The Rotation of The Earth

a thesis for the degree of Ph.D. presented to
the University of Liverpool

by

Andrew Young

May 1959

C O N T E N T S

PREFACE

The main work contained in this thesis is to be found in the published papers attached as appendices. These are concerned with the determination of the angular momentum vector of the Earth and the explanation of changes in this vector. The variation of latitude and the secular variation in the axis of rotation are two aspects of the problem of the position of the vector in space and in the Earth, and the seasonal and secular variations in the rate of rotation are aspects of the problem of the magnitude of the vector.

A short historical introduction is given and also a commentary on the papers. These are not intended to do more than place the papers in the context of the work done by others on the subject and to indicate the views of other authors where this seems to be advisable. Some outstanding problems and possible lines for further investigation are suggested.

All the papers contain material which was novel at the time of publication. In the two papers written with A.M.Walker the very closest collaboration took place since the statistical work depended so much on the observational problems. It is difficult to separate our individual contributions, but in general it can be said that Walker did the Statistical development and I was responsible for formulating the problem and interpreting the results; in addition, I did all the computation.

The papers written with Walker have made a considerable impact on the work of other geophysicists.

I give in the commentary a hitherto unpublished explanation of how the effects of glacial eustasy may explain a discrepancy between the theoretical and the observed ratios of the secular decelerations of the Moon and the Sun.

HISTORICAL INTRODUCTION

The variation of latitude

A rigid body acted upon by no forces can rotate permanently about any of its principal axes of inertia, the motion being stable if it is about either the axes of greatest or least moments. In the special case of a body whose two least moments are equal, the rotation is stable about the axis of greatest moment. If, in this case, the moments of inertia are (A, A, C), the axis of rotation completes its revolution within the body in a period A/(C-A) times the period of rotation. This wobble is, of course, superposed on any motion due to external forces.

In the case of the Earth, A/(C-A) is accurately known from precession and is about 305. Astronomers therefore sought for evidence of such a period in the rotation of the Earth since Euler first enunciated the theory. By definition, the latitude of an astronomical observatory is the mean of the altitudes of a circumpolar star when it crosses the meridian above and below the pole so the colatitude of the observatory is the angle between local gravity and the Earth's instantaneous axis of rotation. Any motion of this axis therefore causes changes in the latitudes of all observatories - hence the term 'variation of latitude'.

The variation was not however detected until the end of the 19th century. Chandler then found a very small variation which unlike the simple Eulerian variation, had two components with periods of one year and fourteen months respectively, both of which had amplitudes of the order of 0.1". These became known as the annual and the Chandlerian periods.

Newcomb pointed out that the non-rigidity of the Earth causes a lengthening of the Eulerian period and so the Chandlerian period may be identified with the lengthened Eulerian period. The elastic properties of the Earth are described by three Love's numbers, h k and l, and it was shown by Love and Larmor that the lengthening of the Eulerian period provides a means of estimating k to an accuracy of about one part in 300. This is one of the only observational methods of getting any one of the three Love's numbers by itself so that accurate determination of the Chandler period is of prime importance to geophysics.

The annual period was explained as being due to a forced motion, which was looked for in changes in the products of inertia of the Earth not due to elastic deformation. The chief cause of such changes was found in the annual change in the distribution of air over the Earth's surface.

Full historical accounts can be found in Lambert (1931) and Melchior (1954). An extended theoretical account is to be found in Jeffreys' "The Earth" (4th edition 1959, chapter 7).

Periodic changes in the rate of rotation of the Earth

The interval of time during which the Earth makes one complete revolution about its axis is defined as one day, which was formerly the standard of astronomical time-keeping by which clocks were regulated. Improvements made in the accuracy of artificial time-keepers - crystal and atomic clocks - which are now accurate to 1 part in 10^9 or better have shown that in fact the Earth's rate of rotation is not uniform so the day is not of fixed length.

Using batteries of very accurate pendulums, Stoyko had in 1937 detected an annual variation in the length of the day. The more accurate clocks now available show that in addition to an annual variation there are irregular fluctuations. Van den Dungen, Cox and van Mieghem (1949) and Munk and Miller (1950) found that the annual movement of air masses caused changes in the moment of inertia of the atmosphere sufficient to explain 15 - 20% of the observed annual changes in the rate of rotation. Mintz and Munk (1951, 1954) found a better explanation in the seasonal changes of the angular momentum of the air masses relative to the solid Earth.

Secular changes in the rate of rotation

A secular deceleration in the rate of rotation of the Earth was historically the first of all these various phenomena to be discovered. It was suspected by Halley in 1693 and in 1749 Dunthorne showed that the Moon was moving in advance of its calculated position when records of ancient and modern eclipses were compared. Many investigators - notably Fotheringham, Schoch, de Sitter - have confirmed this. Spencer Jones (1939) showed that the accelerations of various bodies in the solar system were in the ratios necessary to prove that the accelerations are in fact the reflections of the Earth's deceleration. Heiskanen (1921), Taylor (1919) and Jeffreys have ascribed the deceleration to the effect of tidal friction generated principally in shallow seas. Jeffreys gives a full account in "The Earth" (4th edition Chapter8).

Secular Changes in the Axis of rotation

The fourth problem concerns the position of the axis of rotation over long periods of time. The subject has been highly controversial, but in the past decade palaeomagnetic evidence has pointed to the movement of the pole at least relative to the surface of the Earth. Various authors e.g. Bondi and Gold (1956) Jeffreys (1956) have studied the motion of a crust over a liquid core, which introduces the damping of the variation of latitude. Munk (1958) has shown that with values of damping found in Paper 6 the motion of the Earth's crust over the core, is "embarassing" - the difficulty is not to explain the secular change but to explain its smallness ! This subject is in its infancy.

Fluctuations

Secular changes may in fact include long term fluctuations over periods which are long compared with the intervals for which observations are available. I considered such fluctuations in relation to the secular decelerations of the Earth in Paper 3.

All the phenomena have in common the fact that the observable effects are small - the observations are made very near the limits of the instruments and observational errors are bound to be large. Moreover, it is doubtful if any of the processes are smooth, so all of them must be treated as stochastic processes and refined statistical techniques are required to do any effective analysis of the observations. Despite the years of effort put into the study of these phenomena, finality has not yet been reached in the understanding of any of them.

COMMENTS ON PUBLISHED PAPERS

The variation of latitude

Paper 2 is the result of work which Professor E. Findlay Freundlich suggested I might undertake. It is generally accepted that the annual term in the variation of latitude is due to seasonal changes in the distribution of atmospheric masses over the Earth's surface, but previous authors - Spitaler (1897), Jeffreys (1916) Schweydar (1919) Rosenhead (1929) - who had calculated the products of inertia of the atmosphere had all done so with more or less unsatisfactory data. Rosenhead, for example, used isobar charts compiled by Napier Shaw and had to apply corrections for the fact that these charts showed sea-level values of the pressure. The isobars are drawn through "spot diagrams" and subsequent reading and correction for height may bear no relation to the height of the stations from which the actual lines were deduced. Moreover, as Napier Shaw himself said, isobar maps are often fictional, the isobars beginning and ending at the whim of the compiler especially at the coasts of continents. Schweydar used Gorczinski's charts and applied no recorrections for height. Similar objections can be raised to other works. Schweydar in fact recognized these defects and suggested that station-level values of the pressure should be used to calculate the products of inertia for a number of years.

In Paper 2 therefore, I used station level data for each of some 340 land stations in Reseau Mondial, for which a continuous six-year sequence of pressure readings was available, (1925-1930 inclusive)

It is to be noted from the results given in Table II that the product F is more regular and has greater amplitude than G - perhaps due to the fact that F is more influenced by the very large regular changes in the Central Asian pressure system. But in both F and G there are wide variations from year to year.

The work does confirm the accepted theory. I think it is proper to point out, however, that I would not now present so elaborate an analysis of the variation of F and G, nor would I so confidently integrate the differential equations of the motion after inserting the fitted formulae for the products of inertia.

After this work, I was convinced that classical methods of Fourier Analysis are not applicable to the observations of the variation of latitude so with A.M. Walker I undertook the work leading to Papers 5 and 6. These give a thorough examination of the whole question of the analysis of a two-dimensional time series in which the process is subject to auto-correlated random disturbances as well as observational error.

The investigation of bias with which Paper 6 ends is particularly important because it shows evidence of particular kinds of systematic errors which in my view make it impossible adequately to estimate the damping factor. The estimate of the Chandler period is probably as good as can be obtained from the data.

The method of these papers is open to some objection. I believe that Paper 2 indicates such variability of F and G from year to year that the forcing terms should themselves be treated as auto-regressive series. This might affect the decision reached in Paper 5 that ordinary Fourier Analysis is suitable for removing the annual term and if so, the imperfect removal of this term would bias the data from which we estimated the parameters of the Chandlerian motion. This is a possible problem for future investigations. Further, as Fellgett (1959, unpublished paper) points out, the use of the correlogram necessarily imposed on us a choice of lags so that we did not use all the information. Fellgett reverts to the use of the periodogram and attempts to avoid all the innate difficulties of this use by a powerful technique of determining from the whole periodogram the parameters of the motion. He claims far more consistent results than we got; in particular he finds no real evidence of bias. He agrees substantially with our estimate of the Chandler period and also concludes that it will be several hundred years before either his or our method will accurately determine the damping.

Melchior (1958) has objected to the whole basis of the method on the ground, substantially, that the results are good estimates of the parameters of a hypothetical mathematical model which is not however a good model of the physical system. My reply is simply this: If damping did not exist our model would include the possibility and produce the answer $k = o$ within statistical limits. Imperfect as our own estimates are, all are statistically different from zero.

Russian astronomers at the IAU meeting in Moscow in 1958 strongly supported our view against Melchior and claimed that when our method was used with more homogeneous Soviet data, good results were obtained - the damping had a relaxation time of about 100 years compared with the ranges 10 - 30 years found by us and also by Fellgett.

For the purposes of the Time-Service, which requires values of the variation of latitude more quickly than the I.L.S. at present supplies them A.Stoyko and N.Stoyko (1958) have proposed the adoption of a formula. They say -

En utilisant les équations du mouvement du pôle de rotation données par Helmert et l'hypothèse de Newcomb qui admet qu'en cas de déformation de la Terre le pôle d'inertie se déplace dans la direction du pôle de rotation, on peut representer les coordonées du pôle instanté par les formules suivantes :

$$
\begin{aligned}
x &= 0'',088 \cos(18^\circ t + 112^\circ) + C \cos(15^\circ t + c) \\
y &= 0'',075 \sin(18^\circ t + 112^\circ) - C \sin(15^\circ t + c)
\end{aligned}
\tag{8}
$$

où t est exprimé en vintièmes d'année comme unité.

Ainsi, le mouvement du pôle de rotation de la Terre peut être représenté par la somme de deux composantes harmoniques : 1° une composante annuelle de phase et amplitude constantes, et 2° une composante ayant pour période environ 1,2 d'année, de phase et amplitude variables. L'expérience montre que pendant un intervalle ne dépassant pas 1,2 d'année on peut admettre avec une précision suffisante pour les buts pratiques que l'amplitude C et la phase c de la période chandlérienne sont constantes.

Ces propriétés donnent la possibilité de calculer les coordonnées du pôle instantané d'après un seul service de latitude.

I have actually compiled the amplitudes and phases of a motion with the two periods given using successive runs of 24 successive pairs of values of l and m (Jan.1900 - Dec.1901, Feb.1900 - Jan. 1902, March 1900 - Feb. 1902 and so on). The results which are too voluminous to give in detail ~~but~~ clearly show that Stoyko's statements are just not borne out. The amplitudes and phases fluctuate wildly and are of no use for rapid interpolation.

Variation of the length of the day

When I was computing F and G for Paper 2, I realised that the changing distribution of air-masses would also cause an annual change in C, the moment of inertia, and hence from conservation of angular momentum one should expect a change in the length of the day of annual period. Before completing Paper 2 therefore, I computed the values of C and also A and B the other moments of inertia and published the results in Paper 1.

Clearly this is a contributory cause of the observed variation of the length of the day, but as it has now been shown, the angular momentum associated with the air circulation which builds up and disperses the air masses is the major cause.

Paper 1 revealed a conflict of opinion with van den Dungen, Cox and van Mieghem (1949) who did not make the same assumption as I did about the hydrostatic compensation over the oceans. They (1952) have challenged this assumption but I do not see how the challenge can really be sustained. The effect is important because it reverses the sign of C - and incidentally of F and G. The internal evidence of the agreement of cause and effect in the case of the variation of latitude seems to be satisfactory evidence of the correctness of the assumption of hydrostatic compensation.

In Paper 1 I pointed out for the first time that the corrections made in different ways by Jeffreys and Schweydar for the elastic yielding of the Earth's surface under the changing atmospheric load are both wrong. Jeffreys (The Earth 4th edition p.227) accepts this criticism which is essentially that neither way of correcting ensures conservation of mass. No-one has yet produced a satisfactory method of correction.

Motion of a rigid body with a mobile envelope

Having established in Papers 1 and 2 the orders of magnitude of the moments and products of inertia of the atmosphere, I used them in Paper 4 with Munk's estimates of the relative angular momentum of the atmosphere to consider the rotation of the Earth as if it were a rigid body with a mobile envelope. I found that variation of latitude and variation of the length of the day can be treated separately and cleared up certain doubts about the applicability of the simple Eulerian theory expressed by Jeffreys. In this paper the elasticity of the Earth is ignored so that a proviso should be added that the theory given is true only if the solid Earth does not change its shape. To the orders of magnitudes and the time-scales concerned, I believe that this is near enough to the truth for the purpose of the present theory.

Long term fluctuations in the rate of rotation of the Earth

The effect of tidal friction is to slow down the Earth and thus to lengthen the day. This is seen in an apparent acceleration of the Moon, Sun and planets. Suppose the Moon moves among the stars with mean angular velocity n, and the angular speed of the Earth is ω. Then if tidal interaction between the Earth and the Moon causes a deceleration of the Earth there will also be a change in n and the apparent acceleration of the Moon will be ν_1 given by

$$\nu_1 = \frac{dn}{dt} - \frac{n}{\omega}\frac{d\omega}{dt}$$

If however, the cause of the Earth's changed angular speed is not due to any dynamical effect attributable to the Moon, the apparent acceleration ν_o will be given by

$$\nu_o = \frac{-n}{\omega}\frac{d\omega}{dt}$$

since the Moon's motion will be unaffected. The suffixes o and 1 distinguish the "unlinked" acceleration from the "linked".

Similarly for the Sun the "linked" and "unlinked" apparent accelerations ν_1' and ν_0' will be given by

$$\nu_1' = \frac{dn^1}{dt} - \frac{n^1}{\omega}\frac{d\omega}{dt}$$

and

$$\nu_0' = \frac{-n^1}{\omega}\frac{d\omega}{dt}$$

where n^1 is the mean angular velocity of the Sun. The total apparent accelerations will be

$$\nu = \nu_0 + \nu_1 \qquad \text{and} \qquad \nu' = \nu_0' + \nu_1' \qquad \text{respectively}$$

De Sitter calculated ν and ν' as

$$\tfrac{1}{2}\nu = (5.22 \pm 0.45)''/\text{century}^2$$

$$\tfrac{1}{2}\nu' = (1.80 \pm 0.24)''/\text{century}^2$$

giving a ratio $\nu/\nu' = 2.9$. The observations covered the last 2,000 years.

Spencer Jones obtained

$$\tfrac{1}{2}\nu = (3.11 \pm 0.85)''/\text{century}^2$$

$$\tfrac{1}{2}\nu' = (1.07 \pm 0.09)''/\text{century}^2$$

from observations covering the last 200 years. He assumed the ratio ν/ν' to be the same as de Sitter's.

Brouwer (1952) obtained

$$\tfrac{1}{2}\nu = (2.20 \pm 3.8)''/\text{century}^2$$

$$\tfrac{1}{2}\nu' = (1.01\pm0.28)''/\text{century}^2$$

but this method is criticised by Murray (1957) who found that the present acceleration of the Moon is at least twice as great as the average acceleration since the time of Hipparchus.

Spencer Jones states that his result clearly indicates a reduction in tidal friction. In Paper 3 I challenged this view and showed that the rise and fall in sea level due to changes in average ice-coverage causes large fluctuations in the rate of rotation - in fact plausible glacial evidence gives average values of varying from -130" to + 150"/century^2. These rates are very sensitive to changes in climatic conditions, but the fluctuations take place over periods of centuries and would be measured as pseudo-secular effects. I wish to emphasise that my estimates in Paper 3 are estimates of what I now call ν_0 and that De Sitter and others estimate not ν_0 but [illegible] which includes ν_0 and ν_1

Jeffreys (The Earth 4th edition Chapter 8) gives a dynamical theory from which he obtains the theoretical ratios

$$\frac{\nu_1}{\nu_1'} = 6.3 \text{ or } 7.2 = a \text{ say}$$

according to whether the friction is taken as linear or non linear. (Jeffreys of course does not use the suffix 1). He remarks succinctly "The observed ratio is much less". So far as I know the discrepancy has never previously been explained.

However, the distinction between the linked and unlinked accelerations affords a possible explanation, for the ratio of the unlinked accelerations is

$$\frac{\nu_0}{\nu_0'} = \frac{n}{n_1} = 13.3 = b \text{ say}$$

Since $\quad \nu' = \nu_0' + \nu_1'$

and $\quad \nu = \nu_0 + \nu_1$

$$= a\nu_0' + b\,\nu_1'$$

it follows that

$$\nu_0' = (\nu - b\,\nu')/(a-b)$$

$$\nu_1' = (\nu - a\,\nu')/(b-a)$$

so if ν and ν' are known $\nu_0', \nu_1', \nu_0, \nu_1$ can be determined. The following results are obtained from the different authors' estimates of ν and ν':-

	a = 6.3		a = 7.2	
	ν_0 (unit	ν_1 seconds of	arc/century)^2 ν_0	ν_1
De Sitter	−23.2±7.5	+33.7±6.6	−33.8±9.5	+44.2±8.6
Spencer Jones	−13.8±5.4	+20.0±3.7	−20.0±6.5	+26.3±4.8
Brouwer	−15.8±8.2	+20.2±7.4	−22.1±10.5	+26.5±9.7

Jeffreys estimated a deceleration of about $\nu_1 = 7''$/century^2 due to tidal friction in shallow seas. He says (The Earth 4th edition p.244) that the estimate of tidal friction in the Bering Sea might be wrong by half its amount. Jeffreys concluded that there is no reason to suppose that tidal friction is not sufficient to account for the secular deceleration but the above rough calculations point to the opposite conclusions since the values of ν_0 are within the bounds of possible eustatic and isostatic effects given in Paper 3.

Murray (1957) distinguishes ν_0 and ν_1 (which he calls q and s). In a note on this paper, Jeffreys (The Earth 4th edition p.397) gives values of q and s in tolerable agreement with those I obtain from Spencer Jones' and Brouwer's' results and concludes that Murray's results introduce new difficulties with no obvious explanation.

However, there are various possible explanations of the fact that ν has changed considerably in recent centuries as Spencer Jones and Murray contend. If the above theory is accepted then it is necessary to find a further cause of "linked" deceleration - possibly in bodily friction. Then either 1) tidal friction has changed, 2) the other cause has changed or 3) the change has taken place in eustatic or isostatic effects. To justify a belief in the first cause one must bring forward evidence of considerable changes in conditions in shallow seas; this I find quite unacceptable as the general configurations of shallow seas have scarcely changed in 2,000 years and the effect of the rise and fall in sea-level on tidal friction is negligible. Sea-level however immediately responds to climatic change and does have a marked influence on ν_0. There is ample qualititive evidence of climatic changes to suggest that it is ν_0 that changes rather than ν_1 – for example, between 1900 and 1945 the condition in Spitzbergen harbours changed from being ice-bound for eight months to four months in every year. Such evidence is widespread.

As Jeffreys points out, the melting of polar ice-caps consequent on a climatic amelioration would increase the moment of inertia and thus produce an immediate effect in the wrong direction. But, as I pointed out in Paper 3, after removal of the ice-cap, there is a far slower period of isostatic compensation during which the sea-floor is depressed under the new load of water and the land rises after the removal of the load of ice. Ultimately when the compensation is complete the rate of rotation is the same as it was when compensation was complete during the Ice Age. Over the last 2,000 years on the average, it is the isostatic compensation which has had the dominating effect (thus explaining the sign of ν_0). The change in the last few hundred years has been reduced because of the eustatic effect of climatic fluctuations during that period.

It is perhaps apposite at this point to suggest other effects of climatic fluctuation. In the last few decades there has been an eastward shift of the isobars of the Eurasian continental land-mass. Whereas formerly the wind system of northwestern Europe was circumpolar, winds are now entering the Arctic basin which is warming up. Also, the ocean currents are changing direction - in the case of the Gulf Stream this happened on several occasions in historical times e.g. the Dutch Wars have been attributed to the fact that herring shoals followed the Stream from the East to the West coastlines of the North Sea (Tyne 1952) - and the general effect is to change ice conditions and the general air circulation in the Arctic. These must have geodynamical consequences:-

a) the change in the location of air masses in Central Asia must affect F and G and hence the amplitude and phase of the forced motion in the variation of latitude,

b) the change from zonal to meriodonal wind systems must tend to reduce the component of the relative angular momentum of the atmosphere which causes variation in the length of the day.

c) changed average position of air masses must lead to a new position of the mean pole and hence an apparent (small) polar drift.

d) changing ocean currents also affect the length of the day.

There is evidently here a wide field of research which may be worth investigating.

....

REFERENCES

Brouwer, D. - Astron, J., 57. 125 - 146.

Bondi, H, and Gold, T. 1956 - M.N.R.A.S. 115. 14 - 46.

van den Dungen, F.H., Cox, J.F. and van Mieghem, J.

1949 - Bull.Acad.Roy.Belg.Cl.Sci., 35 642 - 655.

1952 - Bull.Acad.Roy.Belg.Cl.Sci., 38. 1 - 4.

Heiskanen, W. 1921 - Annal.Acad.Sci.Fenn.18A. 1 - 84.

Jeffreys, H. 1916 - M.N.R.A.S. 76. 499 - 525.

1956 - M.N.R.A.S. 116. 361 - 364.

1959 - "The Earth" 4th edition, Cambridge University Press.

Lambert, W.D. 1931 - Bull.U.S. Nat.Res.Coun. No.78. Chapter 16.

Melchior, P. 1954 - Observatoire Royal de Belgique. Monograph 3.

1958 - Mimeographed contribution to Symposium on the Rotation of the Earth. Xth General Assembly of I.A.U., Moscow.

Mintz, Y. and Munk, W.H. 1951 - Tellus.3. 117 - 121.

1954 - M.N.G.S. 6. 566 - 578.

Munk, W.H. 1958 - Geophysica.6. 335 - 355.

Munk, W.H. and Miller, R.L. 1950 - Tellus.2. 93 - 101.

Murray, C.A. 1957 - M.N.R.A.S. 117. 478 - 482.

Rosenhead, L. 1929 - M.N.G.S. 2. 171 - 196.

Spitaler, R. 1897 - Denkschr.Akad.Wiss.Wien.64. 633 - 642.

Schweydar, W. 1919 - Preuss. Akad. Wiss.Ber. 20. 357 - 366.

Spencer Jones, H. 1939 - M.N.R.A.S. 99. 541 - 558.

Stoyko, A. and Stoyko, N. 1958 - Bull.Astron.21. 205 - 259.

Taylor, G.I. 1919 - Philos,Trans. A 220. 1 - 33.

Toyne, S.M. 1952 - History Today. 2. 686 - 695.

(1)

Académie royale de Belgique — Koninklijke Belgische Academie

BULLETIN DE LA CLASSE DES SCIENCES — MEDEDELINGEN VAN DE KLASSE DER WETENSCHAPPEN

5e Série. — Tome XXXVII — 5de Reeks. — Boek XXXVII

1951

EXTRAIT — UITTREKSEL

Two causes contributing to the periodic fluctuation in the length of the day

PAR

Andrew YOUNG

BRUXELLES
PALAIS DES ACADÉMIES
RUE DUCALE, 1

BRUSSEL
PALEIS DER ACADEMIËN
HERTOGELIJKESTRAAT, 1

1951

Extrait du *Bulletin de l'Académie royale de Belgique* (Classe des Sciences)
Séance du 1er août 1951.

ASTRONOMIE

Two causes contributing to the periodic fluctuation in the length of the day,

par Andrew YOUNG. (*)

Summary. — Variations in the moment of inertia of the Earth are calculated from atmospheric pressure variations, and the resulting variations in the length of the day are obtained.

The effect of polar waters released by differential seasonal melting is estimated.

1. By studying the performance of a number of clocks at Paris, Washington and Berlin during the years 1934-37, N. Stoyko (1,2) shewed that there exists a variation with an annual period in the speed of rotation of the Earth. These results were confirmed by the same author (3) from the study of clocks in the years 1946-47. H. F. Finch (4) found a result in close agreement with those of Stoyko by analysing the apparent performance of quartz crystal clocks used in the Greenwich time service between 1943 and 1949. The results show that if the conventional astronomical clock and a crystal clock keeping mean time are started together in February, the astronomical clock reaches a maximum of $0^s.06$ slow about the beginning of June and a maximum of about $0^s.06$ fast in October relative to the quartz clock.

2. According to W. Schweydar (5), atmospheric pressure changes explain quantitatively the variation of latitude which depends on the products of inertia of the Earth. The same cause must give rise to changes in the moment of inertia and hence to fluctuations in the length of the day. A number of other

(*) Présenté par MM. J. F. Cox et F. H. VAN DEN DUNGEN.

writers have already given estimates of this effect. F. H. van den Dungen, J. F. Cox and J. van Mieghem (6) estimated the change in the moment of inertia at opposite seasons and concluded that the effect only in part accounts for the observed seasonal fluctuation in the length of the day. A conflicting numerical estimate has been given by W. H. Munk and R. L. Miller (7).

More exact determination of the moment of inertia is possible, and in this paper the results of detailed computations are given. These are based on pressure data actually observed at a large number of weather stations during the years 1927-30 (the latest for which complete data were found to be readily available). By a method used by both Schweydar and H. Jeffreys (8) in variation of latitude calculations, monthly values of the changes in the moment of inertia are obtained and the fluctuations in the length of the day are deduced.

The effect of the redistribution of polar waters is also examined.

3. Let h be the variable part of the mass per unit area at the point on the Earth's surface at which the colatitude is θ and the longitude is λ. Let R be the radius of the Earth and let dS be the element of area. The variable contribution of this mass to the moment of inertia of the earth about the polar axis is

$$\Delta C = R^4 \int_0^{2\pi} \int_0^{\pi} h \sin^3 \theta d\theta d\lambda. \qquad (1)$$

Over the sea, it can be shown that h is independent of position, varying only with the time. Let its value on the sea be h_0. Since the total mass is constant,

$$\iint h dS = 0$$

over the surface, and so

$$h_0 \iint_{sea} \sin\theta \; d\theta d\lambda = - \iint_{land} h \sin\theta \; d\theta d\lambda.$$

But

$$\iint_{sea} \sin\,\theta d\theta d\lambda = 4\pi - \iint_{land} \sin\,\theta d\theta d\lambda \tag{2}$$

and

$$\iint_{sea} \sin^3\,\theta d\theta d\lambda = \frac{8}{3}\,\pi - \iint_{land} \sin^3\,\theta d\theta d\lambda. \tag{3}$$

Equations (1), (2) and (3) give

$$\Delta \mathrm{C} = \mathrm{R}^4 \left\{ -\,a_c \iint_{land} h \sin\,\theta d\theta d\lambda + \iint_{land} h \sin^3\,\theta d\theta d\lambda \right\} \tag{4}$$

where

$$a_c = \left\{ \frac{8}{3}\pi - \iint_{land} \sin^3\,\theta d\theta d\lambda \right\} \Big/ \left\{ 4\pi - \iint_{land} \sin\,\theta d\theta d\lambda \right\}. \tag{5}$$

If the atmospheric pressure excess at a given point be p millibars, then

$$h = 1.020\,p \text{ gms.} \tag{6}$$

A correction has to be made for elastic deformation of the Earth. In their variation of latitude calculations, Jeffreys and Schweydar differed in that the former applied this correction to the whole surface, whereas the latter applied it only to the land surface.

Using (4) and (6), the equations are then

$$\Delta \mathrm{C} = 1.020\,\mathrm{R}^4 k \left\{ -\,a_c \iint_{land} p \sin\,\theta d\theta d\lambda + \iint_{land} p \sin^3\,\theta d\theta d\lambda \right\} \tag{7}$$

according to Jeffreys and

$$\Delta \mathrm{C} = 1.020\,\mathrm{R}^4 \left\{ -\,a_c \iint_{land} p \sin\,\theta d\theta d\lambda + k \iint_{land} p \sin^3\,\theta d\theta d\lambda \right\} \tag{8}$$

according to Schweydar, k being the correction constant for elastic yielding.

Schweydar's formula appears to ignore the yielding of the ocean floor under the changing load above it, and so might be expected to be incorrect. That it is so can be seen by examining the other moments of inertia ΔA and ΔB which can be derived in the same way as ΔC. They involve constants α_a and α_b analagous to α_c given by (5). On adding $\alpha_a + \alpha_b + \alpha_c = 2$. Since mass is conserved, $\Delta A + \Delta B + \Delta C = 0$, and on adding the three formulae, it is found that this condition is satisfied for any value of k in Jeffreys' formula but only by $k = 1.0$ in Schweydar's. This corresponds to the uncorrected formula and so Schweydar's is certainly incorrect.

It is evident that only land values of p are required for the computation of ΔC.

4. We take the mean position of the Earth as that in which it rotates about the polar axis in 24 hours and with the atmosphere everywhere arranged so that its pressure at any point has its mean value P_0. In this position J, the moment of inertia about the axis of rotation, is equal to C. The other moments of inertia are A and B. In the disturbed position, the atmospheric pressure at any point is $P_0 + p$. The moments and products of inertia are $A + \Delta A$, $B + \Delta B$, $C + \Delta C$, ΔF, ΔG and ΔH.

Hence

$$J + \Delta J = (A + \Delta A)l^2 + (B + \Delta B)m^2 + C + \Delta C + 2m\,\Delta F + 2l\Delta G + 2lm\Delta H,$$

where $(l, m, 1)$ are the direction cosines of the axis of rotation.

It is found that all the variable parts of the moments and products of inertia are of the order 10^{35} gm. cm^2. Since A, B and C are of order 10^{45}, and l and m are of order 10^{-6}, it follows that this equation may be written

$$\Delta J = \Delta C + (Al^2 + Bm^2). \tag{9}$$

5. Atmospheric pressure values were taken from the tables in « Reseau Mondial » published yearly by H. M. Meteorological Office up to 1932. Owing to the number of Central Asian and South American values missing, it was not possible to use the tables for 1931 and 1932. The tables for 1927-1930 are complete and the calculations were made for these years.

In general « Reseau Mondial » lists two stations in each land area bounded by lines of latitude and longitude at 10° intervals.

At each station the mean pressures for each month and over the year are given, and for comparison the averages over a long period of years are added. In the majority of cases the values are given both at station level and reduced to sea level.

Stations were used for the present calculations provided complete series of pressure values at *station* level were available. In a very small number of cases, however, stations were used for which only sea level values were available. In all cases where this was done, the station was at a very low altitude and the correction required to revert the variable pressure value to station level was negligible. In such cases a station was used only if its inclusion was required on grounds of geographical isolation. For a similar reason, or where, as in North Africa, stations are widely separated, a station was occasionally included for which monthly values had to be interpolated. Interpolation was never done at any station for more than four months. Average values were taken from the maps in Napier Shaw's « Manual of Meteorology » vol. 2. (9) for the land in the South Polar Regions for which no results are given in « Reseau Mondial ». In all, the numbers of stations used were 341, 339, 337 and 331 in 1927, 1928, 1929 and 1930 respectively.

For each station, factors δs_r and δc_r were calculated in the following way. « Squares » bounded by lines of latitude and longitude at 5° intervals were grouped together into subareas around each station. For example, the four « squares » in the area between latitudes 60° and 70° N and longitudes 10° and 20° E contain two Reseau Mondial stations. — Trondheim (n° 239) and Bodo (n° 240). In the years considered, full ranges of pressure values for both these stations are available and so both were used, the two southern squares being allotted to Trondheim and the two northern ones to Bodo.

In each square the fraction μ of land was estimated from large scale maps and

$$\iint_{square} \sin\theta d\theta d\lambda$$

and

$$\iint_{square} \sin^3\theta d\theta d\lambda$$

were evaluated by actual integration. Then

$$\delta s_r = \Sigma\mu \iint_{square} \sin\,\theta d\theta d\lambda \tag{10}$$

and

$$\delta c_r = \Sigma\mu \iint_{square} \sin^3\,\theta d\theta d\lambda,$$

the summations extending over the sub-area allotted to the particular station. In the area cited above we find

$$\iint_{square} \sin\,\theta d\theta d\lambda$$

and

$$\iint_{square} \sin^3\,\theta d\theta d\lambda$$

to be $(0.04028)\,\frac{\pi}{36}$ and $(0.00864)\,\frac{\pi}{36}$ respectively in both southern squares, and μ is 0.95 and 0.68 in the south west and south east squares respectively. Hence for Trondheim to which the squares are allotted,

$$\delta s_{239} = (0.95 + 0.68) \times 0.04028 \times \frac{\pi}{36} = 0.0057,$$

$$\delta c_{239} = (0.95 + 0.68) \times 0.00864 \times \frac{\pi}{36} = 0.0012.$$

The factors calculated in this way were used to compute

$$\iint p \sin^3\,\theta d\theta d\lambda\,, \quad \iint p \sin\,\theta d\theta d\lambda\,, \quad \iint \sin^3\,\theta d\theta d\lambda$$

and

$$\iint \sin\,\theta d\theta d\lambda$$

over the land area by the approximations

$$\iint p \sin^3\,\theta d\theta d\lambda = \sum_r p_r \delta c_r \tag{11}$$

etc.

Writing equations (2), (3) and (5) in the approximate form

$$\iint_{sea} \sin\,\theta d\theta d\lambda = 4\pi - \sum_r \delta s_r \tag{2a}$$

$$\iint_{sea} \sin^3\,\theta d\theta d\lambda = \frac{8}{3}\pi - \sum_r \delta c_r \tag{3a}$$

and

$$\alpha_c = \left\{\frac{8}{3}\pi - \sum_r \delta c_r\right\} \Big/ \left\{4\pi - \sum_r \delta s_r\right\} \tag{5a}$$

equation (7) becomes

$$\Delta C = 1.020\ R^4\ k\ (-\alpha_c\ \Sigma p\delta s + \Sigma p\delta c) \tag{7a}$$

in which form it was used for the computations.

6. The computed values of $\Sigma p\delta s$ and $\Sigma p\delta c$ are given in the first two columns of Table I. The values of $4\pi - \Sigma\delta s$, $\frac{8}{3}\pi - \Sigma\delta c$ and α_c were found to be 8.9396, 6.1620 and 0.6893 respectively. The following values based on those given by Bartels (12) of the other constants were used.

$R = 6.371 \times 10^8$ cms.
$A = B = 8.050 \times 10^{44}$ gm. cm^2.
$C = 8.077 \times 10^{44}$ gm. cm^2.
$1.020\ R^4 = 1.6807 \times 10^{35}$.

L. Rosenhead (13) has shown that $k = 0.6$, and this value has been used here. The values of ΔC arc given in Table I.

Values of $Al^2 + Bm^2$ were calculated from the observed values of l and m given in H. Kimura's « Report on the work of the International Latitude Service » (14). The maximum value of the ratio of $Al^2 + Bm^2$ to ΔC during the period considered was about $\frac{1}{4}$ of 1 %. The effect of variation of latitude was therefore ignored in the calculation of the change in the length of the day, and for this purpose ΔJ was taken equal to ΔC.

7. If ω is the angular velocity of the earth, then

$$J\omega = \text{constant.} \tag{12}$$

Also, if D is the length of the day,

$$D\omega = \text{constant} \tag{13}$$

by conservation of angular momentum.
As the quantities vary,

$$\frac{\Delta J}{J} = -\frac{\Delta\omega}{\omega} = \frac{\Delta D}{D}. \tag{14}$$

Ignoring the variation of latitude term this gives

$$\Delta D = D \times \Delta C/C = 1.070 \times 10^{-40} \Delta C \text{ seconds.} \tag{15}$$

ΔD is, of course, the mean difference of the length of the day during the month.

As the mean values of ΔD thus obtained for each month of the four year period can easily be inferred from the values of ΔC given in Table I, they are not given in detail here. A graph of the average monthly means calculated from the four yearly sets of values for $k = 0.6$ is given in Fig. 1. In the same figure there is also given the graph of the observed values for 1934-37 according to Stoyko. (1) The scale of the calculated curve is 100 times that of the observed curve. It should be noted that ΔD used throughout this paper is the $-\Delta t_r$ of Stoyko's work. (3)

TABLE I.

Mean monthly variation in moment of inertia computed from atmospheric data.

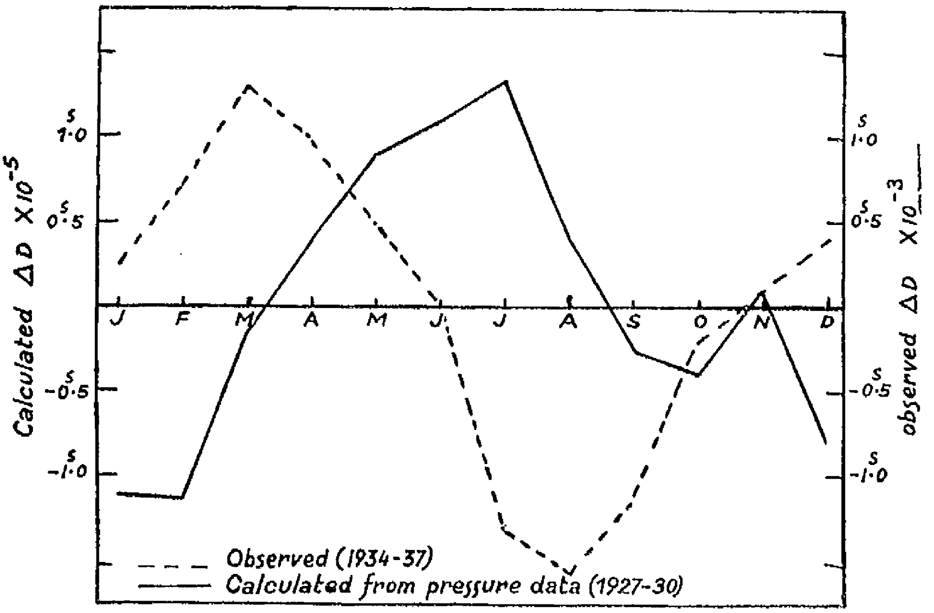
(Unit = 10^{35} gm.cm^2).

		$\Sigma p\delta c$	$\Sigma p\delta s$	ΔC
1927	Jan	+ 2.4189	+ 4.4135	− 0.629
	Feb	+ 2.5603	+ 5.1849	− 1.023
	Mar	+ 0.5374	+ 1.4310	− 0.453
	Apr	− 1.3369	− 2.1895	+ 0.174
	May	− 0.5986	− 1.1221	+ 0.176
	Jun	− 2.2249	− 4.6009	+ 0.955
	Jul	− 1.9784	− 4.0517	+ 0.822
	Aug	− 2.3137	− 3.7901	+ 0.302
	Sep	− 1.3308	− 2.3336	+ 0.280
	Oct	+ 0.8847	+ 1.7173	− 0.302
	Nov	+ 1.6636	+ 2.4503	− 0.026
	Dec	+ 1.3338	+ 3.4634	− 1.063

TABLE I *(cont.)*.

		$\Sigma p\delta c$	$\Sigma p\delta s$	ΔC
1928	Jan	+ 2.8286	+ 3.5781	+ 0.365
	Feb	+ 2.8779	+ 4.6659	− 0.341
	Mar	− 0.0336	+ 0.7379	− 0.547
	Apr	− 2.3225	− 4.0760	+ 0.491
	May	− 1.2763	− 3.1378	+ 0.894
	Jun	− 4.6553	− 7.9832	+ 0.855
	Jul	− 1.6871	− 4.1968	+ 1.216
	Aug	− 0.5818	− 1.6642	+ 0.570
	Sep	− 0.3375	+ 0.0955	− 0.407
	Oct	+ 1.1788	+ 2.3466	− 0.442
	Nov	+ 1.9355	+ 1.9860	+ 0.571
	Dec	+ 2.4637	+ 3.9251	− 0.244
1929	Jan	+ 2.5744	+ 7.9160	− 2.906
	Feb	+ 0.5949	+ 4.7120	− 2.675
	Mar	− 0.2784	− 1.7976	+ 0.969
	Apr	− 1.4926	− 3.4150	+ 0.869
	May	− 1.2408	− 2.7200	+ 0.639
	Jun	− 2.9974	− 5.9488	+ 1.112
	Jul	− 2.4837	− 5.5728	+ 1.369
	Aug	− 2.0987	− 3.6814	+ 0.442
	Sep	− 0.1793	+ 0.2868	− 0.380
	Oct	+ 1.1593	+ 1.9655	− 0.197
	Nov	+ 1.3687	+ 2.5122	− 0.366
	Dec	+ 2.6227	+ 5.8474	− 1.420
1930	Jan	+ 3.7837	+ 6.9493	− 1.015
	Feb	+ 2.0727	+ 3.2599	− 0.176
	Mar	− 0.1918	− 1.2115	+ 0.649
	Apr	− 1.3687	− 1.8917	− 0.065
	May	− 0.3953	− 2.8672	+ 1.594
	Jun	− 2.7909	− 5.7388	+ 1.175
	Jul	− 3.8401	− 7.8438	+ 1.580
	Aug	− 1.6115	− 2.6416	+ 0.211
	Sep	+ 0.5554	+ 1.3858	− 0.403
	Oct	+ 1.5748	+ 3.0061	− 0.501
	Nov	+ 3.5318	+ 4.8291	+ 0.205
	Dec	+ 2.6998	+ 4.3322	− 0.289

FIG. 1. — Variation in Length of Day.



(Scale of calculated curve is 100 times that of observed.)

The result due to this cause is seen to be approximately one hundredth of the observed result, and is seriously out of phase.

8. The order of the magnitude of the variation found here is about the same as that estimated by van den Dungen, et. al. who gave the estimated value of $(\Delta C_{\mathrm{Feb}} - \Delta C_{\mathrm{Aug}})/C$ as 2×10^{-9}. This was corrected to 4×10^{-10} by Mme Mariette Laurent (15). The average value found here is -1.8×10^{-10}.

The disagreement as to sign is explained by the sea-land distribution effect on the calculations. In this paper we have used the fact that h is independent of position on the sea. As a result the computations are reduced to a reliance on actual land observations of the pressure. Further, since at each station, $\int\int \sin\theta d\theta d\lambda$ is numerically greater than $\int\int \sin^3 \theta d\theta d\lambda$ and since α_c has a value just less than 0.7, it follows that the term $-\alpha_c \Sigma p\ \delta s$ due to the sea contribution in equation (8) usually determines the sign of ΔC. Van den Dungen et al. ignored this effect and took the pressures given by the isobar charts of Napier Shaw both on land and sea. Thus, in our notation, the sign of ΔC was determined by $\Sigma p \delta c$ i. e. the land contribution, and their estimate reversed the sign.

9. N. Stoyko (2) suggested that polar waters moving from the poles towards the equator and raising the surface of the ocean by 16 cms. over the area of 25° on each side of the equator would vary the speed of rotation of the Earth by $0^s.001$ per day.

The release of polar water is made possible by the climatic consequences of the fact that the Earth passes through perihelion during the northern winter and through aphelion during the southern winter, thus giving rise to a greater climatic range in the southern hemisphere than in the northern.

The ultimate source of all the energy required to melt polar ice is solar radiation. The ice melts when the excess of incoming over outgoing energy provides the necessary latent heat. In principle it is only necessary to estimate the heat excess or deficiency at points within the polar regions in order to estimate ice changes. The ice may be melted by heat gained from a number of sources:

(i) solar and sky radiation,

(ii) conduction through the underlying material from outside the region,

(iii) warm air currents,

(iv) warm ocean currents.

Of these, the first is to be investigated here. The second source cannot have a substantial effect since the heat will be absorbed in raising the temperature of the lower layers of ice without actually melting them. Although at the periphery of the region the effect may be measurable, over the region as a whole the effect must be small. The third source will also be ignored since it is well established that cyclonic systems are unable to invade the polar high pressure wind systems and are diverted around the polar margins. This is particularly true of the southern hemisphere in the belt of the « Roaring Forties ». In the north the effect is not so marked, being modified by the continental pressure systems of the large land masses, but it seems reasonable to suppose that no substantial amount of ice is melted by warm air currents. Ocean currents do play an important part in determining ice extent. In the south, the main current is the « West Wind Drift » in the latitudes of the Roaring Forties. From the point of view of varying the ice area this cannot be important as there is nowhere any large scale pole-wards drift of warm water. In the north, on the other hand, the marked effect of the Gulf Stream and the somewhat lesser effect of the Kuro Siwo Drift are seen in the northward deflection of the isotherms. This must be accounted for in the estimate of the ice melting.

Apart from actual melting the amount of ice may be reduced by drifting out of the area in the form of floes and icebergs. According to Sir Napier Shaw (10), Greenland contains about 90 % of the land ice of the North Polar region and is the main source of icebergs in the north. The average annual wastage of the ice from the West coast of Greenland is about 7-10 cubic miles, whereas ice melted in the area of Baffin Bay and Davis Strait amounts to 467 cubic miles. Compared with the total amount of ice melted in the north, drift ice is not great, especially when it is remembered that much of it will melt within the polar area before drifting out of the area.

10. The only important way in which heat is lost in the area is by terrestial radiation. This leads to a difficulty which has to be overcome indirectly, for as Napier Shaw (11) says, « Not much progress has yet been made in the effective study of the relation of terrestial radiation to the radiation from the sun and sky ». The loss can be left out of the estimate by using the fact that the year may be divided, for practical purposes, into freezing and melting semesters each of twenty six weeks duration, and since ice conditions are the same for a number of years on the average, the ice formed in the freezing semester is equal to that unfrozen in the melting semester. If the total incoming radiation per unit time falling on a given area within the icefield is E and if T is the terrestial radiation per unit time, then the gain per unit time at the area is $(1-\alpha)$ E — T where α is the albedo of the surface. E, being mainly solar radiation, is independent of the nature of the surface conditions but T is almost certainly dependent on the temperature of the surface and the difference between that temperature and that of the air immediately above the surface. Hence in the icefield, where surface temperatures are zero or below and where the temperature difference between the surface and the air is not great, conditions at a point in the north are almost identical for terrestial radiation with conditions at the corresponding southern point either six months sooner or later i. e. the curve of T_n for a northern point very closely follows that of T_s for the corresponding southern point six months before or after. The difference must be substantially less than that between the curves of E_n and E_s which depend only on the Earth's position relative to the Sun. (Suffices n and s refer to the respective hemispheres).

The balance of radiation available for ice melting is

$$R = (1-\alpha)\ E - T.$$

In the course of a year (from $t=0$ to $t=y$ say)

$$\int_0^y R\,dt = 0$$

on the average. It is convenient to take the beginning of the year for the present purpose as the date on which the southern melting semester starts. Since the two semesters are equal,

$$-\int_0^{y/2} R_n dt = +\int_{y/2}^{y} R_n dt = p \text{ say } (>0)$$

and

$$+\int_0^{y/2} R_s dt = -\int_{y/2}^{y} R_s dt = q \text{ say } (>0).$$

The water freed to raise the sealevel is proportional to the excess of ice melted in the south over that formed in the north during the first semester and is so proportional to $q-p$. This water will return to the pole in the next semester, giving the result that the maximum and minimum moments of inertia occur at the end and beginning of the southern melting semester i. e. about February and August respectively.

Now, $$q - p = \int_0^{y/2} R_s dt - \int_{y/2}^{y} R_n dt$$

$$= (1 - a)\left\{\int_0^{y/2} E_s dt - \int_{y/2}^{y} E_n dt\right\} - \left\{\int_0^{y/2} T_s dt - \int_{y/2}^{y} T_n dt\right\}.$$

As we have seen, the curvès of T_n and T_s must follow each other closely and so the difference of the terms involving them is bound to be very small and may be ignored.

11. The terms E are not directly known but can be inferred from calculated values of solar radiation incident on the outer limits of the atmosphere and using atmospheric absorption data obtained from other areas.

Let I_0 be the incident direct solar radiation on the outer limit of the atmosphere per unit time per unit surface area in latitude ϕ. Ignoring the small loss due to the extra distance travelled, I_0 is the amount of direct radiation which would reach the Earth's surface if the atmosphere were transparent. The total daily radiation falling on unit horizontal surface would then be $\int I_0 \sin h dt$ where h is the Sun's altitude and the integration is taken over the hours of daylight. When atmospheric absorption is taken into account, it is strictly necessary to consider the absorption of different wavelengths of radiation separately; other agencies of depletion such as scattering should also be accounted for independently. A full account of the various forms and degrees of depletion has been given by J. M. Stagg (16) from whose paper most of the absorption data used here has been taken. However, for the purposes of this estimate, the simplest measure of depletion has been adopted i. e. the so-called Transmission Coefficient q' derived from the relation $I = I_0 q'^{m}$ where I is the total direct radiation measured at the Earth's surface and m is the air mass traversed by the radiation, the unit of m being the optical length through the atmosphere when the Sun is at zenith. The amount of radiation falling on unit horizontal surface per day is reduced by depletion in the atmosphere to $\int I_0 q'^{m} \sin h dt$, the integral again being taken over the hours of daylight. I_0, although varying with time, as the Earth progresses around the Sun, is sensibly constant over the course of a day. Hence the ratio of the actual radiation received to that which would be received through a transparent atmosphere is

$$\beta = \frac{\int q'^m \sin h dt}{\int \sin \ hdt}.$$

This factor will be referred to as the « atmospheric reduction factor ». It varies with ϕ and with the Sun's declination δ since

$$\sin h = \sin \phi \ \sin \delta + \cos \phi \ \cos \delta \ \cos \tau,$$

τ being the Sun's hour angle.

Stagg gives specimen values of m which, within the required range of h in polar regions and to the accuracy required, are found to fit the interpolation formula

$$m^2 + 1159.7 \, m \sin h - 1159.1 = 0.$$

Stagg points out that q' is not independent of m ; he gives values of q' calculated from observations made at Kew which show that for $m = 2$, q' varies between 0.767 and 0.788 and for $m = 4$, between 0.798 and 0.813. These results are based on very few recordings of ceiling radiation intensities made on days of maximum sunshine and minimum cloud cover. As clouds are much less transparent to radiation than clear sky, the average value of q' derived from all observations is much less. It is probably better to allow specially for the relative opacity of clouds and use a ceiling value of q' and then apply a cloud factor based on average cloud conditions. For this reason the value $q' = 0.8$ has been adopted. With this value sin h, m and $q'^m \sin h$ have been computed for $\phi = 60^o$, 70^o, 80^o and 90^o, for δ at unit intervals in the range $\pm 24^o$ and for τ at hourly intervals. From these quantities the integrals $\int \sin hdt$ and $\int q'^m \sin hdt$ have been calculated by numerical quadrature and from them values of β have been determined. Values of the daily totals of radiation which would be received hypothetically through a transparent atmosphere have been calculated by a number of writers; those of A. Angot (17) have been adopted here. The following tables give Angot's radiation values, the computed values of β and values of the reduced radiation (E_n and E_s). The tables show the values of the daily totals of radiation for the middle day of successive weeks of the year starting from Jan. 4th. with Dec. 31st. inserted as an intercalary day.

TABLES OF SOLAR RADIATION.

a) Energy, in kilowatt-hours, which would be received, if there were no atmosphere, upon a hundred square metres of horizontal surface by direct radiation from the sun with a solar constant of 135 kilowatts per square dekametre.

b) Atmospheric reduction factors (β).

TABLE II *a*

		$\phi = 90°$ N			$\phi = 90°$ S.		
Wk.	Sun's Dec., ° ′	$\int I_0 \sin h dt$	β	E_n	$\int I_0 \sin h dt$	β	E
1	— 22 45				1296	.5637	730.5
2	— 21 50				1246	.5510	686.5
3	— 20 35				1177	.5326	626.8
4	— 19 01				1089	.5071	552.2
5	— 17 10				984	.4731	465.5
6	— 15 04				865	.4285	370.6
7	— 12 45				733	.3703	271.4
8	— 10 16				590	.2955	174.3
9	— 7 40				440	.2025	89.1
10	— 4 58				285	.0974	27.7
11	— 2 14				128	.0185	2.3
12	0 32	30	—	—			
13	3 17	186	.0411	7.6			
14	5 59	338	.1371	46.3			
15	8 36	482	.2373	114.3			
16	11 05	617	.3217	198.4			
17	13 26	743	.3885	288.6			
18	15 36	857	.4403	377.3			
19	17 33	957	.4087	460.0			
20	19 16	1045	.5112	534.2			
21	20 43	1116	.5347	596.7			
22	21 53	1175	.5517	648.2			
23	22 44	1215	.5634	684.5			
24	23 15	1239	.5701	706.3			
25	23 27	1249	.5726	715.1			
26	23 18	1239	.5707	707.0			
27	22 49	1215	.5646	685.9			
28	22 01	1176	.5536	651.0			
29	20 54	1119	.5375	601.4			
30	19 30	1049	.5151	540.3			
31	17 50	964	.4858	468.3			
32	15 56	865	.4477	387.2			
33	13 50	756	.3988	301.4			
34	11 32	633	.3352	212.1			
35	9 06	504	.2554	128.7			
36	6 33	365	.1597	58.2			
37	3 54	217	.0600	13.0			
38	1 12	68	.0029	0.1			
39	— 1 32				86	.0076	0.7
40	— 4 15				240	.0719	17.2
41	— 6 56				393	.1738	68.3
42	— 9 32				541	.2706	146.3
43	— 12 01				683	.3497	238.8
44	— 14 21				815	.4115	335.3
45	— 16 30				938	.4596	431.1
46	— 18 26				1048	.4967	520.5
47	— 20 05				1141	.5247	598.6
48	— 21 27				1219	.5455	664.6
49	— 22 28				1276	.5598	714.3
50	— 23 08				1314	.5686	747.1
51	— 23 26				1331	.5724	761.8
52	— 23 21				1328	.5713	758.6
Dec 31 st	— 23 07				1316	.5684	748.0

c) Energy in kilowatt-hours per hundred square metres of horizontal surface actually received (E_n and E_s).

Totals for the middle day of successive weeks of the year.

Multiply by .86 to express the results in gramme calories per square centimetre.

TABLE II *b*

	$\phi = 80^o$ N.			$\phi = 80^o$ S.		
Wk.	$\int I_0 \sin hdt$	β	E_n	$\int I_0 \sin hdt$	β	E_s
1				1277	.5681	725.4
2				1227	.5570	683.4
3				1158	.5412	626.7
4				1072	.5201	557.5
5				969	.4939	478.5
6				852	.4634	394.8
7				721	.4323	311.6
8				581	.4061	235.9
9	20	.0140	0.2	454	.3729	169.2
10	63	.0663	4.1	344	.3251	111.8
11	123	.1460	17.9	248	.2661	65.9
12	196	.2225	43.6	166	.1938	32.1
13	281	.2904	81.6	97	.1152	11.1
14	377	.3437	129.5	45	.0416	1.8
15	482	.3861	186.1	7	.0048	—
16	608	.4136	251.4			
17	732	.4408	322.6			
18	844	.4711	397.6			
19	944	.4996	471.6			
20	1029	.5235	538.6			
21	1100	.5430	597.3			
22	1157	.5577	645.2			
23	1196	.5679	679.2			
24	1220	.5739	700.1			
25	1229	.5762	708.1			
26	1222	.5745	702.0			
27	1197	.5689	680.9			
28	1157	.5593	647.1			
29	1102	.5454	601.0			
30	1033	.5267	544.0			
31	949	.5034	477.7			
32	852	.4759	405.4			
33	744	.4460	331.8			
34	624	.4182	260.9			
35	500	.3929	196.4	4	—	—
36	393	.3547	139.3	35	.0313	1.0
37	297	.3029	89.9	82	.0961	7.8
38	212	.2396	50.7	146	.1749	25.5
39	138	.1657	22.8	223	.2483	55.3
40	77	.0861	6.6	313	.3101	97.0
41	30	.0244	0.7	416	.3617	150.4
42	1	—	—	535	.3983	213.0
43				672	.4236	284.6
44				803	.4533	363.9
45				923	.4841	446.9
46				1031	.5119	527.7
47				1123	.5346	600.3
48				1200	.5523	662.7
49				1257	.5647	709.8
50				1295	.5726	741.5
51				1311	.5760	755.1
52				1308	.5751	752.2
Dec 31 st.				1296	.5724	741.8

TABLE II *c*

	ϕ = 70° N.			ϕ = 70° S.		
Wk.	$\int I_0 \sin hdt$	β	E_n	$\int I_0 \sin hdt$	β	E_s
1				1218	.6018	732.9
2				1170	.5990	700.8
3				1106	.5966	659.8
4	4	—	—	1027	.5934	609.4
5	19	.0186	0.3	945	.5855	553.2
6	45	.0671	3.0	857	.5756	493.2
7	78	.1301	10.1	767	.5593	428.9
8	123	.2024	24.8	676	.5400	365.0
9	174	.2724	47.4	587	.5172	303.5
10	234	.3393	79.3	501	.4927	246.8
11	298	.3847	114.6	419	.4544	190.3
12	370	.4305	159.2	340	.4139	140.7
13	444	.4692	208.3	270	.3671	99.1
14	524	.5032	263.6	207	.3160	65.4
15	603	.5253	316.7	151	.2475	37.4
16	683	.5465	373.2	104	.1783	18.5
17	763	.5649	431.0	65	.1122	7.2
18	841	.5780	486.0	36	.0525	1.8
19	915	.5872	537.2	15	.0132	0.1
20	984	.5940	584.4	3	—	—
21	1050	.5968	626.6			
22	1103	.5991	660.8			
23	1142	.6017	687.1			
24	1165	.6037	703.3			
25	1173	.6046	709.1			
26	1165	.6039	703.5			
27	1142	.6020	687.4			
28	1104	.5995	661.8			
29	1052	.5971	628.1			
30	987	.5946	586.8	1	—	—
31	918	.5884	540.1	12	.0093	0.1
32	845	.5195	489.6	31	.0434	1.3
33	770	.5683	437.5	59	.1022	6.0
34	691	.5499	379.9	95	.1650	15.6
35	612	.5297	324.1	139	.2341	32.5
36	533	.5079	270.7	192	.3019	57.9
37	456	.4780	217.9	251	.3570	89.6
38	382	.4398	168.0	319	.4024	128.3
39	312	.3966	123.7	393	.4445	174.6
40	246	.3512	86.3	473	.4828	228.3
41	186	.2919	54.2	556	.5110	284.1
42	135	.2226	30.0	643	.5337	343.1
43	90	.1506	13.5	732	.5535	405.1
44	54	.0878	4.7	821	.5714	469.1
45	27	.0313	0.8	907	.5824	528.2
46	8	.0039	—	992	.5910	586.2
47				1072	.5959	638.8
48				1145	.5982	684.9
49				1199	.6009	720.4
50				1235	.6032	744.9
51				1251	.6045	756.2
52				1247	.6041	753.3
Dec 31 st.				1237	.6031	746.0

TABLE II *d*

Wk.	$\int I_0 \sin h dt$	$\phi = 60°$ N. β	E_n	$\int I_0 \sin h dt$	$\phi = 60°$ S. β	E_s
1	68	.1385	9.4	1189	.6650	790.6
2	80	.1670	13.3	1160	.6632	769.3
3	99	.1962	19.4	1118	.6581	735.7
4	124	.2345	29.0	1067	.6523	696.0
5	155	.2823	43.7	1008	.6457	650.8
6	193	.3317	64.0	944	.6371	601.4
7	238	.3786	90.1	872	.6268	546.5
8	289	.4266	123.2	799	.6175	493.3
9	344	.4665	160.4	725	.6054	438.9
10	404	.4922	198.8	651	.5840	380.1
11	467	.5208	243.2	576	.5625	324.0
12	532	.5491	292.1	506	.5491	277.8
13	599	.5708	341.9	437	.5096	222.6
14	667	.5923	395.0	374	.4823	180.3
15	733	.6112	448.0	316	.4552	143.8
16	798	.6203	494.9	263	.4111	108.1
17	860	.6297	541.5	216	.3649	78.8
18	918	.6397	587.2	176	.3205	56.4
19	971	.6470	628.2	142	.2723	38.6
20	1018	.6531	664.8	113	.2283	25.7
21	1058	.6587	696.9	92	.1930	17.7
22	1092	.6634	724.4	74	.1658	12.2
23	1115	.6650	741.4	63	.1390	8.7
24	1130	.6656	752.1	57	.1219	6.9
25	1135	.6657	755.5	54	.1151	6.2
26	1130	.6656	752.1	56	.1202	6.7
27	1115	.6651	741.5	62	.1363	8.4
28	1091	.6639	724.3	73	.1626	11.8
29	1057	.6594	696.9	88	.1886	16.6
30	1017	.6540	665.1	109	.2225	24.2
31	971	.6480	629.2	135	.2649	35.7
32	918	.6414	588.8	167	.3135	52.3
33	861	.6314	543.6	207	.3569	73.8
34	801	.6219	498.1	251	.4025	101.0
35	737	.6137	452.2	301	.4480	134.8
36	672	.5967	400.9	356	.4769	169.7
37	606	.5757	348.8	417	.5032	209.8
38	541	.5544	299.9	482	.5319	256.3
39	477	.5283	251.9	551	.5570	306.9
40	414	.4994	206.7	622	.5785	359.8
41	355	.4733	168.0	695	.5998	416.8
42	300	.4401	132.0	768	.6151	472.3
43	250	.3932	98.3	841	.6237	524.5
44	205	.3464	71.0	911	.6337	577.3
45	166	.2994	49.7	979	.6434	629.8
46	132	.2495	32.9	1040	.6502	676.2
47	107	.2081	22.2	1095	.6561	718.4
48	85	.1758	14.9	1141	.6616	754.8
49	72	.1478	10.6	1176	.6646	781.5
50	62	.1258	7.7	1200	.6654	798.4
51	58	.1157	6.7	1211	.6657	806.1
52	59	.1185	6.9	1210	.6656	805.3
Dec 31 st.	62	.1264	7.8	1202	.6654	799.8

12. The area to be included in the polar regions must contain all areas within the circumpolar 0°C. isotherm. In the south this is clearly delineated and lies between 55°S latitude in midwinter and about 60°S. in midsummer, practically following circles of latitude. In the north it is not so symmetrical, but by taking the same boundary as in the south, the error involved in neglecting the effect of warm water currents will be more or less corrected. Mountain ice areas outside polar regions and areas lying to the south of 55°N. in the northern land masses having a subzero temperature are excluded from the estimate as the former are relatively small in extent and the latter areas will only have a small amount of snow cover in any case.

Snow and ice have been treated as the same. This is substantially true as ordinary seawater, when it freezes, crystallises out of the solution and leaves the salts behind so that sea ice is mostly pure ice. The latent heat of fusion has been taken as 80 calories per gram and the water formed on melting has been assumed to have unit density. Radiation falling on the surface at 60°, 70°, 80° and 90° has been assumed typical of the areas included in the latitude belts 55-65°, 65-75°, 75-85° and 85-90°, respectively. Because of difficulties arising out of the method by which terrestial radiation effects are eliminated, it is not possible to work with contracting areas, and so estimates have been made for the boundary at both 55° and 60°, the mean of the two being taken as a true estimate.

Area and moment of inertia factors for the sea surface have been obtained directly from the factors calculated as in § **5**.

Stoyko suggested the redistribution of the released water in the area between latitudes 25° north and south of the equator but analysis of the sealevel data given in The Admiralty Tide Tables (18) shows that the ports for which the seasonal variation exceeds six inches on any day in the year are of world wide distribution suggesting that the general sea-level changes are not confined to any particular locality. The results have therefore been calculated on the assumption that the polar waters are distributed in such a way as to raise sea-level uniformly over the entire sea surface.

The date of commencement of the southern melting semester has been taken as occurring at the start of week 34 (Aug. 20th.) since that date gives the maximum energy excess. This corresponds with the northern melting semester beginning about Feb. 19th.

The excess radiation per square centimetre in the zones 85°-90° is

$$0.86\,(1-a)\left[\int_0^{\frac{1}{2}y} \mathrm{E}_s dt - \int_{\frac{y}{2}}^{y} \mathrm{E}_n dt\right]_{\phi=90^\circ} \text{ calories,}$$

the factor .86 being necessary to convert the radiation from kilowatts per square dekametre (in which units E_n and E_s are given above) to calories per square centimetre. If it is assumed that the ice is at freezing point no heat is required to raise its temperature and with this assumption the excess energy melts an amount of ice in the south in excess of that frozen in the north equal to a depth over the whole southern zone of

$$h_{90} = 0.01075\,(1-a)\left[\int_0^{\frac{1}{2}y} \mathrm{E}_s dt - \int_{\frac{y}{2}}^{y} \mathrm{E}_n dt\right]_{\phi=90^\circ} \text{ cms.,}$$

since 80 calories per square centimetre melts ice to a depth of 1 centimetre. When redistributed, this covers the sea surface to a depth

$$h_0 = \frac{\delta S_{90}}{\delta S_0} h_{90} \text{ cms.,}$$

where δS_0 and δS_{90} are the area factors, for the sea surface over the whole Earth and the area factors for the whole surface in the 85°-90° zone respectively. The change in the moment of inertia due to the transfer of this water is

$$h_0 \delta C_0 - h_{90} \delta C_{90}$$

where δC_0 and δC_{90} are the corresponding moment of inertia factors. This gives the difference between the variable parts of the moments of inertia in February and August as

$$(\Delta C_{\text{Feb}} - \Delta C_{\text{Aug}})_{90^\circ} = h_{90}\delta S_{90}\left[\frac{\delta C_0}{\delta S_0} - \frac{\delta C_{90}}{\delta S_{90}}\right]$$

$$= 0.01075\,(1-a)\delta S_{90}\left[\frac{\delta C_0}{\delta S_0} - \frac{\delta C_{90}}{\delta S_{90}}\right]\left[\int_0^{\frac{1}{2}y} E_s dt - \int_{\frac{y}{2}}^{y} E_n dt\right]_{\phi=90^\circ}.$$

Similarly the contributions from the other polar latitude zones are found, the sum of the results giving the total change in the moment of inertia.

The results which have been obtained by this method are as follows :

$$\frac{\delta C_0}{\delta S_0} = 0.6843\ R^2 \text{ where R is the radius of the Earth.}$$

(R is taken as 6.371×10^8 cms.)

Zone	$\delta S \times R^2$ cm²	$\frac{\delta C}{\delta S} \times R^2$	$0.1075\left[\frac{\delta C_0}{\delta S_0} - \frac{\delta C}{\delta S}\right] \times 10^{32}$	Excess Radiation Calories $\times (1-a)$	$\Delta C_{\text{Feb}} - \Delta C_{\text{Aug}} \times 10^{32}$ gm. cm². $\times (1-a)$
85°-90°	.003805	.0038	0.2882	657.0	0.02
75°-85°	.030269	.0374	2.1793	797.8	0.17
65°.75°	.059618	.1234	3.7217	1249.3	0.47
55°-65°	.087156	.2551	4.1633	1912.8	0.80
60°-65°	.040282	.2146	2.1058	1912.8	0.40
			Total	55°-90°	1.46
			Total	60°-90°	1.06
			Mean		1.26

$$\text{Mean } (\Delta C_{\text{Feb}} - \Delta C_{\text{Aug}}) = 1.26\,(1-a) \times 10^{36}\text{gm. cm}^2.$$

and

$$\frac{\Delta C_{\text{Feb}} - \Delta C_{\text{Aug}}}{C} = 1.6\,(1-a) \times 10^{-9}.$$

It is seen that the mean value adopted gives

$$(\Delta C_{\text{Feb}} - \Delta C_{\text{Aug}})/C = +\,1.6\,(1-a) \times 10^{-9}$$

which has yet to be corrected for the cloud absorption of radiation. The cloud factor is not easily obtainable, and the albedo also gives difficulty as it may be any value between that of ice and snow and that of water since pools of water will form in the icefields and there may be considerable stretches of open sea. Since it is know that 467 cubic miles of ice are dissipated annually in the Davis Strait-Baffin Bay seas it is possible to estimate the factor by which the results have to be reduced. The area is fortunately a good one for this purpose as it stretches from 60°-85°N. Its area is such that the radiation as calculated here would, if entirely absorbed, melt 1.3×10^{19} c. c. s. of ice. Since 467 cubic miles is only 1.9×10^{18} c. c. s it follows that about 15 % of the estimated available energy is used to melt ice, the remainder being lost in the cloud absorption, reflected at the surface or reradiated as terrestial radiation.

With this reduction factor $\dfrac{\Delta C_{\mathrm{Feb}} - \Delta C_{\mathrm{Aug}}}{C} = +\,2.5 \times 10^{-10}$.

Acknowledgements.

The author would like to thank Professor Freundlich of the University of St. Andrews who suggested the original investigation and whose interest and helpful criticism has been very much appreciated.

The author has also to acknowledge with thanks the financial assistance provided by the commitee on Research of the University of Liverpool who made a grant to defray the cost of computational assistance.

Institute of Mathematics,
The University,
Liverpool 3.

REFERENCES

1. N. Stoyko — *Comptes Rendus*, 203, p. 39, 1936.
2. — *Ibid.*, 205, p. 79, 1937.
3. — *Ibid.*, 130, p. 507, 1950. and *Acad. r. de Belgique, Bull. Cl. des Sci.*, 5e Sér., 35, 1949.

4. H. F. Finch. — *M. N. R. A. S.*, 110, 1950.
5. W. Schweydar. — *Sitz. der Pr. Ak. Wiss.*, 20, 1919.
6. F. H. van den Dungen, J. F. Cox and J. van Mieghem. — *Acad. r. de Belgique, Bull. Cl. des Sci.*, 5e Sér. 35, 1949.
7. W. H. Munk and R. L. Miller. — *Tellus*, 2, 2, 1950.
8. H. Jeffreys. — *M. N. R. A. S.*, 76, 1916 and « *The Earth* » (2nd. ed.) Cambridge, 1924.
9. Sir Napier Shaw. — *Manual of Meteorology*. vol. 2, Cambridge, 1928.
10. — *Ibid.*, p. 406-7.
11. — *Ibid.*, p. 8.
12. F. Bartels. — « Geophysics ». vol. 1. F. I. A. T. *Reviews of German Science*, 1950.
13. L. Rosenhead. — *M. N. R. A. S. Geophysical Supp.* 1929.
14. H. Kimura. — *Travaux de la Section de Géodesie de l'Union Géodésique et Géophysique Internationale.* Tome II (III), 1933.
15. M. Laurent. — *Acad. r. de Belgique, Bull. Cl. des Sci.* 5e Sér., 35. 1949.
16. J. M. Stagg. — « Solar Radiation at Kew Observatory ». *Geophysical Memoirs.* no 86. H. M. S. O., 1950.
17. A. Angot. — see Napier Shaw., *Op. cit.*, p. 3-5.
18. Admiralty Tide Tables Part. 2, p. 162-5. London, 1938.

(2)

Académie royale de Belgique

Koninklijke Belgische Academie

BULLETIN
DE LA
CLASSE
DES SCIENCES

MEDEDELINGEN
VAN DE
KLASSE DER
WETENSCHAPPEN

5e Série. — Tome XXXVIII

5de Reeks. — Boek XXXVIII

1952

EXTRAIT — UITTREKSEL

The Contribution of the Seasonal Movements of Air Masses to the Variation of Latitude

BY

Andrew YOUNG

BRUXELLES
PALAIS DES ACADÉMIES
RUE DUCALE, 1

BRUSSEL
PALEIS DER ACADEMIËN
HERTOGELIJKESTRAAT, 1

1952

Extrait du *Bulletin de l'Académie royale de Belgique* (Classe des Sciences)
Séance du 5 août 1952

ASTRONOMY

The Contribution of the Seasonal Movements of Air Masses to the Variation of Latitude,

by ANDREW YOUNG (*).

Summary. — Variations in the products of inertia of the Earth are calculated from atmospheric pressure variations. Expressions for the variation of latitude are derived taking damping and possible resonance into account. It is found that, in general character, the variation of latitude is explainable on the assumption that the free motion is subject to random disturbances and that the forced motion is due to seasonal movement of air masses.

1. INTRODUCTION.

Although it is frequently asserted that the forced component of the variation of latitude is due to the seasonal concentration and dispersion of large masses of air, especially over the Central Asian land mass, the evidence for this is inconclusive. H. Jeffreys (1) assumed that the forced component of the motion (due to changes in the products of inertia of the Earth) was of yearly period and estimated the effects of various putative causes on the products of inertia. His results, revised and extended by L. Rosenhead (2), left some doubt as to the complete explanation of the annual part of the motion. On the other hand, W. Schweydar (3) making no assumptions about the period of the forced motion, made a much more detailed calculation of the average effect of atmospheric mass movements and by a numerical integration of the equations of the motion, reached the conclusion that in order of magnitude as well as in general form, the path of the pole could be adequately explained by the movement

(*) Présenté par MM. J. F. Cox et F. H. VAN DEN DUNGEN.

of air masses alone. Since Jeffreys' estimate of the effect of the movement of air masses gave a serious phase discrepancy these two results are conflicting. None of these investigations attempted to account for damping effects in the free motion.

Of the many efforts which have been made to determine the periods and amplitudes of the components of the motion by a direct analysis of the observed data, an important feature has been the discovery of sudden, inexplicable changes in the amplitude and phase of the free motion. In these analyses it has been usual to assume that the forced motion contains only a yearly period, although Wanach (4) gives a theoretical solution of the equations of motion with a series of forced components including a resonance term. This, on physical grounds, is necessarily an abstraction unless damping is taken into account. L. W. Pollak (5) and K. Stumpff (6) both found the forced component to have a period of about 0.9 years. Pollak used a method, due to Fuhrich, which is related to the modern conceptions of auto-regressive time series. Jeffreys (7) analysed the data from this point of view which represents the free motion as a damped harmonic regenerated by a series of erratic disturbances at irregular intervals. He first removed the forced motion, assumed to be undamped, disturbance free, and of truly annual period, and then estimated the length of the free period and the damping coefficient by finding the auto-regressive equation of the free motion. This seems to be the only published attempt to allow for damping and the irregularities of the free motion.

Although Schweydar saw the necessity for verifying his conclusion by performing the calculations over an extended number of years using actual rather than average atmospheric pressure values, this never seems to have been done. The object of this paper is to extend Schweydar's work in order to test his conclusion. The equations of motion, modified for damping, are solved, the products of inertia computed from actual pressure data and the path of the pole computed. The possibility of a near resonance effect is examined.

2. Equations of motion.

Let l and m be the angular displacements of the axis of rotation of the Earth towards and perpendicular to the Greenwich meridian. The direction cosines of the axis are then $(l, m, 1)$ and the equations of motion are

$$\frac{dl}{dt} + kl + \frac{n}{\tau} m = - Fn/A_0,$$
$$\frac{dm}{dt} + km - \frac{n}{\tau} l = Gn/A_0, \tag{1}$$

where

n = angular velocity of the Earth,

A_0, C_0 = moments of inertia of Earth if undisturbed by elastic strain,

$\tau = A_0/(C_0 - A_0)$ practically,

F, G = products of inertia,

k = damping factor.

Writing $\frac{n}{\tau} = \gamma$, the solution of (1) is found to be

$$l = e^{-kt}(P \cos \gamma t - Q \sin \gamma t),$$
$$m = e^{-kt}(P \sin \gamma t + Q \cos \gamma t), \tag{2}$$

where

$$P = l_0 + \frac{n}{A_0}\int_0^t e^{kt} (G \sin \gamma t - F \cos \gamma t)\, dt,$$
$$Q = m_0 + \frac{n}{A_0}\int_0^t e^{kt} (F \sin \gamma t + G \cos \gamma t)\, dt, \tag{3}$$

l_0 and m_0 being the values of l and m at $t = 0$.

Assuming that F and G reflect the periodicities of the surface phenomena causing their variation, they may be expanded as series of harmonic terms. It will be further assumed that they are undamped and disturbance free. Let the *pth* harmonics of F and G be

$$F_p = f_{1p} \cos pt + f_{2p} \sin pt,$$
$$G_p = g_{1p} \cos pt + g_{2p} \sin pt. \tag{4}$$

With these assumptions, (2) and (3) give

$$l =$$

$$e^{-kt}\left[\left\{l_0 - \frac{n}{A_0}\sum_p (c_{2p} + c_{4p})\right\}\cos\gamma t - \left\{m_0 - \frac{n}{A_0}\sum_p (c_{1p} - c_{3p})\right\}\sin\gamma t\right]$$

$$+ \sum_p \frac{n}{A_0}\left\{ (c_{1p} + c_{3p}) \sin pt + (c_{2p} + c_{4p}) \cos pt \right\},$$

$$m = \tag{5}$$

$$e^{-kt}\left[\left\{l_0 - \frac{n}{A_0}\sum_p (c_{2p} + c_{4p})\right\}\sin\gamma t + \left\{m_0 - \frac{n}{A_0}\sum_p (c_{1p} - c_{3p})\right\}\cos\gamma t\right]$$

$$+ \sum_p \frac{n}{A_0}\left\{ (c_{1p} - c_{3p}) \cos pt - (c_{2p} - c_{4p}) \sin pt \right\},$$

where

$$\begin{aligned} c_{1p} &= \tfrac{1}{2}\{k(g_{1p} - f_{2p}) - (p+\gamma)(g_{2p} + f_{1p})\}/\{k^2 + (p+\gamma)^2\},\\ c_{2p} &= -\tfrac{1}{2}\{k(g_{2p} + f_{1p}) + (p+\gamma)(g_{1p} - f_{2p})\}/\{k^2 + (p+\gamma)^2\},\\ c_{3p} &= -\tfrac{1}{2}\{k(g_{1p} + f_{2p}) - (p-\gamma)(g_{2p} - f_{1p})\}/\{k^2 + (p-\gamma)^2\},\\ c_{4p} &= \tfrac{1}{2}\{k(g_{2p} - f_{1p}) + (p-\gamma)(g_{1p} + f_{2p})\}/\{k^2 + (p-\gamma)^2\}. \end{aligned} \tag{6}$$

These constants may be regarded as being formed by amplification factors such as $k/\{k^2 + (p+\gamma)^2\}$ acting on the harmonic coefficients of F and G.

It is apparent that the motion thus obtained contains a damped « circular » component of period $2\pi/\gamma$ which includes both the true free motion and a contribution from the forced motion, and an undamped part which is the sum of the elliptic components due to each harmonic in F and G.

The free motion would eventually be damped out completely. If, as in the auto-regressive model, it is regenerated by irregular disturbances, the solution will hold only between consecutive disturbances. However, the forced elliptical components remain unaffected.

3. Amplification factors.

Jeffreys found $k = 0.0663$ when the unit of time is a year, and $2\pi/\gamma = 1.202$ years. For the purpose of this investigation the value $\gamma = 25^o$ per month has been adopted. In order to see how the results depend on the value of k, amplification factors have been computed with $k = .000555$, $.00555$ and $.0555$, and also for $k = 0$, corresponding to undamped motion.

The amplification factors appearing in (6) are tabulated for these values and various values of p in Table I. It is observed that $(p + \gamma) / \{ k^2 + (p + \gamma)^2 \}$ is insensitive to changes in k, and $k / \{ k^2 + (p + \gamma)^2 \}$ is practically linear in k. These results are also true for values of $(p - \gamma) / \{ k^2 + (p - \gamma)^2 \}$ and $k / \{ k^2 + (p - \gamma)^2 \}$ for values of p corresponding to periods of 6 months or less, (and for periods exceeding 20 months which are not given in the table). The values corresponding to periods of 12, 16 and 18 months show that resonance effects do materially depend on the value of k.

TABLE I. — *Amplification Factors.*

	$(p + \gamma) / \{ k^2 + (p + \gamma)^2 \}$				$k / \{ k^2 + (p + \gamma)^2 \}$		
$k =$	0	.000555	.00555	.0555	.000555	.00555	.0555
p (deg. per m.)							
20	1.2732	1.2732	1.2732	1.2669	0.0009	0.0090	0.0895
$22\frac{1}{2}$	1.2062	1.2062	1.2062	1.2008	.0008	.0081	.0804
30	1.0417	1.0417	1.0417	1.0383	.0006	.0060	.0600
60	.6741	.6741	.6741	.6731	.0003	.0025	.0252
90	.4982	.4982	.4982	.4978	.0001	.0014	.0138
120	.3951	.3951	.3951	.3950	.0001	.0009	.0087
150	.3274	.3274	.3274	.3273	.0001	.0006	.0059
180	.2795	.2795	.2795	.2794	—	.0004	.0043

	$(p - \gamma) / \{ k^2 + (p - \gamma)^2 \}$				$k / \{ k^2 + (p - \gamma)^2 \}$		
$k =$	0	.000555	.08555	0.555	.000555	.00555	.0555
p (deg. per m.)							
20	—11.4592	—11.4588	—11.4131	—8.1591	0.0729	0.7259	5.1890
$22\frac{1}{2}$	—22.9184	—22.9147	—22.5536	—8.7545	.2915	2.8688	11.1354
30	11.4592	11.4588	11.4131	8.1591	.0729	0.7259	5.1890
60	1.6370	1.6370	1.6369	1.6236	.0015	.0149	0.1475
90	0.8815	0.8815	0.8815	0.8794	.0004	.0043	.0430
120	.6031	.6031	.6031	.6024	.0002	.0020	.0202
150	.4584	.4584	.4584	.4581	.0001	.0012	.0117
180	.3697	.3696	.3696	.3695	.0001	.0008	.0076

4. THE NUMERICAL VALUES OF THE PRODUCTS OF INERTIA.

Values of F and G have been calculated for each month of years 1925-1930 using the atmospheric data for some 340 stations listed in « Reseau Mondial ». The calculations were performed by the method described in detail in a previous paper (8) the only modification being that the factors used were those relevant to the products instead of the moments of inertia. The results, uncorrected for the elastic yielding of the Earth, are given in Table II. It is noticeable that F is much more regular than G.

TABLE II.

Values of F *and* G *calculated from atmospheric pressure data.*

(uncorrected for elastic yielding.).

(unit 10^{34}gm. cm^2)

	Year	1925	1926	1927	1928	1929	1930	Average
F	Jan.	25.32	49.14	32.49	42.72	54.94	30.96	39.26
	Feb.	46.92	49.99	47.10	34.48	41.61	38.91	43.17
	Mar.	17.64	7.78	14.89	23.71	23.83	15.23	17.18
	Apr.	— 2.83	— 5.03	—12.22	— 7.45	— 5.59	— 5.67	— 6.47
	May.	—24.50	—17.90	—14.63	—27.22	—23.89	—24.80	—22.16
	Jun.	—45.79	—41.25	—41.97	—50.36	—46.44	—39.37	—44.16
	Jul.	—50.91	—60.69	—59.59	—48.75	—56.48	—67.11	—57.26
	Aug.	—49.32	—48.97	—46.93	—36.41	—41.35	—43.65	—44.44
	Sep.	—14.88	—17.84	—18.45	1.03	—12.21	1.40	—10.16
	Oct.	10.79	14.93	14.77	21.84	18.64	11.59	15.43
	Nov.	12.22	28.86	19.94	30.64	35.70	33.34	26.78
	Dec.	36.83	43.96	29.33	50.61	55.48	44.90	43.52
G	Jan.	12.11	9.04	6.60	3.76	15.74	— 4.88	7.06
	Feb.	— 1.40	8.71	10.78	2.14	3.04	1.37	4.11
	Mar.	— 4.29	— 5.11	— 5.43	0.75	0.42	—14.47	— 4.69
	Apr.	— 1.45	— 2.98	—15.80	—11.16	— 5.37	— 5.20	— 6.99
	May.	— 5.77	— 3.76	0.58	— 1.46	0.06	2.76	— 1.27
	Jun.	— 6.64	— 4.95	— 1.59	— 7.05	— 3.04	— 2.68	— 4.33
	Jul.	— 8.22	—12.89	—12.81	— 6.82	— 4.88	—12.70	— 9.72
	Aug.	— 4.78	— 5.65	— 2.72	— 8.00	— 0.30	— 1.63	— 3.65
	Sep.	— 0.18	8.61	— 3.23	4.58	5.49	3.03	3.05
	Oct.	— 5.37	— 2.17	2.31	— 0.04	3.05	3.19	0.16
	Nov.	— 1.92	6.77	7.27	1.91	3.54	1.60	3.20
	Dec.	— 2.39	1.64	0.53	9.81	9.60	13.96	5.53

5. The periodic components of the products of inertia.

These series have been analysed in three ways. Firstly ordinary Fourier Analysis of the twelve monthly averages gave the result given as hypothesis A of Table III. The insertion of these representations of F and G in (5) with $k = 0$ give a result which can be directly compared with Schweydar's work.

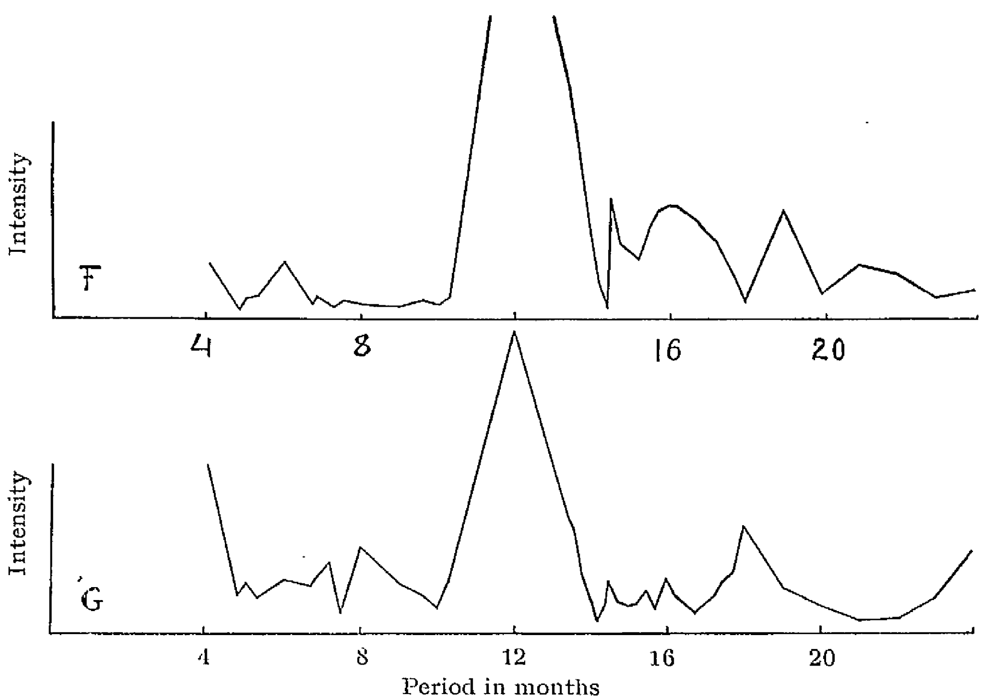


Fig. I. Periodograms of F and G.

Secondly periodogram analyses of the two series are depicted in the periodograms of Fig. I. Since only components whose amplitudes are significant after amplification are of interest, and no such components were detected with periods in the range 19-36 months, only the parts of the periodograms with trial periods in the range 4-18 months are shown. From the periodogram of F, it appears that the peaks corresponding to periods 4, 6, 12 and 16 months are of the breadth required theoretically for significance. Other peaks appear to be either interference effects or are the side peaks which usually occur near true periods. It is, however, difficult to decide whether the pronounced peak at 18 months reflects a genuine period or not. Two least square solutions have therefore been made to fit components of 4, 6, 12, 16 and 18 month periods and 4, 6, 12 and 16 month periods to the data, the results being given as hypotheses B and C of Table III respectively. The periodogram of G is less well defined, as is to be expected from the greater irregularity of G. However all the peaks of F appear in G and so least square solutions have been obtained for G as for F.

TABLE III.

Coefficients of the Trigonometric Representations of F *and* G.
(unit 10^{34} gm. cm^2)

HYPOTHESIS A : — *Fourier Analysis of Monthly Means.*

p	f_{1p}	f_{2p}	g_{1p}	g_{2p}
0°	0.0557		— 0.6449	
30°	47.7481	— 6.7756	5.0934	— 3.1110
60°	— 5.6256	0.2603	0.7940	— 0.6507
90°	3.3731	3.6306	3.3433	0.8763
120°	— 2.3136	— 0.4311	— 1.7276	0.3794
150°	— 2.8629	— 0.5397	— 0.0459	0.4096
180°	— 1.1132		0.2504	

HYPOTHESIS B : — *Least Square Fit of Five Periodic Components.*

p	f_{1p}	f_{2p}	g_{1p}	g_{2p}
0	0.0557		— 0.6449	
20°	— 0.3044	— 0.2011	1.0707	— 1.2071
22½°	— 0.2141	— 1.1046	— 0.6288	0.3273
30°	47.5504	— 6.7241	5.1710	— 2.9602
60°	— 5.6473	0.2724	0.8186	— 0.6150
90°	3.3663	3.6370	3.3646	0.8953

HYPOTHESIS C : — *Least Square Fit of Four Periodic Components.*

p	f_{1p}	f_{2p}	g_{1p}	g_{2p}
0	0.0557		— 0.6449	
22½°	— 0.0927	— 1.3173	0.1730	1.0678
30°	47.5079	— 6.7532	5.2848	— 3.1527
60°	— 5.6560	0.2655	0.8149	— 0.6605
90°	3.3605	3.6333	3.3508	0.8712

6. Expressions for the resulting variation of latitude.

In evaluating F and G, only the variable parts of the pressure at each of the meteolorogical stations should have been used, and the values of f_{10} and g_{10} corresponding to $p = 0$ should be zero. The values shown in Table III result from the accumulate errors of the adopted annual mean pressures of all the individual stations, and have therefore been ignored. This, however, in no case affected the final results by an amount exceeding $0''.001$. The remaining coefficients shown are uncorrected for elastic yielding of the Earth. According to Rosenhead, the correction factor is $\kappa = 0.6$ and this value has been adopted. With this correction applied, the coefficients have been inserted into equations (5) and (6) with the results given in Table IV. The entries under Hypothesis C′, showing the effect of not applying the elastic yielding factor, are derived as for Hypothesis C but without the application of an elastic yielding factor.

The series have been evaluated fully in the cases of Hypothesis A for $k = 0$ and 0.00555, and for Hypotheses B and C for $k = .00555$ and .0555. The values of the standard error

$$\left(= \sqrt{\frac{\Sigma(obs. - calc.)^2}{72}} \right)$$

for each of these series is given in Table V and two of the series are illustrated in Figure II.

TABLE IV.

Calculated Coefficients of the Variation of Latitude (unit 0″.0001).

1. HYPOTHESIS A : — (*Fourier Analysis of Monthly Mean Values of* F *and* G *with elastic yielding factor* $\kappa = 0.6$)

Series	*l*			*m*		
Damping factor *k*	0	.00555	.0555	0	.00555	.0555
e^{-kt} cos 25°*t*	3953	4512	7781	2402	2452	4976
e^{-kt} sin 25°*t*	— 2402	— 2452	— 4976	3953	4512	7781
cos 30°*t*	— 465	— 1010	— 4299	7879	7828	5297
sin 30°*t*	— 9245	— 9191	— 6638	— 101	— 639	— 3858
cos 60°*t*	20	22	33	— 58	— 57	— 54
sin 60°*t*	182	182	179	31	31	39
cos 90°*t*	92	92	90	1	2	5
sin 90°*t*	— 63	— 64	— 68	88	147	87
cos 120°*t*	— 11	— 11	— 11	— 13	— 13	— 13
sin 120°*t*	35	35	35	— 26	— 26	— 26
cos 150°*t*	— 7	— 6	— 5	— 10	— 11	— 10
sin 150°*t*	34	34	34	— 2	— 2	— 1
cos 180° *t*	0	1	1	— 1	— 1	— 1

Tables III and IV. The author has followed the usual practice in Fourier Analysis in carrying two extra decimals which are used in the synthesis, in the expectation that the errors will cancel out.

2. HYPOTHESIS B : — (*Least Square Fit of* 5 *Components with elastic yielding factor* $\kappa = 0.6$)

Series	*l*			*m*		
Damping factor *k*	0	.00555	.0555	0	.00555	.0555
e^{-kt} cos 25°*t*	3525	4061	7626	2381	2350	5212
e^{-kt} sin 25°*t*	— 2381	— 2350	— 5212	3525	4061	7626
cos 20°*t*	— 170	— 179	— 195	— 124	— 113	— 12
sin 20°*t*	180	170	72	— 122	— 133	— 151
cos 22½°*t*	575	589	303	180	250	— 215
sin 22½°*t*	— 184	— 254	212	592	605	320
cos 30°*t*	— 443	— 985	— 4258	7821	7771	5266
sin 30°*t*	— 9185	— 9128	— 6605	— 79	— 613	— 3816
cos 60°*t*	21	22	34	— 59	— 59	— 56
sin 60°*t*	183	183	180	32	32	40
cos 90°*t*	92	92	90	1	1	5
sin 90°*t*	— 63	— 64	— 68	89	89	88

3. HYPOTHESIS C : — (*Least Squatre Fit of 4 Components with elastic yielding factor* $\kappa = 0.6$)

Series	l			m		
Damping factor k	0	.00555	.0555	0	.00555	.0555
e^{-kt} cos 25°t	3560	4059	7427	2034	2136	5008
e^{-kt} sin 25°t	— 2034	— 2136	— 5008	3560	4059	7427
cos 22½°t	359	401	310	373	319	— 53
sin 22½°t	— 408	353	22	412	455	364
cos 30°t	— 431	— 974	— 4261	7850	7801	5294
sin 30°t	— 9207	— 9156	— 6626	— 63	— 598	— 3815
cos 60°t	20	22	34	— 58	— 58	— 54
sin 60°t	182	182	179	31	32	40
cos 90°t	92	92	90	1	2	5
sin 90°t	— 63	— 64	— 68	88	88	88

4. HYPOTHESIS C′ : — (*As* C *without elastic yielding factor*).

Series	l			m		
Damping factor k	0	.00555	.0555	0	.00555	.0555
e^{-kt} cos 25°t	3533	4365	9979	— 3411	— 3240	1546
e^{-kt} sin 25°t	3411	3240	— 1546	3533	4365	9979
cos 22½°t	598	669	516	622	532	— 89
sin 22½°t	— 680	589	37	686	758	607
cos 30°t	— 719	— 1624	— 7101	13083	13001	8824
sin 30°t	—15345	—15260	—11044	— 105	— 997	— 6359
cos 60°t	34	36	56	— 96	— 96	— 90
sin 60°t	304	304	299	52	54	66
cos 90°t	154	154	150	2	3	9
sin 90°t	— 105	— 106	— 113	147	147	146

TABLE V.

Standard Errors of the Series obtained.

(unit 0″.001)

Damping factor k	Hypothesis	Standard Error	
		l	m
0	A	81.4	73.5
.00555	A	78.1	71.7
	B	77.2	71.4
	C	75.9	72.1
.0555	B	56.3	56.6
	C	56.9	57.0

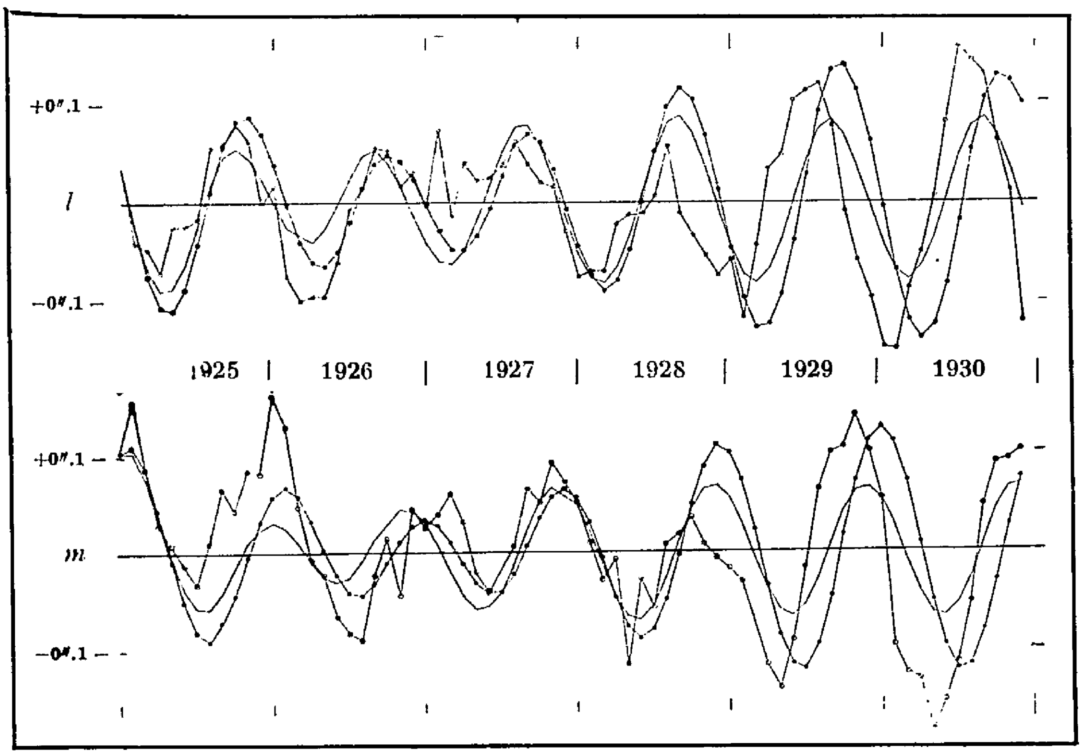


FIG. II. Variation of Latitude.

——o—— Observed Motion.
——•—— Calculated Motion : Hypothesis A ; undamped.
———— Calculated Motion : Hypothesis C, damping factor $k = 0.0555$.

The results on the other hypotheses described in the text are intermediate between the two results depicted.

7. Discussion of results.

The curves shown in Fig. II clearly demonstrate the irregularity of the actual variation of latitude. The curves obtained in this paper are very much more regular than the observed ones ; it is interesting to note that, after apparently disturbed periods in 1925 and 1927, the observed curves are very closely approximated to by the calculated ones during the later part of 1927, but after a very obviously disturbed period in 1928, the phase of the observed motion differs by as much as eight weeks from that calculated.

The amplitude of the actual motion is reasonably well approximated to by the calculated values until 1928, after which the continuance of the damping in the calculated values shows its effect. This would be even more marked if the results were to be extrapolated forward to later years.

It does seem that, in the six year period investigated, the effect of adopting a heavier damping coefficient than the one obtained by Jeffreys, is to improve the phase agreement but to worsen the amplitude agreement.

The contribution of the components other than the annual term in F and G is not great, and the results would not be materially altered if there components were ignored.

The adopted value of the elastic yielding factor, $\kappa = 0.6$, is a theoretical lower limit and the actual value may well be higher. This would materially affects the values of F and G, and owing to the way in which these quantities contribute to the « circular » motion, the phase and amplitude of the synthesised motion would change with κ. This is made obvious by comparing the phases and amplitudes of the dominant components shown in Table IV under hypothesis C and C′.

Better approximations might be arrived at by allowing for the possibility of a fairly substantial error in the values of l_0 and m_0 which have been adopted here, and by fitting the curves to three two-year periods rather than to one six-year period. This has not been done, however, for when the uncertainties in the constants k, κ and γ are taken into consideration, the results obtained seem to be quite in accordance with what is to be expected under the auto-regressive hypothesis discussed by Jeffreys.

In conclusion, it can be said that the general character of the motion is explicable on the basis that the free motion is subject to random disturbances and the forced motion is due to the seasonal movements of large air masses.

8. Acknowledgements.

All the heavy computation involved in the calculation of F and G was done on Hollerith punched card equipment. The author is indebted to the management and staff of the Royal Insurance Co. Ltd., Liverpool, who generously made this equipment available for the purpose. The author also wishes to thank Mr. C. V. Jones of the Department of Electrical Engineering of this University who prepared the figures appearing in the text.

The Institute of Mathematics,
The University, Liverpool 3.

REFERENCES

1. H. Jeffreys. — *M. N. R. A. S.*, **76**. 1916 and « The Earth » (2nd. ed.), Cambridge, 1924.
2. L. Rosenhead. — *M. N. R. A. S., Geophysical Supp.* 1929.
3. W. Schweydar. — *Sitz. der. Pr. Ak. Wiss.*, **20**. 1919.
4. B. Wanach. — « Resultate des Internationalen Breitendienstes », Bd. 5. Berlin 1916.
5. L. W. Pollak. — *Meteor Zeitschr.*, 1935, pp. 330-333 and *Statisticky Obzor* (Prague), **16**, 1935.
6. K. Stumpff. — *Astron. Nachr.*, **5536**, 1927.
7. H. Jeffreys. — *M. N. R. A. S.*, **100**, 1939.
8. *Acad. roy. Belgique.* Cl. Sci. 5e sér., **37**; 1951.

3

Reprinted from the MONTHLY NOTICES OF THE ROYAL ASTRONOMICAL SOCIETY, GEOPHYSICAL SUPPLEMENT, *Vol.* 6, *No.* 7, 1953

GLACIAL EUSTASY AND THE ROTATION OF THE EARTH

by

ANDREW YOUNG

Reprinted from the Monthly Notices of the Royal Astronomical Society, Geophysical Supplement, *Vol.* 6, *No.* 7, 1953

GLACIAL EUSTASY AND THE ROTATION OF THE EARTH

Andrew Young

(Received 1952 December 29)

Summary

A rise in sea-level at the rate of 1 cm per century following a reduction in the extent of ice in glaciated regions results in an increase of the moment of inertia of the Earth about its axis of rotation sufficient to lengthen the day by 10^{-4} second per century provided there is no isostatic compensation. The consequent apparent secular acceleration of the Moon is 2 seconds of arc per century per century. The changes however are negligible if immediate and complete isostatic compensation occurs. A tentative discussion of climatic and glaciological evidence indicates that fluctuations of sea-level do occur and that their effect on the Moon's apparent acceleration may be appreciable. This re-opens the question of the cause of the Moon's apparent acceleration which is usually attributed to the deceleration of the Earth by tidal friction. The change in tidal friction due to a rise in sea-level is very small and has not been sufficient to produce the observed change in the Moon's acceleration.

1. *Introduction.*—The theory of eustasy is a geological theory to explain variations of sea-level. Tectonic movements of the Earth's crust, processes of mountain building, isostatic adjustments, etc., lead to local, or at most, zonal changes in sea-level, but in some cases ancient shore lines can be traced horizontally over hundreds of miles regardless of local stratigraphical and tectonic units. Such cases appear to be well explained by glacial eustasy, i.e. the rise and fall of sea-level due to the release and absorption of water in glaciated regions.

Table I

Estimates of the volume of ice at present

Author	Date of publication	Estimated Volume ($\times 10^6$ km^3)	Rise in Sea-level if entirely melted (metres)
E. Antevs	1929	16·1–23·2	40–60
W. Ramsay	1930	22·5	55
R. A. Daly	1934	20·9	50
S. Thorarinsson *	1940	9·5	24

At present, of the total land surface of the Earth, about 10 per cent is covered with ice. More than 95 per cent of this glaciated area is in Antarctica (13×10^6 km^2) and Greenland ($1{\cdot}65 \times 10^6$ km^2). At the maximum extents of the Pleistocene and Fourth Glacial Ages, 32 per cent and 27 per cent respectively of the present land area was covered with ice. Flint (**1**, pp. 429–437) summarizes a number of writers' estimates of the total volume of ice on the Earth, and the data given in Tables I and II show the main conclusions.

Earlier and more severe glaciations may have reduced sea-level by 200 metres.

2. *Isostatic up-doming after deglaciation.*—Following deglaciation of a given region, isostatic compensation results in an up-doming of the crust which formerly

* Thorarinsson's estimate is based on more up-to-date knowledge of the thickness of the ice-sheets, and is probably a better estimate than the others.

carried the ice load. Niskanen (2) estimates the former Fenno-Scandinavian ice-sheet thickness as 2650 metres. In the period from the dissipation of this ice-sheet to 6800 B.C., the up-doming was 270 metres; since 6800 B.C. the up-doming has been 250 metres; a further 210 metres is still to be expected.

It is important to notice that this isostatic compensation occurs at a very much slower rate than the regression or advance of the ice-sheet. Moreover, isostatic compensation seems to occur only when a minimum change in the surface load is exceeded, and minor glacial fluctuations can occur without isostatic compensation. Hence when changes in sea-level occur owing to changes in glacial conditions, there is an immediate change in the moment of inertia of the Earth followed by a slower compensatory change of opposite sign.

TABLE II

Estimates of the volume of ice at maximum extent of fourth glacial age *

Author	Date of publication	Estimated Volume ($\times 10^6$ km^3)	Sea-level fall compared with present day (metres)
E. Antevs	1928	36·9	90
R. A. Daly	1934	34·3	85

3. *Change in moment of inertia.*—Consider now the actual changes in the moment of inertia of the Earth about its polar axis consequent on a change in glacial conditions. In the recent times which will be considered, nearly all the ice available for melting has been in the Antarctic zone from 70° S. to the pole, and it will be sufficiently accurate to consider a model in which all the ice is there. Further it will be assumed that ice is melted to a uniform depth over all the zone. The error involved in neglecting the areas in the North Polar zone is small because of the symmetry about the polar axis of the geographical locations of the northern and southern zones, and also because of the fact that North Polar ice is not land ice. The melting of sea ice *in situ* does not affect sea-level, and although the melting does change the moment of inertia it is a second-order effect. Two extreme cases have to be considered.

Case (i): No isostatic compensation.

Let S, A = areas of ocean and glaciated surfaces respectively,
d = thickness of ice-cap melted,
x = consequent rise in ocean level,
ρ_i, ρ_w = densities of ice and water respectively,
θ, λ = coordinates of colatitude and longitude,
R = radius of Earth.

Conservation of mass gives

$$Sx\rho_w = Ad\rho_i. \tag{1}$$

The change in the moment of inertia, C, is

$$\Delta C = -d\rho_i R^4 \iint_{\text{glac}} \sin^3\theta \, d\theta \, d\lambda + x\rho_w R^4 \iint_{\text{ocean}} \sin^3\theta \, d\theta \, d\lambda, \tag{2}$$

the first integration being carried out over the glaciated surface and the second over the ocean surface. In (1)

$$S = R^2 \iint_{\text{glac}} \sin\theta \, d\theta \, d\lambda$$

* Flint's nomenclature is retained. Glaciologists differ in this matter, not all recognizing four phases in the last series of glaciations. This does not affect the argument in this paper.

and
$$A = R^2 \iint_{\text{ocean}} \sin\theta \, d\theta \, d\lambda.$$

All the integrals involving θ and λ have been evaluated elsewhere (**3**). Taking the values previously found, with $\rho_i = 0{\cdot}92\rho_w$ (thus ignoring salinity effects) and $R = 6{\cdot}37 \times 10^8$ cm, it is found that

$$d = 25{\cdot}6\,x \text{ cm} \tag{3}$$

when the glaciated area is taken as 70° S. to the pole. For a rise in sea-level of 1 cm

$$\frac{\Delta C}{C} = 1{\cdot}15 \times 10^{-9},$$

C being taken as $8{\cdot}08 \times 10^{44}$ g cm^2.

Case (ii): Complete and immediate isostatic compensation.—In this case when the ice melts and flows into the ocean the crust adjusts itself until hydrostatic equilibrium is restored everywhere. There is therefore no resultant change in the surface load per unit area and the only change in the moment of inertia is a second-order effect due to changes in the density distribution at the surface. If isostatic compensation occurs immediately and completely there is no first-order change in the moment of inertia.

4. *Theoretical change in Earth's angular speed and apparent secular acceleration of Moon.*—By conservation of angular momentum,

$$C\omega = \text{constant},$$

and by definition,

$$D\omega = 2\pi,$$

where ω is the Earth's angular speed and D is the length of the day. It is assumed that the polar axis is the axis of rotation. Hence

$$\frac{\Delta C}{C} = -\frac{\Delta\omega}{\omega} = \frac{\Delta D}{D}.$$

An isostatically uncompensated rise in sea-level of amount 1 cm would therefore increase the length of the day by

$$\Delta D = 10^{-4} \text{ seconds}.$$

The corresponding change in ω can be inferred, but it is of more interest to relate this to the average apparent secular acceleration of the Moon ($\bar{\nu}$) which would be observed as a consequence of the Earth's deceleration.

Jeffreys (**4**, p. 223) shows

$$\nu = \frac{dn}{dt} - \frac{n}{\omega}\frac{d\omega}{dt},$$

where n is the mean motion of the Moon. Since a change in sea-level is a purely terrestrial event, there would be no change in n. Hence the average acceleration,

$$\bar{\nu} = -\frac{n}{\omega}\frac{\Delta\omega}{\Delta t} = +\frac{n}{C}\frac{\Delta C}{\Delta t}.$$

Taking $\Delta t = 1$ century, a rise in sea-level at the average rate of 1 cm per century would, if uncompensated, give an average secular acceleration

$$\bar{\nu} = 2''/(\text{century})^2. \tag{4}$$

5. *Observed lunar acceleration and tidal friction.*—Actual observations of the Moon's secular acceleration (**5**) show that over the last 2000 years the acceleration has been

$$\tfrac{1}{2}\nu = (5{\cdot}22 \pm 0{\cdot}45)''/(\text{century})^2,$$

and over the last 200 years,

$$\tfrac{1}{2}\nu = (3{\cdot}11 \pm 0{\cdot}85)''/(\text{century})^2.$$

Hitherto the accepted explanation of the lunar acceleration has been that it is due to the dissipation of energy by tidal friction. Jeffreys (**4**, p. 231) says that this cause is certainly enough to account for a large fraction of the secular acceleration, and there is no reason to suppose it cannot account for all of it.

Spencer Jones states that there seems to be no escape from the conclusion that the effects of tidal friction are appreciably less at the present time than the average effects over the past two thousand years.

About 70 per cent of the tidal friction is generated in the Bering Sea, which has an average depth of about 50 metres and which shows no signs of having experienced any crustal movement for a very long time. Other factors being unchanged, tidal friction varies as the depth of the sea in which it is generated.* It follows that a rise of sea-level of 1 cm would increase tidal friction in the Bering Sea by only 0·02 per cent. Although it is known that isostatic uplifting is decreasing the depth of some shallow seas, this can hardly reduce the proportion of the total friction generated in the Bering Sea substantially below 70 per cent, and a rise in sea-level of 1 cm would therefore increase ν by less than $0''{\cdot}002/(\text{century})^2$.

6. *Tentative discussion of evidence of sea-level fluctuations.*—Without a reasonably accurate reconstruction on a world-wide scale of climatic changes and of fluctuations of sea-level, the actual contribution of eustatic changes to the Earth's rotation must remain unknown. Unfortunately data for the immediate past are scanty. The following discussion is given, therefore, only to demonstrate the possible order of magnitude of the effects.

It is generally accepted that following the last glacial phase, climatic conditions ameliorated until the "Climatic Optimum" which lasted from 6 000 to 4 000 years ago. Since then a gradual regression has occurred, interrupted by minor fluctuations. Stearns (**6**) reports the widespread occurrence of a well-preserved shore line at +5 ft. among Pacific islands. Its wide distribution leads Flint to suggest that this marks sea-level during the Climatic Optimum. A reconstruction of more recent climate based on ecological studies of Alaskan glacier regions by Laurence (**7**) gives strong evidence of the ending of the climatic regression about A.D. 1750 since when glacier retreat has been almost everywhere general. Half of the Alaskan retreat since 1750 has actually occurred since 1910. This correlates well with the independent evidence of statistical analyses of tide-gauge records. Flint (**1**, p. 428) reports that Gutenberg and Marmer have independently found that sea-level has been rising at the rate of 2·5 inches per century for the last few decades, a rate which is greater than during the preceding century. If this connection is real, since 1750 sea-level has risen at a rate of 2·5 cm per century, and since 1910 at the rate of 6·4 cm per century, and if this has occurred, as is probable, without isostatic compensation, then applying (4)

$$\bar{\nu} = +\ 5''/(\text{century})^2 \qquad \text{since 1750,}$$

and

$$\bar{\nu} = +13''/(\text{century})^2 \qquad \text{since 1910.}$$

* A referee has pointed out that owing to magnification of the currents in shallow seas it might be more reasonable to suppose that tidal friction would vary inversely with depth. To a first approximation this alters the sign but not the magnitude of the change in $\bar{\nu}$ so long as the change in depth remains small compared with the total depth.

Zeuner (**8**, pp. 93–94) quotes results of work by Godwin, confirmed by others, of sea-level fluctuations in Southern Britain since 8000 B.C. Successive maxima and minima are shown on a graph as having occurred at about 1600 B.C. (+2 ft.), 800 B.C. (−15 ft.), A.D. 0 (+5 ft.), A.D. 700 (−7 ft.), followed by a rise to the present-day level. These data, if typical of world-wide fluctuations, yield fluctuations of $\bar{\nu}$ between $-130''$ and $+150''/(\text{century})^2$ with an average value of about $+34''/(\text{century})^2$ since A.D. 700. The fluctuations would almost certainly occur too rapidly for isostatic compensation to have much effect. Accepting Flint's suggestion regarding the +5 ft. shore line, and taking this as being uncompensated, it follows that

$$\bar{\nu} = -5''/(\text{century})^2 \qquad \text{from 60 to 2 centuries ago.}$$

Niskanen's data show, however, that the isostatic compensation for the final phase of the last major series of glaciations is still going on. A very rough estimate of its effect may be derived by assuming that the Fenno-Scandinavian data are applicable generally, and that the rate of compensation for the last 9000 years has been enough to compensate one-third of the rise of sea level, i.e. it is equivalent to a fall in sea-level of 27 metres in 90 centuries. This gives an estimate of $\bar{\nu} = -60''/(\text{century})^2$ which has to be superimposed on the values found for changes due to very recent uncompensated fluctuations in sea-level.

7. *Conclusions.*—These results serve to show that changes of sea-level can contribute significant amounts to the secular acceleration of the Moon, and recent geological records, however imperfect, do indicate that quite violent fluctuations are likely to have occurred. Reliable results, however, can only be obtained when the geological information is sound enough to allow the full interplay of isostasy and eustasy to be deduced.

As far as tidal friction is concerned, it seems reasonable to assert that it is not the only factor involved in the secular deceleration of the Earth and the acceleration of the Moon. The evidence points to a rise in sea-level at the present time, and as a consequence, tidal friction is changing, but only slightly.

8. *Acknowledgments.*—The author wishes to acknowledge with thanks the help given by Professor Gordon Manley of Bedford College, London, who kindly provided a very detailed bibliography of climatological studies.

The Institute of Mathematics,
The University,
Liverpool 3 :
1952 *December* 24.

References

(1) Flint, R. F., *Glacial geology and the Pleistocene epoch*, New York, 1947.
(2) Niskanen, E., *Ann. Acad. Sci. Fenn.* **53,** 10, 1939; and A**3,** 21, 1949.
(3) Young, A., *Bull. Acad. Belg., Cl. Sci., ser.* 5, **37,** 728–751, 1951.
(4) Jeffreys, H., *The Earth*, 3rd edn., Cambridge, 1952.
(5) Jones, H. Spencer, *M.N.*, **99,** 541–558, 1939.
(6) Stearns, H. T., *Bull. Geol. Soc. Amer.*, **56,** 1071–1078, 1945.
(7) Laurence, D. B., *Geogr. Rev.*, **10,** 191–223, 1950.
(8) Zeuner, F., *Dating the Past*, London, 1946.

General reviews of eustatic changes are given in Flint and Zeuner which have been used as suitable collations. Other works, such as the well-known books by C. E. P. Brooks on climatology, have also been consulted.

(4)

Reprinted from the Monthly Notices of the Royal Astronomical Society, Geophysical Supplement, *Vol.* 6, *No.* 8, 1953

THE EFFECT OF THE MOVEMENT OF SURFACE MASSES ON THE ROTATION OF THE EARTH

by

ANDREW YOUNG

Printed in England by Taylor & Francis, Ltd., London

Reprinted from the MONTHLY NOTICES OF THE ROYAL ASTRONOMICAL SOCIETY, GEOPHYSICAL SUPPLEMENT, *Vol.* 6, *No.* 8, 1953

THE EFFECT OF THE MOVEMENT OF SURFACE MASSES ON THE ROTATION OF THE EARTH

Andrew Young

(Received 1953 March 16)

Summary

When account is taken of the angular momentum of surface masses moving relative to the solid Earth it is found that the equations governing the variation of latitude are not sensibly affected by changes in the rate of rotation and that the equations previously used are adequate if correction terms are added to account for the angular momentum which may be large enough to have an observable effect.

1. Sir Harold Jeffreys (**1**, pp. 215, 216) has expressed doubt concerning the validity of the equations of motion governing the variation of latitude. An observer on the solid surface of the Earth cannot observe the changing motion of the whole Earth, since the axes fixed in the solid Earth to which he is attached do not necessarily continue to coincide with the axes of reference used in the equations as they are usually derived. Moreover, with the seasonal redistribution of surface masses which gives rise to the forced motion in the variation of latitude, there is associated a seasonal variation of angular momentum relative to the solid Earth which causes an observable variation in the rate of rotation.

The purpose of this note is to show that, in fact, the theories of the variation of latitude and the variation of the length of the day may be treated independently and that apart from a correction for the angular momentum, the usual variation of latitude equations are adequate.

2. Consider the Earth as an ellipsoidal body (the solid Earth) with mass centre O at which the principal moments of inertia are A_0, B_0 and C_0 and principal axes $Oxyz$, surrounded by a mobile envelope of particles whose mass- and relative velocity-distributions are known. The total mass and dimensions of the envelope are small compared with those of the solid Earth. The Earth as a whole rotates about a variable axis near the polar axis Oz. An observer attached to the solid Earth actually observes the motion of Oz provided the solid Earth moves as if rigid.

Let a, b, c, f, g, h be the instantaneous values of the moments and products of inertia of the envelope relative to the axes $Oxyz$ and let the relative angular momentum of the envelope have components (X, Y, Z). Let $(\omega_x, \omega_y, \omega_z)$ be the components of the angular velocity of the axes $Oxyz$ in space.

Then if $A=A_0+a$, etc., the equations of motion are

$$\left.\begin{aligned}
&\frac{d}{dt}(A\omega_x-h\omega_y-g\omega_z+X)\\
&\qquad+\omega_y(-g\omega_x-f\omega_y+C\omega_z+Z)-\omega_z(-h\omega_x+B\omega_y-f\omega_z+Y)=0,\\
&\frac{d}{dt}(-h\omega_x+B\omega_y-f\omega_z+Y)\\
&\qquad+\omega_z(A\omega_x-h\omega_y-g\omega_z+X)-\omega_x(-g\omega_x-f\omega_y+C\omega_z+Z)=0,\\
&\frac{d}{dt}(-g\omega_x-f\omega_y+C\omega_z+Z)\\
&\qquad+\omega_x(-h\omega_x+B\omega_y-f\omega_z+Y)-\omega_y(A\omega_x-h\omega_y-g\omega_z+X)=0.
\end{aligned}\right\}\quad(1)$$

3. The following data are relevant to the discussion. The Earth is so nearly spheroidal that two axes can be chosen anywhere in the equatorial plane and if Ox is chosen through the meridian of Greenwich, Oy through the meridian 90° E., then for seasonal variations in the distribution of atmospheric air masses

$$\begin{aligned} a &= 2{\cdot}9\cos\tfrac{1}{6}\pi t - 0{\cdot}2\sin\tfrac{1}{6}\pi t,\\ b &= -1{\cdot}4\cos\tfrac{1}{6}\pi t + 0{\cdot}2\sin\tfrac{1}{6}\pi t,\\ c &= -1{\cdot}5\cos\tfrac{1}{6}\pi t,\\ f &= 4{\cdot}8\cos\tfrac{1}{6}\pi t - 0{\cdot}7\sin\tfrac{1}{6}\pi t,\\ g &= 0{\cdot}5\cos\tfrac{1}{6}\pi t - 0{\cdot}3\sin\tfrac{1}{6}\pi t, \end{aligned}$$

when t is measured in months from January 15, the unit being 10^{35} g cm². These values are taken from analyses of sequences of monthly values which have been found for the period 1927–1932 by a method described elsewhere (2). h which has not been calculated must be of about the same order of magnitude. In every case a semi-annual term with an amplitude about one-eighth of that of the annual term is present but this component does not survive the subsequent integration and is ignored here (3). W. H. Munk and R. L. Miller (4) give the annual range $Z_{\text{Jan}}-Z_{\text{July}}$ as of order $1{\cdot}75\times10^{-8}\,C\omega_z$ based on seasonal changes in the atmospheric circulation with a small contribution from oceanic current changes. No estimates of X and Y are available. The observed order of magnitude of ω_x and ω_y is about $5\times10^{-7}\omega_z$, A_0, B_0, C_0 are of order 8×10^{44} g cm² and

$$\frac{C_0-A_0}{A_0} \doteqdot \frac{1}{305}.$$

With these values, typical terms in the equations of motion have magnitudes of the following orders:

$$C_0\omega_z\sim 8\times10^{44}\omega_z,\qquad A_0\omega_x\sim 4\times10^{38}\omega_z,\qquad Z\sim 1{\cdot}4\times10^{37}\omega_z,$$
$$c\omega_z\sim 1{\cdot}5\times10^{35}\omega_z,\qquad a\omega_x\sim 1{\cdot}4\times10^{29}\omega_z.$$

4. Retaining quantities as small as $10^{-10}\,C\omega_z$, the equations of motion are

$$\left.\begin{aligned} \dot{\xi}+\frac{\eta+f\omega_z}{B_0}(C\omega_z+Z)-\eta\,\omega_z &= -\dot{X}+Y\omega_z,\\ \dot{\eta}+\xi\,\omega_z-\frac{\xi+g\omega_z}{A_0}(C\omega_z+Z) &= -\dot{Y}-X\omega_z,\\ \frac{d}{dt}(C\omega_z+Z) &= 0, \end{aligned}\right\}\tag{2}$$

where

$$\left.\begin{aligned} \xi &= A_0\omega_x-g\omega_z,\\ \eta &= B_0\omega_y-f\omega_z, \end{aligned}\right\}\tag{3}$$

and a dot denotes differentiation with respect to t. X and Y have been retained as being possibly of the order of magnitude of the otherwise smallest term.

The final equation immediately gives

$$C\omega_z+Z=\text{constant}=k\text{ say},\tag{4}$$

thus justifying the usual approach to the theory of the variation of the length of the day. The first two equations of (2) become

$$\left.\begin{aligned} \dot{\xi}+\dot{\alpha}\eta &= -\dot{X}+\left(Y-\frac{kf}{A_0}\right)\left(\frac{k-Z}{C}\right)=\phi,\\ \dot{\eta}-\dot{\alpha}\xi &= -\dot{Y}-\left(X-\frac{kg}{A_0}\right)\left(\frac{k-Z}{C}\right)=\psi, \end{aligned}\right\}\tag{5}$$

where

$$\dot{\alpha} = \frac{k}{A_0} - \frac{k-Z}{C}. \tag{6}$$

At this stage it is assumed that the solid Earth is truly spheroidal. If this is not so, it can be shown that if $A_0 \doteqdot B_0$ then the motion is given to a high degree of approximation by that of a spheroid for which the equatorial moments of inertia are $\frac{1}{2}(A_0+B_0)$ and the above equations will still hold provided it is understood that A_0 is the mean of the two moments of inertia.

Writing $u = \xi + \mathrm{i}\eta$ and $\chi = \phi + \mathrm{i}\psi$, the solution for the motion is formally

$$u = \mathrm{e}^{\mathrm{i}\alpha} \int \mathrm{e}^{-\mathrm{i}\alpha} \chi \, dt. \tag{7}$$

5. In (6) let $Z = z \cos(\frac{1}{6}\pi t + \epsilon)$ and to the order of accuracy so far retained

$$\alpha = \alpha_0 t + \alpha_1 \sin(\tfrac{1}{6}\pi t + \epsilon),$$

where $\alpha_0 = k\dfrac{(C_0 - A_0)}{A_0 C_0}$ and $\alpha_1 = \dfrac{6z}{\pi C_0}$.

With the seasonal data quoted above, $\alpha_1 \sim 3 \times 10^{-6}$ and so in (7)

$$\cos\alpha = \cos\alpha_0 t - \alpha_1 \sin(\tfrac{1}{6}\pi t + \epsilon)\sin\alpha_0 t,$$

$$\sin\alpha = \sin\alpha_0 t + \alpha_1 \sin(\tfrac{1}{6}\pi t + \epsilon)\cos\alpha_0 t.$$

The integrals in (7) may now be evaluated, the ultimate result being that no terms containing α_1 are large enough to be observable. The conclusion is that for all observable purposes (6) may be written

$$\dot{\alpha} = k\frac{(C_0 - A_0)}{A_0 C_0},$$

and ω_z may be treated as constant in the variation of latitude equations.

Reverting to equation (1) let $\omega_z = n$, $A_0/(C_0 - A_0) = \tau$, $\omega_x/n = l$, $\omega_y/n = m$. Then $(l, m, 1)$ are the direction cosines of the axis of rotation referred to $Oxyz$. The variation of latitude equations now become

$$\left.\begin{aligned} \dot{l} + \frac{n}{\tau} m &= \{-\dot{f}n + g + Y - \dot{X}/n\}/A_0, \\ \dot{m} - \frac{n}{\tau} l &= \{gn + \dot{f} - X - \dot{Y}/n\}/A_0, \end{aligned}\right\} \tag{8}$$

which are the familiar equations with corrections added for the angular momentum of the moving surface masses. In the usual derivation $\dot{f}$ and $\dot{g}$ are discarded as being small compared with f and g. In this derivation the terms $\dot{X}$ and $\dot{Y}$ may probably be discarded for the same reason.

The X and Y terms are probably of importance, however, since, from the seasonal data for the atmospheric circulation, $Z/fn \sim 30$ and $Z/gn \sim 240$. If X/Z and Y/Z are as large as 10^{-3}, account must be taken of the angular momentum of the atmosphere in the variation of latitude. It is conjectured that observed irregularities in the variation of latitude may be in part due to this effect since the angular momentum of the atmosphere may vary considerably without substantially changing the actual atmospheric mass-distribution.

6. For sustained secular changes in surface mass-distributions, equations (4) and (7) still describe the motion. The effect on the length of the day of changes in sea level due to the partial deglaciation of the Antarctic ice cap has already been

studied (5). The following data apply in the case of a rise in sea level at the rate of 2·5 inches/century for which there is at present some evidence:

$$\dot{a} = -2{\cdot}3 \qquad \dot{b} = -2{\cdot}5 \qquad \dot{c} = +4{\cdot}8$$

$$\dot{f} = -2{\cdot}1 \qquad \dot{g} = -1{\cdot}4.$$

the units being $10^{33}\,\mathrm{g\,cm^2\,month^{-1}}$ when t is measured in months. In this case the associated angular momentum is negligible. For periods of decades, equation (4) becomes

$$(C_0+c)\,\omega_z = k \tag{9}$$

and the effect on the variation of latitude is found adequately by putting f and g into (8). This leads to a secular movement of the pole parallel to the meridian $\tan^{-1}\dot{f}/\dot{g}$ East. When the sea level changes persist over centuries, however, it may be necessary to retain A and B instead of A_0 and B_0 in (3), and to solve (4) and (5) with X, Y, Z all zero. Second-order differential equations have been derived for ξ and η in this case but it is not considered necessary to give them here.

7. *Conclusion.*—It is concluded that in investigating the effect of surface mass movements on the general rotation of the Earth, the change in the length of the day may be treated independently of the variation of latitude. Further, the seasonal variation of latitude is described by the equations usually formulated, but account may have to be taken of corrections for the angular momentum of the surface masses. In the equations now derived, however, the axes of reference are fixed in the solid Earth and so are fixed relative to an observer on the solid Earth, thus disposing of a doubt expressed by Jeffreys. The fact that sudden fluctuations in angular momentum of the atmosphere occur may account in part at least for the irregularities of the observed variation of latitude. The equations remain valid for secular changes in mass distributions provided such changes persist for periods of decades; for periods of centuries the equations possibly need modification.

The Institute of Mathematics,
The University,
Liverpool 3:
1953 *March* 13.

References

(1) Harold Jeffreys, *The Earth*, 3rd edn., Cambridge, 1952.
(2) A. Young, *Bull. Acad. Roy. Belg., Cl. Sci.*, ser. 5, **37**, 728–751, 1951.
(3) A. Young, *Bull. Acad. Roy. Belg., Cl. Sci.*, ser. 5, **38**, 824–837, 1952.
(4) W. H. Munk and R. L. Miller, *Tellus*, **2**, 93–101, 1950.
(5) A. Young, *M.N., Geophys. Suppl.*, **6**, 453–457, 1953.

(5)

Reprinted from the Monthly Notices of the Royal Astronomical Society,
Vol. 115, *No.* 4, 1955

THE ANALYSIS OF THE OBSERVATIONS OF THE VARIATION OF LATITUDE

by

A. M. WALKER *and* ANDREW YOUNG

Reprinted from the MONTHLY NOTICES OF THE ROYAL ASTRONOMICAL SOCIETY,
Vol. 115, *No.* 4, 1955

THE ANALYSIS OF THE OBSERVATIONS OF THE VARIATION OF LATITUDE

A. M. Walker and Andrew Young

(Received 1955 April 2*)

Summary

A statistical model of the variation of latitude is formulated and a detailed analysis of its properties is made. Difficulties arising from the necessity to account for observational errors are discussed.

A computational procedure is obtained and applied to a 35-year series of observations in which the z-term has not been taken into account. It is found that the damping factor is

$$\kappa = 0{\cdot}322 \pm 0{\cdot}103 \text{ year}^{-1},$$

giving a relaxation time of about 3·1 years; and the free period is

$$2\pi/\gamma = 1{\cdot}267 \pm 0{\cdot}039 \text{ years}$$
$$= 462{\cdot}8 \pm 14{\cdot}2 \text{ mean solar days.}$$

The value of κ is several times that of the only previous determination, and the free period is longer than has hitherto been determined.

These results are provisional pending the completion of a full numerical analysis.

Part I. *Introduction*

1. If l, m are direction cosines of the instantaneous axis of rotation of the Earth with respect to right-handed axes fixed in the Earth, the dynamical equations of motion may be written

$$\left.\begin{aligned} \dot{l} + \gamma m &= -\omega_0 F/A, \\ \dot{m} - \gamma l &= \quad \omega_0 G/A, \end{aligned}\right\} \tag{1}$$

the solution of which, if l_0, m_0 are the initial values of l and m, is

$$\left.\begin{aligned} l &= l_0 \cos\gamma t - m_0 \sin\gamma t + l_1, \\ m &= l_0 \sin\gamma t + m_0 \cos\gamma t + m_1. \end{aligned}\right\} \tag{2}$$

Here, $2\pi/\gamma$ is the so-called Chandler period of the (circular) free component of the motion, ω_0 is the mean angular velocity of the Earth, (A, A, C) are the moments of inertia and F and G are the products of inertia of the Earth. Assuming that the products of inertia are periodic with periodicities of a year and sub-multiples of a year, the forced motion, l_1, m_1, will have elliptical components corresponding to each period in F and G.

This dynamical theory suggests that harmonic analysis of values of l and m, which are observed as the variation of latitude, should yield the value of the Chandler period and the various periods of the forced motion. However, very many investigations over different intervals have been made by a number of workers and have all yielded different values of the Chandler period. As early as 1917, Kimura gave an empirical formula for $2\pi/\gamma$ containing components with periods of 22, 26·67, 50 and 80 years from which Melchior (1) has calculated a table of values of the Chandler period during the interval 1900–1960. The maximum value

* Received in original form 1954 October 18.

is 1·2066 years in 1914 and the minimum is 1·1123 years in 1929. Different authors have found, by harmonic analysis of different intervals, Chandler periods within this range of values. Melchior has developed a theory relating variations of the Chandler period to changes in the axes of the elliptic forced motion.

In 1940, Harold Jeffreys (2) rejected the validity of harmonic analysis as a means of determining the Chandler period on the ground that undamped free vibrations in the sense of elementary dynamics probably have no physical existence. Irregular disturbances of the motion may well repeatedly change the phase of the motion; despite this there exists a natural free period.

The appearance of the series of observations suggests that statistical fluctuations are not negligible, and harmonic analysis cannot take such fluctuations adequately into account. Jeffreys's paper, in fact, contains the only example so far published of an analysis which is based on a proper statistical model. Jeffreys's treatment is re-examined in this paper in the light of recent developments in the theory of time series analysis. The problems of estimation and statistical inference associated with his model are discussed in detail, account being taken of the fact that the effect of observational errors may not be negligible. The theory of the statistical analysis is given below in Part II. The principal steps of the computational procedure based on the statistical analysis are outlined at the beginning of Part III, which contains the results of carrying out this analysis on a certain series of observations.

An analysis of the residual motion is given in Part IV.

Part II. *The Statistical Model*

2. *Specification of the statistical model.*—The equations of motion (1), including a damping term, may be written

$$\left.\begin{aligned} \dot{l}+\kappa l+\gamma m &= -F_1, \\ \dot{m}+\kappa m-\gamma l &= G_1, \end{aligned}\right\} \tag{3}$$

where $F_1=\omega_0 F/A$, $G_1=\omega_0 G/A$ and κ is the coefficient of damping.

Now let

$$\left.\begin{aligned} F_1(t) &= F_S(t)+f_1(t), \\ G_1(t) &= G_S(t)+g_1(t), \end{aligned}\right\} \tag{4}$$

where F_S, G_S are the systematic parts of F_1 and G_1 respectively and $f_1(t)$, $g_1(t)$ are the fluctuation terms, $\{f_1(t), g_1(t)\}$ being in fact a stochastic process in two variables with zero means. We make the usual assumption that F_S and G_S have a yearly period, and can be represented to a sufficiently good approximation by the first few terms of their Fourier series, e.g.

$$F_S(t)=c_f+\sum_{r=1}^{p}(a_{fr}\cos\nu rt+b_{fr}\sin\nu rt),$$

where p is a small integer and $2\pi/\nu=1$ year.

The simplest statistical assumption as regards the fluctuation terms is that $\{f_1(t)\}$ and $\{g_1(t)\}$ are independent stationary impulse processes. This means that only the integrals $I(t)=\int^t f_1(u)\,du$ and $J(t)=\int^t g_1(u)\,du$ strictly exist, $\{I(t)\}$ and $\{J(t)\}$ being statistically independent homogeneous additive processes, characterized by the property that the change of I (or J) in any time interval depends only on the length of the interval, and changes in non-overlapping time intervals are statistically independent of one another (see e.g. Bartlett (3), Chapter III, Section 6, and Moyal (4), page 165). If the impulses are discrete, the changes in I or J are discontinuous.

From (3), F_S and G_S give rise to forced components $l_s(t)$ and $m_s(t)$ of the same form (the damped components may be ignored since the motion may be assumed to have started a very long time ago); and for the fluctuation terms we have

$$x(t)+\mathrm{i}y(t)=\int^{t}\mathrm{e}^{(\mathrm{i}\gamma-\kappa)(t-u)}(\mathrm{i}dJ(u)-dI(u)), \tag{5}$$

where

$$\left.\begin{aligned} x(t)&=l(t)-l_s(t),\\ y(t)&=m(t)-m_s(t). \end{aligned}\right\} \tag{6}$$

Thus x, y are the contributions to the motion of the statistical fluctuations in the products of inertia of the Earth.

If we now consider observations at times separated by a constant interval h, and write

$$l_n=\ l(t_0+nh)=l_{sn}+x_n,$$
$$m_n=m(t_0+nh)=m_{sn}+y_n,$$

we find that

$$\left.\begin{aligned} x_{n+1}-\alpha x_n+\beta y_n&=\epsilon_{n+1},\\ y_{n+1}-\beta x_n-\alpha y_n&=\eta_{n+1}, \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} \alpha&=\mathrm{e}^{-\kappa h}\cos\gamma h,\\ \beta&=\mathrm{e}^{-\kappa h}\sin\gamma h,\\ \epsilon_{n+1}&=-\int_0^h \mathrm{e}^{-\kappa u}\{\cos\gamma u\,dI(\overline{n+1}h-u)+\sin\gamma u\,dJ(\overline{n+1}h-u)\},\\ \text{and}\quad \eta_{n+1}&=\int_0^h e^{-\kappa u}\{\cos\gamma u\,dJ(\overline{n+1}h-u)-\sin\gamma u\,dI(\overline{n+1}h-u)\}. \end{aligned}\right\} \tag{8}$$

The residuals $\{\epsilon_n, \eta_n\}$ form an independent stationary process, i.e. its properties are invariant under translation of the time axis, and $\{\epsilon_{n+r}, \eta_{n+r}\}$, $(r=0,\ 1,\ 2\ldots)$ are all statistically independent. If we make the further natural assumption that the rates of increase of variance per unit time for I and J are equal, we have also

$$\left.\begin{aligned} \operatorname{var}\epsilon_n=\operatorname{var}\eta_n&=\sigma^2 \text{ say},\\ \text{and}\quad \operatorname{cov}(\epsilon_n, \eta_n)&=0. \end{aligned}\right\} \tag{9}$$

Thus we have arrived at a model identical with that postulated by Jeffreys (2, page 143) apart from his additional assumption that the residuals follow a normal distribution.

Our assumptions for the fluctuation terms can of course give only a very approximate representation of the actual mechanism, but an analysis based on them may nevertheless still be sufficiently accurate to be useful. That Jeffreys's model follows from our statistical treatment may be regarded as an argument for using this as a working hypothesis. In (2) the model is introduced by analogy with Yule's treatment of the problem of a pendulum bombarded by randomly arriving particles, but this has since been shown to be incorrect (see e.g. Bartlett (5), page 35).

The assumption of normality of the residuals is more difficult to justify. This is approximately true only provided that the mean number of impulses during the interval of observation, h, is large, a condition which is perhaps not very realistic for values of h which are usually considered in practice (of the order of a month). Jeffreys even claims (**2**, page 148) to have identified a period of more than 12 years during which the effects of fluctuations were negligible. Hence it is advisable to dispense with the normality assumption if at all possible. It is not difficult to show that we can do so provided that the effect of observational error can be ignored, so that equations (7) can be used as they stand. However, as Jeffreys points out, observational errors are not necessarily small compared with x_n and y_n, so that they should be allowed for in the model. Unfortunately when this is done an important part of the analysis becomes very complicated unless the residuals are normally distributed, and indeed is tractable only when both the residuals and observational errors are normally distributed.

We therefore re-define (x_n, y_n) to include the effect of errors of observation and assume that

$$\left.\begin{aligned} x_n &= x'_n + u_n, \\ y_n &= y'_n + v_n, \end{aligned}\right\} \tag{10}$$

where both $\{x_n', y_n'\}$, which satisfies (7), and the errors of observation, $\{u_n, v_n\}$, are normal. It seems reasonable to assume that

$$\left.\begin{aligned} \operatorname{var} u_n = \operatorname{var} v_n &= \sigma'^2, \text{ say,} \\ \operatorname{cov}(u_n, v_n) &= 0, \end{aligned}\right\} \tag{11}$$

and $\{u_{n+r}, v_{n+r}\}$, $(r=0, 1, 2, \ldots)$, are all statistically independent and also that $\{u_n, v_n\}$ is statistically independent of the series $\{x'_n, y'_n\}$.

These properties, together with

$$\left.\begin{aligned} l_{sn} &= c_l + \sum_{r=1}^{p} (a_{lr} \cos \nu rnh + b_{lr} \sin \nu rnh), \\ m_{sn} &= c_m + \sum_{r=1}^{p} (a_{mr} \cos \nu rnh + b_{mr} \sin \nu rnh), \end{aligned}\right\} \tag{12}$$

and

$$\left.\begin{aligned} x_{n+1} - \alpha x_n + \beta y_n &= \epsilon_{n+1} + u_{n+1} - \alpha u_n + \beta v_n = \epsilon'_{n+1}, \text{ say,} \\ y_{n+1} - \beta x_n - \alpha y_n &= \eta_{n+1} + v_{n+1} - \beta u_n - \alpha v_n = \eta'_{n+1}, \text{ say,} \end{aligned}\right\} \tag{13}$$

completely specify the model. Equation (12) specifies the true dynamical contributions to the motion as opposed to the statistical fluctuations; equation (13) results from inserting (10) in (7).

3. *Correlational and spectral properties of the model.*—It is not difficult to show from (13) that the autocovariance matrix for lag s, defined by

$$V(s) = \begin{bmatrix} \overline{x_n x_{n+s}} & \overline{x_n y_{n+s}} \\ \overline{y_n x_{n+s}} & \overline{y_n y_{n+s}} \end{bmatrix},$$

is given by

$$\left.\begin{aligned} V(s) &= \left\{\frac{\sigma^2 e^{-s\kappa h}}{1 - e^{-2\kappa h}}\right\} \begin{bmatrix} \cos s\gamma h & \sin s\gamma h \\ -\sin s\gamma h & \cos s\gamma h \end{bmatrix}, \qquad (s>0), \\ V(0) &= \left\{\sigma'^2 + \frac{\sigma^2}{1 - e^{-2\kappa h}}\right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}\right\} \tag{14}$$

The autocorrelation matrix for lag s, $(s>0)$, obtained by replacing x_n, y_n in $V(s)$ by $x_n/\sigma(x)$, $y_n/\sigma(y)$, where $\sigma^2(x)=\operatorname{var} x$, $\sigma^2(y)=\operatorname{var} y$, is therefore

$$R(s)=\lambda e^{-s\kappa h}\begin{bmatrix}\cos s\gamma h & \sin s\gamma h\\ -\sin s\gamma h & \cos s\gamma h\end{bmatrix}, \tag{15}$$

where

$$\lambda=\sigma^2/\{\sigma^2+\sigma'^2(1-e^{-2\kappa h})\}. \tag{16}$$

Equation (15) summarizes the main correlational properties of the model and was in effect given by Jeffreys. The theoretical correlograms for the autocorrelations of the $\{x\}$ and $\{y\}$ series are identical, consisting of an exponentially damped simple harmonic oscillation, and a phase shift of 90° yields the theoretical correlogram for the cross-correlations.

This result expresses the physical fact that, apart from the damping, the expected motion is given by a simple rotation of the pole about its mean position through the angle $s\gamma h$ in time sh. The effect of the observational errors is seen in the reduction of all the lag correlations by the factor λ.

We also note that the spectral density matrix $F(\omega)=(1/2\pi)\sum_{s=-\infty}^{\infty} V(s)e^{-i\omega s}$, i.e. the Fourier transform of the autocorrelation matrix, is given by

$$2\pi F(\omega)=\sigma'^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\sigma^2\{(1+\alpha^2+\beta^2-2\alpha\cos\omega)^2-4\beta^2\sin^2\omega\}^{-1}$$
$$\times\begin{bmatrix}1+\alpha^2+\beta^2-2\alpha\cos\omega & -2i\beta\sin\omega\\ 2i\beta\sin\omega & 1+\alpha^2+\beta^2-2\alpha\cos\omega\end{bmatrix}. \tag{17}$$

4. *Analysis of the systematic components of the variation.*—We now have to consider methods of estimating the unknown parameters in the model, and of calculating the precision of our estimates, given an actual series of observations $\{l_r, m_r\}$ $(r=1, 2, \ldots N)$. There is also the associated problem of testing whether the model is consistent with the observations. The parameters fall naturally into two groups, the first consisting of those which occur in the equations (12), i.e. c_l, a_{lr}, b_{lr}, c_m, a_{mr}, b_{mr}, defining the systematic components of the variation, and the second of those associated with the equations (8) and (13), i.e. α, β (or κ and γ), σ^2 and σ'^2, which define the behaviour of the fluctuations. The two groups can be dealt with to a large extent independently of each other; we must however begin by considering estimates of the parameters in the first group, since these are required before we can calculate the estimates for the second group. We shall assume from now on that N is sufficiently large for an asymptotic theory to be used; this may not necessarily be true in practice (see Section 6 below), but no exact theory has yet been developed.

It can be shown by using an approach due to Whittle (**6, 7**) that the maximum likelihood estimates are asymptotically the same as those obtained by standard harmonic analysis i.e.

$$\left.\begin{aligned} c^*_l&=\frac{1}{N}\sum_{i=1}^{N} l_i, & a^*_{lr}&=\frac{2}{N}\sum_{i=1}^{N} l_i\cos\nu rih, & b^*_{lr}&=\frac{2}{N}\sum_{i=1}^{N} l_i\sin\nu rih,\\ c^*_m&=\frac{1}{N}\sum_{i=1}^{N} m_i, & a^*_{mr}&=\frac{2}{N}\sum_{i=1}^{N} m_i\cos\nu rih, & b^*_{mr}&=\frac{2}{N}\sum_{i=1}^{N} m_i\sin\nu rih.\end{aligned}\right\} \tag{18}$$

They are asymptotically unbiased (and are in fact exactly unbiased if the observations cover a number of complete years). Their asymptotic variances and covariances are given by the inverse of the information matrix as in classical maximum likelihood theory, but are easily obtained directly. We find that the estimates fall into asymptotically uncorrelated groups (c^*_l, c^*_m), $(a^*_{l1}, a^*_{m1}, b^*_{l1}, b^*_{m1})$, $(a^*_{l2}, a^*_{m2}, b^*_{l2}, b^*_{m2})$, etc., and that if we write $\mathbf{c}^{*\prime}=(c^*_l, c^*_m)$, $\mathbf{a}^*_1{}'=(a^*_{l1}, a^*_{m1})$, $\mathbf{b}^*_1{}'=(b^*_{l1}, b^*_{m1})$, etc., these being row vectors and $\mathbf{c}^*$, $\mathbf{a}^*_1$, $\mathbf{b}^*_1$, etc. the corresponding column vectors,

$$\left.\begin{aligned} &\operatorname{cov}(\mathbf{c}^*, \ \mathbf{c}^{*\prime}) \sim 2\pi F(0)/N, \\ &\operatorname{cov}(\mathbf{a}^*_r, \mathbf{a}^*_r{}') \sim \operatorname{cov}(\mathbf{b}^*_r, \mathbf{b}^*_r{}') \sim 2\mathscr{R}\{2\pi F(\nu r h)\}/N, \\ &\operatorname{cov}(\mathbf{a}^*_r, \mathbf{b}^*_r{}') \sim -2\mathscr{I}\{2\pi F(\nu r h)\}/N, \end{aligned}\right\} \tag{19}$$

where $F(\omega)$ is the spectral density matrix given by (17), and the symbols $\mathscr{R}$ and $\mathscr{I}$ denote real and imaginary parts. Thus c^*_l, c^*_m are asymptotically uncorrelated and have variance

$$\left\{\sigma'^2 + \frac{\sigma^2}{1+\alpha^2+\beta^2-2\alpha}\right\}\Big/N$$

and the asymptotic covariance matrix for the group $(a^*_{lr}, a^*_{mr}, b^*_{lr}, b^*_{mr})$ is

$$\begin{array}{c|c} & \begin{array}{cccc} a^*_{lr} & a^*_{mr} & b^*_{lr} & b^*_{mr} \end{array} \\ \begin{array}{c} a^*_{lr} \\ a^*_{mr} \\ b^*_{r} \\ b^*_{mr} \end{array} & \begin{bmatrix} P_r & 0 & 0 & -Q_r \\ 0 & P_r & Q_r & 0 \\ 0 & Q_r & P_r & 0 \\ -Q_r & 0 & 0 & P_r \end{bmatrix} \end{array} \tag{20}$$

where

$$\left.\begin{aligned} P_r &= \frac{2}{N}\left\{\sigma'^2 + \frac{\sigma^2(1+\alpha^2+\beta^2-2\alpha\cos\nu r h)}{(1+\alpha^2+\beta^2-2\alpha\cos\nu r h)^2-4\beta^2\sin^2\nu r h}\right\}, \\ \text{and}\quad Q_r &= \frac{4}{N}\left\{\frac{-\sigma^2\beta\sin\nu r h}{(1+\alpha^2+\beta^2-2\alpha\cos\nu r h)^2-4\beta^2\sin^2\nu r h}\right\}. \end{aligned}\right\} \tag{21}$$

All these estimates are also distributed asymptotically in the multivariate normal form, so that we can not only calculate their estimated standard errors, once estimates of α, β, σ^2, σ'^2, with standard errors which tend to zero as $N\to\infty$, have been obtained (estimates which have standard errors proportional to $1/\sqrt{N}$ are given in Section 5), but can also derive approximate tests of significance and confidence intervals for the true values of the parameters. For example, a^*_{lr} would be regarded as differing from zero at a level of probability approximately equal to 0·95 if it was greater in magnitude than $1{\cdot}96\sqrt{P^*_r}$, where P^*_r denotes the estimated value of P_r. Again the significance of the whole set of estimates $(a^*_{lr}, a^*_{mr}, b^*_{lr}, b^*_{mr})$ of the $(r-1)$th harmonic components could be judged by referring the expression

$$\{P^*_r(a^*{}_r{}^2+a^*{}_{mr}{}^2+b^*{}_{lr}{}^2+b^*{}_{mr}{}^2)+2Q^*_r(a^*_{lr}b^*_{mr}-a^*_{mr}b^*_{lr})\}/(P^*{}_r{}^2-Q^*{}_r{}^2)$$

to a χ^2 distribution with four degrees of freedom; this follows from the fact that the inverse of the matrix (20) is

$$\frac{1}{(P_r^2-Q_r^2)}\begin{bmatrix} P_r & 0 & 0 & Q_r \\ 0 & P_r & -Q_r & 0 \\ 0 & -Q_r & P_r & 0 \\ Q_r & 0 & 0 & P_r \end{bmatrix}.$$

The latter test is useful in the problem of deciding on the appropriate value of p, i.e. on the order of the highest harmonics which should be included in the systematic variation. This cannot be specified exactly theoretically although it may be assumed that the amplitudes of the harmonic terms will, on the whole, decrease as their order increases. Thus one might proceed by first allowing for fundamental terms only, i.e. taking $p=1$, and testing whether their amplitudes differ significantly from zero; then if they do so (which should certainly be the case here) taking $p=2$ and testing whether the amplitudes of the first harmonic terms differ significantly from zero, and so on, the value of p finally adopted corresponding to the order of the last harmonic terms whose amplitudes were significant. This method is somewhat laborious, since at each stage the estimates α^*, β^*, σ^{*2}, $\sigma^{*\prime 2}$ occurring in P^*_r and Q^*_r have to be recalculated, and will of course be misleading if some high-order harmonic has a fairly large amplitude; but it becomes more satisfactory if we increase the value of p by more than unity at each stage, and also continue as a check for at least one stage beyond that which first yields no significant harmonic terms. However, if the interval between consecutive observations is a rational fraction of the period (i.e. a year), there will be a maximum number of distinguishable harmonic terms irrespective of the number of observations, and if this number is not too large, it will be best to include them all in the specification (12) and carry out a complete harmonic analysis, even though some terms are found subsequently not to be significantly different from zero (see Section 9).

It may be noted that the above analysis can be shown to be asymptotically valid even when the residual fluctuations $\{\epsilon_n, \eta_n\}$ and the observational errors $\{u_n, v_n\}$ are not normally distributed. It is sufficient that the statistical independence properties stated in Section 2 should hold. The estimates (18) are of course then no longer maximum likelihood estimates.

There remains the question as to whether equations of the type (12) give a satisfactory representation of the systematic variation. Here, however, there is the difficulty that to test whether the observations are consistent with the model we must in general postulate some alternative hypothesis for the form of the systematic variation: for example, that we require additional terms $d_l n$ and $d_m n$ in l_{sn} and m_{sn} respectively to take account of a linear trend (corresponding perhaps to variation with a very long period). The only practicable way of avoiding this difficulty seems to be to divide the series into several consecutive sub-series, from each of which estimates of the form (18) are obtained. Then if the sub-series are sufficiently long for the asymptotic formulae (19) to apply, and for the correlations between the estimates for the different sub-series to be neglected, it is fairly easy to test whether corresponding estimates differ significantly. It is of course useful to examine the graphs of the estimated l and m residuals

$$\left.\begin{aligned} l_n^* &= l_n - c^*_l - \sum_{r=1}^{p} \{a^*_{lr}\cos \nu rnh + b^*_{lr}\sin \nu rnh\}, \\ m_n^* &= m_n - c^*_m - \sum_{r=1}^{p} \{a^*_{mr}\cos \nu rnh + b^*_{mr}\sin \nu rnh\}, \end{aligned}\right\} \tag{22}$$

plotted against n, in case these reveal any unusual features (but see Section 12).

5. *Analysis of the fluctuation components.*—Estimates of the second group of parameters α, β, σ^2 and σ'^2 are obtained from the auto-covariances and cross-covariances of the series $\{l_n^*, m_n^*\}$, which are defined by

$$\left.\begin{aligned} c_{11}(r) &= \frac{1}{N-r}\sum_{n=1}^{N-r} l_n^* l_{n+r}^*, \quad & c_{22}(r) &= \frac{1}{N-r}\sum_{n=1}^{N-r} m_n^* m_{n+r}^*, \\ c_{12}(r) &= \frac{1}{N-r}\sum_{n=1}^{N-r} l_n^* m_{n+r}^*, \quad & c_{21}(r) &= \frac{1}{N-r}\sum_{n=1}^{N-r} m_n^* l_{n+r}^*. \end{aligned}\right\} \tag{23}$$

The effect of the differences between l_n^* and x_n, and between m_n^* and y_n will become negligible for sufficiently large N, so that we can regard $c_{11}(r)$ as equivalent to

$$\frac{1}{N-r}\sum_{n=1}^{N-r} x_n x_{n+r}, \text{ and so on.}$$

If we do not take account of observational error, i.e. assume that $\sigma'^2=0$, the procedure is straightforward. Maximum likelihood estimates of α and β are asymptotically equivalent to those obtained by the method of least squares, i.e. by minimizing

$$\sum_n [(l_{n+1}^* - \alpha l_n^* + \beta m_n^*)^2 + (m_{n+1}^* - \beta l_n^* - \alpha m_n^*)^2],$$

which gives

$$\left.\begin{aligned} \alpha^* &= \{c_{11}(1)+c_{22}(1)\}/\{c_{11}(0)+c_{22}(0)\}, \\ \beta^* &= \{c_{12}(1)-c_{21}(1)\}/\{c_{11}(0)+c_{22}(0)\}. \end{aligned}\right\} \tag{24}$$

Also σ^{*2} is obtained as the mean square residual, equivalent to

$$\sigma^{*2} = \tfrac{1}{2}\{c_{11}(0)+c_{22}(0)\}\{1-\alpha^{*2}-\beta^{*2}\}. \tag{25}$$

It is easily shown that α^* and β^* are asymptotically unbiased and that

$$\operatorname{var}\alpha^* \sim \operatorname{var}\beta^* \sim \{1-\alpha^2-\beta^2\}/2N = (1-e^{-2\kappa h})/2N \tag{26}$$

$\operatorname{cov}(\alpha^*, \beta^*) \sim 0$, whence with estimates of κ and γ obtained from

$$e^{-2\kappa^* h} = \alpha^{*2}+\beta^{*2}, \qquad \cot\gamma^* h = \alpha^*/\beta^*, \tag{27}$$

these being also asymptotically unbiased, we find that

$$\operatorname{var}(\kappa^* h) \sim \operatorname{var}(\gamma^* h) \sim \{e^{2\kappa h}-1\}/2N. \tag{28}$$

These formulae remain valid when the ϵ_n and η_n are not normally distributed, although the estimates are not then equivalent to those obtained by the method of maximum likelihood.

In the general case, however, the maximum likelihood approach becomes very complicated. It is possible to proceed by the method of Whittle (7) but this involves very lengthy algebraic calculations, and yields a set of equations which are difficult to solve. A more practical method consists of using covariances for lags h and rh, as suggested by Jeffreys. The above estimate of γ, given by

$$\cot\gamma^* h = \{c_{11}(1)+c_{22}(1)\}/\{c_{12}(1)-c_{21}(1)\} = p(1)/q(1), \text{ say,} \tag{29}$$

where

$$p(r) = c_{11}(r)+c_{22}(r), \qquad q(r) = c_{12}(r)-c_{21}(r), \tag{30}$$

is easily seen (remembering equation (15)) to be consistent, i.e. to differ from γ by not more than any pre-assigned quantity ϵ, however small, with probability arbitrarily near unity, if N is taken to be sufficiently large. Similarly $\{p(r)\}^2+\{q(r)\}^2$ is found to be a consistent estimate of $4\sigma^4 e^{-2r\kappa h}/(1-e^{-2\kappa h})^2$, and so a consistent estimate of κ is given by

$$e^{2\kappa^* h(r-1)} = [\{p(1)\}^2+\{q(1)\}^2]/[\{p(r)\}^2+\{q(r)\}^2]. \tag{31}$$

The best value of r to take is not obvious. Clearly for large r the precision of κ^* will be low since the expected value of the denominator in (31) tends to zero exponentially with r, but there is also the effect of the final division by $r-1$. Jeffreys states that r should be chosen so that $e^{-r\kappa h}$ is about 0·45, but does not explain how he arrives at this result, and it is incorrect at least in the case of small κh, for which it can be shown that when $\lambda \simeq 1$, i.e. the observational error variance is small, r should be as small as possible (see Section 6). In view of this, and the fact that the precision of κ^* can be calculated most accurately for small r, it seems best in general to use (31) with $r=2$.

It is easy to see that γ^* and κ^* are asymptotically unbiased, and that their asymptotic standard errors are proportional to $1/\sqrt{N}$. Formulae for these may be obtained by expressing $\gamma^*-\gamma$, $\kappa^*-\kappa$ as asymptotic linear functions of the quantities $\delta p(1)$, $\delta q(1)$, $\delta p(r)$, $\delta q(r)$ (where $\delta p(1)=p(1)-\overline{p(1)}$ and $\overline{p(1)}$ is the mathematical expectation of $p(1)$, etc.), and then using the expressions for the asymptotic variances and covariances of the p's and q's. The latter follow from formulae of the type

$$\operatorname{cov}\{c_{12}(r),\ c_{34}(r+s)\}=\left\{\frac{1}{(N-r)(N-r-s)}\right\}\sum_{i,j}\operatorname{cov}\{X_1(i)X_2(i+r),\ X_3(j)X_4(j+r+s)\}$$

$$\sim\left\{\frac{\sigma_1\sigma_2\sigma_3\sigma_4}{N-r}\right\}\sum_{v=-\infty}^{\infty}\{\rho_{13}(v)\rho_{24}(v+s)+\rho_{23}(v)\rho_{14}(v+2r+s)\}, \tag{32}$$

where e.g.

$$\operatorname{cov}\{X_1(i),\ X_3(i+v)\}=\sigma_1\sigma_3\rho_{13}(v). \tag{33}$$

(In this paper the sign $\sim$ means " is asymptotically equal to ".)
For instance we find that

$$(N-r)\operatorname{var}p(r)\sim 2\{\sigma'^2+\sigma^2/(1-e^{-2\kappa h})\}^2$$
$$\times\left[\frac{\lambda^2}{1-e^{-2\kappa h}}\left(1+e^{-2\kappa h}+e^{-2r\kappa h}\cos 2r\gamma h\{2r+1-(2r-1)e^{-2\kappa h}\}\right)\right.$$
$$\left.+(1-\lambda)\{1+\lambda(1+2e^{-2r\kappa h}\cos 2r\gamma h)\}\right].$$

After some rather heavy algebra, we arrive at the results

$$\operatorname{var}\gamma^* h\sim\left(\frac{e^{2\kappa h}}{2N}\right)\left[1-e^{-2\kappa h}+\frac{1-\lambda}{\lambda^2}\left(1+\lambda\{1+2e^{-2\kappa h}\}\right)\right], \tag{34}$$

$$\operatorname{var}\{\kappa^* h(r-1)\}\sim\frac{1}{2N}\left[\frac{e^{2r\kappa h}+e^{2(r-1)\kappa h}-2+(2r-1)(1-e^{2\kappa h})}{1-e^{-2\kappa h}}\right.$$
$$\left.+\frac{(1-\lambda^2)e^{2r\kappa h}+(1-\lambda)(1-3\lambda)e^{2\kappa h}}{\lambda^2}\right]. \tag{35}$$

In particular for $r=2$, (35) gives

$$\operatorname{var}\kappa^* h=\frac{e^{2\kappa h}}{2N\lambda^2}\{e^{2\kappa h}-1+2(1-\lambda)^2\}. \tag{36}$$

To estimate λ, we may use the equation

$$\lambda^{*2}e^{-2\kappa^* h}=[\{p(1)\}^2+\{q(1)\}^2]/[\{p(0)\}^2+\{q(0)\}^2]. \tag{37}$$

From (34) and (35) we obtain estimated standard errors of κ^* and γ^* on substituting κ^* and λ^* for κ and λ, and hence approximate confidence limits for κ and γ, since κ^* and γ^* can be shown to be asymptotically normally distributed. λ^* is also asymptotically normally distributed, with mean λ and variance

$$\operatorname{var}\lambda^*\sim[\lambda^2(e^{2\kappa h}-1)^2+2\lambda(1-\lambda)(e^{4\kappa h}-1)+(1-\lambda)^2(e^{4\kappa h}+4e^{2\kappa h}+2)]/2N. \tag{38}$$

We can therefore assess approximately the significance of the difference of λ^* from unity. If λ^* was considered not to be significantly lower than unity at an appropriate probability level, we should regard the observations as consistent with the hypothesis that observational errors can be neglected. Finally a consistent estimate of σ^2, which is required in the estimation of the variances and covariances (19) of the harmonic components, may be obtained from the equation

$$\tfrac{1}{2}\{c_{11}(0)+c_{22}(0)\}=\sigma^{*\prime 2}+\sigma^{*2}/(1-e^{-2\kappa^* h})=\sigma^{*2}/\{\lambda^*(1-e^{-2\kappa^* h})\}. \tag{39}$$

It should be noted that the normality assumptions of Section 2 are really used only in the derivation of equations (34), (35) and (38). Without these assumptions it is in general very troublesome to derive expressions for the asymptotic variances of γ^*, κ^* and λ^*, and the estimation of the variances is no longer straightforward.

The validity of the correlational structure given by the model (13) may be examined by comparing the sample covariances $c_{ij}(r)$ with the theoretical covariances $V_{ij}(r)$ defined by (14), or the sample correlations (obtained by dividing the corresponding covariances by $\sqrt{\{c_{ii}(0)c_{jj}(0)\}}$) with the theoretical correlations $R_{ij}(r)$ defined by (15). We can, for example, use the goodness-of-fit test associated with the "*G*-operators" introduced by Bartlett and Rajalakshman (**8**); this can be shown to be applicable here although the model is more general than those considered in their paper. In addition we can test whether $c_{11}(0)/c_{22}(0)$ and $c_{12}(0)$, which are asymptotically uncorrelated with the "*G*-forms" and with each other, differ significantly from 1 and 0 respectively, using the expressions for their asymptotic standard errors obtained by means of the general formula (32). It seems, however, difficult to construct a suitable test of the assumption that the residuals are normally distributed. Some indication of the appropriateness of the assumption may be obtained by constructing frequency distributions of $l_{n+1}^*-\alpha^* l_n^*+\beta^* m_n^*$ and $m_{n+1}^*-\beta^* l_n^*-\alpha^* m_n^*$, but it is not known how the significance of deviations of these from the normal form should be assessed; the standard χ^2 distribution certainly cannot be used.

6. *The effect of a small damping factor.*—Difficulties may occur in the above analysis if κh, the product of the damping factor and the interval between consecutive observations, is small. We then have from (35), provided that $r\kappa h$ is also small,

$$\operatorname{var}\{\kappa^* h(r-1)\}\simeq\frac{1}{N\lambda^2}\,[\lambda^2(r-1)^2\kappa h+(1-\lambda)^2+(1-\lambda)\kappa h\{2\lambda(r-1)+(1-\lambda)(r+1)\}], \tag{40}$$

so that if $V(\kappa^*)$ denotes the coefficient of variation of κ^*,

$$\{V(\kappa^*)\}^2\simeq\frac{1}{N\kappa h}+\frac{(1-\lambda)^2+(1-\lambda)\kappa h\{2\lambda(r-1)+(1-\lambda)(r+1)\}}{N\lambda^2\kappa^2h^2(r-1)^2}. \tag{41}$$

If $r\kappa h$ is not necessarily small, the first term on the right-hand side of (41) is replaced by $(e^{2y}-1-2y)/2N\kappa h y^2$, where $y=(r-1)\kappa h$. Since this increases with y, $V(\kappa^*)$ will be approximately minimized by taking r to be as small as possible, i.e. $r=2$, when $1-\lambda$ is sufficiently small for the remaining terms to be neglected. If the latter are taken into account, the choice of y to minimize $V(\kappa^*)$ depends on κ and λ in a rather complicated way; however, as was indicated in Section 5, the accuracy of the approximation (35) tends to decrease as r increases, and the value $r=2$ seems suitable in general.

Since, to a good approximation, $V(\kappa^*)\geqslant(N\kappa h)^{-1/2}$ it is desirable that $N\kappa h$ should be appreciably greater than unity. Indeed, when this is not so, the approximate lower confidence limit for κ, given e.g. by $\kappa^*\{1-1\cdot 96V^*(\kappa^*)\}$ for a confidence probability of 0·95, where $V^*(\kappa^*)$ is the estimated coefficient of variation, is likely to be negative, no matter how small the observational errors may be. Thus, since negative values of κ are inadmissible, we can only consider the upper confidence limit; the precision of κ^*, or the estimate $1/\kappa^*$ of the "relaxation time", which has to a good approximation the same coefficient of variation $V(\kappa^*)$, cannot then be regarded as satisfactory.

There is, however, a more serious difficulty when the condition $N\kappa h \gg 1$ is not satisfied. This is that the asymptotic formulae for var $(\kappa^* h)$ and var $(\gamma^* h)$, the latter being approximately $(1/2N\lambda^2)[2\kappa h + (1-\lambda)(1+3\lambda)]$ for small κh, may no longer be good approximations. For the representation of $\gamma^* - \gamma$ and $\kappa^* - \kappa$ as linear functions of the $\delta p(r)$ and $\delta q(r)$, used in their derivation, is valid only on the assumption that the coefficients of variation of $p(r)$ and $q(r)$ are small, and it can be shown that (provided λ is not small) these coefficients of variation are both approximately equal to $(N\kappa h)^{-1/2}$. (This assumes that the asymptotic formulae for var $p(r)$ and var $q(r)$ are still applicable; it is difficult to estimate the relative error of these, but a rough argument suggests that they are of the same order of magnitude as that caused by replacing $\sum_{v=0}^{N} e^{-2v\kappa h}$ by $\sum_{v=0}^{\infty} e^{-2v\kappa h}$, i.e. as $e^{-2(N+1)\kappa h}$, which is e.g. less than 0·05 when $N\kappa h > 1{\cdot}5$.) The same difficulty arises in connection with the formula (38) for var λ^*. A further point is that the distributions of $\kappa^* h$, $\gamma^* h$ and λ^* may then no longer be approximately normal.

These difficulties are really due to the fact that when κh is sufficiently small the elements of the autocorrelation matrix decrease very slowly as the lag increases. In particular, when the effect of observational errors is small, so that we have $\lambda \simeq 1$, this can be shown to imply that the observations are equivalent to a very small number of independent observations, and hence we should not expect the asymptotic theory to apply.

A more refined method of statistical analysis which avoids these difficulties does not seem to be available at present; in any case it is conjectured that when $N\kappa^* h$ is of order unity, i.e. the time Nh covered by the observations is comparable with the estimated "relaxation time", it is not possible to obtain much information about the true values of the parameters in the model.

Jeffreys (2) estimated κ to be about 0·06, the unit of time being a year. Part of the evidence for this estimate is provided by a statistical analysis of a series of 154 observations at intervals of 0·3 year. However, with $N = 154$, $h = 0{\cdot}3$, $\kappa = 0{\cdot}06$, $N\kappa h$ is only 2·8, so that the above difficulties certainly arise. In fact if we substitute these values of N, κ, h together with the value $\lambda = 0{\cdot}933$ given by Jeffreys (2, p. 145) in (41) we obtain $V(\kappa^*) = 0{\cdot}72$, which is sufficiently large to indicate that the analysis is not satisfactory If Jeffreys's estimate were accurate we should have the same situation arising in the analysis of the series of observations used in Part III below, for which $N = 420$, $h = \frac{1}{12}$. However, our estimate of κ turns out to be much larger, i.e. $\kappa^* = 0{\cdot}322$, giving $N\kappa^* h = 11{\cdot}3$, and an estimated coefficient of variation $V^*(\kappa^*) = 0{\cdot}31$. This value of $N\kappa^* h$ is not particularly large, and some doubt may therefore be felt as to the accuracy of the asymptotic formulae for standard errors and confidence limits. Despite this, however, the full analysis, the results of which are summarized, was thought to be justified.

Part III. *Computational Procedure and Preliminary Results*

7. *Computational procedure.*—Estimates c^*_l, c^*_m, a^*_{l1}, a^*_{m1}, etc., defined by (18), are first obtained, it being supposed that the question of the order of the highest harmonic to be included has been settled.

These estimates are used to calculate the estimated residual series $\{l_n{}^*\}$ and $\{m_n{}^*\}$ from (22). From these series, auto-covariances and cross-covariances for lags $s=0$, 1 and 2 are calculated and substituted in formulae (29), (31), (37) and (39) of Section 5 to give estimates of κ, γ, σ^2 and σ'^2. Further auto-covariances and cross-variances for higher lags are computed at this stage if the goodness-of-fit test mentioned at the end of Section 5 is to be applied to the results.

The estimated standard errors of κ^*, γ^* and λ^* are then calculated from (34), (35) and (38), and finally the estimated standard errors of $c^*{}_l$, $c^*{}_m$, etc. from (19), which also gives the covariances required for carrying out the approximate test of the significance of a set of harmonic coefficients of given order described in Section 4.

8. *Data used in the analysis.*—As yet we have only applied the results of this analysis to a series of 420 "monthly" values of l and m covering the 35 years 1900–1935 taken from the various publications of the International Variation of Latitude Service. In some cases data have been readjusted after original publication; the latest available values have always been taken in such cases.

For the reason discussed in Section 6 the longest possible series of results are needed. Unfortunately the data available have not been obtained in a uniform manner since 1900. Firstly, the number of stations at which observations have been made declined throughout the period. From the inception of the programme, at the beginning of 1900, to 1915 January, six stations (Carloforte, Tschardjui or Kitab, Mizusawa, Ukiah, Cincinnati and Gaithersburg) were in operation. In 1915 January, Gaithersburg ceased to operate, as did Cincinnati in 1916 January and Kitab in 1919 May. From 1919 May onwards only three stations were used in the determinations, this being the minimum number necessary for determining l, m and the so-called z-term. Secondly, the method of reducing the observations was changed at 1922·7.

Observations are made on twelve star fields, consecutive pairs of which overlap by about a month, and the values of l and m pertain to the mean date of each group-combination. For example, in the early years, star Group I was used for observation from September 23 to December 6, Group II from November 2 to January 4 and the published values of l and m for the group combination I–II refer to the mean date of the overlapping period November 2 to December 6. This gives 12 irregular intervals per year. After 1922·7 the group combinations were altered to make the intervals equal. A good account of the actual procedures and programmes is given by Melchior (**10**, Chapter 1). The irregularities in the intervals may be regarded as leading to slight, irregular phase changes in the motion and their effect will be to increase the uncertainty of the results. We feel that these series of results are better than the $\frac{1}{10}$ yearly series which are usually quoted and which Jeffreys used in his analysis. The reason for this preference is that the most essential requirement of the series examined is that successive terms should be in no way correlated except by the physical processes underlying the motion. This is true of the $\frac{1}{12}$ yearly series but the $\frac{1}{10}$ yearly series are derived from the former by graphical interpolation and in the process of drawing smooth curves through the rather irregular looking curves of the $\frac{1}{12}$ yearly series, correlations between successive runs of 3 or even 4 values must be introduced. Comparison of the graphs of the $\frac{1}{12}$ and $\frac{1}{10}$ yearly series do in fact indicate a high degree of smoothing. There must therefore be bias introduced into the auto-covariances for small lags. In the fuller analysis which we hope to make later this will be examined more closely, particularly as it has a bearing also on a point raised by Professor Jeffreys (private communication) regarding the effect of errors introduced by changes in the star fields occasioned by their proper motions, and the effects of precession.

We have used the results from which the so-called z-term has *not* been removed, as the results so obtained appear to be possibly more homogeneous than those obtained when the z-term is taken into account. They are certainly unsmoothed. This point will require examination in our fuller analysis, as will also the effect of the important change in the method of reduction made in 1922·7.

9. *Preliminary results.*—For ease of computing, the 35-year means for each month have been calculated and analysed by the usual simple method of numerical Fourier analysis. This yields all terms up to the cosine term for the fifth harmonic. The results are given in Table I. As expected, the fundamental is the dominant term.

TABLE I

Results of harmonic analysis
(Unit 0″·01)

	c^*	a^*_1	b^*_1	a^*_2	b^*_2
l	+11·062	−55·105	−70·286	−2·138	+6·914
m	−10·057	+74·790	−45·276	−5·548	−3·105
Estimated standard error	±3·05	±10·77		±2·40	

	a^*_3	b^*_3	a^*_4	b^*_4	a^*_5	b^*_5	a^*_6
l	+2·348	−4·014	−1·757	+1·114	−0·905	+2·086	−0·771
m	+2·238	+3·943	+1·024	+1·219	−1·000	+1·257	−1·562
Estimated standard error	±1·70		±1·48		±1·39		±0·97

The series $\{l_n^*\}$ and $\{m_n^*\}$ were easily obtained by deducting from each value of l and m the 35-year mean for the appropriate month. This of course implies that all the components given in Table I have been removed whether or not they later turn out to be significant.

The auto-covariances and cross-covariances, $c_{ij}(r)$, $r=0, 1, 2$, calculated from the series $\{l_n^*\}$ and $\{m_n^*\}$ are given in Table II (taking 0″·01 as the unit of measurement).

TABLE II

Covariances of $\{l_n^*\}$ and $\{m_n^*\}$

Auto-covariances		Cross-covariances	
$c_{11}(0)$	11 289·114 3	$c_{12}(1)$	4 583·685 0
$c_{11}(1)$	9 972·436 8	$c_{12}(2)$	8 192·782 3
$c_{11}(2)$	7 080·925 8		
$c_{22}(0)$	12 377·847 6	$c_{21}(1)$	−4 480·670 6
$c_{22}(1)$	10 716·248 2	$c_{21}(2)$	−8 071·636 4
$c_{22}(2)$	7 718·382 8		

From these were obtained the estimates and standard errors given in Table III.

Finally we calculated the estimated errors of the harmonic coefficients. These are given in the bottom line of Table I. The first and second harmonic terms, though relatively small, clearly differ significantly from zero, but it appears that the higher-order terms could be omitted. This is confirmed by applying the χ^2 test described in Section 4, the results of which are given in Table IV. Ideally, the numerical analysis should now be repeated omitting the insignificant harmonics, but as their contributions to $\{l^*\}$ and $\{m^*\}$ are trivial this has not been considered necessary.

TABLE III

Constants for free motion

	Estimated standard error
$\kappa^* = 0{\cdot}322$	$\pm 0{\cdot}103$
$\gamma^* h = 23^\circ\ 40'$	$\pm 44'$
$\lambda^* = 0{\cdot}9803$	$\pm 0{\cdot}00347$
$\sigma^* = 24{\cdot}6$	...
$\sigma^{*\prime} = 15{\cdot}3$	...

TABLE IV

Significance of harmonic constants : χ^2 test

Order of harmonic	1	2	3	4	5
Value of χ^2	10·95	12·30	3·54	3·94	3·22
Degrees of freedom	4	4	4	4	2
$P(\chi^2)$	$<0{\cdot}05$	$<0{\cdot}02$	$>0{\cdot}30$	$>0{\cdot}30$	$>0{\cdot}20$

10. *Free period and damping coefficient.*—From Table III it is seen that
(i) the estimated damping coefficient is

$$\kappa^* = 0{\cdot}322 \pm 0{\cdot}103 \text{ year}^{-1}$$

giving a relaxation time of 3·1 years;
(ii) the estimated free period is

$$2\pi/\gamma^* = 1{\cdot}267 \pm 0{\cdot}039 \text{ year}$$
$$= 462{\cdot}8 \pm 14{\cdot}2 \text{ mean solar days.}$$

The period is better described as a 15-monthly rather than 14-monthly period.

11. *Discussion of the results.*—We have already remarked in Section 6 that the estimate of the damping coefficient κ is much larger than that given by Jeffreys.

The estimated free period is greater than those obtained by previous analyses, although it is in fact still compatible with the traditional value $2\pi/\gamma = 1{\cdot}2$ year. From the analysis of data for the years 1892–1933 Jeffreys obtained $2\pi/\gamma = 1{\cdot}223 \pm 0{\cdot}019$ years, which is in quite good agreement with the present result. He preferred a result, however, based on the analysis of data for the much shorter period 1908·3–1921·5 from which he obtained $2\pi/\gamma = 1{\cdot}202 \pm 0{\cdot}016$ year.

The need to take account of observational error in the model is clearly shown by the value of $1 - \lambda^*$, 0·0197, which is more than 5·6 times its estimated standard error and certainly differs significantly from zero. When the analysis of Section 5 ignoring observational error is carried out, we obtain the value $\kappa^* = 0{\cdot}560$ for the damping.

There appears to be only one other occasion on which so long a period has been suggested. Jeffreys (**9**, pp. 213–4) has made a solution, using Takeuchi's theory of the shell and an approximate treatment of the core of the Earth. It predicts a period of 460 days for the free motion—in close agreement with our present determination. We understand, however, that this theory gives an unsatisfactory value for the nutation and that further work is still in progress.

The standard deviation of the observational errors seems to be quite large, our estimate, 15·3 units, being more than four times that suggested by Jeffreys.

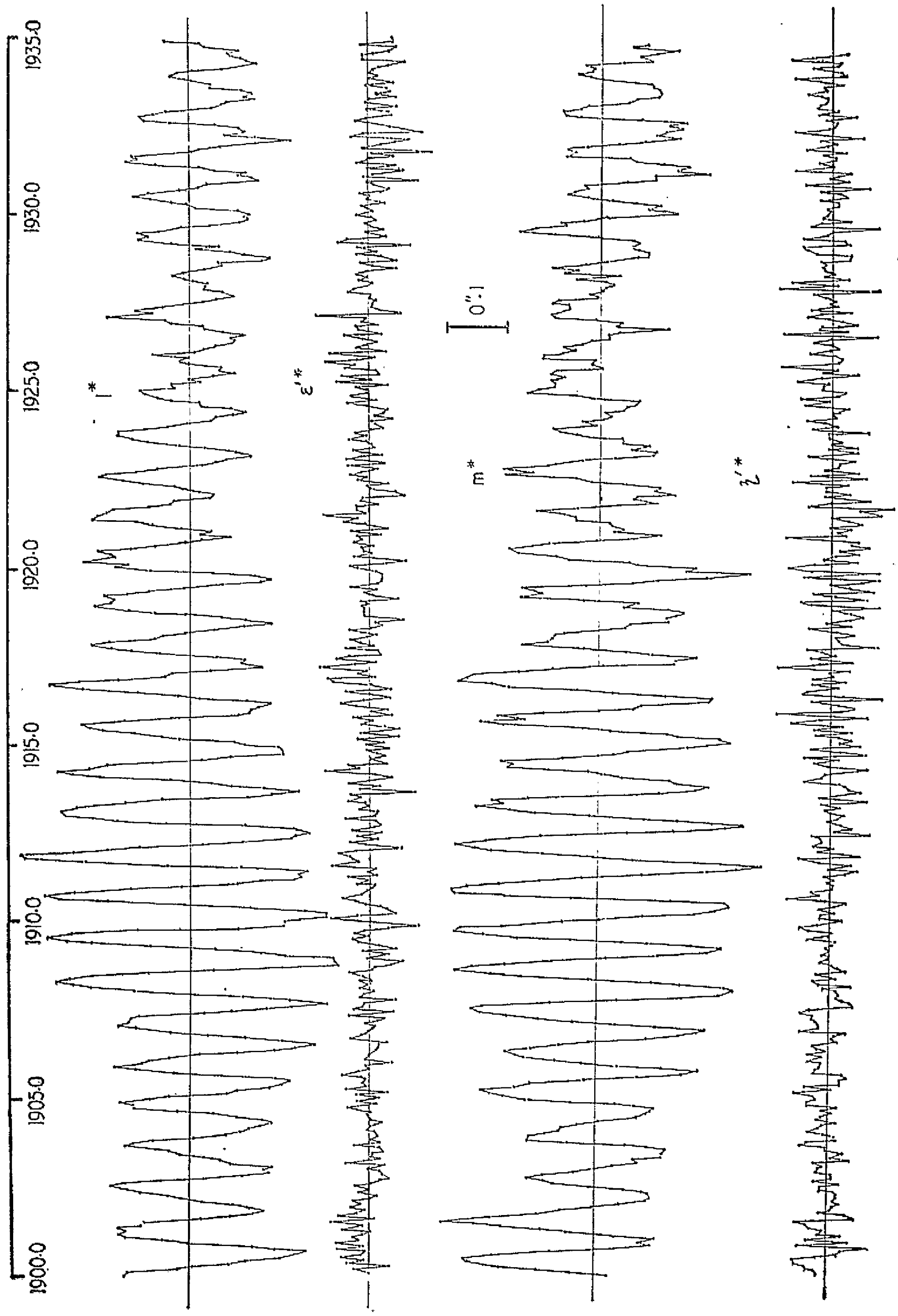


FIG. 1.—*Free motion (l^*, m^*) and residual motion (ϵ'^*, η'^*).*

This may be due in part to the uncertainties being increased by the slight irregularities of the intervals as suggested in Section 8. The increased uncertainty may be due however to the non-removal of the z-term.

In Sections 4 and 5 we have suggested other points which bear examination. We hope to report on these later. They deal mainly with the adequacy of the model but we may mention that the application of the Bartlett–Rajalakshman goodness-of-fit test requires a very much larger amount of numerical computation than has even been involved in the analysis so far completed!

Part IV. *The Residual Motion*

12. *The distribution of the residuals.*—Reverting to equation (13), it is seen that when α^* and β^* have been evaluated, estimates ϵ'^*, η'^* of ϵ' and η' can be obtained by replacing x and y by their estimates l^* and m^*. We do not see how to separate the fluctuation components (ϵ, η) from the observational error (u, v) and so cannot test them separately for normality and randomness. However, we have computed the series $\{\epsilon'^*\}$ and $\{\eta'^*\}$ and examined them. They are both symmetrically distributed with zero means and have standard deviations which do not differ appreciably from those calculated from the values of the estimated parameters. Both distributions are flattened compared with the normal distribution, but not much so. The residuals are displayed in Fig. 1.

We conclude therefore that the evidence, as far as it goes, gives no reason for disbelieving the hypotheses of Section 2, namely, that both $\{\epsilon, \eta\}$ and $\{u, v\}$ are normally distributed and form independent stationary processes.

Jeffreys (2, p. 147) inspected the series corresponding to our $\{l^*\}$ and $\{m^*\}$ and concluded that there were abrupt changes in amplitude and phase at certain times. He chose the period 1908·3–1921·5 for further analysis since he found it to be one in which the motion was almost undisturbed. Inspection of Fig. 1 shows similar phenomena, the period up to about 1915 being apparently disturbance free. Thereafter the $\{l^*\}$ and $\{m^*\}$ series seem increasingly disturbed. To what extent this is due to physical causes, or to the smaller number of stations employed, or to the change in the method of reduction made in 1922 is not clear. The point is, however, not that the $\{l^*\}$ and $\{m^*\}$ series should follow a certain pattern but that the mean number of "impulses" during the intervals h should be large (Section 2) and this cannot be decided by a rather subjective examination of the actual series. The real test of departure from the assumptions made in the model rests in the analysis of the residuals $\{\epsilon, \eta\}$. Although these cannot be isolated, examination of the series $\{\epsilon'^*\}$ and $\{\eta'^*\}$, as stated above, does not reveal any non-random effects. There are nowhere any systematic runs of values longer than one would expect in random series. For this reason we do not regard as satisfactory any procedure for isolating periods for preferential analysis solely on the basis of the appearance of the series $\{l^*\}$ and $\{m^*\}$.

Dept. of Mathematics,
The University,
Manchester, 13 :

Dept. of Applied Mathematics,
The University,
Liverpool, 3.

1955 *March* 25.

References

(1) P. Melchior, *Bull. Acad. Roy. Belg., Cl. Sci.*, ser. 5, **37**, 158–183, 1951.
(2) H. Jeffreys, *M.N.*, **100**, 139–155, 1940.
(3) M. S. Bartlett, *Stochastic Processes*, Mimeographed North Carolina lecture notes, 1946.
(4) J. E. Moyal, *J. Roy. Statist. Soc.* B, **11**, 150–210, 1949.
(5) M. S. Bartlett, *J. Roy. Statist. Soc.* B, **8**, 27–41, 1946.
(6) P. Whittle, *Trabajos de Estadistica*, **3**, 43–57, 1952.
(7) P. Whittle, *J. Roy. Statist. Soc.* B, **15**, 125–139, 1953.
(8) M. S. Bartlett and D. V. Rajalakshman, *J. Roy. Statist. Soc.* B, **15**, 107–124, 1953.
(9) H. Jeffreys, *The Earth*, 3rd edn., Cambridge, 1952.
(10) P. Melchior, *Observatoire Royal de Belgique, Monographie*, **3**, 1954.

(6)

Reprinted from the Monthly Notices of the Royal Astronomical Society,
Vol. 117, *No.* 2, 1957

FURTHER RESULTS ON THE ANALYSIS OF THE VARIATION OF LATITUDE

by

A. M. WALKER *and* ANDREW YOUNG

Printed in England by Taylor & Francis, Ltd., London.

and thanks are due to Dr Melchior of Brussels, Professor Cecchini and Dr Vergnano of Turin, and Professor Nicolini of Naples for useful discussions and assistance.

We also wish to thank Mr Alan Kirk who has prepared the figures for this and our previous paper.

Unit of Biometry,
6 *Keble Road, Oxford:*
1956 *December.*

Department of Applied Mathematics,
The University,
Liverpool, 3.

References

(1) A. M. Walker and Andrew Young, *M.N.*, **115**, 443–459, 1955.
(2) H. Jeffreys, *M.N.*, **100**, 139–155, 1940.
(3) P. Melchior, *Observatoire Royal de Belgique, Monographie*, 3, 1954,
(4) G. Cecchini, *Nuova Programma di Osservazione delle Stazioni Internazionali Boreali*. Typescript. Turin, 1954.
(5) P. Melchior, *Communications de L'Obs. Royal de Belgique*, No. 92, 1955.
(6) B. Gutenberg, *Nature*, **177**, 887, 1956.
(7) R. Revelle and W. Munk, *Annales de Geophysique*, **11**, 104–108. 1955.

Reprinted from the MONTHLY NOTICES OF THE ROYAL ASTRONOMICAL SOCIETY,
Vol. 117, *No.* 2, 1957

FURTHER RESULTS ON THE ANALYSIS OF THE VARIATION OF LATITUDE

A. M. Walker and Andrew Young

(Received 1957 January 1)

Summary

In this paper we give further results of our analysis of the observations of the variation of latitude. We find that the best results are those derived from the unsmoothed series of values given at monthly intervals by the International Service.

There is evidence of inhomogeneity in the data because of which it is difficult to obtain a reliable estimate of the damping factor. We discuss the bias introduced into the estimates of the period and damping by the inhomogeneity of the data and by other causes.

---

1. In our previous paper (1) we formulated a statistical model of the variation of latitude and gave a detailed analysis of its properties. We obtained a computational procedure which we applied to a 35-year series of observations given by the International Service. These particular observations were examined because they were unsmoothed and we hoped that the effect of smoothing might be seen when the results were compared with those obtained by Jeffreys (2) who used smoothed observations. However, the two sets of observations differed in another important respect—the z-term had not been taken into account in the set which we analysed but had been in the set analysed by Jeffreys.

In this paper we give much more detailed results and examine some of the points raised in the previous paper.

2. Through the good offices of Professor T. Nicolini of Naples we have obtained much more extensive sets of observations, one set of which forms series of unsmoothed results covering the interval from 1900 to 1955 at monthly intervals, the other forming series of smoothed results for the interval from 1890 to 1955 at 1/10 yearly intervals. In both sets the z-term has been removed. These sets are collated from several sources and, as some of the results are as yet unpublished in easily accessible form, they are given in Tables I *a* and I *b*. The first series (Table I *a*) are exhibited graphically in Fig. 1 (*a, c*).

Observations of the variation of latitude were made at a number of uncoordinated observatories prior to the inception of the International Service. The results were collated by Albrecht and form the basis for the values given in Table I *b* for the interval 1890·0 to 1899·8. The international programme started at the end of 1899. Until 1922·7, the work was under the direction firstly of Albrecht and then Wanach and Mahnkopf at Potsdam. From 1922·7 to 1934·9 the work was directed by Kimura at Mizusawa; he was succeeded in 1935 by Professor Carnera and in 1948 by Professor Cecchini of Turin, the present director. The methods of reduction used by the different directors have not been the same and according to Melchior (3, p. 38) the results require

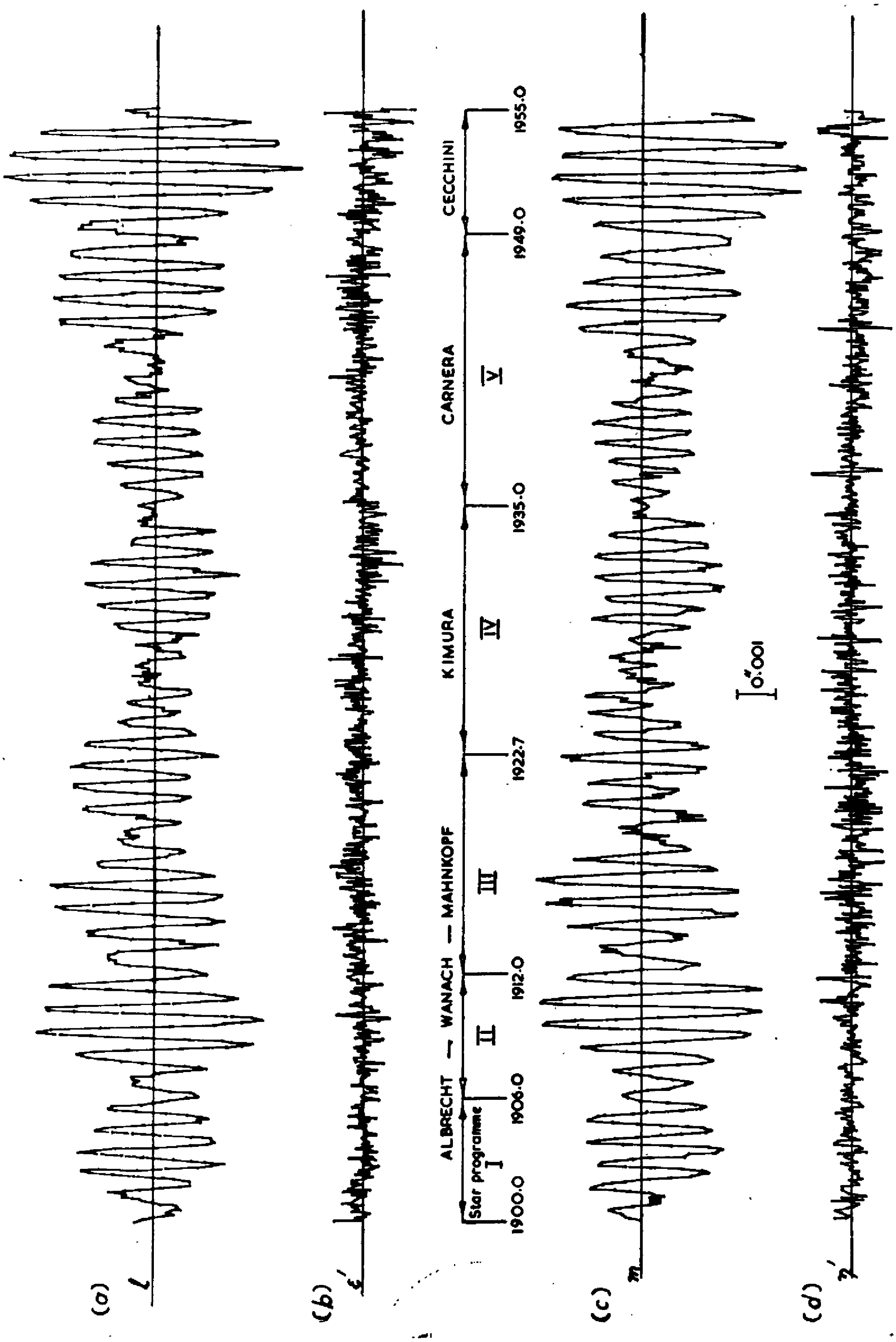


FIG. 1.—*Variation of latitude. Motion (l, m) at monthly intervals and residual motion* (ε′, η′).

corrections; the published results can, he states, be brought into agreement by applying the following corrections*

$$\begin{aligned} x_w &= x_k + 0''\cdot048 = x_c + 0''\cdot040 = x_{ce} + 0''\cdot004, \\ y_w &= y_k + 0''\cdot072 = y_c + 0''\cdot060 = y_{ce} - 0''\cdot002, \end{aligned} \tag{1}$$

where (x, y) are the coordinates of the pole† as published and w, k, c, ce refer to Wanach, Kimura, Carnera and Cecchini respectively. As the values of x and y seldom exceed $\pm 0''\cdot300$, these corrections seem to be large.

3. For the international programme, observations are made at several observatories on the parallel of latitude 39° 8′ N. Ideally there should be a large number of stations distributed symmetrically on the parallel. However, at most, six stations have been in use, and for long periods only three have functioned. The stations with details of the intervals during which they functioned are given in Table II.

If $\Delta\phi_i$ is the value of the variation of latitude of the station, i, measured from its mean latitude, then (x, y) are given by

$$x \cos \lambda_i + y \sin \lambda_i = \Delta\phi_i - z \tag{2}$$

where λ_i is the longitude of the station and z is Kimura's z-term which arises from a number of causes which are not all perfectly understood. These causes include errors in the adopted values of the proper motions and declinations of the groups of stars observed, and also perhaps local effects such as refraction and oscillations of the vertical.

When more than three stations were included, equations (2) were solved by least squares. For example, from 1901·7 to 1906·0 the values of x were found from the formula

$$x = -0\cdot437\Delta\phi_1 + 0\cdot139\Delta\phi_2 + 0\cdot426\Delta\phi_3 + 0\cdot101\Delta\phi_4 + 0\cdot042\Delta\phi_5 - 0\cdot272\Delta\phi_6,$$

whereas, when only three stations were used from 1922·7 to 1935·0, the formula became

$$x = -0\cdot396\Delta\phi_1 + 0\cdot591\Delta\phi_3 - 0\cdot195\Delta\phi_6.$$

There are evidently quite considerable changes in the relative weights of the contributions of different stations.

Because of precession and proper motion, no individual star can be used in the observing programme indefinitely and the programme has been changed on a number of occasions. From 1899·9 to 1905·9 each of the twelve groups of stars included eight pairs, of which two were "refraction-pairs" of larger declination than the others. At 1906·0 these were replaced by normal pairs, and, in effect, the number of stars used was thus increased from 144 to 192, of which 132 were in the original programme. Further changes were made at 1912·0, 1922·7 and 1935·0. At each change the number of new stars introduced was not great. In fact, in the programme used since 1955·0, there are still ten of the original pairs of stars used since 1899 (see Cecchini, 4). From conversations which one of us (A. Y.) has had with Professors Cecchini and

* If these suggested corrections are valid they should agree with the means of c, c' given by Jeffreys (**2**, p. 146.) The latter however average to $0''\cdot01$ at most over the intervals mentioned.

† In (1) (x, y) are the coordinates of the pole referred to a left-handed set of axes. We follow Jeffreys, and use a right-handed set with $l = x$ and $m = -y$. We use (x, y) to denote the free component of the motion after the systematic, annual component (l_s, m_s) has been deducted from (l, m)—cf. **1**, equation (6).

TABLE I*a* *Results of Variations of Latitude :*

l

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1899												−7
1900	69	49	40	15	−15	−46	−32	−44	−70	−64	−65	−8
1	8	44	53	68	102	94	143	91	47	−10	−71	−99
2	−102	−76	−48	−15	71	128	203	216	167	53	−34	−119
3	−162	−199	−136	−103	−46	37	95	165	226	175	116	−3
4	−104	−153	−172	−163	−161	−102	−14	67	133	183	153	117
1905	30	−30	−130	−122	−146	−134	−90	−27	61	115	144	132
6	44	12	−20	−42	−94	−117	−110	−96	−36	−9	20	40
7	66	71	52	67	66	10	1	−11	−57	−147	−122	−77
8	−35	35	99	153	205	223	232	177	79	−71	−209	−246
9	−276	−223	−193	−85	43	189	280	347	309	163	−45	−166
1910	−205	−253	−307	−277	−216	−78	96	244	324	322	230	95
1	−47	−145	−197	−237	−207	−172	−126	25	144	249	314	263
2	148	73	9	−72	−114	−154	−116	−96	−64	−16	82	116
3	116	143	138	136	136	123	108	8	−26	−106	−83	−72
4	−84	−36	74	144	196	182	198	168	92	8	−115	−158
1915	−196	−158	−150	−34	24	107	205	270	289	183	71	−86
6	−144	−168	−191	−186	−120	−2	80	191	289	310	222	116
7	−5	−38	−112	−172	−126	−70	−30	59	160	174	172	160
8	88	−9	−28	−43	−66	−76	−74	12	58	82	99	100
9	112	68	52	92	80	86	70	50	−9	−58	−68	−42
1920	21	59	28	126	152	198	212	203	170	82	22	−72
1	−76	−102	−62	12	65	176	214	244	236	146	72	22
2	−82	−104	−90	−38	2	92	172	254	210	167	95	−26
3	−89	−125	−178	−127	−90	−11	79	176	207	198	160	72
4	−9	−62	−111	−96	−97	−78	−9	84	94	109	94	71
1925	36	−41	−47	−70	−24	−24	−17	54	58	76	61	−2
6	16	−75	−99	−95	−95	−61	−8	16	54	52	16	28
7	−5	71	−14	39	21	24	36	59	37	19	15	−29
8	−75	−70	−68	−22	−13	−12	6	56	−11	−59	−53	−73
9	−58	−116	−44	33	46	101	110	117	75	−9	−59	−97
1930	−146	−148	−88	−50	5	79	154	173	127	60	11	−121
1	−151	−148	−172	−95	−41	70	165	193	206	72	25	−56
2	−168	−238	−186	−111	−119	−69	15	137	187	159	100	36
3	−85	−120	−138	−162	−117	−91	−28	68	73	74	55	27
4	−26	−72	−130	−167	−123	−74	−11	37	28	51	20	37
1935	−2	9	11	8	11	23	50	42	10	−22	−46	−60
6	−67	−62	−44	−8	45	92	102	89	50	7	−33	−81
7	−129	−133	−80	−12	33	89	128	139	75	30	−22	−95
8	−134	−125	−102	−66	−22	37	129	176	172	100	36	−76
9	−104	−115	−112	−48	−14	36	100	172	179	163	105	22
1940	−42	−92	−122	−133	−84	8	65	99	121	120	58	22
1	55	15	3	−31	7	16	19	57	67	79	93	23
2	16	−7	1	−7	12	18	3	−24	−12	13	0	22
3	43	72	107	98	147	132	98	120	38	−23	7	−67
4	−33	−7	10	76	191	278	274	279	151	−19	−55	−152
1945	−153	−174	−137	−66	92	171	234	279	305	239	100	74
6	−115	−176	−192	−135	−82	−39	114	166	270	283	244	208
7	73	−33	−106	−183	−194	−129	−20	73	148	226	254	253
8	245	198	164	79	−18	−75	−58	−74	−112	−48	−14	120
9	124	234	240	238	205	204	115	−5	−101	−144	−178	−200
1950	−138	−67	76	169	266	352	370	307	132	−18	−185	−281
1	−324	−328	−199	−55	139	259	397	454	377	259	95	−112
2	−308	−369	−420	−247	−158	45	216	364	428	415	289	66
3	−80	−199	−336	−353	−351	−192	−42	120	280	336	298	12[illegible]
4	−12	−93	−186	−222	−269	−144	−80	27	34	70	88	

Sources:
1899 Dec.–1905 Dec. Albrecht and Wanach
1906 Jan. –1911 Dec. Wanach
1912 Jan. –1922 July Wanach and Mahnkop
1922 Aug.–1934 Dec. Kimura
1935 Jan. –1940 Dec. Carnera
1941 Jan. –1948 Dec. Carnera

1949 Jan. –1954 Dec. Cecchini

(*Monthly interval*) : *unit* 0″·001.

m

Jan.	Feb.	Mar.	Apr.	May	June	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	
											−61	1899
−4	33	35	54	63	80	86	63	19	−.29	−58	−26	1900
−61	−26	−67	−41	−25	33	73	143	130	113	82	40	1
−30	−94	−147	−169	−197	−184	−122	−42	37	125	115	71	2
17	−34	−126	−137	−220	−234	−238	−197	−91	55	149	146	3
116	66	7	−56	−116	−189	−209	−193	−142	−61	6	79	4
147	150	122	102	25	−21	−117	−178	−192	−162	−83	−3	1905
40	51	55	31	23	−22	−76	−101	−129	−131	−130	−147	6
−94	−47	−33	1	29	56	77	79	33	−26	−113	−177	7
−211	−208	−206	−167	−112	−9	40	132	170	167	120	23	8
−73	−179	−257	−298	−319	−290	−210	−61	95	218	284	266	9
190	74	−46	−157	−275	−320	−349	−314	−163	68	192	286	1910
280	235	130	32	−92	−197	−311	−346	−311	−190	21	76	1
156	210	186	189	104	54	1	−124	−134	−166	−170	−117	2
−70	−44	−4	54	64	114	66	102	109	45	−32	−89	3
−122	−98	−137	−87	−87	−75	−12	79	136	210	129	47	4
14	−116	−198	−270	−284	−224	−210	−92	18	204	268	196	1915
231	124	−19	−108	−152	−276	−282	−184	−30	82	222	251	6
298	253	188	84	−6	−98	−220	−201	−171	−42	24	98	7
190	118	61	24	−23	−94	−139	−126	−122	−66	−57	−68	8
−53	−9	−82	−23	50	4	14	57	14	−10	−29	−107	9
−188	−112	−176	−114	−132	−40	−21	62	143	156	154	77	1920
6	−70	−42	−90	−104	−109	−53	−32	54	150	107	70	1
−34	−24	−80	−174	−173	−190	−140	−61	109	204	185	233	2
150	106	12	−76	−172	−187	−147	−158	−47	7	83	111	3
129	109	−10	−22	−84	−110	−102	−114	−53	−12	4	82	4
102	153	85	27	−6	−11	−31	10	64	43	83	81	1925
157	127	46	−6	−21	−64	−80	−88	−22	14	−43	44	6
25	38	59	31	−20	−39	−40	6	64	50	89	62	7
51	9	−27	−8	−114	−27	−56	6	17	34	7	−6	8
−17	−31	−72	−115	−138	−89	−17	61	99	104	137	100	9
50	−95	−124	−129	−182	−152	−113	−52	47	89	92	101	1930
18	−59	−109	−229	−209	−238	−198	−134	−58	102	114	126	1
84	4	−91	−180	−174	−201	−229	−188	−150	−22	105	131	2
125	67	3	−97	−141	−188	−193	−173	−96	−38	14	71	3
89	71	−4	−37	−55	−130	−177	−141	−100	−82	17	−5	4
10	25	17	−2	−15	−21	−21	−17	−9	9	19	9	1935
−29	−57	−74	−81	−76	−53	−18	20	53	74	81	33	6
−17	−61	−4	−11	−113	−100	−43	4	45	74	90	92	7
67	−14	−77	−119	−147	−145	−96	−43	18	81	130	127	8
95	40	−30	−64	−104	−126	−137	−54	−2	37	85	140	9
117	70	37	−35	−97	−140	−109	−56	−8	35	61	80	1940
25	8	10	−2	−42	−90	−112	−136	−124	−68	−71	−9	1
−9	−20	29	−31	−20	−35	−27	−29	−96	−109	−102	−137	2
−134	−76	−68	−45	−60	−7	42	32	54	30	−14	−51	3
−103	−144	−153	−141	−119	−90	22	90	168	180	134	78	4
19	53	−148	−219	−232	−236	−119	−36	65	183	217	187	1945
158	86	4	−118	−208	−267	−284	−257	−174	−59	25	87	6
172	132	92	−6	−90	−161	−190	−229	−216	−176	−158	−103	7
−30	54	63	123	96	62	40	−41	−90	−161	−247	−249	8
−259	−228	−157	−69	15	67	86	138	125	35	−114	−216	9
−267	−347	−339	−274	−203	−100	−2	117	186	229	200	29	1950
−118	−241	−366	−460	−411	−317	−201	−12	126	238	258	246	1
141	−65	−210	−348	−440	−468	−445	−352	−189	−5	119	229	2
225	171	69	−51	−210	−328	−400	−391	−407	−260	−77	41	3
189	224	240	141	87	−62	−173	−259	−315	−290	−271	−201	4

Resultate des Internationalen Breitendienstes Band III, pp. 227–228.
Resultate . . . . Band V, pp. 184–185.
Ergebnisse . . . Band VI, pp. 219–221.
Results . . . . . . Vol. VIII, pp. 170–172.
Definitive results communicated to Dublin meeting of I.A.U. 1955.
Contribute Astronomico Capodimonte II, vol. IV no. 1, p. 7, no. 2, p. 6, no. 2, p. 18 and no. 6, p. 44. (These results are *not* definitive).
Typescript. Definitive results communicated to Dublin meeting of I.A.U. 1955.

TABLE I*b* *Results of Variation of Latitude*

l

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
1890	−271	−258	−188	−73	55	154	216	202	90	−55
1	−200	−260	−260	−210	−70	65	184	257	254	186
2	61	−72	−189	−249	−228	−90	61	160	179	158
3	104	14	−68	−139	−197	−182	−98	9	94	144
4	126	84	34	−17	−53	−79	−108	−119	−91	−51
1895	−33	3	−7	−11	8	43	71	18	−43	−95
6	−115	−111	−84	−23	53	139	181	162	91	−8
7	−107	−186	−216	−177	−77	51	153	204	225	148
8	53	−65	−172	−202	−174	−105	27	158	199	151
9	91	26	−38	−70	−80	−104	−80	−17	63	12
1900	39	60	37	−8	−40	−39	−55	−62	−63	−53
1	−1	26	58	83	106	122	88	30	−25	−80
2	−102	−78	−51	38	134	205	200	139	44	−56
3	−144	−179	−142	−68	29	114	188	209	171	86
4	−43	−146	−170	−162	−94	1	87	151	181	148
1905	92	−11	−121	−144	−131	−82	1	84	122	148
6	97	24	−26	−71	−110	−109	−68	−33	−2	17
7	51	64	57	57	28	−4	−29	−84	−135	−115
8	−63	15	112	183	226	215	150	33	−104	−212
9	−270	−249	−171	−6	174	295	338	277	122	−56
1910	−185	−254	−295	−239	−73	121	275	323	300	199
1	36	−117	−208	−227	−167	−79	59	175	251	297
2	246	102	−22	−109	−140	−123	−91	−44	20	85
3	121	136	141	137	123	85	8	−60	−99	−100
4	−80	−20	81	171	197	192	151	70	−24	−112
1915	−174	−181	−109	0	115	223	284	271	164	28
6	−102	−167	−193	−152	−43	95	231	306	301	210
7	85	−30	−130	−151	−80	0	89	159	182	174
8	131	26	−33	−61	−76	−51	26	66	87	101
9	108	77	63	89	87	69	30	−27	−66	−63
1920	−21	28	80	138	198	213	197	147	72	−10
1	−76	−103	−55	42	153	238	250	210	140	61
2	−28	−100	−80	−8	92	191	272	217	167	75
3	−46	−108	−164	−131	−77	19	134	199	199	148
4	46	−36	−94	−102	−90	−55	44	94	107	90
1925	59	−4	−49	−65	−27	−22	24	60	71	53
6	8	−32	−90	−97	−87	−43	5	45	52	25
7	15	32	27	21	25	27	49	42	24	6
8	−44	−73	−65	−25	−13	−6	34	6	−49	−62
9	−67	−88	−61	14	64	104	114	86	−3	−65
1930	−113	−147	−102	−51	17	104	165	139	63	−6
1	−130	−150	−162	−95	−23	104	180	202	77	11
2	−89	−200	−197	−132	−94	−39	85	174	160	89
3	−3	−103	−134	−161	−117	−70	27	72	74	50
4	9	−49	−116	−167	−116	−54	14	36	40	33
1935	−14	4	12	8	11	52	20	19	−20	−46
6	−63	−66	−52	−14	49	98	98	67	12	−37
7	−94	−137	−96	−22	40	102	140	94	35	−25
8	−112	−133	−110	−70	−18	62	160	180	108	28
9	−85	−110	−116	−55	−10	54	140	179	166	98
1940	2	−68	−114	−134	−76	28	84	115	122	54
1	70	38	3	−13	−4	21	40	66	82	74
2	23	7	−4	1	9	9	−7	−8	2	14
3	30	56	87	118	145	129	104	63	7	−49
4	−66	−26	4	60	210	268	272	182	−10	−84
1945	−144	−173	−152	−60	92	191	280	305	252	115
6	12	−154	−193	−142	−74	−5	140	249	285	242
7	170	10	−94	−183	−194	−87	32	132	226	256
8	251	224	170	90	−28	−70	−75	−104	−49	114
9	68	178	244	248	220	160	54	−74	−152	−189
1950	−180	−105	34	174	282	361	347	170	−19	−204
1	−302	−332	−256	−58	140	302	445	408	260	58
2	−196	−358	−392	−298	−135	104	302	420	420	254
3	19	−129	−317	−372	−308	−140	48	245	336	283
4	100	0	−100	−180	−240	−140	−40	0	40	70

Sources:
1890·0–1899·8 Albrecht
1899·9–1905·9 Albrecht and Wanach
1906·0–1911·9 Wanach
1912·0–1922·6 Wanach and Mahnkopf
1922·7–1934·9 Kimura
1935·0–1940·0 Carnera
1940·0–1948·9 Carnera

1949·0–1954·9 Cecchini

					m					
·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	
83	−110	−246	−279	−221	−126	22	144	232	279	1890
240	100	− 50	−190	−240	−166	− 34	95	157	168	1
134	71	19	− 4	− 50	−156	−199	−160	− 32	46	2
107	121	111	87	43	− 11	− 71	−111	−112	− 78	3
− 49	− 16	12	43	95	123	118	70	10	− 51	4
− 87	−117	− 87	− 28	37	106	101	110	90	53	1895
− 29	−102	−139	−153	−140	− 95	11	110	222	195	6
117	22	− 83	−143	−181	−171	−139	− 73	− 7	130	7
147	107	55	− 10	− 87	−146	−176	−140	− 65	29	8
99	150	140	98	25	− 45	−119	−136	−118	− 73	9
− 30	15	40	57	78	76	50	6	− 29	− 42	1900
− 47	− 52	− 54	− 28	27	100	141	130	110	77	1
11	− 77	−149	−186	−180	−110	− 22	61	118	113	2
51	− 22	−110	−191	−240	−228	−165	− 51	76	143	3
141	86	− 5	− 97	−183	−211	−175	−123	− 55	28	4
101	144	125	54	− 38	−127	−182	−189	−145	− 68	1905
10	50	45	19	− 20	− 78	−113	−127	−130	−143	6
−119	− 60	− 16	15	53	75	67	22	− 41	−123	7
−192	−215	−198	−124	− 17	67	141	169	160	96	8
− 15	−152	−261	−312	−290	−174	− 8	127	228	282	9
237	94	− 73	−230	−328	−343	−266	−105	81	227	1910
288	239	122	− 36	−205	−321	−342	−286	−158	2	1
112	199	200	138	55	− 31	−109	−160	−171	−150	2
−100	− 48	10	57	91	109	111	91	24	− 46	3
− 99	−116	−114	−101	− 66	− 1	80	170	210	137	4
23	− 92	−205	−282	−260	−175	− 63	70	211	271	1915
235	129	1	−132	−260	−273	−144	− 10	112	212	6
283	271	154	28	− 99	−199	−207	−135	− 44	40	7
123	132	55	− 20	− 93	−132	−129	−100	− 70	− 61	8
− 60	− 57	− 38	− 3	22	30	30	20	− 9	− 62	9
−124	−159	−154	−119	− 55	20	90	149	165	134	1920
49	− 29	− 81	−110	− 99	− 59	3	80	130	107	1
39	− 40	−117	−185	−181	−128	− 60	98	187	210	2
199	128	34	− 71	−172	−172	−151	− 74	4	87	3
117	118	18	− 36	− 87	−107	−106	− 69	− 24	24	4
82	130	100	30	0	− 26	− 8	37	60	77	1925
109	142	64	5	− 33	− 70	− 87	− 38	14	− 43	6
28	38	44	32	− 22	− 40	− 14	42	65	78	7
64	29	− 6	− 42	− 72	− 45	− 18	16	33	5	8
− 10	− 24	− 62	−112	−130	− 65	24	81	113	131	9
83	− 28	−107	−143	−178	−139	− 79	19	83	97	1930
73	− 22	− 99	−207	−228	−216	−161	− 76	90	118	1
112	43	− 69	−168	−190	−212	−207	−159	− 29	100	2
129	96	18	− 94	−148	−190	−182	−118	− 41	23	3
75	76	17	− 26	− 77	−143	−157	−112	− 73	14	4
8	20	23	1	− 16	− 22	− 20	− 12	6	20	1935
3	− 44	− 70	− 80	− 76	− 44	1	44	72	79	6
20	− 40	− 86	−110	−113	− 89	− 18	34	71	90	7
88	29	− 60	−115	−149	−133	− 71	− 2	73	131	8
119	74	− 14	− 61	−107	−130	− 99	− 16	33	87	9
139	93	50	− 25	−100	−138	− 84	− 23	31	63	1940
− 16	13	12	− 6	− 49	− 93	−124	−121	− 81	− 47	1
− 18	− 6	− 1	− 16	− 27	− 28	− 34	− 76	−103	−120	2
− 46	−112	− 70	− 55	− 36	5	37	56	33	− 12	3
− 58	−125	−156	−144	−116	− 84	60	150	180	130	4
62	− 12	−135	−211	−237	−228	− 65	38	78	215	1945
176	129	18	−104	−215	−276	−276	−200	− 72	34	6
125	160	97	− 1	−100	−169	−214	−218	−182	−146	7
− 82	15	63	113	96	58	10	− 66	−160	−252	8
−283	−252	−181	− 74	21	83	128	131	35	−122	9
−237	−324	−348	−280	−186	− 72	74	176	230	190	1950
− 18	−182	−336	−460	−406	−286	−100	102	238	259	1
225	40	−170	−334	−447	−467	−403	−240	− 14	142	2
237	204	95	− 48	−228	−358	−413	−386	−254	− 60	3
95	190	200	140	40	−100	−210	−285	−290	−250	4

Astron. Nach. 3333 and 3633.
Resultate des Internationalen Breitendienstes Band III, pp. 223–224.
Resultate . . . . Band V, pp. 180–182.
Ergebnisse . . . Band VI, p. 222.
Results . . . . . . Band VIII, p. 232.
Definitive results reported to Dublin meeting of the I.A.U. 1955.
Contribute Astronomico Capodimonte II, vol. IV no. 1, p. 10, no. 2, p. 11 and p. 20, and no. 6, p. 46. (These results are *not* definitive.)
Definitive results reported to Dublin meeting of I.A.U. 1955.

Nicolini, it appears that the observers are satisfied that the overlap in the programmes has been sufficient to minimize any inhomogeneity of the results because of these changes.

TABLE II

Stations used in International Programme

i	Station	Longitude (λ)	Dates of functioning
1.	Mizusawa	141° 8′ E.	1899 Dec. to date.
2.	Tschardjui	(i) 63° 29′ E.	1899 Dec. to 1909 July.*
	Tschardjui	(ii) 63° 35′ E.	1909 July to 1919 May.
	Kitab	(iii) 66° 53′ E.	1930 Nov. to date.†
3.	Carloforte	8° 19′ E.	1899 Dec. to 1943.
			1946 to date.
4.	Gaithersburg	77° 12′ W.	1899 Dec. to 1914 Dec.
			1932 Apr. to 1953 Aug.†
5.	Cincinnati	84° 25′ W.	1899 Dec. to 1915 Dec.
6.	Ukiah	123° 13′ W.	1899 Dec. to date.

* The site of the observatory was moved in 1909 July.

† Kimura did not include the results of these stations when observations recommenced there, but Carnera did so after 1935.

However, the star catalogue used from 1899 to 1935 was that of Cohn in which some of the declinations are in error. Since 1935, the more accurate Boss's catalogue has been used but, even in this, errors large enough to affect the accuracy of the results are known to exist.

The reliability of the recorded results might be affected by the fact that equal weight cannot really be given to each station. Because of their favourable climates, Carloforte and Ukiah usually contribute many more observations than the other stations. Between 1900 and 1924 from two to three thousand pairs of stars were observed at Carloforte every year, but in 1925, 1926 and 1927 the numbers of pairs observed were respectively only 1248, 928 and 1358. At that particular time only three stations, Mizusawa, Carloforte and Ukiah, were in operation, so the results for this period might well be less reliable than those for other periods. There have been other intervals when there have been relatively few observations from particular stations.

There is, finally, the "closing error" of the observations (see Melchior, **3**, p. 17). This has always been systematically negative and positive respectively at stations in the Northern and Southern Hemispheres but, following the modifications in the observing procedure made by Kimura in 1922·7, the error was reduced in magnitude.

We see, therefore, that the series of observations really cover a fairly large number of sub-intervals of time, each of which differs from the others in one or more of the number of stations in action, the star-programme used, the method of observing used and the method of reduction followed.

We might expect that the sub-intervals are sufficient in number for our statistical theory to remain approximately true despite this inhomogeneity, systematic irregularities arising in a sub-interval due to the organization of the programme persisting only for a short time and changing sufficiently frequently for us to neglect the resulting bias in the estimates of the harmonic components comprising the assumed systematic part of the variation (see **1**, equation (18)). In (**1**, p. 449), however, we suggested that the adequacy of our representation of the systematic variation might be tested by dividing the series of observations into several consecutive sub-series and obtaining estimates from these which could be compared with estimates for the whole series. This no longer seems reasonable. If, on the one hand, we choose several short sub-series it seems probable that they would not include enough of the sub-intervals described above for our theory to be applicable. On the other hand, the inclusion of enough sub-intervals would give too few sub-series for useful comparisons to be made.

4. The computational procedure follows that described in our previous paper. There we discussed, at some length, the factors governing the choice of the lags which ought to be adopted in evaluating the covariances of the various series and argued (pp. 450–451) that it seems best in general to use the smallest possible lags. We have, however, made a number of analyses using higher lags; for these we need formulae which are slight generalizations of those given before. In (1, p. 450) equation (29) generalizes to

$$\cot r\gamma^* h = p(r)/q(r) \tag{3}$$

where $p(r)$ and $q(r)$ are as given in equation (30), r being the lag, $2\pi/\gamma$ the period and h the interval between observations. The damping factor, κ, is estimated from

$$e^{2\kappa^* h(r-1)} = \frac{\{p(1)\}^2+\{q(1)\}^2}{\{p(r)\}^2+\{q(r)\}^2}. \tag{4}$$

The equation

$$e^{2\kappa^* h(r_1-r_2)} = \frac{\{p(r_2)\}^2+\{q(r_2)\}^2}{\{p(r_1)\}^2+\{q(r_1)\}^2} \tag{4 a}$$

may also be used, but detailed discussion of this is given later when we examine the question of bias (Sections 15 to 18).*

5. We have analysed the series and sub-series which are listed in Table III. They include unsmoothed and smoothed series in which the z-term has been taken into account and unsmoothed series in which the z-term has been ignored. The series have been given code numbers for ease of reference. The first digit of the code denotes the type of series (e.g. unsmoothed series in which the z-term has been taken into account are type-5 series), and the second digits distinguish the series according to the interval they cover.

As one of the major purposes of the work is to determine the period and damping factor of the free motion of the pole, values of these constants are given in the table, but it should be noted that the values given are those which have been obtained by using the smallest lags.

For the sake of completeness we have added the series which we considered in our previous paper, together with those considered by Jeffreys (2).

6. First we considered in more detail the data which we originally examined in (1). In the reduction of these observations, the z-term was ignored, and as it seems certain that this term is physically real we did not think it necessary to try to locate data for the years after 1934 (if indeed such data are to be found). The thirty-five year interval is covered by series 31. Series 32 cover the twenty-two complete years, 1900 Jan. to 1921 Dec. when the programme was directed at Potsdam and series 33 cover the twelve complete years 1923 Jan.

* An asterisked quantity is the estimated value of the unasterisked one, except that l^* and m^* are estimates of $l-l_s\,(=x)$ and $m-m_s\,(=y)$. In the tables asterisks are dropped because all the tabular values are estimates.

TABLE III

Description of data and principal results

Code	Length	Description	Free period ($2\pi/\gamma$) in years	Damping factor (κ) in years^{-1}
00	1890·0–1955·0	Smoothed; 0·1 year interval; z-term taken into account. Data of Table I*b*.	1·304 ± 0·024	0·506 ± 0·095
01	1900·0–1934·9		1·287 ± 0·028	0·361 ± 0·107
02	1900·0–1920·9		1·249 ± 0·025	0·198 ± 0·100
15	1892·0–1932·9	Smoothed; 0·1 year interval; z-term taken into account. Data reduced by taking non-overlapping means of three consecutive values. Jeffreys's data.	1·223 ± 0·019	0·058 ± 0·015
16	1908·3–1921·5		1·202 ± 0·016	0·066 ± 0·007
31	1900 Jan.–1934 Dec.	Unsmoothed; monthly interval; z-term *not* taken into account.	1·267 ± 0·039	0·322 ± 0·103
32	1900 Jan.–1921 Dec.		1·248 ± 0·034	0·261 ± 0·113
33	1923 Jan.–1934 Dec.		1·355 ± 0·099	0·661 ± 0·340
34	1900 Jan.–1905 Dec.		1·241 ± 0·054	0·366 ± 0·221
50	1899 Dec.–1954 Dec.	Unsmoothed; monthly interval; z-term taken into account. Data of Table I*a*.	1·287 ± 0·026	0·441 ± 0·095
51	1900 Jan.–1934 Dec.		1·267 ± 0·041	0·356 ± 0·109
52	1900 Jan.–1920 Dec.		1·238 ± 0·033	0·239 ± 0·111

to 1934 Dec., during which Kimura was director. The change in the method of making the observations was made in 1922 July, so this year has been omitted. Series 33 should be homogeneous in every respect, but in 32, although the methods of reduction and observation remained unchanged, there were changes in the star-programmes and numbers of stations in operation. We have also examined series 34 which cover the first six years of the work of the International Service and should be homogeneous. The main results are given in Table IV. Series 33 appear to give results which are not in accord with the other three series of this group although it is unlikely that the difference would be highly significant. The physical explanation of this apparent anomaly is possibly to be found in the fact that the interval covered was short and seriously affected by the paucity of observations to which we have referred earlier. Series 34 give a value of the free period in reasonable agreement with the remaining two but the value of κ^* obtained from these series appears to be subject to considerable uncertainty. In this case, the fact that λ^* exceeds unity by more than three times its estimated standard error may well be attributed to the inapplicability of the asymptotic theory (developed in Sections 4 and 5 of 1) to such short series. (It will be recalled that λ is the parameter that measures the effect of observational error.)

7. Next we examined the smoothed series in which account was taken of the z-term. We have analysed the longest available series, 00, covering the sixty-five years from 1890·0 to 1954·9 and also the series 01 and 02 which correspond to the series 31 and 32. The results for the type-0 series are given in Table V.

In this group of results there is a striking decrease of both the estimated values of the free period and the damping factor as the series shorten. The results are not particularly consistent with each other, especially in the case of values of κ^*.

TABLE IV

Results for series of type-3

(*a*) Harmonic components of systematic annual motion (unit 0″·001)

	Series 31			Series 32		Series 33		Series 34		
	l	m	est. stand. error	l	m	l	m	l	m	est. stand. error
c	+11·06	−10·06	± 3·05	+18·57	− 3·59	− 7·59	−21·23	+ 0·32	+11·90	± 6·01
a_1	−55·11	+74·79	± 10·77	−52·35	+63·91	−53·78	+92·28	−42·35	+61·96	± 24·35
b_1	−70·29	−45·28		−61·57	−39·48	−82·42	−46·03	−67·64	−28·72	
a_2	− 2·14	− 5·55	± 2·40	− 2·34	− 8·51	− 1·11	+ 2·31	− 1·96	− 2·44	± 4·51
b_2	− 6·91	− 3·11		+ 7·80	− 5·11	+ 4·11	+ 2·53	+ 6·28	− 5·29	
a_3	+ 2·35	+ 2·24	± 1·70	+ 3·21	+ 2·20	+ 1·06	+ 4·40	− 0·58	− 1·22	± 2·76
b_3	− 4·01	+ 3·94		− 1·37	− 4·42	− 8·71	+ 3·50	− 1·83	− 1·14	
a_4	− 1·76	+ 1·02	± 1·48	− 1·95	+ 1·99	− 1·31	+ 0·81	− 1·07	+ 2·06	± 2·15
b_4	+ 1·11	+ 1·22		+ 0·20	+ 2·95	+ 2·61	− 1·00	− 0·79	+ 0·48	
a_5	− 0·91	− 1·00	± 1·39	− 2·23	+ 2·47	+ 0·97	+ 0·72	− 4·73	− 0·40	± 1·90
b_5	+ 2·09	+ 1·26		+ 1·95	+ 1·89	+ 1·83	+ 2·44	− 0·75	+ 3·33	
a_6	− 0·77	− 1·56	± 0·97	− 0·28	− 1·89	− 1·74	+ 0·27	− 0·29	− 3·68	± 1·83

(*b*) Free Motion

	Series 31	Series 32	Series 33	Series 34
γh	23° 40′ ± 44′	24° 2′ ± 39′	22° 8′ ± 1° 37′	24° 10′ ± 1° 3′
period	1·267 ± 0·039 years	1·248 ± 0·034 years	1·355 ± 0·099 years	1·241 ± 0·054 years
κ	0·322 ± 0·103 year^{-1}	0·261 ± 0·113 year^{-1}	0·661 ± 0·340 year^{-1}	0·366 ± 0·221 year^{-1}
λ	0·9803 ± 0·0035	0·9939 ± 0·0025	0·9639 ± 0·0149	1·013 ± 0·004
σ	24·6	25·83	20·56	21·14
σ'	15·3	9·72	12·32	...

The most serious feature of these results is that $\lambda^* > 1$. Taking $\lambda = \lambda^*$ has little effect on the estimate of var $\kappa^* h$, but makes a great deal of difference to those of var λ^* and var $\gamma^* h$, the latter being in fact useless since it is then negative. The calculation of estimated variances is particularly suspect for this group of smoothed series in view of the values of the ratio of $\lambda^* - 1$ to $\sqrt{\{(\text{var}\,\lambda^*)_{\lambda=1}\}}$. These are 12·6 for series 00, 10·4 for series 01, and 10·3 for series 02, all of which are so highly significant as to make the results unacceptable. The discrepancy must of course be due to the behaviour of the auto-covariances and cross-covariances with lags 0, 1, 2 since only these are used in calculating the ratio.

Since the three series overlap, the estimates of γh, κ and σ are related to each other; because of this it is very troublesome to carry out even approximate large sample tests for the significance of the differences between the estimates. The extent of the correlation could be worked out but the calculations would be so tedious that we do not think they would be justified, in view of the behaviour of λ^*.

TABLE V

Results for series of type-0

(*a*) Harmonic components of systematic annual motion (unit 0″·001)

	Series 00		Series 01		Series 02	
	l	*m*	*l*	*m*	*l*	*m*
c	+17·81	−28·10	+19·00	−20·92	+27·40	−22·33
a_1	−36·51	+75·25	−39·76	+79·87	−41·12	+66·92
b_1	−89·30	−28·97	−77·36	−30·69	−64·80	−31·24
a_2	− 0·90	− 0·36	+ 1·05	− 1·33	+ 3·18	− 2·43
b_2	+ 1·32	− 0·70	+ 1·68	+ 0·25	+ 2·19	+ 0·06
a_3	+ 0·61	+ 0·36	+ 2·62	− 0·96	+ 1·80	− 1·03
b_3	− 1·93	+ 0·78	− 1·50	+ 1·42	− 0·08	− 0·11
a_4	− 1·58	− 0·49	− 1·15	− 0·04	0·00	− 1·35
b_4	+ 0·06	+ 0·55	− 0·21	+ 0·53	+ 0·22	+ 0·61
a_5	− 0·69	− 0·03	+ 0·01	− 0·23	+ 0·45	− 0·59

(*b*) Free Motion (calculated with least possible values of lag)

	Series 00	Series 01	Series 02
γh	27° 37′ ± 31′	27° 58′ ± 36′	28° 50′ ± 34′
period	1·304 ± 0·24 years	1·287 ± 0·028 years	1·249 ± 0·025 years
κ	0·506 ± 0·095 year^{-1}	0·361 ± 0·107 year^{-1}	0·198 ± 0·100 year^{-1}
λ	1·037 ± 0·003	1·029 ± 0·003	1·020 ± 0·002
σ	37·3	28·6	25·0
σ'	...	...	...

N.B. The standard errors were calculated using $\lambda = 1$.

8. Finally, we examined the unsmoothed series in which account was taken of the z-term. The longest suitable series, 50, cover the fifty-five years 1900 Jan. to 1954 Dec. Series 51 and 52 correspond to series 31 and 32 respectively. The results derived from this group of type-5 series are given in Table VI. Here we see again that the estimates of the length of the free period and of the damping factor decrease with the length of the interval. This group of results shows much more consistency than do either the groups of type-0 or type-3 series and there is no trouble over the values of λ^* which are all significantly less than unity.

Again the overlapping of the intervals makes comparisons difficult but the estimates of the standard errors are now acceptable and so do give a rough idea of the consistency of the estimates. We can use the inequality $\sigma(X-Y) \leqslant \sigma(X) + \sigma(Y)$

(for any random variables X, Y) to obtain an under-estimate of the significance of a difference—though admittedly this is very much of an under-estimate if there is a fair degree of positive correlation between the two variables. Thus we would probably be justified in concluding that the difference between the λ^*'s for series 50 and 51 was significant since this is 0·0182 compared with an estimated standard error of *at most* $0{\cdot}0022+0{\cdot}0038=0{\cdot}0060$.

TABLE VI

Results for series of type-5

(*a*) Harmonic components of systematic annual motion (unit 0″·001)

	Series 50			Series 51			Series 52		
	l	m	est. stand. error	l	m	est. stand. error	l	m	est. stand. error
c	+18·80	−31·72	± 3·17	+12·92	−22·02	± 3·20	+19·00	−23·30	± 3·86
a_1	−64·45	+69·51	± 10·58	−55·33	+74·99	± 11·36	−48·42	+65·67	± 15·75
b_1	−71·41	−45·59		−69·59	−45·89		−59·68	−37·05	
a_2	− 1·29	− 4·91	± 2·28	− 2·15	− 6·00	± 2·51	− 2·51	− 7·95	± 3·00
b_2	+ 5·66	− 0·39		+ 6·79	− 3·05		+ 8·37	− 5·45	
a_3	− 0·16	+ 1·38	± 1·44	+ 1·84	+ 2·36	± 1·77	+ 2·91	+ 2·29	± 1·96
b_3	− 2·59	+ 2·79		− 4·15	+ 3·93		− 1·19	+ 4·38	
a_4	− 0·78	+ 1·47	± 1·14	− 1·02	+ 0·55	± 1·53	− 0·65	+ 0·89	± 1·62
b_4	+ 1·62	+ 1·52		+ 1·25	+ 1·36		+ 0·81	+ 4·28	
a_5	+ 0·21	+ 0·77	± 1·02	+ 0·58	+ 0·41	± 1·45	− 1·85	+ 1·85	± 1·49
b_5	+ 1·59	+ 0·63		+ 2·12	+ 1·77		+ 2·11	+ 2·94	
a_6	− 0·71	− 0·85	± 0·70	− 0·72	− 1·63	± 1·00	+ 0·19	− 2·65	± 1·03

(*b*) Free Motion (calculated with least possible values of lag)

	Series 50	Series 51	Series 52
γh	23° 18′ ± 28′	23° 41′ ± 46′	24° 14′ ± 39′
period	1·287 ± 0·026 years	1·267 ± 0·041 years	1·238 ± 0·033 years
κ	0·441 ± 0·095 year^{-1}	0·356 ± 0·109 year^{-1}	0·239 ± 0·111 year^{-1}
λ	0·9975 ± 0·0022	0·9793 ± 0·0038	0·9942 ± 0·0024
σ	32·4	25·8	25·2
σ'	6·0	15·6	9·8

9. The components of the systematic annual motion shown in Tables IV, V and VI vary according to type of series and interval, there being noticeable differences in both the amplitudes and phases of the several components. When the tests of significance used in (**1**) were applied, we found that on the whole only the annual and semi-annual components are highly significant. When we have actually evaluated the estimated standard errors of the components, they appear to be relatively large. The differences of corresponding harmonic components of type-5 series are given in Table VII, together with the estimated standard errors of the pairs of components. The overlapping of the three series has again the effect of making any precise assessment of the significance of the difference very troublesome, but in view of the magnitudes of the estimated standard errors no startling discrepancy is indicated, although the differences 50–51 and 50–52 for the constant terms in the m components are suggestive.

10. There may be systematic trends in the observational data caused by genuine secular trends in the systematic terms in F_1 and G_1 (cf. **1**, equation (3)).

TABLE VII

Comparison of harmonic components of type-5 series

Differences (l-components)
(unit 0″·001)

	50–51			50–52			51–52		
r	a_r	Est. s.e.'s	b_r	a_r	Est. s'e.'s	b_r	a_r	Est. s.e.'s	b_r
0	5·88	3·17, 3·20		− 0·20	3·17, 3·86		−6·08	3·20, 3·86	
1	−9·12	10·58, 11·36	−1·82	−16·03	10·58, 15·75	−11·73	−6·91	11·36, 15·75	−9·91
2	0·86	2·28, 2·51	−1·13	1·22	2·28, 3·00	− 2·71	0·36	2·51, 3·00	−1·58
3	−2·00	1·44, 1·77	1·56	− 3·07	1·44, 1·96	− 1·40	−1·07	1·77, 1·96	−2·96
4	0·24	1·14, 1·53	0·37	− 0·13	1·14, 1·62	0·81	−0·37	1·53, 1·62	0·44
5	0·79	1·02, 1·45	−0·53	2·06	1·02, 1·49	− 0·52	1·27	1·45, 1·49	0·01
6	0·01	0·70, 1·00		− 0·90	0·70, 1·03		−0·91	1·00, 1·03	

Differences (m-components)
(unit 0″·001)

	50–51			50–52			51–52		
r	a_r	Est. s.e.'s	b_r	a_r	Est. s.e.'s	b_r	a_r	Est. s.e.'s	b_r
0	−9·70	3·17, 3·20		− 8·42	3·17, 3·86		1·28	3·20, 3·86	
1	−5·48	10·58, 11·36	0·30	3·84	10·58, 15·75	− 8·54	9·32	11·36, 15·75	−8·84
2	1·09	2·28, 2·51	2·66	3·04	2·28, 3·00	5·06	1·95	2·51, 3·00	2·40
3	−0·98	1·44, 1·77	−1·14	− 0·91	1·44, 1·96	− 1·59	0·07	1·77, 1·96	−0·45
4	0·92	1·14, 1·53	0·16	0·58	1·14, 1·62	− 2·76	−0·34	1·53, 1·62	−2·92
5	0·36	1·02, 1·45	−1·14	− 1·08	1·02, 1·49	− 2·31	−1·44	1·45, 1·49	−1·17
6	0·78	0·70, 1·00		1·80	0·70, 1·03		1·02	1·00, 1·03	

We have examined the series of type-5 for such trends by fitting curves of the type

$$l_s = c_0 + c_1 t + a\cos\nu t + b\sin\nu t \qquad (2\pi/\nu = 1 \text{ year})$$

to each interval covered by only one director or only one star-programme. The results give values of c_0 markedly different from those of c recorded in Table VI and the values of c_1 are so large that the removal of systematic trends of the kind assumed here yields series for the free motion which are visibly irregular, with large discontinuities at each change in star-programme. If the trends are real, we cannot explain them satisfactorily. However, the frequent changes in programme make it difficult to determine trends accurately, and the large discontinuities found at each change in star-programme are likely to be due more to accidental rather than true physical causes.

11. We have examined the residual motion $\{\epsilon'^*, \eta'^*\}$ which includes the effects of both observational errors and the random disturbances which maintain the free motion. We have restricted our attention to those of type-5 series. The residuals have been estimated in the way described in (1, p. 458) for each of the series 50, 51 and 52, those for 50 being displayed in Fig. 1 (*b*, *d*).

If the series were heterogeneous and corrections such as those given by (1) should be applied, then the numerical harmonic analysis would leave in l^*, m^* errors which remain constant through each homogeneous sub-interval, but which change between such sub-intervals. As a result $\{\epsilon'^*\}$ and $\{\eta'^*\}$ would also be affected by similar errors. We have therefore determined the means of $\{\epsilon'^*\}$ and $\{\eta'^*\}$ over various sub-intervals during which star-programme and

TABLE VIII

Mean residuals: series of type-5 (unit 0″·001)

Series 50			
Star Programme	Interval	$\bar{\epsilon}'$	$\bar{\eta}'$
I	1900–05	2·89 ± 2·94	6·89 ± 2·90
II	1906–11	− 6·00 ± 3·22	5·39 ± 3·03
III	1911–22 July	8·83 ± 2·60	− 7·00 ± 3·38
I-III	1900–22 July	3·31 ± 1·71	− 0·02 ± 1·97
IV	1922 July–34	2·67 ± 2·60	9·89 ± 2·67
V	1935–48	3·73 ± 2·30	− 5·24 ± 2·24
VI	1949–54	−24·15 ± 5·44	− 9·86 ± 4·42
I–VI	1900–54	0·28 ± 1·28	− 0·19 ± 1·28
Series 51			
Star Programme		$\bar{\epsilon}'$	$\bar{\eta}'$
I		− 0·06 ± 2·89	3·58 ± 2·93
II		− 9·11 ± 3·12	2·21 ± 2·94
III		5·90 ± 2·59	−10·43 ± 3·35
I–III		0·33 ± 1·70	− 3·35 ± 1·95
IV		− 1·57 ± 2·71	6·17 ± 2·65
I–IV		− 0·35 ± 1·46	0·03 ± 1·59
Series 52			
Star Programme		$\bar{\epsilon}'$	$\bar{\eta}'$
I		− 0·26 ± 2·86	6·35 ± 2·93
II		− 9·57 ± 3·07	5·08 ± 2·88
III		5·44 ± 2·80	− 6·60 ± 3·41
I–III		− 0·48 ± 1·73	0·44 ± 1·91

direction remained uniform. The results are given in Table VIII. The uncertainties given in the table are the standard errors calculated in the normal way but, because of correlations introduced by the processes by which the estimated annual and forced motions were removed, and possibly also correlations between the true residuals ϵ'_n and ϵ'_{n+1} and η'_n and η'_{n+1}, they are not unbiased estimates of the true standard errors. The bias is very troublesome to calculate but may be expected in all cases to be positive, so that they are useful as estimates of upper limits to the uncertainties. The results suggest that the data do suffer from inhomogeneity.

12. When the results are grouped according to the intervals covered by the series it is immediately evident that the results for the group 02, 32 and 52 are very consistent, and those for the group 01, 31 and 51 are slightly less so. The agreement for the longer series 00 and 50 appears to be less good.

The evidence of the statistical analysis so far presented does not give any conclusive reasons for preferring or rejecting any particular set of results, although these for type-0 series are suspected to be unacceptable because $\lambda^* > 1$. When the results are considered in conjunction with the known facts about the initial data it seems reasonable to draw certain inferences.

The data might be inhomogeneous, and if the corrections given in (1) are applicable in full the long series might be expected to yield unreliable results. The results for series 00 and 50 seem to support this, and indeed, the results derived from series 50 appear to be less untypical compared with the main body of results than do those for series 00 which differ from 50 by the inclusion of the results of the uncoordinated observations made in the decade preceding the beginning of the international programme.

It might also be expected that the series covering the first twenty-one years of the international programme would give better results than those covering the first thirty-five years. We have already recorded that the differences in the constant terms of the harmonic analyses of the m components between the series 50 and 51 and between the series 50 and 52 are suggestive, and although these differences are not what they should be if they were due to the non-application of corrections of the order given by (1) they still support the opinion that the data which we have analysed are inhomogeneous to some extent.

It also appears that the variability of the results depends more on the interval covered by the series than on the type of series. The fact that $\lambda^* > 1$ in the case of type-0 series does suggest that smoothing has had an effect, but the differences between the results of type-3 and type-5 series suggest that the inclusion or exclusion of the z-term has had little influence. On observational grounds, however, the z-term must be accounted for and we conclude that the most reliable results are those derived from type-5 series.

The results of the analysis of the short series 33 and 34 give periods and damping factors which seem unsatisfactory, but their estimated standard errors are large enough to account for the discrepancies. The low precision of the estimates and the probable inaccuracy of the asymptotic theory for short series are likely to be general and we have not made any other analyses of such short series. This has led to the disadvantage, already noted, that the intervals are overlapping, but that has not proved to be too serious a defect.

13. Jeffreys's results are given in Table III as series 15 and 16. He used essentially the same data as those of our type-0 series. The periods which he

obtained are in fair agreement with those for our series 01 and 02 but the damping factors are markedly different. His value of λ for series 15 was less than unity. However his method of computing κ and λ was different from ours, being equivalent to solving (in our notation)

$$\lambda^2 e^{-2\kappa h r} = \frac{\{p(r)\}^2 + \{q(r)\}^2}{\{p(0)\}^2}$$

by least squares using three values of the lag r. He advocated the use of fairly large values of r, whereas we have argued that the smallest possible lags should give greatest accuracy (1, p. 450). Jeffreys also reduced the volume of data by taking non-overlapping means of three consecutive observations so that his lags 0, 1 and 2 correspond roughly to our 0, 3 and 6.

In order to make a direct comparison with Jeffreys's work we have examined two series, 05 and 06, for the intervals 1892·0 to 1932·9 and 1908·3 to 1921·5 which were the intervals of Jeffreys's series 15 and 16 in Table III. The data differ only in that we have not grouped them. We have calculated covariances only for every third lag and have used equation (4) in the modified form

$$e^{6\kappa^* h(r-1)} = \frac{\{p(3)\}^2 + \{q(3)\}^2}{\{p(3r)\}^2 + \{q(3r)\}^2} \tag{5}$$

and (4 *a*) with $r_1 = 3r$ and $r_2 = 3r - 3$, so that

$$e^{6\kappa^* h} = \frac{\{p(3r-3)\}^2 + \{q(3r-3)\}^2}{\{p(3r)\}^2 + \{q(3r)\}^2}. \tag{5 a}$$

The results given in Table IX show that for small lags, the estimates of the period agree favourably with those given by Jeffreys so it seems that his grouping of the data has had no serious effect. The damping factors, except those for series 06 computed by formula (5), seem to be irregular, but for both period and damping factor it can be said that as the lag increases there is initially a marked decrease in the estimates followed by a more or less marked oscillation.

TABLE IX

Analysis of series 05 and 06 for various lags

$3r$	Period in years		Damping factor in year^{-1}			
	Series 05	Series 06	Series 05		Series 06	
			Formula (5)	Formula (5*a*)	Formula (5)	Formula (5*a*)
3	1·230	1·216				
6	1·193	1·206	+0·169	+0·169	+0·064	+0·064
9	1·178	1·202	+0·037	−0·096	+0·028	−0·008
12	1·190	1·201	−0·005	−0·088	+0·024	+0·017
15	1·209	1·201	+0·036	+0·159	+0·022	+0·015
18	1·189	1·203	+0·060	+0·159	+0·022	+0·021
21	1·184	1·202	+0·035	−0·090	+0·022	+0·022
24	1·187	1·200	+0·021	−0·063	+0·021	+0·012

14. As we believe that the data of the type-5 series are superior for the purposes of analysis, we have carried out more calculations for the series 50, 51 and 52. The results are given in Table X. (Formula (4 *a*) was used with $r_1 = r$, $r_2 = r - 1$.)

The estimated period is clearly seen to be a damped oscillatory function of the lag. For the damping, the estimates follow a similar pattern but there is an appreciable difference between the results according to which formula is used.

TABLE X

Analysis of series 50, 51 and 52 for various lags.

	Period in years			Damping factor in year^{-1}					
	Series	Series	Series	Series 50		Series 51		Series 52	
r	50	51	52	Form. 4	Form. 4a	Form. 4	Form. 4a	Form. 4	Form. 4a
1	1·288	1·268	1·237						
2	1·265	1·253	1·227	0·443	+0·443	0·352	+0·352	0·231	+0·231
3	1·241	1·239	1·219	0·457	+0·471	0·379	+0·406	0·241	+0·251
4	1·220	1·226	1·212	0·436	+0·394	0·389	+0·409	0·234	+0·220
5	1·205	1·215	1·207	0·400	+0·291	0·387	+0·380	0·223	+0·190
6	1·194	1·203	1·203	0·348	+0·142	0·350	+0·200	0·199	+0·105
7	1·185	1·193	1·199	0·315	+0·150	0·315	+0·140	0·174	+0·045
8	1·178	1·186	1·196	0·270	−0·003	0·254	−0·107	0·138	−0·074
9	1·172	1·181	1·194	0·221	−0·118	0·200	−0·184	0·105	−0·127
10	1·169	1·179	1·193	0·166	−0·276	0·151	−0·239	0·077	−0·152
11	1·169	1·178	1·192	0·117	−0·318	0·109	−0·263	0·051	−0·180
12	1·171	1·180	1·193	0·083	−0·257	0·077	−0·243	0·034	−0·138
13	1·177	1·183	1·194	0·066	−0·128	0·066	−0·057	0·033	+0·024
14	1·182	1·185	1·195	0·070	+0·118	0·069	+0·097	0·039	+0·114
15	1·185	1·187	1·195	0·080	+0·216	0·075	+0·165	0·045	+0·126
16	1·187	1·189	1·195	0·092	+0·254	0·081	+0·158	0·050	+0·108

15. Professor Jeffreys has suggested to us that the use of higher lags may be advisable because of the possibility that the disturbances ϵ and η are not uncorrelated as we assume in our model. There are good physical reasons for believing that correlations between the disturbances do exist; e.g. the seasonal climatic changes which give rise, at least in part, to the systematic terms $F_s(t)$ and $G_s(t)$ (1, equation (4)) are not entirely smooth, and irregularities in them tend to persist over intervals of several weeks, so that there may well be appreciable positive correlation between values of ϵ or η at times separated by intervals up to 2 or 3 months.

Such correlation may cause our estimates of the period and of the damping factor to be biased. We have investigated the bias on the assumption that $\operatorname{cov}(\epsilon_t, \epsilon_{t+rh}) = \operatorname{cov}(\eta_t, \eta_{t+rh}) = g_r$, say, while $\operatorname{cov}(\epsilon_t, \eta_{t+rh})$ remains equal to zero for all r. It is then easily shown that

$$b(r\gamma^* h) \doteqdot \tan^{-1}\frac{B(r)}{A(r)} - r\gamma h \tag{6}$$

and for the estimate κ^* given by equation (4 a)

$$b\{(r_1 - r_2)\kappa^* h\} \doteqdot \log\left|\frac{e^{r_2\kappa h}\{A(r_2) + iB(r_2)\}}{e^{r_1\kappa h}\{A(r_1) + iB(r_1)\}}\right| \tag{7}$$

where

$$A(r) + iB(r) = \sum_{s,t=0}^{\infty} e^{-\kappa h(s+t) + i\gamma h(s-t)} g_{r-s+t}$$

and we use the notation $b(\theta^*)$ for the bias in an estimate θ^* of a quantity θ.

In particular, when $g_r = g_0 \mu^r$ $(r > 0)$, where $|\mu| < 1$, i.e. the auto-correlation function of the series $\{\epsilon_t\}$, $\{\eta_t\}$ is that of a first-order autoregressive process,

(6) and (7) give

$$b(r\gamma^*h) \doteqdot \frac{-2\mu\kappa h \sin\gamma h}{1+\mu^2-2\mu\cos\gamma h} + \left(\frac{2\kappa h\mu^{r+1}}{1-\mu^2}\right) \sin\{(r+1)\gamma h - \phi\} \tag{8}$$

and

$$b\{(r_1-r_2)\kappa^*h\} \doteqdot \frac{2\kappa h}{1-\mu^2} [\mu^{r_1+1}\cos\{(r_1+1)\gamma h-\phi\} - \mu^{r_2+1}\cos\{(r_2+1)\gamma h-\phi\}] \tag{9}$$

where

$$\phi = 2\tan^{-1}\left(-\frac{\mu\sin\gamma h}{1-\mu\cos\gamma h}\right)$$

and it is assumed that r, r_1, r_2 are sufficiently small for $e^{r\kappa h}$, $e^{r_1\kappa h}$, $e^{r_2\kappa h}$ to be taken approximately equal to unity. Thus as r increases the bias in γ^* tends to zero like $1/r$ after an initial damped oscillation. When κ^* is calculated by formula (4) the bias is given by (9) with $r_1 = r$ and $r_2 = 1$. This behaves in the same way as the bias in γ^*. When κ^* is calculated by formula (4 *a*), however, the bias is given by (9) with $r_1 = r$ and $r_2 = r-1$ and oscillates with an amplitude that decreases like μ^r. The behaviour of the biases should in general not be very different from this since $g_0, g_1, g_2, \ldots$ may be expected to be a decreasing sequence of positive numbers. In fact, if g_r can be put equal to zero for $r > q$, it follows from (6) and (7) that for $r > q$, the biases of γ^* and κ^* calculated from equation (4) are proportional to $1/r$, while the bias in κ^* calculated from equation (4 *a*) with $r_1 = r$, $r_2 = r-1$ is negligibly small. This should be the case for $q = 3$ or 4 since correlations between the disturbances should not extend over intervals exceeding a few months.

We conclude that correlation of this type would suffice to explain the behaviour of the estimates of $T = 2\pi/\gamma$ and κ in Table X for small values of r. However, it cannot reasonably be considered responsible for the persistence of oscillations for the larger values of r (those of κ^* in the last three columns being particularly marked), since that would require the auto-correlations g_r/g_0 to remain appreciable for these values of r.

16. The imperfect removal of systematic variations from the residual series used for the calculation of $p(r)$ and $q(r)$ will also produce bias in our estimates. Suppose that $\{\xi_t, \zeta_t\}$ represent systematic variations which have not been allowed for in determining the free motion. Then $p(r)$ and $q(r)$ are calculated from $\{X_t, Y_t\}$ instead of $\{x_t, y_t\}$ where

$$X_t = x_t + \xi_t, \qquad Y_t = y_t + \zeta_t.$$

If this is so, it can easily be shown that the consequential biases in $p(r)$ and $q(r)$ are

$$b(p(r)) = \frac{1}{N-r}\sum_t (\xi_t\zeta_{t+rh} + \zeta_t\zeta_{t+rh}), \tag{10}$$

$$b(q(r)) = \frac{1}{N-r}\sum_t (\xi_t\zeta_{t+rh} - \xi_{t+rh}\zeta_t), \tag{11}$$

N being the total number of observations.

The results of Section 9 suggest that the annual term is different in different sub-intervals so that the process which we adopted of removing a uniform annual term throughout the whole interval must lead to the presence of terms ξ_t and ζ which take the form

$$\xi_t = a_0 + a_1\cos(\nu t + \phi),$$
$$\zeta_t = b_0 + b_1\cos(\nu t + \psi),$$

where a_0, a_1, b_0, b_1, ϕ and ψ are constants and $2\pi/\nu = 1$ year; there may also be small harmonic terms. In the true situation, the constants change from one sub-interval to another, but for simplicity we will first consider them unchanged over the whole interval. We find that the biases in $p(r)$ and $q(r)$ lead to

$$b(r\gamma^* h) \doteqdot \frac{-e^{r\kappa h}\sin r\gamma h}{2V_{11}(0)}(a_0^2+b_0^2+\tfrac{1}{2}(a_1^2+b_1^2)\cos\nu r h) \qquad (12)$$

provided that λ is nearly equal to unity. ($V_{11}(0)$ is the expected value of $\frac{1}{2}p(0)$ in the absence of bias—see (1, equation (14)). If the damping is estimated by using equation (4), the bias is given by

$$b\{2(r-1)\kappa^* h\} \doteqdot \{\cos\gamma h[a_0^2+b_0^2+\tfrac{1}{2}(a_1^2+b_1^2)\cos\nu h] \\ -e^{(r-1)\kappa h}\cos r\gamma h[a_0^2+b_0^2+\tfrac{1}{2}(a_1^2+b_1^2)\cos\nu r h]\}/V_{11}(0) \quad (13)$$

and if it is obtained by using equation ($4a$) with $r_1 = r_2 + 1 = r$, it is given by

$$b\{2\kappa^* h\} \doteqdot \frac{e^{(r-1)\kappa h}}{V_{11}(0)}\{(a_0^2+b_0^2)[\cos(r-1)\gamma h - \cos r\gamma h] \\ +\tfrac{1}{2}(a_1^2+b_1^2)[\cos(r-1)\gamma h\cos\nu(r-1)h - \cos r\gamma h\cos\nu r h]\}. \quad (13a)$$

In each of these formulae we have omitted "cross-product" terms involving $a_1 b_1 \sin(\phi-\psi)$ which appear in the detailed working. Their inclusion would, at worst, double the estimated bias, but should have a much smaller effect.

If the period $2\pi/\gamma$ is denoted by T, then

$$\frac{b(T^*)}{T} \doteqdot -\frac{b(r\gamma^* h)}{r\gamma h},$$

and the bias in T, regarded as a function of r, is an oscillating one with an amplitude which varies with $e^{r\kappa h}/r$, which decreases as r increases, at least initially.

For the damping, formula (4) leads to a bias in κ^* which consists partly of a term decreasing with $1/(r-1)$ and partly of an oscillation with an amplitude which decreases (initially) with $e^{(r-1)\kappa h}/(r-1)$. Formula ($4a$) leads to an oscillatory bias whose amplitude varies with $e^{(r-1)\kappa h}$.

The formulae given are not valid for very large lags, when the second order terms, which have been neglected, may become more important than the terms given.

When this analysis is extended to deal with ξ_t and ζ_t whose amplitudes change from sub-interval to sub-interval, the results are similar but, because of the increasing overlap of sub-intervals as r increases, the biases tend to decrease more and for sufficiently long series with sufficiently many sub-intervals the biases could become negligible—thus giving a justification for a remark made to this effect in Section 3 above.

17. Values of a_0, b_0, a_1 and b_1 consistent with the corrections given by (1) yield values of $b(p(r))$ which are comparable with $p(r)$ itself, and our results do not indicate such serious effects. However, the results of harmonic analysis given in Tables IV, V and VI show that the inhomogeneity could give values of the constants of order of magnitude 10 units (the unit being 0″·001). Putting $a_0 = b_0 = a_1 = b_1 = 10$ units, the largest $b(p(r))$ is about 50 units and hence the largest bias in $r\gamma^* h$ is about 0·02 in the range of values of r up to $r = 16$. The results correspond to a bias in the period which is approximately $0{\cdot}06/r$ years, which is in rough agreement with the behaviour of our estimates of T. Again,

using formula (4), the bias in κ^* given by the above analysis is at most about $0{\cdot}36/(r-1)$ which also compares with the behaviour of the estimates of κ.

18. It can also be shown that other kinds of systematic inhomogeneity in the data, not necessarily directly connected with the amplitudes of the annual terms, lead to biases similar to that arising from the sinusoidal terms (ξ_t, ζ_t). For example, systematic errors in individual star-places could give rise to a systematic contribution to the observational error which would repeat annually until the particular stars were replaced at a subsequent change in the observing programme; such an error would contribute to the bias in much the same way as considered above.

We may conclude that the behaviour of our estimates is explained by biases introduced by inhomogeneity and auto-correlation of the disturbing function.

19. We believe that the sub-intervals during which the data can be assumed to be homogeneous are too short ever to give a reasonable prospect of eliminating the systematic motion adequately; this is really a problem which is better tackled in the design of the observing programme rather than in the subsequent analysis of the results. Despite the complexities introduced by the inhomogeneity and the auto-correlation of the disturbances, however, it seems practicable to make a reasonable estimate of the period. First we have to detect a lag beyond which it is unlikely that auto-correlation of the disturbances is effective. Secondly, considering the form of $b(r\gamma^*h)$ in equation (12), we must judge where the oscillations have decreased to an extent such that a mean value is well determined. Inspection of the results in Table X suggests that both these requirements are met by taking $r \geqslant 6$. The maximum variation in the estimated period thereafter is less than about 10 days for series 50 and 51 and 4 days for series 52. The means of the estimated values of $T=2\pi/\gamma$ for $r=1$ to 16 and $r=6$ to 16 are given below.

	Series 50	Series 51	Series 52
$r=1$ to 16	1·198 years	1·202 years	1·201 years
$r=6$ to 16	1·179 years	1·186 years	1·193 years

On observational grounds, series 52 appear more likely to be homogeneous than the others. Series 50 contain an interval for which the published results are not yet definitive, and both series 50 and 51 cover intervals before and after 1922·7 when a major change in the method of reducing the observations was made. This view is supported by the fact that the estimates of the period made from series 52 show markedly less variation than the others. We therefore consider the best estimate of the period to be

$$1{\cdot}193 \text{ years} = 435{\cdot}8 \text{ mean solar days.}$$

Unfortunately there does not seem to be any acceptable way of estimating κ. As far as freedom from bias is concerned, the most reliable estimate would be obtained from formula (4*a*), choosing r_2 to be sufficiently large to eliminate bias due to the auto-correlation between the disturbances $\{\epsilon, \eta\}$, and $r_1 - r_2$ sufficiently large to eliminate the bias due to ignoring part of the systematic variation. On the other hand, to keep the standard error of κ^* small, r_1 and r_2 should be small, so that it becomes a question of balancing conflicting requirements. To make a theoretical estimate of the best combination of values of r_1 and r_2 is, in our opinion, impossible in view of all the unknown factors.

The results in Table X show that an oscillation persists in the estimates of κ which is such as to make it very difficult (even bearing in mind the probable form of the bias) to decide which value to adopt. For series 52, formula (4) gives small estimated values, but it is hardly justifiable to say more than that, in the absence of bias, there is some evidence that the relaxation time lies in the range 10 to 30 years. To attach a measure of probability to this statement seems to be impossible.

20. None of the results will be acceptable if the model should be proved to be completely invalid, and it seems fair to point out that Melchior (5) and Gutenberg (6) have both expressed doubts over the question of the damping of the free motion. The case for the model was, however, very well stated by Jeffreys in (2); our model is a generalization of the one used by Jeffreys and is designed to take further account of observational error. As we showed in (1), it contains Jeffreys's model as a special case, so that we have confidence in the model at least from this aspect.

Jeffreys, however, isolated what we have called series 16 for special attention because they covered an apparently disturbance-free interval. We do not wish to add to the comments we have made on this isolation (1, p. 458) so far as the numerical analysis is concerned, but it should be understood that, in establishing the validity of our model, as well as that of Jeffreys, we used the assumption that there is a fairly large number of disturbances in the interval between successive observations (apart from the limiting case of completely disturbance-free motion, when its validity can be seen by putting $f_1(t)$ and $g_1(t)$ and consequently ϵ and η all equal to zero in (1, equations (4) and (13)). The model will certainly not be valid if there are, as Jeffreys thought, quiescent periods and highly disturbed periods, except perhaps if the observations were extended over a very large number of years indeed. This is important in the light of current geophysical theories of the Earth's core (see e.g. Revelle and Munk, 7). If the compensations envisaged in such theories at the interface between the crust and core occur only at infrequent intervals—and such seems to be believed—then doubt must arise over the validity of our model.

21. We conclude:

(*a*) The harmonic analysis of the data is unlikely to do more than reveal qualitatively that inhomogeneity exists in the data because changes in the programme have occurred too frequently to permit the statistical theory to apply over the short sub-intervals involved.

(*b*) The unsmoothed data at monthly intervals (series of type-5) appear to be best, and of these the short series 52 are to be preferred.

(*c*) Analysis of type-5 series using high lags indicates that the estimates are biased. Qualitatively, the bias is consistent with the hypotheses that the data are inhomogeneous and that auto-correlation exists in the series of disturbances.

(*d*) The period of the free motion may be taken to be

$$1\cdot 193 \text{ years} = 435\cdot 8 \text{ mean solar days.}$$

(*e*) The bias is fatal for the accurate estimation of the damping factor for which, however, there is some evidence suggesting that the relaxation time is between about 10 and 30 years.

Acknowledgments.—We wish especially to thank Professor Sir Harold Jeffreys for his continued interest in this work over a number of years.

One of us (A. Y.) has been assisted by a Leverhulme Research Award which is gratefully acknowledged. This award enabled him to visit observers in Europe