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Optimal Hedging and Reinsurance Strategies under Risk Measures

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To my grandparents up in heaven.

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Abstract

This thesis aims to address some optimal problems using risk measure for different purpose.

First topic is considering to find the optimal hedging strategy in general to replace the fixed minimum capital reserve. This problem, that minimises the cost of the financial position, is generated from ruin theory by using the properties of risk measures. Under the cash invariance, positive homogeneity and sub-additivity of the general risk measure, a generalised minimum capital (GMC) is introduced. By selecting the cumulative risk measure and cumulative pricing rule, an example is demonstrated.

Since a reinsurance contract can be considered as a hedging approach, the work in this thesis then aim to find the optimal reinsurance design under various framework.

Secondly, the distortion risk measure is selected to avoid insolvent of the cedent and to calculate the premium of reinsurance policy. To work on this problem, we construct a discrete-time dynamic surplus model. To our knowledge, the work in this thesis is the first instance of a proposal of a framework for an optimal reinsurance contract wherein the insurance company's life-time dividends are maximised while addressing the risk of moral hazard. We considered different types of conditions and observed that if there is no specific dividend policy in place, the optimal design of the reinsurance contract in static and dynamic frameworks is similar. Meanwhile, the problem with a specific dividend policy, in its general form, can be different for dynamic and static frameworks. However, for particular cases (for example, if the premium principle is Value at Risk) the optimal design for dynamic and static frameworks is similar.

With the similar aim in mind when choosing the objective function in the last work, reciprocal set-up is used in the final topic in this thesis. We propose a new approach to this problem by considering a multi-objective optimisation problem. We optimise one party's overall risk while regulating the risk of the other party under the distortion risk measures. We also proposed a more realistic global risk position in contrast to the existing over-simplified assumptions in the literature. We demonstrated that the problem can be solved in a manner similar to solving a straightforward optimal reinsurance design problem. We studied different examples and determined their solutions.

In the last two problems, we utilised the Marginal Indemnity Function formulation method (Assa, 2015a; Zhuang et al., 2016). Under the no–moral–hazard assumption and the unique form of DRM we used, the MIF formulation ensures that the multi-layer type reinsurance is optimal and is accessed to. In this thesis, we demonstrate how this method can aid the introduction of more sophisticated problems in the literature on reinsurance contract design.

Contents

Chapter 1

Introduction and Literature Review

The research carried out in this thesis is concerned with determining optimal reinsurance strategies to satisfy two types of criteria individually. One is to maximise the expectation of discounted cumulative dividends, whereas the other is to minimise the risk of certain parties. In the literature on the optimal reinsurance problem, most of the studies focus on the perspective of the cedent and omitted the benefit of reinsurance company. This is particularly so when the research used the first two criteria reviewed in Section 1.3, which are minimising the ruin probability and the ceding risks respectively. This is the reason that the results of existing studies are likely to be optimal for the cedent and not for the reinsurer. Balbás et al. (2013) emphasised the motivation behind this is that the ceding company makes the final decision on buying the reinsurance contract. However, as another participating party of the reinsurance agreement, the reinsurer needs to approve the chosen strategy. Therefore, our work presented in Chapter 4 brings the shareholders as the third party into consideration by maximising the discounted total dividends. Additionally, in Chapter 5, we investigate the optimal reinsurance problem under a so-called reciprocal objective, which implies that the interests of both the insurer and reinsurer are considered. All the risk-related assessments in our work shown in Chapter 4 and 5 are under the Distortion Risk Measure (DRM) framework; see Section 1.1.2, Definition 2.2.6 in Section 2.2.

This thesis is organised as follows: in the rest of this chapter, Section 1.1 reviews the development of risk measures, while Section 1.3 describes the existing research on reinsurance optimisation under different criteria and set-ups. Chapter 2 serves as a mathematical and theoretical foundation of the main topics demonstrated in Chapter 3, 4 and 5. Their results are summarised in Chapter 6 with the potential areas of interest indicated.

1.1 Risk Measure

While uncertainty is with regard to events that are likely to occur in the future under different scenarios, risk is a means to translate the (negative) impact of uncertainty to today's value. Whereas uncertainty is objective and is concerned only with the modelling issues, risk can be subjective; moreover, its assessment can vary among different risk-averse individuals. From the financial and actuarial perspective, risk is not always a negative entity as investments with high risks are associated with higher returns. Therefore, in risk management, decision-makers are divided into two types: risk-seeking and risk-averse. However, regardless of the different preferences of individuals, actuaries analyse the available information to provide professional recommendations. Risk measures are mostly used as the criteria to assess the risk and rank different products. Risk measures are also used to price the insurance products and are generally called insurance premium principles. A variety of risk measures have been introduced and studied in the literature. Similar to mathematical modelling, it is not feasible to identify one among all the families of risk measures and/or their properties as the most effective. However, for particular purposes, one can identify a family of risk measures that can be more effectively applied to our problems.

The most popular risk measure is likely to be the Value at Risk (VaR). VaR is a quantile of the loss distribution. According to Dowd and Blake (2006), the concept underlying VaR was first formulated in the early 70s; however, the term VaR has not been commonly used until the 1990s. At that time, the financial institutes were expanding rapidly and they needed to develop certain mathematical models to manage their risks. VaR is convenient to use and straightforward to understand. Additionally, VaR is applicable to various risk and capital losses with their associated probabilities. Numerous researches have been done to develop the methods of risk management based on VaR. As a result, VaR has been considered as the benchmark to manage financial risks (see Duffie and Pan (1997) and Jorion (2002) for more detailed reviews of VaR). Notwithstanding these advantages, VaR has certain significant limitations, such as the deficiency of information on tail losses. From VaR, the maximum losses are estimated in optimistic situations, and this is likely to result in moral hazard problems (Dowd and Blake, 2006). Another shortcoming is the deficiency of sub-additivity (Artzner et al., 1997, 1999), which is incompatible with the observations in reality wherein the diversification of investments aids risk reduction or at least prevents increase in risk. Moreover, Balbás et al. (2017) summarised that optimisation of VaR is more complicated comparing to other risk measures.

Following the study of this celebrated VaR, researchers developed two main methods to generate new risk measures: providing a group of axioms and defining the functional forms.

1.1.1 Axiom-based Risk Measure

After addressing the issues caused by VaR, Artzner et al. (1997) applied the methods of defining new terminology in geometry to the measures of risk. They proposed the coherent risk measure as a family of risk measures satisfying four properties: sub-additivity, cash invariance (also called translation invariance), positive homogeneity, and monotonicity. This work then was further developed in Artzner et al. (1999). In these two seminal papers, the coherent risk measures were defined on a finite-probability space, which was extended to a general probability space in Delbaen (2002). As VaR does not satisfy sub-additivity, it is not coherent. As an improvement, Conditional Value at Risk (CVaR), which accounts the losses exceeding VaR and can be derived from VaR, is introduced as an alternative risk measure. Similar to VaR, the concept of CVaR has been used since the 1960s (Artzner et al., 1999; Dowd and Blake, 2006); however, it was formally introduced and discussed in Artzner et al. (1999). CVaR is an extension of VaR because it takes into consideration the average over the losses exceeding a quantile for estimating the worse cases [1]. Another example of coherent risk measure is the entropic value at risk (EVaR), introduced by Ahmadi-Javid (2012), which is the upper bound for VaR and CVaR derived from the Chernoff inequality.

As we stated, '*the best*' set of axioms applicable to all purposes is not available. Föllmer and Schied (2002) argues that the positive homogeneity may not always hold; for example, the value is likely to increase at different rates or be non-linear, with the constant increment in market risks. Therefore, they recommended the use of convexity to replace both the sub-additivity and positive homogeneity in the coherent risk measure family. In the next chapter, it is revealed that these two properties together are the sufficient but not necessary conditions of convexity. All the risk measures satisfying convexity, cash invariance and monotonicity are called convex risk measures. Similar concepts are expressed in Frittelli and Gianin (2002).

There are a large number of studies with a similar underlying concept providing different combinations of new properties that a '*good*' risk measure needs to satisfy for certain applications. For practical purposes of analysing empirical data, the property of law invariant is required (Kusuoka, 2001; Acerbi, 2002). Acerbi (2002) introduces the spectral risk measures by adding both law invariant and comonotone additivity to coherent risk measures. Further details of the discussions and development in this topic are available in Balzer (2001); Goovaerts et al. (2003); Teugels and Sundt (2004); Rockafellar et al. (2006); Dhaene et al. (2006, 2008) and Rachev et al. (2008).

[1] It is important to note that in continuous distributions, CVaR is equal to yet another risk measure called Expected Shortfall, which is not coherent when the probability distribution of the losses is discrete (Artzner et al., 1999; Dowd and Blake, 2006).

1.1.2 Distortion Risk Measure

Distortion Risk Measure (DRM) is a family of risk measures which can be written as an integral of VaR (more detailed terms will be explained in Section 1.1.2). Yaari (1987) modifies the independence axiom of the expected utility theory in Von Neumann and Morgenstern (1944) to develop the dual theory with a distortion on survival functions. Young (1999) states that this distortion probability could be considered as a risk-neutral probability in the area of the pricing of financial derivatives with the non-additive property (Denneberg, 1994). Wang (1996) applies this concept into the Choquet integral and defines the DRM. Both VaR and CVaR belong to this family of risk measures, while DRM could also be coherent under certain constraints (Balbás et al., 2009). For example, the DRM is coherent if the corresponding distortion functions are concave or if and only if they are continuous (Wang et al., 1997; Goovaerts et al., 2003). DRM is reviewed in Pflug (2006) and discussed in Balbás et al. (2009). DRM is the core risk measure used in Chapter 4 and 5; therefore, we will explain it in further detail in Chapter 2 on the preliminaries.

Although we addressed some disadvantages of VaR, it still crucially participates in risk management by financial institutions and academic researchers (Balbás et al., 2017). This is why we choose DRM to include VaR in our general optimization results. Especially, examples under VaR are given in 4 to demonstrate the main results. Additionally, DRM has been applied to a variety of problems in the financial area, such as the asset allocation problem in Hamada et al. (2006) and catastrophe bond pricing in Wang (2004). More general discussions of the applications of DRM for portfolio optimisation is available in Chapter 25 of Sereda et al. (2010). Another application of DRM is in the determination of insurance premiums, i.e. they can be straightforwardly considered as insurance premium principles. When DRM is used to calculate premiums, it is called the distortion premium principle (DPP). Insurance premium is the compensation paid by the contract buyer to the insurer for shifting the covered risks. The calculation of the premium is one of the most important topics that insurance companies are interested in. There are three main methods to develop the premium principles: ad hoc method, characterisation method, and economic method. While the expected value premium principle is typically an ad hoc approach to risk premium, the axiomatic approach to generating risk measures provides the necessary characterisation to premium principles. Furthermore, by applying existing theory to generate a new set of premium principles, we can introduce an economic method of risk measurement (for further details, see Teugels and Sundt (2004)). Researchers are impelled to improve the theories in risk measures by seeking a more effective premium principle, e.g. the introduction of the DRM. Wang (1995) introduced the proportional hazards premium principle, and then, the concept of distortion premium principle (DPP) was derived in Wang (1996). Wang Transform as a distortion function was recommended by Wang (2000), and the corresponding DRM named Wang's premium principle was defined. Note that Wang's premium principle belongs to DPP. We will provide more mathematical definitions in the next chapter, and DPP is selected to calculate the reinsurance premium in Chapter 4.

1.2 Hedging in Financial Markets

Hedging is a common practice in banking, finance and insurance. In the literature on financial mathematics, parametric and non-parametric are two main approaches of hedging. The parametric method is in nature as it assumes that the market price follows a particular diffusion process. This approach includes the intrinsic risk hedging of Schweizer (1992), the super-hedging of El Karoui and Quenez (1995) and the efficient hedging of Föllmer and Leukert (2000). The second approach is non-parametric since it does not make use of the structure of the model that drives the underlying price dynamics. The robust pricing and hedging strategies of Cox and Obłój (2011a, 2011b) serve as an example of this approach. A different line of research in model-free hedging is based directly on the concepts of hedging and minimising the risk (see Xu (2006), Balbás et al. (2009), Balbás et al. (2009), Balbás et al. (2010) and Assa and Balbás (2011)). In this setting, the investor or portfolio manager minimises the risk of a global position given the budget constraint on a set of manipulatable positions (a set of accessible portfolios, for instance). We account for market incompleteness and frictions by minimising aggregate hedging costs that consist of the market price associated and the so-called risk margin. As a general method, this non-parametric or robust hedging approach is used in different ways such as hedging contingent claims and economic risk variables. For the development of the approaches on the sub-additive risk evaluators and pricing rules, more discussions can be found in Jaschke and Küchler (2001), Staum (2004), Xu (2006), Balbás et al. (2009), Balbás et al. (2009), Balbás et al. (2010), Assa and Balbás (2011) and Arai and Fukasawa (2014).

1.3 Reinsurance Optimisation

A reinsurance contract is an insurance agreement written by the reinsurance company to protect the ceding insurance company (or cedent). Similar to an insurance policy, reinsurance reduces the risk of the cedent arising from the insurance claims, and the reinsurer is compensated by a premium. As a consequence, the ceding company has the opportunity to sell a more flexible range of insurance contracts to the policyholders while retaining the risk at an acceptable level. Through this cooperation of a risk-sharing relation, the small-scaled cedents are rendered more competitive in the financial markets. Another effect on the ceding companies is that a reinsurance contract aids them in decreasing the ruin probability from unexpected losses. Refer Teugels and Sundt (2004) for further material covering discussions on reinsurance terms and their functioning.

Inspired by the general equilibrium theory established by Arrow and Debreu (1954), Borch (1960a) first formally applied the concepts related to uncertainty in economics to the field of insurance and initially solved the optimal risk-sharing problem. As a result of this fundamental work, a stop-loss contract is proven to be the optimal reinsurance strategy to minimise the variance of cedent losses under the expected value premium principle. Arrow (1963) obtained similar results for different optimisation criteria by maximising an expected utility of the terminal surplus for the insurance company.

In more recent literature, three main objective criteria are selected to determine the optimal reinsurance retention levels: ruin probability in risk theory, losses estimated by risk measures and expected utilities (including the discounted cumulative dividends, although eventually, they are not utilities).

1.3.1 Optimal Criteria

The optimal problem of minimising the probability of ruin is popular since the birth of the risk theory (Lundberg, 1926, 1903), which was developed by Cramér (1930). One of the core interests in risk theory is the relation between the ruin probability and the initial reserve of the insurance company. Following the development under this fundamental Cramér–Lundberg surplus setting, ruin probability estimates the likelihood of insurer bankruptcy and is popular in the optimisation literature as the objective function. Ruin probability minimisation provides a criterion to lower the bankruptcy risk; however, it omits the economic value of the cedent. Therefore, dividend maximisation is selected to take the shareholders' perspectives into consideration. Asmussen and Taksar (1997) applied HJB equation to solve the general optimal aggregate discounted dividends problem in 1997. Three years later, Asmussen et al. (2000) used the results from that work to solve the optimal reinsurance problem with excess-of-loss reinsurance as an example. Mnif and Sulem (2005) also used HJB equation to obtain the optimal excess-of-loss reinsurance contract and general dividend strategy. A few new papers that extend Asmussen et al. (2000) have been published with the general form of the dividend payments; for example, Bai et al. (2010) with transaction costs and taxes, Wu and Guo (2012) with capital injections and Liu and Hu (2014) with equity. Most of the available literature on the optimal reinsurance problem is discussed under the continuous time dynamic setting. In this case, Bellman Principle is commonly applied and extended to the Hamilton–Jacobi–Bellman (HJB) equation to minimise the ruin probability under the expected value principle to determine the optimal solutions of excess of loss reinsurance; e.g. see Hipp and Vogt (2003) and Dickson and Waters (2006). Schmidli et al.

(2002) extended the problem by finding the optimal investment policy additional on the problem of minimising the ruin probability to find the optimal proportional reinsurance policies by the HJB equation in the work of Schmidli (2001). In 2007, Zhang et al. (2007) worked on a similar topic, albeit with a focus on the combination of quota-share and excess-of-loss reinsurance. Zhang and Siu (2009) then incorporated the investment part into the setting and used Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations to solve the problem under two objective functions.

Another objective commonly used in the literature is to minimise the total risk of an insurance company by using a risk measure. Gajek and Zagrodny (2004) used a general risk measure on a combination of a quota share and stop-loss contract to analyse a general form of reinsurance optimisation problem. Cai and Tan (2007) initially obtained the optimal stop-loss reinsurance retention level under the VaR and CVaR minimisation. Inspired by this work, a series of extensions of risk minimisation under VaR and CVaR have been investigated. Cai et al. (2008) determined that the optimal reinsurance type is a stop-loss reinsurance in certain cases, whereas in other ones, quota-share or a combination of the two are optimal. To extend this work, Cheung (2010) solved the problem with a geometric method and provided an example of minimising VaR under Wang's premium principle. Another work in this series by Chi and Tan (2011) demonstrated that the stop-loss reinsurance, capped stop-loss, or truncated stop-loss reinsurance is optimal owing to different properties of the ceded loss functions. Balbás et al. (2009) studied the different selected reinsurance contract types under a general risk measure and under VaR. Following the development in risk measures, researchers such as Zheng and Cui (2014) changed the objective to the ceding risk estimated using the DRM. Note that most of the literature mentioned before applies the expected value premium principle. However, with the development of risk measures, the problem of constructing the optimal reinsurance strategy has been studied under different types of risk-measure-based premium; for example, Kaluszka (2005) studies the problem using the convex premium principles, and Chi and Tan (2013) uses a three-axiom-specified premium principle. Similar to Cui et al. (2013), Assa (2015a), Zheng et al. (2015) and

Zhuang et al. (2016), our research discussed in Chapter 4 calculates the reinsurance premium under the DPP; however, both focused on minimising the risks under the DRM of ceding losses, while we target to maximise the shareholders' benefits. In particular, Assa (2015a) introduces a general framework where the optimal problem can be constructed under a general DRM and premium, from the perspective of insurance, reinsurance and social planer. The major technical advantage of this paper was to introduce the so-called marginal indemnification function method. An extension to problems with constraints on the budget was achieved in Zhuang et al. (2016).

The expected total dividends work as the objective function among the optimal reinsurance problems as well. To render the classical model in risk theory more realistic, De Finetti (1957) recommended the addition of more criteria such as dividend payments to limit the higher bound of the insurer's surplus. In his work, he determined that the barrier strategy is the optimal dividend policy to maximise the expectation of the discounted dividends under a discrete time surplus model. There are two main dividend policies studied in the literature: the band strategy and the barrier strategy. The barrier policy is a special case of the band strategy. Numerous researchers aimed to identify the optimal dividend policy under different settings. Højgaard and Taksar (1999) discussed two types of dividend strategies under the continuous time models with the proportional reinsurance contract. One is the barrier policy with an upper limit, and another one is the standard barrier strategy. However, because the band strategy does not satisfy the monotonicity with respect to the remaining balance, it is demonstrated to be optimal only in more general settings. In the work of Albrecher and Thonhauser (2008), band strategy is obtained as the optimal policy when maximising the expected cumulatively discounted dividend payments until ruin under a risk model with a compound Poisson process. By adding the capital injection into the objective function, Kulenko and Schmidli (2008) found that the barrier policy is optimal under the risk theory framework. Eisenberg and Schmidli (2009) stated that barrier policy is typically the optimal dividend strategy in a framework of collective risk theory but it leads to ruin almost surely which could be proven similarly as the classical one from ruin theory. Whereas Miyasawa (1961), Morrill (1966) and Claramunt et al. (2003) studied the optimal barrier dividend problem under the discrete-time set-up, the research of Jeanblanc-Picqué and Shiryaev (1995), Siegl and Tichy (1999), Asmussen and Taksar (1997), Paulsen and Gjessing (1997), Gerber and Shiu (1998, 2003, 2004), Højgaard (2002), Irbäck (2003) and Eisenberg and Schmidli (2009) focused on the continuous models for constant barrier strategy, and Albrecher and Kainhofer (2002) were interested in the non-linear barrier dividend agreement. The first text book stating the fundamental concepts of both discrete and continuous dividend policies was written by Bühlmann (1970). Albrecher and Thonhauser (2009) reviewed the optimal dividend policy under classical risk models and suggested some potential problems.

1.3.2 Reciprocal Reinsurance

The term reciprocal reinsurance was first used in Borch (1960b) and recently regained centre-stage owing to studies such as Cui et al. (2013) and Balbás et al. (2013). Reciprocal reinsurance implies placing equal emphasis on both the parties involved in a reinsurance contract. The mainstream literature is concerned only with the insurance company and omits the reinsurers' interests. The most optimal reinsurance policy for a cedent might be the least optimal for the reinsurance company (see Borch (1969) and in particular, Assa (2015a)). One approach is studied in (Cui et al., 2013), wherein the objectives are to maximise the joint survival probability and the joint profitable probability. In addition, they derived the existence conditions of different optimal reinsurance contracts under the expected value premium principle and then extended them to a general premium principle. Two specific examples are provided to explain the applications of the theorems. By considering a group of insurance and reinsurance companies, Balbás et al. (2013) address more than one cedent and diversify the risks among several parties under both the deviation measures and coherent risk measures. More recent studies related to this topic are investigated by Dimitrova and Kaishev (2010); Castañer et al. (2013); Fang and Qu (2014) and Castañer and Claramunt (2016).

The general concept of this thesis is related to determining the optimal reinsurance policies under DRM. Therefore, in this chapter, we reviewed the literature on the history of risk measures in Section 1.1, and in Section 1.3, we illustrated the beginning and growth of the studies of reinsurance optimisation with three objective criteria and under the reciprocal set-up. The rest of this thesis is organised as follows. In Chapter 2, we will demonstrate the key concepts supporting our research with mathematical explanations. The results from Chapter 4 are under the maximisation of dividend payments, and the premium calculation is under the DPP. A dynamic DRM is derived in Section 3.1 to match the discrete time surplus set-up used in the same chapter. Chapter 5 discusses the optimal reinsurance agreements profiting both the cedent and reinsurer, with examples under the selected reinsurance contract related to stop-loss and quota-share policies. Chapter 6 concludes the results and provides potential work topics for future study.

Chapter 2

Preliminary

The available theories and outcomes from the literature listed in this chapter serve as the foundation for this thesis.

Section 2.1 states certain useful concepts of actuarial science along with the classical risk model and the extensions of it with the terms related to the reinsurance and dividend policies. The classical Cramér–Lundberg surplus model is considered in Chapter 3 while the one with reinsurance and dividend components is used in Chapter 4.

Section 2.2 describes certain important properties of risk measures and provides the definition of *Distortion Risk Measure* (DRM) and *Distortion Premium Principle* (DPP). In Section 2.3, the no–moral–hazard assumption is stated, which leads to *Marginal Indemnification Function* (MIF) formulation method used in our studies.Section 2.4 demonstrates the theory of primal optimisation problem and its Lagrangian dual problem and supports our work in Chapter 4. These three sections provide crucial background and method for Chapters 4 and 5.

Before going further, let us state the fundamental mathematical setting underlying our work. Consider a non-atomic filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\{\mathbb{N}\cup\infty\}}, \mathbb{P})$, where Ω represents the space of all scenarios (or states of the world), $\mathcal{F}$ is a σ-field consisting of all events, $\{\mathcal{F}_t\}_{t\in\{\mathbb{N}\cup\infty\}}$ is a filtration, and $\mathbb{P}$ is a probability measure. Non-atomic measure is a measure with no atoms, i.e. for any set $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, there exists a subset $B \subseteq A \in \mathcal{F}$ such that $\mathbb{P}(A) > \mathbb{P}(B) > 0$. Recall that a random variable is an $\mathcal{F}$-measurable function and that two random variables are equivalent if they coincide $\mathbb{P}$-almost surely. Let us denote the *Cumulative Distribution Function* (CDF) of X by $F_X(x) = \mathbb{P}(X \leq x)$ and its *Probability Density Function* (PDF) by f_X. Furthermore, we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_\infty = \mathcal{F}$.

2.1 Actuarial Concept

We explain certain fundamental concepts in the field of actuarial science.

- *Reserve* is the initial capital that covers the risk of losses and retains the company's overall risk at an acceptable level.
- *Claims* are the losses paid by insurers that are covered by certain insurance contracts. The value of each claim is referred to as the *claim size*.
- Insurance *premium* is the amount of money paid by the insured to acquire an insurance policy. A *premium (calculation) principle* is a method or a rule to determine the premium of an insurance or reinsurance policy. There are three premium principles mentioned later in this chapter: the net premium principle, the expected value premium principle and the distortion premium principle(DPP).
- The *surplus* of an insurance company is the balance left after receiving premiums and paying claims. If the surplus drops below zero, the insurer is *ruined* or *insolvent*.
- A *reinsurance* agreement is an insurance contract sold by the reinsurer to an insurance company, also called the *cedent* or ceding company.

Lundberg (1926, 1903) and Cramér (1930) first introduced the Cramér–Lundberg surplus model as

$$U_t = u + ct - \sum_{i=1}^{N_t^\lambda} Y_i, \tag{2.1}$$

where u is the initial capital; c is the premium rate; N_t^λ is the number of claims in $(0, t]$, which is a counting process with a Poisson distribution with a parameter λ; and Y_i is the individual claim and independent of N_t^λ. For further reading of Risk theory, see the books by Asmussen and Albrecher (2010), Grandell (2012) and Schmidli (2017).

Assume the cedent has access to buy reinsurance contract to transfer part of the risk from the claims. Let X denote the total loss from the claims. The loss covered by the reinsurance policy is a deterministic function of the total loss for cedent, denoted by $R(X)$. Adding the reinsurance components into the surplus of the insurer and consider a discrete-time surplus model for time period $[t-1, t)$, we obtain

$$U_t = U_{t-1} + c - c^R - (X - R(X)) = U_{t-1} + c - c^R - I(X), \tag{2.2}$$

where c^R is the reinsurance premium paid by cedent and $I(X)$ denotes the remaining loss for the ceding company.

2.1.1 Reinsurance Type

There are numerous popular reinsurance contracts discussed in the literature. However, only the reinsurance types acting on the total claims presented are taken into consideration in this thesis. Specifically, these four types of reinsurance are discussed and demonstrated in Figure 2.1.

- **Quota-share** contract with proportion p: $R(X) = pX$.
- **(Limited or capped) Stop-loss** reinsurance with retention levels $a \geq 0$ and $b > 0$: the claims covered by reinsurance is a function of total claims given by

$$R(X) = \min\{(X-a)^+, b\} = \begin{cases} 0, & X < a \\ x - a, & a \leq X \leq a + b \cdot \\ b, & X > a + b \end{cases}$$

- **Stop-loss after quota-share** reinsurance

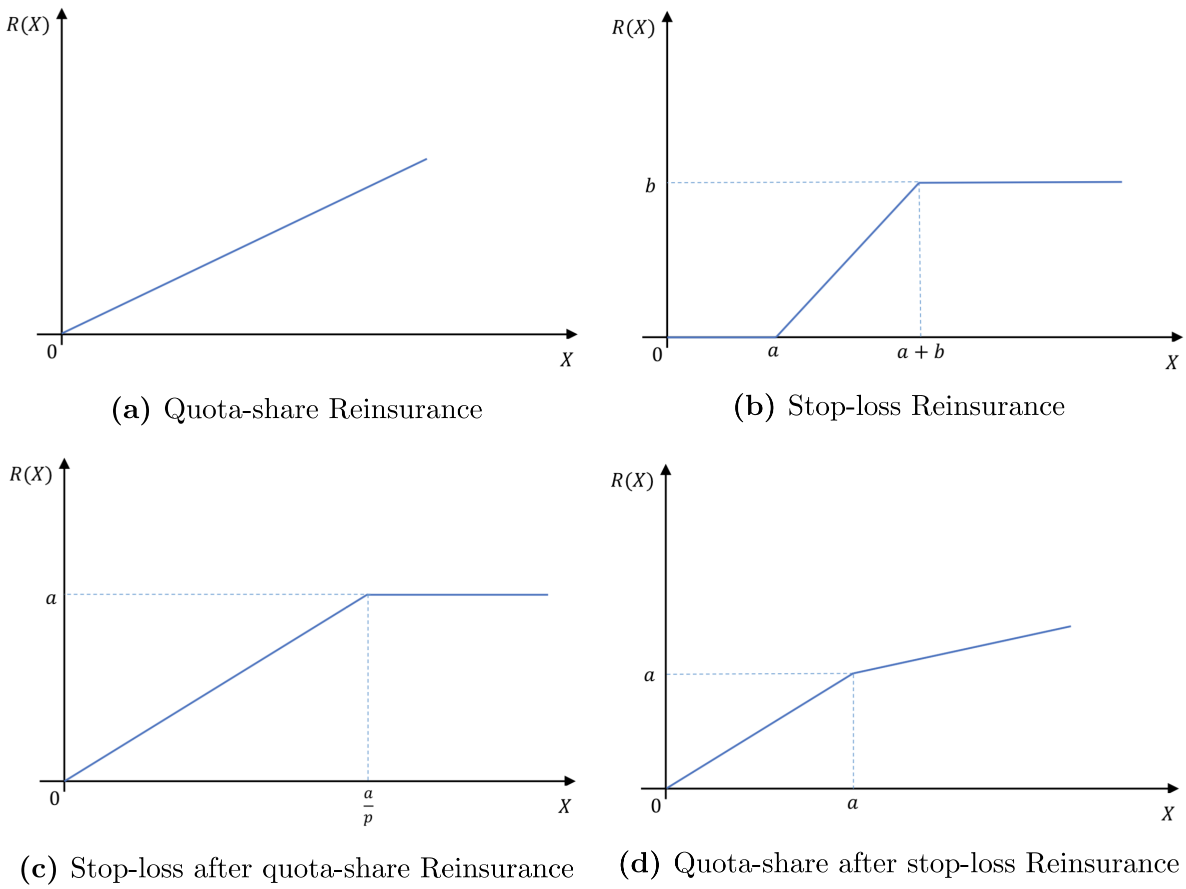
$$R(X) = \begin{cases} pX, & X < \frac{a}{p} \\ a, & X \geq \frac{a}{p} \end{cases}.$$

- **Quota-share after stop-loss** reinsurance

$$R(X) = \begin{cases} X, & X < a \\ pX + (1-p)\,a, & X \geq a \end{cases}.$$

From the definitions and figures, another common property of these four reinsurance types considered could be found is the continuity of the reinsurance claims function $R(X)$. This is not necessary and left-continuity is enough for the Proposition 2.3.2 of VaR proven later in Section 2.3.

Figure 2.1: Reinsurance Types



(a) Quota-share Reinsurance **(b)** Stop-loss Reinsurance

(c) Stop-loss after quota-share Reinsurance **(d)** Quota-share after stop-loss Reinsurance

In addition, let us assume that the cedent needs to pay the dividends, denoted by D_t, to the shareholders. As mentioned in the last chapter, this set-up brings the third party into consideration. The admissible dividend strategies are assumed to be a function of surplus, i.e. $D_t = H(U_t)$. One example of dividend policies is the band dividend strategy defined in Chapter 5.1 of the book from Azcue and Muler (2014).

Definition 2.1.1. For non-negative a_i, b_i and finite n such that $0 = b_0 \leq a_0 < b_1 < a_1 < b_2 < \cdots < b_n < a_n < b_{n+1} = \infty$, the band dividend payments at the time t, D_t^n, is defined as a function of the surplus U_t as $D_t^n = f(U_t)$, where

$$f(x) = (x - a_i)^+ \mathbb{1}_{(a_i, b_{i+1}]}(x), i = 0, 1, \ldots, n. \tag{2.3}$$

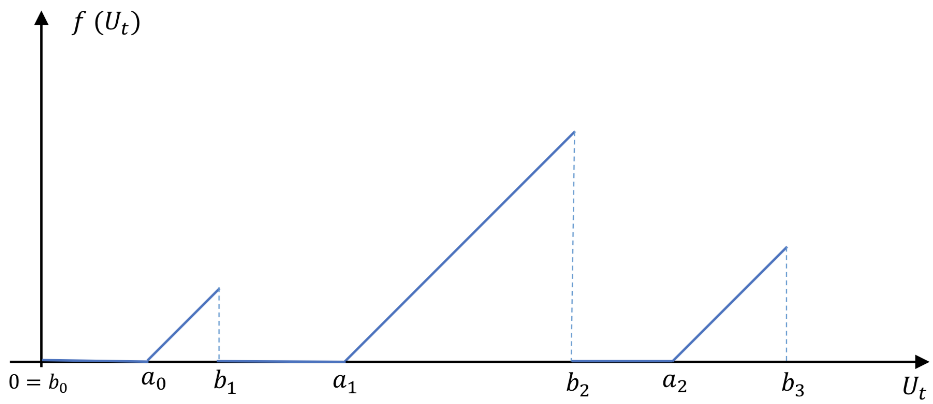


Figure 2.2: Band Dividend Policy with $n = 3$.

If we set $n = 0$, this particular band strategy is called the barrier dividend policy which is more popular to use in literature due to its monotonicity and without jump as the band strategy.

Definition 2.1.2. A barrier dividend policy at time t denoted by D_t is a function of the surplus U_t such that $D_t = f(U_t)$, where $f(x) = (x - a_0)^+$ and a_0 is a non-negative constant number.

Figure 2.2 and Figure 2.3 illustrate the band dividend strategy with $n = 3$ and the barrier dividend policy, respectively. The general form and the barrier dividend strategy will be used in Chapter 4 to find the optimal reinsurance policy to maximising the expectation of cumulative dividends.

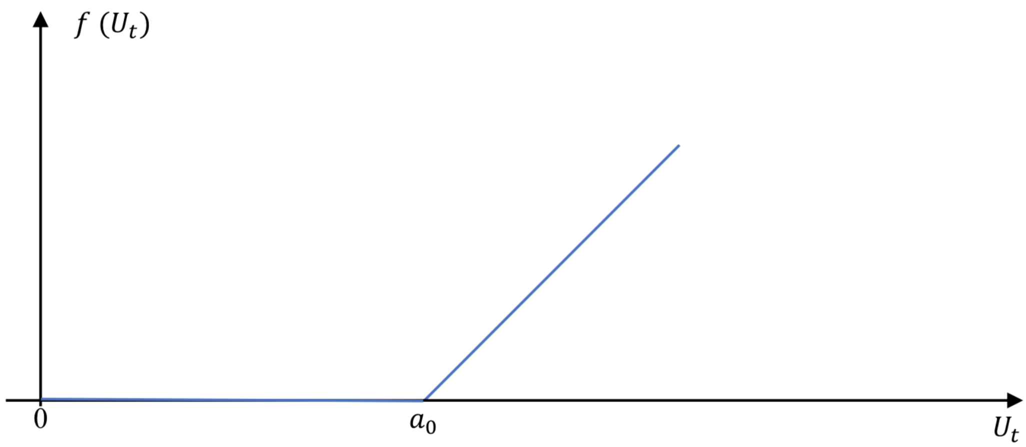


Figure 2.3: Barrier Dividend Policy

2.2 Risk Measure

Consider a set $\mathcal{X}$ containing all real-valued loss variables. A risk measure is defined as a mapping $\rho : \mathcal{X} \to \mathbb{R}$ that maps each risk variable to a real number. Here are two popularly used risk measures related to our research in this thesis. As reviewed in last chapter, the first risk measure used in literature the Value at Risk (VaR) which is the quantile of the risk distribution and introduced below.

Definition 2.2.1. The VaR of random variable X with confidence level $\alpha \in [0, 1]$ is defined as

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} | \, \mathbb{P}(X \leq x) \geq \alpha\}. \tag{2.4}$$

Following the work of Cheung and Lo (2017), denote the generalised left-continuous inverse function of any non-decreasing and right-continuous function $\zeta : \mathbb{R} \to \mathbb{R}$ by ζ^{-1}

which is defined as

$$\zeta^{-1}(p) = \inf\{x \in \mathbb{R} |\, \zeta(x) \geq p\}. \tag{2.5}$$

Recall that the CDF of a random variable X is denoted as $F_X(x) = \mathbb{P}(X \leq x)$ which is a non-decreasing and right-continuous function.Therefore, VaR is equivalent to the generalised left-continuous inverse function of CDF, i.e., the left-continuous inverse distribution function written as

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha). \tag{2.6}$$

See Dhaene et al. (2002) for more information of inverse distribution function. As discussed in Section 1.1.1, Conditional Value at Risk (CVaR) is introduced with the advantage of satisfying the property of sub-additivity comparing to VaR. Because of long-history of use, several forms of it has been derived. In this thesis, only the version consistent to the distortion risk measure form is used which is defined as

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_z(X)dz. \tag{2.7}$$

Now let us move to more general way to describe a set of risk measures. As mentioned in Section 1.1.1, Artzner et al. (1997) proposed the axiomatic method to introduce new risk measures; specifically, they introduced the family of coherent risk measures as follows:

Definition 2.2.2. A risk measure is coherent if it satisfies the following four axioms:

1. Sub-additivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in \mathcal{X}$;
2. Cash invariance: $\rho(X+a) = \rho(X) + a$ for a fixed $X \in \mathcal{X}$ and any $a \in \mathbb{R}$;
3. Positive homogeneity: $\rho(aX) = a\rho(X)$ for any $X \in \mathcal{X}$ and $a \geq 0$;
4. Monotonicity: If $X \leq Y \in \mathcal{X}$, then $\rho(X) \leq \rho(Y)$.

After arguing the possibility of non-linear growth in reality, Föllmer and Schied (2002) relaxed the constraints by replacing both positive homogeneity and monotonicity by convexity. Hence, the convex risk measures can be defined as follows:

Definition 2.2.3. A risk measure is convex if it satisfies both axioms 1 and 2 in Definition 2.2.2 and also complies with the convexity stated as

$$\rho\left(aX + (1-a)Y\right) \leq a\rho\left(X\right) + (1-a)\rho\left(Y\right),$$

for any $X, Y \in \mathcal{X}$ and $a \in [0,1]$.

Remark 2.2.4. It is important to note that in Artzner et al. (1997), Artzner et al. (1999) and Föllmer and Schied (2002), risk measures are considered on a profit variable (instead of a loss variable defined in the beginning of this section). In this scenario, $Y < 0$ means loss whereas a loss variable X represents loss when it is positive. As a result, a risk measure of a profit variable Y is cash-invariant if $\rho(Y + a) = \rho(Y) - a$ for any $a \in \mathbb{R}$. This implies that by adding cash as a profit to a risky position, the risk is reduced. In Chapter 3, the variables inside risk measure are defined as the profit variables while the ones defined in Chapters 4 and 5 are loss variables.

Another strand of literature introducing and using risk measures is based on Choquet integral theory (see Yaari (1987)), from which, the form of DRM used in this thesis is derived.

Definition 2.2.5. (Yaari, 1987) A distortion function denoted by g is a non-decreasing function from $[0,1]$ to $[0,1]$ such that $g(0) = 1 - g(1) = 0$.

Let $\mathbb{P}^*(A) := g\left(\mathbb{P}\left(X \in A\right)\right)$, which is called a distorted probability measure in the literature (Balbás et al., 2009). By using the extended definition of the Choquet integral discussed in Denneberg (1994), the Choquet integral of X with respect to $\mathbb{P}^*$ could be expressed in the form of Riemann integral as

$$(C)\int X d\mathbb{P}^* = \int_{-\infty}^{0} \left[\mathbb{P}^*\left(\{x | x \geq z\}\right) - \mathbb{P}^*(\Omega)\right] dz + \int_{0}^{\infty} \mathbb{P}^*\left(\{x | x \geq z\}\right) dz \tag{2.8}$$

$$= \int_{-\infty}^{0} \left[g\left(\bar{F}_X(z)\right) - g(1)\right] dz + \int_{0}^{\infty} g\left(\bar{F}_X(z)\right) dz, \tag{2.9}$$

where $\bar{F}_X(x) = 1 - F_X(x)$ is the survival distribution function of the risk X; $(C)\int$ inside the first integral represents a Choquet integral, and all the other integrals above are Riemann integrals. By stating this relation, Wang (1996) introduced the DRM as the Choquet integral determined by the chosen distortion function.

Definition 2.2.6. A *Distortion Risk Measure* (DRM) ρ^g of the loss variable X with the corresponding distortion function g satisfies

$$\rho^g(X) := \int_{-\infty}^{0} (g(\bar{F}_X(z)) - 1)dz + \int_{0}^{\infty} g(\bar{F}_X(z))dz. \tag{2.10}$$

If $g\left(\bar{F}_X(z)\right)$ is re-expressed as an integral $\int_0^{\bar{F}_X(z)} dg(\alpha)$ and recall VaR as the inverse distribution function in Equation (2.6), we have the equivalent form of DRM, which is the integral of VaR shown below (Dhaene et al., 2006):

$$\rho^g(X) = \int_0^1 \mathrm{VaR}_{1-z}(X)dg(z). \tag{2.11}$$

Define the *dual distortion function* of g as $\Lambda(u) := 1 - g(1-u)$. Hence, $\Lambda : [0,1] \to [0,1]$ is also non-decreasing. Replacing the distortion function g by the dual Λ in (2.11) and applying change of variables, we obtain

$$\rho^g(X) = \int_0^1 \mathrm{VaR}_{1-z}(X)d\left[1 - \Lambda(1-z)\right] \tag{2.12}$$

$$= \int_0^1 \mathrm{VaR}_z(X)d\Lambda(z) \tag{2.13}$$

$$=: \rho^{\Lambda}(X). \tag{2.14}$$

Therefore, the form of DRM with respect to the dual distortion function demonstrated in Lemma 2.2.7 is obtained.

Lemma 2.2.7. The formula of DRM in Definition 2.2.6 is equivalent to

$$\rho^{\Lambda}(X) := \int_0^1 \mathrm{VaR}_z(X)d\Lambda(z), \tag{2.15}$$

where Λ:$[0,1] \to [0,1]$ is a non-decreasing dual distortion function; VaR with confidence level α has the forms in Equation (2.4) and (2.6).

In this thesis, we mainly apply the form (4.2) in Lemma 2.2.7 for DRM and DPP. Although DRMs are not directly generated from axioms, they are proven to fulfil three properties of coherent risk measures: cash invariance, positive homogeneity and monotonicity; furthermore, they are comonotonely additive. DRMs are sub-additive if and only if the corresponding distortion function is concave. Hence, DRMs are coherent if and only if they have concave distortion functions (Wirch and Hardy, 2001). Other axioms discussed in the literature are available in Dhaene et al. (2006); Balbás et al. (2009); Dowd and Blake (2006) and Guerard Jr (2009).

Specific DRMs are obtained by selecting different distortion functions. For example, VaR in Equation (2.4) belongs to DRM when the distortion function is an indicator function defined as $\Pi^{\mathrm{VaR}_\alpha}(u) := \mathbb{1}_{[\alpha,1]}(u)$, while CVaR could be derived to the form in Equation (2.7) with distortion function $\Pi^{\mathrm{CVaR}_\alpha} := \frac{u-\alpha}{1-\alpha}\mathbb{1}_{[\alpha,1]}(u)$. Furthermore, one of the popular DRMs is generated by Wang's transformation defined in (2.16) as the distortion function. This DRM introduced by Wang (1996) is commonly used for financial derivatives pricing and insurance premium calculation.

$$g^\eta(u) := \Phi\left(\Phi^{-1}(u) + \eta\right), \tag{2.16}$$

where Φ is the CDF of standard normal distribution and η is a real number that stands for the market price of the risk term. Using the properties of Φ, the dual distortion function can be expressed as

$$\Pi^\eta(u) = 1 - g^\eta(1-u) = \Phi\left(\Phi^{-1}(u) - \eta\right). \tag{2.17}$$

Denote the premium principle by $\pi : \mathcal{X} \to \mathbb{R}$. The net premium principle is defined as $\pi[X] = \mathbb{E}[X]$, which is under the assumption that there is no risk if the insurer sells an adequate number of independent and identically distributed contracts Teugels and Sundt (2004). By introducing the term called safety loading or risk loading denoted by $\theta > 0$, a more commonly used method is the expected value premium principle defined as $\pi[X] = (1+\theta)\mathbb{E}[X]$. This principle is more reasonable as an insurer should expect profits on an average from selling the policies. Similarly, one can introduce the *Distortion Premium Priciple* (DPP) by incorporating the safety loading as follows:

Definition 2.2.8. For a safety loading $\theta \in \mathbb{R}_+$ and a dual distortion function $\Lambda : [0,1] \to [0,1]$, a distortion risk premium π^Λ is introduced as

$$\pi^\Lambda(X) := (1+\theta)\int_0^1 \mathrm{VaR}_z(X)d\Lambda(z). \tag{2.18}$$

Lemma 2.2.9. Given the safety loading $\theta > 0$, a DPP could be represented by a DRM with the same distortion function as $\pi^\Lambda(X) = \rho^\Lambda\big((1+\theta)X\big)$. Therefore, DPPs could be technically treated as DRMs.

Proof. By applying the cash invariance property of VaR on Definition 2.2.8, we obtain the following relation:

$$\begin{aligned}\pi^\Lambda(X) &= (1+\theta)\int_0^1 \mathrm{VaR}_z(X)d\Lambda(z)\\ &= \int_0^1 \mathrm{VaR}_z\big((1+\theta)X\big)d\Lambda(z)\\ &= \rho^\Lambda\big((1+\theta)X\big).\end{aligned}$$

□

When the Wang's Transform in (2.16) is selected to be the distortion function, this corresponding DPP is called Wang's premium principle derived by Wang (1996). Notice that the expected value premium principle is a special case of the DPP when the dual distortion function satisfies $\Lambda^{\mathbb{E}}(x) := x$.

2.3 Moral Hazard and MIF Formulation

In this section, we discuss the key assumption and the main formulation method that we use in this thesis. More precisely, one of the basic assumptions in reinsurance optimal contract design problems is the so-called no–moral–hazard assumption, on whose basis we can introduce the *Marginal Indemnification Function* (MIF). Moral hazard is explained in Dowd (2009) as the risk caused by one of the decision-making parties who puts its own benefits above those of the other one. He discussed the role of moral hazard in the financial crisis of 2008 and recommended more prudential measures to overcome this problem in the financial markets. Assa (2015b) explained the moral hazard problem in a banking system, where tougher prudential risk management measures are applied to remove the risk of moral hazard.

In the literature of actuarial science, this problem is addressed by considering that both of the parties need to perceive the risk of losses (Heimer, 1989; Bernard and Tian, 2009). Therefore, we make the following assumption and call it no-moral-harzard assumption:

Assumption 1. We assume that both the loss functions of the claims covered by the insurance and reinsurance companies are non-decreasing functions of the total losses.

This assumption also enables us to employ the MIF formulation introduced in Assa (2015a) and later developed in Zhuang et al. (2016). Following the work of Assa (2015a), the set of admissible reinsurance contracts under Assumption 1 is defined as the space of indemnification functions:

$$\mathbb{C} = \{0 \leq R(x) \leq x | R(x) \text{ and } x - R(x) \text{ are non-decreasing}\}. \tag{2.19}$$

Here $R(X)$ is a Lipschitz function so it is absolutely continuous; therefore, it is differentiable almost everywhere, and its derivative is essentially bounded by the Lipschitz constant. As a result, we have the following expressions:

Proposition 2.3.1. The space of the indemnification functions in (2.19) is equivalent to

$$\mathbb{C} = \left\{ R : \mathbb{R}_+ \to \mathbb{R}_+ | R(x) = \int_0^x h(t)dt, 0 \leq h \leq 1 \right\}. \tag{2.20}$$

Therefore, we define the space of marginal indemnification functions as

$$\mathbb{D} = \{h : \mathbb{R}_+ \to \mathbb{R}_+ | 0 \leq h \leq 1\}. \tag{2.21}$$

Assumption 1 along with the following two key Propositions 2.3.2 and 2.3.3 form the basis of the MIF formulation. The property of static VaR with a left-continuously non-decreasing function, $R(X)$, stated below is commonly used in the past papers regarding

VaR. It is proven in Theorem 1-(a) by Dhaene et al. (2002) and listed in Lemma 2.1 by Dhaene et al. (2006) as the quantile of transformed random variables.

Proposition 2.3.2. For a non-decreasing and left-continuous function $R : \mathcal{X} \to \mathbb{R}$, VaR with the confidence level $\alpha \in [0, 1]$ satisfies

$$\mathrm{VaR}_\alpha\left(R(X)\right) = R\left(\mathrm{VaR}_\alpha\left(X\right)\right). \tag{2.22}$$

Proof. This proof follows the steps shown in Theorem 1-(a) by Dhaene et al. (2002). Recall that CDF of a random variable X is denoted by F_X and its left-continuous inverse function is denoted by F_X^{-1}. For any $x \in \mathbb{R}$, CDF satisfies the following two equivalent inqualities:

$$F_X^{-1}(\alpha) \leq x \Leftrightarrow \alpha \leq F_X(x). \tag{2.23}$$

Therefore, the equivalence below holds for any $x \in \mathbb{R}$.

$$F_{R(X)}^{-1}(\alpha) \leq x \Leftrightarrow \alpha \leq F_{R(X)}(x). \tag{2.24}$$

Since R is left-continuous and non-deceasing,

$$R(z) \leq x \Leftrightarrow z \leq \sup\{y|R(y) \leq x\} \tag{2.25}$$

holds for any $x, z \in \mathbb{R}$. Thus, take it back into the equivalence (2.24) to obtain

$$\alpha \leq F_{R(X)}(x) \Leftrightarrow \alpha \leq F_X\left(\sup\{y|R(y) \leq x\}\right). \tag{2.26}$$

There are three possible scenarios for the supremum.

1. If $\sup\{y|R(y) \leq x\}$ is finite, then by equivalence (2.23),

$$\alpha \leq F_X\left(\sup\{y|R(y) \leq x\}\right) \Leftrightarrow F_X^{-1}(\alpha) \leq \sup\{y|R(y) \leq x\} \tag{2.27}$$

2. If $\sup\{y|R(y) \leq x\} = +\infty$, then $\alpha \leq F_X(+\infty) = 1 \Leftrightarrow F_X^{-1}(\alpha) \leq +\infty$ holds and is consistent to the equivalence in the first scenario.

3. If $\sup\{y|R(y) \leq x\} = -\infty$, then $\alpha \leq F_X(-\infty) = 0 \Leftrightarrow F_X^{-1}(\alpha) \leq -\infty$ holds and is also consistent to the equivalence in the first scenario.

Therefore, $F_{R(X)}^{-1}(\alpha) \leq x$ is equivalent to $F_X^{-1}(\alpha) \leq \sup\{y|R(y) \leq x\}$. As R is left-continuous and non-deceasing,

$$F_X^{-1}(\alpha) \leq \sup\{y|R(y) \leq x\} \Leftrightarrow R(F_X^{-1}(\alpha)) \leq R\left(\sup\{y|R(y) \leq x\}\right) = x \tag{2.28}$$

holds for any real x. As a consequence, $F_{R(X)}^{-1}(\alpha) \leq x \Leftrightarrow R(F_X^{-1}(\alpha)) \leq x$ holds for any $x \in \mathbb{R}$ and $F_{R(X)}^{-1}(\alpha) = R(F_X^{-1}(\alpha))$ Based on the relation shown in Equation (2.6), rewrite the left-continuous inverse distribution function by VaR on both sides to get $\mathrm{VaR}_\alpha\left(R(X)\right) = R\left(\mathrm{VaR}_\alpha\left(X\right)\right)$ as required. □

Applying the MIF in Chapters 4 and 5, we will use another useful proposition stated here. It is first established and proved in Lemma 2.1 by Zhuang et al. (2016).

Proposition 2.3.3. For a DRM ρ^Λ and a function $R \in \mathbb{C}$ with its derivative R', we have

$$\rho^\Lambda\left(R\left(X\right)\right) = \int_0^\infty \left(1 - \Lambda\left(F_X\left(z\right)\right)\right) R'\left(z\right) dz. \tag{2.29}$$

In straightforward terms, the MIF formulation is used to work with the marginal indemnity functions rather than the indemnity functions because based on the discussions above, DRM and DPP are linear over the space of marginal indemnities with the no–moral–hazard assumption. This fact can significantly simplify the mathematical technology that we use to address optimal reinsurance problems.

2.4 Duality

The aim of Chapter 4 is to determine the optimal reinsurance policy that maximises the expectation of discounted total dividends under budget constraints and solvency conditions. In addition, Chapter 5 focus on the optimal reinsurance strategy design that minimises total risk of cedent, reinsurer and both. To solve these optimal problems, we use the method called duality or duality principle. In this section, we will explain the existing terminology of the duality for solving the optimisation problems. It will be demonstrated in (4.32) of Chapter 4 that our optimal problem is technically equivalent to a minimisation problem although the initial goal is concerned with maximising the total dividends. Moreover, one can conveniently shift between the maximisation and minimisation problems by changing the associated signs of the objective functions and inequality constraints below in (2.30); thus, only the minimisation problem will be explained in this section. Further details are available in Chapters 4 and 5 of the book written by Boyd and Vandenberghe (2004).

An optimisation problem is a setting for determining an optimal solution that minimises the *objective function* and satisfies all the conditions. Let $J^0 : \mathbb{R}^n \to \mathbb{R}$ denote the objective function and $x \in \mathbb{R}^n$ denote the *control variable* or *optimisation variable*. There are two types of conditions that the control variable needs to satisfy: equality and inequality constraints. Such a problem is called a *primal problem* and we denote it as follows:

$$\begin{aligned} \inf_{x} \quad & J^0(x) \\ \text{Subject to} \quad & g_j(x) = 0, \quad j = 1, 2, \ldots, p \\ & g_j(x) \leq 0, \quad j = p+1, p+2, \ldots, p+m \end{aligned} \tag{2.30}$$

where p, m are positive integers and $g_j : \mathbb{R}^n \to \mathbb{R}$ are *constraint functions*. Define the *optimal value* J^* of this minimisation problem as

$$J^* := \inf \left\{ J^0(x) \mid g_j(x) = 0, \ j = 1, \ldots, p, \text{ and } g_j(x) \leq 0, \ j = p+1, \ldots, p+m \right\}. \tag{2.31}$$

Therefore, the set of all the optimal points x^* is referred to as the *optimal set* and is denoted by

$$\begin{aligned} X_{opt} := \left\{ x^* \mid J^0(x^*) = J^*, \ g_j(x^*) = 0, \ j = 1, \ldots, p, \right. \\ \left. \text{and } g_j(x^*) \leq 0, \ j = p+1, \ldots, p+m \right\}. \end{aligned} \tag{2.32}$$

The Lagrangian function is introduced in the following definition.

Definition 2.4.1. Let $\boldsymbol{\lambda} := \{\lambda_j\}_{j=1}^{p}$ and $\boldsymbol{\mu} := \{\mu_j\}_{j=1}^{m}$ denote the Lagrangian multiplier vectors, where λ_j and μ_j are called the Lagrangian multipliers of the equality and inequality constraints, respectively. The *Lagrangian function* is defined as a function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ such that

$$\mathcal{L}(x, \boldsymbol{\lambda}, \boldsymbol{\mu}) := J^0(x) + \sum_{j=1}^{p} \lambda_j g_j(x) + \sum_{j=1}^{m} \mu_j g_{j+p}(x). \tag{2.33}$$

The use of the Lagrangian function is occasionally called the Lagrangian relaxation method. It is generally used when certain constraints make the optimal problem complicated and hinder its solution. Therefore, by minimising the Lagrangian function, the lower bound on the optimal value of the primal problem could be obtained.

Definition 2.4.2. The *Lagrangian dual function*, denoted by $\mathcal{L}^D : \mathbb{R}^p \times \mathbb{R}^{m+} \to \mathbb{R}$, is defined as the infimum of the Lagrangian function, i.e.

$$\mathcal{L}^D(\boldsymbol{\lambda}, \boldsymbol{\mu}) := \inf_x \mathcal{L}(x, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_x \left(J^0(x) + \sum_{j=1}^{p} \lambda_j g_j(x) + \sum_{j=1}^{m} \mu_j g_{j+p}(x) \right), \tag{2.34}$$

where the Lagrangian multipliers, μ_j, functioning on inequality constraint functions are non-negative.

Here, the non-negative Lagrangian multiplier ensure that the Lagrangian dual function gives a lower bound of the optimal value.

Lemma 2.4.3. By Definition 2.4.2 with non-negative $\boldsymbol{\mu}$, the Lagrangian dual function yields lower bounds of the optimal value, J^*, i.e.

$$\mathcal{L}^D(\boldsymbol{\lambda}, \boldsymbol{\mu}) \leq J^*. \tag{2.35}$$

Proof. Assume that $\tilde{x}$ satisfies all the constraints in the primal problem. Because $\mu_j \geq 0$, we have $\sum_{j=1}^{p} \lambda_j g_j(\tilde{x}) = 0$ and $\sum_{j=1}^{m} \mu_j g_{j+p}(\tilde{x}) \leq 0$; as a result,

$$\mathcal{L}(\tilde{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = J^0(\tilde{x}) + \sum_{j=1}^{p} \lambda_j g_j(\tilde{x}) + \sum_{j=1}^{m} \mu_j g_{j+p}(\tilde{x}) \leq J^0(\tilde{x}). \tag{2.36}$$

Consequently,

$$\mathcal{L}^D(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \boldsymbol{\lambda}, \boldsymbol{\mu}) \leq \mathcal{L}(\tilde{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \leq J^0(\tilde{x}). \tag{2.37}$$

holds for any feasible point $\tilde{x}$; thus, $\mathcal{L}^D(\boldsymbol{\lambda}, \boldsymbol{\mu}) \leq J^*$, which implies that the Lagrangian dual function yields lower bounds of the optimal value. $\square$

Note that irrespective of whether the primal problem is a convex problem or not, the Lagrangian dual function is concave. As the constraint $\boldsymbol{\mu} \geq \mathbf{0}$ is a convex set, the associated Lagrangian dual problem defined below is convex. The *Lagrangian dual problem* is denoted by

$$\sup_{\boldsymbol{\lambda},\boldsymbol{\mu}} \quad \mathcal{L}^D(\boldsymbol{\lambda},\boldsymbol{\mu}) \tag{2.38}$$

$$\text{Subject to} \quad \boldsymbol{\mu} \geq \mathbf{0}. \tag{2.39}$$

Let $\mathcal{L}^{D^*}$ denote the optimal value of this dual problem; therefore, (2.38) implies that it is the largest lower bound, i.e. $\mathcal{L}^{D^*} \leq J^*$ always holds. The difference between them is called the *duality gap*: $J^* - \mathcal{L}^{D^*}$. If this gap is equal to zero, the *strong duality* holds and the optimal value of the primal problem could be obtained by determining the optimal value of the Lagrangian dual problem. We assume that the strong duality holds in Chapters 4 and 5. Numerous methods are used to ensure that this holds, which are called the *constraint qualifications*. Slater's condition is one of them which holds when the inequality constraints hold with strict inequalities and the primal problem is convex.

Proposition 2.4.4. The Slater's condition states that the strong duality holds if the primal problem is convex and there exists an x in the relative interior of its domain such that

$$g_j(x) < 0, \quad j = p+1, p+2, \ldots, p+m. \tag{2.40}$$

(See Section 4.2.5 in Boyd and Vandenberghe (2004) for the proof.)

The strong duality implies the *complementary slackness* as follows:

Proposition 2.4.5. When there is no duality gap, for any primal optimal point x^* and dual optimal points $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, the conditions $\mu_j^* g_{j+p}(x^*) = 0,\ j = 1, 2, \ldots, m$ hold, which is called the complementary slackness.

Proof. Using the equality and inequality constraints of the primal problem and Definition 2.4.2 with the non-negative $\boldsymbol{\mu}$, we have

$$\begin{aligned} J^0(x^*) &= \mathcal{L}^D(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \\ &= \inf_x \left(J^0(x^*) + \sum_{j=1}^{p} \lambda_j^* g_j(x^*) + \sum_{j=1}^{m} \mu_j^* g_{j+p}(x^*) \right) \\ &\leq J^0(x^*) + \sum_{j=1}^{p} \lambda_j^* g_j(x^*) + \sum_{j=1}^{m} \mu_j^* g_{j+p}(x^*) \\ &\leq J^0(x^*) \end{aligned}$$

As a result, the duality gap is zero when $\mu_j^* g_{j+p}(x^*) = 0$, $j = 1, 2, \ldots, m$. □

Another condition to be commonly used under strong duality is called the *Karush-Kuhn-Tucker* (KKT) optimality conditions. Kuhn and Tucker (1951) extended and completed the KKT theorem for nonlinear optimization problems with inequality constraints which was first introduced by Karush in 1939 in his unpublished master's thesis. If the objective function and all the constraint functions are differentiable, then the *Karush-Kuhn-Tucker* (KKT) optimality conditions below are the necessary conditions of the primal and dual optimal problems with no duality gap (See more discussions of KKT condtions in Boyd and Vandenberghe (2004) and Aragón et al. (2019)).

Proposition 2.4.6. For any primal and dual optimal problems defined above, if the duality gap is zero with differentiable objective and constraint functions, then any primal optimal point x^* and dual optimal points $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ satisfy the following conditions called the *Karush-Kuhn-Tucker* (KKT) optimality conditions:

$$\begin{aligned} g_j(x^*) &= 0, \quad j = 1, 2, \ldots, p \\ g_j(x^*) &\leq 0, \quad j = p+1, p+2, \ldots, p+m \\ \mu_j^* &\geq 0, \quad j = 1, 2, \ldots, m \\ \mu_j^* g_{j+p}(x^*) &= 0, \quad j = 1, 2, \ldots, m \\ \nabla J^0(x^*) + \sum_{j=1}^{p} \lambda_j^* \nabla g_j(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_{j+p}(x^*) &= 0, \end{aligned} \tag{2.41}$$

where J^0 and g_j, $j = 1, 2, \dots, p+m$ are differentiable.

The KKT condition is used in Chapter 5 under the strong duality condition.

Chapter 3

From Ruin Theory to a Hedging Problem

Hedging is a common practice that financial institutions such as commercial banks, investment corporations, and insurance companies use to reduce risk. As reviewed in Chapter 1, there are two main types of hedging methods in existing studies: parametric and non-parametric approaches. In this chapter, a non-parametric method is used, which is also called a robust hedging approach. We account for the market incompleteness and friction by minimising the aggregate hedging costs that consist of the associated market price and so-called *risk margin*. This approach is fairly general and can be used for various purposes such as hedging contingent claims and economic risk variables. Research in this area has provided methods for sub-additive risk evaluation and pricing rules. The novelty of the study mainly lies in incorporating the concept of hedging within a market-consistent valuation framework. Although the focus of this chapter is on the pricing part of the hedging problem and the extension of a pricing rule to the space of all financial and economic variables in imperfect markets, we also construct a set of market principles that are used to determine the existence of a solution to the hedging problem.

Because any financial position in a complete market can be replicated by definition, perfect hedging is always an option (while not the only one). However, in an incomplete market, replication and complete perfect hedging are not an option for all positions (see El Karoui and Quenez (1995)). The reason for incompleteness is the underlying asset liquidity. It is clear that we also cannot have liquidity in the insurance market because the insurance contracts or underlying risks are not traded in the market.

In actuarial applications, a main hedging activity is to buy reinsurance, where the risk of claims would be transferred from the insurance company to the reinsurance company. However, with a larger perspective, one can think of hedging in a larger scope, where it can be generalised to a large range of activities that can reduce the risk up front. Interestingly, this generalisation can include other risk-mitigating practices such as setting a minimum capital requirement, as we will discuss shortly. In line with the other chapters, we will discuss hedging strategies in the insurance industry in this chapter.

3.1 Motivation and Set-up of the Hedging Problem

Let us begin by reviewing the setup in ruin theory. In actuarial science, risk theory is widely used to measure the amount of initial capital that insurance companies need to reserve in order to limit the probability of bankruptcy below a bearable level. Lundberg (1903) modelled the surplus process of an insurance company for the first time on the basis of a compound Poisson process, which was later known as the Cramér–Lundberg model:

$$U_t = u + ct - \sum_{i=1}^{N_t^\lambda} Y_i, \tag{3.1}$$

where u is the initial capital; c is the premium rate; N_t^λ is the number of claims in $(0,t]$, which is a counting process with a Poisson distribution with a parameter λ; and Y_i is the individual claim and independent of N_t^λ. Recall that the minimum initial capital denoted by u^* can be found by controlling the ruin probability as follows:

$$u^* := \inf \left\{ u \in \mathbb{R} \,\middle|\, \mathbb{P}\left[\inf_{t \geq 0} \left(u + ct - \sum_{i=1}^{N_t^\lambda} Y_i < 0 \right) \right] \leq 1 - \alpha \right\}. \tag{3.2}$$

Because the value at risk (VaR) can be considered as a left-continuous inverse distribution function of the loss, the minimal initial capital can be interpreted in the risk measure framework as

$$u^* = \mathrm{VaR}_\alpha \left(\sup_{t \geq 0} \left(\sum_{i=1}^{N_t^\lambda} Y_i - ct \right) \right). \tag{3.3}$$

Essentially, this equation can be interpreted as the minimum capital needed to obtain no deficit risk, where the risk is measured by VaR_α. Thus, the problem can be rewritten as

$$\inf \left\{ u \in \mathbb{R} \,\middle|\, \mathrm{VaR}_\alpha \left(\sup_{t \geq 0} \left(\sum_{i=1}^{N_t^\lambda} Y_i - ct - u \right) \right) \leq 0 \right\}. \tag{3.4}$$

Now, we can generalise this problem in four steps:

1. We consider a more general surplus process $S = \{S_t\}_{0 \leq t \leq T}$ for some $0 \leq T \leq \infty$ (instead of $\left\{ ct - \sum_{i=1}^{N_t^\lambda} Y_i \right\}_{t \geq 0}$):

$$\inf \left\{ u \in \mathbb{R} \,\middle|\, \mathrm{VaR}_\alpha \left(\sup_{0 \leq t \leq T} (-S_t - u) \right) \leq 0 \right\}. \tag{3.5}$$

2. The simple capital reserve strategy can be extended to a more sophisticated one by including other risky capital, e.g. considering a process $X = \{X_t\}_{t \geq 0}$ instead of the constant (process) u. If the cost of this risky capital is measured by a functional $\pi : \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ is the set of all risky capital accessible to us, then we have the following generalisation of the previous problem:

$$\inf \left\{ \pi(X) \,\middle|\, \mathrm{VaR}_\alpha \left(\sup_{0 \leq t \leq T} (-S_t - X_t) \right) \leq 0, X \in \mathcal{X} \right\}. \tag{3.6}$$

3. Recall that the VaR is a special case of distortion risk measures (DRMs). Thus, VaR_α can be replaced by the general form of a distortion risk measure denoted by

ϱ to obtain

$$\inf\left\{\pi\left(X\right)\middle|\varrho\left(\sup_{0\le t\le T}\left(-S_t-X_t\right)\right)\le 0, X\in\mathcal{X}\right\}. \tag{3.7}$$

where ϱ is a DRM on a set of random variables denoted by $\mathcal{Y}$.

4. Finally, we assume a general risk measure ρ on random processes denoted by $\mathcal{Y}=\{X|\rho(X)<\infty\}$. In particular, one can see that for the set $S+X:=\left\{\{S_t+X_t\}_{t\ge 0}\,|X\in\mathcal{X}\right\}$, the final set-up of the problem is as follows:

$$\inf\left\{\pi\left(X\right)|\rho\left(S+X\right)\le 0, X\in\mathcal{X}\right\}. \tag{3.8}$$

As one can see, the problem is thoroughly generalised to find the least costly risky position to be added to control the risk of insolvency. Following the steps above, a general problem is set up, and the two examples below demonstrate particular cases belonging to our set-up.

Example 3.1.1. The obvious example is the minimum initial capital problem introduced in ruin theory, which is therefore a special case in our problem set-up when $\mathcal{X}=\mathbb{R}$, $T=\infty$, the cost function π is an identity function, $S_t=ct-\sum_{i=1}^{N_t^\lambda}Y_i$, and

$$\rho\left(X\right)=\mathrm{VaR}_\alpha\left(\sup_{t\ge 0}\left(-X_t\right)\right).$$

Example 3.1.2. Consider a financial market consisting of n assets denoted by X^1 to X^n. Let $F_t^1,\dots,F_t^m$ be m independent processes modelling the main economic factors (economic forces) that drive the prices in the economy (one can think of the infamous Fama and French three-factor model plus some default factors). Then, the value of each asset X_t^i would be a linear combination of some of these F_t^m processes. Mathematically, we let $F_t^1,\dots,F_t^m$ be m independent processes. Each X_t^i is a linear combination of some or all of $F_t^1,\dots,F_t^m$, i.e.

$$\begin{pmatrix} X_t^1 \\ X_t^2 \\ \vdots \\ X_t^n \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} F_t^1 \\ F_t^2 \\ \vdots \\ F_t^m \end{pmatrix} + \begin{pmatrix} \epsilon_t^1 \\ \epsilon_t^2 \\ \vdots \\ \epsilon_t^n \end{pmatrix}, \tag{3.9}$$

where the values of a_{ij} are real numbers for $1 \leq i \leq n$ and $1 \leq j \leq m$ and ϵ_t^i is an asset-specific risks independent of any other process and with zero mean. In this case, one can think of all linear combinations of $\sum_{j=1}^{m} w_j F^j$, where w_j is given as

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{3.10}$$

Furthermore, we assume that F_t^j follows a simple affine model as follows:

$$F_t^j = \exp\left(v_j B_t^j\right),$$

where $(B_t^1, \ldots, B_t^n)$ is a vector of independent Brownian motions, $v_1 = 0$ (representing the risk-free scenario), and v_j, $j = 2, \ldots, n$ are positive numbers. In this case,

$$\begin{aligned} \mathbb{E}\left[F_t^j\right] &= \exp\left(\frac{1}{2} v_j^2 t\right), \\ \sigma^2\left(F_t^j\right) &= \exp\left(t^2 v_j^2\right)\left(\exp(t^2 v_j^2) - 1\right). \end{aligned}$$

For the assessment of our model, we have to compare the minimum value of the quadratic risk measure with $\rho(S)$, which is the minimum value to keep S solvent. For instance, we can assume a three-factor model: the first one is the risk-free $v_1 = 0$, the second one represents the financial market (e.g. a major index), and the third represents a major macro-economic force such as inflation or consumption growth, which is not captured by the first and second factors.

3.1.1 Restructuring Dilemma and Extension to Sub-additive Risks

As before, let S denote the surplus and $n \in \mathbb{N}$ be a natural number greater than $\frac{1}{\alpha}$. Let us assume that $\{\Omega_1, \ldots, \Omega_n\}$ is a partition of Ω, where $P(\Omega_i) = \frac{1}{n}, i = 1, \ldots, n$. We restructure the risk capital in the form of $S^i = 1_{\Omega_i} S$. It is clear that $\sum_{i=1}^{n} S^i = S$. Now, observe that for an arbitrary positive number $x > 0$, we obtain $P(x + S^i > 0) \geq P(\Omega \setminus \Omega_i) = 1 - \frac{1}{n}$ for any $x > 0$ because $\{S^i = 0\} = \{1_{\Omega_i} S = 0\} = \{S = 0\} \cup (\Omega \setminus \Omega_i) \supseteq \Omega \setminus \Omega_i$, which implies that $P(x + S^i < 0) \leq \frac{1}{n} \leq \alpha$. Because $x > 0$ is arbitrarily chosen, the capital reserve for the process S_i will be equal to zero.

This means that ruin theory essentially leads us to capital restructuring that can reduce risk with no cost. This would not occur if we had the so-called 'sub-additivity' property for ρ. Let us recall the sub-additivity below.

Definition 3.1.3. A risk measure ρ is sub-additive if

1. $\forall X, Y \in \mathcal{Y}$, $X + Y \in \mathcal{Y}$;
2. $\rho\,(X + Y) \leq \rho\,(X) + \rho\,(Y)$, for all $X, Y \in \mathcal{Y}$.

Proposition 3.1.4. If a risk measure ϱ is sub-additive on random variables, then the following risk measure is sub-additive on processes:

$$\rho\,(X) = \varrho\left(\sup_{0 \leq t \leq T} \{-X_t\}\right).$$

As a reminder of the Remark 2.2.4, the property of cash invariance, under a risk measure on a profit variable, is defined as follows:

Definition 3.1.5. A risk measure ρ of a profit variable X is cash-invariant if $\rho\,(X + c) = \rho\,(X) - c$ for any $c \in \mathbb{R}$.

Note the differences between signs of this cash invariance and that used in other chapters. This is simply to accommodate the cash invariance for the risk measure induced by ruin theory, i.e.

$$\rho_\alpha\,(X + c) = \mathrm{VaR}_\alpha\left(\sup_{t \geq 0}\,(-X_t - c)\right) = \mathrm{VaR}_\alpha\left(\sup_{t \geq 0}\,(-X_t)\right) - c = \rho_\alpha\,(X) - c.$$

Definition 3.1.6. A risk measure is positive homogeneous if $\forall X \in \mathcal{Y}$ and $\lambda > 0$, $\lambda X \in \mathcal{Y}$ and $\rho\,(\lambda X) = \lambda \rho\,(X)$.

3.2 Problem Set-up

In this section, we set up a hedging problem to find the generalised solvency problem. Let us denote the net value of an insurance company by S. Consider a cedent that has given the freedom to choose either to keep a minimum capital requirement or to hedge the company's risk with a set of accessible positions. We denote this set by $\mathcal{X}$ and assume that it is a positive cone that contains the risk-free case and is contained in $\mathcal{Y}$, i.e.

$$\begin{cases} \mathcal{X} + \mathcal{X} \subseteq \mathcal{X} \\ \lambda\mathcal{X} \subseteq \mathcal{X}, \forall \lambda \geq 0 \\ \mathbb{R} \subseteq \mathcal{X} \subseteq \mathcal{Y} \end{cases}. \tag{3.11}$$

The cost function $\pi : \mathcal{X} \to \mathbb{R}$ is now the price of forming any financial position from the set $\mathcal{X}$, which is positive homogeneous and translation-invariant. Note that pi is not a risk measure; therefore, the translation-invariant property means that $\pi(X + c) = \pi(X) + c, \forall c \in \mathbb{R}$. Given that the insurance company has access to a set of valid positions $\mathcal{X}$, the solvency problem can be regarded as a hedging problem following the steps from the previous section and can be rewritten as

$$\begin{cases} \min \pi(X) \\ \rho\left(X + S\right) \leq 0 \\ X \in \mathcal{X} \end{cases}. \tag{3.12}$$

Now, the major question we need to answer is whether this is a viable practice. To obtain the results, we first need to prove a short lemma.

Lemma 3.2.1. Let us fix a random process S such that $-S \in \mathcal{Y}$. Then, $\rho\left(\rho\left(-S\right) + X\right) \leq 0$ for any $X \in \mathcal{Y}$ with $\rho\left(X + S\right) \leq 0$.

Proof. Because $\rho\left(X + S\right) \leq 0$, we have

$$\begin{aligned} \rho\left(\rho\left(-S\right) + X\right) &= \rho\left(\left(\rho\left(-S\right) - S\right) + \left(X + S\right)\right) \\ &\leq \rho\left(\rho\left(-S\right) - S\right) + \rho\left(X + S\right) \leq 0. \end{aligned}$$

□

Now, we provide the answer for the case where the risk measure holds under certain conditions.

Theorem 3.2.2. Let us assume that both ρ and π are positive homogeneous and cash-invariant and particularly assume that ρ is sub-additive. Then, problem (3.12) is finite if and only if $\pi\left(X+\rho\left(X\right)\right)\geq 0\ \forall X\in\mathcal{X}$.

Proof. First, we assume that $\exists X_0\in\mathcal{X}$ such that $\pi\left(X_0+\rho(X_0)\right)<0$ and show that (3.12) does not have a solution. By cash invariance, $\pi\left(X_0+\rho(X_0)\right)<0$ is equivalent to $\rho\left(X_0-\pi(X_0)\right)<0$. To this end, we have to show that there exists a sequence X_n of positions that satisfy the constraints in (3.12) and that $\pi(X_n)\to-\infty$.

Because ρ is cash-invariant, $\rho\left(X_0-\pi(X_0)-\epsilon\right)<0$, where $\epsilon=\frac{-\rho(X_0-\pi(X_0))}{2}$. By our assumptions on S, we also know that there exist $c\in\mathbb{R}$ such that $\rho\left(c+S\right)\leq 0$. Let us introduce $X_n=c+n(X_0-\pi(X_0)-\epsilon)$ where $n>0$. Because $\rho\left(X_0-\pi(X_0)-\epsilon\right)<0$, $\rho\left(c+S\right)\leq$ and by sub-additivity and positive homogeneity of ρ, we have

$$\begin{aligned}\rho\left(X_n+S\right)&=\rho\left(c+n(X_0-\pi(X_0)-\epsilon)+S\right)\\&\leq\rho\left(c+S\right)+\rho\left(n(X_0-\pi(X_0)-\epsilon)\right)\\&=\rho\left(c+S\right)+n\rho\left((X_0-\pi(X_0)-\epsilon)\right)<0.\end{aligned}$$

On the other hand, $X_n\in\mathcal{X}$ by (3.11). Hence, X_n satisfies the conditions in (3.12), and we also have

$$\begin{aligned}\pi(X_n)=&\quad\pi(c+n(X_0-\pi(X_0)+\epsilon)\\&=c+n(\pi(X_0)-\pi(X_0)-\epsilon)=c-n\epsilon,\end{aligned}$$

which tends to $-\infty$ as $n\to\infty$.

Let us assume that X is a random process such that $X\in\mathcal{X}$ and $\rho\left(X+S\right)\leq 0$. By Lemma 3.2.1, we have

$$\rho\left(X-\pi(X)+\left(\pi(X)+\rho\left(-S\right)\right)\right)=\rho\left(\rho\left(-S\right)+X\right)\leq 0.$$

Because $\rho\left(X-\pi(X)\right)\geq 0$, we must have $\pi(X)+\rho\left(-S\right)\geq 0$, implying that $\pi(X)\geq-\rho\left(-S\right)$. $\square$

There is a nice interpretation behind this theorem. If we look at $\pi\left(X+\rho\left(X\right)\right)\geq 0$, one can see that it indicates that hedging is possible if no solvent position is free. In more detail, any position $X\in\mathcal{X}$ can be solvent if we add $\rho\left(X\right)$ to it, i.e. $\rho\left(X+\rho\left(X\right)\right)=\rho\left(X\right)-\rho\left(X\right)=0$. However, making the position X solvent by adding a sufficient amount of capital does not make it free; that is, $\pi\left(X+\rho\left(X\right)\right)<0$ cannot occur.

We also can look at this from a different perspective. $\pi\left(X+\rho\left(X\right)\right)\geq 0$ is equivalent to $\rho\left(X-\pi\left(X\right)\right)\geq 0$ for cash invariance. This means no free position is insolvent. More precisely, if $X\in\mathcal{X}$, the position $X-\pi\left(X\right)$ is free by cash invariance, i.e. $\pi\left(X-\pi\left(X\right)\right)=0$. However, this free position cannot be solvent; otherwise, our hedging problem would not be bounded.

It is also interesting to note that the boundedness of the hedging problem is totally independent of the position S and fully depends on the interrelation between $\mathcal{X}$, π, and ρ.

Now, we want to establish a new definition for the minimum capital requirement based on our hedging problem. Because we assume that $\mathbb{R}\subseteq\mathcal{X}$, we can replace X with $X+\rho(X+S)$, in which case problem (3.12) is reduced to

$$\begin{cases}\min\pi(X)+\rho(X+S)\\ X\in\mathcal{X}\end{cases}. \tag{3.13}$$

For the particular case of finding the solvency value when $\mathcal{X}=\mathbb{R}$, the minimum in (3.12) is equal to $\rho(S)$. This proves the following definition.

Definition 3.2.3. The generalised minimum capital (GMC) requirement for the solvency of an insurance company is defined by

$$\mathrm{GMC}(S)=\min_{X\in\mathcal{X}}\left\{\pi(X)+\rho(X+S)\right\}. \tag{3.14}$$

There are two other interesting facts. First, constant capital cannot make any difference in the hedging strategy:

$$\begin{aligned}
\mathrm{GMC}(S) &= \min_{X\in\mathcal{X}} \{\pi(X) + \rho(X+S)\} \\
&= \min_{X\in\mathcal{X}} \left\{\pi(X) + (ct)_0^T - (ct)_0^T + \rho(X+S)\right\} \\
&= \min_{X\in\mathcal{X}} \left\{\pi(X + (ct)_0^T) + \rho(X + (ct)_0^T + S)\right\}.
\end{aligned}$$

Second, if the conditions of the theorem hold, i.e. the hedging problem is bounded, then the GMC can be regarded as a risk measure on S. If both π and ρ are positive homogeneous and ρ is cash-invariant, then so is the GMC. Furthermore, if both are sub-additive, then the GMC is also sub-additive.

Now, let us use a specific risk measure and a pricing rule to show an example of the GMC. Following Assa (2011), the risk of a random process can be measures by defining a *cumulative risk measure* ρ as follows:

$$\rho(X) = \frac{1}{T}\int_0^T \rho_0(X_t)dt\ , \tag{3.15}$$

where ρ_0 is a cash invariant and positive homogeneous risk measure that is finite on X_t, for all $t \in [0,T]$. We also assume that $t \mapsto \rho_0(X_t)$ is integrable on interval $[0,T]$.

In Assa et al. (2016), the use of *Entropic Value at Risk* measure (EVaR_α) of Ahmadi-Javid (2012) as ρ_0, is a suitable choice for studying the risk processes.

We also assume that a market uses a *cumulative pricing rule* π:

$$\pi(X) = \frac{1}{T}\int_0^T \pi_0(X_t)dt.$$

Now, if we want to verify if the hedging problem is finite, we need to check

$$\rho(X) + \pi(X) = \frac{1}{T}\int_0^T \left(\rho_0(X_t) + \pi_0(X_t)\right)dt \geq 0, \forall X \in \mathcal{X}.$$

This is possible if the static problem $\rho_0(X_t) + \pi_0(X_t) \geq 0$ can be verified.

Example 3.2.4. Let $\rho_0\left(X_t\right) = \sigma\left(X_t\right) - \mathbb{E}\left[X_t\right]$ (σ is the standard deviation) and $\pi\left(X_t\right) = \mathbb{E}\left[X_t\right]$. In this case, we have

$$\begin{aligned}\rho(X) + \pi(X) &= \frac{1}{T}\int_0^T \left(\rho_0(X_t) + \pi_0(X_t)\right) dt \\ &= \frac{1}{T}\int_0^T \left(\sigma(X_t)\right) dt \geq 0.\end{aligned}$$

This shows that the hedging problem always has a solution (regardless of the set $\mathcal{X}$). To find the hedging strategy, we need to solve the following problem:

$$\mathrm{GMC}(S) = \min_{X\in\mathcal{X}} \left\{\pi(X) + \rho(X+S)\right\} \tag{3.16}$$

$$= \min_{X\in\mathcal{X}} \left\{ \frac{1}{T}\int_0^T \mathbb{E}(X_t)dt + \frac{1}{T}\int_0^T \left(\sigma\left(X_t+S_t\right) - \mathbb{E}\left[X_t+S_t\right]\right) dt \right\} \tag{3.17}$$

$$= \min_{X\in\mathcal{X}} \left\{ \frac{1}{T}\int_0^T \left(\sigma\left(X_t+S_t\right) - \mathbb{E}\left[S_t\right]\right) dt \right\}. \tag{3.18}$$

If one needs to look at the benefit that hedging can have on the solvency, we can compare the GMC with the value of the solvency without any hedging strategy $\rho\left(S\right)$, i.e.

$$\rho\left(S\right) - \mathrm{GMC}(S) = \min_{X\in\mathcal{X}} \left\{ \frac{1}{T}\int_0^T \left(\sigma\left(S_t\right) - \sigma\left(X_t+S_t\right)\right) dt \right\}.$$

This shows all the benefits depend on how the values of S_t are correlated, which makes perfect sense.

3.3 Discussion

In this chapter, we extended the classical problem of finding the minimum initial capital from ruin theory. First, we demonstrated the process of deducing a hedging problem from the original problem established on the basis of risk models to a general risk measure. By replacing the constant initial capital reserve with the risky positions with a cost, we introduced a hedging problem to find the generalised minimum capital (GMC) requirement, which is more realistic in a financial market. In addition, the problem was discussed for certain properties of the risk measure and cost function, i.e. positive homogeneity, cash invariance, and sub-additivity. In the last section, an example was shown for the chosen risk measures and cost functions by the cumulative risk measure and cumulative pricing rule, respectively.

Chapter 4

Optimal Reinsurance Policy Design under a Dynamic Framework

The optimal reinsurance problem has been extensively discussed in the literature by considering different objective criteria under different premium principles with different types of reinsurance contracts. Borch (1960a) and Arrow (1963) are among the first papers that address the optimal reinsurance policy design in a utility-maximisation framework. Later, this problem was further developed by considering set-ups minimising the ruin probability, maximising the expected utility of terminal surplus, and maximising the cumulative expected discounted dividends, these constituting the three main objective criteria chosen to design for the optimal reinsurance retention level. The relevant literature has been reviewed in Chapter 1. The constant-barrier dividend policy is proven to be optimal in several discrete-time settings (e.g. De Finetti (1957), Miyasawa (1961), Morrill (1966), Claramunt et al. (2003) and Mármol et al. (2005)).

The problem of optimal reinsurance design has been studied in both dynamic and static frameworks. In a continuous-time dynamic set-up, the Bellman Principle is the main principle used for finding optimal reinsurance contracts. In a dynamic programming problem, a Hamilton–Jacobi–Bellman (HJB) equation that minimises the ruin probability under the expected value principle is used to find the optimal solutions of excess-of-loss reinsurance; seeHipp and Vogt (2003) and Dickson and Waters (2006). Zhang et al. (2007) worked on a similar topic but focused on the combination of quota-share and excess-of-loss reinsurance. Zhang and Siu (2009) added the investment part into the setting and used Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations to solve the problem under two objective functions. In a different strand of the literature, ruin probability minimisation that provides a criterion to reduce the bankruptcy risk and stands for the ceding company, along with a dividend maximisation problem, adding the shareholders' point of view, has been considered. Asmussen and Taksar (1997) applied the HJB equation to solve the general optimal aggregate discounted dividends problem. Using the results from that paper, Asmussen et al. (2000) solved the optimal reinsurance problem considering excess-of-loss policies. Mnif and Sulem (2005) used the HJB equation to obtain the optimal excess-of-loss reinsurance for a general dividend strategy. Recently, the method of Asmussen et al. (2000) has been extended in various ways; for instance, Bai et al. (2010) added transaction costs and taxes, Wu and Guo (2012) added capital injections, and Liu and Hu (2014) added the equity.

As mentioned, another strand of the literature on optimal reinsurance design is the static set-up. Although the optimal design based on utility maximisation in Borch (1960a) and Arrow (1963) falls in this category, in recent years most of the problems in this branch have been developed by minimising the risk of the insured's global position as measured by a risk measure. Gajek and Zagrodny (2004) and Cui et al. (2013) considered a general risk measure. In contrast, Cai and Tan (2007) worked on stop-loss reinsurance under two specific chosen risk measures, Value at Risk (VaR) and Conditional Tail Expectation (CTE). Most of the studies mentioned worked under the expected value premium principle. By the development of risk measures, the problem of constructing reinsurance strategy has been studied under different types of risk measures as both the objective function and the premium principle. One family of risk measures is the family of distortion risk measures (DRMs), which includes Wang's premium principle (Wang, 2000), Value at Risk (VaR), and Conditional Value at Risk (CVaR) (see Pflug (2006) and Balbás et al. (2009)). In Assa (2015a), it is shown that under a very general framework, when the risk measure and the risk premium are both from the family of DRMs,the ceding, reinsurance, and social planner problems can be considered in a unified set-up. By introducing the marginal indemnification functions (MIFs), the optimal solution could be obtained via MIF formulation and it has a multilayer structure.

Although a dynamic set-up better reflects the reality of the world, there are numerous technical restrictions involved, causing researchers to focus their attention on a limited number of reinsurance strategies in a dynamic setting, such as stop-loss or excess-of-loss policies. The static framework, on the other hand, has the flexibility to consider a wider range of policies. In this chapter, we combine the dynamic and static frameworks and take advantage of both set-ups at the same time. Indeed, because reinsurance contracts are written at the beginning of each year, we need only consider a discrete-time model. Therefore, inspired by the celebrated Cramér–Lundberg model, we consider a surplus process in a discrete-time model with independent increments. While being dynamic, this set-up also allows us to take advantage of the static approach over the course of a year. To the best of our knowledge, this study is the first to introduce dynamic distortion risk measures to fit in the proposed setting. We will see that the assumption of independent increments will help to simplify the problem using dynamic distortion risk measures. In this work, we show the optimal results under the general distortion risk measures and applications with some well-known risk measures in the DRM family as specific cases.

This chapter is organised as follows. Section 4.1 lists two common expressions of DRMs found in the literature and introduces the dynamic DRM. The optimisation problems with respect to different conditions are set up and explained in Section 4.2, and the results are presented in Section 4.3. In Section 4.4, we apply the results with particular DRMs as examples. Section 4.5 summarise this chapter.

4.1 Dynamic Distortion Risk Measures and Premiums

Let us first review the basic setting stated in Chapter 2. Consider a non-atomic filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\{\mathbb{N}\cup\infty\}}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ-field, $\{\mathcal{F}_t\}_{t\in\{\mathbb{N}\cup\infty\}}$ is a filtration, and $\mathbb{P}$ is a probability measure. Assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_\infty = \mathcal{F}$. We will introduce the specific filtration $\mathcal{F}_t$ in Section 4.2.

In this chapter, we work first in a static framework and then in a discrete-time dynamic set-up. For $t = 0, 1, 2, \dots, \infty$, let $L^0(\Omega, \mathcal{F}_t)$ be the vector space consisting of all real-valued random variables that are $\mathcal{F}_t$-measurable. We denote the set of all random variables in $L^0(\Omega, \mathcal{F}_t)$ that have a finite second momentum by $L^2(\Omega, \mathcal{F}_t)$. In a discrete-time dynamic set-up, we consider a claim process $X_t \in L^2(\Omega, \mathcal{F}_t)$ that is simply the aggregate claims received from the insurance company's clients over the time period $[t-1, t)$. Now let us consider $\mathbf{L}^2 = L^2(\Omega, \mathcal{F}_t) \times L^2(\Omega, \mathcal{F}_{t+1}) \times L^2(\Omega, \mathcal{F}_{t+2}) \times \cdots$. We introduce the Hilbert space $\left(\mathbf{L}^2_\beta, \langle .,. \rangle_{\mathbf{L}^2_\beta}\right)$, for a constant number β, that consists of all members of $\mathbf{L}^2$ that have a finite norm induced by the following inner product:

$$\left\langle \{Y_s\}_{s\geq t}, \{Y'_s\}_{s\geq t} \right\rangle_{\mathbf{L}^2_\beta} = \sum_{s\geq t} \beta^{s-t} \left\langle Y_s, Y'_s \right\rangle_{L^2(\Omega,\mathcal{F}_s)}.$$

Throughout this chapter, we further assume that $\{X_s\}_{s\geq t} \in \left(\mathbf{L}^2_\beta, \langle .,. \rangle_{\mathbf{L}^2_\beta}\right)$.

Recall that Value at Risk (VaR), as one of the most popular risk measures both in the literature and in industry, is the quantile of the risk variable distribution. It is also referred to as the left-continuous inverse distribution function. Let us rewrite the definition below.

Definition 4.1.1. The VaR of a random variable X with confidence level $\alpha \in [0, 1]$ is defined as

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} | \, \mathbb{P}(X \leq x) \geq \alpha\}. \tag{4.1}$$

Throughout this chapter, we denote the left inverse of a CDF by $F_X^{-1}(\alpha)$, which implies $\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$. As a reminder from Section 2.2, the expression of a distortion risk measure derived by introducing the dual distortion function $\Lambda(u) = 1 - g(1-u)$, is written as follows:

Definition 4.1.2. A *Distortion Risk Measure (DRM)* ρ^Λ can be rewritten as

$$\rho^\Lambda(X) := \int_0^1 \mathrm{VaR}_z(X) d\Lambda(z), \tag{4.2}$$

where $\Lambda : [0,1] \to [0,1]$ is a non-decreasing and càdlàg function and $\mathrm{VaR}_\alpha(X)$ is as defined in Definition 4.1.1.

Again, this form of DRM will be used throughout this chapter. Similarly, a distortion premium principle (DPP) can be introduced as follows:

Definition 4.1.3. For a safety load $\theta \in \mathbb{R}^+$ and a distortion function $\Lambda : [0,1] \to [0,1]$, a *Distortion Premium Principle* (DPP) π^Λ is introduced as

$$\pi^\Lambda[X] := (1+\theta) \int_0^1 \mathrm{VaR}_z(X) d\Lambda(z).$$

In this chapter, a dynamic set-up will be studied, which leads us to consider a dynamic risk measure. This is because for an insurance company, the risk measure used for controlling risk and calculating premiums could be determined anew at the beginning of each accounting year.

For this purpose, let us first review the literature for the dynamic Value at Risk defined in Cheridito and Stadje (2009). First, however, in order to introduce a dynamic Value at Risk, we need to introduce some notation. For a family of random variables F, the essential infimum of F, which is the greatest lower bound of the family, is denoted by $\operatorname{ess\,inf}(F)$. For the reader's convenience, we provide the following proposition from Neveu (1975, p. 121).

Proposition 4.1.4. For every family F of real-valued measurable functions $f : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists one and up to equivalence only one measurable function $g : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that

1. $g \leq f$ almost surely(a.s.) for all $f \in F$;

2. if h is a measurable function such that $h \leq f$ a.s. for all $f \in F$, then $h \leq g$ a.s.

This function g is the essential infimum of the family F.

Therefore, the definition of dynamic VaR can be extended from the infimum in (4.1) to the essential infimum as follows:

Definition 4.1.5. For any risk $X \in L^0(\mathcal{F}_\infty)$ and confidence level $\alpha \in (0, 1)$, a *dynamic Value at Risk* at time t is defined as

$$\mathrm{VaR}_t^\alpha(X) := \operatorname{ess\,inf}\{Z_t \in L^0(\mathcal{F}_t) : \mathbb{P}(X \leq Z_t|\mathcal{F}_t) \geq \alpha\}, \tag{4.3}$$

where $t \in \mathbb{N}$.

Notice here that the subscript t of Z_t inside $\operatorname{ess\,inf}$ only indicates that it is in the set $L^0(\mathcal{F}_t)$. The following proposition is a result of Definition 4.1.5:

Proposition 4.1.6. We have the following statements:

1. For any $X \in \mathcal{F}_t$ and Y, $\mathrm{VaR}_t^\alpha(Y + X) = \mathrm{VaR}_t^\alpha(Y) + X$.

2. For any non-decreasing and left-continuous real function f, we have

$$\mathrm{VaR}_t^\alpha\left(f\left(X\right)\right) = f\left(\mathrm{VaR}_t^\alpha\left(X\right)\right).$$

The first statement is simply an immediate result of the definition. To show the second one, first, we can prove the statement for an increasing function f; then, we can consider a sequence of increasing functions f_n that uniformly converge to f, and take the limit as $n \to \infty$.

Now for the first time, we introduce a dynamic DRM in a manner similar to that for Definition 4.1.2.

Definition 4.1.7. For a distortion function $\Lambda : [0,1] \to [0,1]$, a *Dynamic Distortion Risk Measure* at time t is a mapping $\rho_t^\Lambda : L^0(\mathcal{F}_\infty) \to L^0(\mathcal{F}_t)$ for $t \in \mathbb{N}$ such that

$$\rho_t^\Lambda(X) := \int_0^1 \mathrm{VaR}_t^z(X) d\Lambda(z). \tag{4.4}$$

Notice that from Definitions 4.1.5 and 4.1.7, both mappings $\mathrm{VaR}_t^\alpha(X)$ and $\rho_t^\Lambda(X)$ are from $L^0(\mathcal{F}_\infty)$ to $L^0(\mathcal{F}_t)$; thus, $\mathrm{VaR}_t^\alpha(X)$ and $\rho_t^\Lambda(X)$ are both $\mathcal{F}_t$-measurable. Similarly, we can introduce a dynamic distortion premium principle as $\Pi_t^\Lambda(X) = (1 + \theta) \int_0^1 \mathrm{VaR}_t^z(X) d\Lambda(z)$.

4.2 Problem Set-up

In this section, we set up the problem, discuss the main objective, and present several constraints.

First, however, we need to introduce the surplus process of the insurance company for the context of the following discussions. As before, let us denote the aggregate claims over the period $[t, t+1)$ by X_{t+1}. We consider a filtration $\mathcal{F}_t$ as the smallest σ-field generated by $X_1, \ldots, X_t$.

The amount of losses covered by the reinsurance contract for the same period is given by $R_t(X_{t+1})$, where R_t is known as the indemnity function at time t. The terms of the reinsurance policy (the indemnity function) are agreed between the cedent and the reinsurer before the period of coverage, $[t, t+1)$. R_t is $\mathcal{F}_t$-measurable, and the reinsurance premium is calculated by the distortion premium principle, denoted as $\pi_t^\Pi[R_t(X_{t+1})]$. To keep the problem manageable, we assume that the insurance premium per unit time, c_t, is a piece-wise constant function over the time period $[t, t+1)$. Finally, we assume that a dividend D_t is paid at time t according to the dividend policy chosen.

The aim in this chapter is to find the optimal reinsurance strategy to maximise the shareholders' lifetime dividend. Thus, the objective function is defined as the conditional expectation on discounted aggregate dividends given information $\mathcal{F}_t$, i.e.

$$\mathbb{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} D_s | \mathcal{F}_t\right],$$

where β is the discounting factor per unit time.

To achieve this, we assume that dividend payments are made before the cedent pays the reinsurance premium. Thus, the surplus of the insurer at time $s+1$, before dividends and reinsurance premiums are paid, follows the following dynamic:

$$U_{s+1} = \frac{U_s - D_s + c_s}{\beta} - I_s(X_{s+1}) - \pi_s^{\Pi}\left[R_s(X_{s+1})\right], \tag{4.5}$$

where

$$I_s(X_{s+1}) = X_{s+1} - R_s(X_{s+1}) \tag{4.6}$$

is the amount of the remaining claims covered by the cedent for period $[s, s+1)$, and $s = t, t+1, \ldots, \infty$. This is the budget constraint of the insurance company.

Next, let us consider the condition for keeping the cedent solvent:

$$\rho_s^{\Gamma}(-U_{s+1}) \leq 0. \tag{4.7}$$

This condition means that the insurance company cannot accept any shortfall risk over a period $[s, s+1)$. Using the budget constraint, we can rewrite this condition as follows:

$$\rho_s^{\Gamma}\left(-\frac{U_s - D_s + c_s}{\beta} + I_s(X_{s+1}) + \pi_s^{\Pi}\left[R_s(X_{s+1})\right]\right) \leq 0. \tag{4.8}$$

Now, using Proposition 4.1.6, we can rewrite the solvency condition as

$$\rho_s^{\Gamma}\left(I_s(X_{s+1})\right) \leq \frac{U_s - D_s + c_s}{\beta} - \pi_s^{\Pi}\left[R_s(X_{s+1})\right], \tag{4.9}$$

for $s = t, t+1, \ldots, \infty$.

Notice that $I_s\left(X_{s+1}\right)$ is the insurance loss variable, and the insurance company's balance is equal to $\frac{U_s - D_s + c_s}{\beta} - \pi_s^{\Pi}\left[R_s(X_{s+1})\right]$. Therefore, the company is solvent if the company's capital requirement, that is, $\rho_s^{\Gamma}\left(I_s(X_{s+1})\right)$, is less than the company's balance.

There are two main assumptions that need to be considered. The first one is inspired by the Cramér–Lundberg model. In this classical model, the aggregate claim at time t is given by $\sum_{i=1}^{N(t)} Y_i$, where $\{Y_i\}_{i=1,2,\ldots,\infty}$ is an independent process that is independent of the Poisson process $\{N(t)\}_{t\geq 0}$. Therefore, the increments over the course of $[t-1, t)$ are given by $X_t = \sum_{i=N(t-1)}^{N(t)} Y_i$, which is an independent sequence. Therefore, we consider the following assumption on the claim size X_t.

Assumption 2. The total claims X_t are independent and have the cumulative distribution function F_{X_t}.

Under this assumption, we have the following results.

Proposition 4.2.1. Under Assumption 2 (by which the period total claims X_t are independent and have cumulative distribution function F_{X_t}), the dynamic VaR satisfies

$$\mathrm{VaR}_t^{\alpha}(Y) = \mathrm{VaR}_{\alpha}(Y), \text{ a.s.,} \tag{4.10}$$

for any Y independent of $\mathcal{F}_t$.

Proof. Let $B = \bigcap_{i=1}^{t}\{a_i \leq X_i \leq b_i\} \in \mathcal{F}_t$ and let Z_t be an $\mathcal{F}_t$-measurable random variable. Assume that the joint CDF of $(Y, Z_t, X_t, \ldots, X_1)$ is denoted by F. Thus, by the definition of the conditional expectation, we have

$$\begin{aligned}&\mathbb{E}\left[\mathbb{P}\left(Y \leq Z_t|\mathcal{F}_t\right)\mathbb{1}_B\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Y\leq Z_t\}}|\mathcal{F}_t\right]\mathbb{1}_B\right] = \mathbb{E}\left[\mathbb{1}_{\{Y\leq Z_t\}}\mathbb{1}_B\right]\\ &= \int_{a_1}^{b_1}\cdots\int_{a_t}^{b_t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{1}_{\{y\leq z_t\}}dF(y, z_t, x_t, \ldots, x_1).\end{aligned}$$

By Assumption 2, X_i, Y are independent of B. If we denote the joint CDF of $Z_t, X_t, \ldots, X_1$

by F_2, we obtain

$$
\begin{aligned}
&\mathbb{E}[\mathbb{P}(Y \le Z_t|\mathcal{F}_t)\mathbb{1}_B] \\
=&\int_{a_1}^{b_1} \cdots \int_{a_t}^{b_t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{1}_{\{y \le z_t\}} \\
&\qquad\qquad \times dF_Y(y) dF_2(z_t, x_t, \dots, x_1) \\
=&\int_{a_1}^{b_1} \cdots \int_{a_t}^{b_t} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z_t} dF_Y(y)\right] dF_2(z_t, x_1, \dots, x_t) \\
=&\int_{a_1}^{b_1} \cdots \int_{a_t}^{b_t} \int_{-\infty}^{\infty} F_{X_{s+1}}(z_t) dF_2(z_t, x_1, \dots, x_t) \\
=&\mathbb{E}[F_Y(Z_t)\mathbb{1}_B].
\end{aligned}
$$

Given that $\mathcal{F}_t$ is the smallest σ-field generated by sets like B, it can be deduced that $\mathbb{P}(Y \le Z_t|\mathcal{F}_t) = \mathbb{E}[F_Y(Z_t)|\mathcal{F}_t] = F_Y(Z_t)$ a.s. Following the definition of dynamic VaR, we have

$$
\begin{aligned}
\mathrm{VaR}_t^\alpha(Y) &= \operatorname{essinf}\{Z_t \in L^0(\mathcal{F}_t) : \mathbb{P}(Y \le Z_t|\mathcal{F}_t) \ge \alpha\} \\
&= \operatorname{essinf}\{Z_t \in L^0(\mathcal{F}_t) : F_Y(Z_t) \ge \alpha\} \\
&= \operatorname{essinf}\{Z_t \in L^0(\mathcal{F}_t) : Z_t \ge F_Y^{-1}(\alpha)\}.
\end{aligned}
$$

By Definition 4.1.1, $F_Y^{-1}(\alpha) = \inf\{x \in \mathbb{R}|\, \mathbb{P}(Y \le x) \ge \alpha\} = \mathrm{VaR}_\alpha(Y)$. As a result,

$$
\mathrm{VaR}_t^\alpha(Y) = \mathrm{VaR}_\alpha(Y) \text{ a.s.;}
$$

i.e. under Assumption 2, the dynamic VaR is no longer dynamic. □

This proposition shows that VaR is no longer dynamic under Assumption 2. Hence, similar results can be applied to the general DRMs as follows:

Lemma 4.2.2. Following Proposition 4.2.1, for any distortion function Λ and a measurable real function f, we obtain

$$
\rho_t^\Lambda \left(f\left(X_{t+i}\right)\right) = \rho^\Lambda \left(f\left(X_{t+i}\right)\right), \text{ a.s.}, \; i \in \mathbb{N}. \tag{4.11}
$$

Using this lemma, we can rewrite the solvency condition as follows:

$$\rho^{\Gamma}\left(I_s(X_{s+1})\right) \leq \frac{U_s - D_s + c_s}{\beta} - \pi_s^{\Pi}\left[R_s(X_{s+1})\right], \tag{4.12}$$

for $s = t, t+1, \ldots, \infty$.

In a manner similar to the discussion regarding ρ_s^{Γ}, we can see that since X_{s+1} is independent of $\mathcal{F}_s$, we have that $\pi_s^{\Pi}\left[R_s(X_{s+1})\right] = \pi^{\Pi}\left[R_s(X_{s+1})\right]$. Again, therefore, we can rewrite the solvency condition as follows:

$$\rho^{\Gamma}\left(I_s(X_{s+1})\right) \leq \frac{U_s - D_s + c_s}{\beta} - \pi^{\Pi}\left[R_s(X_{s+1})\right], \tag{4.13}$$

for $s = t, t+1, \ldots, \infty$. The same holds for the budget constraint, which is rewritten as follows:

$$U_{s+1} = \frac{U_s - D_s + c_s}{\beta} - I_s(X_{s+1}) - \pi^{\Pi}\left[R_s(X_{s+1})\right]. \tag{4.14}$$

4.2.1 MIF formulation

In this section, let us recall the MIF-based formulation method from Section 2.3 under the no–moral–hazard assumption formulation. As a reminder, the risk of moral hazard is ruled out under Assumption 1 because both cedent and reinsurer will be affected by any increase in the global loss. This is consistent with the construction of the space of indemnification functions defined as

$$\mathbb{C} = \left\{0 \leq R(x) \leq x | R(x) \text{ and } x - R(x) \text{ are non-decreasing}\right\}. \tag{4.15}$$

Or equivalently,

$$\mathbb{C} = \left\{R : \mathbb{R}^+ \to \mathbb{R}^+ | R(x) = \int_0^x h(t)dt,\ 0 \leq h \leq 1\right\}. \tag{4.16}$$

Additionally, the space of MIFs as

$$\mathbb{D} = \{h : \mathbb{R}^+ \to \mathbb{R}^+ | 0 \leq h \leq 1\}. \tag{4.17}$$

From Proposition 2.3.3, a DRM denoted by ρ^{Λ} can be written as

$$\rho^{\Lambda}\left(R\left(X\right)\right) = \int_0^{\infty} \left(1 - \Lambda\left(F_X\left(z\right)\right)\right) R'\left(z\right) dz, \tag{4.18}$$

where $R \in \mathbb{C}$ is a function with its derivative R'.

Before applying MIF formulation in the next section, we need to state an useful lemma here. Recalling the notation for positive and negative parts of a random variable Y as $Y^+ = \max(Y, 0)$ and $Y^- = -\min(Y, 0)$, respectively, we have the following lemma.

Lemma 4.2.3. For any real function $R \in \mathbb{C}$ and random variables $X \geq 0$ and Y with $\mathbb{E}[Y] \neq 0$, if $\mathbb{E}[Y^+] \neq 0$ and $\mathbb{E}[Y^-] \neq 0$, then there exist two random variables X_1, X_2 with probability density functions $f_{X_1}(x) = \frac{\int_0^\infty y f_{X,Y}(x,y)dy}{\mathbb{E}[Y^+]}$ and $f_{X_2}(x) = \frac{\int_{-\infty}^0 y f_{X,Y}(x,y)dy}{\mathbb{E}[-Y^-]}$, respectively, such that

$$\begin{aligned}\mathbb{E}\left[R(X)Y\right] =& \mathbb{E}\left[Y^+\right] \int_0^\infty (1 - F_{X_1}(z))\, R'(z)\, dz \\ &- \mathbb{E}\left[Y^-\right] \int_0^\infty (1 - F_{X_2}(z))\, R'(z)\, dz. \end{aligned} \tag{4.19}$$

If $\mathbb{E}[Y^+] = 0$ and $\mathbb{E}[Y^-] \neq 0$, then X_2 exists with the density above such that $\mathbb{E}[R(X)Y] = -\mathbb{E}[Y^-] \int_0^\infty (1 - F_{X_2}(z))\, R'(z)\, dz$.

If $\mathbb{E}[Y^+] \neq 0$ and $\mathbb{E}[Y^-] \neq 0$, then X_1 exists with the density above such that $\mathbb{E}[R(X)Y] = \mathbb{E}[Y^+] \int_0^\infty (1 - F_{X_1}(z))\, R'(z)\, dz$.

Proof. Assume first that the joint density $f_{X,Y}$ exists. To prove this lemma, we distinguish four cases.

Case 1: If $Y = 0$ a.s., then $\mathbb{E}[Y] = 0$, and the problem is trivial.

Case 2: Assume that Y is non-negative and $Y > 0$ on a positive measure set. This implies that $\mathbb{E}[Y] > 0$. We have

$$\begin{aligned}
&\mathbb{E}[YR(X)]\\
=&\int_0^\infty \int_0^\infty yR(x)\, f_{X,Y}(x,y)\, dydx\\
=&\int_0^\infty \left(\int_0^\infty y f_{X,Y}(x,y)\, dy\right) R(x)\, dx\\
=&\left(\int_0^\infty \int_0^\infty y f_{X,Y}(x,y)\, dydx\right) \int_0^\infty \frac{\left(\int_0^\infty y f_{X,Y}(x,y)\, dy\right)}{\left(\int_0^\infty \int_0^\infty y f_{X,Y}(x,y)\, dydx\right)} R(x)\, dx\\
=&\mathbb{E}[Y] \int_0^\infty f_{X_1}(x)\, R(x)\, dx = \mathbb{E}[Y]\, \mathbb{E}[R(X_1)],
\end{aligned}$$

where X_1 is a random variable with probability density function equal to

$$f_{X_1}(x) = \frac{\left(\int_0^\infty y f_{X,Y}(x,y)\, dy\right)}{\left(\int_0^\infty \int_0^\infty y f_{X,Y}(x,y)\, dydx\right)} = \frac{\int_0^\infty y f_{X,Y}(x,y)\, dy}{\mathbb{E}[Y]}$$

and corresponding cumulative distribution function F_{X_1}. Therefore, using Lemma 2.3.3, we have $\mathbb{E}[R(X_1)] = \int_0^\infty (1 - F_{X_1}(z))\, R'(z)\, dz$.

Case 3: Consider the scenario opposite of that in Case 2; assume that Y is non-positive and $Y < 0$ on a positive measure set. This implies that $\mathbb{E}[Y] = -\mathbb{E}[Y^-] < 0$ and $\mathbb{E}[-Y^-] = \left(\int_0^\infty \int_{-\infty}^0 y f_{X,Y}(x,y)\, dydx\right)$. Thus, similar to Case 2, we obtain

$$\begin{aligned}
&\mathbb{E}[YR(X)]\\
=&\int_0^\infty \int_{-\infty}^0 yR(x)\, f_{X,Y}(x,y)\, dydx\\
=&\left(\int_0^\infty \int_{-\infty}^0 y f_{X,Y}(x,y)\, dydx\right) \int_0^\infty \frac{\left(\int_{-\infty}^0 y f_{X,Y}(x,y)\, dy\right)}{\left(\int_0^\infty \int_{-\infty}^0 y f_{X,Y}(x,y)\, dydx\right)} R(x)\, dx\\
=&\mathbb{E}[Y] \int_{-\infty}^0 f_{X_2}(x)\, R(x)\, dx = \mathbb{E}[Y]\, \mathbb{E}[R(X_2)],
\end{aligned}$$

where X_2 is again a random variable with probability density function equal to $f_{X_2}(x) = \frac{\int_{-\infty}^0 y f_{X,Y}(x,y) dy}{\mathbb{E}[-Y^-]}$ and corresponding cumulative distribution function F_{X_2}. Therefore, using Lemma 2.3.3, we have

$$\mathbb{E}[R(X_2)] = \int_0^\infty (1 - F_{X_2}(z))\, R'(z)\, dz.$$

Case 4: Consider a scenario in which Cases 1, 2, and 3 do not apply. Here, we need to notice that $\mathbb{E}\left[R\left(X\right)Y\right]=\mathbb{E}\left[R\left(X\right)Y^{+}\right]-\mathbb{E}\left[R\left(X\right)Y^{-}\right]$, and then we can use Cases 2 and 3 to derive the result as

$$\begin{aligned}&\mathbb{E}\left[R\left(X\right)Y\right]\\=&\mathbb{E}\left[Y^{+}\right]\int_{0}^{\infty}\left(1-F_{X_1}\left(z\right)\right)R'\left(z\right)dz-\mathbb{E}\left[Y^{-}\right]\int_{0}^{\infty}\left(1-F_{X_2}\left(z\right)\right)R'\left(z\right)dz,\end{aligned}$$

where the corresponding probability density functions of random variables X_1, X_2 are

$$f_{X_1}\left(x\right)=\begin{cases}0, & \text{if } Y^{+}=0\\ \frac{\int_0^\infty y f_{X,Y}(x,y)dy}{\mathbb{E}[Y^{+}]}, & \text{otherwise}\end{cases}$$

and

$$f_{X_2}\left(x\right)=\begin{cases}0, & \text{if } Y^{-}=0\\ \frac{\int_{-\infty}^0 y f_{X,Y}(x,y)dy}{\mathbb{E}[-Y^{-}]}, & \text{otherwise}\end{cases},$$

respectively, for the cumulative distribution functions F_{X_1} and F_{X_2}.

Now consider the case where the joint distribution does not exist. In this case, we can consider sequences (X_n, Y_n) that have joint distributions and converge uniformly to (X, Y). Then the result can be derived by obtaining a limit as $n\to\infty$ and using the fact that R is Lipschitz and that uniform convergence implies L^1 convergence. $\square$

Proposition 2.3.3 and Lemma 4.2.3 will be used throughout the proofs of our main theorems on the optimal results in the next section.

Finally, we want to mention that since $\{X_s\}_{s\geq t}\in\mathbf{L}^2_\beta$ and since R and I are Lipschitz, it is easy to see that $\{U_s\}_{s\geq t}\in\mathbf{L}^2_\beta$. In order to formalise this, we state it as a proposition as follows:

Proposition 4.2.4. If $\{X_s\}_{s\geq t}\in\mathbf{L}^2_\beta$, then $\{U_s\}_{s\geq t}\in\mathbf{L}^2_\beta$.

4.3 Optimal Solutions

In this section, we discuss the optimal results in three scenarios. The objective function is the conditional expectation of cumulative dividends. To maximize the shareholders benefits, the supremum of the objective function is taken as follows:

$$\sup_{D_s} \mathbb{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} D_s | \mathcal{F}_t\right]. \tag{4.20}$$

Then, we consider the budget constraints, the solvency condition, and dividend rules as follows:

$$\begin{cases} U_{s+1} = \dfrac{U_s - D_s + c_s}{\beta} - I_s(X_{s+1}) - \pi^{\Pi}[R_s(X_{s+1})] & [i] \\ \dfrac{U_s - D_s + c_s}{\beta} - \rho^{\Gamma}\left[I_s(X_{s+1})\right] - \pi^{\Pi}[R_s(X_{s+1})] \geq 0 & [ii] \\ D_s = H(U_s) & [iii] \end{cases} \tag{4.21}$$

for $s = t, t+1, \ldots, \infty$. Thus, we have the set-ups for maximising the objective function subject to constraints as

- **Problem I: Dynamic Setting without Solvency Condition under General Dividend Policy.** To maximise the shareholders' profits, we only consider the budget constraint $[i]$ in (4.21) and look for the optimal reinsurance contract in this set-up.

- **Problem II: Extension with Solvency Condition under General Dividend Policy.** We extend the settings of the first problem by adding the solvency condition, $[ii]$ in (4.21). As a consequence, the cedents benefit is also taken care of.

- **Problem III: Dynamic Setting with a Specific Dividend Policy.** Both Problems I and II are under the general dividend policy without any restrictions. In this setting, the specific condition $[iii]$ in (4.21) of the dividend strategies is given as explained in Section 4.3.3.

4.3.1 Problem I

In this setting, we only consider the maximisation of the discounted total dividends with the budget constraint stated in (4.21)-[i].

Theorem 4.3.1. Using the notation defined above, if $\{X_s\}_{s\geq t} \in \mathbf{L}^2_\beta$ and if Assumptions 2 and 1 hold, 'the optimal solution of Problem I (dynamic setting with budget constraint only) is myopic, and the optimal reinsurance contract is given by

$$R_s^*(x) = \int_0^x (R_s^*)'(z)\,dz,$$

where

$$(R_s^*)'(z) = \mathbb{1}_{\left\{1-F_{X_{s+1}}(z)>(1+\theta)\left(1-\Pi\left[F_{X_{s+1}}(z)\right]\right)\right\}}.$$

Proof. Let us rearrange the budget constraint [i] in (4.21) so that the dividend policy is represented in terms of $s = t, t+1, \ldots, \infty$, as

$$D_s = U_s - \beta U_{s+1} + c_s - \beta\left(I_s(X_{s+1}) + \pi^\Pi\left[R_s(X_{s+1})\right]\right). \tag{4.22}$$

Taking this into the objective function (denoted by J^0) and establishing the notation $\mathbf{RU}_t := \{\mathbf{R}_t, \mathbf{U}_t\}$, where $\mathbf{R}_t = \{R_s\}_{s\geq t}$ and $\mathbf{U}_t = \{U_s\}_{s\geq t}$, we have

$$\begin{aligned}
&J^0(\mathbf{RU}_t)\\
=&\mathbb{E}\left[\sum_{s=t}^\infty \beta^{s-t} D_s\middle|\mathcal{F}_t\right]\\
=&\mathbb{E}\left[\sum_{s=t}^\infty \beta^{s-t}U_s - \beta^{s-t+1}U_{s+1} + \beta^{s-t+1}\left(\frac{c_s}{\beta} - I_s(X_{s+1}) - \pi^\Pi\left[R_s(X_{s+1})\right]\right)\middle|\mathcal{F}_t\right]\\
=&\mathbb{E}\left[\sum_{s=t}^\infty \beta^{s-t}U_s - \beta^{s-t+1}U_{s+1}\middle|\mathcal{F}_t\right]\\
+&\mathbb{E}\left[\sum_{s=t}^\infty \beta^{s-t+1}\left(\frac{c_s}{\beta} - I_s(X_{s+1}) - \pi^\Pi\left[R_s(X_{s+1})\right]\right)\middle|\mathcal{F}_t\right].
\end{aligned} \tag{4.23}$$

Now let us deal with the two summations. First, we consider the one on the left. Based on Proposition 4.2.4, $\{U_s\}_{s\geq t} \in \mathbf{L}^2_\beta$, which gives us in the limit

$$\begin{aligned}&\sum_{s=t}^{\infty}\left(\beta^{s-t}U_s - \beta^{s-t+1}U_{s+1}\right)\\ =&(U_t - \beta U_{t+1}) + (\beta U_{t+1} - \beta^2 U_{t+2}) + (\beta^2 U_{t+2} - \beta^3 U_{t+3}) + \cdots\\ =&U_t.\end{aligned}$$

Thus, (4.23) becomes

$$J^0(\mathbf{RU}_t) = \mathbb{E}\left[\sum_{s=t}^{\infty}\beta^{s-t+1}\left(-I_s(X_{s+1}) - \pi^{\Pi}\left[R_s(X_{s+1})\right]\right)\Big|\mathcal{F}_t\right] + U_t + \sum_{s=t}^{\infty}\beta^{s-t+1}c_s.$$

Because U_t is known at time t and the summation at the end of the equation does not contain any control variable, $\mathbf{R}_t$ is now the only remaining control variable. As a result, maximising $J^0(\mathbf{RU}_t)$ is equivalent to minimising

$$J(\mathbf{R}_t) := \mathbb{E}\left[\sum_{s=t}^{\infty}\beta^{s-t+1}\left(I_s(X_{s+1}) + \pi^{\Pi}\left[R_s(X_{s+1})\right]\right)\Big|\mathcal{F}_t\right], \tag{4.24}$$

where $I_s(X_{s+1}) = X_{s+1} - R_s(X_{s+1})$. Notice that to minimise the objective function of the summation with respect to set $\mathbf{R}_t$ is now equivalent to minimising the conditional expectation of each term with the corresponding R_s: for $s = t, t+1, \ldots, \infty$,

$$\inf_{\mathbf{R}_t} J(\mathbf{R}_t) \Longleftrightarrow \inf_{R_s}\mathbb{E}\left[\beta^{s-t+1}\left(I_s(X_{s+1}) + \pi^{\Pi}\left[R_s(X_{s+1})\right]\right)|\mathcal{F}_t\right].$$

Therefore, the optimal solution of the problem is myopic in this case. Given the independence of $\mathcal{F}_t$ and X_{s+1}, $s \geq t$, we find that solving the previous problems is equivalent to finding the answer to the following optimisations:

$$\inf_{R_s}\left\{\mathbb{E}\left[I_s(X_{s+1})\right] + \pi^{\Pi}\left[R_s(X_{s+1})\right]\right\} \quad \forall s = t, t+1, \ldots, \infty. \tag{4.25}$$

Now, the MIF formulation can be used to solve this problem. Note that expectation is a DRM with distortion function $g^{\mathbb{E}}(x) = x$. Therefore, $\Pi^{\mathbb{E}}(x) = x$, which with Lemma 2.3.3 yields

$$\mathbb{E}[I_s(X_{s+1})] = \int_0^{\infty}\left(1 - F_{X_{s+1}}(z)\right)I_s'(z)\,dz = \int_0^{\infty}\left(1 - F_{X_{s+1}}(z)\right)\left(1 - R_s'(z)\right)dz. \tag{4.26}$$

By again applying Lemma 2.3.3, we obtain

$$\pi^{\Pi}\left[R_s(X_{s+1})\right] = (1+\theta)\int_0^{\infty}\left(1-\Pi\left[F_{X_{s+1}}(z)\right]\right)R_s'(z)\,dz. \tag{4.27}$$

Now, using (4.26) and (4.27) in (4.25), we obtain, $\forall s = t, t+1, \ldots, \infty$,

$$\inf_{R_s}\int_0^{\infty}\left[\left(1-F_{X_{s+1}}(z)\right)\left(1-R_s'(z)\right)+(1+\theta)\left(1-\Pi\left[F_{X_{s+1}}(z)\right]\right)R_s'(z)\right]dx. \tag{4.28}$$

From Proposition 2.3.1, the derivative $R_s'(z)$ is between 0 and 1. Therefore, the optimal solution satisfies

$$R_s^{*\prime}(z) = \mathbb{1}_{\left\{1-F_{X_{s+1}}(z)>(1+\theta)\left(1-\Pi\left[F_{X_{s+1}}(z)\right]\right)\right\}}, \tag{4.29}$$

and the minimum is equal to

$$\int_0^{\infty}\inf\left\{\left(1-F_{X_{s+1}}(z)\right),(1+\theta)\left(1-\Pi\left[F_{X_{s+1}}(z)\right]\right)\right\}dz.$$

As a consequence, the reinsurance contract

$$R_s^*(x) = \int_0^{x}(R_s^*)'(z)\,dz$$

is a multilayer reinsurance policy. □

4.3.2 Problem II

It is interesting to notice that adding any general solvency condition, such as $S(U_s, R_s, D_s) \geq 0$, does not change the solution. To see this, note that by following the same argument made above, the set-up of the optimisation Problem II for $s = t, t+1, \ldots, \infty$ can be written as

$$\begin{cases} \inf_{R_s}\mathbb{E}\left[\sum_{s=t}^{\infty}\beta^{s-t+1}\left(I_s(X_{s+1})+\pi^{\Pi}\left[R_s(X_{s+1})\right]\right)|\mathcal{F}_t\right] \\ S(U_s, R_s, D_s) \geq 0 \end{cases}. \tag{4.30}$$

As can be seen, we can first solve for optimal reinsurance R_s and afterwards find a proper D_s that makes the solvency condition hold. Note that since the function S is decreasing with respect to D_s, in order to maximise the dividend it suffices to find the maximum D_s and make the constraint binding; i.e. $S(U_s, R_s, D_s) = 0$.

4.3.3 Problem III

In this problem set-up, we focus on the same objective function with one additional condition, one regarding the dividend payments. This section is divided into two parts. First, we explain the characteristics of acceptable dividend policies and show the optimal results satisfying these restrictions. Second, we show the results under the barrier dividend strategy.

We assume that the dividend policy chosen is a function of the surplus; thus, we have the third constraint stated in (4.21)-[iii] as

$$D_s := H(U_s), \text{ for } s = t, t+1, \ldots, \infty. \tag{4.31}$$

We also assume that the admissible dividend strategies satisfies following property.

Property 4.3.2. The mapping $x \mapsto x - H(x)$ is bounded above. This means If the surplus at the end of one period is large the cedent benefit is almost as large as the surplus.

Since $\left(\mathbf{L}^2_\beta, \langle ., . \rangle_{\mathbf{L}^2_\beta}\right)$ is a Hilbert space, we can use the Lagrangian on infinite dimensional spaces to find the dual problem; see Zalinescu (2002). We denote the Lagrangian multipliers associated with conditions [i] and [ii] by $\boldsymbol{\mu} = \{\mu_{s+1}\}_{s\geq t}$ and $\boldsymbol{\lambda} = \{\lambda_s \geq 0\}_{s\geq t}$ where $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \left(\mathbf{L}^2_\beta, \langle ., . \rangle_{\mathbf{L}^2_\beta}\right)$.

Let $G(x, s) := \frac{x - H(x) + c_s}{\beta}$, and recall that $\mathbf{RU}_t := \{\mathbf{R}_t = \{R_s\}_{s\geq t}, \mathbf{U}_t = \{U_s\}_{s\geq t}\}$. We can write Problem III as stated in (4.21) in an easier way as follows:

$$\begin{cases} \inf\limits_{\mathbf{RU}_t} \mathbb{E}\left[\sum\limits_{s=t}^{\infty} \beta^{s-t+1} \left(I_s(X_{s+1}) + \pi^{\Pi}\left[R_s(X_{s+1})\right]\right) | \mathcal{F}_t\right] \\ G(U_s, s) - I_s(X_{s+1}) - \pi^{\Pi}[R_s(X_{s+1})] - U_{s+1} = 0 & [i] \\ G(U_s, s) - \rho^{\Gamma}\left[I_s(X_{s+1})\right] - \pi^{\Pi}[R_s(X_{s+1})] \geq 0 & [ii] \end{cases} \tag{4.32}$$

for $s = t, t+1, \ldots, \infty$.

Theorem 4.3.3. Assume $\{X_s\}_{s\geq t} \in \mathbf{L}^2_\beta$ and that Assumptions 2 and 1 hold. If we assume that there is no duality gap, then for optimisation Problem III as stated in (4.32),there exist Lagrangian multipliers $\boldsymbol{\mu} = \{\mu_{s+1}\}_{s\geq t}$ and $\boldsymbol{\lambda} = \{\lambda_s \geq 0\}_{s\geq t}$, where $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \left(\mathbf{L}^2_\beta, \langle .,.\rangle_{\mathbf{L}^2_\beta}\right)$, and the following statements hold true.

1. $\mu_{s+1} \geq 0$, $\mu_{s+2} + \lambda_{s+1} \geq \mu_{s+1}$, $s = t, t+1, t+2, \ldots, \infty$.

2. The dual problem is given as follows

$$
\begin{aligned}
&\mathcal{L}^D(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \\
&-\beta G(U_t, t)\left[\lambda_t + \mathbb{E}_t\left(\mu_{t+1}\right)\right] + \mathbb{E}_t\left[M_G\left(\mu_{s+1}, \mu_{s+2} + \lambda_{s+1}\right)\right] \\
&+ \sum_{s=t}^{\infty} \beta^{s-t+1}\Bigg[\int_0^\infty \min\Bigg\{\left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\left(1 - \Pi[F_{X_{s+1}}(z)]\right), \\
&(1 - F_{X_{s+1}}(z)) + \mathbb{E}_t\left(\mu_{s+1}\right)(1 - F_{X^1_{s+1}}(z)) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma[F_{X_{s+1}}(z)]\right)\Bigg\}dz\Bigg],
\end{aligned}
\tag{4.33}
$$

   where $f_{X^1_{s+1}}(x) = \frac{\int_0^\infty y f_{X_{s+1},\mu_{s+1}}(x,y)dy}{\mathbb{E}_t(\mu_{s+1})}$, and $M_G(a,b) = \min_x ax - bG(x,s)$.

3. The optimal reinsurance contract has the form $R^*_s(x) = \int_0^x \mathbb{1}_{\{z\in\mathbb{B}_s\}}dz$, where

$$
\begin{aligned}
\mathbb{B}_s = \Big\{z \in \mathbb{R}^+ \,\Big|\, &\left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\left(1 - \Pi\left[F_{X_{s+1}}(z)\right]\right) \\
&\leq \mathbb{E}_t\left(\mu_{s+1}\right)\left(1 - F_{X^1_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma[F_{X_{s+1}}(z)]\right) + \left(1 - F_{X_{s+1}}(z)\right)\Big\}.
\end{aligned}
$$

Proof. Let us find the Lagrangian dual of the problem stated in (4.32). Denoting $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$, we obtain the Lagrangian function as follows:

$$
\begin{aligned}
\mathcal{L}(\mathbf{RU}_t, \boldsymbol{\mu}, \boldsymbol{\lambda}) =&\mathbb{E}_t\left[\sum_{s=t}^{\infty} \beta^{s-t+1}\left(I_s(X_{s+1}) + \pi^\Pi\left[R_s(X_{s+1})\right]\right)\right] \\
&-\mathbb{E}_t\left[\sum_{s=t}^{\infty} \beta^{s-t+1}\mu_{s+1}\left(G(U_s, s) - I_s(X_{s+1}) - \pi^\Pi[R_s(X_{s+1})] - U_{s+1}\right)\right] \\
&-\mathbb{E}_t\left[\sum_{s=t}^{\infty} \beta^{s-t+1}\lambda_s\left(G(U_s, s) - \rho^\Gamma\left[I_s(X_{s+1})\right] - \pi^\Pi[R_s(X_{s+1})]\right)\right].
\end{aligned}
\tag{4.34}
$$

Note that all the constraints need to be discounted with constant rate β. Then the Lagrangian dual function is $\mathcal{L}^D(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \inf_{\mathbf{RU}_t} \mathcal{L}(\mathbf{RU}_t, \boldsymbol{\mu}, \boldsymbol{\lambda})$, and the Lagrangian dual problem becomes

$$\sup_{\boldsymbol{\mu},\boldsymbol{\lambda}} \mathcal{L}^D(\boldsymbol{\mu}, \boldsymbol{\lambda}) \Leftrightarrow \sup_{\boldsymbol{\mu},\boldsymbol{\lambda}} \inf_{\mathbf{RU}_t} \mathcal{L}(\mathbf{RU}_t, \boldsymbol{\mu}, \boldsymbol{\lambda})$$

subject to $\mu_{s+1} \in \mathbb{R}$, $\lambda_s \geq 0$ for $s = t, t+1, \ldots, \infty$.

Reorganising the Lagrangian function (4.34) to put the terms with I_s together, and recognising that $\mathbb{E}_t$ is continuous in L^2 space, we have that

$$\begin{aligned}
&\mathcal{L}(\mathbf{RU}_t, \boldsymbol{\mu}, \boldsymbol{\lambda}) \\
=&\mathbb{E}_t\Bigg\{ \sum_{s=t}^{\infty} \beta^{s-t+1} \Big[I_s(X_{s+1}) + \mu_{s+1} I_s(X_{s+1}) + \lambda_s \rho^{\Gamma} \left[I_s(X_{s+1}) \right] \\
&+ (1 + \mu_{s+1} + \lambda_s)\, \pi^{\Pi} \left[R_s(X_{s+1}) \right] + \mu_{s+1} U_{s+1} - (\mu_{s+1} + \lambda_s) G(U_s, s) \Big] \Bigg\} \\
=& \sum_{s=t}^{\infty} \beta^{s-t+1} \Big\{ \mathbb{E}_t \left[I_s(X_{s+1}) \right] + \mathbb{E}_t \left[\mu_{s+1} I_s(X_{s+1}) \right] + \mathbb{E}_t \left[\lambda_s \right] \rho^{\Gamma} \left[I_s(X_{s+1}) \right] \\
&+ (1 + \mathbb{E}_t \left[\mu_{s+1} \right] + \mathbb{E}_t \left[\lambda_s \right])\, \pi^{\Pi} \left[R_s(X_{s+1}) \right] + \mathbb{E}_t \Big[\mu_{s+1} U_{s+1} - (\mu_{s+1} + \lambda_s) G(U_s, s) \Big] \Big\}.
\end{aligned} \tag{4.35}$$

Before we continue further with our discussion, we need to work on the term $\mathbb{E}_t \left[\mu_{s+1} I_s(X_{s+1}) \right]$. This term is apparently different from the others as it is the only one that has the product of the Lagrangian and the decision variable inside the expectation. By using Lemma 4.2.3, however, we have

$$\begin{aligned}
&\mathbb{E}_t \left[\mu_{s+1} I_s(X_{s+1}) \right] \\
=&\mathbb{E}_t \left[\mu_{s+1}^{+} \right] \int_0^{\infty} \left(1 - F_{X_{s+1}^1}(x) \right) I_s'(x)\, dx - \mathbb{E}_t \left[\mu_{s+1}^{-} \right] \int_0^{\infty} \left(1 - F_{X_{s+1}^2}(x) \right) I_s'(x)\, dx,
\end{aligned} \tag{4.36}$$

where $F_{X_{s+1}^1}(x)$ and $F_{X_{s+1}^2}(x)$ are two distributions with densities

$$f_{X_{s+1}^1}(x) = \begin{cases} 0, & \text{if } \mu_{s+1}^{+} = 0 \\ \frac{\int_0^{\infty} y f_{X_{s+1}, \mu_{s+1}^{+}}(x,y) dy}{\mathbb{E}_t\left(\mu_{s+1}^{+}\right)}, & \text{otherwise} \end{cases}$$

and

$$f_{X^2_{s+1}}(x) = \begin{cases} 0, & \text{if } \mu^-_{s+1} = 0 \\ \frac{\int_{-\infty}^0 y f_{X_{s+1},\mu^-_{s+1}}(x,y)dy}{\mathbb{E}_t\left(-\mu^-_{s+1}\right)}, & \text{otherwise} \end{cases},$$

respectively. Then the Lagrangian dual function changes to

$$\begin{aligned}\mathcal{L}^D(\boldsymbol{\mu},\boldsymbol{\lambda}) &= \inf_{\mathbf{RU}_t} \mathcal{L}(\mathbf{RU}_t,\boldsymbol{\mu},\boldsymbol{\lambda}) \\ &= \inf_{\mathbf{RU}_t} \sum_{s=t}^{\infty} \beta^{s-t+1} \Big\{ \mathbb{E}_t\left[I_s(X_{s+1})\right] + \mathbb{E}_t\left[\mu^+_{s+1}\right]\mathbb{E}_t\left[I_s\left(X^1_{s+1}\right)\right] \\ &\qquad - \mathbb{E}_t\left[\mu^-_{s+1}\right]\mathbb{E}_t\left[I_s\left(X^2_{s+1}\right)\right] + \mathbb{E}_t\left[\lambda_s\right]\rho^{\Gamma}\left[I_s(X_{s+1})\right] \\ &\qquad + \big(1+\mathbb{E}_t\left[\mu_{s+1}\right]+\mathbb{E}_t\left[\lambda_s\right]\big)\pi^{\Pi}\left[R_s(X_{s+1})\right] \\ &\qquad + \mathbb{E}_t\Big[\mu_{s+1}U_{s+1} - (\mu_{s+1}+\lambda_s)G(U_s,s)\Big]\Big\}.\end{aligned}$$

The last term inside the sum does not directly contain the control vectors $\mathbf{R}_t$, and because we use Lagrangian multipliers to separate the conditions and periods, we can split the dual function into two parts, $\mathcal{L}^D(\boldsymbol{\mu},\boldsymbol{\lambda}) = \mathcal{L}^D_1(\boldsymbol{\mu},\boldsymbol{\lambda}) + \mathcal{L}^D_2(\boldsymbol{\mu},\boldsymbol{\lambda})$, where

$$\begin{aligned}\mathcal{L}^D_1(\boldsymbol{\mu},\boldsymbol{\lambda}) := \inf_{\mathbf{R}_t} \sum_{s=t}^{\infty} \beta^{s-t+1} \Big\{ & \mathbb{E}_t\left[I_s(X_{s+1})\right] + \mathbb{E}_t\left[\mu^+_{s+1}\right]\mathbb{E}_t\left[I_s\left(X^1_{s+1}\right)\right] \\ & - \mathbb{E}_t\left[\mu^-_{s+1}\right]\mathbb{E}_t\left[I_s\left(X^2_{s+1}\right)\right] + \mathbb{E}_t\left[\lambda_s\right]\rho^{\Gamma}\left[I_s(X_{s+1})\right] \\ & + \left(1+\mathbb{E}_t\left[\mu_{s+1}\right]+\mathbb{E}_t\left[\lambda_s\right]\right)\pi^{\Pi}\left[R_s(X_{s+1})\right]\Big\}. \end{aligned} \tag{4.37}$$

Since U_t is $\mathcal{F}_t$-measurable, U_t is known at time t, and we only need to make a decision on each U_s for all $s \geq t+1$. Hence, we replace $\mathbf{U}_t$ by $\mathbf{U}_{t+1}$ in $\mathcal{L}^D_2$:

$$\mathcal{L}^D_2(\boldsymbol{\mu},\boldsymbol{\lambda}) := \inf_{\mathbf{U}_{t+1}} \sum_{s=t}^{\infty} \beta^{s-t+1} \left\{ \mathbb{E}_t\Big[\mu_{s+1}U_{s+1} - (\mu_{s+1}+\lambda_s)G(U_s,s)\Big]\right\}. \tag{4.38}$$

Now for the first part of the dual function, using Lemma 2.3.3 we can rewrite (4.37) as

$$\begin{aligned}&\mathcal{L}^D_1(\boldsymbol{\mu},\boldsymbol{\lambda}) \\ &= \inf_{\mathbf{R}_t} \sum_{s=t}^{\infty} \beta^{s-t+1} \int_0^{\infty} \Big\{\Big[\left(1-F_{X_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1-\Gamma[F_{X_{s+1}}(z)]\right) \\ &+ \mathbb{E}_t\left(\mu^+_{s+1}\right)\left(1-F_{X^1_{s+1}}(z)\right) - \mathbb{E}_t\left(\mu^-_{s+1}\right)\left(1-F_{X^2_{s+1}}(z)\right)\Big]\left(1-R'_s(z)\right) \\ &+ \left(1+\mathbb{E}_t\left(\mu_{s+1}\right)+\mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\left(1-\Pi\left[F_{X_{s+1}}(z)\right]\right)R'_s(z)\Big\}dz. \end{aligned} \tag{4.39}$$

Since each term is now non-negative, in order to find the optimal reinsurance contract, we need to minimise each part in the summation; i.e. the minimum is found where each of the following term holds, for all $s = t, t+1, \ldots, \infty$:

$$\begin{aligned}\inf_{R_s} \int_0^\infty \Big\{ \Big[& \left(1 - F_{X_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma[F_{X_{s+1}}(z)]\right) \\ & + \mathbb{E}_t\left(\mu_{s+1}^+\right)\Big(1 - F_{X_{s+1}^1}(z)\Big) - \mathbb{E}_t\left(\mu_{s+1}^-\right)\Big(1 - F_{X_{s+1}^2}(z)\Big)\Big]\left(1 - R_s'(z)\right) \\ & + \left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\left(1 - \Pi[F_{X_{s+1}}(z)]\right) R_s'(z)\Big\} dz. \end{aligned} \tag{4.40}$$

Then, recalling the no–moral–hazard assumption again, $R_s^*(x) \in \mathbb{C}$, the marginal optimal indemnity is given by

$$\left(R_s^*\right)'(z) = \mathbb{1}_{\{z \in \mathbb{B}_s\}}, \tag{4.41}$$

where set $\mathbb{B}_s$ satisfies

$$\begin{aligned}\mathbb{B}_s = \Big\{ z \in \mathbb{R}^+ \Big| & \left(1 - F_{X_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma\left[F_{X_{s+1}}(z)\right]\right) \\ & + \mathbb{E}_t(\mu_{s+1}^+)\Big(1 - F_{X_{s+1}^1}(z)\Big) - \mathbb{E}_t\left(\mu_{s+1}^-\right)\Big(1 - F_{X_{s+1}^2}(z)\Big) \\ & \geq \left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\Big(1 - \Pi\left[F_{X_{s+1}}(z)\right]\Big)\Big\}. \end{aligned} \tag{4.42}$$

Therefore, the optimal solution is $R_s^*(x) = \int_0^x \left(R_s^*\right)'(z)\, dz = \int_0^x \mathbb{1}_{\{z \in \mathbb{B}_s\}} dz$. Notice that when the equality holds, the optimal MIF could have any value between 0 and 1 and the result would be the same.

Next, considering the second part (4.38) of the dual function, we rearrange the problem as follows:

$$\begin{aligned}\mathcal{L}_2^D(\boldsymbol{\mu}, \boldsymbol{\lambda}) :=& \inf_{\mathbf{U}_{t+1}} \sum_{s=t}^{\infty} \beta^{s-t+1} \mathbb{E}_t\left[\mu_{s+1} U_{s+1} - \left(\mu_{s+1} + \lambda_s\right) G(U_s, s)\right] \\ =& -\beta G(U_t, t)\left[\lambda_t + \mathbb{E}_t\left(\mu_{t+1}\right)\right] \\ & + \inf_{\mathbf{U}_{t+1}} \sum_{s=t}^{\infty} \beta^{s-t+1} \mathbb{E}_t\left[\mu_{s+1} U_{s+1} - \beta(\mu_{s+2} + \lambda_{s+1}) G(U_{s+1}, s+1)\right]. \end{aligned} \tag{4.43}$$

To obtain the maximum with respect to the corresponding surplus for each $s = t, t+1, \ldots, \infty$, we need to find

$$\inf_{U_{s+1}} \mathbb{E}_t\left[\mu_{s+1} U_{s+1} - \beta(\mu_{s+2} + \lambda_{s+1}) G(U_{s+1}, s+1)\right]. \tag{4.44}$$

For any two numbers a, b, let us define $M_G(a,b) = \inf_x ax - bG(x,s)$. For each ω, the infimum in (4.44) is equal to

$$\begin{aligned}&\mathbb{E}_t\left[\inf_{U_{s+1}(\omega)}\left(\mu_{s+1}(\omega)\,U_{s+1}(\omega) - \beta\left[\left(\mu_{s+2}(\omega) + \lambda_{s+1}(\omega)\right]G[U_{s+1}(\omega), s]\right)\right]\\ =&\mathbb{E}_t\left[M_G\left(\mu_{s+1}, \mu_{s+2} + \lambda_{s+1}\right)\right]. \end{aligned} \tag{4.45}$$

Therefore, we have

$$\mathcal{L}_2^D(\boldsymbol{\mu}, \boldsymbol{\lambda}) = -\beta G(U_t, t)\left[\lambda_t + \mathbb{E}_t\left(\mu_{t+1}\right)\right] + \mathbb{E}_t\left[M_G\left(\mu_{s+1}, \mu_{s+2} + \lambda_{s+1}\right)\right]. \tag{4.46}$$

By Property 4.3.2, it is clear that the function G is bounded from above. Let us assume that $G(x,s) \leq L\,\forall x$. Now, if there exists $\epsilon > 0$ such that the set $A_\epsilon = \{\mu_{s+1} < -\epsilon\}$ is of positive measure, then, by choosing $U_{s+1} = M$ on A_ϵ, we have

$$\begin{aligned}&\mathbb{E}_t\left[\mathbb{1}_{A_\epsilon}\left[\mu_{s+1}U_{s+1} - \beta\left(\mu_{s+2} + \lambda_{s+1}\right)G(U_{s+1}, s+1)\right]\right]\\ \leq& -\mathbb{E}_t\left[\mathbb{1}_{A_\epsilon}\mu_{s+1}M\right] + \beta L\mathbb{E}_t\left[\mathbb{1}_{A_\epsilon}\left|\mu_{s+2} + \lambda_{s+1}\right|\right]\\ \leq& -\epsilon M\mathbb{P}\left(A_\epsilon\right) + \beta L\mathbb{E}_t\left[\left|\mu_{s+2} + \lambda_{s+1}\right|\right]. \end{aligned} \tag{4.47}$$

If M goes to $+\infty$ in the right-hand side of the inequality above, it can be seen that (4.45) does not have a finite infimum. This contradicts the assumption that there is no duality gap. Since $\epsilon > 0$ is an arbitrary positive number, we have

$$\mu_{s+1} \geq 0 \text{ almost surely for all } s = t, t+1, \ldots, \infty. \tag{4.48}$$

As a result, $\mu_{s+1}^- = 0$; thus, (4.42) can be written as

$$\begin{aligned}\mathbb{B}_s = \Big\{z \in \mathbb{R}^+ \Big| &\left(1 - F_{X_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma\left[F_{X_{s+1}}(z)\right]\right)\\ &+ \mathbb{E}_t(\mu_{s+1})\left(1 - F_{X^1_{s+1}}(z)\right)\\ &\geq \left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\left(1 - \Pi\left[F_{X_{s+1}}(z)\right]\right)\Big\}, \end{aligned} \tag{4.49}$$

and (4.40) is equal to

$$\begin{aligned}\inf_{R_s}\int_0^\infty \Big\{\Big[&\left(1 - F_{X_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma[F_{X_{s+1}}(z)]\right)\\ &+ \mathbb{E}_t\left(\mu_{s+1}\right)\left(1 - F_{X_{s+1}}(z)\right)\Big]\left(1 - R_s'(z)\right)\\ &+ \left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)(1+\theta)\left(1 - \Pi[F_{X_{s+1}}(z)]\right)R_s'(z)\Big\}dz, \end{aligned} \tag{4.50}$$

which can be reorganised as

$$\int_0^\infty \min\Big\{(1-F_{X_{s+1}}(z)) + \mathbb{E}_t(\mu_{s+1})(1-F_{X^1_{s+1}}(z)) + \mathbb{E}_t(\lambda_s)\left(1-\Gamma[F_{X_{s+1}}(z)]\right), \\ (1+\mathbb{E}_t(\mu_{s+1})+\mathbb{E}_t(\lambda_s))(1+\theta)\left(1-\Pi[F_{X_{s+1}}(z)]\right)\Big\}dz. \tag{4.51}$$

Consequently, the first part of the dual function is a summation of the discounted integral obtained above, so we have

$$\begin{aligned}&\mathcal{L}_1^D(\boldsymbol{\mu},\boldsymbol{\lambda})\\ =&\sum_{s=t}^{\infty}\beta^{s-t+1}\Big[\int_0^\infty \min\Big\{(1+\mathbb{E}_t(\mu_{s+1})+\mathbb{E}_t(\lambda_s))(1+\theta)\left(1-\Pi[F_{X_{s+1}}(z)]\right),\\ &(1-F_{X_{s+1}}(z))+\mathbb{E}_t(\mu_{s+1})(1-F_{X^1_{s+1}}(z))+\mathbb{E}_t(\lambda_s)\left(1-\Gamma[F_{X_{s+1}}(z)]\right)\Big\}dz\Big].\end{aligned} \tag{4.52}$$

By adding back two parts of the dual function, (4.52) and (4.46), we have

$$\begin{aligned}&\mathcal{L}^D(\boldsymbol{\mu},\boldsymbol{\lambda}) = \mathcal{L}_1^D(\boldsymbol{\mu},\boldsymbol{\lambda}) + \mathcal{L}_2^D(\boldsymbol{\mu},\boldsymbol{\lambda})\\ =&\sum_{s=t}^{\infty}\beta^{s-t+1}\Big[\int_0^\infty \min\Big\{(1+\mathbb{E}_t(\mu_{s+1})+\mathbb{E}_t(\lambda_s))(1+\theta)\left(1-\Pi[F_{X_{s+1}}(z)]\right),\\ &(1-F_{X_{s+1}}(z))+\mathbb{E}_t(\mu_{s+1})(1-F_{X^1_{s+1}}(z))+\mathbb{E}_t(\lambda_s)\left(1-\Gamma[F_{X_{s+1}}(z)]\right)\Big\}dz\Big]\\ &-\beta G(U_t,t)\left[\lambda_t+\mathbb{E}_t(\mu_{t+1})\right]+\mathbb{E}_t\left[M_G(\mu_{s+1},\mu_{s+2}+\lambda_{s+1})\right].\end{aligned}$$

Now, for an arbitrary positive number $\epsilon > 0$, let $B_\epsilon = \{\mu_{s+2}+\lambda_{s+1}-\mu_{s+1} < -\epsilon\}$. We claim that B_ϵ must be of measure zero. If it is not, then by choosing $U_{s+1} = -M$ on B_ϵ, for a large number $M > 0$, by Property 4.3.2, we get

$$\begin{aligned}&\mathbb{E}_t\left[\mathbb{1}_{B_\epsilon}\left(\mu_{s+1}(-M)-\beta(\mu_{s+2}+\lambda_{s+1})\left(\frac{-M+c_s}{\beta}\right)\right)\right]\\ =&-M\mathbb{E}_t\left[\mathbb{1}_{B_\epsilon}(\mu_{s+1}-(\mu_{s+2}+\lambda_{s+1}))\right]-c_s\mathbb{E}_t\left[\mathbb{1}_{B_\epsilon}(\mu_{s+2}+\lambda_{s+1})\right]\\ \le&-\epsilon M\mathbb{P}_t(B_\epsilon)+c_s\mathbb{E}_t\left[|\mu_{s+2}+\lambda_{s+1}|\right].\end{aligned} \tag{4.53}$$

However, by sending M to ∞, the minimum of the objective is not finite, and there is a duality gap, which contradicts the assumption. Since $\epsilon > 0$ was chosen arbitrarily, it can be deduced that

$$\mu_{s+2}+\lambda_{s+1} \ge \mu_{s+1} \text{ almost surely for all } s = t, t+1, \ldots, \infty. \tag{4.54}$$

□

Now, let us discuss the particular case of dividend strategy as the barrier dividend policy denoted by

$$H\left(U_t\right)=\left(U_t-b\right)^{+}, \tag{4.55}$$

where $b \in \mathbb{R}$ is the known constant dividend barrier, and U_t is the balance of the insurance company before the reinsurance retention level covering the claims of the next period is determined and the dividends are paid out. Let us prove one lemma to simplify the proof of the following corollary with barrier dividend policy selected.

Lemma 4.3.4. For non-negative numbers a, d, let $K(x, s)=aG(x, s)-dx$. Then $\max_x K(x, s)=b\left(\frac{a}{\beta}-d\right)^{+}+\frac{ac_s}{\beta}$.

Proof. By construction, we have

$$\begin{aligned} K(x, s)= & aG(x, s)-dx=a\left(\frac{x-(x-b)^{+}+c_s}{\beta}\right)-dx \\ = & \frac{a}{\beta}\left(\min(x, b)+c_s\right)-dx=\begin{cases}\left(\frac{a}{\beta}-d\right) x+\frac{ac_s}{\beta}, & x \leq b \\ -dx+\left(\frac{ab}{\beta}\right)+\frac{ac_s}{\beta}, & x>b\end{cases}. \end{aligned} \tag{4.56}$$

If $\frac{a}{\beta}<d$, then the maximum of $K(x, s)$ is taken at $x=0$, and the maximum value is equal to $\frac{ac_s}{\beta}$. If $\frac{a}{\beta}=d$, then the maximum can be taken at any $x \in[0, b]$, and the value of the maximum is equal to $\frac{ac_s}{\beta}$. Finally, if $\frac{a}{\beta}>d$, then the maximum of $K(x, s)$ is taken at $x=b$, and the maximum value is equal to $\left(\frac{a}{\beta}-d\right) b+\frac{ac_s}{\beta}$. Therefore, we have $\max_x K(x, s)=b\left(\frac{a}{\beta}-d\right)^{+}+\frac{ac_s}{\beta}$. □

By applying barrier dividend strategy, the results in Theorem 4.3.3 can be further derived in the following corollary.

Corollary 4.3.5. Let us assume that $H\left(x\right)=(x-b)^{+}$ for some positive number $b>0$ and assume that the duality gap is zero in optimisation Problem III. Then the optimisation Problem III is now simplified to be the problem below:

$$\begin{cases} \inf_{R_s} \left(\mathbb{E}_t \left[I_s(X_{s+1}) \right] + \pi^{\Pi} \left[R_s(X_{s+1}) \right] \right) \\ I_t(X_{t+1}) + \pi^{\Pi}[R_t(X_{t+1})] \leq G\left(U_t, t\right) - b, & \text{if } s = t \\ I_s(X_{s+1}) + \pi^{\Pi}[R_s(X_{s+1})] \leq \frac{b+c_s}{\beta} - b, & \text{if } s = t+1, t+2, \ldots, \infty \end{cases}.$$

Proof. Let $\mathbb{A}$ be a set of measure 1 so that the inequalities in (4.48) and (4.54) hold everywhere on $\mathbb{A}$. For a single scenario $\omega \in \mathbb{A}$, let $\frac{a}{\beta} = \mu_{s+2}(\omega) + \lambda_{s+1}(\omega)$ and $d = \mu_{s+1}(\omega)$. Then we have $\frac{a}{\beta} \geq 0$, $d \geq 0$, and $\frac{a}{\beta} - d \geq 0$. As a result, from the previous lemma, we obtain

$$\begin{aligned} \max_x K(x, s+1) = & b\left(\frac{a}{\beta} - d\right)^+ + \frac{ac_{s+1}}{\beta} \\ = & b\left(\mu_{s+2} + \lambda_{s+1} - \mu_{s+1}\right) + \left(\mu_{s+2} + \lambda_{s+1}\right) c_{s+1}. \end{aligned}$$

By applying this to (4.44) inside the second part of the dual function, we obtain

$$\begin{aligned} & \inf_{U_{s+1}} \mathbb{E}_t \left[\mu_{s+1} U_{s+1} - \beta(\mu_{s+2} + \lambda_{s+1}) G(U_{s+1}, s+1) \right] \\ = & \sup_{U_{s+1}} \mathbb{E}_t \left[K(U_{s+1}, s+1) \right] \\ = & \mathbb{E}_t \left[b\mu_{s+1} - \left(b + c_{s+1}\right) \mu_{s+2} - \left(b + c_{s+1}\right) \lambda_{s+1} \right]. \end{aligned} \tag{4.57}$$

The solution of U_{s+1} is also given by

$$U_{s+1} = \begin{cases} b, & \mu_{s+2} + \lambda_{s+1} > \mu_{s+1} \\ l, & \mu_{s+2} + \lambda_{s+1} = \mu_{s+1} \end{cases},$$

where l can be any number in $[0, b]$. Putting this term back into the summation and reorganising the second part of the dual function, we have

$$\begin{aligned}
\mathcal{L}_2^D(\boldsymbol{\mu},\boldsymbol{\lambda}) = &\sum_{s=t}^{\infty}\beta^{s-t+1}\mathbb{E}_t\left[b\mu_{s+1}-(b+c_{s+1})\,\mu_{s+2}-(b+c_{s+1})\,\lambda_{s+1}\right] \\
=&\beta\mathbb{E}_t\left[b\mu_{t+1}-(b+c_{t+1})\,\mu_{t+2}-(b+c_{t+1})\,\lambda_{t+1}\right] \\
&+\beta^2\mathbb{E}_t\left[b\mu_{t+2}-(b+c_{t+2})\,\mu_{t+3}-(b+c_{t+2})\,\lambda_{t+2}\right] \\
&+\beta^3\mathbb{E}_t\left[b\mu_{t+3}-(b+c_{t+3})\,\mu_{t+4}-(b+c_{t+3})\,\lambda_{t+3}\right] \\
&+\cdots \\
=&-\sum_{s=t+1}^{\infty}\beta^{s-t}\,(b+c_s)\,\lambda_s \\
&+\beta b\mathbb{E}_t\left(\mu_{t+1}\right) \\
&-\beta\,(b+c_{t+1})\,\mathbb{E}_t\left(\mu_{t+2}\right)+\beta^2 b\mathbb{E}_t\left(\mu_{t+2}\right) \\
&-\beta^2\,(b+c_{t+2})\,\mathbb{E}_t\left(\mu_{t+3}\right)+\beta^3 b\mathbb{E}_t\left(\mu_{t+3}\right) \\
&-\cdots.
\end{aligned} \tag{4.58}$$

Now, let us see the dual problem in (4.35) for optimal solutions I_s^* and R_s^*:

$$\begin{aligned}
\mathcal{L}^D(\boldsymbol{\mu},\boldsymbol{\lambda}) = &\sum_{s=t}^{\infty}\beta^{s-t+1}\left\{\mathbb{E}_t\left(I_s^*(X_{s+1})\right)+\mathbb{E}_t\left(\mu_{s+1}I_s^*(X_{s+1})\right)+\mathbb{E}_t\left(\lambda_s\right)\rho^{\Gamma}\left[I_s^*(X_{s+1})\right]\right. \\
&\left.(1+\mathbb{E}_t\left(\mu_{s+1}\right)+\mathbb{E}_t\left(\lambda_s\right))\,\pi^{\Pi}\left[R_s^*(X_{s+1})\right]\right\} \\
&-\beta G(U_t,t)\left[\lambda_t+\mathbb{E}_t\left(\mu_{t+1}\right)\right]+\beta b\mathbb{E}_t\left(\mu_{t+1}\right) \\
&-\beta\,(b+c_{t+1})\,\mathbb{E}_t\left(\mu_{t+2}\right)+\beta^2 b\mathbb{E}_t\left(\mu_{t+2}\right) \\
&-\beta^2\,(b+c_{t+2})\,\mathbb{E}_t\left(\mu_{t+3}\right)+\beta^3 b\mathbb{E}_t\left(\mu_{t+3}\right) \\
&-\cdots \\
&-\sum_{s=t+1}^{\infty}\beta^{s-t}\,(b+c_s)\,\lambda_s.
\end{aligned}$$

Therefore, by concatenating the terms with respect to multipliersλ_s and μ_{s+1}, for

$s = t, t+1, \ldots, \infty$, we obtain

$$
\begin{aligned}
\mathcal{L}^D(\boldsymbol{\mu}, \boldsymbol{\lambda}) = & \sum_{s=t}^{\infty} \beta^{s-t+1} \left\{ \mathbb{E}_t \left(I_s^*(X_{s+1}) \right) + \pi^{\Pi} \left[R_s^*(X_{s+1}) \right] \right\} \\
& + \beta \mathbb{E}_t \left(\left(b - G\left(U_t, t\right) + \pi^{\Pi} \left[R_t^*(X_{t+1}) \right] + I_s^*(X_{t+1}) \right) \mu_{t+1} \right) \\
& + \sum_{s=t+1}^{\infty} \beta^{s-t+1} \mathbb{E}_t \left(\left(b - \left(\frac{b + c_s}{\beta} \right) + \pi^{\Pi} \left[R_s^*(X_{s+1}) \right] + I_s^*(X_{s+1}) \right) \mu_{s+1} \right) \\
& + \beta \mathbb{E}_t \left(\left(-G\left(U_t, t\right) + \pi^{\Pi} \left[R_t^*(X_{t+1}) \right] + \rho^{\Gamma} \left[I_t^*(X_{t+1}) \right] \right) \lambda_t \right) \\
& + \sum_{s=t+1}^{\infty} \beta^{s-t+1} \mathbb{E}_t \left(\left(- \left(\frac{b + c_s}{\beta} \right) + \pi^{\Pi} \left[R_s^*(X_{s+1}) \right] + \rho^{\Gamma} \left[I_s^*(X_{s+1}) \right] \right) \lambda_s \right).
\end{aligned}
\tag{4.59}
$$

Now let us minimise $\mathcal{L}^D(\boldsymbol{\mu}, \boldsymbol{\lambda})$ in terms of $\boldsymbol{\mu}, \boldsymbol{\lambda}$. In order to avoid a duality gap, from the second and third lines above as well as condition $[i]$, we derive

$$
\begin{cases}
G\left(U_t, t\right) - U_{t+1} = \pi^{\Pi} \left[R_t^*(X_{t+1}) \right] + I_t^*(X_{t+1}) \leq G\left(U_t, t\right) - b, \\
G\left(U_s, s\right) - U_{s+1} = \pi^{\Pi} \left[R_s^*(X_{s+1}) \right] + I_s^*(X_{s+1}) \leq \frac{b+c_s}{\beta} - b.
\end{cases}
\tag{4.60}
$$

A simple implication of this inequality is

$$
\begin{cases}
\pi^{\Pi} \left[R_t^*(X_{t+1}) \right] + I_t^*(X_{t+1}) < G\left(U_t, t\right), \\
\pi^{\Pi} \left[R_s^*(X_{s+1}) \right] + I_s^*(X_{s+1}) < \frac{b+c_s}{\beta}.
\end{cases}
\tag{4.61}
$$

Now, using the strict inequalities in (4.61), to avoid a duality gap, by the fourth and fifth lines in (4.59) above, we need to have

$$
\lambda_s = 0, s = t, t+1, \ldots, \infty.
$$

Furthermore, (4.60) implies

$$
b = G\left(U_t, t\right) - G\left(U_t, t\right) + b \leq U_{t+1},
$$

and

$$
\begin{aligned}
b - \frac{\left(U_s - b\right)^-}{\beta} = & \left(\frac{U_s - \left(U_s - b\right)^+ + c_s}{\beta} \right) - \left(\frac{b + c_s}{\beta} \right) + b \\
= & G\left(U_s, s\right) - \left(\frac{b + c_s}{\beta} \right) + b \leq U_{s+1}.
\end{aligned}
$$

By induction, these two together imply that $U_{s+1} \geq b$, $s = t, t+1, \ldots, \infty$. This shows that we can search for the optimal solution in a more restricted set consisting of $\mathbf{R_t}$ and satisfying conditions $[i]$ and $[ii]$ in (4.32) and $\{U_{s+1} \geq b\}$, $s = t, t+1, \ldots, \infty$. We show the last new condition as $[iv]$. Given that for $x \geq b$, $G(x, s) = \frac{b+c_s}{\beta}$, we simplify to obtain

$$\begin{cases} \inf\limits_{R_s} \mathbb{E}_t \left[\sum\limits_{s=t}^{\infty} \beta^{s-t+1} \left(I_s(X_{s+1}) + \pi^{\Pi} \left[R_s(X_{s+1}) \right] \right) \right] \\ G(U_t, t) - I_t(X_{t+1}) - \pi^{\Pi}[R_t(X_{t+1})] - U_{t+1} = 0 & [i1] \\ \frac{b+c_s}{\beta} - I_s(X_{s+1}) - \pi^{\Pi}[R_s(X_{s+1})] - U_{s+1} = 0 & [i2] \\ G(U_t, t) - \rho^{\Gamma} \left[I_t(X_{t+1}) \right] - \pi^{\Pi}[R_t(X_{t+1})] \geq 0 & [ii1] \\ \frac{b+c_s}{\beta} - \rho^{\Gamma} \left[I_s(X_{s+1}) \right] - \pi^{\Pi}[R_s(X_{s+1})] \geq 0 & [ii2] \\ U_{s+1} \geq b & [iv] \\ s = t, t+1, \ldots, \infty. \end{cases} \tag{4.62}$$

However, conditions $[i1]$, $[i2]$, and $[iv]$ imply $[ii1]$ and $[ii2]$, meaning that $[ii1]$ and $[ii2]$ can be removed:

$$\begin{cases} \inf\limits_{R_s} \mathbb{E}_t \left[\sum\limits_{s=t}^{\infty} \beta^{s-t+1} \left(I_s(X_{s+1}) + \pi^{\Pi} \left[R_s(X_{s+1}) \right] \right) \right] \\ G(U_t, t) - I_t(X_{t+1}) - \pi^{\Pi}[R_t(X_{t+1})] - U_{t+1} = 0 & [i1] \\ \frac{b+c_s}{\beta} - I_s(X_{s+1}) - \pi^{\Pi}[R_s(X_{s+1})] - U_{s+1} = 0 & [i2] \\ U_{s+1} \geq b & [iv] \\ s = t, t+1, \ldots, \infty. \end{cases}$$

Since U_{s+1} does not exist in the objective function, we obtain

$$\begin{cases} \inf\limits_{R_s} \sum\limits_{s=t}^{\infty} \beta^{s-t+1} \left(\mathbb{E}_t \left[I_s(X_{s+1}) \right] + \pi^{\Pi} \left[R_s(X_{s+1}) \right] \right) \\ I_t(X_{t+1}) + \pi^{\Pi}[R_t(X_{t+1})] \leq G\left(U_t, t\right) - b \\ I_s(X_{s+1}) + \pi^{\Pi}[R_s(X_{s+1})] \leq \frac{b+c_s}{\beta} - b \\ s = t, t+1, \ldots, \infty. \end{cases}$$

Now, as can be seen, none of the conditions are linked to each other. Therefore, we can simply minimise any of the following expressions:

$$\begin{cases} \inf\limits_{R_s} \left(\mathbb{E}_t\left[I_s(X_{s+1})\right] + \pi^{\Pi}\left[R_s(X_{s+1})\right]\right) \\ I_t(X_{t+1}) + \pi^{\Pi}[R_t(X_{t+1})] \leq G\left(U_t, t\right) - b, & \text{if } s = t \\ I_s(X_{s+1}) + \pi^{\Pi}[R_s(X_{s+1})] \leq \frac{b+c_s}{\beta} - b, & \text{if } s = t+1, t+2, \ldots, \infty. \end{cases}$$

□

4.4 Examples

In this section, we show how to use the results presented in the previous section. Corollary 4.4.1 will reveal that under VaR, the optimal reinsurance is consistent with the form of a two-layer stop-loss reinsurance for any distribution of claims. Furthermore, using the exponential distribution for claims at each period, we show three examples with different DRMs chosen for risk control and reinsurance premium principles.

Corollary 4.4.1. If VaR is chosen as the reinsurance premium principle, i.e. $\pi^{\Pi}(\cdot) = (1+\theta)\mathrm{VaR}_{\alpha}(\cdot)$ for some $\alpha \in (0,1)$, and the optimal solutions exist, and either $\theta > 0$ or $F_{X_{s+1}}(z) > 0, z \neq 0$, we have $R_s^*(x) = \min\left\{\mathrm{VaR}_{\alpha}\left(X_{s+1}\right), X_{s+1}\right\}$.

Proof. Recall the set in (4.49),

$$\begin{aligned}\mathbb{B}_s = \Big\{ z \in \mathbb{R}^+ \Big| & \left(1 + \mathbb{E}_t\left(\mu_{s+1}\right) + \mathbb{E}_t\left(\lambda_s\right)\right)\left(1+\theta\right)\left(1 - \Pi\left[F_{X_{s+1}}(z)\right]\right) \\ & \leq \left(1 - F_{X_{s+1}}(z)\right) + \mathbb{E}_t\left(\mu_{s+1}\right)\left(1 - F_{X^1_{s+1}}(z)\right) + \mathbb{E}_t\left(\lambda_s\right)\left(1 - \Gamma[F_{X_{s+1}}(z)]\right)\Big\}.\end{aligned}$$

Now, when $\Pi\left[F_{X_{s+1}}(z)\right] = 1$, the left-hand side of the above inequality is zero and is not greater than or equal to the right-hand side. On the other hand, if $\Pi\left[F_{X_{s+1}}(z)\right] = 0$, since $1 - F_{X_{s+1}}(z) \leq 1$ and $1 - F_{X^1_{s+1}}(z) \leq 1$ then either $\theta > 0$ or $F_{X_{s+1}}(z) > 0$, $z \neq 0$, implies that the left-hand side is strictly greater than the right-hand side. As a result,

$$\mathbb{B}_s = \left\{ z \in \mathbb{R}^+ \,\middle|\, F_{X_{s+1}}\left(z\right) \leq \alpha \right\}. \tag{4.63}$$

Therefore, by taking the integral of the MIF as the indicator function on $\mathbb{B}_s$, the reinsurance contract is a multilayer policy satisfying

$$R_s^*(x) = \int_0^x R_s^{*'}(z)\,dz = \begin{cases} F_{X_{s+1}}^{-1}(\alpha), & x > F_{X_{s+1}}^{-1}(\alpha) \\ x, & \text{otherwise} \end{cases}. \tag{4.64}$$

□

Now consider the total claims over each period $[s, s{+}1)$ for $s = t, t{+}1, t{+}2, \ldots, /infty$, all following an exponential distribution but with different constant parameters depending only on time $\kappa_{s+1} > 0$. Thus, the CDF of X_{s+1} can be denoted as $F_{X_{s+1}}^{Exp}(x) = 1 - e^{-\kappa_{s+1}x}$. By using different DRMs, we show the following examples.

Example 4.4.2. Consider DRMs for both the solvency condition and reinsurance premium as Value at Risk with different confidence levels $1-\gamma$ and $1-\alpha$, respectively, i.e. $\rho^\Gamma(.) = \mathrm{VaR}_\gamma(.)$ and $\pi^\Pi(.) = (1+\theta)\rho^\Pi(.) = (1+\theta)\,\mathrm{VaR}_\alpha(.)$; the corresponding dual distortion functions are $\Gamma^{VaR}(u) = \mathbb{1}_{[\gamma,1]}(u)$ and $\Pi^{VaR}(u) = \mathbb{1}_{[\alpha,1]}(u)$, respectively. In addition, the total claims in each time period $[s, s{+}1)$, with $s = t, t{+}1, t{+}2, \ldots, \infty$, follow an exponential distribution with parameter $\kappa_{s+1} > 0$, so the CDF is $F_{X_{s+1}}^{Exp}(x) = 1 - e^{-\kappa_{s+1}x}$. Therefore, the optimal reinsurance contract of

Problem I is a one-layer stop-loss reinsurance contract with retention level $-\frac{\ln\alpha}{\kappa_{s+1}}$ for all $s = t, t+1, \ldots, \infty$, written as

$$R_s^*(x) = \min\left\{x, -\frac{\ln\alpha}{\kappa_{s+1}}\right\},$$

and that of

Problem III has the form, for all $s = t, t+1, \ldots, \infty$,

$$R_s^*(x) = \begin{cases} x + \frac{\ln(1-\alpha)}{\kappa_{s+1}}, & -\frac{\ln(1-\alpha)}{\kappa_{s+1}} \le x < \infty \\ 0, & \text{otherwise} \end{cases}, \tag{4.65}$$

or, equivalently, $R_s^*(x) = \left(x + \frac{\ln(1-\alpha)}{\kappa_{s+1}}\right)^+$.

Proof. Problem I: Following the results in Theorem 4.3.1 and being aware that safety loading θ is a non-negative number, we know that the optimal MIF has the form

$$
\begin{aligned}
(R_s^*)'(x) &= \mathbb{1}_{\left\{1-F^{Exp}_{X_{s+1}}(x)>(1+\theta)\left(1-\Pi^{VaR}[F^{Exp}_{X_{s+1}}(x)]\right)\right\}} \\
&= \mathbb{1}_{\{e^{-\kappa_{s+1}x}>(1+\theta)\mathbb{1}_{[\alpha,1]}(e^{-\kappa_{s+1}x})\}} \\
&= \mathbb{1}_{[0,\alpha)}(e^{-\kappa_{s+1}x}) \\
&= \mathbb{1}_{[0,-\frac{\ln\alpha}{\kappa_{s+1}})}(x).
\end{aligned}
$$

Therefore, the reinsurance contract becomes

$$
R_s^*(x) = \int_0^x (R_s^*)'(z)\,dz = \min\left\{x, -\frac{\ln\alpha}{\kappa_{s+1}}\right\}.
$$

Problem III: Taking the specific dual distortion functions into Theorem 4.3.3, we can rewrite (4.41) and (4.49) as

$$
(R_s^*)'(z) = \mathbb{1}_{\{z\in\mathbb{B}_s\}}, \tag{4.66}
$$

where

$$
\begin{aligned}
\mathbb{B}_s = \Big\{z\Big|&(1-F_{X_{s+1}}(z))+\mathbb{E}_t(\mu_{s+1})(1-F_{X^1_{s+1}}(z))+\mathbb{E}_t(\lambda_s)\mathbb{1}_{[0,\gamma)}\left(F_{X_{s+1}}(z)\right) \\
&\geq (1+\mathbb{E}_t(\mu_{s+1})+\mathbb{E}_t(\lambda_s))(1+\theta)\mathbb{1}_{[0,\alpha)}\left(F_{X_{s+1}}(z)\right)\Big\}.
\end{aligned} \tag{4.67}
$$

Denote the complement of set $\mathbb{B}_s$ as $\bar{\mathbb{B}}_s$. If $F_{X_{s+1}}(z)\in[0,\alpha)$, then the inequality condition inside $\bar{\mathbb{B}}_s$ becomes

$$
\begin{aligned}
&(1-F_{X_{s+1}}(z))+\mathbb{E}_t(\mu_{s+1})(1-F_{X^1_{s+1}}(z))+\mathbb{E}_t(\lambda_s)\mathbb{1}_{[0,\gamma)}\left(F_{X_{s+1}}(z)\right) \\
<&(1+\theta)\left[1+\mathbb{E}_t(\mu_{s+1})+\mathbb{E}_t(\lambda_s)\right].
\end{aligned}
$$

This is equivalent to

$$
\theta+F_{X_{s+1}}(z)+\mathbb{E}_t(\mu_{s+1})\left[\theta+F_{X^1_{s+1}}(z)\right]+\mathbb{E}_t(\lambda_s)\left\{1+\theta-\mathbb{1}_{[\gamma,1]}\left(F_{X_{s+1}}(z)\right)\right\}<0.
$$

Since $F_{X_{s+1}}$ and $F_{X^1_{s+1}}$ are CDFs, $F_{X_{s+1}}\in[0,1]$ and $F_{X^1_{s+1}}\in[0,1]$ always hold. Moreover, the safety loading θ is non-negative, and one minus an indicator function is still an indicator function with the complement condition, so the term $1+\theta-\mathbb{1}_{[\gamma,1]}\left(F_{X_{s+1}}(z)\right)$

is non-negative. In addition, based on the discussion in the proof of Theorem 4.3.3, $\mu_{s+1} \geq 0$ and $\lambda_s \geq 0$ for all $s = t, t+1, \ldots, \infty$, so their expectations are also non-negative. Hence, the set $[0, \mathrm{VaR}_\alpha(X)) \subseteq \bar{\mathbb{B}}_s$ despite the choices of Lagrangian multipliers and time period. This is equivalent to $\mathbb{B}_s \subseteq [\mathrm{VaR}_\alpha(X), \infty)$ for all $s = t, t+1, \ldots, \infty$.

On the other hand, if $F_{X_{s+1}}(z) \in [\alpha, 1]$, then the inequality condition in $\mathbb{B}_s$ becomes $(1 - F_{X_{s+1}}(z)) + \mathbb{E}_t(\mu_{s+1})(1 - F_{X^1_{s+1}}(z)) + \mathbb{E}_t(\lambda_s)\mathbb{1}_{[0,\gamma)}\big(F_{X_{s+1}}(z)\big) > 0$. Again, following similar reasoning, we know that this holds when all three non-negative terms equal zero at the same time, i.e.

$$\begin{cases} (1 - F_{X_{s+1}}(x)) = 0 & (i) \\ \mathbb{E}_t(\mu_{s+1})(1 - F_{X^1_{s+1}}(x)) = 0 & (ii) \\ \mathbb{E}_t(\lambda_s)\mathbb{1}_{[0,\gamma)}\big(F_{X_{s+1}}(x)\big) = 0. & (iii) \end{cases} \tag{4.68}$$

Condition (4.68-i) holds when $F_{X_{s+1}}(x) = 1 \in [\alpha, 1]$, which leads to $\mathbb{1}_{[0,\gamma)}\big(F_{X_{s+1}}(x)\big) = 0$ since $\gamma \leq 1$. As a result, (4.68-iii) holds. Because $F_{X_{s+1}}(x)$ is a CDF, x tends to infinity. Consequently, $F_{X^1_{s+1}}(x)$ as a CDF itself also equals 1, so condition (4.68-iii) holds for all $s = t, t+1, \ldots, \infty$. Therefore, $\big\{z | F_{X_{s+1}}(z) \in [\alpha, 1)\big\} \subseteq \mathbb{B}_s$ and $\big\{z | F_{X_{s+1}}(z) = 1\big\} \subseteq \bar{\mathbb{B}}_s$ for any time period.

To summarise, we can rewrite the condition set (4.67) of the MIF as

$$\mathbb{B}_s = \big\{z | F_{X_{s+1}}(z) \in [\alpha, 1)\big\} \text{ for all } s = t, t+1, \ldots, \infty.$$

As a consequence, $(R_s^*)'(x) = \mathbb{1}_{[\alpha,1)}\big(F_{X_{s+1}}(x)\big) = \mathbb{1}_{[F^{-1}_{X_{s+1}}(\alpha),\infty)}(x)$. Notice that the set $\mathbb{B}_s$ here is no longer related to time s. Therefore, the reinsurance contract as the integral of the MIF has the form

$$\begin{aligned} R_s^*(x) &= \int_0^x (R_s^*)'(z)\,dz = \int_0^x \mathbb{1}_{[F^{-1}_{X_{s+1}}(\alpha),\infty)}(z)\,dz \\ &= \begin{cases} x - F^{-1}_{X_{s+1}}(\alpha), & F^{-1}_{X_{s+1}}(\alpha) \leq x < \infty \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

or, equivalently, $R_s^*(x) = \Big(x - F^{-1}_{X_{s+1}}(\alpha)\Big)^+$. When we take the exponential distribution for the total claims, we have $R_s^*(x) = \Big(x + \frac{\ln(1-\alpha)}{\kappa_{s+1}}\Big)^+$. □

Example 4.4.3. Consider an insolvency risk measured by Value at Risk at confidence level $1-\gamma$. Thus, $\rho^{\Gamma}(.) = \mathrm{VaR}_{\gamma}(.)$ with dual distortion functions $\Gamma^{VaR}(u) = \mathbb{1}_{[\gamma,1]}(u)$. The reinsurance premium is estimated by Conditional Value at Risk at confidence level $1-\alpha$ as

$$\pi^{\Pi}(.) = (1+\theta)\mathrm{CVaR}_{\alpha}(X) = \frac{1+\theta}{1-\alpha}\int_{\alpha}^{1} \mathrm{VaR}_s(X)ds,$$

with dual distortion function $\Pi^{\mathrm{CVaR}}(u) := \frac{u-\alpha}{1-\alpha}\mathbb{1}_{[\alpha,1]}(u)$.

In addition, the total claims in one period follow an exponential distribution with parameter $\kappa_{s+1} > 0$, so the CDF is $F^{Exp}_{X_{s+1}}(x) = 1 - e^{-\kappa_{s+1}x}$. Therefore, the optimal reinsurance policy to pay more dividends in the set-ups of both Problem I and Problem III is not to spend money to buy reinsurance; i.e. $R^*_s(x) = 0$ for all $s = t, t+1, \ldots, \infty$.

Proof. Problem I: Following the results in Theorem 4.3.1, we know that the optimal MIF has the form

$$\begin{aligned}(R^*_s)'(x) &= \mathbb{1}_{\left\{1 - F^{Exp}_{X_{s+1}}(x) > (1+\theta)\left(1 - \frac{F^{Exp}_{X_{s+1}}(x) - \alpha}{1-\alpha}\mathbb{1}_{[\alpha,1]}(F^{Exp}_{X_{s+1}}(x))\right)\right\}} \\ &= \mathbb{1}_{\left\{1 - \frac{e^{-\kappa_{s+1}x}}{1+\theta} < \left(1 - \frac{e^{-\kappa_{s+1}x}}{1-\alpha}\right)\mathbb{1}_{[0,\alpha)}(e^{-\kappa_{s+1}x})\right\}}.\end{aligned}$$

If $e^{-\kappa_{s+1}x} \in [0,\alpha)$, then since $\theta \geq 0$ but $\alpha \in [0,1]$ and $e^{-\kappa_{s+1}x} \in [0,1]$, $\frac{1}{1+\theta} \leq 1 < \frac{1}{1-\alpha}$. As a result, $\frac{e^{-\kappa_{s+1}x}}{1+\theta} > \frac{e^{-\kappa_{s+1}x}}{1-\alpha}$ cannot hold for any x, so the MIF $R^{*\prime}_s(x) = 0$. Otherwise, if $e^{-\kappa_{s+1}x} \in [\alpha, 1]$, then $e^{-\kappa_{s+1}x} > 1+\theta \geq 1$ cannot hold for any x, so the MIF $(R^*_s)'(x) = 0$. In this set-up, the optimal reinsurance policy is not to buy reinsurance in order to pay more dividends.

Problem III: Taking the distortion function of CVaR back to the results in Theorem 4.3.3, we have $\Pi^{\mathrm{CVaR}}(u) := \frac{u-\alpha}{1-\alpha}\mathbb{1}_{[\alpha,1]}(u) = \max(\frac{u-\alpha}{1-\alpha}, 0)$, and only when $u = 1$ does $\Pi^{\mathrm{CVaR}}(u) = 1$. Thus, the optimal MIF set $\mathbb{B} = \left\{z | \Pi^{\mathrm{CVaR}}[F^{Exp}_{X_{s+1}}(z)] = 1\right\} = \left\{z | F^{Exp}_{X_{s+1}}(z) = 1\right\} = \{z | 1 - e^{-\kappa_{s+1}z} = 1\} = \{0\}$. Therefore, the optimal MIF is

$$(R^*_s)'(z) = \begin{cases} 1, & F^{Exp}_{X_{s+1}}(z) = 1 \\ 0, & F^{Exp}_{X_{s+1}}(z) \in [0,1) \end{cases}, \tag{4.69}$$

and the optimal reinsurance contract satisfies $R_s^*(x) = \int_0^x \mathbb{1}_{\{z\in\mathbb{B}\}}dz = 0$. Therefore, as with the set-up in Problem I, the optimal policy of the insurance company is not to buy reinsurance to maximise the dividends paid. $\square$

Example 4.4.4. Consider an insolvency risk measured by Value at Risk at confidence level $1-\gamma$. Thus, $\rho^\Gamma(.) = \mathrm{VaR}_\gamma(.)$ with dual distortion functions $\Gamma^{VaR}(u) = \mathbb{1}_{[\gamma,1]}(u)$. The reinsurance premium is estimated by Wang's premium principle with dual distortion function as

$$\Pi^\eta(u) = \Phi(\Phi^{-1}(u) - \eta),$$

where $\eta \in \mathbb{R}$ is a real parameter. The safety loading satisfies $\theta^\eta = 0$.

In addition, the total claims in one period follow an exponential distribution with parameter $\kappa_{s+1} > 0$, so the CDF is $F^{Exp}_{X_{s+1}}(x) = 1 - e^{-\kappa_{s+1}x}$. Therefore, the optimal reinsurance policy to pay more dividends in the set-up of Problem I is not to spend money to buy reinsurance; i.e. $R_s^*(x) = 0$ for all $s = t, t+1, \ldots, \infty$.

Proof. Problem I: Following the results in Theorem 4.3.1, we know that the optimal MIF has the form

$$\begin{aligned}
(R_s^*)'(x) &= \mathbb{1}_{\left\{1-F^{Exp}_{X_{s+1}}(x)>(1+\theta)\left(1-\Pi_\iota[F^{Exp}_{X_{s+1}}(x)]\right)\right\}} \\
&= \mathbb{1}_{\left\{e^{-\kappa_{s+1}x}>(1+\theta)\left[1-\Phi(\Phi^{-1}(1-e^{-\kappa_{s+1}x})-\iota)\right]\right\}} \\
&= \mathbb{1}_{\left\{\Phi^{-1}(1-e^{-\kappa_{s+1}x})-\Phi^{-1}(1-\frac{e^{-\kappa_{s+1}x}}{1+\theta})<\iota\right\}}.
\end{aligned}$$

Since the safety loading θ is a non-negative value, $\Phi^{-1}(1-e^{-\kappa_{s+1}x}) \le \Phi^{-1}(1-\frac{e^{-\kappa_{s+1}x}}{1+\theta})$. Therefore, if the fixed parameter is positive, then for all x and s, the MIF is equal to zero, and the reinsurance contract is also equal to zero for all x and s. $\square$

4.5 Discussion

In this chapter, we have introduced dynamic DRM to support our discrete-time surplus model and demonstrate how the assumption of independent periodic total claims simplifies all the dynamic settings. Various constraints were discussed for the maximisation of the expectation of cumulatively discounted dividend payments, but similar conclusions were obtained as multilayer reinsurance contracts are the optimal choice for the shareholders' benefit. The examples under specifically chosen DRMs in the last section can be considered a demonstration of how to use the general results that we obtained in Section 4.3 with exponentially distributed and independent claims.

The assumption of independence was convenient in this work, but with the application of DRM properties such as comonotonicity, this assumption may be removed, and the results could potentially be extended to more realistic and general set-ups in the future.

Chapter 5

Reciprocal Optimisation

In this chapter, we consider the problem of reciprocal optimal reinsurance design when the risk is measured by a distortion risk measure (DRM) and the premium is given by a distortion premium principle (DPP). First, we point out that, in a very common situation, the most optimal reinsurance contracts from an insurance point of view are the least optimal contracts from a reinsurance point of view. In light of this observation, we introduce three different reciprocal reinsurance contracting problems:

1. *Ceding Optimal/Reinsurance Control,* where the insurance company minimises its risk of losses while the reinsurance company controls the risk.
2. *Reinsurance Optimal/Ceding Control,* where the reinsurance company minimises its risk of losses while the insurance company controls the risk.
3. *Ceding Reinsurance/Optimal Control,* where the sum of the insurance and reinsurance total risks is minimised while both parties control their level of risk.

On the basis of these three problems, we find their Lagrangian dual and show that strong duality holds for all of them. Using this and the MIF method demonstrated in Section 2.3, we can characterise the optimal solutions by solving the dual problem and show that the solutions have a multi-layered structure. Second, we consider three types of contracts that are usually traded in the reinsurance market—namely, the stop-loss, stop-loss after quota-share, and quota-share after stop-loss. We show how to find the retention levels for each contract with respect to each of the three problems introduced above. We will show how we can find the solutions within by using the Karush–Kuhn–Tucker (KKT) conditions.

The rest of this chapter is organised as follows. In Section 5.1, we introduce our reciprocal reinsurance problems and show that the most optimal solution for one party in a common situation is the least optimal for the other. In Section 5.2.1, we solve the reciprocal problems in their most general forms. In Section 5.2.2, we consider the stop-loss, stop-loss after quota-share, and quota-share after stop-loss of particular policies and show how to find the retention levels for each contract. Section 5.2.2 summarises the results of this chapter.

5.1 Problem Set-up

In this section, we will set up the problem framework and find the optimal solution. First, we have to know the total loss of each party in the market.

Let us denote the annual risk of an insurance company by a non-negative loss variable X. In general, a reinsurance company will accept to cover part of the risk in exchange for receiving a premium. Let us denote the part of the risk covered by the reinsurer by X^R and the part covered by the insurance company by $X^I = X - X^R$. We assume that $0 \leq X^R \leq X$. The premium received by the reinsurance company is $(1+\theta)\pi(X^R)$, where $\theta \geq 0$ is a relative-safety-loading factor. Therefore, the global loss of the ceding company, denoted by T^I, can be expressed as

$$T^I = X^I + (1+\theta)\pi\left(X^R\right). \tag{5.1}$$

In the same manner, for the reinsurer, we have

$$T^R = X^R - (1+\theta)\pi\left(X^R\right). \tag{5.2}$$

Note that in this chapter, more aspects (the cedent, reinsurance, and both as a group) are involved in the objectives; to simplify the notation and avoid the confusion of superscripts, we remove the symbol of the distortion function from the DRM and DPP, i.e. $\rho(X) := \rho^{\Lambda}(X)$ and $\pi(X) := \pi^{\Lambda}(X)$. Therefore, the risks with respect to each party for chosen risk measures are expressed below.

- **Total risk of the ceding company.** If the insurance company chooses ρ^I to measure the risk, then its total risk will be

$$TR_I := \rho^I\left(X^I + (1+\theta)\pi\left(X^R\right)\right).$$

- **Total risk of the reinsurance company.** Likewise, given that ρ^R denotes the reinsurance company's risk measure, the total risk is defined as

$$TR_R := \rho^R\left(X^R - (1+\theta)\pi\left(X^R\right)\right).$$

- **Total risk.** The total risk of both the ceding and reinsurance companies is

$$\rho^R\left(X^R - (1+\theta)\pi\left(X^R\right)\right) + \rho^I\left(X^I + (1+\theta)\pi\left(X^R\right)\right),$$

  where ρ^I and ρ^R are the risk measures used by the insurance and reinsurance companies, respectively.

By applying the cash invariance property, they are respectively equivalent to the following three expressions:

- **Total risk of the ceding company.**

$$TR_I = \rho^I\left(X^I\right) + (1+\theta)\pi\left(X^R\right).$$

- **Total risk of the reinsurance company.**

$$TR_R = \rho^R\left(X^R\right) - (1+\theta)\pi\left(X^R\right).$$

- **Total risk.**

$$\rho^R\left(X^R\right) + \rho^I\left(X^I\right).$$

Therefore, we introduce three problem setups.

1. *Ceding Optimal/Reinsurance Control (CORC).* We assume that the insurance company wants to minimise its total risk; at the same time, the reinsurance company has to control its total risk below a tolerable level $b^R > 0$. Hence, the first setup is as follows:
$$\begin{cases} \min \rho^I\left(X^I\right) + (1+\theta)\pi\left(X^R\right) \\ \rho^R\left(X^R\right) - (1+\theta)\pi\left(X^R\right) \leq b^R \end{cases}.$$

2. *Reinsurance Optimal/Ceding Control (ROCC).* The position between the insurance and reinsurance companies in the last setup are exchanged, and it is assumed that the total risk of the insurance company cannot exceed a level $b^I > 0$. The second problem is set up as
$$\begin{cases} \min \rho^R\left(X^R\right) - (1+\theta)\pi\left(X^R\right) \\ \rho^I\left(X^I\right) + (1+\theta)\pi\left(X^R\right) \leq b^I \end{cases}.$$

3. *Ceding Reinsurance/Optimal Control (CROC).* In this case, we assume a total risk, given that both the insurance and reinsurance risks are controlled, and the total risk of both sides is minimised; therefore, we have
$$\begin{cases} \min \rho^R\left(X^R\right) + \rho^I\left(X^I\right) \\ \rho^R\left(X^R\right) - (1+\theta)\pi\left(X^R\right) \leq b^R \\ \rho^I\left(X^I\right) + (1+\theta)\pi\left(X^R\right) \leq b^I \end{cases}.$$

5.2 Optimal Solution

In this section, we find the optimal policies for the three problems in the previous section. Here, we discuss a few assumptions that will be used in the following.

Assumption 3. We assume that the DRM and DPP satisfy a particular regularity condition. Let ρ be either a risk measure or risk premium. We assume that the following limit holds:

$$\lim_{n\to\infty} \rho(X \wedge n) = \rho(X). \tag{5.3}$$

Recall the assumption of no moral hazard risk from Section 2.3 (see Assumption 1). The interpretation of a marginal indemnification function is as follows: if $R(x) = \int_0^x h(z)dz$ is a contract, then at each value $X = x$, a marginal change Δ in the value of the global loss will result in a marginal change in the cedent risk of size $\Delta h(x)$. We will see in the following that the marginal change of an optimal contract is either 0 or Δ in our framework, i.e. $h = 0$ or 1.

Now, let us prove two lemmas that will be used in further discussions.

Lemma 5.2.1. Let us assume that $X^R = R(X)$, where $R \in \mathbb{C}$ and $\rho(X)$ is a DRM; then, $\rho\left(X^I\right) = \rho\left(X\right) - \rho\left(X^R\right)$.

Proof. Because $\rho(X)$ is a DRM, the corresponding distortion function Λ has the form shown in (4.2) such that

$$\rho(X) = \int_0^1 \mathrm{VaR}_z\left(X\right) d\Lambda\left(z\right).$$

Given that $x \mapsto x - R(x)$ and R are non-decreasing and that VaR_z commutes with any non-decreasing function, we have the following simple calculations:

$$\begin{aligned}
\rho\left(X - R(X)\right) &= \int_0^1 \mathrm{VaR}_z\left(X - R(X)\right) d\Lambda(z) \\
&= \int_0^1 \left(\mathrm{VaR}_z(X) - R(\mathrm{VaR}_z(X))\right) d\Lambda(z) \\
&= \int_0^1 \left(\mathrm{VaR}_z(X) - \mathrm{VaR}_z(R\left(X\right))\right) d\Lambda(z) \\
&= \rho\left(X\right) - \rho\left(R\left(X\right)\right).
\end{aligned}$$

□

By applying the notation used in this chapter to Proposition 2.3.3 from Chapter 2, we have the following expression for the DRM or DPP on the reinsurance contract: $\rho\left(X^R\right) = \int_0^\infty \left(1 - \Pi\left(F_X(z)\right)\right) h\left(z\right) dz$, where $X^R = R\left(X\right) \in \mathbb{C}$.

Remark 5.2.2. To simplify the notation, the following symbols are defined and used in this chapter. Given the DRM or DPP as $\rho(X) = \int_0^1 \mathrm{VaR}_t(X)\, d\Pi(t)$ and the CDF of X as F_X, we have

$$\begin{aligned}
&\Pi^\rho := \Pi,\\
&\Pi^\rho_X := \Pi^\rho \circ F_X,\\
&\Phi^\rho(t) := \Pi^\rho(1) - \Pi^\rho(t) = 1 - \Pi^\rho(t),\\
&\Phi^\rho_X(t) := \Pi^\rho(1) - \Pi^\rho_X(t) = 1 - \Pi^\rho_X(t).
\end{aligned}$$

In addition, let $\Phi^I_X(t) := \Phi^{\rho^I}_X(t)$ and $\Phi^R_X(t) := \Phi^{\rho^R}_X(t)$. We also define the following notation

$$\begin{aligned}
\Psi^I_X(t) &:= (1+\theta)\,\Phi^\pi_X(t) - \Phi^I_X(t),\\
\Psi^R_X(t) &:= \Phi^R_X(t) - (1+\theta)\Phi^\pi_X(t).
\end{aligned}$$

Now, we are in a position to show that in certain scenarios, the most optimal reinsurance contract from an insurance point of view can be the worst contract for the reinsurance company.

Theorem 5.2.3. If Assumptions 1 and 3 hold and $\rho^I = \rho^R = \rho$, then the most optimal solution for each party is the least optimal for the other one.

Proof. As a reminder, the total risks of the ceding and reinsurance companies are as follows:

$$TR_I = \rho\left(X^I\right) + (1+\theta)\pi\left(X^R\right)$$

and

$$TR_R = \rho\left(X^R\right) - (1+\theta)\pi\left(X^R\right).$$

If we sum them, we have $TR_I + TR_R = \rho\left(X^I\right) + \rho\left(X^R\right)$. If $X^R = R(X)$ and $X^I = X - R(X)$, where $R(x) = \int_0^x h(z)dz \in \mathbb{C}$, we have the following using Lemma 5.2.1 and

Proposition 2.3.3:

$$\begin{aligned} TR_I\left(X^R\right)+TR_R\left(X^R\right) &= \int_0^X \Phi^\rho(t)\, h(t)\, dt + \int_0^X \Phi^\rho(t)\,(1-h(t))\, dt \\ &= \int_0^X \Phi^\rho(t)\, dt = \rho(X). \end{aligned}$$

By using Lemma 5.2.1, we have

$$\begin{aligned} TR_I + TR_R = \rho\left(X^I\right)+\rho\left(X^R\right) &= \rho\left(X-X^R\right)+\rho\left(X^R\right) \\ &= \rho(X)-\rho\left(X^R\right)+\rho\left(X^R\right)=\rho(X). \end{aligned}$$

Because the sum of the insurance and reinsurance total risks are constant, any policy X^R that minimises TR_I will maximise TR_R and vice versa. □

Remark 5.2.4. The main assumption of this theorem, i.e. $\rho^I=\rho^R=\rho$, is not very unusual if the companies control their risk for regulatory purposes. For instance, in Solvency II, all companies are asked to keep the company solvent under a VaR of 99.5%.

5.2.1 General Case

In this section, we find the optimal solutions of CORC, ROCC, and CROC by solving their Lagrangian dual problems.

Theorem 5.2.5. If Assumption 1 and 3 hold, then the reinsurance optimal solutions to CORC, ROCC, and CROC are given as

$$\begin{aligned} \left(X^R\right)^{(CORC)} &= \int_0^X h^{*(CORC)}(t)\, dt, \\ \left(X^R\right)^{(ROCC)} &= \int_0^X h^{*(ROCC)}(t)\, dt, \\ \left(X^R\right)^{(CROC)} &= \int_0^X h^{*(CROC)}(t)\, dt, \end{aligned}$$

respectively, where

$$h^{*(CORC)}(t) = \begin{cases} 1 & : \ \Psi_X^I(t) + \lambda^{*R}\Psi_Y^R(t) < 0 \\ 0 & : \ \Psi_X^I(t) + \lambda^{*R}\Psi_Y^R(t) > 0 \end{cases} \text{ and } \int_0^\infty \Psi_Y^R(t) h^{*(CORC)}(t)dt = b^R,$$

$$h^{*(ROCC)}(t) = \begin{cases} 1 & : \ \Psi_X^R(t) + \lambda^{*I}\Psi_Y^I(t) < 0 \\ 0 & : \ \Psi_X^R(t) + \lambda^{*I}\Psi_Y^I(t) > 0 \end{cases} \text{ and } \int_0^\infty \Psi_Y^I(t) h^{*(ROCC)}(t)dt = B^I,$$

and

$$\begin{cases} h^{*(CROC)}(t) = \begin{cases} 1 & : \ \Psi_X^R(t) + \Psi_X^I(t) + \lambda^{**^R}\Psi_X^R(t) + \lambda^{**^I}\Psi_X^I(t) < 0 \\ 0 & : \ \Psi_X^R(t) + \Psi_X^I(t) + \lambda^{**^R}\Psi_X^R(t) + \lambda^{**^I}\Psi_X^I(t) > 0 \end{cases} \\ \int_0^\infty \Psi_X^R(t)\, h^{*(CROC)}(t)dt = b^I \\ \int_0^\infty \Psi_X^I(t)\, h^{*(CROC)}(t)dt = b^R \end{cases}.$$

Proof. Given Lemma 5.2.1 and Proposition 2.3.3, one can rewrite CORC, ROCC, and CROC as follows:

$$(CORC) \begin{cases} \min \int_0^\infty \Psi_X^I(t)\, h(t)\, dt \\ \int_0^\infty \Psi_X^R(t)\, h(t)\, dt \le b^R \\ 0 \le h \le 1 \end{cases}, \quad (ROCC) \begin{cases} \min \int_0^\infty \Psi_X^R(t)\, h(t)\, dt \\ \int_0^\infty \Psi_X^I(t)\, h(t)\, dt \le B^I \\ 0 \le h \le 1 \end{cases}, \tag{5.4}$$

and

$$(CROC) \begin{cases} \min \int_0^\infty \left(\Psi_X^I(t) + \Psi_X^R(t)\right) h(t)\, dt \\ \int_0^\infty \Psi_X^R(t)\, h(t)\, dt \le b^R \\ \int_0^\infty \Psi_X^I(t)\, h(t)\, dt \le B^I \\ 0 \le h \le 1 \end{cases}, \tag{5.5}$$

where $B^I = b^I - \rho^I(X)$. As one can see, the objective functions in all three problems are smooth functions of the retention levels; therefore, one can find their Lagrangian duals.

The Lagrangian functions are given by

$$\begin{aligned}(CORC)\ \mathcal{L}^R\left(\lambda^R,h\right) &= -\textstyle\int_0^\infty \Psi_X^I(t)\,h(t)\,dt+\lambda^R\left(b^R-\int_0^\infty \Psi_X^R(t)\,h(t)\,dt\right),\\ (ROCC)\ \mathcal{L}^I\left(\lambda^I,h\right) &= -\textstyle\int_0^\infty \Psi_X^R(t)\,h(t)\,dt+\lambda^I\left(B^I-\int_0^\infty \Psi_X^I(t)\,h(t)\,dt\right),\end{aligned}$$

and

$$(CROC)\ \mathcal{L}(\lambda^I,\lambda^R,h)=\mathcal{L}^I\left(\lambda^I,h\right)+\mathcal{L}^R\left(\lambda^R,h\right),$$

where $\lambda^I,\lambda^R\geq 0$ are Lagrangian multipliers. We focus our attention only on CORC and its Lagrangian because the other problems can be solved in a similar way. With a simple rearrangement, we have

$$\mathcal{L}^R\left(\lambda^R,h\right)=-\textstyle\int_0^\infty\left(\Psi_X^I(t)+\lambda^R\Psi_X^R(t)\right)h(t)\,dt+\lambda^R b^R\ ,$$

which yields

$$\max_{0\leq h\leq 1}\mathcal{L}^R\left(\lambda^R,h\right)=\textstyle\int_0^\infty\left(\Psi_X^I(t)+\lambda^R\Psi_X^R(t)\right)_-dt+\lambda^R b^R\ .$$

Let us denote the minimal point of the following optimisation by λ^{*R}, i.e.

$$\lambda^{*R}=\operatorname*{argmin}_{\lambda^R\geq 0}\int_0^\infty\left(\Psi_X^I(t)+\lambda^R\Psi_X^R(t)\right)_-dt+\lambda^R b^R.$$

One can easily check that $h(t)=\epsilon$ for a sufficiently small ϵ is within the interior of the optimisation problem above, satisfying Slater's conditions; therefore, strong duality holds. This implies that the following min-max problem has a solution:

$$\min_{\lambda^R\geq 0}\max_{0\leq h\leq 1}\mathcal{L}^R\left(\lambda^R,h\right)=\max_{0\leq h\leq 1}\min_{\lambda^R\geq 0}\mathcal{L}^R\left(\lambda^R,h\right).$$

Let us denote the solution to this problem by $\left(\lambda^{*R},h^{*(CORC)}\right)$. It is clear that $h^{*(CORC)}$ is an solution to the following optimisation problem:

$$\max_{0\leq h\leq 1}-\int_0^\infty\left(\Psi_X^I(t)+\lambda^{*R}\Psi_X^R(t)\right)h(t)\,dt+\lambda^{*R}b^R. \tag{5.6}$$

From (5.6), the optimal solution has the following form

$$h^{*(CORC)}(t)=\begin{cases}1 & : \ \Psi_X^I(t)+\lambda^{*R}\Psi_Y^R(t)<0\\ 0 & : \ \Psi_X^I(t)+\lambda^{*R}\Psi_Y^R(t)>0\end{cases}.$$

Note that $\lambda^{*R} = 0$ is impossible; otherwise, the optimal solution will be $h^{*(CORC)} = 0$, which is a contradiction. Therefore, the complementary slackness condition implies that

$$\int_0^\infty \Psi_Y^R(t) h^{*(CORC)}(t) dt = b^R.$$

This completes the proof for CORC. ROCC and CROC can be solved in the same manner. □

Now, we discuss another assumption and study optimal solutions under it.

Assumption 4. We assume that the following set is of measure zero in $\mathbb{R}$ for any $(a, b, c) \in \mathbb{R}^3$, where either $ab \neq 0$ or $ac \neq 0$:

$$\left\{ t \in \mathbb{R} \,|\, a\Phi_X^\pi(t) + b\Phi_X^I(t) + c\Phi_X^R(t) = 0 \right\}. \tag{5.7}$$

For instance, by Assumption 4,

$$\left\{ \Psi_X^I(t) + \lambda^R \Psi_X^R(t) = 0 \right\} = \left\{ \Phi_X^R(t) - \lambda^R \Phi_X^I(t) + (1+\theta)\left(1 - \lambda^R\right) \Phi_X^\pi(t) = 0 \right\}$$

is of measure zero. This implies that, although the function $\lambda^R \mapsto \left(\Psi_X^I(t) + \lambda^R \Psi_X^R(t) \right)_-$ is not differentiable, the following function is differentiable:

$$A\left(\lambda^R\right) = \int_0^\infty \left(\Psi_X^I(t) + \lambda^R \Psi_X^R(t) \right)_- dt + \lambda^R b^R,$$

and its derivative is equal to

$$A\left(\lambda^R\right) = \int_0^\infty \Psi_X^R(t)\, \mathbb{1}_{\left\{\Psi_X^I + \lambda^R \Psi_X^R < 0\right\}} dt + b^R.$$

Therefore, λ^{*R} will be the root of this function.

Theorem 5.2.6. If in addition to Assumption 1, 3 and 4 hold, then the reinsurance optimal solutions are almost surely unique and given by

$$\begin{aligned}
(CORC)\;\; h^{(CORC)}(t) &= \mathbb{1}_{\left\{\Psi_X^I + \lambda^{*R} \Psi_X^R < 0\right\}}(t), \\
(ROCC)\;\; h^{(ROCC)}(t) &= \mathbb{1}_{\left\{\Psi_X^R + \lambda^{*I} \Psi_X^I < 0\right\}}(t),
\end{aligned}$$

and

$$(CROC)\ \ h^{(CROC)}(t) = \mathbb{1}_{\left\{\Psi_X^R+\Psi_X^I+\lambda^{**R}\Psi_X^R+\lambda^{**I}\Psi_X^I<0\right\}}(t)\,.$$

Furthermore, λ^{*I}, λ^{*R}, and $\left(\lambda^{**I}, \lambda^{**R}\right)$ can be found by solving the following problems:

$$\begin{aligned} &(CORC)\ \ \int_0^\infty \Psi_X^R(t)\,\mathbb{1}_{\left\{\Psi_X^I+\lambda^{*R}\Psi_X^R<0\right\}}(t)\,dt + b^R = 0,\\ &(ROCC)\ \ \int_0^\infty \Psi_X^I(t)\,\mathbb{1}_{\left\{\Psi_X^R+\lambda^{*I}\Psi_X^I<0\right\}}(t)\,dt + B^I = 0, \end{aligned} \tag{5.8}$$

and

$$(CROC)\ \ \begin{cases} \int_0^\infty \left(\Psi_X^R(t)\right)\mathbb{1}_{\left\{\Psi_X^R+\Psi_X^I+\lambda^{**R}\Psi_X^R+\lambda^{**I}\Psi_X^I<0\right\}}(t)\,dt + b^R = 0\\ \int_0^\infty \left(\Psi_X^I(t)\right)\mathbb{1}_{\left\{\Psi_X^R+\Psi_X^I+\lambda^{**R}\Psi_X^R+\lambda^{**I}\Psi_X^I<0\right\}}(t)\,dt + B^I = 0 \end{cases}. \tag{5.9}$$

Proof. The theorem is easily a result of the fact that the following sets are of measure zero in $\mathbb{R}$ by Assumption 4:

$$\begin{aligned} &(CORC)\ \ \left\{\Psi_X^I+\lambda^{*R}\Psi_X^R=0\right\},\\ &(ROCC)\ \ \left\{\Psi_X^R+\lambda^{*I}\Psi_X^I=0\right\}, \end{aligned}$$

and

$$(CROC)\ \ \left\{\Psi_X^R+\Psi_X^I+\lambda^{**R}\Psi_X^R+\lambda^{**I}\Psi_X^I=0\right\}.$$

□

Example 5.2.7. Let us assume that α_1 and α_2 are two risk aversion parameters. Let us assume that Π^π is strictly increasing and that ρ^I and ρ^R belong to the set $\{\mathrm{VaR}_{\alpha_1}, \mathrm{CVaR}_{\alpha_2}\}$. Then, it is easy to see that Assumption 1 holds; therefore, the solutions are either in the form of (5.8) or (5.9).

5.2.2 Stop-Loss and Quota-Share Policies

The methods developed above can characterise the optimal solutions to the optimal reinsurance problem. However, there are not many different forms of contracts or multi-layered policies that can be traded in the real world. Usually, contracts are either stop-loss, stop-loss after quota-share, or quota share after stop loss. In this section, we focus our attention on these contracts. Recall the definitions of these three reinsurance agreements in Section 2.1.1. We denote $0 \leq l \leq u < \infty$ as the lower and the upper retention levels and $\beta \in (0,1]$ as a proportion of the quota-share reinsurance. The contracts could be represented as follows:.

1. **Stop-loss reinsurance:**

$$R(x) = \min\{(x-l)_+, u\} = (x-l)\,\mathbb{1}_{[l,u)}(x) + (u-l)\,\mathbb{1}_{[u,\infty)}(x)\,. \tag{5.10}$$

2. **Stop-loss after quota-share reinsurance:**

$$R(x) = \begin{cases} \beta x & , x < \frac{u}{\beta} \\ u & , x \geq \frac{u}{\beta} \end{cases}$$

$$= \beta x \mathbb{1}_{\left[0,\frac{u}{\beta}\right)}(x) + u\mathbb{1}_{\left[\frac{u}{\beta},\infty\right)}(x)\,. \tag{5.11}$$

3. **Quota-share after stop-loss reinsurance:**

$$R(x) = \begin{cases} x & , x < u \\ \beta x + (1-\beta)\,u & , x \geq u \end{cases}$$

$$= x\mathbb{1}_{[0,u)}(x) + (\beta x + (1-\beta)\,u)\,\mathbb{1}_{[u,\infty)}(x)\,. \tag{5.12}$$

One can easily find the associated derivatives as $h(t) = \mathbb{1}_{[l,u)}(t)$, $h(t) = \beta\mathbb{1}_{\left[0,\frac{u}{\beta}\right)}(t)$, and $h(t) = \mathbb{1}_{[0,u)}(t) + \beta\mathbb{1}_{[u,\infty)}(t)$, respectively. As a reminder, the main objective problems are (5.4) and (5.5); by substituting h in all three cases, we have

1. **Stop-loss policy:**

$$(CORC)\quad \begin{cases} \min \int_l^u \Psi_X^I(t)\,dt \\ \int_l^u \Psi_X^R(t)\,dt \leq b^R \end{cases} \quad , \quad (ROCC)\quad \begin{cases} \min \int_l^u \Psi_X^R(t)\,dt \\ \int_l^u \Psi_X^I(t)\,dt \leq B^I \end{cases} \quad ,$$

and

$$(CROC)\quad \begin{cases} \min \int_l^u \left(\Psi_X^I(t) + \Psi_X^R(t)\right) dt \\ \int_l^u \Psi_X^R(t)\, dt \leq b^R \\ \int_l^u \Psi_X^I(t)\, dt \leq B^I. \end{cases}$$

2. **Stop-loss after quota-share policy:**

$$(CORC)\quad \begin{cases} \min \beta \int_0^{\frac{u}{\beta}} \Psi_X^I(t)\, dt \\ \beta \int_0^{\frac{u}{\beta}} \Psi_X^R(t)\, dt \leq b^R \end{cases}, \quad (ROCC)\quad \begin{cases} \min \beta \int_0^{\frac{u}{\beta}} \Psi_X^I(t)\, dt \\ \beta \int_0^{\frac{u}{\beta}} \Psi_X^R(t)\, dt \leq B^I \end{cases},$$

and

$$(CROC)\quad \begin{cases} \min \beta \int_0^{\frac{u}{\beta}} \left(\Psi_X^I(t) + \Psi_X^R(t)\right) dt \\ \beta \int_0^{\frac{u}{\beta}} \Psi_X^R(t)\, dt \leq b^R \\ \beta \int_0^{\frac{u}{\beta}} \Psi_X^I(t)\, dt \leq B^I. \end{cases}$$

3. **Quota-share after stop-loss policy:**

$$(CORC)\quad \begin{cases} \min \int_0^u \Psi_X^I(t)\, dt + \beta \int_u^\infty \Psi_X^I(t)\, dt \\ \int_0^u \Psi_X^R(t)\, dt + \beta \int_u^\infty \Psi_X^R(t)\, dt \leq b^R \end{cases},$$

$$(ROCC)\quad \begin{cases} \min \int_0^u \Psi_X^R(t)\, dt + \beta \int_u^\infty \Psi_X^R(t)\, dt \\ \int_0^u \Psi_X^I(t)\, dt + \beta \int_u^\infty \Psi_X^I(t)\, dt \leq B^I \end{cases},$$

and

$$(CROC)\quad \begin{cases} \min \int_0^u \left[\Psi_X^I(t) + \Psi_X^R(t)\right] dt + \beta \int_u^\infty \left[\Psi_X^I(t) + \Psi_X^R(t)\right] dt \\ \int_0^u \Psi_X^R(t)\, dt + \beta \int_u^\infty \Psi_X^R(t)\, dt \leq b^R \\ \int_0^u \Psi_X^I(t)\, dt + \beta \int_u^\infty \Psi_X^I(t)\, dt \leq B^I. \end{cases}.$$

If the Lagrangian multiplier is denoted by λ, then we have to solve the following KKT optimality conditions under the setup of CORC:

1. **Stop-loss policy:**

$$\begin{cases} \Psi_X^I(u) + \lambda \Psi_X^R(u) = 0 \\ \Psi_X^I(l) + \lambda \Psi_X^R(l) = 0 \\ \int_l^u \Psi_X^R(t) dt \le b^R \\ \lambda \ge 0 \\ \lambda \left(b^R - \int_l^u \Psi_X^R(t)\, dt \right) = 0. \end{cases}$$

2. **Stop-loss after quota-share policy:**

$$\begin{cases} \Psi_X^I\left(\frac{u}{\beta}\right) + \lambda \Psi_X^R\left(\frac{u}{\beta}\right) = 0 \\ \beta \int_0^{\frac{u}{\beta}} \Psi_X^R(t) dt \le b^R \\ \lambda \ge 0 \\ \lambda \left(b^R - \int_0^{\frac{u}{\beta}} \Psi_X^R(t) dt \right) = 0. \end{cases}$$

3. **Quota-share after stop-loss policy:**

$$\begin{cases} (1-\beta)\left(\Psi_X^I(u) + \lambda \Psi_X^R(u)\right) = 0 \\ \int_u^\infty \left(\Psi_X^I(t) + \lambda \Psi_X^R(t)\right) dt = 0 \\ \int_0^u \Psi_X^R(t) dt + \beta \int_u^\infty \Psi_X^R dt \le b^R \\ \lambda \ge 0 \\ \lambda \left(b^R - \int_0^u \Psi_X^R(t) dt + \beta \int_u^\infty \Psi_X^R(t) dt \right) = 0. \end{cases}$$

We will provide further information about these particular cases in the following example, which demonstrates the optimality condition for three policies explained above can be derived and only focuses on the CORC framework. The other two set-ups can be solved in the same manner. This instance illustrates how a mathematical derivation of the solution can be feasibly found through a finite number of basic computations.

Example 5.2.8. We assume that both of the selected DRMs of the insurer and reinsurer are VaRs but with different confidence levels, denoted by $\rho^I(X) = \mathrm{VaR}_{\alpha_1}(X)$ and $\rho^R(X) = \mathrm{VaR}_{\alpha_2}(X)$, where $\alpha_1 \le \alpha_2$. In addition, we keep the DPP, π, general. To simplify the notation, let us denote $s := F_X(t)$ and apply the selected VaR to Remark 5.2.2 to obtain

$$\Psi^I(s) = (1+\theta)\,\Phi^\pi(t) - \mathbb{1}_{[0,\alpha_1)}(s)\,,$$
$$\Psi^R(s) = \mathbb{1}_{[0,\alpha_2)}(s) - (1+\theta)\,\Phi^\pi(t)\,.$$

1. **Stop-loss policy:** If we denote $\delta := F_X(u)$ and $\gamma_1 := F_X(l)$, the optimality conditions are represented as

$$\begin{cases} (1+\theta)\,\Phi^\pi(\delta)(1-\lambda) - \mathbb{1}_{[0,\alpha_1)}(\delta) + \lambda\mathbb{1}_{[0,\alpha_2)}(\delta) = 0 \\ (1+\theta)\,\Phi^\pi(\gamma_1)(1-\lambda) - \mathbb{1}_{[0,\alpha_1)}(\gamma_1) + \lambda\mathbb{1}_{[0,\alpha_2)}(\gamma_1) = 0 \\ (u \wedge \mathrm{VaR}_{\alpha_2}(X) - l \wedge \mathrm{VaR}_{\alpha_2}(X)) - (1+\theta)(\pi(X\wedge u) - \pi(X\wedge l)) \le b \\ \lambda \ge 0 \\ \lambda\,(b - (u \wedge \mathrm{VaR}_{\alpha_2}(X) - l \wedge \mathrm{VaR}_{\alpha_2}(X)) + (1+\theta)(\pi(X\wedge u) - \pi(X\wedge l))) = 0. \end{cases}$$

In the third and the fifth lines, we used the fact that $\pi(X\wedge x) = \int_0^x \Phi^\pi(t)\,dt$. Now, we have the following six cases to consider.

Case 1. $\gamma_1 < \delta < \alpha_1$. In this case, the optimality conditions become

$$\begin{cases} ((1+\theta)\,\Phi^\pi(\delta) - 1)(1-\lambda) = 0 \\ ((1+\theta)\,\Phi^\pi(\gamma_1) - 1)(1-\lambda) = 0 \\ (u-l) - (1+\theta)(\pi(X\wedge u) - \pi(X\wedge l)) \le b \\ \lambda \ge 0 \\ \lambda\,(b - (u-l) + (1+\theta)(\pi(X\wedge u) - \pi(X\wedge l))) = 0 \end{cases}.$$

If $\lambda \neq 1$, then $\delta = \gamma_1$, which is a contradiction.

If $\lambda = 1$, then all $0 \le \gamma_1 < \delta < \alpha_1$ can be solutions to two equations above. This implies that $0 \le l < u < \mathrm{VaR}_{\alpha_1}(X)$. In this case, the complementary slackness condition also has to hold. Therefore, the following optimality conditions have to hold:

$$\begin{cases} 0 \leq l < u < \mathrm{VaR}_{\alpha_1}(X) \\ (u-l) - (1+\theta)\left(\pi(X \wedge u) - \pi(X \wedge l)\right) = b \end{cases}.$$

<u>*Case 2.* $\alpha_1 \leq \gamma_1 < \delta < \alpha_2$.</u> In this case, we have

$$\begin{cases} (1+\theta)\,\Phi^\pi(\delta)(1-\lambda) + \lambda = 0 \\ (1+\theta)\,\Phi^\pi(\gamma_1)(1-\lambda) + \lambda = 0 \\ (u-l) - (1+\theta)\left(\pi(X \wedge u) - \pi(X \wedge l)\right) \leq b \\ \lambda \geq 0 \\ \lambda\left(b - (u-l) + (1+\theta)\left(\pi(X \wedge u) - \pi(X \wedge l)\right)\right) = 0 \end{cases}.$$

If $0 \leq \lambda \leq 1$, then $\Phi^\pi(\delta) \leq 0$, which contradicts $\alpha_1 \leq \delta$. If $\lambda > 1$, then $\Phi^\pi(\gamma_1) = \Phi^\pi(\delta)$, implying that $\gamma_1 = \delta$, which is a contradiction again.

<u>*Case 3.* $\alpha_2 \leq \gamma_1 < \delta$.</u> In this case, we have

$$\begin{cases} (1+\theta)\,\Phi^\pi(\delta)(1-\lambda) = 0 \\ (1+\theta)\,\Phi^\pi(\gamma_1)(1-\lambda) = 0 \\ -(1+\theta)\left(\pi(X \wedge u) - \pi(X \wedge l)\right) \leq b \\ \lambda \geq 0 \\ \lambda\left(b + (1+\theta)\left(\pi(X \wedge u) - \pi(X \wedge l)\right)\right) = 0 \end{cases}.$$

If $\lambda \neq 1$, then $\gamma_1 = \delta = 1$, which is a contradiction. If $\lambda = 1$, then all $\alpha_1 \leq \gamma_1 < \delta$ can be solutions to first two equations above. From $\alpha_2 \leq \gamma_1 < \delta$, we also obtain $\mathrm{VaR}_{\alpha_2}(X) \leq \gamma_1 < \delta$. Because the complementary slackness condition also has to hold, we have the following optimality conditions:

$$\begin{cases} \mathrm{VaR}_{\alpha_2}(X) \leq l < u \\ -(1+\theta)\left(\pi(X \wedge u) - \pi(X \wedge l)\right) = b \end{cases}.$$

<u>*Case 4.* $\gamma_1 < \alpha_1 \leq \delta < \alpha_2$.</u> In this case, the first two optimality equations become

$$\begin{cases} (1+\theta)\,\Phi^{\pi}(\delta)\,(1-\lambda)+\lambda=0 \\ ((1+\theta)\,\Phi^{\pi}(\gamma_1)-1)\,(1-\lambda)=0 \\ (u-l)-(1+\theta)\,(\pi\,(X\wedge u)-\pi\,(X\wedge l))\leq b \\ \lambda\geq 0 \\ \lambda\,(b-(u-l)+(1+\theta)\,(\pi\,(X\wedge u)-\pi\,(X\wedge l)))=0 \end{cases}.$$

If $0 \leq \lambda \leq 1$, then $\Phi^{\pi}(\delta) \leq 0$, which contradicts $\alpha_1 \leq \delta$. If $\lambda > 1$, then

$$\begin{cases} \delta = (\Phi^{\pi})^{-1}\left(\frac{\lambda}{(1+\theta)(\lambda-1)}\right) \\ \gamma_1 = (\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right) \end{cases}.$$

However, these need to be checked with our conditions. The optimality conditions become

$$\begin{cases} \frac{1}{1+\theta} < (\Phi^{\pi})^{-1}(\alpha_1) \leq u < \mathrm{VaR}_{\alpha_2}(X) \\ l = \mathrm{VaR}_{(\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)}(X) \\ \left(u - \mathrm{VaR}_{(\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)}(X)\right)(1+\theta)\left(\pi\,(X\wedge u) - \pi\left(X \wedge \mathrm{VaR}_{(\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)}(X)\right)\right) = b \end{cases}.$$

<u>*Case 5.* $\alpha_1 \leq \gamma_1 < \alpha_2 \leq \delta$.</u> In this case, the two first lines of the optimality conditions become

$$\begin{cases} (1+\theta)\,\Phi^{\pi}(\delta)\,(1-\lambda)=0 \\ (1+\theta)\,\Phi^{\pi}(\gamma_1)\,(1-\lambda)+\lambda=0 \\ (\mathrm{VaR}_{\alpha_2}(X)-l)-(1+\theta)\,(\pi\,(X\wedge u)-\pi\,(X\wedge l))\leq b \\ \lambda\geq 0 \\ \lambda\,(b-(\mathrm{VaR}_{\alpha_2}(X)-l)+(1+\theta)\,(\pi\,(X\wedge u)-\pi\,(X\wedge l)))=0 \end{cases}.$$

If $\lambda \neq 1$, then $\Phi^{\pi}(\delta) = 0$, which is a contradiction. If $\lambda = 1$, then the second line of the equations above implies that $1 = 0$, which is a contradiction again. Therefore, we are left with nothing to check.

Case 6. $\gamma_1 < \alpha_1 \leq \alpha_2 \leq \delta$. In this case, the first two lines of the optimality conditions are expressed as

$$\begin{cases} ((1+\theta)\,\Phi^{\pi}(\delta) - 1)\,(1-\lambda) = 0 \\ (1+\theta)\,\Phi^{\pi}(\gamma_1)\,(1-\lambda) = 0 \\ (\mathrm{VaR}_{\alpha_2}(X) - l) - (1+\theta)\,(\pi\,(X \wedge u) - \pi\,(X \wedge l)) \leq b \\ \lambda \geq 0 \\ \lambda\,(b - (\mathrm{VaR}_{\alpha_2}(X) - l) + (1+\theta)\,(\pi\,(X \wedge u) - \pi\,(X \wedge l))) = 0 \end{cases}.$$

If $\lambda \neq 1$, then $\delta = 1$ and $\gamma_1 = 0$ must hold. This implies that both $u = \infty$ and $l = 0$. If $\lambda \neq 0$, then

$$\begin{cases} u = \infty \text{ and } l = 0 \\ \mathrm{VaR}_{\alpha_2}(X) \leq b \end{cases}.$$

If $\lambda \neq 0$ and $\lambda \neq 1$, then we have

$$\begin{cases} u = \infty \text{ and } l = 0 \\ \mathrm{VaR}_{\alpha_2}(X) = b \end{cases}.$$

If $\lambda = 1$, then the optimality conditions become

$$\begin{cases} \frac{\theta}{1+\theta} < \alpha_1 \\ l = \mathrm{VaR}_{\frac{\theta}{1+\theta}}(X) \text{ and } u = \infty \end{cases}.$$

If $\lambda = 1$, then all $\gamma_1 < \alpha_1 \leq \alpha_2 \leq \delta$ could be the solutions to the first and the second optimality equations. Therefore, we have

$$\begin{cases} l < \mathrm{VaR}_{\alpha_1}(X) \leq \mathrm{VaR}_{\alpha_2}(X) \leq u \\ (\mathrm{VaR}_{\alpha_2}(X) - l) - (1+\theta)\,(\pi\,(X \wedge u) - \pi\,(X \wedge l)) = b \end{cases}.$$

2. **Stop-loss after quota-share policy:** Let us denote $\gamma_2 := F_X\left(\frac{u}{\beta}\right)$ under this

contract. The optimality conditions become

$$\begin{cases} (1+\theta)\,\Phi^{\pi}(\gamma_2)\,(1-\lambda) - \mathbb{1}_{[0,\alpha_1)}(\gamma_2) + \lambda \mathbb{1}_{[0,\alpha_2)}(\gamma_2) = 0 \\ \left(\frac{u}{\beta} \wedge \mathrm{VaR}_{\alpha_2}(X)\right) - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right) \leq b \\ \lambda \geq 0 \\ \lambda\left(b - \left(\frac{u}{\beta} \wedge \mathrm{VaR}_{\alpha_2}(X)\right) - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right)\right) = 0 \end{cases}.$$

As a result, we need to discuss the following three cases.

Case 1. $\gamma_2 < \alpha_1$. In this case, the optimality conditions are

$$\begin{cases} ((1+\theta)\,\Phi^{\pi}(\gamma_2) - 1)\,(1-\lambda) = 0 \\ \frac{u}{\beta} - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right) \leq b \\ \lambda \geq 0 \\ \lambda\left(b - \left(\frac{u}{\beta}\right) - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right)\right) = 0 \end{cases}.$$

If $\lambda = 0$, then $\gamma_2 = (\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)$, and the optimality conditions are

$$\begin{cases} \frac{u}{\beta} < \mathrm{VaR}_{\alpha_1}(X) \\ \frac{u}{\beta} = \mathrm{VaR}_{(\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)}(X) \\ \frac{u}{\beta} - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right) \leq b \end{cases}.$$

With the same argument, if $\lambda \neq 0$ and $\lambda \neq 1$, then

$$\begin{cases} \frac{u}{\beta} < \mathrm{VaR}_{\alpha_1}(X) \\ \frac{u}{\beta} = \mathrm{VaR}_{(\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)}(X) \\ \frac{u}{\beta} - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right) = b \end{cases}.$$

If $\lambda = 1$, then γ_2 can be any number less than α_1 and can be solution to the first equation of the optimality conditions. Therefore, we have

$$\begin{cases} \frac{u}{\beta} < \mathrm{VaR}_{\alpha_1}(X) \\ \frac{u}{\beta} - (1+\theta)\,\pi\left(X \wedge \frac{u}{\beta}\right) = b \end{cases}.$$

Case 2. $\alpha_1 \le \gamma_2 < \alpha_2$. The optimality conditions are

$$\begin{cases} (1+\theta)\,\Phi^{\pi}(\gamma_2)(1-\lambda)+\lambda=0 \\ \frac{u}{\beta}-(1+\theta)\,\pi\left(X\wedge\frac{u}{\beta}\right)\le b \\ \lambda\ge 0 \\ \lambda\left(b-\frac{u}{\beta}-(1+\theta)\,\pi\left(X\wedge\frac{u}{\beta}\right)\right)=0 \end{cases}.$$

If $0\le\lambda\le 1$, we obtain $\Phi^{\pi}(\gamma_2)\le 0$ from the first line of the optimality conditions, which contradicts $\alpha_1\le\gamma_2$. If $\lambda>1$, we obtain $\gamma_2=(\Phi^{\pi})^{-1}\left(\frac{\lambda}{(\lambda-1)(1+\theta)}\right)$. Therefore,

$$\begin{cases} \mathrm{VaR}_{\alpha_1}(X)\le\frac{u}{\beta}<\mathrm{VaR}_{\alpha_2}(X) \\ \frac{u}{\beta}-(1+\theta)\,\pi\left(X\wedge\frac{u}{\beta}\right)=b \end{cases}.$$

Case 3. $\alpha_2\le\gamma_2$. In this case, we have

$$\begin{cases} (1+\theta)\,\Phi^{\pi}(\gamma_2)(1-\lambda)=0 \\ \mathrm{VaR}_{\alpha_2}(X)-(1+\theta)\,\pi\left(X\wedge\frac{u}{\beta}\right)\le b \\ \lambda\ge 0 \\ \lambda\left(b-\mathrm{VaR}_{\alpha_2}(X)-(1+\theta)\,\pi\left(X\wedge\frac{u}{\beta}\right)\right)=0 \end{cases}.$$

If $\lambda\ne 1$, then $\gamma_2=0$, which contradicts $\alpha_2\le\gamma_2$. If $\lambda=1$, we have

$$\begin{cases} \mathrm{VaR}_{\alpha_2}(X)\le\frac{u}{\beta} \\ \mathrm{VaR}_{\alpha_2}(X)-(1+\theta)\,\pi\left(X\wedge\frac{u}{\beta}\right)=b \end{cases}.$$

3. **Quota-share after stop-loss policy:** Let us denote $\delta = F_X(u)$. Hence, we have

$$\begin{cases} (1-\beta)\left((1+\theta)\,\Phi^{\pi}(\delta)(1-\lambda) - \mathbb{1}_{[0,\alpha_1)}(\delta) + \lambda \mathbb{1}_{[0,\alpha_2)}(\delta)\right) = 0 \\ \begin{cases} (1+\theta)(\pi(X) - \pi(X\wedge u))(1-\lambda) \\ -\left(\mathrm{VaR}_{\alpha_1}(X) - u\wedge \mathrm{VaR}_{\alpha_1}(X)\right) + \lambda\left(\mathrm{VaR}_{\alpha_2}(X) - u\wedge \mathrm{VaR}_{\alpha_2}(X)\right) = 0 \end{cases} \\ \begin{cases} u\wedge \mathrm{VaR}_{\alpha_2}(X) - (1+\theta)\,\pi(X\wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_2}(X) - u\wedge \mathrm{VaR}_{\alpha_2}(X) - (1+\theta)(\pi(X) - \pi(X\wedge u))\right) \le b \end{cases} \\ \begin{cases} \lambda\,(u\wedge \mathrm{VaR}_{\alpha_2}(X) - (1+\theta)\,\pi(X\wedge u) \\ +\, \beta\left(\mathrm{VaR}_{\alpha_2}(X) - u\wedge \mathrm{VaR}_{\alpha_2}(X) - (1+\theta)(\pi(X) - \pi(X\wedge u))\right) - b) = 0 \end{cases} \end{cases}.$$

As a consequence, there are three cases to discuss under this policy.

<u>*Case 1.* $\delta < \alpha_1$.</u> We have

$$\begin{cases} (1-\beta)\left((1+\theta)\,\Phi^{\pi}(\delta) - 1\right)(1-\lambda) = 0 \\ \begin{cases} (1+\theta)(\pi(X) - \pi(X\wedge u))(1-\lambda) \\ -\left(\mathrm{VaR}_{\alpha_1}(X) - u\right) + \lambda\left(\mathrm{VaR}_{\alpha_2}(X) - u\right) = 0 \end{cases} \\ \begin{cases} u - (1+\theta)\,\pi(X\wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)(\pi(X) - \pi(X\wedge u))\right) \le b \end{cases} \\ \begin{cases} \lambda\,(u - (1+\theta)\,\pi(X\wedge u) \\ +\, \beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)(\pi(X) - \pi(X\wedge u))\right) - b) = 0 \end{cases} \end{cases}.$$

If $\lambda = 0$, then $\delta = (\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)$, and

$$\begin{cases} (\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right) < \alpha_1 \\ u = \mathrm{VaR}_{(\Phi^{\pi})^{-1}\left(\frac{1}{1+\theta}\right)}(X) \\ (1+\theta)(\pi(X) - \pi(X\wedge u)) - \left(\mathrm{VaR}_{\alpha_1}(X) - u\right) = 0 \\ \begin{cases} u - (1+\theta)\,\pi(X\wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)(\pi(X) - \pi(X\wedge u))\right) \le b \end{cases} \end{cases}.$$

If $\lambda = 1$, then the second line of the optimality condition implies that $\mathrm{VaR}_{\alpha_2}(X) = \mathrm{VaR}_{\alpha_1}(X)$. Therefore, this case occurs if $\alpha_1 = \alpha_2$.

$$\begin{cases} u < \mathrm{VaR}_{\alpha_1}(X) \\ \begin{cases} u - (1+\theta)\,\pi(X \wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_1}(X) - u - (1+\theta)\left(\pi(X) - \pi(X \wedge u)\right)\right) = b \end{cases} \end{cases}.$$

If $\lambda \neq 0$ and $\lambda \neq 1$, then δ can be solved from the first equation. After substituting the second equation, λ could be found.

$$\begin{cases} (\Phi^\pi)^{-1}\left(\frac{1}{1+\theta}\right) < \alpha_1 \\ u = \mathrm{VaR}_{(\Phi^\pi)^{-1}\left(\frac{1}{1+\theta}\right)}(X) \\ \begin{cases} u - (1+\theta)\,\pi(X \wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)\left(\pi(X) - \pi(X \wedge u)\right)\right) = b \end{cases} \end{cases}.$$

Case 2. $\alpha_1 \leq \delta < \alpha_2$. We have

$$\begin{cases} (1-\beta)\left((1+\theta)\,\Phi^\pi(\delta)(1-\lambda) + \lambda\right) = 0 \\ \begin{cases} (1+\theta)\left(\pi(X) - \pi(X \wedge u)\right)(1-\lambda) \\ +\lambda\left(\mathrm{VaR}_{\alpha_2}(X) - u\right) = 0 \end{cases} \\ \begin{cases} u - (1+\theta)\,\pi(X \wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)\left(\pi(X) - \pi(X \wedge u)\right)\right) \leq b \end{cases} \\ \begin{cases} \lambda\left(u - (1+\theta)\,\pi(X \wedge u)\right. \\ \left. +\ \beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)\left(\pi(X) - \pi(X \wedge u)\right)\right) - b\right) = 0 \end{cases} \end{cases}.$$

If $0 \leq \lambda < 1$, then the first line implies that $\Phi^\pi(\delta) \leq 0$, which contradicts $\alpha \leq \delta$. If $\lambda = 1$, then the first line implies that $1 = 0$. If $\lambda > 1$, then λ could be obtained from the first line according to the choice of u. Therefore,

$$\begin{cases} \mathrm{VaR}_{\alpha_1}(X) \leq u < \mathrm{VaR}_{\alpha_2}(X) \\ \begin{cases} u - (1+\theta)\,\pi(X \wedge u) \\ +\beta\left(\mathrm{VaR}_{\alpha_2}(X) - u - (1+\theta)\left(\pi(X) - \pi(X \wedge u)\right)\right) = b. \end{cases} \end{cases}$$

Case 3. $\alpha_2 \leq \delta$. We have

$$\begin{cases} (1-\beta)(1+\theta)\,\Phi^{\pi}(\delta)(1-\lambda) = 0 \\ (1+\theta)(\pi(X) - \pi(X \wedge u))(1-\lambda) = 0 \\ u - (1+\theta)\,\pi(X \wedge u) - \beta(1+\theta)(\pi(X) - \pi(X \wedge u)) \leq b \\ \lambda(u - (1+\theta)\,\pi(X \wedge u) - \beta(1+\theta)(\pi(X) - \pi(X \wedge u)) - b) = 0 \end{cases}.$$

If $\lambda \neq 1$, then $\delta = 0$ holds from the first line, which contradicts $\alpha_2 \leq \delta$. If $\lambda = 1$, we have

$$\begin{cases} \mathrm{VaR}_{\alpha_2}(X) \leq u \\ u - (1+\theta)\,\pi(X \wedge u) - \beta(1+\theta)(\pi(X) - \pi(X \wedge u)) = b. \end{cases}$$

5.3 Discussion

In this chapter, we investigated the optimal reinsurance problems under three types of reciprocal frameworks: CORC, ROCC, and CROC. We first characterised the optimal solutions to these three problems in general set-ups by using the marginal indemnification function formulation and Lagrangian duality theory. An example of solving the CORC optimal problem was presented with three selected reinsurance policies: stop-loss, stop-loss after quota-share or quota-share after stop-loss. We showed how the optimal retention levels of each reinsurance type can be found and discussed some particular interesting cases.

Chapter 6

Conclusion

In this thesis, we focus on hedging against risk to meet minimum capital reserve requirement for insurance company and design reinsurance strategies from different perspectives including shareholders and reinsurer.

Chapter 3 is devoted to develop a hedging problem for the minimum capital requirement. We demonstrate the steps to construct a problem of finding the minimum cost risky position to avoid insolvency of the insurance company. By discussing the properties of risk measures in general form, we introduce the generalised minimum capital (GMC). An example is given with selected cumulative risk measure and cumulative pricing rule.

Buying reinsurance contract is a common approach of hedging against risk by cedent. Reinsurance policy design is popular in literature but the majority of the existing research on this topic is within the sector of the insurance companies. In reality, as the contract seller, the reinsurance companies play a more important role. The motivation for this thesis is therefore to provide a view from the reinsurers' perspective. We analyse the optimal reinsurance problems in our research from two aspects: dividend maximisation in Chapter 4 and risk minimisation in Chapter 5. The duality with the Lagrangian functions is used in both chapters and the basic concepts are reviewed in Section 2.4.

To achieve the aim that the insolvency risks are maintained under a tolerable level, the distortion risk measures (DRMs) are used in reinsurance optimisation throughout this thesis. As the advantage of the DRM, the distortion premium principle (DPP) is generated similarly to calculate the insurance and reinsurance premium. This enables our analysis under the DRM and DPP to be mathematically consistent. The core techniques used in this thesis are the properties of the DRM along with the marginal indemnity function method introduced by Assa (2015a) and developed by Zhuang et al. (2016). The MIF based formulation is under the assumption that both the ceding and reinsurance companies are sensitive to the change in total losses which rules out the risk of moral hazard in the actuarial area (see Assumption 1 and Section 2.3).

In Chapter 4, a discrete time surplus model of the ceding company is constructed and modified. In order to apply the DRM and DPP to this dynamic optimization framework and fill the blank in the literature, we propose the dynamic DRM and DPP in Section 4.1 which are generated from the well-defined dynamic VaR. We assume that the total claims in each period are independent in this chapter. As a result, Proposition 4.2.1 and Lemma 4.2.2 are proven to solve the technical issues caused by the dynamic DRM. The aim of this chapter is to find the optimal reinsurance contracts to maximize the expectation of the discounted and cumulative dividends paid to shareholders. For this purpose, we consider the objective subject to three sets of constraints formed by the budget constraint, the solvency condition and the dividend policy property. From the MIF method, the MIF is often proven to be an indicator function which implies a multilayer reinsurance policy including the simplest example as the stop-loss reinsurance contract. The optimal reinsurance contracts obtained in Theorems 4.3.1 and 4.3.3 satisfy the property as a multilayer reinsurance agreement.

The study in Chapter 5 is investigated by introducing three optimal problems with different objectives and constraints. In order to make the problems manageable, the risks of one party (insurer, reinsurer or both) are minimised while the risks of another one are assumed to be under an acceptable level. Similar to the results in Chapter 4, the optimal results contain the indicator function as the MIF thus the form of the multilayer reinsurance contracts are satisfied in Theorem 5.2.5 in the general case. Particular cases are discussed in Section 5.2.2 with stop-loss, stop-loss after quota-share and quota-share after stop-loss reinsurance policies. In Example 5.2.8, VaR is selected to estimate both insurance and reinsurance risks and the discussions are demonstrated for the set-up of minimising the risk of the cedant and controlling the risk of the reinsurer.

The hedging problem set up in Chapter 3 can be discussed under different risk measure and pricing rules. The area regarding the reinsurance policy optimisation also has huge potential. As extension of the study in Chapter 4, the prospective research of applying the dynamic DRM and DPP on the framework with correlated claims is closer to the reality. The applications of the outcomes in Chapter 4 with other specific DRMs may result in other interesting findings. With the development of the reinsurance design, the future popular policies could be considered into the optimal problems introduced in Chapter 5. Moreover, the MIF method was introduced in recent years which could be applied to a wide range of the existing setups in the literature.

Appendix A

Major Correction Report

By the requirement from both examiners, the thesis of first submission has been edited following the suggestions from Viva. Additionally, a new chapter is added and one paper has been submitted to journal of Insurance: Mathematics and Economics.

A.1 Corrections in thesis

The core corrections I have done for the thesis are listed below. Page numbers follow the first submission version.

1. Correct some typos and notations including changing "corollaryresponding" to "corresponding", missing subscripts.

2. Add the control variables under the $\min, \max, \sup$ and $\inf$.

3. Add comma to the notation of joint density, i.e., $f_{X,Y}$.

4. Change all the parentheses to square brackets after expectations, $\mathbb{E}[.]$, to be more consistent.

5. All the MIF method are changed to MIF formulation. I checked the literature citing the work of Zhuang et al. (2016) and realised the confusion when using the term MIF method. It is referred to as a MIF-based formulation method.

6. Edit with consistency such as the spelling style in British English, the double slash in text, "set-up" as noun while "set up: for verb and etc.
7. Chapter 1:reference papers are added.
8. P.11: $\{\mathcal{F}_t\}_{t\in\{\mathbb{N}\}}$ is changed to $\{\mathcal{F}_t\}_{t\in\{\mathbb{N}\cup\infty\}}$.
9. P.14: The whole Section 2.1.1 is edited.
10. P.17: Add the definition of VaR and the inverse distribution function form. Edit the Section 1.1 related to VaR introduced in the beginning of the section.
11. P.18: Remark 2.2.4 about cash invariance of a profit variable is updated.
12. P.19: Note that Section 2.1.1 shows that the claims covered by reinsurer is denoted by a continuous function working on the total claims for all four reinsurance types considered in this thesis. Under this stronger condition, the Proposition 2.3.2 could be proven more easily by the one in the first submission.
13. P.20: Move the definition of CVaR to the beginning.
14. P.21: The dynamic VaR was moved and edited to Section 4.1 in Chapter 4. The Proposition 4.1.6 was also moved. Add the note that Z_t is not dynamic and the subscript only indicates that it is in $L^0(\mathcal{F}_t)$.
15. P.23: In Assumption 1, the "the total" is added before the last "losses".
16. P.23: Found the reference of the Proposition 2.3.2 Add condition of left-continuous and update the proof of Proposition 2.3.2 with the new reference papers.
17. P.26: Edit the Section 2.4.
18. P.33: Proposition 4.2.1 change $\mathcal{F}_{t+i}$ to σ_{t+i}- measurable.
19. P.34: Proof of Proposition 4.2.1 is edited.
20. P.35: Rephrase the whole Lemma 4.2.3 and change the pdf of X_2 to $f_{X_2}(x) = \frac{\int_{-\infty}^{0} y f_{X,Y}(x,y)dy}{\mathbb{E}[-Y^-]}$, because $\mathbb{E}[-Y^-] = \left(\int_0^\infty \int_{-\infty}^{0} y f_{X,Y}(x,y)\,dydx\right)$.

21. P.42: Update the Hilbert space notations.

22. P.46: Explain why $\min(\mathbb{E}[]) = \mathbb{E}[\min()]$.

23. The proofreading is done for Chapter 5 by Elsevier.

24. Chapter 4 has been edited along with the submitted paper and the comments of the proofreading of the paper are also taken for editing it.

A.2 Paper submission

The paper titled 'Dynamic Set-up for Designing Optimal Reinsurance Contracts' has been done proofreading by Elsevier and submitted to the journal of Insurance: Mathematics and Economics (current status: "Under Review"). It is also submitted to SSRN (url to abstract page: http://ssrn.com/abstract=3420090).

A.3 New Chapter

Chapter 3 titled 'From Ruin Theory to a Hedging Problem' has been added. The concept studied in this chapter is the first topic that my supervisor, Dr. Hirbod Assa, suggested for my PhD research which is consistent to my PhD application proposal. I researched this topic of the hedging insolvency in my first one and half a year. It is a working in process paper finished in the past year of major correction. I didn't included it in the first submission because it is not related to reinsurance and distortion risk measure. However, it is related to risk measure in general and reinsurance is a common way used as hedging against risk by insurance company so it is added as the new chapter required by examiners. The proofreading has been done for this chapter by Elsevier. The literature review of hedging approach is added as Section 1.2.

1. Add the reference into literature in Chapter 1.

2. Add into abstract

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