ÉTALE MONODROMY AND RATIONAL EQUIVALENCE FOR 1-CYCLES ON CUBIC HYPERSURFACES IN $\mathbb{P}^5$

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Abstract. Let k be an uncountable algebraically closed field of characteristic 0, and let X be a smooth projective connected variety of dimension $2p$, appropriately embedded into $\mathbb{P}^m$ over k. Let Y be a hyperplane section of X, and let $A^p(Y)$ and $A^{p+1}(X)$ be the groups of algebraically trivial algebraic cycles of codimension p and $p+1$ modulo rational equivalence on Y and X respectively. Assume that, whenever Y is smooth, the group $A^p(Y)$ is regularly parametrized by an abelian variety A and coincides with the subgroup of degree 0 classes in the Chow group $CH^p(Y)$. In the paper we prove that the kernel of the push-forward homomorphism from $A^p(Y)$ to $A^{p+1}(X)$ is the union of a countable collection of shifts of a certain abelian subvariety A_0 inside A. For a very general section Y either $A_0 = 0$ or A_0 coincides with an abelian subvariety A_1 in A whose tangent space is the group of vanishing cycles $H^{2p-1}(Y)_{\mathrm{van}}$. Then we apply these general results to sections of a smooth cubic fourfold in $\mathbb{P}^5$.

Contents

1. Introduction

Let X be a smooth projective variety over an algebraically closed field. The Picard-Lefschetz theory yields that the monodromy action on the $(n-1)$-th vanishing cohomology of a smooth section of the variety X is irreducible. The proof of this fact in terms of étale cohomology groups was given by N. Katz

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in his second article (Exposé XVIII) in [12], and by P. Deligne in [7]. It is also explained in terms of singular cohomology in Section 7.3 of Lamotke's paper [14].

The irreducibility of the monodromy action plays an important role in the Hodge theoretical study of algebraic varieties over $\mathbb{C}$, and it was amply utilized in the work by C. Voisin, see, for example, [28], [29], [30], [33] and [35]. More importantly, the irreducibility of monodromy action on cohomology affects algebraic cycles through Hodge theory, see pages 304 - 305 in the second volume of the book [31], and Proposition 2.4 on page 854 in [32].

To explain the latter idea, let X be a smooth projective complex surface, embedded into a projective space, let Y be a general hyperplane section of X with the Jacobian $A = \mathrm{Jac}(Y)$, and let $A_0(X)$ be the Chow group of 0-cycles of degree 0 on X. Let, furthermore, A_1 be an abelian subvariety in A which corresponds, via Hodge theory, to the vanishing cohomology in $H^1(Y)$. Then the kernel of the push-forward homomorphism from A to $A_0(X)$ is a countable union of shifts of a certain abelian subvariety A_0 inside A_1. The cohomological monodromy argument implies that, for a general Y, either $A_0 = A_1$ or $A_0 = 0$, see the top of page 305 in the second volume of [31]. Clearly, this alternative for A_0 must play an important role in the study of 0-cycles on surfaces, especially in the context of Bloch's conjecture.

The aim (and novelty) of the present paper is twofold. First, we will extend the cycle-theoretic monodromy argument from 0-cycles on surfaces to algebraic cycles of dimension $p-1$ on a smooth projective variety X of even dimension $2p$, embedded into a projective space, such that, if Y is a smooth hyperplane section of X, the "continuous" Chow group $A^p(Y)$ is weakly representable in the sense of Spencer Bloch, see Definition 1.1 in [2] or Definition 3.3. in [4]. The main case for us will be the case of 1-cycles on a 4-fold X with representable 1-cycles on its section Y (take, for example, a general cubic hypersurface X in $\mathbb{P}^5$). Second, we will develop a cycle-theoretic monodromy argument working in terms of étale cohomology groups over an abstract field of definition. This is of high importance because the nature of the abelian variety A_0, appearing also in the general $(p-1)$-dimensional case, has actually nothing to do with complex-analytic arguments. Though we were not able to avoid the uncountability and 0 characteristic of the ground field in the present paper, we strongly believe that these two requirements can be effectively omitted in a more subtle arithmetic theory, in which the intrinsic nature of the abelian variety A_0 will be revealed (work in progress by the second named author).

Let us now describe the results of the paper. Let k be an uncountable algebraically closed field of characteristic 0, and let

$$r : Y \to X$$

be a codimension e closed embedding of smooth projective varieties over k. The morphism r induces a push-forward homomorphism

$$r_* : A^p(Y) \to A^{p+e}(X)$$

on the Chow groups of algebraically trivial algebraic cycles, and our aim is to study the kernel

$$K = \ker(r_*)$$

under the assumption that the group $A^p(Y)$ is weakly representable, or, equivalently, it can be regularly parametrized by an abelian variety A over k.

Fix an embedding of the variety X into a projective space $\mathbb{P}^m$, such that X is not contained in a hyperplane. For simplicity, and as it is sufficient for applications we have in mind, we will also assume that the group $CH^p(Y)_{\deg=0}$, defined by the embedding of Y into $\mathbb{P}^m$, coincides with the group $A^p(Y)$. Clearly, this assumption is satisfied if $p = 1$ and the Néron-Severi group is of rank 1, or if p is the dimension of Y (i.e. we study 0-cycles). They are also satisfied when Y is a Fano threefold of Picard number 1 inside X and $p = 2$. For example, when Y is a smooth section of a smooth cubic fourfold in $\mathbb{P}^5$.

Our first result is a generalization of the Mumford-Roitman countability lemma for 0-cycles, see [17] and [19], to cycles of positive dimension.

> THEOREM A. *Under the assumptions above, there exists an abelian subvariety A_0 in A, and a countable set Ξ of closed points on A, such that*
>
> $$K = \cup_{x\in\Xi}(x + A_0)$$
>
> *inside the abelian variety A.*

The presentation of the group $A^p(Y)$ by the abelian variety A provides a homomorphism from $H^1(A)$ to $H^{2p-1}(Y)$, in terms of l-adic cohomology groups. Assuming that this homomorphism is an isomorphism, which is known to be true for $p \leq 2$, and also using the Tate conjecture for abelian varieties proven by Faltings, one can construct an abelian subvariety A_1 in A whose tangent space is the kernel of the Gysin homomorphism from $H^{2p-1}(Y)$ to $H^{2(p+e)-1}(X)$. We prove in the paper that the abelian variety A_0 is a subvariety in A_1.

Let now S be an integral algebraic scheme over k, let $\bar{\eta}$ be the geometric generic point of S, and choose a c-open subset U in S, such that for any closed point P in U there is a scheme-theoretical isomorphism between $\bar{\eta}$ and P over $\mathbb{Q}$. Consider a closed embedding $\mathscr{Y} \to \mathscr{X}$ of smooth families over S. The scheme-theoretic isomorphisms $\bar{\eta} \simeq P$ induce scheme-theoretic isomorphisms $\varkappa_P$ between $\mathscr{Y}_P$ and $\mathscr{Y}_{\bar{\eta}}$ over $\mathbb{Q}$. Assume that $A^p(\mathscr{Y}_{\bar{\eta}})$ is presented by an abelian variety $A_{\bar{\eta}}$ and $A^p(\mathscr{Y}_P)$ is presented by an abelian variety A_P, for every closed point P in U. Then the isomorphisms $\varkappa_P$ induce isomorphisms κ_P between A_P and $A_{\bar{\eta}}$ compatible with the isomorphisms on Chow groups induced by the isomorphisms $\varkappa_P$. In the paper we show that $\kappa_P(A_{P,0}) = A_{\bar{\eta},0}$ and $\kappa_P(A_{P,1}) = A_{\bar{\eta},1}$ for every k-point P in U. In other words, one can study the varieties A_0 in a family either working at the geometric generic point or at a very general closed point on the base scheme S. The c-open set U is not unique, of course, and all remains the same over any of them. What happens to A_0 beyond the union of such c-open sets, is a big and important question which deserves a separate research programme.

Within this paper we are interested in the case where the family in question is a family of smooth hyperplane sections, so that we can enhance the study of the abelian variety A_0 by the monodromy action. So let X be a smooth projective variety of even dimension $2p$ over the ground field k, embedded into a projective space $\mathbb{P}^m$, let $\mathscr{X} = X \times \mathbb{P}^{m\vee}$, where $\mathbb{P}^{m\vee}$ is the dual projective space, and let $\mathscr{Y}$ be the intersection of $\mathscr{X}$ with the universal hyperplane inside $\mathbb{P}^m \times \mathbb{P}^{m\vee}$. Let

also T be the complement to the discriminant locus inside the dual projective space, and consider the family $\mathscr{Y}_T \to T$ of smooth hyperplane sections of the variety X over k. Clearly, $\mathscr{Y}_T$ is embedded into $\mathscr{X}_T$ over T. Let, furthermore, ξ be the generic point of T, let $\bar{\xi}$ be the corresponding geometric generic point, and choose a c-open subset U in T, so that k-points P in U are scheme-theoretically isomorphic to $\bar{\xi}$. Now again, assuming our standard assumptions for the fibres $Y_{\bar{\xi}}$ and Y_P, and applying the l-adic monodromy argument in Lefschetz pencils passing through U, in conjunction with Theorem A, we obtain

> THEOREM B. *In terms above, we have the following alternative: either* $A_{\bar{\xi},0} = 0$ *or* $A_{\bar{\xi},0} = A_{\bar{\xi},1}$. *Respectively, either* $A_{P,0} = 0$ *or* $A_{P,0} = A_{P,1}$, *for any closed point* P *in* U.

Notice that the assumptions of Theorem B are satisfied, for example, for all smooth projective fourfolds X whose very general hyperplane sections are Fano varieties of Picard number 1. In all such cases Theorem B brings new information about rational equivalence of algebraic 1-cycles on the fourfold X.

Let now $\tilde{Y}_P$ be the resolution of singularities on Y_P, and set $\tilde{Y}_P = Y_P$ whenever the section Y_P is smooth. Assume that $p \leq 2$ and the group $A^p(\tilde{Y}_P)$ is weakly representable for every section Y_P having at worst one ordinary double point. The next theorem generalizes Voisin's result on page 305 in Volume II of [31].

> THEOREM C. *If the group* $A^{p+1}(X)$ *is not rationally weakly representable, it follows that the kernel of the push-forward homomorphism from* $A^p(Y_P)$ *to* $A^{p+1}(X)$ *is countable, for a very general hyperplane section* Y_P.

The practical meaning of Theorem C is as follows. Suppose we want to prove that $A^{p+1}(X)$ is weakly representable. Then, by Theorem C, "all we need" is to find a positive-dimensional variety in the kernel of the homomorphism from $A^p(Y)$ to $A^{p+1}(X)$, for a very general ample section Y on X.

Our original motivation for proving Theorem C was to understand more about the structure of the huge Chow group $CH^3(X)$ for a general cubic hypersurface X in $\mathbb{P}^5$, whence the title of the paper. Recall that $CH^3(X)$ is generated by lines, see Theorem 1.1 in [24], and $A^3(X)$ is not weakly representable by Theorem 0.5 in [21]. Then, as a particular case of Theorem C, we obtain

> COROLLARY D. *Let* X *be a smooth cubic hypersurface in* $\mathbb{P}^5$, *and let* Y *be a very general hyperplane section of* X. *Then the kernel of the push-forward homomorphism from* $A^2(Y)$ *to* $A^3(X)$ *is countable.*

Intuitively, this corollary tells us that one can think of $A^3(X)$ as a collection of Prymians of smooth hyperplane sections modulo countable kernels generated by 1-cycles rationally equivalent to 0 on X. We expect that these countable kernels are of deep arithmetical nature relevant to the famous non-rationality conjecture.

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2. Problem setting and standard assumptions

The purpose of the first section is to set up the main problem and justify three basic assumptions which we will keep throughout the paper.

Let k be an algebraically closed field, and let Y be an algebraic variety over k. Denote by $CH^p(Y)$ the Chow group of codimension p algebraic cycles modulo rational equivalence on Y, and let $A^p(Y)$ be the subgroup in $CH^p(Y)$ generated by algebraically trivial algebraic cycles on Y. Let, furthermore, V be another algebraic variety over k, and let $Z = \sum m_i Z_i$ be an algebraic cycle on the product $Y \times V$, such that the composition $g_i : Z_i \to V$ of the closed embedding $Z_i \to Y \times V$ and the projection $Y \times V \to V$ is surjective for each i. If P is a closed point on V, the scheme-theoretic preimage $g_i^{-1}(P)$ is also a closed subscheme in the fibre $Y \times \{P\} = Y$. Let $Z_i(P)$ be the corresponding fundamental cycle, and define $Z(P)$ to be the cycle $\sum_i m_i Z_i(P)$ on Y. If the relative dimension of Z over V is n, then $Z(P)$ is a codimension $p = \dim(Y) - n$ algebraic cycle on the variety Y, for each closed point P on V. Fix a closed point P_0 on V. The cycle

$$Z(P) - Z(P_0)$$

is algebraically trivial, and we obtain a map

$$V(k) \to A^p(Y)$$

sending P to the class of the cycle $Z(P) - Z(P_0)$ on Y. This map can be considered as a family of codimension p algebraically trivial cycle classes on Y, induced by the cycle Z on $Y \times V$ and the fixed point P_0 on V.

In terms of relative cycles (see [13] or [25]) the same can be expressed by saying that a relative cycle Z of relative dimension n on $Y \times V$ over V, and a closed point P_0 on V, induce a family of algebraically trivial cycles of codimension $p = \dim(Y) - n$ on Y.

The next definition appears in Murre's paper [18], and is important for what follows.

Definition 1. Let A be an abelian variety over k. A group homomorphism $A^p(Y) \to A(k)$ is said to be *regular* if its pre-composition with any family of algebraic cycles $V(k) \to A^p(Y)$, in the sense above, is a regular morphism of varieties over k. A regular homomorphism

$$\psi : A^p(Y) \to A(k)$$

to an abelian variety A over k is said to be *universal* if, having another regular homomorphism $A^p(Y) \to B(k)$, there exists a unique homomorphism of abelian varieties $A \to B$, such that the obvious diagram commutes, see [18], page 981. Clearly, if ψ exists, then it is an epimorphism in the category of abelian groups.

Let

$$r : Y \to X$$

be a closed embedding of smooth projective connected varieties over k of codimension e, let

$$r_* : A^p(Y) \to A^{p+e}(X)$$

be the push-forward homomorphism induced by the proper morphism r, and let K be the kernel of the homomorphism r_*. Our aim is to study the kernel K of the push-forward homomorphism r_*. Certainly, it is difficult to study K in general, and therefore we need to impose some reasonable assumptions on the Chow group $A^p(Y)$.

Assumption 1

Our first assumption is that the universal regular epimorphism

$$\psi : A^p(Y) \to A(k)$$

exists, and that the group $A^p(Y)$ is weakly representable in the sense of Spencer Bloch, in which case the universal regular homomorphism ψ is an isomorphism of abelian groups, so that we can identify $A^p(Y)$ and $A(k)$ by means of ψ, see [2] or [4].

Remark 2. Clearly, the universal homomorphism ψ exists and is an isomorphism if $p = 1$. The main result of [18] asserts that ψ exists in case $p = 2$. Therefore, Assumption 1 is satisfied whenever $p = 2$ and $A^2(Y)$ is weakly representable in the sense of Bloch, see [2] and [4]. Notice that if the group $A^3(Y)$ of 0-cycles on Y is weakly representable, so is the group $A^2(Y)$, see Lemma 3.1 in [11]. Therefore, Assumption 1 is satisfied whenever $A^3(Y)$ is representable. In particular, it is satisfied when Y is rationally connected. This is so, for example, if Y is a Fano threefold inside a smooth projective variety X over k.

Assumption 2

Fix an embedding of the variety X into a projective space $\mathbb{P}^m$. Since Y is a subvariety in X, it induces the embedding of Y into the same space $\mathbb{P}^m$. Obviously, $A^p(Y) \subset CH^p(Y)_{\deg=0}$. For simplicity, and as it is sufficient for applications we have in mind, we will also assume that

$$A^p(Y) = CH^p(Y)_{\deg=0} \ .$$

Remark 3. If $p = 1$, this assumption is obviously satisfied. If $p = 2$, Assumption 2 is satisfied, for example, for all Fano threefolds Y inside a smooth projective fourfold X over k, such that the Picard number of Y is equal to 1. In particular, the second assumption is satisfied for smooth sections Y of a smooth cubic fourfold in $\mathbb{P}^4$.

Assumption 3

Since ψ is an isomorphism, and hence the group $A^p(Y)$ is weakly representable, there exists a smooth projective curve Γ, a cycle Z of codimension p on $\Gamma \times Y$, and an algebraic subgroup $G \subset J_\Gamma$ in the Jacobian variety J_Γ, such that the induced homomorphism

$$z_* : J_\Gamma = A^1(\Gamma) \to A^p(Y) \simeq A$$

is surjective, and its kernel is the group G. Here z is the cycle class of Z in the Chow group $CH^p(\Gamma \times Y)$. Furthermore, the class z gives the morphism

$$z : M(\Gamma) \otimes \mathbb{L}^{p-1} \to M(Y) \ ,$$

where $M(-)$ is the functor from smooth projective varieties over k to (contravariant) Chow motives over k, $\mathbb{L}$ is the Lefschetz motive and $\mathbb{L}^n$ is the n-fold tensor power of $\mathbb{L}$. Fix a point on the curve Γ and consider the induced embedding

$$i_\Gamma : \Gamma \to J_\Gamma \ .$$

Let also

$$\alpha : J_\Gamma \to A$$

be the projection from the Jacobian J_Γ onto the abelian variety $A = J_\Gamma/G$. Define w to be the composition

$$z \circ (M(\alpha \circ i_\Gamma) \otimes \mathrm{id}_{\mathbb{L}^{p-1}})$$

in the category of Chow motives with coefficients in $\mathbb{Z}$. Then w is a morphism from the motive $M(A) \otimes \mathbb{L}^{p-1}$ to the motive $M(Y)$, which induces the homomorphism

$$w_* : H^1_{\text{ét}}(A, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l) \ , \tag{1}$$

see [11]. Our third assumption is that the homomorphism w_* is an isomorphism of cohomology groups.

Remark 4. If $p = 1$ and $\dim(Y) = 1$, then w_* is an isomorphism by the standard argument. If $p = 2$ and $\dim(Y) = 3$, Assumption 3 is satisfied by Lemma 4.3 in [11]. Conjecturally, w_* is an isomorphism for any $p > 2$ as well, but we did not prove that in the paper. The reason for that is that Lemma 4.3 from [11] uses some result by Merkurjev and Suslin on the injectivity of Bloch's map λ^2_l, see [16]. Thought we believe that the Bloch-Kato conjecture, which is now a theorem due to Voevodsky and Rost, can help us to prove that w_* is an isomorphism for an arbitrary p, this may well be quite a big piece of work, deserving a separate paper to be worked it out.

Of course, if $k = \mathbb{C}$, then the isomorphism between $H^1(A, \mathbb{C})$ and $H^{2p-1}(Y, \mathbb{C})$ can be easily achieved by the Hodge-theoretical methods, and Assumption 3 is always satisfied.

Consider the Gysin homomorphism

$$H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l) \xrightarrow{r_*} H^{2(p+e)-1}_{\text{ét}}(X, \mathbb{Q}_l) \ . \tag{2}$$

induced by the closed embedding r on the l-adic cohomology groups. Assumption 3 gives us an advantage that we now can describe the kernel of the homomorphism (2) in terms of an abelian subvariety in A, transporting Hodge-theoretical arguments in to l-adic representations.

Indeed, let

$$A_{l^n} = \ker(A \xrightarrow{l^n} A)$$

be the l^n-torsion subgroup in the abelian variety A over k_0, let

$$T_l(A) = \lim A_{l^n}$$

be the Tate module of A, and let

$$V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \ .$$

Let G be the image in the group $H^1_{\text{ét}}(A, \mathbb{Q}_l(1-p))$ of the kernel of the Gysin homomorphism (2) under the isomorphism w_*^{-1}, inverse to (1). Then G induces a $\mathbb{Q}_l$-vector subspace in $H^1_{\text{ét}}(A, \mathbb{Q}_l)$. But the group $H^1_{\text{ét}}(A, \mathbb{Q}_l)$ is isomorphic to the dual vector space $\text{Hom}_{\mathbb{Q}_l}(V_l(A), \mathbb{Q}_l)$. Therefore, the kernel G induces a vector subspace G' in $\text{Hom}_{\mathbb{Q}_l}(V_l(A), \mathbb{Q}_l)$. Since the space $V_l(A)$ is finite-dimensional, the dual space $\text{Hom}_{\mathbb{Q}_l}(V_l(A), \mathbb{Q}_l)$ is isomorphic to $V_l(A)$, and therefore G' induces a vector subspace in $V_l(A)$. The latter vector subspace determines an idempotent e_G in the associative ring $\text{End}_{\mathbb{Q}_l}(V_l(A))$.

Now, without loss of generality, one can temporarily assume that k is the algebraic closure of a field which is finitely generated over $\mathbb{Q}$. In such a case, the Tate conjecture for abelian varieties, proven by Faltings, tells us that the canonical l-adic representation

$$\text{End}(A) \otimes \mathbb{Q}_l \to \text{End}(V_l(A))$$

is an isomorphism, see the article [9] (or page 72 in [26], or page 74 in [22]). Therefore, the idempotent e_G induces an idempotent $\tilde{e}_G$ in the associative ring $\text{End}(A) \otimes \mathbb{Q}_l$. This idempotent determines a unique, up to an isogeny, abelian subvariety

$$A_1 \subset A$$

in the abelian variety A, such that the image of the injective homomorphism

$$H^1_{ét}(A_1, \mathbb{Q}_l)(1-p) \to H^1_{ét}(A, \mathbb{Q}_l)(1-p) \ ,$$

induced by the inclusion $A_1 \subset A$, coincides with the kernel of the composition of the isomorphism w_* with the Gysin homomorphism r_* from $H^{2p-1}_{ét}(Y, \mathbb{Q}_l)$ to $H^{2p+1}_{ét}(X, \mathbb{Q}_l)$.

Remark 5. If $p = 1$, $\dim(Y) = 1$ and $\dim(X) = 2$, then A_1 is the connected component of the kernel of the induced homomorphism from the Jacobian A of the curve Y to the Albenese variety $\text{Alb}(X)$ of the surface X.

Remark 6. In the applications below we will be dealing with the case when X is embedded into a projective space, the dimension of X is $2p$ and Y is a smooth hyperplane section of X, so that $\dim(Y) = 2p-1$ and $e = 1$. Assume also that $p = 1$ or 2, in order to have that w_* is an isomorphism by Remark 4. If $H^{2p+1}_{\text{ét}}(X, \mathbb{Q}_l)$ vanishes, then the primitive cohomology group coincides with $H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l)$, in which case $A_1 = A$. This is so, for example, when X is a smooth hypersurface in $\mathbb{P}^{2p+1}$. If the group does not $H^{2p+1}_{\text{ét}}(X, \mathbb{Q}_l)$ vanish, the abelian variety A_1 can be smaller than the variety A.

Remark 7. If, moreover, $k = \mathbb{C}$, then A_1 can be described Hodge-theoretically. Indeed, for any algebraic variety V over $\mathbb{C}$ and any non-negative integer n the étale cohomology group $H^n_{\text{ét}}(V, \mathbb{Q}_l)$ is functorially isomorphic to the singular cohomology group $H^*(V(\mathbb{C}), \mathbb{Q}_l) = H^*(V(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_l$. The étale cohomology groups with coefficients in $\mathbb{Q}_l$ can be further tensored with the algebraic closure $\bar{\mathbb{Q}}_l$ of the l-adic field over $\mathbb{Q}_l$. Fixing an isomorphism between $\bar{\mathbb{Q}}_l$ and $\mathbb{C}$, the étale

cohomology groups $H^*_{\text{ét}}(-, \bar{\mathbb{Q}}_l)$ are functorially isomorphic to the singular cohomology groups $H^*(-, \mathbb{C})$. The Gysin homomorphism r_* from $H^{2p-1}(Y, \mathbb{C})$ to $H^{2p+1}(X, \mathbb{C})$ is a morphism of Hodge structures, so that its kernel H_1 is a Hodge substructure in $H^{2p-1}(Y, \mathbb{C})$. Suppose $p = 2$. By Remark 4 the group $H^3(Y, \mathbb{C})$ is isomorphic to the group $H^1(A, \mathbb{C})$ via the homomorphism w_*, and w_* is obviously a morphism of Hodge structures too. It follows that H_1 is of weight 1. This gives the abelian subvariety A_1 in A, where $A = J^2(Y)_{\text{alg}}$ is the intermedian Jacobian of the threefold Y (see [18]).

3. The generalization of the Mumford-Roitman theorem

In this section we generalize a certain result due to Mumford and Roitman, appeared first in [17] and then in [19], to algebraic cycles of positive dimension. For that purpose we shall use the theory of relative cycles developed by Suslin and Voevodsky in [25] and, independently, by Kollár in [13].

So, let N be the category of locally Noetherian schemes over k. Let X be a scheme of finite type over k and consider the presheaf of monoids[1] $\mathscr{C}_r(X)$ on N, where for any scheme S in N the value $\mathscr{C}_r(X)(S)$ is the monoid $\mathscr{C}_r(X \times S/S)$ freely generated by relative cycles on $X \times S$ of relative dimension r over S, in the sense of Suslin and Voevodsky, and the pullbacks are the pullbacks as constructed in Section 3 in [25]. To understand why the monoids $\mathscr{C}_r(X)(S)$ are free, see Corollary 3.1.6, or Corollaries 3.4.5 and 3.4.6, in loc.cit. The presheaf $\mathscr{C}_r(X)$ is actually a sheaf in cdh, and hence in the Nisnevich topology on N, see Theorem 4.2.9 in [25]. If X is equidimensional, we will also write $\mathscr{C}^p(X)(S)$, or $\mathscr{C}^p(X \times S/S)$, for the same monoids of relative cycles of relative codimension p, where $p = \dim(X) - r$. If X is projective over k, we fix a closed embedding of X into a projective space $\mathbb{P}^m$ over k and consider the subsheaf $\mathscr{C}^p_d(X)$ of relative cycles of degree d in $\mathscr{C}^p(X)$, whose sections $\mathscr{C}^p_d(X \times S/S)$ are generated by relative cycles of relative codimension p and degree d, where the degree is understood with regard to the closed embedding of X into $\mathbb{P}^m$.

Since now we will always assume that X is equidimensional and projective over k, and that the embedding of X into $\mathbb{P}^m$ is fixed. Corollary 4.4.13 in [25] says then that the sheafification of the presheaf $\mathscr{C}^p_d(X)$ in h-topology on the category N is representable by a *Chow scheme* $C^p_d(X)$, projective over k. Let

$$C^p(X) = \coprod_{d \geq 0} C^p_d(X)$$

be the *total Chow scheme*, where the coproduct is taken in the category N.

Now, let N_0 be the full subcategory of seminormal schemes in N. Since k is of characteristic 0, the above h-representability can be replaced by the usual representability, if we restrict the presheaves on the category N_0, see Theorem 3.21 in [13], and notice that the Suslin-Voevodsky's pullbacks of relative cycles coincide with the Kollár's ones in our case. Thus, for each seminormal S, one has the bijection

$$\Upsilon^p_d(X, S) : \mathscr{C}^p_d(X \times S/S) \xrightarrow{\sim} \text{Hom}(S, C^p_d(X)) \ ,$$

[1] all monoids in this paper will be commutative monoids

functorial in S. Moreover, these bijections are also functorial in X by Corollary 3.6.3 in [25].

Notice that if $d = 0$, then, by convention, $C_0^p(X) = \mathrm{Spec}(k)$ and the unique k-point of $\mathrm{Spec}(k)$ is the neutral element 0 of the free monoid $\mathscr{C}^p(X)$. Since 0 can be also considered as the empty codimension p relative cycle over $\mathrm{Spec}(k)$, we in fact identify the unique k-point of $\mathrm{Spec}(k)$ with the empty relative cycle.

It is trivial but worth noticing that if k' is another field and $\alpha : k \xrightarrow{\sim} k'$ is an isomorphism of fields, the scheme $C_r(X')$ is the pull-back of the scheme $C_r(X)$ with respect to the morphism $\mathrm{Spec}(\alpha)$, where X' is the pull-back of X, and the corresponding morphism from $C^p(X')$ to $C^p(X)$ is an isomorphism of schemes. The bijections $\Upsilon^p(X', S)$ and $\Upsilon^p(X, S)$ commute through the obvious isomorphisms on monoids and Hom-sets, induced by the isomorphism α.

Next, for a commutative (additive) monoid M, its completion M^+ can be constructed as the quotient of $M \oplus M$ by the image of the diagonal embedding. Let

$$\tau : M \oplus M \to M^+$$

be the corresponding quotient homomorphism, and let

$$\nu : M \to M^+$$

be the composition of the embedding of M as one of the two direct summands and the homomorphism τ. Then ν possesses the obvious universal property, and for any (a, b) in $M \oplus M$ the value $\tau(a, b)$ is the difference $\nu(a) - \nu(b)$. If M is a cancellation monoid then ν is injective, and we can identify M with its image in M^+. Modulo this identification,

$$\tau(a, b) = a - b\ .$$

In particular, we can consider the presheaf $\mathscr{Z}^p(X)$ of abelian groups on N, such that for each S the group of sections

$$\mathscr{Z}^p(X \times S/S) = \mathscr{C}^p(X \times S/S)^+$$

is the group completion of the monoid $\mathscr{C}^p(X \times S/S)$.

Identifying schemes with representable presheaves, we identify the presheaf $\mathscr{C}_d^p(X)$ with the Chow scheme $C_d^p(X)$. Looking at $\mathscr{C}^p(X) = \coprod_d \mathscr{C}_d^p(X)$ as a monoid object in the category of presheaves, we may also interpret the presheaf $\mathscr{Z}^p(X)$ as the group completion $\mathscr{C}^p(X)^+$ of this monoid in the category of presheaves.

Let S and Y be two Noetherian schemes over k. Define a functor $\mathscr{H}om(S, Y)$ on N sending a scheme T to the set $\mathscr{H}om(S, Y)(T)$ of morphisms from $S \times T$ to $Y \times T$ over T. The graphs of such morphisms give us an embedding of $\mathscr{H}om(S, Y)$ into the Hilbert functor $\mathscr{H}ilb(S \times Y)$. If S and Y are projective over k, the latter functor is representable by the projective Hilbert scheme $\mathrm{Hilb}(S \times Y)$ and $\mathscr{H}om(S, Y)$ is representable by an open subscheme $\mathrm{Hom}(S, Y)$ in $\mathrm{Hilb}(S \times Y)$. By the universal property of fibred products over k one has the natural bijection between $\mathscr{H}om(S, Y)(T)$ and $\mathrm{Hom}(T \times S, Y)$. This yields the adjunction

$$\mathrm{Hom}(T \times S, Y) \simeq \mathrm{Hom}(T, \mathrm{Hom}(S, Y))$$

and the corresponding regular evaluation morphism

$$e_{S,Y} : \mathrm{Hom}(S, Y) \times S \to Y \ .$$

Within this paper we are interested in the case when $S = \mathbb{P}^1$. Also, fixing a very ample sheaf $\mathscr{O}(1)$ on Y allows us also to define an appropriate functor $\mathscr{H}om^d(\mathbb{P}^1, Y)$, and the corresponding quasi-projective scheme $\mathrm{Hom}^d(\mathbb{P}^1, Y)$ parametrizing morphisms from $\mathbb{P}^1$ to Y whose graphs are of degree d in Y, with regard to the corresponding projective embedding of $\mathbb{P}^1 \times Y$. Then we obtain a regular evaluation morphism of quasi-projective schemes

$$e_{\mathbb{P}^1,Y} : \mathrm{Hom}^d(\mathbb{P}^1, Y) \times \mathbb{P}^1 \to Y \ ,$$

for each positive integer d. If P is a closed point of $\mathbb{P}^1$, then one has the evaluation-at-P morphism

$$e_P : \mathrm{Hom}^d(\mathbb{P}^1, Y) \to Y \ ,$$

sending f to $f(P)$. More details about schemes of morphisms can be in Kollár's book [13], or in [6].

Let now X be a smooth projective variety embedded into $\mathbb{P}^n$ over k. Let A and A' be two algebraic cycles of codimension p on X. The cycle A is rationally equivalent to the cycle A' if and only if there exists an effective codimension p algebraic cycle Z on $X \times \mathbb{P}^1$ and an effective codimension p algebraic cycle B on X, such that

$$Z(0) = A + B \qquad \text{and} \qquad Z(\infty) = A' + B \ .$$

Assume that A is rationally equivalent to A', and let

$$f_Z = \Upsilon(Z)$$

and

$$f_{B\times\mathbb{P}^1} = \Upsilon(B \times \mathbb{P}^1)$$

be two regular morphisms from $\mathbb{P}^1$ to $\mathscr{C}^p(X)$, where

$$\Upsilon = \Upsilon^p(X, \mathbb{P}^1) : \mathscr{C}^p(X \times \mathbb{P}^1/\mathbb{P}^1) \to \mathrm{Hom}(\mathbb{P}^1, C^p(X))$$

is the functorial bijection considered above. Let also

$$f = f_Z \oplus f_{B\times\mathbb{P}^1} : \mathbb{P}^1 \to C^p(X) \oplus C^p(X)$$

be the morphism generated by f_Z and $f_{B\times\mathbb{P}^1}$. Since $C^p(X)$ is a cancellation monoid, for any two elements $a, b \in C^p(X)$ the value $\tau(a, b)$ in $C^p(X)^+$ is $a - b$, after the identification of $C^p(X)$ with its image in $C^p(X)^+$ under the injective homomorphism ν from $C^p(X)$ to $C^p(X)^+$. Then

$$\tau f(0) = \tau(f_Z(0), f_{B\times\mathbb{P}^1}) = f_Z(0) - f_{B\times\mathbb{P}^1}(0) = Z(0) - B = A$$

and

$$\tau f(\infty) = \tau(f_Z(\infty), f_{B\times\mathbb{P}^1}) = f_Z(0) - f_{B\times\mathbb{P}^1}(\infty) = Z(\infty) - B = A' \ .$$

Vice versa, suppose there is a regular morphism $f = f_1 \oplus f_2$ from $\mathbb{P}^1$ to the direct sum $C^p(X) \oplus C^p(X)$, such that $\tau f(0) = A$ and $\tau f(\infty) = A'$. Let Z_1 and Z_2 be two algebraic cycles in $\mathscr{C}^p(X \times \mathbb{P}^1/\mathbb{P}^1)$, such that $\Upsilon(Z_i) = f_i$ for $i = 1, 2$, and let $Z = Z_1 - Z_2$. Then $Z(0) = A$ and $Z(\infty) = A'$, which means that A is rationally equivalent to A'.

For any non-negative integers $d_1, \ldots, d_s$ let

$$C^p_{d_1,\ldots,d_s}(X) = C^p_{d_1}(X) \times \cdots \times C^p_{d_s}(X)$$

be the fibred product over the ground field k. For any degree $d \geq 0$ let

$$W_d = \{(A, B) \in C^p_{d,d}(X) \mid A \overset{\text{rat}}{\sim} B\}$$

be the Zariski closed subset in $C^p_{d,d}(X)$ determined by ordered pairs (A, B) of closed points in $C^p_d(X)$, such that the cycle A is rationally equivalent to the cycle B on X. For any non-negative u and positive v let also

$$W^{u,v}_d =$$

$$\{(A, B) \in C^p_{d,d}(X) \mid \exists f \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)), \text{ s.th. } \tau f(0) = A, \tau f(\infty) = B\}$$

Then

$$W^{u,v}_d \subset W_d \qquad \text{and} \qquad W_d = \cup_{u,v} W^{u,v}_d \ .$$

Let also $\bar{W}^{u,v}_d$ be the Zariski closure of the set $W^{u,v}_d$ in the scheme $C^p_{d,d}(X)$.

Proposition 8. *For any d, u and v, the set $W^{u,v}_d$ is a quasi-projective subscheme in $C^p_{d,d}(X)$ whose Zariski closure $\bar{W}^{u,v}_d$ is contained in W_d.*

Proof. To prove the proposition all we need is to extend the arguments in [19] from zero-cycles and symmetric powers to codimension p cycles and Chow varieties.

Let

$$e : \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \to C^p_{d+u,u,d+u,u}(X)$$

be the evaluation morphism sending f to the ordered pair $(f(0), f(\infty))$, and let

$$s : C^p_{d,u,d,u}(X) \to C^p_{d+u,u,d+u,u}(X)$$

be the regular morphism given by the formula

$$s(A, C, B, D) = (A + C, C, B + D, D) \ .$$

The two morphisms e and s allow us to take the product

$$V = \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \times_{C^p_{d+u,u,d+u,u}(X)} C^p_{d,u,d,u}(X) \ ,$$

which is a closed subvariety in the product

$$\mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \times C^p_{d,u,d,u}(X)$$

over $\mathrm{Spec}(k)$ consisting of quintuples (f, A, C, B, D) with $e(f) = s(A, C, B, D)$, i.e.

$$(f(0), f(\infty)) = (A + C, C, B + D, D) \ .$$

The latter equality gives that

$$\mathrm{pr}_{2,3}(V) \subset W^{u,v}_d \ ,$$

where $\mathrm{pr}_{2,3}$ is the projection of $\mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \times C^p_{d,u,d,u}(X)$ onto $C^p_{d,d}(X)$.

Vice versa, if (A, B) is a closed point of $W^{u,v}_d$, there exists a regular morphism

$$f \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X))$$

with

$$\tau f(0) = A \qquad \text{and} \qquad \tau f(\infty) = B \ .$$

Let $f(0) = (C', C)$ and $f(\infty) = (D', D)$. Then

$$\tau f(0) = C' - C = A$$

and

$$\tau f(\infty) = D' - D = B$$

in the completed monoid $\mathscr{Z}^p(X) = C^p(X)^+$. This means that there exist effective codimension p algebraic cycles M and N on X, such that

$$C' + M = C + A + M \qquad \text{and} \qquad D' + N = D + B + N$$

in $C^p(X)$. Since $C^p(X)$ is a free monoid, it possesses the cancellation property. Therefore, the latter two equalities imply the equalities

$$C' = C + A \qquad \text{and} \qquad D' = D + B$$

respectively. This yields

$$e(f) = s(A, C, B, D) \ ,$$

whence

$$(f, A, C, B, D) \in V \ .$$

It means that (A, B) is in $\mathrm{pr}_{2,3}(V)$.

Thus,

$$\mathrm{pr}_{2,3}(V) = W_d^{u,v} \ .$$

Being the image of a quasi-projective variety under the projection $\mathrm{pr}_{2,3}$ the set $W_d^{u,v}$ is itself a quasi-projective variety.

Let

$$\tilde{s} : C^p_{d,d,u,u}(X) \to C^p_{d+u,d+u,u,u}(X)$$

be the morphism obtained by composing and precomposing s with the transposition of the second and third factors in the domain and codomain of s. Then

$$W_d = \mathrm{pr}_{1,2}(\tilde{s}^{-1}(W_{d+u} \times W_u)) \ .$$

Let (A, B, C, D) be a closed point in $C^p_{d,d,u,u}(X)$, such that the value

$$\tilde{s}(A, B, C, D) = (A + C, B + D, C, D)$$

is in $W^{0,v}_{d+u} \times W^{0,v}_u$. It means that there exist two regular morphisms

$$g \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u}(X)) \qquad \text{and} \qquad h \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_u(X))$$

with

$$g(0) = A + C \ , \quad g(\infty) = B + D \ , \quad h(0) = C \quad \text{and} \quad h(\infty) = D \ .$$

Then

$$f = g \times h \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \ ,$$

$$f(0) = (A + C, C) \qquad \text{and} \qquad f(\infty) = (B + D, D) \ .$$

Hence, $\tau f(0) = A$ and $\tau f(\infty) = B$. It means that $(A, B) \in W_d^{u,v}$. We have shown that

$$\mathrm{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_u)) \subset W_d^{u,v} \ .$$

Vice versa, suppose $(A,B) \in W_d^{u,v}$, and let f be a morphism from $\mathbb{P}^1$ to $C^p_{d+u,u}(X)$ with $\tau f(0) = A$ and $\tau f(\infty) = B$. Composing f with the projections of $C^p_{d+u,u}(X)$ onto $C^p_{d+u}(X)$ and $C^p_u(X)$ one can easily show that

$$W_d^{u,v} \subset \mathrm{pr}_{1,2}(\tilde{s}^{-1}(W_{d+u}^{0,v} \times W_u^{0,v}))\ .$$

Thus,

$$W_d^{u,v} = \mathrm{pr}_{1,2}(\tilde{s}^{-1}(W_{d+u}^{0,v} \times W_u^{0,v}))\ .$$

Since $\tilde{s}$ is continuous and $\mathrm{pr}_{1,2}$ is proper, we obtain that

$$\bar{W}_d^{u,v} = \mathrm{pr}_{1,2}(\tilde{s}^{-1}(\bar{W}_{d+u}^{0,v} \times \bar{W}_u^{0,v}))\ .$$

This gives us that in order to prove the second assertion of the proposition it suffices to show that $\bar{W}_d^{0,v}$ is contained in W_d.

Let (A,B) be a closed point of $\bar{W}_d^{0,v}$. If (A,B) is in $W_d^{0,v}$, then it is also in W_d. Suppose that (A,B) is in $\bar{W}_d^{0,v} \smallsetminus W_d^{0,v}$. Let W be an irreducible component of the quasi-projective variety $W_d^{0,v}$ whose Zariski closure $\bar{W}$ contains the point (A,B). Let U be an affine neighbourhood of (A,B) in $\bar{W}$. Since (A,B) is in the closure of W the set $U \cap W$ is non-empty. Let C be an irreducible curve passing through (A,B) in U and let $\bar{C}$ be the Zariski closure of C in $\bar{W}$. The evaluation regular morphisms

$$e_0 : \mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X)) \to C^p_d(X) \quad \text{and} \quad e_\infty : \mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X)) \to C^p_d(X)$$

give the regular morphism

$$e_{0,\infty} : \mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X)) \to C^p_{d,d}(X)\ .$$

Then $W_d^{0,v}$ is exactly the image of the regular morphism $e_{0,\infty}$, and we can choose a quasi-projective curve T in $\mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X))$, such that the closure of the image $e_{0,\infty}(T)$ is $\bar{C}$. Since $\mathrm{Hom}^v(\mathbb{P}^1, \mathscr{C}^p_d(X))$ is a quasi-projective variety, we can embed it into some projective space $\mathbb{P}^m$. Let $\bar{T}$ be the closure of T in $\mathbb{P}^m$, let $\tilde{T}$ be the normalization of $\bar{T}$ and let $\tilde{T}_0$ be the pre-image of T in $\tilde{T}$. Consider the composition

$$f_0 : \tilde{T}_0 \times \mathbb{P}^1 \to T \times \mathbb{P}^1 \subset \mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X)) \times \mathbb{P}^1 \xrightarrow{e} C^p_d(X)\ ,$$

where e is the evaluation morphism $e_{\mathbb{P}^1, C^p_d(X)}$. The regular morphism f_0 defines a rational map

$$f : \tilde{T} \times \mathbb{P}^1 \dashrightarrow C^p_d(X)$$

Since $\tilde{T}$ is a smooth projective curve, the product $\tilde{T} \times \mathbb{P}^1$ is a smooth projective surface over the ground field. Under this condition there exists a finite chain of σ-processes $(\tilde{T} \times \mathbb{P}^1)' \to \tilde{T} \times \mathbb{P}^1$ resolving indeterminacy of f and giving a regular morphism

$$f' : (\tilde{T} \times \mathbb{P}^1)' \to C^p_d(X)\ .$$

The regular morphism $\tilde{T}_0 \to T \to \bar{C}$ extends to the regular morphism $\tilde{T} \to \bar{C}$. Let P be a point in the fibre of this morphism at (A,B). For any closed point Q on $\mathbb{P}^1$ the restriction $f|_{\tilde{T} \times \{Q\}}$ of the rational map f onto $\tilde{T} \times \{Q\} \simeq \tilde{T}$ is regular on the whole curve $\tilde{T}$, because $\tilde{T}$ is smooth. Then

$$(f|_{\tilde{T} \times \{0\}})(P) = A$$

and

$$(f|_{\tilde{T}\times\{\infty\}})(P) = B\ .$$

It means that the points A and B are connected by a finite collection of curves which are the images of rational curves on $(\tilde{T}\times\mathbb{P}^1)'$ under the regular morphism f'. In turn, it follows that A is rationally equivalent to B, whence

$$(A,B)\in W_d\ .$$

□

In what follows, for any equi-dimensional algebraic scheme V over k, let $CH^p(V)$ be the Chow group, with coefficients in $\mathbb{Z}$, of codimension p algebraic cycles modulo rational equivalence on V. If a closed embedding $V\subset\mathbb{P}^m$ is fixed, let $CH^p(V)_{\deg=0}$ be the subgroup generated by cycles classes of degree 0 in $CH^p(V)$. Then, for any two nonnegative integer d, we have a map

$$\theta_d^p : C_{d,d}^p(X)\to CH^p(X)_{\deg=0}$$

sending an ordered pair (A,B) of closed points on the Chow variety $C_d^p(X)$ to the class of the difference $Z_A - Z_B$, where Z_A and Z_B are codimension p algebraic cycles on X corresponding to the points A and B respectively.

Corollary 9. *$(\theta_d^p)^{-1}(0)$ is a countable union of irreducible Zariski closed subsets in the Chow scheme $C_{d,d}^p(X)$.*

Proof. Proposition 8 gives that W_d is the countable union of Zariski closed sets $\bar{W}_d^{u,v}$ over u and v. This completes the proof. □

4. Countability lemmas and the abelian variety A_0

The purpose of this section is to prove Theorem A in Introduction, which generalizes the argument from Section 10.1.2 in the second volume of the book [31] (see also pp 304 - 305 in loc.cit.). Theorem A introduces the abelian variety A_0 which is of key importance to the whole approach. We shall also prove that A_0 is a subvariety of the abelian variety A_1 introduced in Section 2. Since now and throughout the rest of the paper we shall assume that the ground field k is uncountable.

Lemma 10. *Let V be an irreducible quasi-projective algebraic variety over k. Then V cannot be written as a countable union of its Zariski closed subsets, each of which is not the whole V.*

Proof. Since V is supposed to be irreducible, without loss of generality we may assume that V is affine. Let d be the dimension of V and suppose $V=\cup_{n\in\mathbb{N}}V_n$ is the union of closed subsets V_n in V, such that $V_n\neq V$ for each n. By Emmy Noether's lemma, there exists a finite surjective morphism $f: V\to\mathbb{A}^d$ over k. Let W_n be the image of V_n under f. Since f is finite, it is proper. Therefore, W_n are closed in $\mathbb{A}^d$, so that we obtain that the affine space $\mathbb{A}^d$ is the union of W_n's. Since the ground field is uncountable, the set of all hyperplanes in $\mathbb{A}^d$ is uncountable. Therefore, there exists a hyperplane H, such that $W_n\not\subset H$ for any index n. Induction reduces the assertion of the lemma to the case when $d=1$. □

A countable union $V = \cup_{n\in\mathbb{N}} V_n$ of algebraic varieties will be called irredundant if V_n is irreducible for each n and $V_m \not\subset V_n$ for $m \neq n$. In an irredundant decomposition, the sets V_n will be called c-components of V.

Lemma 11. *Let V be a countable union of algebraic varieties over an uncountable algebraically closed ground field. Then V admits an irredundant decomposition, and such an irredundant decomposition is unique.*

Proof. Let $V = \cup_{n\in\mathbb{N}} V'_n$ be a countable union of algebraic varieties over k. For each n let $V'_n = V'_{n,1} \cup \cdots \cup V'_{n,r_n}$ be the irreducible components of V'_n. Ignoring all components $V'_{m,i}$ with $V'_{m,i} \subset V'_{n,j}$ for some n and j we end up with a irredundant decomposition. Having two irredundant decompositions $V = \cup_{n\in\mathbb{N}} V_n$ and $V = \cup_{n\in\mathbb{N}} W_n$, suppose there exists V_m such that V_m is not contained in W_n for any n. Then V_m is the union of the closed subsets $V_m \cap W_n$, each of which is not V_m. This contradicts to Lemma 10. Therefore, any V_m is contained in some W_n. By symmetry, any W_n is in V_l for some l. Then $V_m \subset V_l$. By irredundancy, $l = m$ and $V_m = W_n$. □

Lemma 12. *Let A be an abelian variety over k, and let K be a subgroup which can be represented as a countable union of Zariski closed subsets in A. Then the irredundant decomposition of K contains a unique irreducible component passing through 0, and this component is an abelian subvariety in A.*

Proof. Let $K = \cup_{n\in\mathbb{N}} K_n$ be the irredundant decomposition of K, which exists by Lemma 11. Since $0 \in K$, there exists at least one component in the irredundant decomposition, which contains 0. Suppose there are s components $K_1, \ldots, K_s$ containing 0 and $s > 1$. The summation in K gives the regular morphism from the product $K_1 \times \cdots \times K_s$ into A, whose image is the irreducible Zariski closed subset $K_1 + \cdots + K_s$ in A. By Lemma 10, there exists $n \in \mathbb{N}$, such that $K_1 + \cdots + K_s \subset K_n$ and so $K_i \subset K_n$ for each $1 \leq i \leq n$. By irredundancy, $s = 1$, which contradicts to the assumption $s > 1$.

After renumbering of the components, we may assume that $0 \in K_0$. If $K_0 = \{0\}$, then K_0 is trivially an abelian variety. Suppose $K_0 \neq \{0\}$ and take a non-trivial element x in K_0. Since $-x + K_0$ is irreducible, it must be in some K_n by Lemma 10. As $0 \in -x + K_0$ it follows that $0 \in K_n$ and so $n = 0$. It follows that $-x \in K_0$. Similarly, since $K_0 + K_0$ is irreducible and contains K_0, we see that $K_0 + K_0 = K_0$. Being a Zariski closed abelian subgroup in A, the set K_0 is an abelian subvariety in A. □

Now we are ready to prove Theorem A in Introduction.

Theorem 13. *In terms above, there exists an abelian subvariety A_0 and a countable subset of closed points in A, such that*

$$K = \cup_{x\in\Xi}(x + A_0)$$

inside the abelian variety A.

Proof. Let n be the dimension of Y. Since $\mathscr{C}_{n-p}$ is a functor on Noetherian schemes over k, the closed immersion r induces a morphism from $\mathscr{C}_{n-p}(Y)$ to $\mathscr{C}_{n-p}(X)$ in the category presheaves on seminormal schemes. In upper indices,

this is a morphism from $\mathscr{C}^p(Y)$ to $\mathscr{C}^{p+e}(X)$. Passing to Chow schemes, we obtain a regular morphism

$$r_* : C^p(Y) \to C^{p+e}(X)$$

of projective schemes over k. Since X is embedded into $\mathbb{P}^m$, and since Y is a closed subvariety in X, the morphism r_* induces the morphisms

$$r_* : C^p_d(Y) \to C^{p+e}_d(X) \ ,$$

one for each degree d. Taking in to account Assumption 2, we obtain the obvious commutative diagram

$$\begin{array}{ccc} \coprod_d C^p_{d,d}(Y) & \xrightarrow{\coprod_d r_*} & \coprod_d C^{p+e}_{d,d}(X) \\ \downarrow{\scriptstyle \coprod \theta^p_d} & & \downarrow{\scriptstyle \coprod \theta^{p+e}_d} \\ A^p(Y) & \xrightarrow{r_*} & CH^{p+e}(X)_{\deg=0} \end{array}$$

Since $A^p(Y)$ is weakly representable, there exists a smooth projective curve Γ over k, and an algebraic cycle Z of codimension p on $\Gamma \times Y$, such that the induced homomorphism

$$Z_* : A^1(\Gamma) \to A^p(Y)$$

is surjective. On the other hand, since $A^1(\Gamma)$ is representable by the Jacobian of Γ, the map

$$\theta^1_d : C^1_{d,d}(\Gamma) \to A^1(\Gamma)$$

is surjective for big enough d. Using these two facts one can show that the right vertical arrow of the above commutative diagram is surjective. Then the kernel K of the bottom horizontal homomorphism is the image under the map $\coprod \theta^p_d$ of the preimage of 0 under the composition of the maps $\coprod_d r_*$ and $\coprod \theta^{p+e}_d$. Corollary 9 implies that the latter preimage is the coproduct of countable unions of Zariski closed subsets in the schemes $C^p_{d,d}(Y)$.

Now, consider the composition

$$\psi \circ \theta^p_d : C^p_{d,d}(Y) \to A^p(Y) \to A \ ,$$

for each number d. By the definition of the regularity of ψ, this composition is a regular morphism of schemes. Since these schemes are projective, the composition is proper. It follows that the subgroup $\psi(K)$ is a countable union of Zariski closed subsets in the abelian variety A.

For simplicity of notation, identify $\psi(K)$ and K. By Lemma 11, the set K admits a unique irredundant decomposition inside the abelian variety A, say

$$K = \cup_{n \in \mathbb{N}} K_n \ .$$

Let A_0 be the unique component of that decomposition passing through 0, which is an abelian subvariety in A by Lemma 12. For any x in K the set $x + A_0$ is an irreducible Zariski closed subset in K. Since K coincides with $\cup_{x \in K}(x + A_0)$, ignoring each set $x + A_0$ which is a subsets in $y + A_0$ for some $y \in K$, we can find a subset Ξ in K, such that

$$K = \cup_{x \in \Xi}(x + A_0) \ ,$$

and for any two elements $x, y \in \Xi$ the irreducible sets $x + A_0$ and $y + A_0$ are not contained one in another. Since $x + A_0$ is irreducible, it is contained in K_n for some n by Lemma 10. Then $A_0 \subset -x + K_n$. Similarly, $-x + K_n \subset K_l$ for some l, so that $K_l = A_0$ by Lemma 12. This yields

$$x + A_0 = K_n \ ,$$

for each $x \in \Xi$. It means that the set Ξ is countable. □

Let us also prove that the abelian variety A_0, provided by Theorem 13, is contained in the abelian variety A_1 introduced in Section 2. Choose an ample line bundle L on the abelian variety A. Let

$$i : A_0 \to A$$

be the closed embedding of A_0 into A, and let L_0 be the pull-back of L to A_0 under the embedding i. Define the homomorphism ζ on divisors via the commutative diagram

$$\begin{array}{ccc} A^1(A_0) & \xrightarrow{\ \zeta\ } & A^1(A) \\ \Big\downarrow{\scriptstyle (\lambda_{L_0})_*} & & \Big\uparrow{\scriptstyle \lambda_L^*} \\ A^1(A_0^\vee) & \xrightarrow{\ i^{\vee *}\ } & A^1(A^\vee) \end{array}$$

Similarly, we define the homomorphism $\zeta_{\mathbb{Z}_l}$ on cohomology by means of the commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(A_0, \mathbb{Z}_l) & \xrightarrow{\ \zeta_{\mathbb{Z}_l}\ } & H^1_{\text{ét}}(A, \mathbb{Z}_l) \\ \Big\downarrow{\scriptstyle \lambda_{L_0 *}} & & \Big\uparrow{\scriptstyle \lambda_L^*} \\ H^1_{\text{ét}}(A_0^\vee, \mathbb{Z}_l) & \xrightarrow{\ i^{\vee *}\ } & H^1_{\text{ét}}(A^\vee, \mathbb{Z}_l) \end{array}$$

and analogously for the homomorphism

$$\zeta_{\mathbb{Q}_l/\mathbb{Z}_l} : H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l) \to H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l) \ .$$

The homomorphism $\zeta_{\mathbb{Z}_l}$ induces the injective homomorphisms

$$\zeta_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_0, \mathbb{Q}_l) = H^1_{\text{ét}}(A_0, \mathbb{Z}_l) \otimes \mathbb{Q}_l \to H^1_{\text{ét}}(A_0, \mathbb{Q}_l) = H^1_{\text{ét}}(A, \mathbb{Z}_l) \otimes \mathbb{Q}_l$$

and

$$\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l : H^1_{\text{ét}}(A_0, \mathbb{Z}_l) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^1_{\text{ét}}(A_0, \mathbb{Z}_l) \otimes \mathbb{Q}_l/\mathbb{Z}_l \ .$$

Proposition 14. *The image of the composition*

$$H^1_{\text{ét}}(A_0, \mathbb{Q}_l(1-p)) \xrightarrow{\zeta_{\mathbb{Q}_l}} H^1_{\text{ét}}(A, \mathbb{Q}_l(1-p)) \xrightarrow{w_*} H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l)$$

is contained in the kernel of the Gysin homomorphism

$$H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l) \xrightarrow{r_*} H^{2(p+e)-1}_{\text{ét}}(X, \mathbb{Q}_l) \ .$$

Proof. For the proof we will be using Bloch's l-adic Abel-Jacobi maps. For any abelian group A, a prime l and positive integer n let A_{l^n} be the kernel of the multiplication by l^n endomorphism of A and let $A(l)$ be the l-primary part of A, i.e. the union of the groups A_{l^n} for all n. For any smooth projective variety V over k, there is a canonical homomorphism

$$\lambda_l^p(V) : CH^p(V)(l) \to H^{2p-1}_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(p)) \ ,$$

constructed by Bloch in [3]. The homomorphisms $\lambda_l^p(V)$ are functorial with respect to the action of correspondences between smooth projective varieties over k. Moreover, the homomorphisms

$$\lambda_l^1(V) : CH^1(V)(l) \to H^1_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(1))$$

are all isomorphisms, loc.cit.

Since A_0 and A are abelian varieties, their Néron-Severi groups are torsion free. It follows that $CH^1(A_0)(l) = A^1(A_0)(l)$ and $CH^1(A)(l) = A^1(A)(l)$, so that we actually have the isomorphism $\lambda_l^1(A_0)$ between $A^1(A_0)(l)$ and $H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1))$, and the isomorphism $\lambda_l^1(A)$ between $A^1(A)(l)$ and $H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1))$. Similarly, one has the isomorphisms $\lambda_l^1(A_0^\vee)$ and $\lambda_l^1(A^\vee)$ for the dual abelian varieties.

The functorial properties of Bloch's maps λ_l^1 give us that the diagram

$$\begin{array}{ccc} A^1(A_0)(l) & \xrightarrow{\quad\zeta\quad} & A^1(A)(l) \\ \Big\downarrow{\scriptstyle \lambda_l^1(A_0)}\ \sim & & \Big\downarrow{\scriptstyle \lambda_l^1(A)}\ \sim \\ H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\zeta_{\mathbb{Q}_l/\mathbb{Z}_l}} & H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \end{array} \tag{3}$$

is commutative.

For a smooth projective V over k, one has the homomorphisms

$$\varrho_l^{i,j}(V) : H^i_{\text{ét}}(V, \mathbb{Z}_l(j)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^i_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(j)) \ ,$$

with finite kernels and cokernels, used in [11], and much earlier in [5]. In particular, we have the commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\zeta_{\mathbb{Q}_l/\mathbb{Z}_l}} & H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \\ \Big\uparrow{\scriptstyle \varrho_l^{1,1}(A_0)} & & \Big\uparrow{\scriptstyle \varrho_l^{1,1}(A)} \\ H^1_{\text{ét}}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\zeta_{\mathbb{Z}_l}\otimes\mathbb{Q}_l/\mathbb{Z}_l} & H^1_{\text{ét}}(A, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \end{array} \tag{4}$$

Let $\sigma : A_0 \xrightarrow{\sim} A^1(A_0^\vee)$ and $\sigma : A \xrightarrow{\sim} A^1(A^\vee)$ be the autoduality isomorphisms. The morphism of motives

$$w : M(A) \otimes \mathbb{L}^{p-1} \to M(Y)$$

induces the homomorphism $w_* : A^1(A) \to A^p(Y)$ on Chow groups. A straightforward verification shows that the diagram

$$\begin{array}{ccc} A^1(A)(l) & \xrightarrow{w_*} & A^p(Y)(l) \\ \uparrow{\scriptstyle \lambda_L^*} & & \uparrow{\scriptstyle (\psi_Y^p)^{-1}} \\ A^1(A^\vee)(l) & \xleftarrow{\sigma} & A(l) \end{array} \tag{5}$$

is commutative.

The homomorphism $w_* : A^1(A) \to A^p(Y)$ on Chow groups and the homomorphism $w_* : H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p))$ induced by w on cohomology fit into the commutative diagram

$$\begin{array}{ccc} A^1(A)(l) & \xrightarrow{w_*} & A^p(Y)(l) \\ \downarrow{\scriptstyle \lambda_l^1(A)}\ \sim & & \downarrow{\scriptstyle \lambda_l^p(Y)} \\ H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{w_*} & H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p)) \end{array} \tag{6}$$

The commutativity of the diagrams (3), (4), (5) and (6), the definition of the abelian variety A_0 and easy diagram chase over the obvious commutative diagrams with Gysin mappings show that the image of the triple composition

$$r_* \circ w_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l) : H^1_{\text{ét}}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2(p+e)-1}_{\text{ét}}(X, \mathbb{Z}_l(p+e)) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is contained in the kernel of the homomorphism

$$\varrho_l^{2(p+e)-1,p+e}(X) : H^{2(p+e)-1}_{\text{ét}}(X, \mathbb{Z}_l(p+e)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2(p+e)-1}_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(p+e)) \, .$$

Since the latter kernel is finite, the image of the composition $r_* \circ w_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l)$ is finite too. Since the étale cohomology groups of smooth projective varieties with $\mathbb{Z}_l$-coefficients are finitely generated $\mathbb{Z}_l$-modules, it follows that the image of the triple composition

$$r_* \circ w_* \circ \zeta_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_0, \mathbb{Q}_l(1)) \to H^{2(p+e)-1}_{\text{ét}}(X, \mathbb{Q}_l(p+e))$$

is zero, which finishes the proof of the proposition. □

Corollary 15. *The abelian variety A_0 is contained in the abelian variety A_1.*

Proof. Since the triple composition $r_* \circ w_* \circ \zeta_{\mathbb{Q}_l}$ is 0 by Proposition 14, the homomorphism $\zeta_{\mathbb{Q}_l}$ factorizes through the group $H^1_{\text{ét}}(A_1, \mathbb{Q}_l(1-p))$. □

5. Geometric generic versus very general version of A_0

Since the ground field k is uncountable and algebraically closed, its transcendental degree over the primary subfield is infinite. We will be using the following terminology. If S is an integral algebraic scheme, a (Zariski) c-closed subset in S is a union of a countable collection of Zariski closed irreducible subsets in S. A (Zariski) c-open subset in S is the complement to a c-closed subset in S. A

property **P** of points in S holds for a very general point on S if there exists a c-open subset U in S, such that **P** holds for each closed point in U.

The purpose of this section is to convince the reader that, given a flat family $\mathscr{X} \to S$ over an integral base S over k, there exists a natural c-open subset U in S, such that the fibres $\mathscr{X}_P$, for all closed k-points $P \in U$, are isomorphic to the geometric generic fibre $\mathscr{X}_{\bar{\eta}}$, as schemes over $\mathrm{Spec}(\mathbb{Q})$, and these isomorphisms preserve algebraic and rational equivalence of algebraic cycles. This is certainly a folklore, see, for example, [27], but we give all the proofs here for clarity and completeness. Then we use such isomorphisms to show that the abelian variety A_0 for fibres in a family is of purely algebraic nature, and therefore its very general and geometric generic versions coincide.

Let S be an integral affine scheme of finite type over k, let $k(S)$ be the function field of S, and let $\overline{k(S)}$ be the algebraic closure of the field $k(S)$. Let $I(S)$ be the ideal of S and let $f_1, \ldots, f_n$ be a set of generators of $I(S)$. The polynomials f_i have a finite number of coefficients. Then we can choose a countable algebraically closed subfield k_0 in k, such that there exists an irreducible quasi-projective scheme S_0 over k_0 with $S = S_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$.

Let Z be a closed subscheme of S_0, and let $i_Z : Z \subset S_0$ be the corresponding closed embedding. Then Z is defined by an ideal $\mathfrak{a}$ in $k_0[S_0]$. Since the field k_0 is countable and $\mathfrak{a}$ is finitely generated, there exists only countably many closed subschemes Z in S_0. For each Z let U_Z be the complement $S_0 \smallsetminus \mathrm{im}(i_Z)$, $Z_k = Z \times_{k_0} k$ and $(U_Z)_k = U_Z \times_{k_0} k$. If $(i_Z)_k : Z_k \to S$ is the pull-back of i_Z with respect to the extension k/k_0, then $(U_Z)_k$ is the complement $S \smallsetminus \mathrm{im}((i_Z)_k)$. Let, furthermore,

$$U = S \smallsetminus \cup_Z \mathrm{im}((i_Z)_k) = \cap_Z (U_Z)_k \ ,$$

where the union is taken over closed subschemes Z, such that $\mathrm{im}((i_Z)_k) \neq S$. Notice that the last condition is equivalent to the condition $\mathrm{im}(i_Z) \neq S_0$. The set U is c-open by its construction (see also the proof of Lemma 2.1 in [27]).

Lemma 16. *For any closed k-point P in U, one can construct a field isomorphism between $\overline{k(S)}$ and k, whose value at $f \in k_0[S_0]$ is $f(P)$.*

Proof. If now P is a closed k-point in the above defined subset U in S, defined by the corresponding morphism $f_P : \mathrm{Spec}(k) \to S$, then its image under the projection $\pi : S \to S_0$ belongs to U_Z for each Z, such that $\mathrm{im}(i_Z) \neq S_0$. This means that this image is noting but the generic point $\eta_0 = \mathrm{Spec}(k_0(S_0))$ of the scheme S_0. In other words, there exists a morphism $h_P : \mathrm{Spec}(k) \to \mathrm{Spec}(k_0(S_0)) = \eta_0$, such that $\pi \circ f_P = g_0 \circ h_P$, where g_0 is a morphism from the generic point η_0 to S_0. In terms of commutative rings, it means the following. If $ev_P : k[S] \to k$ is the evaluation at P, i.e. the morphism inducing f_P on spectra, there exists a

morphism of fields ϵ_P making the diagram

$$\begin{array}{ccc} k[S] & \xrightarrow{ev_P} & k \\ \uparrow & & \uparrow{\scriptstyle \epsilon_P} \\ k_0[S_0] & \longrightarrow & k_0(S_0) \end{array} \tag{7}$$

to be commutative, where k in the top right corner is considered as the residue field of the scheme S at P. Since the left vertical morphism is injective, $k_0[S_0] \smallsetminus \{0\}$ is a multiplicative system in $k[S]$. It is not hard to see that the localization $(k_0[S_0] \smallsetminus \{0\})^{-1}k[S]$ is the tensor product of $k[S]$ and $k_0(S_0)$ over $k_0[S_0]$. This is why there exists a unique universal morphism of rings ε_P whose restriction on the ring $k[S]$ is ev_P and the restriction on $k_0(S_0)$ is ϵ_P. Our aim is to construct an embedding of $k(S)$ into k whose restriction to $k_0(S_0)$ would be ϵ_P. Certainly, such an embedding will *not* be over the ring $(k_0[S_0] \smallsetminus \{0\})^{-1}k[S]$.

Let d be the dimension of S_0. By the Noether normalization lemma, there exist d algebraically independent elements, $x_1, \ldots, x_d$ in $k_0[S_0]$, such that the latest ring is integral (i.e. finitely generated) over the ring $k_0[x_1, \ldots, x_d]$, and $k_0(S_0)$ is algebraic over the field of fractions $k_0(x_1, \ldots, x_d)$. Then $k[S]$ is integral over the ring $k[x_1, \ldots, x_d]$ and $k(S)$ is algebraic over $k(x_1, \ldots, x_d)$. Let $b_i = ev_P(x_i)$ for $i = 1, \ldots, d$. Since $P \in U$, the quantities $b_1, \ldots, b_d$ are algebraically independent over k_0. Extend the set $\{b_1, \ldots, b_d\}$ to a transcendental basis B of k over k_0, so that $k = k_0(B)$. Since B is of infinite cardinality, so is the set $B \smallsetminus \{b_1, \ldots, b_d\}$.

Choose and fix a bijection

$$B \xrightarrow{\sim} B \smallsetminus \{b_1, \ldots, b_d\} \ .$$

It gives a field embedding

$$k = k_0(B) \simeq k_0(B \smallsetminus \{b_1, \ldots, b_d\}) \subset k_0(B) = \Omega$$

over k_0, such that the set $\{b_1, \ldots, b_d\}$ is algebraically independent over its image. The latter embedding induces a new field embedding

$$k(x_1, \ldots, x_d) \to k$$

sending x_i to b_i for each i. The restriction of this field embedding on $k_0(x_1, \ldots, x_d)$ is the restriction of ϵ_P on the same field. Since $k(S)$ is the tensor product of $k(x_1, \ldots, x_d)$ and $k_0(S_0)$ over $k_0(x_1, \ldots, x_d)$, we get a uniquely defined embedding

$$k(S) \to k \ ,$$

which can be extended to an isomorphism

$$e_P : \overline{k(S)} \xrightarrow{\sim} k \ .$$

As the square (7) is commutative, $e_P(f) = f(P)$ for each f in $k_0[S_0]$. □

Remark 17. It is important to mention that the above isomorphism e_P is non-canonical depends on the choice of the transcendental basis B containing the quantities $b_1, \ldots, b_d$.

Let now $f : \mathscr{X} \to S$ be a smooth morphism of schemes over k. Extending k_0 if necessary, we may assume that there exists a morphism of schemes $f_0 : \mathscr{X}_0 \to S_0$ over k_0, such that f is the pull-back of f_0 under the field extension from k_0 to k. Let $\eta_0 = \mathrm{Spec}(k_0(S_0))$ be the generic point of the scheme S_0, $\eta = \mathrm{Spec}(k(S))$ the generic point of the scheme S, and $\bar{\eta} = \mathrm{Spec}(\overline{k(S)})$ be the geometric generic point of S. Then we also have the corresponding fibres $\mathscr{X}_{0,\eta_0}$, $\mathscr{X}_\eta$ and $\mathscr{X}_{\bar{\eta}}$.

Pulling back the scheme-theoretic isomorphism $\mathrm{Spec}(e_P)$ to the fibres of the family f, we obtain the Cartesian square

$$\begin{array}{ccc} \mathscr{X}_P & \longrightarrow & \mathrm{Spec}(k) \\ {\scriptstyle \varkappa_P}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(e_P)} \\ \mathscr{X}_{\bar{\eta}} & \longrightarrow & \bar{\eta} \end{array}$$

Since $\mathrm{Spec}(e_P)$ is an isomorphism of schemes over η_0, the morphism $\varkappa_P$ is an isomorphism of schemes over $\mathscr{X}_{0,\eta_0}$.

For any field F, a scheme Y over F and an automorphism σ of F let Y_σ be the fibred product of Y and $\mathrm{Spec}(F)$ over $\mathrm{Spec}(F)$, with regard to the automorphism $\mathrm{Spec}(\sigma)$, and let $w_\sigma : Y_\sigma \xrightarrow{\sim} Y$ be the corresponding isomorphism of schemes over $\mathrm{Spec}(F^\sigma)$, where F^σ is the subfield of σ-invariant elements in F.

Let L be a field subextension of k/k_0. The projection $\mathscr{X} \to \mathscr{X}_0$ naturally factorizes through $\mathscr{X}_{0L} = \mathscr{X}_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(L)$. Composing the embedding of the fibre $\mathscr{X}_P$ into the total scheme $\mathscr{X}$ with the morphism $\mathscr{X} \to \mathscr{X}_{0L}$ we can consider $\mathscr{X}_P$ as a scheme over $\mathscr{X}_{0L}$.

If now P' is another closed k-point in U, let $\sigma_{PP'} = e_{P'} \circ e_P^{-1}$ be the automorphism of the field k, and let $\varkappa_{PP'} = \varkappa_{P'}^{-1} \circ \varkappa_P$ be the induced isomorphism of the fibres as schemes over $\mathrm{Spec}(k^{\sigma_{PP'}})$. In these terms, $(\mathscr{X}_P)_{\sigma_{PP'}} = \mathscr{X}_{P'}$, the isomorphism $w_{\sigma_{PP'}} : \mathscr{X}_{P'} \xrightarrow{\sim} \mathscr{X}_P$ is over $\mathscr{X}_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k^{\sigma_{PP'}})$, and $w_{\sigma_{PP'}} = \varkappa_{P'P}$. To see that we just need to use Lemma 16 and pull-back the scheme-theoretic isomorphisms between points on S to isomorphisms between the corresponding fibres of the morphism $f : \mathscr{X} \to S$.

Remark 18. The assumption that S is affine is not essential, of course. We can always cover S by open affine subschemes, construct the system of isomorphisms $\varkappa$ in each affine chart and then construct "transition isomorphisms" between very general fibres in a flat family over an arbitrary integral base S.

Let now S be an integral scheme of finite type over k, let $\mathscr{X}$ and $\mathscr{Y}$ be two schemes, both smooth, projective and connected over S, and let

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\ r\ } & \mathscr{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

be a closed embedding morphism of schemes over the base S. Extending k_0 appropriately we may assume that there exist models f_0, g_0 and r_0 over k_0 of the

morphisms f, g and r respectively, such that $g_0 \circ r_0 = f_0$. Then, for any closed k-point P in U, the diagram

$$\begin{array}{ccc} \mathscr{Y}_P & \xrightarrow{r_P} & \mathscr{X}_P \\ \downarrow{\scriptstyle \varkappa_P} & & \downarrow{\scriptstyle \varkappa_P} \\ \mathscr{Y}_{\bar{\eta}} & \xrightarrow{r_{\bar{\eta}}} & \mathscr{X}_{\bar{\eta}} \end{array} \tag{8}$$

is commutative, where r_P and $r_{\bar{\eta}}$ are the obvious morphisms on fibres induced by the morphism r. Then, of course, the isomorphisms $\varkappa_{PP'}$ commute with the morphisms r_P and $r_{P'}$, for any two closed k-points P and P' in U. Cutting out more Zariski closed subsets from U we may assume that the fibres of the families f and g over the points from U are smooth.

Lemma 19. *The scheme-theoretic isomorphisms $\varkappa_P$ preserve the algebraic and rational equivalence of algebraic cycles.*

Proof. As we already explained in Section 3, if $\alpha : k \xrightarrow{\sim} k'$ is an isomorphism of fields, the functorial bijections Υ from the representation of Chow monoids by Chow schemes commute through the isomorphisms of monoids and Hom-sets induced by the isomorphism $\operatorname{Spec}(\alpha)$. In particular, if $k' = \overline{k(S)}$ and $\alpha = e_P^{-1}$, the bijections $\Upsilon(\mathscr{X}_{\bar{\eta}})$ over $\overline{k(S)}$ commute with the bijections $\Upsilon(\mathscr{X}_P)$ over k, and the same for $\mathscr{Y}$. The commutativity for the sections of the corresponding presheaves on an algebraic curve C over k and its pull-back C' over k' gives the first assertion of the lemma. If $C = \mathbb{P}^1$, we get the second one. □

Assume now that Assumptions 1, 2 and 3 are satisfied for the geometric generic fibre $\mathscr{Y}_{\bar{\eta}}$, and for the fibre $\mathscr{Y}_P$ for each closed point P in U. Let also

$$\psi_{\bar{\eta}} : A^p(\mathscr{Y}_{\bar{\eta}}) \xrightarrow{\sim} A_{\bar{\eta}}$$

and

$$\psi_P : A^p(\mathscr{Y}_P) \xrightarrow{\sim} A_P$$

be the corresponding regular parametrizations. By Lemma 19, the isomorphism $\varkappa_P$ induces the push-forward isomorphism of abelian groups

$$\varkappa_{P*} : A^p(\mathscr{Y}_P) \to A^p(\mathscr{Y}_{\bar{\eta}}) \, .$$

Let

$$\kappa_P : A_P \to A_{\bar{\eta}}$$

be the composition given by the commutative diagram

$$\begin{array}{ccc} A_P & \xrightarrow{\psi_P^{-1}} & A^p(\mathscr{Y}_P) \\ \downarrow{\scriptstyle \kappa_P} & & \downarrow{\scriptstyle \varkappa_{P*}} \\ A_{\bar{\eta}} & \xleftarrow{\psi_{\bar{\eta}}} & A^p(\mathscr{Y}_{\bar{\eta}}) \end{array} \tag{9}$$

Consider the obvious commutative diagram

$$\begin{array}{ccccc} A_P & \xleftarrow{\psi_P} & A^p(\mathscr{Y}_P) & \xleftarrow{\theta_d} & C^p_{d,d}(\mathscr{Y}_P) \\ \downarrow{\scriptstyle \kappa_P} & & \downarrow{\scriptstyle \varkappa_{P*}} & & \downarrow{\scriptstyle C^p_{d,d}(\varkappa_P)} \\ A_{\bar\eta} & \xleftarrow{\psi_{\bar\eta}} & A^p(\mathscr{Y}_{\bar\eta}) & \xleftarrow{\theta_d} & C^p_{d,d}(\mathscr{Y}_{\bar\eta}) \end{array} \tag{10}$$

The top and bottom horizontal compositions in this diagram are regular morphisms of schemes over k and $\overline{k(S)}$ respectively, and the vertical morphism from the right hand side is a regular morphism of schemes over $\mathbb{Q}$. It follows that the homomorphism $\kappa_P : A_P \to A_{\bar\eta}$ is a regular morphism of schemes over $\mathbb{Q}$ too.

Now, the commutative diagram (8) gives the commutative diagram

$$\begin{array}{ccc} A^p(\mathscr{Y}_P) & \xrightarrow{r_{P*}} & A^{p+e}(\mathscr{X}_P) \\ \downarrow{\scriptstyle \varkappa_{P*}} & & \downarrow{\scriptstyle \varkappa_{P*}} \\ A^p(\mathscr{Y}_{\bar\eta}) & \xrightarrow{r_{\bar\eta *}} & A^{p+e}(\mathscr{X}_{\bar\eta}) \end{array} \tag{11}$$

where e is the codimension of $\mathscr{Y}_{\bar\eta}$ in $\mathscr{X}_{\bar\eta}$. Let $A_{P,1}$ and $A_{\bar\eta,1}$ be the abelian subvarieties in A_P and $A_{\bar\eta}$ respectively, constructed in Section 2. Let, furthermore, $A_{P,0}$ and $A_{\bar\eta,0}$ be the abelian subvarieties in $A_{P,1}$ and $A_{\bar\eta,1}$ respectively, provided by Theorem 13 and Corollary 15.

Proposition 20. *For any closed point P in U,*

$$\kappa_P(A_{P,1}) = A_{\bar\eta,1}$$

and

$$\kappa_P(A_{P,0}) = A_{\bar\eta,0}$$

Proof. The first claim is actually true for any closed point P on S, not only on U, and can be easily deduced using specialization isomorphisms on étale cohomology groups. Let us prove the second claim. Let Ξ_P be the countable subset in A_P and $\Xi_{\bar\eta}$ the countable subset in $A_{\bar\eta}$, such that we have the kernels

$$K_P = \cup_{x\in\Xi_P}(s + A_{P,0}) \quad \text{and} \quad K_{\bar\eta} = \cup_{x\in\Xi_{\bar\eta}}(x + A_{\bar\eta,0})$$

in A_P and $A_{\bar\eta}$ respectively (see Theorem 13). Then

$$\kappa_P(K_P) = \kappa_P(\cup_{x\in\Xi_P}(x + A_{P,0})) = \cup_{x\in\Xi_P}(\kappa_P(x) + \kappa_P(A_{P,0}))$$

The definition of κ_P and the commutative diagram (11) give that

$$\kappa_P(K_P) = K_{\bar\eta}\ .$$

Therefore,

$$\cup_{x\in\Xi_P}(\kappa_P(x) + \kappa_P(A_{P,0})) = \cup_{x\in\Xi_{\bar\eta}}(x + A_{\bar\eta,0})$$

inside the abelian variety $A_{\bar\eta}$. Since the group isomorphisms κ_P are regular morphisms of schemes over $\mathrm{Spec}(\mathbb{Q})$, we obtain that $\kappa_P(A_{P,0})$ is a Zariski closed

subset in $A_{\bar{\eta}}$. Since $\kappa_P(A_{P,0})$ is a subgroup in $A_{\bar{\eta}}$, it is an abelian subvariety in $A_{\bar{\eta}}$. Lemma 11 and Lemma 12 finish the proof. □

Remark 21. Of course, the set U is not uniquely defined, and all the same works well over the union of all such c-open subset in the integral scheme S. The behaviour of A_0 beyond the union of the sets U is an open question of particular importance and deserves a separate big research programme.

6. Étale monodromy argument for cycles of dimension $p-1$

This is the main section of the paper, in which we apply Theorem A in the family of hyperplane sections of a projective variety embedded into a projective space. In such a case one can enhance the study of A_0 by the monodromy argument in terms of étale l-adic cohomology over k.

Let $d = 2p$ and let X be a smooth d-dimensional projective variety over the ground field k. Fix a closed embedding $X \subset \mathbb{P}^m$, such that X is not contained in a smaller linear subspace in $\mathbb{P}^m$. Let

$$\mathscr{H} = \{(P, H) \in \mathbb{P}^m \times \mathbb{P}^{m\vee} \mid P \in H\}$$

be the universal hyperplane, and let p_1 and p_2 be the projections of $\mathscr{H}$ on $\mathbb{P}^m$ and $\mathbb{P}^{m\vee}$ respectively. Let

$$\mathscr{X} = X \times \mathbb{P}^{m\vee}$$

and let

$$\mathscr{Y} = \mathscr{X} \cap \mathscr{H}$$

inside $\mathbb{P}^m \times \mathbb{P}^{m\vee}$.

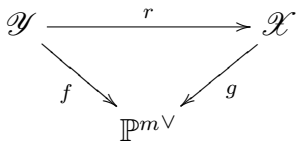
Let

$$f : \mathscr{Y} \to \mathbb{P}^{m\vee}$$

be the composition of the closed embedding $\mathscr{Y} \subset \mathscr{H}$ with the projection p_2, let

$$g : \mathscr{X} \to \mathbb{P}^{m\vee}$$

be the composition of the closed embedding of $\mathscr{X}$ into $\mathbb{P}^m \times \mathbb{P}^{m\vee}$ with the projection onto $\mathbb{P}^{m\vee}$, and let



be the obvious closed embedding over the dual projective space.

For any scheme S and for any morphism of schemes

$$S \to \mathbb{P}^{m\vee}$$

let $\mathscr{H}_S \to S$ be the pull-back of p_2 with respect to the morphism $S \to \mathbb{P}^{m\vee}$, let $\mathscr{Y}_S$ be the fibred product of $\mathscr{Y}$ and $\mathscr{H}_S$ over the universal hyperplane $\mathscr{H}$, and let

$$f_S : \mathscr{Y}_S \to S$$

be the induced projection, i.e. the composition of the closed embedding of $\mathscr{Y}_S$ into $\mathscr{H}_S$ and the morphism $\mathscr{H}_S \to S$. Let also $\mathscr{X}_S \to S$ be the pull-back of trivial

family $\mathscr{X} \to \mathbb{P}^{m\vee}$ with respect to the morphism $S \to \mathbb{P}^{m\vee}$. Then we obtain the closed embedding

$$\begin{array}{ccc} \mathscr{Y}_S & \xrightarrow{r_S} & \mathscr{X}_S \\ & {\scriptstyle f_S}\searrow \quad \swarrow{\scriptstyle g_S} & \\ & S & \end{array}$$

over S.

Assume that the scheme S is integral. Let $k(S)$ be the function field of S,

$$\eta = \mathrm{Spec}(k(S))$$

be the generic point of S, $\overline{k(S)}$ be the algebraic closure of $k(S)$ and

$$\bar{\eta} = \mathrm{Spec}(\overline{k(S)})$$

be the geometric generic point of S. Then we also have the closed embeddings r_η and $r_{\bar{\eta}}$ over η and $\bar{\eta}$ respectively.

As in the previous section, choose an appropriate c-open subset U in S, such that the point $\bar{\eta}$ scheme-theoretically is isomorphic to each closed point in P, and assume that Assumptions 1, 2 and 3 are satisfied for the geometric generic fibre

$$Y_{\bar{\eta}} = \mathscr{Y}_{\bar{\eta}} \ ,$$

and for the fibre

$$Y_P = \mathscr{Y}_P \ ,$$

for each closed point P in U. Let also

$$\psi_{\bar{\eta}} : A^p(Y_{\bar{\eta}}) \xrightarrow{\sim} A_{\bar{\eta}}$$

and

$$\psi_P : A^p(Y_P) \xrightarrow{\sim} A_P$$

be the corresponding regular parametrizations. Then we have the abelian subvarieties

$$A_{\bar{\eta},0} \subset A_{\bar{\eta},1} \subset A_{\bar{\eta}} \qquad \text{and} \qquad A_{P,0} \subset A_{P,1} \subset A_P$$

for each closed point P in U.

Let first $S = D$ be a projective line inside the dual space $\mathbb{P}^{m\vee}$, such that the morphism f_D is a Lefschetz pencil for the variety X. Let L be the minimal subextension of $k(D)$ in $\overline{k(D)}$, such that the abelian varieties $A_{\bar{\eta},0}$, $A_{\bar{\eta},1}$ and $A_{\bar{\eta}}$ are defined over L. Then L is finitely generated and algebraic of finite degree n over $k(D)$. Let D' be an algebraic curve, such that $L = k(D')$ and the embedding of k into $k(D)$ is induced by a generically of degree n morphism from D' onto D. Since the closed embedding of $A_{\bar{\eta},0}$ into $A_{\bar{\eta},1}$ and $A_{\bar{\eta},1}$ into $A_{\bar{\eta}}$ are now defined over L, there exist a Zariski open subset U' in D', spreads $\mathscr{A}_{\bar{\eta},0}$, $\mathscr{A}_{\bar{\eta},1}$ and $\mathscr{A}_{\bar{\eta}}$ of $A_{\bar{\eta},0}$, $A_{\bar{\eta},1}$ and $A_{\bar{\eta}}$ respectively over U', and morphisms

$$\mathscr{A}_{\bar{\eta},0} \to \mathscr{A}_{\bar{\eta},1}$$

and

$$\mathscr{A}_{\bar{\eta},1} \to \mathscr{A}_{\bar{\eta}}$$

over U', such that, when passing to the fibres at the geometric generic point $\bar{\eta}$, we obtain the closed embeddings

$$A_{\bar{\eta},0} \to A_{\bar{\eta},1}$$

and

$$A_{\bar{\eta},1} \to A_{\bar{\eta}}$$

over $\overline{k(D)}$.

Let α be the morphism from $\mathscr{A}$ onto U', and let α_0 and α_1 be the morphism from $\mathscr{A}_0$ and, respectively, $\mathscr{A}_1$ onto U'. Since $\mathscr{A}$ is a spread of $A_{\bar{\eta}}$ over U' and $A_{\bar{\eta}}$ is a projective variety over $L = k(D')$, the morphism α is locally projective and, therefore, proper. Similarly, the morphisms α_0 and α_1 are proper. Cutting more points from D' we may assume that the morphisms α, α_0 and α_1 are all smooth over U'.

Let η' be the generic point of D', let $\bar{\eta}' = \bar{\eta}$ be the geometric generic point of D', let $\pi_1(U', \bar{\eta})$ be the étale fundamental group of D' pointed at $\bar{\eta}$, and let $\pi_1^{\mathrm{tame}}(U', \bar{\eta})$ be the corresponding tame fundamental group. For any scheme V and non-negative integer n let $(\mathbb{Z}/l^n)_V$ be the constant sheaf on V associated to the group $\mathbb{Z}/l^n$.

Since the morphisms α_0, α_1 and α are smooth and proper, the higher direct images

$$R^1\alpha_{0*}(\mathbb{Z}/l^n)_{\mathscr{A}_0}\ , \quad R^1\alpha_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}_1} \qquad \text{and} \qquad R^1\alpha_*(\mathbb{Z}/l^n)_{\mathscr{A}}$$

are locally constant by Theorem 8.9, Ch. I in [10]. Then the fibres of these sheaves at the geometric generic point $\bar{\eta}$ are finite continuous $\pi_1(U', \bar{\eta})$-modules, see Proposition A I.7 in loc. cit. The proper base change (see, for example, Theorem 6.1′ on page 62 in loc. cit.) gives that

$$(R^1\alpha_{0*}(\mathbb{Z}/l^n)_{\mathscr{A}_0})_{\bar{\eta}} = H^1_{\text{ét}}(\mathscr{A}_{0\bar{\eta}}, \mathbb{Z}/l^n)\ ,$$
$$(R^1\alpha_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}_1})_{\bar{\eta}} = H^1_{\text{ét}}(\mathscr{A}_{1\bar{\eta}}, \mathbb{Z}/l^n)$$

and

$$(R^1\alpha_*(\mathbb{Z}/l^n)_{\mathscr{A}})_{\bar{\eta}} = H^1_{\text{ét}}(\mathscr{A}_{\bar{\eta}}, \mathbb{Z}/l^n)\ .$$

Then we obtain that $\pi_1(U', \bar{\eta})$ acts continuously on

$$H^1_{\text{ét}}(\mathscr{A}_{0\bar{\eta}}, \mathbb{Z}/l^n)\ ,$$
$$H^1_{\text{ét}}(\mathscr{A}_{1\bar{\eta}}, \mathbb{Z}/l^n)$$

and

$$H^1_{\text{ét}}(\mathscr{A}_{\bar{\eta}}, \mathbb{Z}/l^n)\ .$$

Passing to limits on n and then tensoring with $\mathbb{Q}_l$ we then obtain that $\pi_1(U', \bar{\eta})$ acts continuously on

$$H^1_{\text{ét}}(\mathscr{A}_{0\bar{\eta}}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l)\ ,$$
$$H^1_{\text{ét}}(\mathscr{A}_{1\bar{\eta}}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{\bar{\eta},1}, \mathbb{Q}_l)$$

and

$$H^1_{\text{ét}}(\mathscr{A}_{\bar{\eta}}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Q}_l)\ .$$

The homomorphism

$$\zeta_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l) \to H^1_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Q}_l)$$

is the composition of the obvious homomorphisms

$$\zeta'_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l) \to H^1_{\text{ét}}(A_{\bar{\eta},1}, \mathbb{Q}_l)$$

and

$$\zeta''_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_{\bar{\eta},1}, \mathbb{Q}_l) \to H^1_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Q}_l) \ .$$

The action of $\pi_1(U', \bar{\eta})$ naturally commutes with both $\zeta'_{\mathbb{Q}_l}$ and $\zeta''_{\mathbb{Q}_l}$.

Without loss of generality, we may assume that U' is the pre-image of a Zariski open subset U in D and all the fibres of the Lefschetz pencil

$$f_D : \mathscr{Y}_D \to D$$

over the closed points of U are smooth. Let

$$f_{D'} : \mathscr{Y}_{D'} \to D'$$

be the pull-back of the pencil f_D with respect to the morphism $D' \to D$, and let

$$f_{U'} : \mathscr{Y}_{U'} \to U'$$

be the pull-back of $f_{D'}$ with respect to the open embedding of U' to D'. Applying the same arguments to the morphism $f_{U'}$, we obtain the continuous action of the étale fundamental group $\pi_1(U', \bar{\eta})$ on the group $H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$, and it is well known that this action is tame, in the sense that it factorizes through the surjective homomorphism from $\pi_1(U', \bar{\eta})$ onto $\pi_1^{\text{tame}}(U', \bar{\eta})$.

For each closed point s in the complement $D \smallsetminus U$ let

$$\delta_s \in H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$$

be the unique up-to conjugation vanishing cycle corresponding to the point s in the standard sense (see Theorem 7.1 on page 247 in [10]), and let

$$E \subset H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$$

be the $\mathbb{Q}_l$-vector subspace generated by all the elements δ_s, $s \in D \smallsetminus U$. In other words, E is the space of vanishing cycles in $H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$. One can show that

$$E = \ker(H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l) \to H^{2p+1}_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_l)) \ ,$$

where $X_{\bar{\eta}} = X \times \bar{\eta}$, see Section 4.3 in [8].

In what follows we will be using the étale l-adic Picard-Lefschetz formula for the monodromy action. For each $s \in D \smallsetminus U$ let

$$\pi_{1,s} \subset \pi_1^{\text{tame}}(U, \bar{\eta})$$

be the so-called tame fundamental group at s, a subgroup uniquely determined by the point s up to conjugation in $\pi_1^{\text{tame}}(U, \bar{\eta})$. In terms of [10], $\pi_{1,s}$ is the image of the homomorphism

$$\gamma_s : \hat{\mathbb{Z}}(1) \to \pi_1^{\text{tame}}(U, \bar{\eta}) \ ,$$

where $\hat{\mathbb{Z}}(1)$ is the limit of all groups μ_n, and μ_n is the group of n-th roots of unity in the algebraically closed field $\overline{k(U)}$ whose exponential characteristic is 1.

The tame fundamental group $\pi_1^{\text{tame}}(U, \bar{\eta})$ is generated by the subgroups $\pi_{1,s}$. If u is an element in $\hat{\mathbb{Z}}(1)$, let $\bar{u}$ be the image of u in $\mathbb{Z}_l(1)$. If now v is an element in the $\mathbb{Q}_l$-vector space $H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$ the Picard-Lefschetz formula says

$$\gamma_s(u)x = x \pm \bar{u}\langle x, \delta_s\rangle \delta_s \ . \tag{12}$$

Proposition 22. *Under the assumptions above, either* $A_{\bar{\eta},0} = 0$ *or* $A_{\bar{\eta},0} = A_{\bar{\eta},1}$.

Proof. By Proposition 14 and the fact that the space E of vanishing cycles coincides with the kernel of the Gysin homomorphism

$$r_{\bar{\eta}*} : H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l) \to H^{2p+1}_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_l) \ ,$$

we see that the image of the composition

$$H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l(1-p)) \xrightarrow{\zeta_{\mathbb{Q}_l}} H^1_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Q}_l(1-p)) \xrightarrow{w_*} H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$$

is contained in E. The homomorphism $\zeta_{\mathbb{Q}_l}$ is injective and compatible with the action of $\pi_1(U', \bar{\eta})$. Since $p \leq 2$, the homomorphism w_* is bijective, see Remark 4. Then

$$E \simeq H^1_{\text{ét}}(A_{\bar{\eta},1}, \mathbb{Q}_l(1-p))$$

via $\zeta''_{\mathbb{Q}_l}$ and w_*.

Since the variety $Y_{\bar{\eta}}$ satisfies Assumption 1, there exists a smooth projective curve Γ and an algebraic cycle Z on $\Gamma \times Y$ over $\bar{\eta}$, such that the cycle class z of Z induces the surjective homomorphism

$$z_* : A^1(\Gamma) \to A^p(Y_{\bar{\eta}}) \ ,$$

whose kernel is G. The homomorphism

$$w_* : H^1_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\text{ét}}(Y_{\bar{\eta}}, \mathbb{Q}_l)$$

is then induced by the composition of the embedding of the curve Γ into its Jacobian J_Γ over $\bar{\eta}$, the quotient map from J_Γ onto the abelian variety $A = J_\Gamma$, also over $\bar{\eta}$, and the homomorphism induced by the correspondence Z (see Section 2). Spreading out the morphisms $\Gamma \to J_\Gamma$ and $J_\Gamma \to A$, as well as the cycle X, over a certain Zariski open subset in D', we can achieve that the homomorphism w_* is compatible with the action of the fundamental group $\pi_1(U', \bar{\eta})$.

This gives that the composition $w_* \circ \zeta_{\mathbb{Q}_l}$ is an injection of the $\pi_1(U', \bar{\eta})$-module $H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l(1-p))$ into the $\pi_1(U', \bar{\eta})$-module of vanishing cycles E. Let

$$E_0 = \text{im}(w_* \circ \zeta_{\mathbb{Q}_l})$$

be the image of this injection.

Since U' is finite of degree n over U, the group $\pi_1(U', \bar{\eta})$ is a subgroup of finite index n in the étale fundamental group $\pi_1(U, \bar{\eta})$, and the latter group acts continuously on E by the standard étale monodromy theory.

We are now going to use the Picard-Lefschetz formula in order to show that E_0 is a $\pi_1^{\text{tame}}(U, \bar{\eta})$-equivariant subspace in E. Obviously, it is enough to show that for each element $\gamma_s(u)$ in $\pi_1^{\text{tame}}(U, \bar{\eta})$ and any element x in E_0 the element $\gamma_s(u)x$ is again in the space E_0.

Indeed, since $\langle \delta_s, \delta_s \rangle = 0$, the Picard-Lefschetz formula (12) and easy induction give that

$$(\gamma_s(u))^m x = x \pm m\bar{u}\langle x, \delta_s \rangle \delta_s \ ,$$

for a natural number m, whence

$$\bar{u}\langle x, \delta_s \rangle \delta_s = \frac{1}{m} \left((\gamma_s(u))^m x \pm x \right) \ .$$

When m is the index of $\pi_1(U', \bar{\eta})$ in $\pi_1(U, \bar{\eta})$, then $(\gamma_s(u))^m$ sits in the subgroup $\pi_1(U', \bar{\eta})$, so that the right hand side of the latter formula is an element of E_0. Applying the Picard-Lefschetz formula again, we see that $\gamma_s(u)x$ is in E_0.

Thus, E_0 is a submodule in the $\pi_1^{\mathrm{tame}}(U, \bar{\eta})$-module E. Since E is known to be an absolutely irreducible (see, for example, Corollary 7.4 on page 249 in [10]), we see that either $E_0 = 0$ or $E_0 = E$. In the first case $H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l) = 0$, whence $A_{\bar{\eta},0} = 0$. In the second case

$$\zeta'_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_{\bar{\eta},0}, \mathbb{Q}_l(1-p)) \to H^1_{\text{ét}}(A_{\bar{\eta},1}, \mathbb{Q}_l(1-p))$$

is an isomorphism, whence $A_{\bar{\eta},0} = A_{\bar{\eta},1}$. □

Let now T be the complement to the discriminant variety of X in $\mathbb{P}^{m\vee}$. In other words, T is the set hyperplanes in $\mathbb{P}^m$ whose intersections with X are smooth. Now we want to consider the global case, when the base scheme S is the scheme T. Let again U be a c-open subset in T constructed as in Section 5. In other words, we define U by removing the images of the pull-backs of all closed embeddings into the model T_0 of T defined over the minimal field of definition of T. Then U is a c-open subset in T, and in the dual projective space $(\mathbb{P}^m)^\vee$, such that, if ξ is the generic point of the projective space $\mathbb{P}^{m\vee}$ and $\bar{\xi}$ the corresponding geometric generic point, for any closed point $P \in U$ one has the isomorphism $\varkappa_P$ between Y_P and $Y_{\bar{\xi}}$, and for any two closed points P and P' in U one has the scheme-theoretic isomorphism $\varkappa_{PP'}$ between Y_P and $Y_{P'}$, constructed in Section 5. As above, we assume that Assumptions 1, 2 and 3 are satisfied for the fibres at $\bar{\xi}$ and at every closed point P of the set U.

The next theorem is Theorem B in Introduction, and it represents the main result in the paper.

Theorem 23. *In terms above, either $A_{\bar{\xi},0} = 0$, in which case $A_{P,0} = 0$, or $A_{\bar{\xi},0} = A_{\bar{\xi},1}$, so that $A_{P,0} = A_{P,1}$, for any closed point P in U.*

Proof. For every closed point P in $\mathbb{P}^{m\vee}$ let H_P be the corresponding hyperplane in $\mathbb{P}^m$. Let Σ be a Zariski closed subset in $\mathbb{P}^{m\vee}$, such that for each point P in the complement to Σ in $\mathbb{P}^{m\vee}$ the hyperplane H_P does not contain X and the scheme-theoretic intersection of X, and $X \cap H_P$ is either smooth or contains at most one singular point, which is double point, see Definition 1.4 and Proposition 1.5 in [10], or read through Exposé XVII in [12]. Let G be the Grassmannian of lines in $\mathbb{P}^{m\vee}$. There is a Zariski open subset W in G, such that for each line D in W the line D does not intersect Σ and the corresponding codimension 2 linear subspace in $\mathbb{P}^m$ intersects X transversally. In other words, any line D from W gives rise to a Lefschetz pencil on X, see the top of page 180 in [10] or Exposé XVII in [12]. Let Z be the complement to the above c-open subset U in $\mathbb{P}^{m\vee}$. Then Z is the union of a countable collection of Zariski closed irreducible subsets in $\mathbb{P}^{m\vee}$, each of which is irreducible. In particular, Z is c-closed. It follows that the condition for a line $D \in G$ to be not a subset in Z is c-open. By Lemma 10, the intersection of the corresponding c-open subset in G with W is non-empty, so that we can choose a line D, such that D gives a Lefschetz pencil $f_D : \mathscr{Y}_D \to D$ and $D \cap U \neq \emptyset$. Let P_0 be a point in $D \cap U$ and let $\bar{\eta}$ be the geometric generic point of D. By Proposition 22, either $A_{\bar{\eta},0} = 0$ or $A_{\bar{\eta},0} = A_{\bar{\eta},1}$. Suppose $A_{\bar{\eta},0} = 0$.

Proposition 20, being applied to the pencil f_D, gives that $A_{P_0,0} = 0$. Applying the same proposition to the family f_T we obtain that $A_{\bar{\xi},0} = 0$ and so for each closed point P in U the abelian variety $A_{P,0}$ is zero. Similarly, if $A_{\bar{\eta},0} = A_{\bar{\eta},1}$ then, by Proposition 20 applied to f_D we obtain that $A_{P_0,0} = A_{P_0,1}$. Applying Proposition 20 to the family f we see that $A_{\bar{\xi},0} = A_{\bar{\xi},1}$ and $A_{P,0} = A_{P,1}$ for each closed point P in U. □

7. Applications in the study of 1-cycles on 4-dimensional varieties

The purpose of this section is to apply Theorem 23 (Theorem B in Introduction) in the study of rational equivalence of 1-dimensional algebraic cycles on 4-dimensional varieties, extending Voisin's idea on page 305 in the second volume of [31]. We keep all the notation and assumptions of the previous section. To enhance and, at the same time, simplify the exposition, we will assume that Assumptions 1, 2 and 3 are satisfied not only for the fibres at closed points of the c-open subset $U \subset T$, but rather for the fibres at all closed points of the set T, i.e. at all smooth sections of the variety X by hyperplanes in $\mathbb{P}^{m\vee}$. Assume, moreover, that $p \leq 2$ and that the group $H^{2p+1}_{\text{ét}}(X, \mathbb{Q}_l)$ vanishes. The latter implies that $A_{\bar{\eta},1} = A_{\bar{\eta}}$ for the generic point η of S, $A_{\bar{\xi},1} = A_{\bar{\xi}}$ and $A_{P,1} = A_P$ for each closed point P in T.

For any closed point $P \in \mathbb{P}^{m\vee}$ let $\tilde{Y}_P$ be the resolution of singularities of the section Y_P. In addition to the assumptions above, we will also require that whenever the section Y_P has at worst one singular point and this point is an ordinary double point, the continuous group $A^p(\tilde{Y}_P)$ is weakly representable.

Since the group $A^p(Y_{\bar{\xi}})$ is weakly representable, we can choose a smooth projective curve C over $\bar{\xi}$ and an appropriate algebraic cycle Z on $C \times Y_{\bar{\xi}}$, such that the induced homomorphism Z_* from $A^1(C)$ to $A^p(Y_{\bar{\xi}})$ is surjective. Then the homomorphism θ^p_d from $C^p_{d,d}(Y_{\bar{\xi}})$ to $A^p(Y_{\bar{\xi}})$ is surjective for big enough d (see the proof of Theorem 13). Since the group $A^p(\tilde{Y}_P)$ is weakly representable, whenever Y_P has at worst one singular point and this point is an ordinary double point, the homomorphism θ^p_d from $C^p_{d,d}(\tilde{Y}_P)$ to $A^p(\tilde{Y}_P)$ is surjective as well.

It is important to stress that, as we now assume that Assumptions 1, 2 and 3 are satisfied for the fibre at every closed point P of T, accordingly we have the abelian varieties $A_{P,0} \subset A_{P,1} \subset A_P$ for every closed point P of T. However, it does not mean that we can extend the coherence provided by Proposition 20 from fibres at closed points of U to fibres at closed points of T. Let $T^\natural$ be the set of closed points in T such that $P \in T^\natural$ if and only if $A_{P,0}$ coincides with $A_{P,1}$.

Lemma 24. *The set $T^\natural$ is constructible.*

Proof. Let

$$\mathscr{V} = \{(Z, P) \in \mathscr{C}^{p+1}_d(X) \times \mathbb{P}^{m\vee} \mid Z \subset H_P\}$$

be the incidence subvariety, where $Z \subset H_P$ means that the codimension $p+1$ algebraic cycle Z of degree d on X is supported on the hyperplane section $X \cap H_P$ for a closed point P in $\mathbb{P}^{m\vee}$. Let

$$v_T : \mathscr{V}_T \to T$$

be the corresponding pull-back of the projection to $\mathbb{P}^{m\vee}$ with respect to the inclusion of T into $\mathbb{P}^{m\vee}$, and let

$$s_T : \mathscr{V}_T \to C^{p+1}_{d,d}(X)$$

be the obvious morphism from $\mathscr{V}_T$ to $C^{p+1}_{d,d}(X)$. Let

$$\mathscr{V}^2_T = \mathscr{V}_T \times_T \mathscr{V}_T$$

be the 2-fold fibred product of $\mathscr{V}_T$ over T, and the consider the corresponding morphisms

$$v^2_T : \mathscr{V}^2_T \to T$$

and

$$s^2_T : \mathscr{V}^2_T \to C^{p+1}_{d,d}(X) \ .$$

By Corollary 9 we have that $(\theta^{p+1}_d)^{-1}(0)$ is the union of a countable collection of irreducible Zariski closed subsets in $C^{p+1}_{d,d}(X)$, say

$$(\theta^{p+1}_d)^{-1}(0) = \cup_{i\in I} Z_i \ .$$

Let

$$W_i = (s^2_T)^{-1}(Z_i)$$

for each $i \in I$. For any closed point P in T the pre-image $(v^2_T)^{-1}(P)$ is the 2-fold product $\mathscr{V}^2_P$ of the fibre $\mathscr{V}_P$ of the morphism v_T at P over $\mathrm{Spec}(k)$. Since the homomorphism θ^p_d from $C^p_{d,d}(Y_P)$ to $A^p(Y_P)$ is surjective, we obtain that the condition $r_{P*} = 0$ is equivalent to the condition that the fibre $\mathscr{V}^2_P$ of the morphism v^2_T at P is a subset of the pre-image $\cup_{i\in I} W_i$ of 0 under the composition $\theta^{p+1}_d \circ s^2_T$. By Lemma 10, this is equivalent to saying that $\mathscr{V}^2_P$ is a subset in

$$W_{i_1} \cup \cdots \cup W_{i_n} \ ,$$

for a finite collection of indices $i_1, \ldots, i_n$ in I. It follows that the set $T^\natural$ is constructible. □

We need one more easy lemma about c-open sets over an uncountable field.

Lemma 25. *Let V be an irreducible quasi-projective variety over k, and let U be a nonempty c-open subset in V. Then the Zariski closure of U in V is V.*

Proof. Indeed, since U is c-open, there exists a countable union $Z = \cup_{i\in I} Z_i$ of Zariski closed irreducible subsets in V, such that $U = V \smallsetminus Z$. Then $\bar{U}$ is nothing but the complement to the interior $\mathrm{Int}(Z)$ of the set Z in V. Assume that that $\mathrm{Int}(Z)$ is nonempty. Then there exists a nonempty subset W in $\mathrm{Int}(Z)$, which is Zariski open in V. By Lemma 10, there exists an index $i_0 \in I$, such that W is contained in Z_{i_0}. This gives that $\mathrm{Int}(Z_{i_0})$ of the set Z_{i_0} is nonempty. This is not possible as Z_{i_0} is a closed proper subset in a Zariski topological space. □

Now, Bloch's definition of weak representability in [2] (see also [4]) can be done for chow groups with coefficients in $\mathbb{Z}$ and with coefficients in $\mathbb{Q}$. In the latter case we will speak about rational weak representability. Keeping the assumptions made in the beginning of this section, we can now prove Theorem C in Introduction.

Theorem 26. *If the group $A^{p+1}(X)$ is not rationally weakly representable, then the kernel of the push-forward homomorphism from $A^p(Y_P)$ to $A^{p+1}(X)$ is countable, for a very general hyperplane section $\mathscr{Y}_P$.*

Proof. By Theorem 23, we have that either $A_{\bar{\xi},0} = 0$ or $A_{\bar{\xi},0} = A_{\bar{\xi}}$. Suppose the latter. By the same Theorem 23, $A_{P,0} = A_P$ for each closed point P in the c-open subset U in T. On the other hand, U is a subset in $T^\natural$, and the set $T^\natural$ is constructible by Lemma 24. Represent U as the complement to a countable union $\cup_{i\in I} D_i$ of irreducible Zariski closed subsets D_i in T, and represent $T^\natural$ as a countable union $\cup_{j\in J} T^\natural_j$, where $T^\natural_j$ is Zariski open in an irreducible Zariski closed subset Z_j in T. Let Z be the union $\cup_{j\in J} Z_j$ and let W be the complement to Z in T. Then W is c-open in T and $W \cap U = \emptyset$. The intersection of W and U is the complement to the union of all D_i and Z_j, $i \in I$, $j \in J$, in T. As $U \neq \emptyset$, it follows that $D_i \neq T$ for each index i. Since $W \cap U = \emptyset$, by Lemma 10, there must exist an index $j_0 \in J$, such that $Z_{j_0} = T$. This gives that $A_{P,0} = A_P$, i.e. $r_{P*} = 0$, for each closed point P in the nonempty Zariski open subset $T^\natural_{j_0}$ in T.

By Lemma 25, the intersection of $T^\natural$ with U is nonempty. Let

$$f_D : \mathscr{Y}_D \to D$$

be a Lefschetz pencil for X, such that the set-theoretic intersection of the line $D = \mathbb{P}^1$ with the set $T^\natural \cap U$ is nonempty. Since the group $A^{p+1}(Y_P)$ is weakly representable for each closed point $P \in T$ and D passes through U, it follows that the group $A^{p+1}(Y_{\bar{\eta}})$ is weakly representable too. Let $\Gamma_{\bar{\eta}}$ be a smooth projective curve and $Z_{\bar{\eta}}$ be an algebraic cycle of codimension 1 on $\Gamma_{\bar{\eta}} \times Y_{\bar{\eta}}$ over $\bar{\eta}$ implementing the weak representability of $A^{p+1}(Y_{\bar{\eta}})$. Let

$$D' \to D$$

be a finite extension of the curve D, such that both $\Gamma_{\bar{\eta}}$ and $Z_{\bar{\eta}}$ are defined over the function field $k(D')$. Spreading out the curve $\Gamma_{\bar{\eta}}$ and the cycle $Z_{\bar{\eta}}$ into a relative curve $\mathscr{G} \to V'$ and a relative cycle $\mathscr{Z}$ on $\mathscr{G} \times_{V'} \mathscr{Y}_{V'}$ over the preimage V' of a certain Zariski open subset V in D under the map $D' \to D$, we obtain a homomorphism

$$\mathscr{Z}_* : A^1(\mathscr{G}) \to A^{p+1}(\mathscr{Y}_{V'}) \ .$$

Compactifying and resolving singularities, we obtain a surface $\mathscr{G}'$, a codimension 1 algebraic cycle $\mathscr{Z}'$ on the variety $\mathscr{G}' \times_{D'} \mathscr{Y}_{D'}$ and the homomorphism

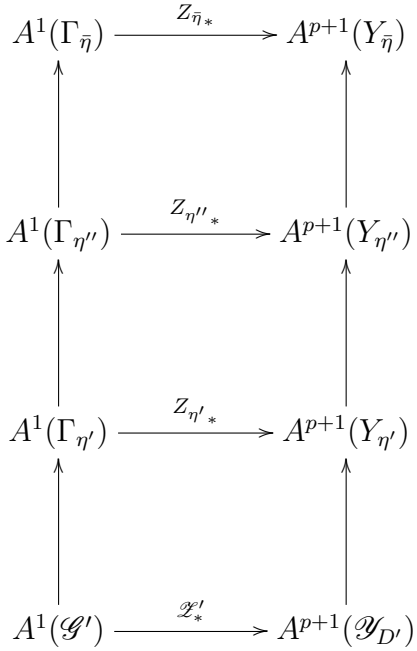
$$\mathscr{Z}'_* : A^1(\mathscr{G}') \to A^{p+1}(\mathscr{Y}_{D'}) \ .$$

Take any element α in the group $A^{p+1}(\mathscr{Y}_{D'})$. Let α' be its image in $A^{p+1}(Y_{\eta'})$, and let $\bar{\alpha}$ be the image of α' in $A^{p+1}(Y_{\bar{\eta}})$. Take a cycle $\bar{\beta} \in A^1(\Gamma_{\bar{\eta}})$ which goes to $\bar{\alpha}$ under the surjective homomorphism $Z_{\bar{\eta}*}$ from $A^1(\Gamma_{\bar{\eta}})$ to $A^{p+1}(Y_{\bar{\eta}})$, and consider a finite extension

$$D'' \to D' \ ,$$

such that $\bar{\beta}$ is defined over the function field $k(D'')$.

Let η' and η'' be the generic points of D' and D'' respectively and consider the following commutative diagram



which illustrates what is going on.

The cycle class $\bar\beta$ comes from a cycle class $\beta'' \in A^1(\Gamma_{\eta''})$ under the pullback from $A^1(\Gamma_{\eta''})$ to $A^1(\Gamma_{\bar\eta})$. Let β' be the image of β'' under the pushforward homomorphism from $A^1(\Gamma_{\eta''})$ to $A^1(\Gamma_{\eta'})$, and let β be a cycle class in $A^1(\mathscr{G}')$ going to β' under the surjective homomorphism from $A^1(\mathscr{G}')$ to $A^1(\Gamma_{\eta'})$. Let γ (respectively, γ' and γ'') be the image of the cycle class β (respectively, β' and β'') under the homomorphism $\mathscr{Z}'_*$ (respectively, ${Z_{\eta'}}_*$ and ${Z_{\eta''}}_*$). Without loss of generality, we may assume that η''/η' is Galois. Let N be the corresponding norm on the group $A^{p+1}(Y_{\eta''})$. Since the kernel of the homomorphism from $A^{p+1}(Y_{\eta''})$ to $A^{p+1}(Y_{\bar\eta})$ is torsion, there exists a positive integer m, such that

$$m(\gamma'' - \alpha'') = 0 \ .$$

Then

$$mN(\gamma'') = mN(\alpha'') = mn\alpha' \ ,$$

where

$$n = [\eta'' : \eta'] \ .$$

It follows that

$$m\gamma - mn\alpha$$

belongs to the kernel of the homomorphism from $A^{p+1}(\mathscr{Y}_{D'})$ to $A^{p+1}(Y_{\bar\eta})$.

We see that, if

$$B_1 = \mathrm{im}(A^1(\mathscr{Y}') \xrightarrow{\mathscr{Z}'_*} A^{p+1}(\mathscr{Y}_{D'})$$

and

$$B_2 = \ker(A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(Y_{\bar\eta}) \ ,$$

the $\mathbb{Q}$-vector space $A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q}$ is generated by the vector subspaces $B_1 \otimes \mathbb{Q}$ and $B_2 \otimes \mathbb{Q}$.

Now, as $A^{p+1}(Y_{\eta'})$ is the colimit of the groups $A^{p+1}(\mathscr{Y}_{W'})$, where W' runs through Zariski open subsets in D', the localization sequences for open embeddings $\mathscr{Y}_{W'} \subset \mathscr{Y}_{D'}$ show that kernel B_2 is generated by the image of the homomorphism

$$\oplus_{P' \in D'} A^p(Y_{P'}) \to A^{p+1}(\mathscr{Y}_{D'}) \ ,$$

induced by the proper push-forward homomorphisms $r_{P'*}$. If $D^\natural$ is the intersection of $T^\natural$ and D, and $D'^\natural$ is the pre-image of $D^\natural$ under the finite map from D' onto D, then $r_{P'*} = 0$ for each closed point P' in $D'^\natural$. It follows that B_2 is generated by the image of the homomorphism

$$\oplus_{P' \in D' \smallsetminus D'^\natural} A^p(Y_{P'}) \to A^{p+1}(\mathscr{Y}_{D'}) \ .$$

Notice that the complement $D' \smallsetminus D'^\natural$ is finite.

Next, if $Y_{P'}$ is smooth, then $Y_{P'} = Y_P$, where P is the image of P' under the finite map from D' onto D, and $A^p(Y_{P'})$ is isomorphic to the abelian variety A_P via the universal regular morphism ψ_P. In particular, $A^p(Y_{P'})$ is weakly representable. If the section $Y_{P'} = Y_P$ is singular, resolving the double point on it we obtain a nonsingualr variety $\tilde{Y}_P$ whose group $A^p(\tilde{Y}_P)$ is weakly representable by our assumption.

Thus, B_2 is covered by the finite direct product of weakly representable groups $A^p(Y_P)$ and $A^p(\tilde{Y}_P)$. It means that B_2 itself is weakly representable. Then, of course, $B_2 \otimes \mathbb{Q}$ is rationally weakly representable. The image B_1 of the homomorphism $\mathscr{Z}'_* \otimes \mathbb{Q}$ is weakly representable because $A^1(\mathscr{Y}')$ is parametrized by the Picard variety of the surface $\mathscr{Y}'$. Since $A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q}$ is generated by rationally weakly representable $\mathbb{Q}$-vector subspaces B_1 and B_2, the whole group $A^{p+1}(\mathscr{Y}_{D'})$ is rationally weakly representable. This contradicts to the third assumption of the theorem.

Hence, $A_{\bar{\xi},0} = 0$, and Theorem 23 finishes the proof. □

8. Application to hyperplane sections of cubic fourfolds in $\mathbb{P}^5$

Let first X be a $K3$-surface embedded appropriately into $\mathbb{P}^m$. Since smooth hyperplane sections of a projective surface are smooth projective curves, Assumptions 1, 2 and 3 are satisfied. It is also well-known that the group $A^2(X)$ is divisible (see Lemma 0.1.1 in [1]). The third cohomology of a $K3$-surface vanishes, so that the Albanese variety is trivial. By Roitman's theorem, [20], the group $A^2(X)$ is uniquely divisible. Then there is no difference between rational and integral weak representability for this group. Moreover, we know that $A^2(X)$ is not representable by Mumford's result in [17]. By Theorem 26, for a very general hyperplane section Y_P of the surface X the kernel of the push-forward homomorphism r_{P*} from $A^p(Y_P)$ to $A^{p+1}(X)$ is countable. This is, of course, a particular case of Proposition 2.4 in [32]. Another application of Theorem 26 is this (Corollary D in Introduction).

Corollary 27. *Let X be a smooth cubic hypersurface in $\mathbb{P}^5$, and let Y be a very general hyperplane section of X. Then the kernel of the push-forward homomorphism from $A^2(Y)$ to $A^3(X)$ is countable.*

Proof. For the cubic X we have that $H^5_{\text{ét}}(X, \mathbb{Q}_l) = 0$ and so $A_1 = A$ for $Y_{\bar{\xi}}$ and every smooth section Y_P of the fourfold X. Any such a section smooth is a cubic 3-fold in $H_P \simeq \mathbb{P}^4$, whose group $A^2(Y_P)$ is well known to be representable by the corresponding Prymian variety $\mathrm{Prym}(Y_P)$, see [1]. Since the Prym construction is of purely algebraic-geometric nature, we can do it over $\bar{\xi}$ getting the Prym variety $\mathrm{Prym}(Y_{\bar{\xi}})$ for the geometric generic fibre $Y_{\bar{\xi}}$. In other words, all the Assumptions 1, 2 and 3 are satisfied for $Y_{\bar{\xi}}$, as well as for all smooth hyperplane sections Y_P.

If a hyperplane section Y_P of the cubic fourfold X has one singular point and this point is an ordinary double point, then the singular cubic Y_P is rational, so that $\tilde{Y}_P$ is rational. It follows that the group $A^2(\tilde{Y}_P)$ is weakly representable. If Y_P is smooth, then it is unirational and so rationally connected. Hence, $A^3(Y_P)$ is trivial. The group $A^3(X)$ is not weakly representable by Theorem 0.5 in [21]. Since, moreover, it is uniquely divisible, see Theorem 4.7(iii) in [23], it is also not rationally weakly representable. Thus, all the assumptions of Theorem 26 are also satisfied.

By Theorem 26, for each closed point P in the c-open subset U of $\mathbb{P}^{m\vee}$ there exists a countable set Ξ_P of closed points in the Prymian $\mathrm{Prym}(Y_P)$ of the hyperplane section Y_P, such that the kernel of the homomorphism r_{P*} from $\mathrm{Prym}(Y_P)$ to $A^3(X)$ is countable. □

In particular, if Σ and Σ' are two linear combinations of lines of the same degree on X, supported on Y_P, then Σ is rationally equivalent to Σ' on X if and only if the point on $\mathrm{Prym}(Y_P)$, represented by the class of $\Sigma - \Sigma'$, occurs in Ξ_P.

Notice also that the group $A^3(Y_\eta)$ can be nonzero, but we know that it is torsion. Since $A^2(Y_P)$ is divisible, any cycle class in $A^3(X)$ is represented, up to torsion, by line configurations supported on hyperplane sections.

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