)}(a+)$ exist for any $0 \leq j \leq i$, and $f^{(j)}(a+) = g^{(j)}(a+)$ for any $0 \leq j \leq i-1$ and $f^{(i)}(a+) \neq g^{(i)}(a+)$. Without loss of generality we assume $f^{(i)}(a+) > g^{(i)}(a+)$. Then there exists a small number $0 < h < a$ such that

$$f(x) > g(x), \quad x \in (a, a+h). \tag{8}$$

Recall that

$$f(x) = g(x), \quad x \in [0, a). \tag{9}$$

By assumption, f and g satisfy that

$$f(x) > 0, \text{ for any } x > 0 \text{ and } f(0) \geq 0; \qquad g(x) > 0, \text{ for any } x > 0 \text{ and } g(0) \geq 0. \tag{10}$$

The equality $H_{n,m}(f, \cdot) = H_{n,m}(g, \cdot)$ yields the convolution equality at $a+h$

$$f^n * g^m(a+h) - g^n * f^m(a+h) = 0.$$

In terms of integrals

$$\begin{aligned}
&\int_0^{a+h} (f(a+h-u)^n g(u)^m - g(u)^n f(a+h-u)^m) du \\
=&\int_0^h f(a+h-u)^m g(u)^m (f(a+h-u)^{n-m} - g(u)^{n-m}) du \\
&\quad + \int_h^{a+h} f(a+h-u)^m g(u)^m (f(a+h-u)^{n-m} - g(u)^{n-m}) du \\
=&I_1 + I_2 = 0
\end{aligned} \tag{11}$$

where I_1 corresponds to the first integral and I_2 to the second. Recall $0 < h < a$. When $u \in (0, h)$, we have $a+h-u \in (a, a+h)$. Then

$$f(a+h-u) > g(a+h-u) \geq g(u), \qquad \text{for any } u \in (0, h),$$

where the first inequality is due to (8) and the second is due to the fact that g is a nondecreasing function. Taking into account (10), we conclude that

$$I_1 > 0.$$

When $u \in (h, a+h)$, we have $a+h-u \in (0, a)$. Then by (9), $f(a+h-u) = g(a+h-u)$. So I_2 becomes

$$\begin{aligned}
I_2 &= \int_h^{a+h} g(a+h-u)^m g(u)^m (g(a+h-u)^{n-m} - g(u)^{n-m}) du \\
&= \int_h^{a+h} \big(g(a+h-u)^n g(u)^m - g(a+h-u)^m g(u)^n\big) du = 0.
\end{aligned}$$

Then we obtain $I_1 + I_2 > 0$ which is in contradiction to (11). □

We now pass on to the proof of the main theorem.

Proof of Theorem 2. If $f^{(i)}(0) = 0$ for all $i \geq 0$ then, by right analyticity, there exists $a > 0$ such that $f(x) = 0$ for all $x \in [0, a)$. This is in contradiction to the assumption that $f(x), g(x) > 0$ for all $x > 0$. Hence $f^{(j)}(0) \neq 0$ for some j. Similarly, $g^{(j)}(0) \neq 0$ for some j. As f, g are nonnegative functions, applying Lemma 1 and Corollary 1, we have

$$f^{(i)}(0) = g^{(i)}(0), \quad i \geq 0.$$

Due to right real analyticity, there exists $a > 0$ such that $f(x) = g(x)$ for any $x \in [0, a)$. Let

$$A := \sup\{a : f(x) = g(x) \text{ for all } \ x \in [0, a)\}.$$

Assume that $A < \infty$. By Lemma 3 and right analyticity

$$f^{(i)}(A) = g^{(i)}(A), \quad i \geq 0.$$

Again by right analyticity, there exists $h > 0$ such that $f(x) = g(x)$ for any $x \in [A, A + h)$. This fact is in contradiction to the definition of A. So we have $A = \infty$ which means $f(x) = g(x)$ for all $x \geq 0$.

□

3 The auction problem

To see why Theorem 2 partially answers the question about auctions, posed in the introduction, consider again the following scenario. Let $\varepsilon_1, \ldots, \varepsilon_N$ be i.i.d. nonnegative random variables with common distribution function $F(x) = \mathbb{P}(\varepsilon \leq x)$ and let X^* be an independent nonnegative random variable. Bidder i offers

$$X_i = X^* + \varepsilon_i.$$

Ordering the X_i is equivalent to ordering the ε_i:

$$X_{(i)} = X^* + \varepsilon_{(i)}.$$

We assume that we know the distributions of the two largest bids, i.e., the distributions of $X_{(N)}$ and $X_{(N-1)}$. Therefore we know the ratio of their Laplace transforms, and this ratio can be expressed in terms of the unknown distribution F:

$$\frac{\mathbb{E}e^{-\lambda X_{(N)}}}{\mathbb{E}e^{-\lambda X_{(N-1)}}} = \frac{\mathbb{E}e^{-\lambda \varepsilon_{(N)}}}{\mathbb{E}e^{-\varepsilon_{(N-1)}}}.$$

Integrating by parts in a Lebesgue-Stieltjes integral we obtain

$$\mathbb{E}e^{-\lambda\varepsilon_{(N)}} = \int_{[0,\infty)} e^{-\lambda x}\mathbb{P}(\varepsilon_{(N)} \in dx) = \int_0^\infty \lambda e^{-\lambda x}\mathbb{P}(\varepsilon_{(N)} \leq x)dx = \int_0^\infty \lambda e^{-\lambda x} F(x)^N dx = \lambda\widehat{F^N}(\lambda),$$

where $\widehat{F^N}$ is the Laplace transform of the function $x \mapsto F(x)^N$ (and not of the measure induced by this function). Since

$$\begin{aligned}
\mathbb{P}(\varepsilon_{(N-1)} \leq x) &= \mathbb{P}(\varepsilon_{(N)} \leq x) - \mathbb{P}(\varepsilon_{(N-1)} < x < \varepsilon_{(N)}) \\
&= F(x)^N - NF(x)^{N-1}(1 - F(x)) \\
&= NF(x)^{N-1} - (N-1)F(x)^N
\end{aligned}$$

we similarly have

$$\mathbb{E}e^{-\lambda\varepsilon_{(N-1)}} = \int_0^\infty \lambda e^{-\lambda x}(NF(x)^{N-1} - (N-1)F(x)^N)dx = \lambda N\widehat{F^{N-1}}(\lambda) - \lambda(N-1)\widehat{F^N}(\lambda).$$

By simple algebra, the quantity

$$H_{N-1,N}(F, \lambda) = \frac{\widehat{F^N}(\lambda)}{\widehat{F^{N-1}}(\lambda)} = N\left(\frac{\mathbb{E}e^{-\lambda X_{(N-1)}}}{\mathbb{E}e^{-\lambda X_{(N)}}} + N - 1\right)^{-1}$$

is known and thus the problem reduces to the one studied above.

Economists [9] are interested in determining F once $H_{N-1,N}(F, \lambda)$ is known. Note that the conditions in Theorem 2 allow the distribution function F to be piecewise smooth; for example, the mixture of a Gamma random variable and a discrete random variable. So, if, say, bidders use a random variable ε that is, say, exponential(θ) with probability p or geometric(α) with probability $1 - p$ then knowledge of the distribution of $X_{(N)}$ and $X_{(N-1)}$ implies knowledge of the distribution of ε uniquely. Of course, nothing has been said about the computation of this distribution in this paper.

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