Optimal stochastic regulators with state-dependent weights

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Abstract

We introduce two optimal regulators for linear stochastic systems. The first is of a linear state-feedback form, and it generalises the linear-quadratic regulator by introducing *state-dependent* weights in the cost functional. The second is a certain *risk-sensitive* version of the first, and it is of a *nonlinear* state-feedback form. Both regulators are applied to the optimal investment problem.

Keywords: Stochastic regulators; Risk-sensitive control; Optimal investment.

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a fixed complete probability space on which a d-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the linear *scalar* stochastic control system:

$$\begin{cases} dx(t) = [a(t)x(t) + u'(t)b(t)]dt + [c'(t)x(t) + u'(t)D(t)]dW(t), \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

for some given $\mathcal{F}(t)$-adapted coefficient processes $a(\cdot), b(\cdot), c(\cdot), D(\cdot)$, and a suitable $\mathcal{F}(t)$-adapted control process $u(\cdot)$ such that (1.1) has a unique strong solution. The stochastic *linear-quadratic (LQ) control* problem is the optimal

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control problem of minimizing the *quadratic* cost functional

$$\mathbb{E}\left\{\int_0^T [q(t)x^2(t) + x(t)k'(t)u(t) + u'(t)R(t)u(t)]dt + sx^2(T)\right\}, \qquad (1.2)$$

subject to (1.1), for some suitable $\mathcal{F}(t)$-adapted *weight* processes $q(\cdot), k(\cdot)$, $R(\cdot)$, and an $\mathcal{F}(T)$-measurable random variable s. The LQ control problem has been studied extensively since its introduction by Kalman [1] for deterministic systems (see, e.g., [2], [3]), and continues to be developed in various stochastic settings (see, e.g., [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]). An important characteristic of this problem is that it admits an explicit closed-form solution in a *linear state-feedback* form through the *Riccati* equation.

In this paper, we generalise the cost functional (1.2) by introducing *state-dependent* weights as follows:

$$J(u(\cdot)) := \mathbb{E}\left\{\int_0^T x^{\gamma-2}(t)[q(t)x^2(t) + x(t)k'(t)u(t) + u'(t)R(t)u(t)]dt + sx^{\gamma}(T)\right\},$$

for some $\gamma \in \mathbb{R}$. Thus, compared with (1.2), instead of $q(t), k(t), R(t), s$, here we have $x^{\gamma-2}(t)q(t)$, $x^{\gamma-2}(t)k(t)$, $x^{\gamma-2}(t)R(t)$, $x^{\gamma-2}(T)s$, respectively. The idea of using state-dependent weights is well-known for both deterministic and stochastic systems (see, e.g., [16], [17], [18], [19]). In these papers the optimal control problems are solved only approximately, and thus no closed-form solutions are given. Two examples of $J(u(\cdot))$ appear in the optimal investment problem: a special case of $J(u(\cdot))$ with $\gamma = 1$ appears in [20] in the setting of deterministic coefficients, whereas the case of $J(u(\cdot)) = \mathbb{E}[-x^{\gamma}(T)]$ with $\gamma \in (0,1)$ is the criterion of the well-known *Merton problem* (see, e.g., [21], [22], [23]). In both of these cases explicit closed-form solutions are found to the corresponding optimal control problem in a linear state-feedback form.

In §2, we formulate the optimal control problem of minimising $J(u(\cdot))$ subject to (1.1). We find an explicit closed form solution in terms of a *new* Riccati backward stochastic differential equation (BSDE). The solution is in a linear state-feedback form. Both the finite and infinite horizon cases are considered. In §3, a certain *risk-sensitive* version of this problem is considered. An explicit closed-form solution is found in this case as well, however, the optimal controller is of a *nonlinear* state feedback form. In §4, we apply our results to the optimal investment problem and thus generalise the result of [20] to the setting of random coefficients, and resolve the optimal investment problem with exponential utility and *random* interest rate, which turns out to be *qualitatively* different from the case of a deterministic interest rate.

The following notation is used: $L^0_{\mathcal{F}}(0,T;E)$ is the set of all $\mathcal{F}(t)$-adapted processes $f:[0,T]\times\Omega\to E$, where E is an Euclidian space; if $f(\cdot)\in L^0_{\mathcal{F}}(0,T;E)$ and $\mathbb{E}\int_0^T |f(t)|^2dt<\infty$, we write $f(\cdot)\in L^2_{\mathcal{F}}(0,T;E)$; if $f(\cdot)\in L^0_{\mathcal{F}}(0,T;E)$ is uniformly bounded we write $f(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^n)$; for $\mathcal{F}(t)$-adapted processes $f:[0,\infty)\times\Omega\to E$ we use all of the above notations with ∞ instead of T; if $\zeta:\Omega\to E$ is an $\mathcal{F}(T)$-measurable uniformly bounded random variable, we write $\zeta\in L^\infty_{\mathcal{F}(T)}(\Omega;E)$.

2. Generalised regulator

In this section, we consider the problem of minimizing $J(u(\cdot))$ subject to (1.1). Unless otherwise stated, we assume throughout that $x_0>0$, $\gamma\in\mathbb{R}$, $R(\cdot)$ is symmetric, and

$$\begin{aligned}&a(\cdot),q(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}),\quad c(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^d),\quad D(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d}),\\&b(\cdot),k(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^m),\quad s\in L^\infty_{\mathcal{F}(T)}(\Omega;\mathbb{R}),\quad R(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m}).\end{aligned}$$

In order to define the admissible set of controls and thus the optimal control problem to be solved, we first consider the following Riccati BSDE:

$$\begin{cases} dp(t)+\left[\gamma a(t)+\dfrac{\gamma(\gamma-1)}{2}c'(t)c(t)\right]p(t)dt+\gamma c'(t)z(t)dt+q(t)dt\\[2ex] -g'(t)\left[R(t)+\dfrac{\gamma(\gamma-1)}{2}D(t)D'(t)p(t)\right]g(t)dt-z'(t)dW(t)=0,\\[2ex] R(t)+\dfrac{\gamma(\gamma-1)}{2}D(t)D'(t)p(t)>0\quad a.e.\quad t\in[0,T]\quad a.s.,\\[2ex] p(T)=s\quad a.s.,\end{cases}\tag{2.1}$$

where the *controller gain* process $g(\cdot)$ is defined as

$$
\begin{aligned}
g(t) &:= -\frac{1}{2}\left[R(t)+\frac{\gamma(\gamma-1)}{2}D(t)D'(t)p(t)\right]^{-1} \\
&\times \quad \{k(t)+[\gamma b(t)+\gamma(\gamma-1)D(t)c(t)]\,p(t)+\gamma D(t)z(t)\}\,.
\end{aligned}
$$

Equation (2.1) is similar to the one of stochastic LQ control problem (see, e.g., [7], [9], [10]), and it contains it as a special case for $\gamma = 2$. However, there are examples of (2.1) which do not have an equivalent in the Riccati BSDE of LQ control, e.g., when $\gamma \in (0,1)$; $D(t)D'(t) > 0$ *a.e.* $t \in [0,T]$ *a.s.*.

Assumption 1. *Equation (2.1) admits a unique solution pair* $(p(\cdot), z(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

The validity of this assumption for $\gamma = 2$ has been considered at least since [4] and [6], and continues to be studied under different assumptions on the coefficients (see, e.g., the more recent papers [24], [25]). The following result covers some new solvability cases of (2.1).

Theorem 1. *Equation (2.1) has a solution pair* $(p(\cdot), z(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ *in the following cases:*

(i) $\gamma = 0$; $R(t) > 0$ *a.e.* $t \in [0,T]$ *a.s.,* $R^{-1}(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m})$,

(ii) $D(t) = 0$, $R(t) > 0$, $q(t) - k'(t)R^{-1}(t)k(t)/4 \geq 0$, *a.e.* $t \in [0,T]$ *a.s.,* $s \geq 0$ *a.s., and* $R^{-1}(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m})$,

(iii) $\gamma \in (0,1)$; $s \leq 0$ *a.s.;* $D(t)D'(t) > 0$, $R(t) > 0$, $q(t)-k'(t)R^{-1}(t)k(t)/4 \leq 0$, *a.e.* $t \in [0,T]$ *a.s.;* $R^{-1}(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m})$; $(DD')^{-1}(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m})$.

(iv) $\gamma \in (-\infty,0) \cap (1,\infty)$; $s \geq 0$ *a.s.;* $D(t)D'(t) > 0$, $R(t) > 0$, $q(t) - k'(t)R^{-1}(t)k(t)/4 \geq 0$, *a.e.* $t \in [0,T]$ *a.s.;* $R^{-1}(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m})$; $(DD')^{-1}$ $(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{m\times m})$.

Proof. (i) If $\gamma = 0$, then (2.1) reduces to following *linear* BSDE:

$$
\begin{cases}
dp(t) + q(t)dt - \dfrac{1}{4}k'(t)R^{-1}(t)k(t)dt - z'(t)dW(t) = 0, \\[2ex]
R(t) > 0 \quad a.e. \quad t \in [0,T] \quad a.s., \\[2ex]
p(T) = s \quad a.s..
\end{cases}
\tag{2.2}
$$

If $R(t) > 0$ a.e. $t \in [0,T]$ a.s., then existence of solution to (2.2) follows from [26] and the unboundedness of coefficients.

(ii) If $D(t) = 0$ $a.e.$ $t \in [0,T]$ $a.s.$, then (2.1) reduces to

$$\begin{cases} dp(t) + \left[\gamma a(t) + \dfrac{\gamma(\gamma-1)}{2}c'(t)c(t) - \dfrac{\gamma}{2}k'(t)R^{-1}(t)b(t)\right]p(t)dt + \gamma c'(t)z(t)dt \\ -\dfrac{\gamma^2}{4}b'(t)R^{-1}(t)b(t)p^2(t)dt + \left[q(t) - \dfrac{1}{4}k'(t)R^{-1}(t)k(t)\right]dt - z'(t)dW(t) = 0, \\ R(t) > 0 \quad a.e. \quad t \in [0,T] \quad a.s., \\ p(T) = s \quad a.s., \end{cases}$$

If $s \geq 0$ $a.s.$, $R(t) > 0$ and $q(t) - k'(t)R^{-1}(t)k(t)/4 \geq 0$ $a.e.$ $\quad t \in [0,T] \quad a.s.$, then conclusion follows from Theorem 5.1 of [7].

(iii) We suppress the argument t below for notational simplicity, and adapt the approach of Theorem 4.1 of [10]. Let $\bar{a} := \gamma a + \gamma(\gamma-1)c'c/2$, $\bar{c}' := \gamma c'$, $\bar{b} := \gamma b + \gamma(\gamma-1)Dc$. Equation (2.1) can now be written as:

$$\begin{cases} dp + \bar{a}pdt + \bar{c}'zdt + qdt - \dfrac{1}{4}(k + \bar{b}p + \gamma Dz)' \\ \times \left[R + \dfrac{\gamma(\gamma-1)}{2}DD'p\right]^{-1}(k + \bar{b}p + \gamma Dz)dt - z'dW = 0, \\ R + \dfrac{\gamma(\gamma-1)}{2}DD'p > 0 \quad a.e. \quad t \in [0,T] \quad a.s., \\ p(T) = s \quad a.s.. \end{cases} \tag{2.3}$$

We first consider the following related equation:

$$\begin{cases} d\underline{p} + \bar{a}\underline{p}dt + \bar{c}'\underline{z}dt + qdt - \dfrac{1}{4}(k + \bar{b}\underline{p}^- + \gamma D\underline{z})' \\ \times \left[R + \dfrac{\gamma(\gamma-1)}{2}DD'\underline{p}^-\right]^{-1}(k + \bar{b}\underline{p}^- + \gamma D\bar{z})dt - \underline{z}'dW = 0, \\ \underline{p}(T) = s \quad a.s., \end{cases} \tag{2.4}$$

where $\underline{p}^- = \min(0, \underline{p})$. If $R > 0$ and $DD' > 0$, then $|(R+\gamma(\gamma-1)DD'\underline{p}^-/2)^{-1}| \leq |R^{-1}|$ and $|(R+\gamma(\gamma-1)DD'\underline{p}^-/2)^{-1}| \leq |(\gamma(\gamma-1)DD'\underline{p}^-/2)^{-1}|$ if $\underline{p} \neq 0$, and thus by Lemma 2.1 of [10] (due to Koblyanski) equation (2.4) admits a solution $(\underline{p}(\cdot), \underline{z}(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$. Since equation

$$\begin{cases} d\underline{q} + \bar{a}\underline{q}dt + \bar{c}'\underline{y}dt + \dfrac{1}{4}k'\left\{R^{-1} - \left[R + \dfrac{\gamma(\gamma-1)}{2}DD'\underline{q}^-\right]^{-1}\right\}kdt \\ -\dfrac{1}{2}k'\left[R + \dfrac{\gamma(\gamma-1)}{2}DD'\underline{q}^-\right]^{-1}(\bar{b}\underline{q}^- + \gamma D\underline{y})dt \\ -\dfrac{1}{4}(\bar{b}\underline{q}^- + \gamma D\bar{y})'\left[R + \dfrac{\gamma(\gamma-1)}{2}DD'\underline{q}^-\right]^{-1}(\bar{b}\underline{q}^- + \gamma D\bar{y})dt - \underline{y}'dW = 0, \\ \underline{q}(T) = 0 \quad a.s., \end{cases}$$

admits a solution $(\underline{q}(\cdot), \underline{y}(\cdot)) = (0,0)$, it follows from the comparison result in Lemma 2.1 (ii) of [10] that $\underline{p} \leq 0$ under the stated assumptions, and thus the pair $(\underline{p}(\cdot), \underline{z}(\cdot))$ is a solution to (2.4).

(iv) The proof follows closely that of part *(iii)* (instead of $\underline{p}^-$ use $\underline{p}^+ := max(0, p)$), and is thus omitted. $\square$

We restrict the set of admissible controls to those that ensure $x(t) > 0$ $\forall t \in [0,T]$ *a.s.*. This is due to our assumption of $\gamma \in \mathbb{R}$ and the weight $x^{\gamma-2}(t)$ appearing in the cost functional $J(u(\cdot))$. This is also desirable in applications such as optimal investment where $x(\cdot)$ represents investors wealth.

Since we use a certain *completion of squares* method to solve the control problem, we need the following *integrability* condition:

$$\mathbb{E}\int_0^T \{p(t)\gamma[x^\gamma(t)c'(t) + x^{\gamma-1}(t)u'(t)D(t)] + x^\gamma(t)z'(t)\}dW(t) = 0. \quad (2.5)$$

The admissible set of controls and the corresponding optimal control problem to be considered are defined, respectively, as:

$\mathcal{A} := \{u(\cdot) \in L^0_{\mathcal{F}}(0,T;\mathbb{R}^m) : (1.1)$ has a unique strong positive solution and (2.5) holds$\}$,

$$\begin{cases} \min_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \\ s.t. \quad (1.1). \end{cases} \tag{2.6}$$

Theorem 2. *Let $u^*(t) := g(t)x(t)$. If $u^*(\cdot) \in \mathcal{A}$, then it is the unique solution to (2.6). The corresponding optimal cost is $J(u^*(\cdot)) = p(0)x_0^\gamma$.*

Proof. We write (2.1) as:

$$dp(t) = p_0(t)dt + z'(t)dW(t),$$

with the definition of $p_0(\cdot)$ being clear from (2.1). For all $u(\cdot) \in \mathcal{A}$ we have

$$\begin{aligned}
\mathbb{E}[sx^\gamma(T)] &= p(0)x_0^\gamma \\
&+ \mathbb{E}\int_0^T x^\gamma(t)\left\{p_0(t) + p(t)\left[\gamma a(t) + \frac{1}{2}\gamma(\gamma-1)c'(t)c(t)\right] + \gamma c'(t)z(t)\right\}dt \\
&+ \mathbb{E}\int_0^T x^{\gamma-1}(t)u'(t)\left\{p(t)[\gamma b(t) + \gamma(\gamma-1)D(t)c(t)] + \gamma D(t)z(t)\right\}dt \\
&+ \mathbb{E}\int_0^T x^{\gamma-2}(t)u'(t)p(t)\frac{\gamma(\gamma-1)}{2}D(t)D'(t)u(t)dt,
\end{aligned}$$

and thus we can write the cost functional $J(u(\cdot))$ as:

$$\begin{aligned}
J(u(\cdot)) &= p(0)x_0^\gamma \\
&+ \mathbb{E}\int_0^T x^\gamma(t)\left\{q(t)p_0(t) + p(t)\left[\gamma a(t) + \frac{1}{2}\gamma(\gamma-1)c'(t)c(t)\right] + \gamma c'(t)z(t)\right\}dt \\
&+ \mathbb{E}\int_0^T x^{\gamma-1}(t)u'(t)\left\{k(t) + p(t)[\gamma b(t) + \gamma(\gamma-1)D(t)c(t)] + \gamma D(t)z(t)\right\}dt \\
&+ \mathbb{E}\int_0^T x^{\gamma-2}(t)u'(t)\left[R(t) + p(t)\frac{\gamma(\gamma-1)}{2}D(t)D'(t)\right]u(t)dt,
\end{aligned}$$

which after completion of squares with respect to $u(\cdot)$ becomes

$$\begin{aligned}
J(u(\cdot)) &= p(0)x_0^\gamma + \mathbb{E}\int_0^T x^{\gamma-2}(t)[u(t) - g(t)x(t)]' \\
&\times \left[R(t) + p(t)\frac{\gamma(\gamma-1)}{2}D(t)D'(t)\right][u(t) - g(t)x(t)]dt \\
&\geq p(0)x_0^\gamma.
\end{aligned}$$

This lower bound is achieved if and only if $u(t) = u^*(t)$ a.e. $t \in [0,T]$ a.s..□

Note that in the above proof it is not necessary to assume that the solution pair of (2.1) is unique. However, such a uniqueness is a consequence of the remaining assumptions of the above theorem, as the following result shows (the proof is omitted as it is almost identical to that of Theorem 3.2 of [10]).

Corollary 1. *Let the generalised Riccati equation (2.1) have two solution pairs* $(p_i(\cdot), z_i(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, $i = 1,2$, *such that for the corresponding controls* $u_i^*(\cdot)$, $i = 1,2$, *the state equation (1.1) has a unique strong positive solution* $x_i^*(\cdot)$, $i = 1,2$, *and the condition (2.5) holds. Then* $p_1(t) = p_2(t)$ $\forall t \in [0,T]$ *a.s., and* $z_1(t) = z_2(t)$ *a.e.* $t \in [0,T]$ *a.s..*

We now consider an *infinite* horizon version of (2.6), and thus assume that coefficients a, b, c, D, q, k, R, are constant for the reminder of this section. Our cost functional now is:

$$J_\infty(u(\cdot)) := \mathbb{E}\int_0^\infty x^{\gamma-2}(t)[qx^2(t) + x(t)k'u(t) + u'(t)Ru(t)]dt.$$

The following *algebraic* Riccati equation appears in the solution to the problem of minimizing $J_\infty(u(\cdot))$ subject to (1.1):

$$\begin{cases} \gamma ap + \dfrac{\gamma(\gamma-1)}{2}c'cp + q - g'\left[R + \dfrac{\gamma(\gamma-1)}{2}DD'p\right]g = 0, \\[2ex] \gamma(a + g'b) + \dfrac{1}{2}\gamma(\gamma-1)(c' + g'D)(c + D'g) \le 0, \\[2ex] R + \dfrac{\gamma(\gamma-1)}{2}DD'p > 0, \end{cases} \tag{2.7}$$

where the constant controller gain g is defined as:

$$g := -\frac{1}{2}\left[R + \frac{\gamma(\gamma-1)}{2}DD'p\right]^{-1}\{k + [\gamma b + \gamma(\gamma-1)Dc]\,p\}.$$

Assumption 2. *Equation (2.7) admits a unique solution.*

Equation (2.7) is similar to the corresponding one of LQ control, and contains it as a special case for $\gamma = 2$. It can be reformulated in terms of linear matrix inequalities (LMIs), and thus the validity of Assumption 2 be checked numerically as in, e.g., [27], [28].

As in the case of a finite time horizon, we restrict controls to the ones that ensure $x(t) > 0$ $a.s.$ $\forall t \in [0, \infty)$, and satisfy the integrability condition

$$\mathbb{E} \int_0^\infty p\gamma[x^\gamma(t)c' + x^{\gamma-1}(t)u'(t)D]dW(t) = 0. \tag{2.8}$$

We further require the following *stability* condition to hold:

$$\lim_{t\to\infty} \mathbb{E}[x^\gamma(t)] = \eta \geq 0, \quad \forall x_0 > 0. \tag{2.9}$$

The admissible set of controls and the corresponding optimal control problem to be solved are thus defined, respectively, as:

$$\begin{aligned} \mathcal{A}_\infty \;\; := \;\; & \{u(\cdot) \in L^0_{\mathcal{F}}(0, \infty; \mathbb{R}^m) : \text{(1.1) has a unique, strong, positive solution} \\ & \text{that satisfies (2.8) and (2.9)}\}. \end{aligned}$$

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}_\infty} J_\infty(u(\cdot)), \\ \\ s.t. \quad (1.1). \end{cases} \tag{2.10}$$

Theorem 3. *Let $u^*_\infty(t) := gx(t)$. If $u^*_\infty(\cdot) \in \mathcal{A}_\infty$, then it is the unique solution to (2.10). The corresponding optimal cost is $J_\infty(u^*_\infty(\cdot)) = p(x_0^\gamma - \eta)$.*

Proof. For all $u(\cdot) \in \mathcal{A}_\infty$ it holds that:

$$\begin{aligned} p\eta &= px_0^\gamma + \mathbb{E}\int_0^\infty x^\gamma(t)p\left[\gamma a + \frac{1}{2}\gamma(\gamma-1)c'c\right]dt \\ &+ \mathbb{E}\int_0^\infty x^{\gamma-1}(t)u'(t)p[\gamma b + \gamma(\gamma-1)Dc]dt \\ &+ \mathbb{E}\int_0^\infty x^{\gamma-2}(t)u'(t)p\frac{\gamma(\gamma-1)}{2}DD'u(t)dt. \end{aligned}$$

The cost $J_\infty(u(\cdot))$ can now be written as

$$\begin{aligned} J_\infty(u(\cdot)) &= p(x_0^\gamma - \eta) + \mathbb{E}\int_0^\infty x^\gamma(t)\left\{q + p\left[\gamma a + \frac{1}{2}\gamma(\gamma-1)c'c\right]\right\}dt \\ &+ \mathbb{E}\int_0^\infty x^{\gamma-1}(t)u'(t)\left\{k + p[\gamma b + \gamma(\gamma-1)Dc]\right\}dt \\ &+ \mathbb{E}\int_0^\infty x^{\gamma-2}(t)u'(t)\left[R + p\frac{\gamma(\gamma-1)}{2}DD'\right]u(t)dt, \end{aligned}$$

which after completion of squares with respect to $u(\cdot)$ becomes

$$\begin{aligned} J_\infty(u(\cdot)) &= p(x_0^\gamma - \eta) + \mathbb{E}\int_0^\infty x^{\gamma-2}(t)[u(t) - gx(t)]' \\ &\times \left[R + p\frac{\gamma(\gamma-1)}{2}DD'\right][u(t) - gx(t)]dt \\ &\geq p(x_0^\gamma - \eta). \end{aligned}$$

This lower bound is achieved if and only if $u(t) = u_\infty^*(t)$ a.e. $t \in [0, \infty)$ a.s..□

3. Risk-sensitive regulator

The *risk-sensitive* control problem was introduced by Jacobson [29] as the optimal control of a linear stochastic system with *additive* noise and an *exponential* of quadratic forms as a cost functional. Jacobson found an explicit closed-form solution in a linear state-feedback form for the case of full observation, whereas for the case of partial observations see, e.g., [30], [31], for discrete-time systems see, e.g., [32], [33], and for the connection with robust control see [34]. The risk-sensitive *maximum principle* for systems with multiplicative noise is given in [35], [36]. Despite the availability of these general methods, the analog to Jacobson's result in the setting of random coefficients and multiplicative noise remains *open*. A partial result in this direction is [37], where the optimal investment problem with exponential utility is considered in a market with random coefficients, which corresponds to an *exponential-linear* cost. The recent paper [38] is another partial result of a system with multiplicative noise and exponential-linear cost.

In this section, we introduce a certain risk-sensitive version of the optimal regulator problem (2.6), and find its explicit closed-form solution. It turns out that the optimal control is of a *nonlinear* state-feedback form. This sheds light into difficulties in finding an analog to Jacobson's result in the setting of random coefficients and multiplicative noise.

We first extend the state equation (1.1) to include another *scalar* state as:

$$\begin{cases} dx(t) = [a(t)x(t) + u'(t)b(t)]dt + [c'(t)x(t) + u'(t)D(t)]dW(t), \\ dy(t) = [\alpha(t)y(t) + v'(t)\beta(t)]dt + [\lambda'(t)y(t) + v'(t)\Sigma(t)]dW(t), \\ x(0) = x_0 > 0 \quad \text{and} \quad y(0) = y_0 \in \mathbb{R} \quad \text{are given.} \end{cases} \tag{3.1}$$

The *exponential* cost functional to be considered is:

$$\begin{aligned} I(u(\cdot), v(\cdot)) \quad := \quad & \mathbb{E}\left\{\exp\left\{\int_0^T x^{\gamma-2}(t)[q(t)x^2(t) + x(t)u'(t)g(t) + u'(t)R(t)u(t)]dt\right.\right. \\ & + \quad \left.\left.\int_0^T [\theta(t)y(t) + \rho'(t)v(t)]dt + sx^{\gamma}(T) + \mu y(T)\right\}\right\}. \end{aligned}$$

Unless otherwise stated, we assume:

$$\begin{aligned} &\alpha(\cdot), \theta(\cdot) \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}), \quad \lambda(\cdot) \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}^d), \quad \Sigma(\cdot) \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}^{n\times d}), \\ &\beta(\cdot), \rho(\cdot) \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}^n), \quad 0 \neq \mu \in L^{\infty}_{\mathcal{F}(T)}(\Omega; \mathbb{R}), \\ &\Sigma(t)\Sigma'(t) > 0 \quad a.e \quad t \in [0, T] \quad a.s.. \end{aligned}$$

The motivation for introducing state $y(\cdot)$, and for it to appear linearly in the cost functional, is that it enables an explicit solution, and it is this case that has an immediate application (see §4.2 below). The derivation of solution to the problem of minimising $I(u(\cdot), v(\cdot))$ subject to (3.1) proceeds similarly to the previous section, although it is considerably more involved. In particular, we now have *three* BSDEs to consider. The following is the first of those BSDEs, and it depends only on the coefficients associated with state $y(\cdot)$.

$$
\begin{cases}
dp_y(t) + \theta(t)dt - \lambda'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}\rho(t)dt \\
+[\alpha(t) - \lambda'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}\beta(t)]p_y(t)dt \\
+[\lambda'(t) - \beta'(t)[\Sigma(t)\Sigma'(t)]^{-1}\Sigma(t) - \lambda'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}]z_y(t)dt \\
-p_y^{-1}(t)z_y'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}\rho(t)dt \\
-p_y^{-1}(t)z_y'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}\Sigma(t)z_y(t)dt - z_y'(t)dW(t) = 0, \\
p_y(t) \neq 0 \quad \forall t \in [0,T) \quad a.s., \\
p_y(T) = \mu \quad a.s..
\end{cases}
\tag{3.2}
$$

This BSDE, due to the term $-p_y^{-1}(t)z_y'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}\rho(t)dt$, is not of the usual Riccati type, but it reduces to it if $\rho(t) = 0$. We further define the process

$$
\ell^*(t) := -p_y^{-1}(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)], \quad t \in [0,T].
$$

Assumption 3. *Equation (3.2) admits a unique solution pair* $(p_y(\cdot), z_y(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ *such that*

$$
\mathbb{E}\left\{\exp\left\{-\frac{1}{2}\int_0^T [\ell^*(\tau)]'\ell^*(\tau)d\tau + \int_0^T [\ell^*(\tau)]'dW(\tau)\right\}\right\} = 1.
$$

If the coefficients associated with the state $y(\cdot)$ are all deterministic with $\rho(t) = 0$, then equation (3.2) is an ordinary linear differential equation and Assumption 3 holds. Our second BSDE is linear with coefficients depending on $\ell^*(\cdot)$:

$$
\begin{cases}
dp_\ell(t) - z_\ell'(t)\ell^*(t)dt - \frac{1}{2}[\ell^*(t)]'\ell^*(t)dt - z_\ell'(t)dW(t) = 0, \\
p_\ell(T) = 0 \quad a.s..
\end{cases}
\tag{3.3}
$$

Assumption 4. *Equation (3.3) admits a unique solution pair* $(p_\ell(\cdot), z_\ell(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

The third BSDE is of a Riccati type considered in the previous section, however, its coefficients depend on the pair $(p_y(\cdot), z_y(\cdot))$, and thus may be unbounded.

$$
\begin{cases}
dp_x(t) + q(t)dt + \left[\gamma a(t) + \dfrac{\gamma(\gamma-1)}{2} c'(t)c(t)\right] p_x(t)dt \\
-p_y^{-1}(t)\gamma c'(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]p_x(t)dt \\
+\gamma c'(t)z_x(t) - z_x'(t)p_y^{-1}(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]dt \\
-\dfrac{1}{4}\{k(t) + p_x(t)\gamma b(t) + p_x(t)\gamma(\gamma-1)D(t)c(t) + \gamma D(t)z_x(t) \\
- p_x(t)p_y^{-1}(t)\gamma D(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]\}' \\
\times \left[R(t) + \dfrac{\gamma(\gamma-1)}{2} D(t)D'(t)p_x(t)\right]^{-1} \\
\times \{k(t) + p_x(t)\gamma b(t) + p_x(t)\gamma(\gamma-1)D(t)c(t) + \gamma D(t)z_x(t) \\
- p_x(t)p_y^{-1}(t)\gamma D(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]\} dt - z_x'(t)dW(t) = 0, \\
R(t) + \dfrac{\gamma(\gamma-1)}{2} D(t)D'(t)p_x(t) > 0 \quad a.e. \quad t \in [0,T] \quad a.s., \\
p_x(T) = s \quad a.s..
\end{cases}
\tag{3.4}
$$

Assumption 5. *Equation (3.4) admits a unique solution pair* $(p_x(\cdot), z_x(\cdot)) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

Before we proceed further, let us remark that the last two BSDEs are with possibly *unbounded* coefficients. In the case of linear equation (3.3), and (3.4) with $\gamma = 0$, the results of [39], [40], [41], can be used to derive sufficient conditions for solvability. On the other hand, we are not aware of general existence results for Riccati BSDE with unbounded coefficients. Some special cases are known (see, e.g., [42], [39]), and they also appear in [43]. The simplest case where this unboundedness issue does no arise, and our conclusions remain novel, is when system and cost coefficients are deterministic.

We can now define the admissible set, and thus consider:

$$\begin{aligned}
\ell'(t) &:= z'_\ell(t) + x^\gamma(t)[p_x(t)\gamma c'(t) + z'_x(t)] + p_x(t)x^{\gamma-1}(t)u'(t)\gamma D(t) \\
&+ y(t)[z'_y(t) + p_y(t)\lambda'(t)] + p_y(t)v'(t)\Sigma(t), \quad t \in [0,T] \\
M(t) &:= \exp\left\{-\frac{1}{2}\int_0^t \ell'(\tau)\ell(\tau)d\tau + \int_0^t \ell'(\tau)dW(\tau)\right\}, \quad t \in [0,T].
\end{aligned}$$

The admissible set of controls and the corresponding optimal control problem to be solved are, respectively:

$$\begin{aligned}
\mathcal{U} := \{&(u(\cdot),v(\cdot)) \in L^0_{\mathcal{F}}(0,T;\mathbb{R}^m) \times L^0_{\mathcal{F}}(0,T;\mathbb{R}^n) : (3.1) \text{ has a unique strong solution,} \\
&x(t) > 0 \ \forall t \in [0,T] \text{ a.s, and } \mathbb{E}[M(T)] = 1\}.
\end{aligned}$$

$$\begin{cases} \min\limits_{(u(\cdot),v(\cdot))\in\mathcal{U}} I(u(\cdot),v(\cdot)), \\ s.t. \quad (3.1). \end{cases} \tag{3.5}$$

The following are the gain processes that define the *candidates* for the optimal controls $u^*(\cdot)$ and $v^*(\cdot)$, also defined below.

$$\begin{aligned}
g_x(t) &:= -\frac{1}{2}\left[R(t) + \frac{\gamma(\gamma-1)}{2}D(t)D'(t)p_x(t)\right]^{-1} \\
&\times \{k(t) + p_x(t)\gamma b(t) + p_x(t)\gamma(\gamma-1)D(t)c(t) + \gamma D(t)z_x(t) \\
&- p_x(t)p_y^{-1}(t)\gamma D(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]\}, \\
g'_\ell(t) &:= [\ell^*(t) - z_\ell(t)]'p_y^{-1}(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}, \\
g'_y(t) &:= -[p_y^{-1}(t)z'_y(t) + \lambda'(t)]\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}, \\
g'_{yx}(t) &:= -p_y^{-1}(t)[p_x(t)\gamma c'(t) + p_x(t)g'_x(t)\gamma D(t) + z'_x(t)]\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}.
\end{aligned}$$

$$
\begin{aligned}
u^*(t) &:= g_x(t)x(t), \\
v^*(t) &:= g_\ell(t) + g_y(t)y(t) + g_{yx}(t)x^\gamma(t).
\end{aligned}
$$

Theorem 4. *If $(u^*(\cdot), v^*(\cdot)) \in \mathcal{U}$, then they are the unique solution to (3.5). The corresponding optimal cost is:*

$$
I(u^*(\cdot), v^*(\cdot)) = \exp[p_x(0)x_0^\gamma + p_y(0)y_0 + p_\ell(0)].
$$

Proof. It is convenient to write equations (3.4), (3.2), (3.3), respectively, as:

$$
\begin{aligned}
dp_x(t) &= p_1(t)dt + z_x'(t)dW(t), \\
dp_y(t) &= p_2(t)dt + z_y'(t)dW(t), \\
dp_\ell(t) &= p_3(t)dt + z_\ell'(t)dW(t),
\end{aligned}
$$

where the definitions of $p_1(\cdot), p_2(\cdot), p_3(\cdot)$, are clear from (3.4), (3.2), (3.3), respectively. Since

$$
\begin{aligned}
sx^\gamma(T) &= p_x(0)x_0^\gamma \\
&+ \int_0^T x^\gamma(t)\left[p_x(t)\gamma a(t) + \frac{p_x(t)}{2}\gamma(\gamma-1)c'(t)c(t) + p_1(t) + \gamma c'(t)z_x(t)\right]dt \\
&+ \int_0^T x^{\gamma-1}(t)u'(t)\left[p_x(t)\gamma b(t) + p_x(t)\gamma(\gamma-1)D(t)c(t) + \gamma D(t)z_x(t)\right]dt \\
&+ \int_0^T \frac{p_x(t)}{2}\gamma(\gamma-1)x^{\gamma-2}(t)u'(t)D(t)D'(t)u(t)dt \\
&+ \int_0^T \left[p_x(t)\gamma x^\gamma(t)c'(t) + p_x(t)x^{\gamma-1}(t)u'(t)\gamma D(t) + x^\gamma(t)z_x'(t)\right]dW(t), \\
\mu y(T) &= p_y(0)y_0 + \int_0^T y(t)[p_2(t) + \alpha(t)p_y(t) + \lambda'(t)z_y(t)]dt \\
&+ \int_0^T v'(t)[p_y(t)\beta(t) + \Sigma(t)z_y(t)]dt \\
&+ \int_0^T [y(t)z_y'(t) + y(t)p_y(t)\lambda'(t) + p_y(t)v'(t)\Sigma(t)]dW(t), \\
0 &= p_\ell(0) + \int_0^T p_3(t)dt + \int_0^T z_\ell(t)dW(t),
\end{aligned}
$$

we can write the cost functional $I(u(\cdot), v(\cdot))$ as:

$$
\begin{aligned}
&I(u(\cdot), v(\cdot)) = \mathbb{E}\{\exp\{p_x(0)x_0^\gamma + p_y(0)y_0 + p_\ell(0) \\
+ &\int_0^T x^\gamma(t)\left[p_x(t)\gamma a(t) + \frac{p_x(t)}{2}\gamma(\gamma-1)c'(t)c(t) + p_1(t) + \gamma c'(t)z_x(t) + q(t)\right]dt \\
- &\int_0^T x^\gamma(t)p_y^{-1}(t)[p_x(t)\gamma c'(t) + z_x'(t)]\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]dt \\
+ &\int_0^T x^{\gamma-1}(t)u'(t)\left[k(t) + p_x(t)\gamma b(t) + p_x(t)\gamma(\gamma-1)D(t)c(t) + \gamma D(t)z_x(t)\right]dt \\
- &\int_0^T x^{\gamma-1}u'(t)p_y^{-1}(t)p_x(t)\gamma D(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]dt \\
+ &\int_0^T x^{\gamma-2}(t)u'(t)\left[R(t) + \frac{p_x(t)}{2}\gamma(\gamma-1)D(t)D'(t)\right]u(t)dt \\
\\
+ &\int_0^T y(t)[\theta(t) + p_2(t) + \alpha(t)p_y(t) + \lambda'(t)z_y(t)]dt \\
- &\int_0^T y(t)[p_y^{-1}(t)z_y'(t) + \lambda'(t)]\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]dt \\
+ &\int_0^T [p_3(t) - z_\ell'(t)\ell^*(t)]dt \\
+ &\int_0^T \left\{\frac{\ell'(t)\ell(t)}{2} + \ell'(t)p_y^{-1}(t)\Sigma'(t)[\Sigma(t)\Sigma'(t)]^{-1}[\rho(t) + p_y(t)\beta(t) + \Sigma(t)z_y(t)]\right\}dt, \\
- &\int_0^T \frac{\ell'(t)\ell(t)}{2} + \int_0^T \ell(t)dW(t)\Big\}\Big\}.
\end{aligned}
$$

The completion of squares with respect to $u(\cdot)$ and $\ell(\cdot)$ results in:

$$\begin{aligned}
I(u(\cdot),v(\cdot)) &= \mathbb{E}\{\exp\{p_x(0)x_0^\gamma + p_y(0)y_0 + p_\ell(0) \\
&+ \int_0^T x^{\gamma-2}[u(t)-g_x(t)x(t)]'\left[R+\frac{p_x}{2}\gamma(\gamma-1)D(t)D'(t)\right][u(t)-g_x(t)x(t)]dt \\
&+ \int_0^T \frac{1}{2}[\ell(t)+\ell^*(t)]'[\ell(t)+\ell^*(t)]dt \\
&- \int_0^T \frac{\ell'(t)\ell(t)}{2} + \int_0^T \ell(t)dW(t)\Big\}\Big\} \\
&\geq \exp\left[p_x(0)x_0^\gamma + p_y(0)y_0 + p_\ell(0)\right]\mathbb{E}\left\{\exp\left[-\int_0^T \frac{\ell'(t)\ell(t)}{2} + \int_0^T \ell(t)dW(t)\right]\right\} \\
&= \exp\left[p_x(0)x_0^\gamma + p_y(0)y_0 + p_\ell(0)\right].
\end{aligned}$$

This lower bound is achieved if and only if $u(t) = u^*(t)$ a.e. $t \in [0,T]$ a.s., and $v(t) = v^*(t)$ a.e. $t \in [0,T]$ a.s.. □

We now formulate and solve an infinite horizon version of (3.5). Thus, let all the coefficients be constant and consider the criterion

$$\begin{aligned}
I_\infty(u(\cdot),v(\cdot)) &:= \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left\{\exp\left\{\int_0^T x^{\gamma-2}(t)[qx^2(t)+x(t)u'(t)k+u'(t)Ru(t)]dt\right.\right. \\
&+ \left.\left.\int_0^T [\theta y(t)+v'(t)\rho]dt\right\}\right\}.
\end{aligned}$$

We assume $\theta - \lambda'\Sigma'(\Sigma\Sigma')^{-1}\rho \neq 0$, $\alpha - \lambda'\Sigma'(\Sigma\Sigma')^{-1}\beta \neq 0$, and define

$$p_y := [\theta - \lambda'\Sigma'(\Sigma\Sigma')^{-1}\rho][\alpha - \lambda'\Sigma'(\Sigma\Sigma')^{-1}\beta]^{-1}.$$

The following Riccati algebraic equation will appear in the solution to the problem of minimizing $I_\infty(u(\cdot),v(\cdot))$ subject to (3.1):

$$\begin{cases} q+\left[\gamma a+\dfrac{\gamma(\gamma-1)}{2}c'c-p_y^{-1}\gamma c'\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)\right]p_x \\ -\dfrac{1}{4}\left[k+p_x\gamma b+p_x\gamma(\gamma-1)Dc-p_xp_y^{-1}\gamma D\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)\right]' \\ \times\left[R+\dfrac{\gamma(\gamma-1)}{2}DD'p_x\right]^{-1} \\ \times\left[k+p_x\gamma b+p_x\gamma(\gamma-1)Dc-p_xp_y^{-1}\gamma D\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)\right]=0, \\ R+\dfrac{\gamma(\gamma-1)}{2}DD'p_x>0, \\ p_x\geq 0. \end{cases} \tag{3.6}$$

Assumption 6. *There exists a solution to (3.6).*

Similarly to the case of algebraic Riccati equation of LQ control, (3.6) can be reformulated in terms of LMIs and the validity of Assumption 6 checked numerically. In order to define the admissible set of controls, we introduce the processes:

$$\begin{aligned} f'(t) &:= p_x\gamma x^{\gamma-1}(t)[c'x(t)+u'(t)D]+p_y[\lambda'y(t)+v'(t)\Sigma], \quad t\in[0,\infty), \\ N(t) &:= \exp\left\{-\frac{1}{2}\int_0^t f'(\tau)f(\tau)d\tau+\int_0^t f'(\tau)dW(\tau)\right\} \quad t\in[0,\infty). \end{aligned}$$

If $\mathbb{E}[N(t)]=1\ \forall t>0$, then

$$\widetilde{\mathbb{P}}_{f,t}(A):=\int_A N(t,\omega)d\mathbb{P}(\omega), \quad \forall A\in\mathcal{F}.$$

defines a probability measure. The stability condition that we impose on the controls is:

$$\lim_{T\to\infty}\frac{1}{T}\log\widetilde{\mathbb{E}}_{f,T}\left[e^{-p_xx^\gamma(T)-p_yy(T)}\right]=\kappa\geq 0, \quad \forall x_0>0, y_0\in\mathbb{R}, \tag{3.7}$$

where $\widetilde{\mathbb{E}}_{f,T}[\cdot]$ is the expectation under $\widetilde{\mathbb{P}}_{f,T}$. The admissible set of controls and the corresponding optimal control problem to be solved are defined as:

$$\begin{aligned}\mathcal{U}_\infty \;:=\; & \{(u(\cdot),v(\cdot)) \in L^0_{\mathcal{F}}(0,\infty;\mathbb{R}^m) \times L^0_{\mathcal{F}}(0,\infty;\mathbb{R}^n) : (3.1) \text{ has a unique strong} \\ & \text{solution } x(t) > 0 \ \forall t \in [0,\infty) \text{ a.s; } \mathbb{E}[N(T)] = 1 \ \forall T \in [0,\infty); \text{ and (3.7) holds}\}.\end{aligned}$$

$$\begin{cases} \min\limits_{(u(\cdot),v(\cdot))\in\mathcal{U}_\infty} I_\infty(u(\cdot),v(\cdot)), \\ \\ s.t. \quad (3.1). \end{cases} \tag{3.8}$$

The following gains will be used to define the optimal controls:

$$\begin{aligned} f^* &:= -\Sigma'(\Sigma\Sigma')^{-1}p_y^{-1}(\rho + p_y\beta), \\ g_x &:= -\frac{1}{2}\left[R + \frac{\gamma(\gamma-1)}{2}DD'p_x\right]^{-1} \\ &\quad \times \ [k + p_x\gamma b + p_x\gamma(\gamma-1)Dc - p_x p_y^{-1}\gamma D\Sigma'(\Sigma\Sigma')^{-1}(\rho + p_y\beta)], \\ g_f' &:= (f^*)'p_y^{-1}\Sigma'(\Sigma\Sigma')^{-1}, \quad g_y' := -\lambda'\Sigma'(\Sigma\Sigma')^{-1}, \\ g_{yx}' &:= -p_y^{-1}p_x\gamma(c' + g_x'D)\Sigma'(\Sigma\Sigma')^{-1}. \end{aligned}$$

Theorem 5. *Let $u^*_\infty(t) := g_x x(t)$ and $v^*_\infty(t) := g_f + g_y y(t) + g_{yx}x^\gamma(t)$. If $(u^*_\infty(\cdot), v^*_\infty(\cdot)) \in \mathcal{U}_\infty$, then they are the unique solution to (3.8). The corresponding optimal cost is $I_\infty(u^*_\infty(\cdot), v^*_\infty(\cdot)) = \kappa - (f^*)'f^*/2$.*

Proof. Since

$$\begin{aligned} p_x x^\gamma(T) &= p_x x^\gamma(0) \\ &+ \int_0^T \left\{ p_x\gamma x^{\gamma-1}(t)[ax(t) + u'(t)b] + \frac{p_x}{2}\gamma(\gamma-1)x^{\gamma-2}[c'x(t) + u'(t)D][cx(t) + D'u(t)] \right\} dt \\ &+ \int_0^T p_x\gamma x^{\gamma-1}(t)[c'x(t) + u'(t)D]dW(t), \\ p_y y(T) &= p_y y_0 + \int_0^T p_y[\alpha y(t) + u'(t)\beta]dt + \int_0^T p_y[\lambda' y(t) + v'(t)\Sigma]dW(t), \end{aligned}$$

we can write the cost functional $I_\infty(u(\cdot),v(\cdot))$ as:

$$\begin{aligned}
I_\infty(u(\cdot),v(\cdot)) \quad &:= \quad \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left\{\exp\left\{p_x x_0^\gamma - p_x x^\gamma(T) + p_y y_0 - p_y y(T)\right.\right.\\
&+ \quad \int_0^T x^\gamma(t)\left[q + p_x\gamma a + \frac{p_x}{2}\gamma(\gamma-1)c'c - p_y^{-1}p_x\gamma c'\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)\right]dt\\
&+ \quad \int_0^T x^{\gamma-1}(t)u'(t)\left[k + p_x\gamma b + \frac{p_x}{2}\gamma(\gamma-1)Dc - p_y^{-1}p_x\gamma D\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)\right]dt\\
&+ \quad \int_0^T x^{\gamma-2}(t)u'(t)\left[R + \frac{p_x}{2}\gamma(\gamma-1)DD'\right]u(t)dt\\
&+ \quad \int_0^T y(t)[\theta + \alpha p_y - \lambda'\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)]dt\\
&+ \quad \int_0^T \left[\frac{1}{2}f'(t)f(t) + f'(t)p_y^{-1}\Sigma'(\Sigma\Sigma')^{-1}(\rho+p_y\beta)\right]dt\\
&- \quad \left.\left.\int_0^T \frac{1}{2}f'(t)f(t)dt + \int_0^T f(t)dW(t)\right\}\right\},
\end{aligned}$$

which after completion of squares with respect to $u(\cdot)$ and $f(\cdot)$ becomes

$$\begin{aligned}
I_\infty(u(\cdot),v(\cdot)) \quad &:= \quad \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left\{\exp\left\{p_x x_0^\gamma - p_x x^\gamma(T) + p_y y_0 - p_y y(T) - \frac{1}{2}(f^*)'f^*T\right.\right.\\
&+ \quad \int_0^T x^{\gamma-2}(t)[u(t)-g_x x(t)]'\left[R + \frac{p_x}{2}\gamma(\gamma-1)DD'\right][u(t)-g_x x(t)]dt\\
&+ \quad \int_0^T \frac{1}{2}[f(t)+f^*]'[f(t)+f^*]dt\\
&- \quad \left.\left.\int_0^T \frac{1}{2}f'(t)f(t)dt + \int_0^T f(t)dW(t)\right\}\right\},\\
&\geq \quad -\frac{1}{2}(f^*)'f^* + \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left\{\exp\left\{-p_x x^\gamma(T) - p_y y(T)\right.\right.\\
&- \quad \left.\left.\int_0^T \frac{1}{2}f'(t)f(t)dt + \int_0^T f(t)dW(t)\right\}\right\}
\end{aligned}$$

$$= -\frac{1}{2}(f^*)'f^* + \lim_{T\to\infty}\frac{1}{T}\log\widetilde{\mathbb{E}}_{f,T}\left[e^{-p_x x^\gamma(T) - p_y y(T)}\right] = -\frac{1}{2}(f^*)'f^* + \kappa.$$

This lower bound is achieved if and only if $u(t) = u^*_\infty(t)$ a.e. $t \in [0,\infty)$ a.s., and $v(t) = v^*_\infty(t)$ a.e. $t \in [0,\infty)$ a.s.. □

Let us finally remark that the case of $\gamma = 2$ is special as it permits a multidimensional generalisation, i.e. with both $x(\cdot)$ and $y(\cdot)$ permitted to be vectors. Such derivations are the same as in this section, and are thus omitted.

4. Applications to optimal investment

In this section, we apply our results to the problem of optimal investment. Thus, consider a market of $m + 1$ assets the prices of which satisfy the following equations (see, for example, [21], [22], [23]):

$$\begin{cases} dS_0(t) = S_0(t)r(t)dt, \\ \\ dS_i(t) = S_i(t)\left\{\mu_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\right\}, \quad i = 1, ..., m, \\ \\ S_i(0) > 0 \text{ is given for all } i = 0, ..., m. \end{cases}$$

Here the *interest rate* $r(\cdot)$, the *appreciation rates* $\mu_i(\cdot)$, and the *volatilities* $\sigma_{ij}(\cdot)$, are assumed to be $\mathcal{F}(t)$-adapted and bounded processes. If we define

$$\begin{aligned} B'(t) &:= [\mu_1(t) - r(t), ..., \mu_m(t) - r(t)], \\ \sigma_i(t) &:= [\sigma_{1i}(t), \sigma_{2i}(t), ..., \sigma_{mi}(t)], \quad i = 1, ..., d, \\ \sigma(t) &:= [\sigma_1'(t), \sigma_2'(t), ..., \sigma_d'(t)], \end{aligned}$$

then the equation describing the value $Y(t)$ of a *self-financing portfolio* is

$$\begin{cases} dY(t) = [r(t)Y(t) + u'(t)B(t)]dt + u'(t)\sigma(t)dW(t), \\ \\ Y(0) > 0 \quad \text{is given}, \end{cases} \tag{4.1}$$

where $Y(0)$ the investors *initial wealth*, whereas the component i of vector $u(\cdot)$ is the value of the holdings in asset $S_i(\cdot)$. The *optimal investment problem* is the optimal control of $Y(\cdot)$ for some suitable cost functional.

4.1. Risk-return criterion

In [20], the following *risk-return* criterion was introduced:

$$H(u(\cdot)) := \mathbb{E}\left[\int_0^T \frac{1}{Y(t)} u'(t)\Gamma(t)u(t)dt - Y(T)\right].$$

Here the *return* is measured with the linear utility of wealth, whereas the *risk* by the quadratic form $u'(t)\Gamma(t)u(t)$ of risky assets. The wealth dependent weight $1/Y(t)$ permits for more risky investments if the wealth is increasing, i.e. a lesser penalty on controls, whereas if the wealth is decreasing, then the controls are penalized more and thus giving preference to investment in the bank account $S_0(\cdot)$. The control problem of minimising $H(u(\cdot))$ subject to (4.1) was solved in [20] in the setting of deterministic coefficients using dynamic programming and the Hamilton-Jacobi-Bellman equation. Since (4.1) is a special case of system (1.1) and $H(u(\cdot))$ is a special case of $J(u(\cdot))$, we can use Theorem 2 to generalise the result of [20] to the setting of random coefficients. In this case we have:

$$\begin{array}{l}\gamma = 1, \quad a(t) = r(t), \quad b(t) = B(t), \quad c(t) = 0, \quad D(t) = \sigma(t), \\ s = -1, \quad q(t) = 0, \quad k(t) = 0, \quad R(t) = \Gamma(t) > 0 \quad a.e. \quad t \in [0, T] \quad a.s..\end{array}$$

If assumptions of §2 hold, then the Riccati BSDE, the optimal control, and the optimal cost now become, respectively:

$$\begin{cases} dp(t) + r(t)p(t)dt - \frac{1}{4}[B(t)p(t) + \sigma(t)z(t)]'\Gamma^{-1}(t) \\ \times[B(t)p(t) + \sigma(t)z(t)]dt - z'(t)dW(t) = 0, \\ p(T) = -1 \quad a.s., \end{cases}$$

$$\begin{aligned} u^*(t) &= -\frac{1}{2}\Gamma^{-1}(t)[B(t)p(t) + \sigma(t)z(t)]Y(t), \\ H(u^*(\cdot)) &= p(0)Y(0). \end{aligned}$$

4.2. Exponential utility

Another criterion for optimal investment is the *expected utility* from terminal wealth (see, for example, [21], [23], [22]). An important example of utility function is the *exponential utility*, given by:

$$V(u(\cdot)) := \mathbb{E}\left[e^{-\xi Y(T)}\right],$$

where $0 < \xi \in \mathbb{R}$. The optimal investment problem of minimising $V(u(\cdot))$ subject to (4.1) was essentially solved by Merton [21] in the setting of deterministic coefficients (see [38] for a recent generalisation). An explicit closed form solution was found by Merton as an *open-loop* control. In [37], the case of random coefficients was considered, and again an open-loop control was found as a solution. A restriction in [37] is that it assumes a *deterministic* interest rate $r(\cdot)$. Since (4.1) is a special case of system (3.1) and $V(u(\cdot))$ is a special case of $I(u(\cdot), v(\cdot))$, we can use Theorem 4 to solve the optimal investment problem with exponential utility and *random* interest rate. As we show below, the optimal control is now *affine* state-feedback, and thus qualitatively different from the case of a deterministic interest rate. In this case we have:

$$\begin{aligned}
&a(t)=0, \quad b(t)=0, \quad c(t)=0, \quad D(t)=0,\\
&q(t)=0, \quad k(t)=0, \quad R(t)=0, \quad s=0,\\
&\alpha(t)=r(t), \quad \beta(t)=B(t), \quad \lambda(t)=0, \quad \Sigma(t)=\sigma(t),\\
&\theta(t)=0, \quad \rho(t)=0, \quad \mu=-\xi.
\end{aligned}$$

If assumptions of §3 hold, then the corresponding BSDEs, the optimal control, and the optimal cost, are, respectively:

$$\begin{cases}
dp_y(t)+r(t)p_y(t)dt-B'(t)[\sigma(t)\sigma'(t)]^{-1}\sigma(t)z_y(t)dt\\
-p_y^{-1}(t)z_y'(t)\sigma'(t)[\sigma(t)\sigma'(t)]^{-1}\sigma(t)z_y(t)dt-z_y'(t)dW(t)=0\\
p_y(t)\neq 0 \quad \forall t\in[0,T] \quad a.s.,\\
p_y(T)=-\xi \quad a.s..
\end{cases} \tag{4.2}$$

$$\begin{cases}
dp_\ell(t)-z_\ell'(t)\ell^*(t)dt-\dfrac{1}{2}[\ell^*(t)]'\ell^*(t)dt-z_\ell'(t)dW(t)=0,\\
\ell^*(t)=-p_y^{-1}(t)\sigma'(t)[\sigma(t)\sigma'(t)]^{-1}[B(t)p_y(t)+\sigma(t)z_y(t)],\\
p_\ell(T)=0 \quad a.s..
\end{cases}$$

$$
\begin{aligned}
u^*(t) &= [\sigma(t)\sigma'(t)]^{-1}\sigma(t)[\ell^*(t) - z_\ell(t)]p_y^{-1}(t) \\
&- [\sigma(t)\sigma'(t)]^{-1}\sigma(t)z_y(t)p_y^{-1}(t)Y(t), \qquad (4.3) \\
V(u^*(\cdot)) &= \exp[p_\ell(0) + p_y(0)Y(0)].
\end{aligned}
$$

Note that if the interest rate $r(\cdot)$ is deterministic, then the solution pair to BSDE (4.2) is:

$$p_y(t) = -\xi \exp\left[\int_t^T r(\tau)d\tau\right], \quad z_y(t) = 0,$$

which makes (4.3) an open-loop control (as expected from [21], [37]).

5. Conclusions

We have introduced two optimal regulators for linear stochastic systems with random coefficients. The first regulator, which is a generalisation of the LQ regulator, is of a linear-state feedback form and its gain is obtained through a new Riccati BSDE. The second regulator is of a risk-sensitive type and is of nonlinear state-feedback form. Both of these results represent *rare* cases of optimal control problems with explicit closed-form solutions. Their applicability is illustrated with applications to optimal investment. Here we have considered only the most basic forms of these regulators, and it would be interesting to consider other settings, such as the case of coefficients with Markovian switching, systems driven by jump-diffusions, systems with constraints, or mean-field systems.

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