Numerical Treatment of Oscillatory Delay and Mixed Functional Differential Equations Arising in Modelling

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by Md. Abdul Malique.

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Declaration

No part of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other institution of learning. However some parts of the materials contained herein have been published previously.

Conference presentations and publications

This thesis contains material which has been the subject of conference and journal papers, the details of which are given below:

1. Material from Chapter 3 and Chapter 4 were presented at the HERCMA International Conference 2007 in Athens, Greece and at the International Workshop on Analysis and Numerical Approximation of Singular Problems (IWANASP) 2008, Ericeira, Portugal.

3. The materials from Chapter 4 and Chapter 6 were presented at the Leverhulme International Conference 2009 in Chester and the paper [47] relating to this Chapter has been published.

4. Parts of the material in Chapter 6 are based on the paper [47] and were presented by the candidate at an International Seminar in Lisbon (IST), Portugal 2010.

5. Material from the last part of Chapter 6 and its application to related material were presented at the Leverhulme International Workshop 2010 in Chester and at the First International Workshop on Differential and Integral Equations with Application in Biology and Medicine 2010, Karlovassi, Samos, Greece and a paper [48] relating to the last part of Chapter 6 has been published.

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Abstract

The pervading theme of this thesis is the development of insights that contribute to the understanding of whether certain classes of functional differential equation have solutions that are all oscillatory.

The starting point for the work is the analysis of simple (linear autonomous) ordinary differential equations where existing results allow a full explanation of the phenomena. The Laplace transform features as a key tool in developing a theoretical background.

The thesis goes on to explore the corresponding theory for delay equations, advanced equations and functional differential equations of mixed type. The focus is on understanding the links between the characteristic roots of the underlying equation, and the presence or otherwise of oscillatory solutions.

The linear $\vartheta-$methods are used as a class of numerical schemes which lead to discrete problems analogous to each of the classes of functional differential equation under consideration. The thesis goes on to discuss the insights that can be obtained for discrete problems in their own right, and then considers those new insights that can be obtained about the underlying continuous problem from analysis of the oscillatory behaviour of the analogous discrete problem.

The main conclusions of the work are some semi-automated computational approaches (based upon the Principle of the Argument) which allow the prediction of oscillatory solutions to be made. Examples of the effectiveness of the approach are provided, and there is some discussion of its theoretical basis. The thesis concludes with some observations about further work and some of the limitations of existing analytical insights which restrict the reliability with which the approach developed can be applied to wider classes of problem.

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Chapter 1

Introduction

In this chapter, we introduce definitions and equations that will be used or referred to later in the thesis; we also introduce various mathematical concepts and varieties of differential equation.

1.1 Ordinary Differential Equations

1.1.1 First order ordinary differential equations

We suppose that $y(t) \in \mathbb{R}^n$ for some $n \in \mathbb{N}$ (scalar equations arise when $n = 1$); we have $t \in \mathbb{R}$ (typically, $t \in [t_0, \infty)$ for some t_0). An ordinary differential equation of first order is an equation of the form

$$y'(t) = f(t, y(t)), \quad t \geq t_0 \tag{1.1.1}$$

with a given function f ($f(t,v) \in \mathbb{R}^n$ being continuous when $t \geq t_0$, for bounded $v \in \mathbb{R}^n$). A function y with absolutely continuous components is called a solution of this equation for $t \geq t_0$ if it satisfies the differential equation (1.1.1) for almost all $t \geq t_0$. In particular, and throughout our discussion here, y is a solution if it is differentiable for all $t \geq t_0$ and satisfies (1.1.1) for all $t \geq t_0$. The derivative $y'(t_0)$ is (by definition) the right-hand derivative, since it is taken at the left-hand point t_0. A particular solution is determined only when an initial value

$$y(t_0) = y_0 \tag{1.1.2}$$

is prescribed. (Uniqueness of a solution follows from, e,g., a Lipschitz condition on f.) The equation (1.1.1) together with the initial condition (1.1.2) is known as a initial-value problem. Notice that in the case where y is a scalar function, the need for a single initial condition corresponds to the problem being $1-$dimensional; the general case is n-dimensional.

Remark 1.1.1 *We shall use the term 'scalar' to refer to either a real or a complex value. We remark that systems of complex-valued equations may arise and be represented in the form* (1.1.1) *with a vector-valued function* $y : [t_0, \infty) \to \mathbb{C}^n$. $f : [t_0, \infty) \times \mathbb{C}^n \to \mathbb{C}^n$.

1.1.2 First-order linear homogeneous and autonomous equations

We recall some well-known material. In this thesis, we shall refer several times to one of the most basic first order ordinary differential equations.

Definition 1.1.1 *The* basic equation

$$y'(t) = ay(t), \; t \geq t_0, \tag{1.1.3}$$

with initial condition $y(t_0) = y_0$ *is a homogeneous first-order linear scalar differential equation where* a *is a constant scalar parameter (we take* $a \in \mathbb{R}$ *unless indicated otherwise),* y *is the solution (a scalar-valued function of the real variable* t*) and* $y'(t)$ *is the value of its derivative.*

Remark 1.1.2 *(a) Each different value of the parameter 'a'gives a different differential equation. The differential equation (1.1.3) is a relationship between the value of a function of time* $y(t)$ *and the value of its derivative* $y'(t)$*. It is the simplest and one of the most fundamental differential equations and can be used as an initial point of reference in this work. (b) There exist a number of directions in which* (1.1.3) *can be generalised: first,* $a \in \mathbb{R}$ *may be changed so that we consider complex-valued* a *(and complex-valued* y*); secondly, we may replace* a *by a matrix* $A \in \mathbb{R}^{n \times n}$ *(and vector-valued* y*, with* $y(t) \in \mathbb{R}^n$*) so that the equation reads* $y'(t) = Ay(t)$*. See Remark 1.1.7 and Example 1.1.3. (The generalisation from* $A \in \mathbb{R}^{n \times n}$ *to the case* $A \in \mathbb{C}^{n \times n}$ *is straightforward; some of the insight for the first case relies on the second case where* A *is in Jordan canonical form.) Later, we shall consider delay- (or retarded-) differential equations which generalise (1.1.3) further.*

Lemma 1.1.1 *The solution of the initial value problem*

$$y'(t) = Ay(t), \quad (t \geq t_0) \text{ where } A = [a_{i,j}] \in \mathbb{R}^{n \times n} \tag{1.1.4}$$

$$y(t_0) = y_0 \in \mathbb{R}^n \tag{1.1.5}$$

is $y(t) = \exp[A(t - t_0)]y(t_0)$ *(*$t \geq t_0$*).*

The reader who seeks a proof of this result can obtain one most simply by verification. If $A = XJX^{-1}$ is the Jordan canonical form where $X, J \in \mathbb{C}^{n\times n}$ (with $\det X \neq 0$) then $y(t) = X \exp[J(t-t_0)]X^{-1}y(t_0)$ and the qualitative behaviour of y can be discussed in terms of the structure of $J \in \mathbb{C}^{n\times n}$ (see Remark 1.1.1). The salient features of J are its eigenvalues (these are in general complex) and their multiplicities.

Definition 1.1.2 *The characteristic function for $y'(t) = Ay(t)$, where $A = [a_{i,j}] \in \mathbb{R}^{n\times n}$ (and $n \geq 1$ is a natural number) is the characteristic polynomial of A:*

$$\chi(\lambda) := \det\{\lambda I - A\} \text{ where } I \text{ is the identity matrix.} \tag{1.1.6}$$

Its zeros $\{\lambda_\ell\} \subset \mathbb{C}$ are termed eigenvalues or characteristic values both of A and of (1.1.4).

Given any constant multiple of χ, say $\widehat{\chi}(\lambda) := \kappa\chi(\lambda)$ (with $\kappa \neq 0$), the equation $\widehat{\chi}(\lambda) = 0$ will be called an auxiliary equation and its roots are the zeros of χ. Suppose that λ_ℓ is a characteristic value; then all multiples of $\exp(\lambda_\ell t)$ satisfy $y'(t) = Ay(t)$, and there may also be solutions $p_\ell(t)\exp(\lambda_\ell t)$ where $p_\ell(t)$ is an appropriate polynomial. Any solution of this form is called a characteristic solution corresponding to λ_ℓ.

Later, we shall amend, in a natural manner, the definitions of characteristic function, characteristic value, and characteristic solution to cover other types of linear homogeneous and autonomous differential equation.

Example 1.1.1 $\det\{A - \lambda I\} = 0$ *is an auxiliary equation for* (1.1.4).

Lemma 1.1.2 *If $\chi(\lambda_\ell) = 0$ and $\overline{\lambda_\ell}$ is the complex conjugate of λ_ℓ then $\chi(\overline{\lambda_\ell}) = 0$ (with $A \in \mathbb{R}^{n\times n}$).*

As a special case of Lemma 1.1.1, one can show very simply that the general form of solution to equation (1.1.3) is $y(t) = ke^{at}$. This observation leads us to the following result on solution behaviour. The function

$$y(t) = ke^{at} \tag{1.1.7}$$

is a solution of the equation (1.1.3) where k is a scalar whose value is determined by $y(t_0)$ (that is, $k = y(t_0)\exp(-at_0)$). For the case $a = 0$, $k = y(t_0)$, the constant solution ($y(t) = y(t_0)$), is called an equilibrium solution or equilibrium point for the equation (1.1.3). For $a \in \mathbb{R}$ and $a \neq 0$, the sign of a is crucial, as the behaviour of solutions is quite different according to whether a is positive or negative. The qualitative behaviour is given in the following self-evident lemma:

Lemma 1.1.3 *Let $y(t)$ be given by* (1.1.7). *Then, (1.) if $a > 0$, $y(t) \to \infty$ as $t \to \infty$ when $k > 0$, and $y(t) \to -\infty$ as $t \to -\infty$ when $k < 0$; (2.) if $a = 0$, $y(t) = y(t_0)$, a constant; (3.) if $a < 0$, $y(t) \to 0$ as $t \to \infty$.*

Remark 1.1.3 *The above qualitative behaviour can be illustrated by sketching the graphs of solutions (see [78]). In this case the solutions of the equation (1.1.3) have no oscillatory behaviour – just exponential growth or decay. One might gather from Remark 1.1.2 that we shall need to generalise the case $a \in \mathbb{R}$ to obtain oscillatory behaviour. Suitable generalisations arise both from considering systems of ordinary differential equations and from considering differential equations with deviating arguments, such as delay-differential equations.*

For its bearing on delay-differential equations later, we include the following result which can be established by verification.

Lemma 1.1.4 *The solution of the inhomogeneous equation $y'(t) = ay(t) + g(t)$ for $t \in [t_*, \infty)$ (where $a \in \mathbb{R}$, $g \in C[t_*, \infty)$) that satisfies the initial condition $y(t_*) = y_*$ is $y(t) = \exp\{a(t - t_*)\}y_* + \int_{t_*}^{t} \exp\{a(t - s)\}g(s)ds$.*

Further remarks on ordinary differential equations

Differential equations arise naturally in many areas of science and the humanities such as biology, physics, chemistry, economics etc. Its applications are diverse. Nowadays, researchers are using differential equations to solve, or try to solve, real life problems such as the diagnosis of diabetes, the spread of gonorrhea and the detection of art forgeries etc. (see [30]).

Remark 1.1.4 *The equation (1.1.1) can be written as an integral equation by integrating both sides with respect to t. Thus for $t > t_0$,*

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))ds \tag{1.1.8}$$

which is a special case of a Volterra integral equation in the classical form

$$y(t) = g(t) + \int_{t_0}^{t} K(t, s, y(s))ds \ (t \geq t_0). \tag{1.1.9}$$

This is a Volterra integral equation of the second kind. *Also of interest are* Volterra integro-differential equations, *e.g.,*

$$y'(t) = G(t, y(t), \int_{t_0}^{t} K(t, s, y(s))ds).$$

Remark 1.1.5 *It is convenient in passing to refer to a different kind of Volterra integral equation:*

$$\int_{t_0}^{t} K(t, s, y(s))ds = g(t) \ (t \geq t_0). \tag{1.1.10}$$

This is a Volterra integral equation of the first kind. *The equation does not have a solution for arbitrary* $g \in C[t_0, \infty)$. *Indeed, the simplest example,* $\int_{t_0}^{t} y(s)ds = g(t)$ *(for* $t \geq t_0$*) has a solution only when* $g(t_0) = 0$ *and* $g'(t)$ *exists for* $t \geq t_0$ *(the solution is then* $g'(t)$*). This is sufficient to demonstrate that the example is* ill-posed in the sense of Hadamard *(see [110]). We shall encounter differential equations with deviating arguments that are ill-posed in this sense.*

1.1.3 Second order scalar ordinary differential equations

In this subsection, we meet some of the simplest differential equations with real coefficients whose solutions may oscillate, namely second-order autonomous homogeneous, linear, and scalar ordinary differential equations.

Definition 1.1.3 (Oscillatory and non-oscillatory real functions) *Let* $\mathcal{T}_0 \subseteq [t_0, \infty)$ *be a set of real numbers with no finite upper bound. A real-valued function* u *of* $t \in \mathcal{T}_0$ *is said to be oscillatory (or oscillatory about zero) if there does not exists a value* $T \in [t_0, \infty)$ *such that either*

$$u(t) > 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty) \text{ or } u(t) < 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty). \tag{1.1.11}$$

A function that is not oscillatory is called non-oscillatory: that is, there does exists a value $T \in [t_0, \infty)$ *such that either*

$$u(t) > 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty) \text{ or } u(t) < 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty). \tag{1.1.12}$$

A function is either oscillatory about a value k *or non-oscillatory about a value* k *if the function with values* $u(t) - k$ *is respectively oscillatory or non-oscillatory.*

Some authors define oscillatory functions that are vector-valued by reference to some cone, say $\mathcal{K} \subset \mathbb{R}^n$; see, for example, [89]. On setting $n = 1$ it is seen that a function that is oscillatory about zero, in our sense, is oscillatory in this sense. Our definition above suffices for a real-valued function defined for $t \in \{t_0 < t_1, < t_2, < t_3, \cdots\}$ as well as for a real-valued function defined on $[t_0, \infty)$. For future use we add the following definition.

Definition 1.1.4 *Any infinite scalar sequence* $\{u_0, u_1, u_2, \cdots\}$, *is called oscillatory if, for arbitrary* $\mathcal{T}_0 = \{t_0 < t_1, < t_2, \cdots\}$, *the function* u *defined by* $u(t_\ell) = u_\ell$ *for* $\ell = 0, 1, 2, \cdots$ *is oscillatory*

Remark 1.1.6 *Concerning notation, there are various traditions in differing branches of mathematics and we shall clarify our various conventions. In some areas, a vector-valued sequence* $\{u_0, u_1, u_2, \cdots\}$ *(with* $u_m \in \mathbb{R}^n$ *for* $m \in \{0, 1, 2, \cdots\}$*) is denoted by* u. *In other areas the convention is to index using the natural numbers (as in* $v := \{v_1, v_2, v_3, \cdots\}$*) or to use integer arguments (as in* $\{u(0), u(1), u(2), \cdots\}$ *and in* $\{v(1), v(2), v(3), \cdots\}$*). Regarding* u, *this latter convention is compatible with that employed in Definition 1.1.4 when we set* $t_m = m$. *From the point of view of our analysis (though not necessarily that of interpretation), our choice of* $\mathcal{T}_0 = \{t_0 < t_1, < t_2, \cdots\}$ *will be in general immaterial. Thus, the symbol* u *will denote either the sequence* $\{u_0, u_1, u_2, \cdots\}$ *or the function defined on* $\mathcal{T}_0$ *with indexing* $\{0, 1, 2, \cdots\}$ *(or the equivalents with indexing* $\{1, 2, 3, \cdots\}$*), as we see fit.*

Example 1.1.2 *By definition, the* null function $u \in [t_0, \infty) \to \mathbb{R}$ *with* $u(t) \equiv 0$ *is called oscillatory about zero. The function* $u \in [t_0, \infty) \to \mathbb{R}$ *with* $u(t) = t^k \sin(\pi r t)$ *(where* k *and* $r \neq 0$ *are integer) is oscillatory about zero, as is the function with* $u(t) = \exp\{\nu t\} \sin(\pi s t)$ *for* $\nu \in \mathbb{R}$, $s \in \mathbb{N}$.

A real differential equation of second order for $y(t)$ is an equation of the form

$$y''(t) = f(t, y(t), y'(t)) \text{ for all } t \geq t_0, \tag{1.1.13}$$

with a given real continuous function f (where $f(t, u, v)$ is continuous for $t \geq t_0$, $u, v \in \mathbb{R}$); a function y with a second derivative is a solution of this equation if it satisfies the differential equation (1.1.13) for all $t \geq t_0$. A particular solution is determined only when initial values

$$y(t_0) = y_0 \in \mathbb{R}, \quad y'(t_0) = y_0' \in \mathbb{R}, \tag{1.1.14}$$

are given. The equation (1.1.13) together with the initial conditions (1.1.14) is known as an initial value problem for a second-order ordinary differential equation.

Remark 1.1.7 *An* n*-th order ordinary differential equation can be rewritten as a system of first-order differential equations (equivalently, as a first-order differential equation for a vector-valued solution with* n *components). In a similar vein, a first-order differential equation for a complex-valued function may be written as a coupled pair of ordinary differential equations for a real vector-valued solution.*

Example 1.1.3 *(a) If we write* $u(t) = [u_1(t), u_2(t)]^T = [y(t), y'(t)]^T$ *then we can recast* (1.1.13) *(wherein all the functions are assumed to be real-valued) in the form*

$$u'(t) = \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ f(t, u_1(t), u_2(t)) \end{bmatrix} \quad (t \geq t_0), \tag{1.1.15}$$

which is clearly (given a change of notation) of the form (1.1.1) *with* $n = 2$.

(b) Consider $w'(t) = g(t, w(t))$ *(for* $t \geq t_0$*) where* w *and* $g(t, w(t)$ *assume complex values. Write* $w(t) = w_1(t) + iw_2(t)$*, and* $g(t, w(t)) = g_1(t, w_1(t), w_2(t)) + ig_2(t, w_1(t), w_2(t))$ *where* w_1*,* w_2*,* g_1*, and* g_2 *are real-valued functions; then*

$$\begin{bmatrix} w_1'(t) \\ w_2'(t) \end{bmatrix} = \begin{bmatrix} g_1(t, w_1(t), w_2(t)) \\ g_2(t, w_1(t), w_2(t)) \end{bmatrix} \quad (t \geq t_0) \tag{1.1.16}$$

which is again (given a change of notation) of the form (1.1.1) *with* $n = 2$.

Definition 1.1.5 *For* $a \neq 0$ *where* a*,* b*,* c *are real constant parameters, the equation*

$$ay''(t) + by'(t) + cy(t) = 0 \tag{1.1.17}$$

with initial conditions $y(t_0) = y_0$*,* $y'(t_0) = y_0'$*,* $t \geq t_0$ *is called a homogeneous second-order linear scalar differential equation. Here,* $y(t)$ *is an unknown real function of a real variable* t *and* $y'(t)$ *and* $y''(t)$ *are its derivatives. Each set of parameter values (*a*,* b*,* c*) yields a different differential equation. Note that the specific solution depends on the specification of two initial conditions and the dynamical system has dimension two.*

We can rewrite (1.1.17) as $y''(t) = -(b/a)y'(t) - (c/a)y(t))$. Following on from Example 1.1.3 we obtain, with the notation ($u(t) = [y(t), y'(t)]^T$) of that example,

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = A \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \text{wherein } A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}. \tag{1.1.18}$$

The remarks for (1.1.4) now apply. Indeed, the equation equation

$$a\lambda^2 + b\lambda + c = 0. \tag{1.1.19}$$

is an auxiliary equation for (1.1.17) and its solutions λ_1, λ_2 are the associated characteristic values or eigenvalues – the eigenvalues of the matrix A in (1.1.18).

Lemma 1.1.5 *Suppose that $a \neq 0$, b, and c are real numbers, and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the roots of* (1.1.19). *(a) If λ_1, λ_2 are distinct then for any k_1, $k_2 \in \mathbb{C}$ the (real- or complex-valued) functions*

$$y(t) = k_1 \exp(\lambda_1 t) + k_2 \exp(\lambda_2 t) \tag{1.1.20}$$

satisfy (1.1.17), Further, (b) if $\lambda_1 = \lambda_2$ then for any $k_{1,2}$ the functions

$$y(t) = k_1 \exp(\lambda_1 t) + k_2 t \exp(\lambda_1 t) \tag{1.1.21}$$

satisfy (1.1.17). (c) If $\lambda_1, \lambda_2 \in \mathbb{R}$ then these functions are real-valued solutions when $k_1, k_2 \in \mathbb{R}$; if $\lambda_1, \lambda_2 \notin \mathbb{R}$ then these functions are real-valued solutions for suitable k_1, $k_2 \in \mathbb{C}$. (d) In all cases, k_1 and k_2 are uniquely determined by $y(t_0)$ and $y'(t_0)$ and the functions (1.1.20)–(1.1.21) *are real-valued solutions for $y(t_0), y'(t_0) \in \mathbb{R}$. Indeed, with $\lambda_1 = \alpha + \mathrm{i}\beta$, $\lambda_2 = \alpha - \mathrm{i}\beta$ ($\alpha, \beta \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$),* (1.1.20) *can be rewritten in the real form*

$$y(t) = \{c_1 \cos(\beta t) + c_2 \sin(\beta t)\} \exp(\alpha t), \quad c_{1,2} \in \mathbb{R}. \tag{1.1.22}$$

(e) All solutions of (1.1.17) *have one of the forms in* (1.1.20) – (1.1.22).

Lemma 1.1.5 can be established by verification, with (e) a consequence of the fact that $k_{1,2}$ or $c_{1,2}$ are determined by $y(t_0)$ and $y'(t_0)$. It is also possible to employ Laplace transform theory. If the values of $\lambda_{1,2}$ are (both) real then the behaviour of solutions is quite different depending whether each λ is positive or negative. The following trivial lemma echoes Lemma 1.1.3 and gives the qualitative behaviour.

Lemma 1.1.6 *Suppose $\lambda \in \{\lambda_1, \lambda_2\}$ and $y(t) = k \exp(\lambda t) \in \mathbb{R}$. Then, (1) if $\lambda > 0$, $|y(t)| \to \infty$ as $t \to \infty$. (2) If $\lambda = 0$ and $y(t) = k \exp(\lambda t)$ then, clearly, $y(t)$ is constant*[1] *for all t. (3) If $\lambda < 0$, $y(t) \to 0$ as $t \to \infty$.*

Obviously none of the solutions of (1.1.17) oscillate if λ_1 and λ_2 are real and all of the solutions of (1.1.17) oscillate if λ_1 and λ_2 have non-zero imaginary part. We should note (since all linear homogeneous equations have the zero function as a solution) that, according to our definition (Definition 1.1.3), the zero solution is oscillatory. Now we are in a position to state for the first time what we will mean by an *oscillatory equation.*

Definition 1.1.6 *A scalar differential equation is said to be an oscillatory equation if and only if all the solutions of the equation are oscillatory functions in the sense of Definition 1.1.3.*

[1] A constant solution $y(t) = 0$ is called an equilibrium solution or equilibrium point for the equation (1.1.17).

From the preceding discussion we obtain the next lemma:

Lemma 1.1.7 *The equation (1.1.17) is an oscillatory equation if and only if the characteristic values λ_1 and λ_2 are complex with non-zero imaginary part.*

Remark 1.1.8 *Second-order differential equations arise quite often in applications in science and engineering. The equation (1.1.17) is sometimes known as an equation of a damped harmonic oscillator (see [78]).*

Some of the most important examples of second-order differential equations are derived from Newton's second law of motion

$$my''(t) = F\big(t, y(t), y'(t)\big) \quad (t \geq t_0) \tag{1.1.23}$$

which describes the motion of a particle of mass m moving under the influence of a force F. In this equation $y(t)$ is its position at time t, $y'(t)$ is its velocity, and F is the total force acting on the particle. F depends on the position, velocity and time (see [30], [78]). If we consider the non-homogeneous form of the equation (1.1.17)

$$ay''(t) + by'(t) + cy(t) = g(t) \tag{1.1.24}$$

this is sometimes known as the equation for a harmonic oscillator where $g(t)$ is an external force at time t.

1.2 First-order delay-differential equations

Here, we consider some first-order linear autonomous homogeneous scalar delay-differential equations. Some of the main equations of interest in this thesis will be differential equations *deviating arguments*, in particular those with *delayed arguments – 'delay-differential equations'* (DDEs). The study of DDEs has been undertaken since the eighteenth century to extend technological insight. Since the last century it has been developing rapidly. DDEs arise in application such as control theory, biology, etc. The study of DDEs is therefore often quite applications oriented.

We suppose that $y(t) \in \mathbb{R}^n$ for some natural number n ($n = 1$ is the scalar case). A delay-differential equation of first order for $y(t)$ is an equation of the form

$$y'(t) = f\big(t, y(t), y(t - \tau)\big) \tag{1.2.1}$$

with $\tau > 0$, and a function f that is a real-valued continuous function having values $f(t, u, v)$ for $t \geq t_0$, and (say) bounded $u, v \in \mathbb{R}^n$.

In analysis, an absolutely-continuous function $y(t)$ is called a solution of this equation for all $t \geq t_0$ if it satisfies the differential equation (1.2.1) for almost all $t \geq t_0$. *In the cases we consider, solutions are differentiable and satisfy* (1.2.1) *for all* $t \geq t_0$. For definiteness the derivative at t_0 is the right-hand derivative. A particular solution is determined only when the initial function $\phi \in C[t_0 - \tau, t_0]$ is given in the condition

$$y(t) = \phi(t) \text{ for } t_0 - \tau \leq t \leq t_0. \tag{1.2.2}$$

The need to specify an initial function to determine a unique solution, rather than a finite number of initial values needed in (1.1.4), indicates that even a scalar delay-differential equation constitutes an infinite-dimensional dynamical system.

Definition 1.2.1 *The characteristic function of the system of equations*

$$y'(t) = Ay(t) + By(t - \tau) \ (t \geq t_0) \ \textit{where } \tau > 0, A, B \in \mathbb{R}^{n \times n}, \ y(t) \in \mathbb{R}^n, \tag{1.2.3}$$

is defined as the function

$$\chi(\lambda) = \det[\lambda I - A - B\exp(-\tau\lambda)], \tag{1.2.4}$$

This function is a quasi-polynomial. *The zeros* λ_ℓ *of* (1.2.4) *are called the characteristic values.*

We may find it convenient to refer to an auxiliary equation:

Definition 1.2.2 *Suppose that* ξ *is an entire function that vanishes nowhere in* $\mathbb{C}$ *and denote by* $\widehat{\chi}$ *the corresponding function*

$$\widehat{\chi}(\lambda) := \xi(\lambda)\chi(\lambda) \ \textit{for all } \lambda \in \mathbb{C}. \tag{1.2.5}$$

Then, we refer to (1.2.5) *as an auxiliary characteristic function or* auxiliary function *(for the corresponding differential equation) and the set of zeros of* $\widehat{\chi}$ *is the set of characteristic values.*

Definition 1.2.2 is compatible with the terminology introduced for recurrence relations (see Definition 1.1.2).

Definition 1.2.3 *The first order linear autonomous scalar delay-differential equation*

$$y'(t) = \mu y(t - \tau) + \nu y(t)(t \geq t_0) \tag{1.2.6}$$

where $y(t) \in \mathbb{R}$, $\tau > 0$ *and* $\mu, \nu \in \mathbb{R}$ *are constants will be called the basic test equation for DDEs. In this case* (1.2.4) *reduces to*

$$\chi(\lambda) := \lambda - \mu \exp(-\tau) - \nu \tag{1.2.7}$$

which is called the characteristic function for (1.2.6). *The zeros* λ_ℓ *of* (1.2.7) *are called the characteristic values of* (1.2.6).

A *particular solution* of (1.2.6) is defined by requiring $y(t) = \phi(t)$ for $t_0 - \tau \leq t \leq t_0$, given an *initial function* $\phi \in C[t_0 - \tau, t_0]$.

Remark 1.2.1 *For the basic equation* (1.2.6), *a change of variables allows us to normalise by taking* $\tau = 1$ *– to do so we replace* μ *by* $\mu_\natural = \tau\mu$ *and* ν *by* $\nu_\natural = \tau\nu$. *We would obtain a generalisation of both* (1.1.4) *and of* (1.2.6) *if we were to consider* $y'(t) = Ay(t) + By(t - \tau)$ $(t \geq t_0)$ *with* $A, B \in \mathbb{R}^{n\times n}$, $y(t) \in \mathbb{R}^n$ *as in* (1.2.3).

1.2.1 A pure delay equation

The equation

$$y'(t) = \mu y(t - \tau) \tag{1.2.8}$$

where $y(t) \in \mathbb{R}$, $t > 1$ and μ is a constant is an example of a first order linear autonomous homogeneous scalar delay-differential equation. Associated with (1.2.8) we require an initial function ϕ and set $y(t) = \phi(t)$ for $t_0 - \tau \leq t \leq t_0$. The equation (1.2.8) is an example of a *'pure delay' equation* (where $y'(t)$ depends only on past values $y(t - \tau)$, $\tau > 0$) and is one of the simplest delay-differential equations.

The next lemma and its corollary relate to the definition of an exponentially bounded function recalled again later in Definition 2.1.2.

Lemma 1.2.1 *All solutions of* (1.2.8) *are continuous, and are exponentially bounded in the sense that there exist* $T \geq t_0$, k, *and* $\gamma \in \mathbb{R}$ *such that*

$$|y(t)| \leq k \quad \exp(\gamma t) \textit{ for } t \geq T. \tag{1.2.9}$$

Proof: The result follows from the *method of steps*: On every interval $[t_0 + m\tau, t_0 + (m+1)\tau]$ the solution satisfies $y(t) = y(t_0 + m\tau) + \mu \int_{t_0+m\tau}^{t} y(s-1)ds$ (which is continuous). It follows that $|y(t)| \leq \exp\{\mu(t - t_0)\} \sup_{[t_0-\tau,t_0]} |\phi(t)|$ and the result follows.

Corollary 1.2.2 *All solutions of* (1.2.6) *are continuous and are exponentially bounded.*

Remark 1.2.2 *It is easy to show that a change of variables can be used to convert (1.2.6) into the form* $v'(t) = \widehat{\mu} v(t-\tau)$, *compare (1.2.8), or (as a particular case, cf. Remark 1.2.1) into*

$$u'(t) = \mu_\natural \, u(t-1) \tag{1.2.10}$$

obtained with $\mu_\natural := \widehat{\mu}\tau$ *on changing the independent variable. To see this, write* (1.2.6) *in the form* $y'(t) - \nu y(t) = \mu y(t-\tau)$ *to obtain* $(d/dt)\exp(-\nu t)y(t) = \mu \exp(-\nu t) y(t-\tau)$ *or* $v'(t) = \widehat{\mu}\, v(t-\tau)$ *with* $v(t) = \exp(-\nu t)\, y(t)$ *and* $\widehat{\mu} = \exp(\nu\tau)\, \mu$.

Equation (1.2.6) is commonly analysed, as a test of various theories or methods. The preceding remarks show that insight can also be obtained by taking $\tau = 1$ in (1.2.6), as in Remark 1.2.1, or by examining (1.2.8) or (1.2.10).

Definition 1.2.4 *The quasi-polynomial*

$$\chi(\lambda) := \lambda - \mu \exp(-\tau\lambda) \ (\textit{defined for } \lambda \in \mathbb{C}) \tag{1.2.11}$$

is the characteristic function for the DDE (1.2.8). *The zeros of* χ *are called the characteristic values (or eigenvalues) of* (1.2.8).

Our next Lemma addresses the behaviour of exponential solutions of (1.2.10) and conditions for (1.2.10) to be an oscillatory equation. With $\mu_\natural = \tau\mu$ we obtain corresponding results for (1.2.8).

Lemma 1.2.3 *(a) Let us suppose the function*

$$y(t) = k \exp \lambda t, \ \ (k \neq 0) \tag{1.2.12}$$

is a solution of the equation (1.2.10). *Then,*

$$\chi(\lambda) = 0 \ \textit{where} \ \chi(\lambda) := \lambda \exp(\lambda\tau) - \mu_\natural. \tag{1.2.13}$$

(b) The behaviour of solutions is dependent on $\mu_\natural$*:*

(1) If $\mu_\natural > 0$, $y(t) \to \infty$ *as* $t \to \infty$.
(2) If $\mu_\natural = 0$, $y(t) = y(t_0)$ *for all* $t \geq t_0$.
(3) If $\mu_\natural < 0$, $y(t) \to 0$ *as* $t \to \infty$.
(4) The equation $\chi(\lambda) = 0$ *has a real root for* $\mu_\natural \in (-1/e, \infty)$ *and has no real roots for* $\mu_\natural \in (-\infty, -1/e)$ *(see [47]).*
(5) The equation (1.2.10) *is non-oscillatory for* $\mu_\natural \geq -1/e$ *and is oscillatory otherwise (see [47]), when* $\mu_\natural \in \mathbb{R}$.

Theorem 1.2.4 (A necessary and sufficient condition) *(See [70, p.37, Proposition 1.3.1] [73].) Equation (1.2.10) is oscillatory (equivalently, every solution of the equation* (1.2.10) *is oscillatory) if and only if the equation (1.2.13) has no real roots.*

The essential result required to provide a proof of is the fact that the eigenfunctions generated by the characteristic values span the required solution space when $\phi \in C[t_0 - \tau, t_0]$. Hale and Verduyn Lunel [83, Chapter 7, p.220], for example, has a section on this.

Remark 1.2.3 *Our approach can be refined through the use of the Laplace transform, which is useful to get a explicit expression as a contour integral in the form of an inverse Laplace transform for a solution $y(t)$.*

1.3 First order advanced differential equations

Suppose, as previously, that the function f with values $f(t, u, v) \in \mathbb{R}^n$ is continuous for $t \geq t_0$, and bounded $u, v \in \mathbb{R}^n$ for some natural number n (scalar equations arise for $n = 1$).

Definition 1.3.1 *We suppose that $\tau > 0$. An equation of the form*

$$y'(t) = f\big(t, y(t), y(t + \tau)\big) \tag{1.3.1}$$

with $y(t) \in \mathbb{R}^n$ is an advanced differential equation of first order for y. A continuous function y is called a solution of equation (1.3.1) *on* $[t_0, T)$ *if it has a derivative y' and $y(t)$ satisfies (1.3.1) for $t \in [t_0, T)$, say.*

A particular solution of (1.3.1) is determined when. e.g., a *suitable* function ϕ is given in the condition

$$y(t) = \phi(t), \text{ for } t_0 \leq t \leq t_0 + \tau. \tag{1.3.2}$$

Definition 1.3.2 *The first order linear autonomous homogeneous scalar advanced differential equation (or differential equation with a deviating argument of advanced type)*

$$y'(t) = \mu y(t + \tau), \quad t \geq t_0, \tag{1.3.3}$$

where $y(t) \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a constant is called a purely-advanced equation.

Plausible constraints that may (we now explore further) define a unique solution to (1.3.3) are of the form (1.3.2).

Lemma 1.3.1 *Suppose that* $y(t) = \phi(t)$, *for* $t_0 \leq t \leq t_0 + \tau$. *Then a continuous solution* y *of* (1.3.3) *exists on* $[t_0 + m\tau, t_0 + (m+1)\tau]$ *only if the* m*-th derivative* $\phi^m(t)$ *is continuous on* $[t_0, t_0+\tau]$ *and* $\mu\phi^m(t_0+\tau) = \phi^{m+1}(t_0)$.

Proof: Write the given equation as

$$y(t) = y'(t-\tau)/\mu, \quad t \in [t_0 + m\tau t_0 + (m+1)\tau], \; m \in \{1, 2, 3, \cdots\} \quad (1.3.4)$$

and the result follows.

Remark 1.3.1 *(a) The preceding result implies that* (1.3.3) *is ill-posed: if one makes non-differentiable perturbations in* ϕ *then a problem with a solution is transformed into a problem without a solution. (b) We also note that if* $y'(t) = \mu y(t+\tau)$, *for* all $t \in \mathbb{R}$, *then* y *must be infinitely-differentiable. (c) If* (1.3.3) *holds and we write* $x(s) = y(T-s)$, *we obtain the problem*

$$x'(s) = -\mu x(s-\tau), s \geq 0, \text{ with } x(s) = \phi(T-s) \text{ for } s \in [t_0-\tau, t_0]; \quad (1.3.5)$$

this problem, cf. (1.2.8), *is a pure-delay problem. When we think of time as running backwards from the interval on which the function* ϕ *is defined, the advanced equation can be considered as a reformulated pure-delay equation problem. This is however, somewhat contrived.*

1.4 First-order differential equations with delayed and advanced arguments

Next we move on to consider problems where there is a mixture of terms having delayed and advanced arguments – so-called mixed-type equations. The general class of problems considered here are termed mixed type functional differential equations (MTFDEs). We suppose that we seek $y(t) \in \mathbb{R}^n$ for some natural number n (with $n = 1$ the scalar case). A basic mixed-type differential equation of first order for y is introduced in the following definition.

Definition 1.4.1 (Mixed-type differential equations) *Suppose that the function* f, *with values* $f(t, u, v, w) \in \mathbb{R}^n$ *(continuous for* $t \geq t_0$, *and bounded* $u, v, w \in \mathbb{R}^n$*) is given. An equation of the form*

$$y'(t) = f\big(t, y(t), y(t-\tau), y(t+\tau)\big) \quad (t \geq t_0) \qquad (1.4.1)$$

where $\tau > 0$ *and a differentiable function* $y(t)$ *is called a solution of this equation if it satisfies the differential equation (1.4.1) for* $t \geq t_0$.

Lemma 1.4.1 *The value of* τ *in equation* (1.4.1) *can be normalised to be* 1. *Thus the equation under consideration can be taken to be* $y'(t) = f(t, y(t), y(t-1), y(t+1))$ *(for* $t \geq t_0$*); it is also possible to normalise so that* $t_0 = 0$.

Proof: Suppose y satisfies (1.4.1), suppose $t_0^\natural \in \mathbb{R}$ and $s - t_0^\natural = \tau \times (t - t_0) + t_0$ and let $y_\natural(s) = y(t)$. Then $y_\natural'(s) = \frac{1}{\tau} y'(t)$ and hence

$$y_\natural'(t) = f_\natural(t, y_\natural(t), y_\natural(t-1), y_\natural(t+1)) \text{ (for } t \geq t_0^\natural) \tag{1.4.2}$$

where $f_\natural(t, u, v, w) = \tau f(t, u, v, w)$. Clearly, we may pick $t_0^\natural = 0$.

Remark 1.4.1 (Particular solutions) *A particular solution of* (1.4.1) *is determined only when suitable initial and/or boundary functions are prescribed. Plausible (or at least possible) conditions are*

$$y(t) = \phi_0(t), \text{ for } t_0 - \tau \leq t \leq t_0, \tag{1.4.3}$$

$$y(t) = \phi_1(t), \text{for } t_0 \leq t \leq t_0 + \tau. \tag{1.4.4}$$

However, the existence of a solution that satisfies the given conditions is not guaranteed. We saw a similar result in our discussion of (1.3.3). *Here, we consider* (1.4.1). *Given* (1.4.1) *is satisfied for* $t \geq t_0$ *and given* (1.4.4), *we must require* $\phi_1(t)$ *to be differentiable for* $t_0 \leq t \leq t_0 + \tau$,

$$\phi_0(t_0) = \phi_1(t_0) \tag{1.4.5}$$

and that

$$\frac{d}{dt}\phi_1(t_0) = f\big(t_0, \phi_0(t_0), \phi_0(t_0 - \tau), \phi_1(t_0 + \tau)\big)(t). \tag{1.4.6}$$

It follows that arbitrary continuous functions (ϕ_0 *and* ϕ_1) *do not necessarily determine a solution,*

The general class of problems considered here involve differential equations with both delayed and advanced terms and (1.4.1) is a mixed type functional differential equation (MTFDE). We next define a basic equation of mixed-type which we will encounter several times in this thesis.

Remark 1.4.2 *Suppose that* $\tau > 0$. *A change of variable transforms the equation*

$$u'(t) = \alpha y(t) + \beta y(t - \tau) + \gamma u(t + \tau) \tag{1.4.7}$$

into the form

$$y'(t) = ay(t) + by(t-1) + cy(t+1), \tag{1.4.8}$$

Definition 1.4.2 *The equation* (1.4.8) *where* $y(t) \in \mathbb{R}$ *and* a, b, c *are constant is called the basic equation of mixed-type. It is a first order linear autonomous homogeneous scalar mixed differential equation with delay and advanced terms or mixed type functional differential equation (MTFDE).*

We shall consider the question of solutions of (1.4.8) subject to appropriate conditions, such as $y(t) = \phi_1(t)$ for $t_0 + 1 \geq t \geq t_0$ and $y(t) = \phi_0(t)$ for $t_0 + 1 \leq t \leq t_0$ in Chapter 4.

Remark 1.4.3 *Both linear and non-linear MTFDEs arise naturally in problems of travelling waves in discrete spacial media such as lattices (see [140]).*

1.5 First order linear autonomous homogeneous integro-differential equationss

All the equations we have considered in the above sections have described the behaviour of $y'(t)$ in terms of the values of y, y' etc. at points in time that are a fixed distance (e.g., τ) from t. It is also possible, through the use of integro-differential equations, to define equations that take into account historical values and future values in a distributed way. Indeed, through the use of distributions and Stieltjes-type measures, one can express DDEs and MTDEs in the form of an integro-differential equation of a type that can be regarded as generalisations of the ones considered previously.

This section introduces ordinary, delay and mixed integro-differential equations IDEs.

1.5.1 First order ordinary integro-differential equations

To avoid repetition below, we state here that we assume, until otherwise stated, that $f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \times\mathbb{R}^n$ denotes a given real function (with continuous $f(t, u, v)$ for $t \geq t_0$ and bounded u, v).

We suppose that $y(t) \in \mathbb{R}^n$ ($n = 1$ is the scalar case). An equation of the form

$$y'(t) = f\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\Big), \quad t \geq t_0, \tag{1.5.1}$$

(with $k(t, s)$ continuous for $t \geq t_0$, $s \in [t_0, t]$) is an *integro-differential equation of the first order* for y (defined for $t \geq t_0$). A differentiable function y is called a solution of this equation if it satisfies (1.5.1) for all $t \geq t_0$. A particular solution is determined when we are given an initial value

$$y(t_0) = y_0. \tag{1.5.2}$$

Example 1.5.1 *The equation*

$$y'(t) = \int_{-1}^{0} y(t+s)ds \ (t \geq t_0), \tag{1.5.3}$$

where $y(t) \in \mathbb{R}^n$, with initial condition $y(t) = \phi(t)$ for $t \in [t_0 - 1, t_0]$ is a first-order linear autonomous homogeneous scalar integro-differential equation. We may clearly rewrite (1.5.3) *in the form*

$$y'(t) = \int_{t-1}^{t} y(\sigma)d\sigma \ (t \geq t_0). \tag{1.5.4}$$

In principle, we can now proceed to find $y'(t)$ on $[t_0, t_0 + 1]$ and thereby find

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)ds \quad (t \in [t_0, t_0 + 1]). \tag{1.5.5}$$

before proceeding, by a method of steps, to compute the derivative and then the solution on successive intervals $[t_0 + m, t_0 + (m+1)]$.

Note that, for suitable ϕ, a solution of (1.5.3) *will satisfy, for $k \in \{1, 2, 3, \cdots\}$, $y^{k+1}(t) = y^{k-1}(t) - y^{k-1}(t-1)$ $(t \geq t_0)$ (obtained by differentiating* (1.5.4)*).*

Remark 1.5.1 *If we replace the Riemann integral in the equation (1.5.1) by a Riemann-Stieltjes integral we obtain the form*

$$y'(t) = f\Big(t, y(t), \int_{t_0}^{t} [d\eta(s)]k(t,s)y(s)\Big), \text{ for } t \geq t_0,$$

where $y(t) \in \mathbb{R}^n$, $t \geq t_0$. If $n = 1$, we can write this as

$$y'(t) = f\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)d\alpha(s)\Big), \text{ for } t \geq t_0. \tag{1.5.6}$$

In (1.5.6)*, we suppose α is function of bounded variation on $[t_0, \infty)$. (A function is of bounded variation if and only if it is the difference between two monotone functions.)*

1.5.2 First order linear autonomous homogeneous integro-differential equations with delay

Suppose that $y(t) \in \mathbb{R}^n$. in

$$y'(t) = f\Big(t, y(t), \int_{t-\tau}^{t} k(t,s)y(s)ds\Big). \tag{1.5.7}$$

This is a delay integro-differential equation of first order for $y(t)$. Here the real function f is given (see Remark 1.5.1) along with a suitable kernel k (with

$$k(t,s) \text{ continuous for } t \geq t_0,\ s \in [t-\tau, t]. \tag{1.5.8}$$

A differentiable function y is called a solution if it satisfies (1.5.7) for all $t \geq t_0$. A particular solution is determined if an initial function ϕ is given with

$$y(t) = \phi(t), \text{ for } t_0 - \tau \leq t \leq t_0. \tag{1.5.9}$$

The equation can have a uniform contribution from history as in

$$y'(t) = \int_{-1}^{0} y(t+s)ds \quad (t \geq t_0), \tag{1.5.10}$$

where $y(t) \in \mathbb{R}^n$, with initial function $y(t) = \phi(t)$ for $t_0 - s \leq t \leq t_0$ is called a first order linear autonomous homogeneous scalar integro-differential equation with delay term.

Remark 1.5.2 *If we replace the Riemann integral in equation (1.5.7) by a Riemann-Stieltjes integral we obtain (with $n = 1$) the form*

$$y'(t) = f\Big(t, y(t), \int_{t-\tau}^{t} d\eta(s)k(t,s)y(s)\Big). \tag{1.5.11}$$

1.5.3 A canonical delay integro-differential equation

An integro-differential equation that reduces in special cases to equations with retarded argument considered earlier is

$$y'(t) = \int_{-1}^{0} y(t-\tau(s))d\alpha(s), \quad (\text{here, } n = 1). \tag{1.5.12}$$

We suppose the delay term $\tau(s)$ is a positive real continuous function on $[-1,0]$ and $\alpha(s)$ is a monotonically increasing real-valued function of bounded variation on $[-1,0]$. Equations of this form have been studied in the literature (see [120] and its references).

1.5.4 Beyond constant-coefficient equations

In the classical literature the characteristic function is defined for homogeneous equations that have constant coefficients. More recently, researchers have defined *generalised characteristic functions.* The following is an example, and it is motivated in much the same manner as the classical characteristic function. We remark that in other work, a generalised characteristic function is motivated through a *comparison equation.*

The following assumption is made in the current discussion concerning

$$y'(t) = \int_{-1}^{0} y(t-\tau(s))d\alpha(s), \quad (\text{ in the case } n=1). \tag{1.5.13}$$

encountered in (1.5.12).
For similar results for a more general equation see [120].

Assumption 1.5.1 *For all $s \in [-1,0]$, $\alpha'(s)$ exists and $\alpha'(s) \geq 0$.*

Definition 1.5.1 *The generalised characteristic equation of (1.5.13) is the function $\chi_{\sharp}$ with*

$$\chi_{\sharp}(\lambda) := \lambda - \int_{-1}^{0} \exp(-\lambda\tau(s))d\alpha(s) \text{ for } \lambda \in \mathbb{C}. \tag{1.5.14}$$

Here, $n=1$. A zero $\lambda_\star$ of $\chi_{\sharp}$ is called a characteristic value of (1.5.13).

Lemma 1.5.1 *Given Assumption 1.5.1,*

1. *If $\chi_{\sharp}(\lambda_\star) = 0$ then $\exp(\lambda_\star t)$ satisfies* (1.5.13)*;*

2. *Every solution of the equation (1.5.12) oscillates if and only if $\chi_{\sharp}(\lambda) > 0$. for all $\lambda \in \mathbb{R}$;*

3. *Some solutions of the equation (1.5.12) do not oscillate if*

$$\chi_{\sharp}(\lambda_\star) \leq 0 \text{ for some } \lambda \in \mathbb{R}. \tag{1.5.15}$$

For a proof, see [120].

1.5.5 First order integro-differential equations with advanced term

We can give the corresponding definitions for advanced equations. Suppose that $y(t) \in \mathbb{R}^n$ for some natural number n. An equation of the form

$$y'(t) = f\Big(t, y(t), \int_{t}^{t+\tau} k(t,s)y(s)ds\Big) \ (t \geq t_0), \tag{1.5.16}$$

is an advanced integro-differential equation of first order for y. Here we are given f, with $f(t,u,v)$ continuous for $t \geq t_0$ and bounded u,v, and a suitable kernel k (with $k(t,s)$ continuous when $t \geq t_0$, $s \in [t+\tau, t]$). A differentiable function y is a solution of this equation when it satisfies the differential equation (1.5.16) for all $t \leq t_0$. We may conjecture that a particular solution is defined when the condition $y(t) = \phi(t)$, for $t_1 + \tau \geq t \geq t_0$ is given.

Example 1.5.2 *The equation*

$$y'(t) = \int_0^1 y(t+s)ds \ (t \geq t_0), \tag{1.5.17}$$

where $y(t) \in \mathbb{R}^n$ *for* $t \geq t_0$*, can be rewritten*

$$y'(t) = \int_t^{t+1} y(\sigma)d\sigma. \tag{1.5.18}$$

This is a first order linear autonomous homogeneous scalar integro-differential equation with advanced term and, on assuming sufficient differentiability,

$$y(t+1) = y''(t) + y(t) \ (t \geq t_0). \tag{1.5.19}$$

Assume we are given a function ϕ *and seek* y *satisfying* (1.5.19) *and also satisfying the condition*

$$y(t) = \phi(t) \ for\ t \in [t_0, t_0+1]. \tag{1.5.20}$$

An application of a step-by-step method (a 'method of steps') based on (1.5.19) *and* (1.5.20) *gives us, in sequence, the following results:*

$$\begin{array}{lll}
for & t \in [t_0, t_0+1], & y(t) = \phi(t); \\
for & t \in [t_0+1, t_0+2], & y(t) = y''(t-1) + y(t-1); \\
 & & \quad = \phi''(t-1) + \phi(t-1); \\
for & t \in [t_0+2, t_0+3], & y(t) = y''(t-1) + y(t-1) \\
 & & \quad = \phi^{iv}(t-2) + 2\phi''(t-2) + \phi(t-2); \\
for & t \in [t_0+3, t_0+4], & y(t) = \{\phi^{vi}(t-3) + 3\phi^{iv}(t-3) + \\
 & & \quad +3\phi''(t-3) + \phi(t-3)\},
\end{array}$$

etc. The general expression requires that ϕ *be arbitrarily differentiable, and the ill-posedness of the problem is clearly indicated.*

Remark 1.5.3 *If, as before, we replace the Riemann integral by a Riemann-Stieltjes integral in the equation (1.5.16) we obtain where* $y(t) \in \mathbb{R}^n$*, the form*

$$y'(t) = f\Big(t, y(t), \int_t^{t+\tau} [d\eta(s)]y(s)\Big), (t \geq t_0). \tag{1.5.21}$$

or $y'(t) = f\big(t, y(t), \int_t^{t+\tau} y(s)d\alpha(s)\big)$ *if* $n = 1$.

1.5.6 First order integro-differential equations with delayed and advanced terms

Given a real continuous function f with values $f(t, u, v, w)$ (for $t \geq t_0$, and bounded u, v, w), an equation of the form

$$y'(t) = f\Big(t, y(t), \int_{t-\tau}^{t} k_1(t,s)y(s)ds, \int_{t}^{t+\tau} k_2(t,s)y(s)ds\Big) \ (t \geq t_0) \quad (1.5.22)$$

is a mixed-type integro-differential equation of first order for the function y (where $y(t) \in \mathbb{R}^n$ for $t \in [t_0, \infty)$. A differentiable function y is a solution of this equation for all $t \geq t_0$ if it satisfies (1.5.22).

Remark 1.5.4 *We might conjecture that a particular solution is determined when* suitable *functions ϕ_1 and ϕ_2 are given and we require*

$$y(t) = \phi_1(t), \ \textit{for } t_0 - \tau \leq t \leq t_0; \quad (1.5.23)$$

$$y(t) = \phi_2(t), \ \textit{for } t_0 \leq t \leq t_0 + \tau. \quad (1.5.24)$$

In fact. arbitrary choices of ϕ_1 and ϕ_2 do not define a solution.

Remark 1.5.5 *If, as before, we replace the Riemann integral by a Riemann-Stieltjes integral in the equation (1.5.16) we obtain the form*

$$y'(t) = f\Big(t, y(t), \int_{t-\tau}^{t} [d\eta(s)]y(s), \int_{t}^{t+\tau} [d\eta(s)]y(s)\Big), \quad (1.5.25)$$

where $y(t) \in \mathbb{R}^n$, for $t \geq t_0$. For $n = 1$, (1.5.25) *reads*

$$y'(t) = f\Big(t, y(t), \int_{t-\tau}^{t} y(s)d\alpha(s), \int_{t}^{t+\tau} y(s)d\alpha(s)\Big).$$

The equation

$$y'(t) = \int_{-1}^{0} [d\eta(s)]y(t - \tau(s)) + \int_{-1}^{0} [d\eta(s)]y(t + \tau(s)), \quad (1.5.26)$$

where τ is a non-negative real continuous function on $[-1, 0]$, $y(t) \in \mathbb{R}^n$, $s \in [-1, 0]$, $t > t_0$ with constraints $y(t) = \phi_1(t)$ for $t_0 - \sup \tau(s) \leq t \leq t_0$, $y(t) = \phi_2(t)$ for $t_0 \leq t \leq t_0 + \sup \tau(s)$ is an example of a first order linear autonomous homogeneous scalar integro-differential equation with delayed and advanced terms.

Example 1.5.3 *The equation*

$$y'(t) = \int_{-1}^{0} y(t+s)ds + \int_{0}^{1} y(t+s)ds, \tag{1.5.27}$$

where $y(t) \in \mathbb{R}^n$, $t > t_0$ *is a basic first order linear autonomous homogeneous scalar integro-differential equation with delayed and advanced terms.*

Remark 1.5.6 *Related to* (1.5.27) *is the delay equation*

$$y'(t) = \int_{-1}^{1} y(t+s)d\alpha(s), \tag{1.5.28}$$

where $\alpha(s)$ *is a real function of bounded variation on* $[-1, 1]$. *Equation* (1.5.27) *can be rearranged to read*

$$y'(t) = \int_{t-1}^{t+1} y(\sigma)d\sigma \tag{1.5.29}$$

and, if differentiation is justified $y''(t) = y(t+1) - y(t-1)$ *or*

$$y(t+1) = y''(t) + y(t-1). \tag{1.5.30}$$

Plausible constraints of the form

$$y(t) = \phi_1(t) \ (t \in [t_0 - 1, t_0]), \quad y(t) = \phi_2(t) \ (t \in [t_0, t_0 + 1]), \tag{1.5.31}$$

with arbitrary $\phi_{1,2}$ *will not in general define a solution; compare Example 1.5.2 where we had* $y(t+1) = y''(t) + y(t)$.

1.6 Discrete equations (recurrence, difference, and summation equations)

1.6.1 First order discrete recurrence equations

In this material, the function f is defined on $\mathbb{N} \times \mathbb{R}^n$ and $f(m, u)$ is continuous for bounded $u \in \mathbb{R}^n$ for each $m \in \mathbb{N}$. We suppose that $y(t_m) \in \mathbb{R}^n$. With a given function f, an equation of the form

$$y(t_{m+1}) = f(m, y(t_m)), \text{ valid for } m \geq m_0 \geq 0, \tag{1.6.1}$$

defines the sequence of values

$$y = \{y(t_{m_0+1}), y(t_{m_0+2}), y(t_{m_0+3}), \cdots\}, \text{ given } y(t_{m_0}) = \widetilde{y}_{m_0}. \tag{1.6.2}$$

Remark 1.6.1 *If we wish to distance ourselves from a particular choice*

$$\mathcal{T}_{m_0} = \{t_{m_0}, t_{m_0+1}, t_{m_0+2}, \cdots\}$$

we may equally discuss the sequence $\{\widetilde{y}_{m_0+1}, \widetilde{y}_{m_0+2}, \widetilde{y}_{m_0+3}, \cdots\}$ *satisfying*

$$\widetilde{y}_{m+1} = f(m, \widetilde{y}_m), \text{ for } m \in \{m_0, m_0+1, m_0+2, \cdots\}, \tag{1.6.3}$$

We shall employ the alias $y_m = y(t_m)$ *and where there is no danger of confusion we denote a sequence* $\{y_{m_0}, y_{m_0+1}, y_{m_0+2}, \cdots\}$ *by* y. *(The same notation is used to denote the function* $y : m \to y(t_m)$, *or* $y : t_m \to y(t_m)$, *where* $t_0 < t_1 < t_2 < \cdots$ *is arbitrary.) There is no loss of generality is taking* $m_0 = 0$.

Definition 1.6.1 *Equation* (1.6.1) *is an* explicit *discrete recurrence equation of first order for the sequence* y *(equivalently, for the function* y *defined on the integers greater than or equal to* m_0*). The equation is* autonomous *if* $f(m, u)$ *is independent of* m. *If the relation* (1.6.1) *can be written*

$$y(t_{m+1}) - y(t_m) = \psi(m, y(t_m)), m \geq m_0 \geq 0, \tag{1.6.4}$$

it is natural to refer to the discrete recurrence as a difference equation *and the latter term is often applied to all relationships of the form* (1.6.1).

By converting scalars to vectors, (1.6.1) can be formulated to include *summation equations.*

Example 1.6.1 *For an example of a summation equation consider*

$$\begin{bmatrix} y_{m+1} \\ \sigma_{m+1} \end{bmatrix} = \begin{bmatrix} 1 & h \\ h & 1 \end{bmatrix} \times \begin{bmatrix} y_m \\ \sigma_m \end{bmatrix}, \text{ given } \begin{bmatrix} y_0 \\ \sigma_0 \end{bmatrix} = \begin{bmatrix} y_\star \\ 0 \end{bmatrix}. \tag{1.6.5}$$

Clearly, $\sigma_m = h\{y_m + y_{m-1} + \cdots y_0\}$ *and* $y_{m+1} = y_m + h^2\{y_m + y_{m-1} + \cdots y_0\}$, *a discrete analogue of the simple integro-differential equation* $y'(t) = \int_0^t y(s)ds$.

A particular solution of (1.6.1) is determined only when a value $y(t_{m_0}) = y_{m_0}$, is given for given m_0. The equation (1.6.1) together with the condition $y(t_{m_0}) = y_{m_0}$ is an initial value problem. As a variant of Definition 1.6.1. we have the following, which differs from the preceding definition.

Definition 1.6.2 *If the relationships*

$$y(t_{m+1}) = f\big(m, y(t_m), y(t_{m+1})\big), \text{ valid for } m \geq m_0 \geq 0, \tag{1.6.6}$$

define, given a particular choice of $y(t_{m_0}) = y_{m_0}$*, a* unique *sequence of values*

$$y = \{y(t_{m_0+1}), y(t_{m_0+2}), y(t_{m_0+3}), \cdots\}, \tag{1.6.7}$$

then (1.6.6) *is an* implicit difference equation *and there exists a function* g *such that*

$$y(t_{m+1}) = g\big(m, y(t_m)\big), \text{ for } m \geq m_0 \geq 0. \tag{1.6.8}$$

Example 1.6.2 *Suppose* a *and* $h > 0$ *are real constants where the step-size* $h > 0$. $\widetilde{y}_m \in \mathbb{R}$,

$$\widetilde{y}_{m+1} = (1 + ah)\widetilde{y}_m \ (m \geq 0) \tag{1.6.9}$$

with initial value y_0. (1.6.9) *is a first order (ordinary) linear autonomous and homogeneous scalar difference equation. It is a discrete analogue of the differential equation* $y'(t) = ay(t)$. *An* implicit *analogue is*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + ah\widetilde{y}_{m+1} \ (m \geq 0). \tag{1.6.10}$$

assuming that $ah \neq 1$. *The* n*-dimensional versions read* $\widetilde{y}_{m+1} = (I + Ah)\widetilde{y}_m$ *and* $\widetilde{y}_{m+1} = \widetilde{y}_m + Ah\widetilde{y}_{m+1}$ *(assuming* $\det\{I - hA\} \neq 0$*) where* $A \in \mathbb{R}^{n\times n}$.

Definition 1.6.3 *For each linear homogeneous recurrence relation we define a corresponding characteristic polynomial:*

$\widetilde{y}_m$	*Recurrence relation*		*Characteristic polynomial*
Scalar	$\widetilde{y}_{m+1} = (1 + ah)\widetilde{y}_m$		$\chi(\lambda) = \lambda - (1 + ah)$
Scalar	$(1 - ah)\widetilde{y}_{m+1} = \widetilde{y}_m$	$(ah \neq 1)$	$\chi(\lambda) = (1 - ah)\lambda - 1$
Scalar	$\widetilde{y}_{m+1} = \sum_{\ell=1}^{k} \gamma_\ell \widetilde{y}_{m+1-\ell}$		$\chi(\lambda) = \lambda^k - \sum_{\ell=1}^{k} \gamma_\ell \lambda^{k-\ell}$
Vector	$\widetilde{y}_{m+1} = (I + Ah)\widetilde{y}_m$		$\chi(\lambda) = \det[\lambda I - (I + Ah)]$
Vector	$(I - Ah)\widetilde{y}_{m+1} = \widetilde{y}_m$	$\det[I - Ah] \neq 0$	$\chi(\lambda) = \det[\lambda(I - Ah) + I)]$

Characteristic polynomials

Lemma 1.6.1 *Refer to the table of characteristic polynomials in Definition 1.6.3, and suppose that* $\chi(\lambda_\ell) = 0$. *Then* $\{\widetilde{y}_m = y_0(\lambda_\ell)^m\}$ *is a solution of the corresponding recurrence. If* λ_ℓ *is a multiple zero then there can be additional solutions of the form* $p(m)(\lambda_\ell)^m$ *where* p *is a polynomial.*

Consider the behaviour of λ^m $(m = 0, 1, 2, \cdots)$ where λ is a scalar. If $|\lambda| > 1$, $|\widetilde{y}_m| \to \infty$ as $n \to \infty$, if $|\lambda| < 1$, $\widetilde{y}_m \to 0$, and if $\Re(\lambda) = 0$, λ^m oscillates. In the latter case the solutions of the equation (1.6.9) have oscillatory behaviour.

Remark 1.6.2 *Suppose* $a < 0$*; the solution* $y(t)$ *of the ordinary differential equation (1.1.3) has no oscillatory behaviour but the solution* $\widetilde{y}_m$ *of the ordinary difference equation (1.6.9) has oscillatory behaviour if* $h > 1/a$.

1.6.2 Generalisations: discrete recurrence equation with delayed or advanced terms

We have seen that differential equations with advanced or retarded arguments provide generalisations of ordinary differential equations, and we consider whether there are corresponding generalisations of recurrences such as (1.6.1). To a certain extent, the distinctions between current and delayed arguments are redundant: thus, in Example 1.6.1 we see a summation equation in which $\widetilde{y}_{m+1}$ is expressed in terms of all preceding values $\{y_0, \widetilde{y}_1, \cdots, \widetilde{y}_m\}$ using (1.6.1) but this is not apparent in the vector form.

In general a *two-stage recurrence*

$$y(t_{m+1}) = f\big(m, y(t_m), y(t_{m-1})\big), \tag{1.6.11}$$

can be re-expressed in the form of a one-stage recurrence between column vectors $u_{m+1} := [y(t_{m+1}), y(t_m)]^T$ in $\mathbb{R}^2$ as:

$$\begin{bmatrix} y(t_{m+1}) \\ y(t_m) \end{bmatrix} = \begin{bmatrix} \psi(m, u_m) \\ e_1^T u_m \end{bmatrix} = \Psi(m, u_m) \in \mathbb{R}^2 \tag{1.6.12}$$

where $e_1^T[v_1, v_2]^T$ is v_1.

Despite the preceding remark, it is sometimes convenient to retain a scalar formulation. Thus, given constants μ and $h > 0$ the relation $\widetilde{y}_{m+1} = \widetilde{y}_m + \mu h \widetilde{y}_{m-1}$, where $\widetilde{y}_m \in \mathbb{R}$, with initial function $\widetilde{y}_m = y(\phi)$ for $n \geq 0$ is a *two-stage* linear autonomous homogeneous scalar difference equation More generally, for integer $M \geq 1$, the recurrence $\widetilde{y}_{m+1} = \widetilde{y}_m + \mu h \widetilde{y}_{m-N}$ provides a discrete analogue of the delay differential equation (1.2.8).

Example 1.6.3 *Consider the two-step discrete equation of the form*

$$\widetilde{y}_{m+2} = \widetilde{y}_{m+1} + \mu h \widetilde{y}_m, \tag{1.6.13}$$

or, equivalently,

$$\begin{bmatrix} \widetilde{y}_{m+2} \\ \widetilde{y}_{m+1} \end{bmatrix} = \begin{bmatrix} 1 & \mu h \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{y}_{m+1} \\ \widetilde{y}_m \end{bmatrix}. \tag{1.6.14}$$

The characteristic polynomial of (1.6.13) is

$$\chi(\lambda) = \lambda^2 - \lambda - \mu h. \tag{1.6.15}$$

The equation $\chi(\lambda) = 0$ is quadratic and the characteristic values are

$$\lambda_1, \lambda_2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\mu h}.$$

If λ_1 and λ_2 are distinct, then the general solution has $\widetilde{y}_m = c_1\lambda_1^m + c_2\lambda_2^m$ and if $\lambda_1 = \lambda_2$ then it has $\widetilde{y}_m = (c_1 + mc_2)\lambda_1^m$. The every solution of the (1.6.13) oscillates if and only if λ_1 and λ_2 are not positive real numbers. every non-trivial solution oscillates if and only if $\mu h < -\frac{1}{4}$ where μ and h are constants.

Oscillatory behaviour of the DDE (1.2.8) and its discrete analogue (1.6.13) is remarkably similar, but comparisons are improved if we consider $\widetilde{y}_{m+1} = \widetilde{y}_m + \mu h\widetilde{y}_{m-N}$ with $M > 1$.

If we wish to reveal the structure, then the scalar recurrence

$$y(t_{m+1}) = f\big(m, y(t_m), y(t_{m-1}, y(t_{m-2}, \cdots, y(t_{m-N})\big) \ (m \in \{N, N+1, \cdots\}) \tag{1.6.16}$$

provides a natural generalisation of (1.6.11). This suggests that we consider *discrete equations with advanced terms* by examining

$$\widetilde{y}_{m+1} = f\big(m, \widetilde{y}_m, \widetilde{y}_{m+1}, \widetilde{y}_{m+2}, \cdots, \widetilde{y}_{m+M}\big), \tag{1.6.17}$$

and *discrete equations with delayed and advanced terms* by examining

$$\widetilde{y}_{m+1} = f\big(m, \widetilde{y}_{m-N} \cdots, \widetilde{y}_{m+2}, \widetilde{y}_{m-1}, \widetilde{y}_m, \widetilde{y}_{m+1}, \widetilde{y}_{m+2}, \cdots, \widetilde{y}_{m+M}\big). \tag{1.6.18}$$

Example 1.6.4 *The equation*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \mu h\widetilde{y}_{m+N}, \tag{1.6.19}$$

where $n = 0, 1, 2, ..., N, N \in n$, $\widetilde{y}_m \in \mathbb{R}$, is a first order linear autonomous homogeneous scalar difference equation with advanced terms. It provides a discrete analogue of the advanced differential equation (1.3.2). Clearly, it can be rewritten as an explicit difference equation

$$\widetilde{y}_{m+N} = \{\widetilde{y}_{m+1} - \widetilde{y}_m\}/\{\mu h\} \ (m \in \{0, 1, \cdots\}). \tag{1.6.20}$$

Example 1.6.5 *The equation*

$$\widetilde{y}_{m+1} = (1 + ah)\widetilde{y}_m + h(b\widetilde{y}_{m-N} + c\widetilde{y}_{m+N}), \tag{1.6.21}$$

is a linear autonomous homogeneous scalar difference equation with delayed and advanced terms. (It provides a discrete analogue of the mixed differential equation with delayed and advanced terms (1.4.8).)

If we write down (1.6.21) *for $m = m_0, m_0 + 1, \cdots, m_0 + M_0 - 2$, and if $\{\widetilde{y}_{m_0-\ell}\}_{\ell=0}^N$ and $\{\widetilde{y}_{m_0+M_0+\ell}\}_{\ell=0}^N$ are given, we obtain a system of linear equations. There are $M_0 - 1$ equations and $2N + 2$ conditions for finding the $M_0 - 1$ unknown values, so that problem may have, one, none or many possible solutions in principle.*

1.7 Conditions for oscillation of discrete equations

In this section criteria for discrete equations to be oscillatory, such as those that have emerged in the previous section, will be discussed. This will focus on linear equations, where the analysis can be based on the zeros of a polynomial and it will refer also to a characterisation for nonlinear problems.

For the linear equation

$$\widetilde{y}_{n+1} = \sum_{j=0}^{N} a_j \widetilde{y}_{n-j}, \tag{1.7.1}$$

by considering the characteristic equation for problems of this type, it is simple to show that the general solution may be written as a linear combination of eigenfunctions. (One needs to take account of any repeated characteristic values in the usual way.) Let the values λ_ℓ be the zeros of the characteristic polynomial,

$$z^{N+1} - a_0 z^N - a_1 z^{N-1} - \ldots - a_N \tag{1.7.2}$$

then assuming all the zeros are distinct the solution takes the form $\widetilde{y}_m = \sum_{j=1}^{N+1} b_j {\lambda_j}^m$. If characteristic values are repeated, a slightly more complicated expression is needed for the solution but for our purposes the conclusions will be the same. Any particular eigenfunction oscillates unless $\lambda_i \in \mathbb{R}^+$ and therefore we can give the result:

Lemma 1.7.1 *The equation (1.7.1) is oscillatory if and only if none of the zeros of (1.7.2) lie on the positive real axis (see [47]).*

This result, based on the zeros of the characteristic polynomial, will prove most useful in the theoretical analysis, which is currently confined to linear equations.

However, a more general theorem that applies also to certain nonlinear discrete problems has been given (see [43], [73]) and it may prove fruitful in the further investigation of non-linear problems. Consider the difference equation (discrete Volterra equation),

$$\widetilde{y}_{m+1} = \widetilde{y}_m - \sum_{i=1}^{m} p_i f_i(\widetilde{y}_{m-k_i}) \tag{1.7.3}$$

where $p_i > 0, k_i$ are positive integers, and f_i are continuous functions on $\mathbb{R}$.

Theorem 1.7.2 (Elaydi, [43].) *Suppose that the following conditions are met: (a)* $yf_i(y) > 0 \quad$ *for* $y \neq 0, 1 \leq i \leq m$*; (b)* $\liminf_{y\to 0} \frac{f_i(y)}{y} \geq 1$*, for* $1 \leq i \leq m$ *(c)* $\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1$*; then every solution of (1.7.3) oscillates.*

Remark 1.7.1 *In the study of stability theory and exponential growth and decay, it is usual to linearise equations and to use the linear analysis as the basis for obtaining a close approximation to the behaviour of a non-linear problem. For the study of oscillation theory, it is clear that the situation is more complicated, and the extent to which a linear analysis provides useful insights into non-linear problems has not been established. Some examples that are considered are non-linear, and there is no experimental evidence that our methods fail in these cases (see [47]).*

1.7.1 A preview of oscillation theory

Oscillation for both continuous and discrete homogeneous linear problems (in particular those which have constant coefficients and deviating arguments) can be investigated by the location of zeros of a *characteristic function.* In the discrete case, this function will be a polynomial whose degree depends upon the step-size chosen for the numerical schemes (The degree increases as the step-size becomes smaller) and approximation of the continuous problem by the discrete schemes become more accurate.) In this context we have available various techniques, such as the 'boundary locus' or D-partition method, and direct polynomial solvers like the NAG Fortran library, MATLAB, Mathematica etc. Here, we wish to develop a method based on the Argument Principle and which will in principle be applicable for any degree of polynomial.

1.8 A brief remark on application

Application of delay differential equations commonly arise in models where there is some time-lag or after-effect. This situation arises for example in modelling in the biosciences and we can refer the reader to [94, 2] for example, for specific models. Mixed-type equations are less well-used in models. However a discrete FitzHugh-Nagumo equation for modelling nerve cells can lead to an equation of the form

$$\frac{dV}{dT} = \kappa(V(T+H) - 2V(T) + V(T+H)) + f(V,T),$$

for the travelling wave solution (see [34]) where $f(V) = V(V-1)(\alpha - V)$ for suitable constants κ,H, $0 < \alpha < 1$.

Chapter 2

Basic Methods and Solutions: The Laplace transform

2.1 Introduction

This Chapter introduces the Laplace transform method and indicates how the method can be used to solve certain differential equations. The Laplace transform is conventionally defined for functions with domain $[0, \infty)$ and without loss of generality we shall suppose that our equations hold on $[t_0, \infty)$ with $t_0 = 0$. We study its basic properties.

Remark 2.1.1 *Clearly, a function u defined on $[t_0, \infty)$ defines a function $u_\sharp$ on $[0, \infty)$ on setting $u_\sharp(t) = u(t + t_0)$.*

2.1.1 The Laplace transform

The formal definition of the Laplace transform is stated below.

Definition 2.1.1 *The Laplace transform (assuming it exists) of the function f defined on $[0, \infty)$ is the function $\mathcal{L}\{f\}$*

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt \tag{2.1.1}$$

where $f(t) \in \mathbb{R}^n$ for $t > 0$.

Let us recall the following concepts.

Definition 2.1.2 *(a) A function f defined on $[0, \infty)$ is said to be exponentially bounded (sometimes, exponentially bounded at ∞ or of exponential order) if there exist $\gamma \in \mathbb{R}$, $k > 0$ and $T \geq t_0$ such that*

$$|f(t)| \leq k \quad \exp(\gamma t) \text{ for } t \geq T. \tag{2.1.2}$$

(b) A function f is said to be piecewise continuous on a bounded interval $\mathcal{I} \subset \mathbb{R}$ if it has a finite number of discontinuities and the left and right limits exist (and are bounded) at each discontinuity. It is said to be piecewise continuous on $[0, \infty]$ *if it is piecewise continuous on every bounded subinterval* $\mathcal{I} \subset [0, \infty)$.

While s is commonly taken to be real, the extension to $s \in \mathbb{C}$ (with $s = s_1 + \mathrm{i}s_2$) as the complex-valued integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-(s_1+\mathrm{i}s_2)t} f(t)dt \text{ where } s_1 = \Re s \in \mathbb{R} \text{ and } s_2 = \Im s, \quad (2.1.3)$$

is straightforward.

Lemma 2.1.1 *If $f(t)$ is piecewise continuous on* $[0, \infty)$ *and exponentially bounded, and let γ be chosen as in* (2.1.2). *Then* $\mathcal{L}\{f\}(s) = F(s)$ *in* (2.1.3) *exists for all* $s > \gamma, \quad s \in \mathbb{R}$.

Remark 2.1.2 *All bounded continuous functions and every polynomial have Laplace transforms. Thus t^n, n=1, 2, 3, ..., has a Laplace transform but e^{t^2} does not. The conditions are not necessary for the existence; e.g., $t^{-1/2}$ do not satisfy the conditions but yet has a Laplace transform.* $\mathcal{L}\{t^{\frac{-1}{2}}\} = \int_0^\infty e^{-st} e^{\frac{-1}{2}} dt = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\left(\frac{\Pi}{s}\right)}$.

Some properties of Laplace transforms

Suppose f and g satisfy the conditions of Lemma 2.1.1. Then,

- $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$.
- $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c.
- $\mathcal{L}\{e^{at} f\}(s) = F(s - a)$.
- $\mathcal{L}\{f'\}(s) = sF(s) - F(0)$.
- $\mathcal{L}\{f''\}(s) = s^2 F(s) - sf(0) - f'(0)$.
- $\mathcal{L}\{f^n\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{n-1}(0)$.
- $\mathcal{L}\{t^n f\}(s) = (-1)^n \frac{d^n}{ds^n} F(s)$.

2.1.2 The inverse Laplace transform

The definition of the inverse Laplace transform follows from that of the Laplace transform, and several notational conventions are employed.

Definition 2.1.3 *The expression $\mathcal{L}^{-1}\{F\}$ denotes a function whose f whose Laplace transform is $F := \mathcal{L}\{f\}$. Thus, if*

$$\mathcal{L}\{f\} = F \text{ then } f = \mathcal{L}^{-1}\{F\}. \tag{2.1.4}$$

With a commonly adopted abuse of notation, we may write:

$$\mathcal{L}\{f\}(t) = F(s);\ f(t) = \mathcal{L}^{-1}\{F\}(s). \tag{2.1.5}$$

Example 2.1.1 *To illustrate the latter notation,*

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \text{ and } \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at). \tag{2.1.6}$$

A fairly extensive table of inverse transform is given in Appendix A (see [31]).

Theorem 2.1.2 (A uniqueness theorem) *Suppose $f(t)$ and $g(t)$ are continuous on $[0,\infty)$ and of exponential order γ. If $\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s)$ for all $s > \gamma$, then $f(t) = g(t)$ for all $t \geq 0$.*

Remark 2.1.3 *In brief, the preceding well-known result (known as Lerch's theorem) indicates that when f is continuous, $\mathcal{L}^{-1}\{F(s)\}$, is unique.*

The simplest inversion formula is given by the so-called Bromwich integral, which provides a useful analytical tool.

Theorem 2.1.3 *The function $\mathcal{L}^{-1}\{F(s)\}$, is given by*

$$f(t) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} F(s)\exp\{st\}ds, \tag{2.1.7}$$

where the integral is evaluated along the path from $s = c - \infty$ to $s = c + i\infty$ for any real c such that this path lies in the region of convergence of the integral.

2.2 Solving differential equations using the Laplace transform: Examples

If not specifically stated otherwise, we shall assume that the solutions of differential equations discussed here are continuous and exponentially bounded. Their Laplace transforms then exist.

2.2.1 Solving ordinary differential equations using the Laplace transform

Let us consider the ordinary differential equation (ODE) $y'(t) = ay(t)$, in (1.1.3), with initial condition $y(t_0) = y_0$. Of course, we know that the solution is $y(t) = \exp(a[t - t_0])y_0$ which is continuous and exponentially bounded so that in consequence it has a Laplace transform. Now, applying the Laplace transform to $y'(t) = ay(t)$ we obtain

$$\int_0^\infty e^{-ts}y'(t)dt = \int_0^\infty ay(t)e^{-ts}dt \tag{2.2.1}$$

We know that

$$\mathcal{L}[y](s) = \int_0^\infty y(t)e^{-ts}dt, \text{ and } \mathcal{L}[y'](s) = \int_0^\infty y'(t)e^{-ts}dt = s\mathcal{L}[y](s) - y(0).$$

From (2.2.1), $s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s)$, so that $(s - a)\mathcal{L}[y](s) = y(0)$, and

$$\mathcal{L}[y](s) = \frac{y(0)}{s - a} \quad (s \neq a). \tag{2.2.2}$$

Taking the inverse Laplace transform of both sides of (2.2.2) yields, as anticipated,

$$y(t) = e^{at}y(0). \tag{2.2.3}$$

2.2.2 Solving delay differential equations with Laplace transforms

Let us consider the delay differential equation (DDE) (1.2.8), that is, $y'(t) = \alpha y(t) + \beta y(t - \tau)$. A change of variables gives the normalised form

$$y'(t) = ay(t) + by(t - 1), \quad (t \geq 0) \tag{2.2.4}$$

(where $a = \alpha\tau$, $b = \beta\tau$). Suppose $\phi \in C[-1, 0]$ and assume the initial condition $y(t) = \phi(t)$, for $t \in [-1, 0]$. We know that it can be established, by the method of steps, that the the solution of (2.2.4) is continuous and exponentially bounded and therefore has a Laplace transform. To illustrate our procedure we shall take

$$\phi(t) = 1 \text{ for } t \in [-1, 0]. \tag{2.2.5}$$

Applying the Laplace transform to both sides of (2.2.4), we obtain

$$\int_0^\infty e^{-ts}y'(t)dt = \int_0^\infty ay(t)e^{-st}dt + \int_0^\infty by(t - 1)e^{-ts}dt \tag{2.2.6}$$

Let $t-1=\sigma$, $t=\sigma+1$ then $dt=d\tau$ and when $t=0$ then $\sigma=-1$ and when $t=\infty$ then $\sigma=\infty$. Using these results in (2.2.6), we get,

$$\int_{-1}^{\infty} e^{-(\sigma+1)s}y'(\sigma+1)d\sigma = \int_{-1}^{\infty} ay(\sigma+1)e^{-(\sigma+1)s}d\sigma + \int_{-1}^{\infty} by(\sigma)e^{-(\sigma+1)s}d\sigma$$

$$s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s) + \int_{-1}^{\infty} by(\sigma)e^{-(\sigma+1)s}d\sigma$$

Here

$$\mathcal{L}[y](s) = \int_{0}^{\infty} y(t)e^{-ts}dt\mathcal{L}[y'](s) = \int_{0}^{\infty} y'(t)e^{-ts}dt = s\mathcal{L}[y](s) - y(0)$$

$$= a\mathcal{L}[y](s) + \int_{-1}^{0} by(\sigma)e^{-(\sigma+1)s}d\sigma + \int_{0}^{\infty} by(\sigma)e^{-(\sigma+1)s}d\sigma$$

$$= a\mathcal{L}[y](s) + \int_{-1}^{0} by(\sigma)e^{-(\sigma+1)s}d\sigma + be^{-s}\int_{0}^{\infty} y(\sigma)e^{-\sigma}d\sigma.$$

Hence,

$$\mathcal{L}[y](s)(s-a-be^{-s}) = y(0) + \int_{-1}^{0} by(\sigma)e^{-(\sigma+1)s}d\sigma.$$

From this, provided the denominator is non-zero,

$$\mathcal{L}[y](s) = \frac{y(0) + \int_{-1}^{0} by(\sigma)e^{-(\sigma+1)s}d\sigma}{s-a-be^{-s}} \tag{2.2.7}$$

where $y(0)$ and $y(\sigma)$ are determined by the initial function. If $y(t)=\phi(t)$ on $[-1,0]$ then we get $y(0)=1$ and $y(\sigma)=1$

$$\int_{-1}^{0} y(\sigma)e^{-(\sigma+1)s}d\sigma = \int_{-1}^{0} y(\sigma)e^{-\sigma s}e^{-s}d\sigma$$

$$= e^{-s}\int_{-1}^{0} e^{-\sigma s}d\sigma = e^{-s}\left[\frac{e^{-\sigma s}}{-s}\right]_{-1}^{0} = \frac{1}{s}\left(1-\frac{1}{e^{s}}\right)$$

Using these results in (2.2.7) we get,

$$\mathcal{L}[y](s) = \frac{1+\frac{1}{s}(1-\frac{1}{e^{s}})}{s-a-be^{-s}} \tag{2.2.8}$$

Applying the inverse Laplace transform to (2.2.8), we get,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1+\frac{1}{s}(1-\frac{1}{e^{s}})}{s-a-be^{-s}}\right\} \tag{2.2.9}$$

The denominator $s-a-be^{-s}$ is $\chi(s)$.

2.2.3 Solving mixed type functional differential equations using Laplace transforms

It is interesting to attempt to solve mixed type functional differential equations (MTFDEs) by the Laplace transform method. We are able to apply the Laplace transform to a function that is exponentially bounded and piecewise continuous. This is not true of all solutions of MTFDEs, so one must proceed with caution.

Assumption 2.2.1 *We suppose that the solution to which we apply the Laplace transform is exponentially bounded and piecewise continuous.*

Where there exists a characteristic function or generalised characteristic function we know that the class of functions satisfying the last assumption is non-empty; the characteristic values correspond to solutions $\exp(\lambda t)$ that satisfy the necessary condition.

Let us consider a MTFDE of the form (1.4.8)

$$y'(t) = ay(t) + by(t-1) + cy(t+1) \tag{2.2.10}$$

We consider (2.2.10) for $t \in [0, T]$ and we investigate the existence of a solution y when we require that $y(t) = \phi_1(t), t \in [-1, 0]$ and $y(t) = \phi_2(t), t \in [T, T+1]$.

We have to prove this general theorem. Taking Laplace transform of both sides of the above equation, we get,

$$\int_0^\infty e^{-ts}y'(t)dt = \int_0^\infty ay(t)e^{-ts}dt + \int_0^\infty by(t-1)e^{-ts}dt + \int_0^\infty cy(t+1)dt \tag{2.2.11}$$

We know that

$$\mathcal{L}[y](s) = \int_0^\infty y(t)e^{-ts}dt, \qquad \mathcal{L}[y'](s) = \int_0^\infty y'(t)e^{-ts}dt,$$

which implies that

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0),$$

Using these results in equation (2.2.11) we get,

$$s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s) + \int_0^\infty by(t-1)e^{-ts}dt + \int_0^\infty cy(t+1)e^{-ts}dt \tag{2.2.12}$$

Let $t - 1 = \sigma_1$, $t = \sigma_1 + 1$ then $dt = d\sigma_1$. When $t = 0$ then $\sigma_1 = -1$ and when $t = \infty$ then $\sigma_1 = \infty$. Again let $t + 1 = \sigma_2$, $t = \sigma_2 - 1$ then $dt = d\sigma_2$.

When $t = 0$ then $\sigma_2 = 1$ and when $t \to \infty$ then $\sigma_2 \to \infty$. Now equation (2.2.12) becomes

$$s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s) + \int_{-1}^{\infty} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1 + \int_{1}^{\infty} cy(\sigma_2)e^{-(\sigma_2-1)}d\sigma_2$$

$$s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s) + \textstyle\int_{-1}^{0} by(\sigma_1)e^{-(1+\tau_1)s}d\sigma_1 + \int_{0}^{\infty} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1$$

$$+ \int_{1}^{0} cy(\sigma_2)e^{-(\sigma_2-1)s}d\sigma_2 + \int_{0}^{\infty} cy(\sigma_2)e^{-(\sigma_2-1)s}d\sigma_2$$

$$s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s) + \textstyle\int_{-1}^{0} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1 + be^{-s}\int_{0}^{\infty} y(\sigma_1)e^{-\sigma_1 s}d\sigma_1$$

$$- \int_{0}^{1} cy(\sigma_2)e^{-(\sigma_2-1)s}d\sigma_2 + ce^{s}\int_{0}^{\infty} y(\sigma_2)e^{-\sigma_2 s}d\sigma_2$$

$$s\mathcal{L}[y](s) - y(0) = a\mathcal{L}[y](s) + \textstyle\int_{-1}^{0} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1 + be^{-s}\mathcal{L}[y](s)$$

$$- \int_{0}^{1} cy(\sigma_2)e-(\sigma_2 - 1)sd\sigma_2 + ce^{s}\mathcal{L}[y](s)$$

$$s\mathcal{L}[y](s) - y(0) = (a + be^{-s} + ce^{s})\mathcal{L}[y](s) + \textstyle\int_{-1}^{0} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1$$

$$- \int_{0}^{1} cy(\sigma_2)e^{-(\sigma_2)s}d\sigma_2$$

$$(s - a - be^{-s} - ce^{s})\mathcal{L}[y](s) = y(0) + \textstyle\int_{-1}^{0} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1$$

$$- \int_{0}^{1} cy(\sigma_2)e^{-(\sigma_2-1)s}d\sigma_2$$

$$\mathcal{L}[y](s) = \frac{y(0) + \int_{-1}^{0} by(\sigma_1)e-(1+\sigma_1)sd\sigma_1 - \int_{0}^{1} cy(\sigma_2)e^{-(\sigma_2-1)s}d\sigma_2}{s - a - be^{-s} - ce^{s}} \tag{2.2.13}$$

If $y(t) = 1$ on $[-1, 0]$ then we get $y(\sigma_1) = 1$, $y(\sigma_2) = 1$ and $y(0) = 1$ then

$$\int_{-1}^{0} by(\sigma_1)e^{-(1+\sigma_1)s}d\sigma_1 = \int_{-1}^{0} be^{-s}e^{-\tau_1 s}d\sigma_1 = be^{-s}\int_{-1}^{0} e^{-\sigma_1 s}d\sigma_1$$

$$= be^{-s}\left[\frac{e^{-\sigma_1 s}}{-s}\right]_{-1}^{0} = \frac{b}{s} - \frac{b}{se^{s}}$$

Again

$$\int_0^1 cy(\sigma_2)e^{-(\sigma_2-1)s}d\sigma_2 = \int_0^1 ce^{-\sigma_2 s}e^s d\sigma_2 = ce^s\int_0^1 e^{-\sigma_2 s}d\sigma_2$$

$$= ce^s\left[\frac{e^{-\sigma_2 s}}{-s}\right]_0^1 = \frac{ce^s}{s} - \frac{c}{s}$$

Now using these results in (2.2.13) we get ,

$$\mathcal{L}[y](s) = \frac{1+\frac{b}{s}-\frac{b}{se^s}-\frac{ce^s}{s}+\frac{c}{s}}{s-a-be^{-s}-ce^s} \tag{2.2.14}$$

$$\mathcal{L}[y](s) = F(s) \tag{2.2.15}$$

where

$$F(s) = \frac{1+\frac{b}{s}-\frac{b}{se^s}-\frac{ce^s}{s}+\frac{c}{s}}{s-a-be^{-s}-ce^s}$$

Taking inverse the Laplace transform of (2.2.14) we get,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1+\frac{b}{s}-\frac{b}{se^s}-\frac{ce^s}{s}+\frac{c}{s}}{s-a-be^{-s}-ce^s}\right\} \tag{2.2.16}$$

$$y(t) = \mathcal{L}^{-1}\{F(s)\}$$

Remark 2.2.1 *Of course, the equation (1.2.1) can be written as integral equation by integrating both sides with respect to t. Thus, for $t' > t \geq t_0$,*

$$y(t) = y(t') + \int_{t'}^{t} f(s, y(s), y(s-\tau))ds \tag{2.2.17}$$

which is a Volterra integral equation in classical form when we set $t' = t_0$.

There are various ways to generalise (1.2.1) to obtain further examples, for example, assuming that $\tau_l(t) \geq 0(l \in [1,2,3,...,m])$, by considering variable or multiple "lags" $(\tau_l(t) \geq 0(l \in [1,2,3,...,m])$, as in

$$y'(t) = f(t, y(t), y(t-\tau_1(t)), y(t-\tau_2(t)), ..., y(t-\tau_m(t))), (t \geq t_0), \tag{2.2.18}$$

or

by introducing a distributed time-lag as in

$$y'(t) = f(t, y(t), \int_{t-\tau}^{t} k(t, s, y(s))ds), (t \geq t_0), \tag{2.2.19}$$

or

$$y'(t) = f(t, y(t), \int_{t_0}^{t} k(t, s, y(s))ds), (t \geq t_0), \tag{2.2.20}$$

Equations (2.2.19)–(2.2.20) are normally termed Volterra integro-differential equations.

Remark 2.2.2 *If we replace the Riemann integral by the Riemann-Stieltjes integral in the equations (2.2.19)–(2.2.20) we obtain, (each for $t \geq t_0$), the forms*

$$y'(t) = f(t, y(t), \int_{t-\tau}^{t} k(t, s, y(s))d\alpha(s)), (t \geq t_0), \tag{2.2.21}$$

and

$$y'(t) = f(t, y(t), \int_{t_0}^{t} k(t, s, y(s))d\alpha(s)), (t \geq t_0), \tag{2.2.22}$$

where $y(t) \in \mathbb{R}$, $s > 0$.

Chapter 3

Numerical Methods

3.1 Introduction

In this chapter we introduce numerical methods for initial-value problems. Our main focus will be on the linear $\vartheta-$ method, which can be thought of as a generalisation of the classical Euler or trapezium rules. These methods provide an important prototype for investigation because they can be classed as either linear multi-step methods or as Runge-Kutta methods; in consequence, properties that are established for $\vartheta-$methods may well be found in more complicated methods.

3.2 Linear multi-step methods

All the numerical methods for initial-value problems that are discussed here are based on the idea of discretisation: we approximate a true solution y defined on a continuous interval $[t_0, T]$ by an approximate solution defined on a set of discrete points by $t_m = t_0 + mh$, $m = 0, 1, 2, \cdots$ (the parameter $h > 0$ is called the step-size). We introduce familiar methods for ODEs and discuss their adaptation to equations with deviating argument later.

Let us consider the ODE (1.1.1);

- suppose $h > 0$, $t = t_0 + mh$, $m = 0, 1, 2, \cdots$.
- write $t_m := t_0 + mh$ and define $\mathcal{T}_0 = \{t_0, t_1, t_2, \cdots\}$
- write $\widetilde{y}(t_m)$ to denote an approximation to $y(t_m)$:

$$\widetilde{y}(t_m) \approx y(t_m) \tag{3.2.1}$$

(in which $y(t_m)$ is the value of the true solution at t_m). In our discussion, $\mathcal{T}_0$ is the domain of definition of $\widetilde{y}$ (though for more general problems we need to extend the domain to $[t_0, \infty)$ using a dense-output process);

- when we seek brevity, we use the abbreviations

$$\widetilde{y}_m := \widetilde{y}(t_m) \text{ and } \widetilde{f}_m := f(t_m, \widetilde{y}(t_m)) \quad (m \in \{0, 1, 2, \cdots\}) \tag{3.2.2}$$

  when discussing the ODE case (1.1.1). (We shall amend (3.2.2) when we consider approximations to solutions of other equations.)

Remark 3.2.1 *The numerical methods indicated here are methods for determining, in sequence, the approximations $\{\widetilde{y}(t_m)\}$ using a chosen formula for the calculations. In a 1-step method, the approximate solution of the ODE (1.1.1) at t_{k+1} is computed using the value at t_m using the true initial value ($m = 0$) or computed at a previous stage ($m = 1, 2, \cdots$ in turn). General Runge-Kutta methods involve the computation of 'stage values' where an approximation is obtained at additional points in $[t_m, t_{m+1}]$ in order to take the full step. In a k-step multi-step method, the approximate solution of the ODE (1.1.1) has to be available at each of the points $t_1, \cdots, t_k$ before the approximation at t_{k+1} is obtained. Such methods require the initial value and starting procedures in order to compute the approximations at $\{t_1, t_2, \cdots, t_{k-1}\}$.*

A general *linear multi-step method* or linear k-step method, using a fixed step $h > 0$, for the ODE (1.1.1) is defined by suitable parameters $\{\alpha_j, \beta_j\}$ in a formula $\widetilde{y}_{m+1} + \sum_{j=1}^{k} \alpha_j \widetilde{y}_{m-j+1} = h \sum_{j=0}^{k} \beta_j \widetilde{f}_{m-j+1}$. ($\alpha_j$, β_j are constants that specify the formula.) The formula defines $\widetilde{y}_{m+1}$ as the solution (if it exists) of

$$\widetilde{y}_{m+1} - h\beta_0 f(t_{m+1}, \widetilde{y}_{m+1}) = \sum_{j=1}^{k} \alpha_j \widetilde{y}_{m-j+1} + \sum_{j=0}^{k} \beta_j \widetilde{f}_{m-j+1}. \tag{3.2.3}$$

for successive values of m. (In the Adams formulae, $\alpha_1 = 1$ and $\alpha_j = 0$ for $j \in \{2, 3, \cdots, k\}$.) Equation (3.2.3) has a solution if $\beta_0 = 0$ or (for a wide class of functions f) if h is sufficiently small. Now, $\widetilde{y}_{m+1} = \widetilde{y}(t_{m+1})$ and we can consider (3.2.3) with m replaced by $m + 1$.

We next indicate the structure of *Runge-Kutta methods*. The general s-stage Runge-Kutta method is defined for the solution of (1.1.1) by a choice of $h > 0$ and suitable parameters in a formula

$$\widetilde{y}(t_{m+1}) = \widetilde{y}(t_m) + h \sum_{i=1}^{s} b_i k_{m,i}, \tag{3.2.4}$$

where the values $\{k_{m,i}\}$ are to satisfy

$$k_{m,i} = f\big(t_m + c_i h, \widetilde{y}(y_m) + \sum_{j=1}^{s} a_{ij} k_{m,j}\big) \quad (i \in \{1, 2, \cdots, s\}). \tag{3.2.5}$$

Equation (3.2.5) has a solution if $a_{ij} = 0$ for $j \geq i$ and, more generally, for a wide class of functions f, whenever h is sufficiently small.

Example 3.2.1 *Consider the approximate solution of (1.1.1). Throughout,* $\widetilde{y}_0 = y(t_0)$ *and* $h > 0$.

- *Euler's method is associated with the formula*

$$\widetilde{y}(t_{m+1}) \equiv \widetilde{y}_{m+1} = \widetilde{y}_m + hf(t_m, \widetilde{y}_m) \quad (m = 0, 1, 2, \cdots). \tag{3.2.6}$$

  It is the simplest numerical method. It is both a one-step method and a one-stage Runge-Kutta (RK) method and is an explicit method.

- *By contrast the* implicit Euler's method *is associated with the formula*

$$\widetilde{y}(t_{m+1}) \equiv \widetilde{y}_{m+1} = \widetilde{y}_m + hf(t_m, \widetilde{y}_{m+1}) \quad (m = 0, 1, 2, \cdots) \tag{3.2.7}$$

  (with $\widetilde{y}_0 = y(t_0)$ *and where* $h > 0$*). This is an implicit formula that must be solved for* $\widetilde{y}_{m+1}$ *for* $m = 0, 1, 2, \cdots$.

- *The trapezium rule method is associated for* $h > 0$ *with the formula*

$$\widetilde{y}(t_{m+1}) \equiv \widetilde{y}_{m+1} = \widetilde{y}_m + \frac{1}{2}hf(t_m, \widetilde{y}_m) + \frac{1}{2}hf(t_m, \widetilde{y}_{m+1}) \tag{3.2.8}$$

  ($m = 0, 1, 2, \cdots$*). with* $\widetilde{y}_0 = y(t_0)$*, It too is an implicit method.*

- *The method defined, where* $h > 0$*, by the explicit formula*

$$\widetilde{y}(t_{m+1}) \equiv \widetilde{y}_{m+1} = \widetilde{y}_m + hf(t_m + \frac{h}{2}, \widetilde{y}_m + \frac{h}{2}f(t_m, \widetilde{y}_m)) \tag{3.2.9}$$

  ($m = 0, 1, 2, \cdots$*), can be rewritten as a two-stage Runge-Kutta method. It is based on the mid-point rule in which the mid-point value* $\widetilde{y}(t_{m+\frac{1}{2}})$ *is approximated using Euler's method with step* $\frac{1}{2}h$.

The implicit formulae are in general solved by an iterative method, but for linear ODEs the implicit equations reduce to linear equations that can be solved explicitly unless h *is an exceptional value.*

3.2.1 The ϑ-method for ODEs

The ϑ-method is a generalisation of methods such as Euler's explicit and implicit methods and the trapezoidal rule. For the ODE (1.1.1) we have the following definition.

Definition 3.2.1 *The general ϑ-method for (1.1.1) is defined for $\vartheta \in [0,1]$ by the formula*

$$\widetilde{y}(t_{m+1}) \equiv \widetilde{y}_{m+1} = \widetilde{y}_m + h[\vartheta \widetilde{f}_m + (1-\vartheta) f(t_{m+1}, \widetilde{y}_{m+1})], \tag{3.2.10}$$

($m = 0, 1, \cdots$), where $\widetilde{f}_m = f(t_m, \widetilde{y}_m)$ and $\widetilde{y}(t_0) = y_o = y(t_0)$.

Remark 3.2.2 *When $\vartheta = 1$ we obtain from (3.2.10) the form* (3.2.6) *– the Euler explicit (or forward) method for (1.1.3). When $\vartheta = 0$ we obtain from the equation (3.2.10) the form* (3.2.7). *When $\vartheta = \frac{1}{2}$ we obtain from the equation (3.2.10) the form* (3.2.8) *– the trapezoidal method for (1.1.3).*

Example 3.2.2 *Let us consider the ODE (1.1.3), $y'(t) = ay(t)$ for scalar a. Applying, in turn, the Euler forward rule (3.2.6), Euler backward rule (3.2.7) and trapezoidal rule (3.2.8) we obtain, with $h > 0$ and for $m = 0, 1, 2, ...,$ three basic discrete formulae – respectively for $\vartheta = 1$, $\vartheta = 0$, and $\vartheta = \frac{1}{2}$:*

$$\widetilde{y}_{m+1} = (1 + ah)\widetilde{y}_m, \quad \widetilde{y}_{m+1} = \frac{1}{(1-ah)}\widetilde{y}_m, \ \text{and} \ \widetilde{y}_{m+1} = \frac{1 + \frac{1}{2}ah}{1 - \frac{1}{2}ah}\widetilde{y}_m, \tag{3.2.11}$$

including (1.6.9), These are discrete analogues of the ODE (1.1.3). The sequences are defined if, respectively,

$$h > 0, \qquad h > 0 \ \text{and} \ ah \neq 1, \qquad h > 0 \ \text{and} \ ah \neq \frac{1}{2}. \tag{3.2.12}$$

Remark 3.2.3 (Dense output) *Although the primitive ϑ-method described here provides approximate solution values on a mesh $\mathcal{T}_0$ ($\widetilde{y}(t_m) = \widetilde{y}_m$), we sometimes require a densely-defined approximation ('dense-output') ($\widetilde{y}(t)$ for general $t \in \mathbb{R}$). There are various possibilities for obtaining such values and we indicate two:*

1. *We may define $\widetilde{y}(t_m + sh)$ as $(1-s)\widetilde{y}(t_m) + s\widetilde{y}(t_{m+1})$ using linear interpolation;*

2. *We may define $\widetilde{y}(t_m + sh)$ as the solution $\widetilde{y}_{m+s}$ of*

$$\widetilde{y}_{m+s} = \widetilde{y}_m + sh[\vartheta \widetilde{f}_m + (1-\vartheta) f(t_{m+s}, \widetilde{y}_{m+s})], \ s \in (0,1). \tag{3.2.13}$$

Both approaches may be generalised, in principle, to functional differential equations (DDEs etc.).

The sequences corresponding to (3.2.11) when a is replaced by a matrix $A \in \mathbb{R}^{n\times n}$ and we consider $y'(t) = Ay(t)$ are easily written down, and are defined, respectively, for all $h > 0$, for all $h > 0$ when $\det[I - hA] \neq 0$, and for all $h > 0$ when $\det[I - \frac{1}{2}hA] \neq 0$.

Lemma 3.2.1 *For the general ϑ-method applied to $y'(t) = Ay(t)$ (where $A \in \mathbb{R}^{n\times n}$) we obtain, provided* $\det[I-(1-\vartheta)hA] \neq 0$*, the sequence $\{\widetilde{y}_m\}_{m\geq 0}$ satisfying $\widetilde{y}_0 = y(t_0)$ and*

$$\widetilde{y}_{m+1} = [I-(1-\vartheta)hA]^{-1}[I-\vartheta hA]\widetilde{y}_m \quad (m = 0, 1, \cdots). \qquad (3.2.14)$$

Then $\widetilde{y} : \mathcal{T}_0 \to \mathbb{R}^n$ is defined by the relation $\widetilde{y}_m = \widetilde{y}(t_m)$ for $m = 0, 1, 2, \cdots$.

Definition 3.2.2 *(a) The characteristic polynomial for a recurrence relation*

$$u_{m+1} = Mu_m \text{ where } M \in \mathbb{R}^{n\times n}, \text{ and } u_0 \in \mathbb{R}^n \text{ is given} \qquad (3.2.15)$$

is defined to be

$$\chi(\lambda) := \det[\lambda I - M], \qquad (3.2.16)$$

and its zeros $\{\lambda_\ell\}$ are called characteristic values. (b) The sequence $\{u_m\}_{m\geq 0}$ in (3.2.15) *is oscillatory if the function u with domain $\mathcal{T}_0$ and $u(t_m) = u_m$ is oscillatory in the sense of Definition 1.1.3. (c) The recurrence* (3.2.15) *is called oscillatory if all its solutions are oscillatory.*

Lemma 3.2.2 *The recurrence* (3.2.15) *is oscillatory if and only if there is no characteristic value that is real and positive.*

3.2.2 The ϑ-method for numerical integration

Consider the issue of calculating $\int_{t_0}^T g(s)ds$ (for $t_0 < T$). This is equivalent to determining $y(T)$ where $y'(t) = g(T)$ $(t_0 \leq t \leq T)$ and $y(t_0) = 0$. If we select an integer N and write $h := (T - t_0)/N$ then we can calculate an approximate value using the ϑ-method and we obtain

$$\int_{t_0}^T g(s)ds \approx h\{\vartheta g(t_0)+g(t_0+h)+g(t_0+2h)+\cdots(1-\vartheta)g(t_0+Nh)\}. \quad (3.2.17)$$

Here $g(t)$ is evaluated at a set of equally-spaced arguments. It is also possible to use non-uniformly space abscissae. Suppose that each value h_m is positive and

$$t_{m+1} = t_m + h_m (m \in \{0, 1, 2, \cdots, N\}) \text{ and } t_{N+1} = T. \qquad (3.2.18)$$

Then (3.2.17) generalises, with $t_\ell = t_0 + \ell h$ ($\ell \in \mathbb{N}$) to

$$\int_{t_0}^{T} g(s)ds = \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} g(s)ds \approx \sum_{m=0}^{N-1} h_m\{\vartheta g(t_m) + (1-\vartheta)g(t_{m+1})\}. \tag{3.2.19}$$

3.3 The ϑ-method for DDEs

The ϑ-method for ODEs can be adapted to the treatment of a general DDE (1.2.1), i.e., $y'T0 = f(y, y(t), y(t-\tau))$ where $\tau > 0$.

Definition 3.3.1 *Choose h such that $\tau = Nh$ where $N \in \mathbb{N}$. The method of the general form*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + h[\vartheta f(t_m, \widetilde{y}_m, \widetilde{y}_{m-N}) + (1-\vartheta) f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m-N+1}), \tag{3.3.1}$$

for $m \in \{0, 1, \cdots\}$ and $\vartheta \in [0,1]$ is called the ϑ-method for (1.2.1).

Using the conditions in (3.3.1) we obtain different versions of the ϑ methods for (1.2.1):

$$y'(t) = f(t, y(t), y(t-\tau)) \quad (t \geq t_0). \tag{3.3.2}$$

Throughout, N is a positive integer with $Nh = \tau$. When $\vartheta = 1$ we obtain from (3.3.1) the form (3.3.3)

$$\widetilde{y}_{m+1} = \widetilde{y}_m + hf(_m, \widetilde{y}_m, \widetilde{y}_{m-N}) \; m = 0, 1, ...,. \tag{3.3.3}$$

This defines the Euler forward (explicit) method for the DDE (3.3.2). When $\vartheta = 0$ we obtain from (3.3.1) the form (3.3.4) –

$$\widetilde{y}_{m+1} = \widetilde{y}_m + hf(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1-N}) \; m = 0, 1, ...,. \tag{3.3.4}$$

the Euler backward (or implicit) method for the DDE (3.3.2). Finally, when $\vartheta = \frac{1}{2}$ we obtain from (3.3.1) the form

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \frac{h}{2}[f(_m, \widetilde{y}_m, \widetilde{y}_{m-N}) + f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1-N})], \tag{3.3.5}$$

for $m = 0, 1, ...,.$ which defines the trapezoidal method for the DDE (3.3.2).

Example 3.3.1 *Consider $y'(t) = \mu y(t-\tau)$, from (1.2.8), or, equivalently,*

$$y'(t) = \mu_{\natural} y(t-1) \tag{3.3.6}$$

with $\mu_{\natural} = \tau\mu$. Applying, respectively, the Euler forward rule (3.3.3), Euler backward rule (3.3.4) and trapezoidal rule (3.3.5) we obtain,

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \mu_{\natural} h \widetilde{y}_{m-N}, \tag{3.3.7}$$

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \mu_{\natural} h \widetilde{y}_{m+1-N}, \tag{3.3.8}$$

and

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \frac{\mu_{\natural} h}{2}(\widetilde{y}_{m-N} + \widetilde{y}_{m+1-N}), \tag{3.3.9}$$

($m \in \mathbb{Z}$) where N is a positive integer, and $h = 1/N$.

3.4 ϑ-methods for advanced differential equations

Consider (1.3.1), namely the advanced differential equation (ADE)

$$y'(t) = f(t, y(t), y(t+\tau)) \quad (\tau > 0). \tag{3.4.1}$$

Definition 3.4.1 *With $\vartheta \in [0,1]$, the general form of the ϑ-formula for* (3.4.1) *is defined by the formula*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + h[\vartheta f(t_m, \widetilde{y}_m, \widetilde{y}_{m+N}) + (1-\vartheta) f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1+N})], \tag{3.4.2}$$

($m \in \mathbb{Z}$) where $Nh = \tau$ ($N \in \mathbb{N}$). If, under suitable conditions, a sequence $\{\widetilde{y}_m\}$ exists then an approximate solution $\widetilde{y}$ is defined on $\mathcal{T}_0$ on setting $\widetilde{y}(t_m) = \widetilde{y}_m$ ($\ell = 0, 1, 2, \cdots$). The associated method is called the ϑ-method for the ADE (3.4.1).

Definition 3.4.2 *When $\vartheta = 1$ we obtain the Euler forward (explicit) method for the advanced differential equation (1.3.1) from the equation (3.4.2). This is of the form*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + h f(t_m, \widetilde{y}_m, \widetilde{y}_{m+N}), \qquad m \in \mathbb{Z}, \tag{3.4.3}$$

where $N \in \mathbb{N}$ and $h = \tau/N$.

Definition 3.4.3 *When $\vartheta = 0$ we obtain from (3.4.2) the Euler backward (implicit) method for the advanced differential equation (1.3.1). This is of the form*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + hf(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1+N}), \tag{3.4.4}$$

where N is a positive integer and $h = \tau/N$.

Definition 3.4.4 *When $\vartheta = \frac{1}{2}$ we obtain from (3.4.2) the trapezoidal method for the advanced differential equation (1.3.1), of the form*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \frac{h}{2}[f(t_m, \widetilde{y}_m, \widetilde{y}_{m+N}) + f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1+N})], \qquad m \in \mathbb{Z}, \tag{3.4.5}$$

where $h = \tau/N$, $n \in \mathbb{N}$.

Example 3.4.1 *Consider the advanced differential equation (1.3.2),*

$$y'(t) = \mu y(t+1).$$

Using $h - 1/N$ with $N \in \mathbb{N}$, we obtain discrete analogues of this advanced differential equation. Applying the Euler forward rule (3.4.3), Euler backward rule (3.4.4) and trapezoidal rule (3.4.5) we obtain,

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \mu h \widetilde{y}_{m+N}, \tag{3.4.6}$$

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \mu h \widetilde{y}_{m+1+N}, \tag{3.4.7}$$

and

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \frac{\mu h}{2}(\widetilde{y}_{m+N} + \widetilde{y}_{m+1+N}). \tag{3.4.8}$$

3.5 The ϑ-method for mixed type differential equations with delay and advanced terms

All of the preceding cases are subsumed in the general one discussed here for a mixed type functional differential equation (MTFDE).

Definition 3.5.1 *For the MTFDE*

$$y'(t) = f(t, y(t), y(t-\tau), y(t+\tau)), \tag{3.5.1}$$

the ϑ-method is defined by a choice of h and of $\vartheta \in [0,1]$ in the equations

$$\widetilde{y}_{m+1} = \widetilde{y}_m +$$

$$h[\vartheta f(t_m, \widetilde{y}_m, \widetilde{y}_{m-N}, \widetilde{y}_{m+N}) + (1-\vartheta) f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1-N}, \widetilde{y}_{m+1+N}), \tag{3.5.2}$$

where $Nh = \tau$ and $m \in \mathbb{Z}$.

Euler's methods and the trapezoidal method are special cases. When $\vartheta = 1$ we obtain Euler's forward (explicit) method for MTFDE (3.5.1) from (3.5.2),

$$\widetilde{y}_{m+1} = \widetilde{y}_m + hf(t_m, \widetilde{y}_m, \widetilde{y}_{m-N}, \widetilde{y}_{m+N}). \tag{3.5.3}$$

When $\vartheta = 0$ we obtain from (3.5.2)

$$\widetilde{y}_{m+1} = \widetilde{y}_m + f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1-N}, \widetilde{y}_{m+1+N}) \tag{3.5.4}$$

the Euler backward (implicit) method for MTFDE (3.5.1). When $\vartheta = \frac{1}{2}$ we obtain from (3.5.2) the trapezoidal method for MTFDE (3.5.1), namely

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \frac{h}{2}[f(t_m, \widetilde{y}_m, \widetilde{y}_{m-N}, \widetilde{y}_{m+N}) + f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{y}_{m+1-N}, \widetilde{y}_{m+1+N})]. \tag{3.5.5}$$

Example 3.5.1 *Let us consider the MTFDE (1.4.8), that is,*

$$y'(t) = ay(t) + by(t-1) + cy(t+1). \tag{3.5.6}$$

Applying the Euler forward rule (3.5.3), the Euler backward (implicit) rule (3.5.4) and the trapezoidal rule (3.5.5) we obtain, respectively,

$$\widetilde{y}_{m+1} = (1 + ah)\widetilde{y}_m + h(b\widetilde{y}_{m-N+N} + c\widetilde{y}_{m+N+N}), \tag{3.5.7}$$

$$\widetilde{y}_{m+1} = (1 + ah)\widetilde{y}_m + h(b\widetilde{y}_{m+1-N+N} + c\widetilde{y}_{m+1+N+N}), \tag{3.5.8}$$

and

$$\widetilde{y}_{m+1} = (a+\frac{ah}{2})\widetilde{y}_m + \frac{h}{2}\left[b(\widetilde{y}_{m-N+N} + \widetilde{y}_{m+1-N+N}) + c(\widetilde{y}_{m+N+N} + \widetilde{y}_{m+1+N+N})\right], \tag{3.5.9}$$

where $Nh = \tau$, $m = 0, 1, 2, \ldots$. These are discrete analogues of the advanced differential equation (1.4.8).

3.6 The ϑ method for integro-differential equations

For our discussion of the IDE

$$y'(t) = f\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\Big) \tag{3.6.1}$$

in (1.5.1), it is convenient to introduce the notation

$$z_m^{\vartheta^*} = h\Big\{\vartheta^* k(t_m, t_0)\widetilde{y}_0 + \sum_{j=1}^{m-1} k(t_m, t_j)\widetilde{y}_j + (1-\vartheta^*)k(t_m, t_m)\widetilde{y}_m\Big\} \qquad (3.6.2)$$

to denote an approximation to $\int_{t_0}^{t_m} k(t_m, s)y(s)ds$ based on repeated ϑ^* quadrature rules and values $\{\widetilde{y}_\ell\}$.

Definition 3.6.1 *The ϑ-method for (3.6.1) is defined for $\vartheta \in [0,1]$ and some $\vartheta^* \in [0,1]$ by the formulae*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + h\{\vartheta f(t_m, \widetilde{y}_m, \widetilde{z}_m^{\vartheta^*}) + (1-\vartheta) f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{z}_{m+1}^{\vartheta^*})\}. \qquad (3.6.3)$$

For definiteness, we take $\vartheta^* = \vartheta$ unless otherwise stated.

If $\vartheta = \vartheta^* = 1$ in (3.6.3) we have the Euler forward rule, if $\vartheta = \vartheta^* = 0$ in (3.6.3) we have the Euler backward rule and if $\vartheta = \vartheta^* = \frac{1}{2}$ in (3.6.3) we have the trapezoidal rule.

Definition 3.6.2 *When $\vartheta = 1$ we obtain the Euler forward (explicit) method for IDE (3.6.1), of the form*

$$\widetilde{y}_{m+1} = f(t_m, \widetilde{y}_m, \widetilde{z}_m^1). \qquad (3.6.4)$$

Definition 3.6.3 *When $\vartheta = 0$ we obtain the Euler backward (implicit) method for IDE (3.6.1):*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{z}_{m+1}^0). \qquad (3.6.5)$$

Definition 3.6.4 *When $\vartheta = \frac{1}{2}$ we obtain the trapezoidal method for (3.6.1)*

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \frac{h}{2}[f(t_m, \widetilde{y}_m, \widetilde{z}_m^{\frac{1}{2}}).f(t_{m+1}, \widetilde{y}_{m+1}, \widetilde{z}_{m+1}^{\frac{1}{2}})]. \qquad (3.6.6)$$

3.6.1 Reformulation of recurrences in matrix-vector form

Let us consider the DDE ,

$$y'(t) + \mu y(t-\tau) = 0$$

Applying the θ method, where $\tau = Nh$, $h = \frac{\tau}{N}$, we have,

$$\widetilde{y}_{m+1} = \widetilde{y}_n + ah(1-\theta)\widetilde{y}_{m-N} + ah\theta\widetilde{y}_{m-N+1}.$$

Reformulating the recurrence relation as matrix-vector form, we have,

$$\mathbf{u_m} = \begin{pmatrix} \widetilde{y}_m \\ \widetilde{y}_{m-1} \\ \vdots \\ \widetilde{y}_{m-N+1} \end{pmatrix} \text{ and } \mathbf{u_{m+1}} = \begin{pmatrix} \widetilde{y}_{m+1} \\ \widetilde{y}_m \\ \vdots \\ \widetilde{y}_{m-N+2} \end{pmatrix} \tag{3.6.7}$$

where

$$\mathbf{u_{m+1}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & ah(1-\theta) & ah\theta \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} u_m$$

$$u_{m+1} = Au_m \tag{3.6.8}$$

In this way the recurrence can be reformulated as a matrix-vector version.

3.7 Difference equations obtained using numerical methods

Examples in the above sections clearly show that any differential equation would be converted into a discrete form (a difference equation) on the application of $\vartheta-$methods. Therefore there is a close link between difference equations and numerical methods for differential equations. So discretisation for differential equations is a potential source of difference equations involving approximate solutions $\widetilde{y}_m$ which can be computed by an appropriate algorithm or computer code.

3.7.1 Discretisation techniques

Consider

$$y'(t) = \int_{-1}^{0} y(t - r(\varsigma))dq(\varsigma) \tag{3.7.1}$$

(where r is a real continuous non-negative function on $[-1, 0]$ and q is a real function of bounded variation on $[-1, 0]$), This equation has been used to illustrate many fundamental ideas. Numerical methods to solve problems of the form (3.7.1) can be based on a simple combination of a differential equation solver and a quadrature rule (see [23], [85], [88], [109]). One could apply, for example, a linear multi-step method or a Runge-Kutta method for solving the differential equation. We use the linear $\vartheta-$methods which can be expressed either as a linear multi-step method or as a Runge-Kutta method. These are convenient because they illustrate key features of both types of method and because they have simple natural quadrature rule analogues. The resulting equations take the form of a discrete Volterra equation or difference equation. We can give theoretical results that cover more general methods too.

To start with, a simple constant step-size discretisation of (3.7.1) is considered. Let us suppose $M \in \mathbb{N}$, $h = \frac{1}{M} > 0$ and $\widetilde{y}_m \approx y(t_m)$ as usual and we write

$$\frac{\widetilde{y}_{m+1} - \widetilde{y}_m}{h} = h \sum_{j=-M}^{0} w_j \tilde{q}(j) \tilde{y}(t_{m-\tilde{k}(j)}). \tag{3.7.2}$$

Here the values w_j are quadrature weights, $\tilde{q}$ is a weight function based on the original measure q in (3.7.1) and $\tilde{k}(j) = k(jh)$. The function $\widetilde{y}$ is a dense output of the solution process. In other words $\widetilde{y}(t_j) = \widetilde{y}_j$ for $j \in \mathbb{N}$ and $\widetilde{y}(t)$ is defined by interpolating the values $\{\widetilde{y}(t_j)\}$ when its argument t is not in the set $\{t_\ell\}$. The interpolation will be based on some combination of the values $\widetilde{y}_j$ at neighboring points.

In the case of multi-delay equations with constant delays, the step length may be chosen so that interpolation becomes unnecessary whenever the different delays are related appropriately. The equation (3.7.2) provides an expression for $\widetilde{y}_{m+1}$ as a function of $\widetilde{y}_m, \widetilde{y}_{t_{m-1}}, \ldots, \widetilde{y}_{m-M}$.

Of course, here a very simple one-step solver can be used for the differential equation. If a multi-step method is chosen then there will be a much more complicated expression, but we will still retain the same overall idea, and a discrete equation of the same overall form will be obtained. The same observation would apply if a backward difference or central difference approach is adopted to approximating the left hand side of (3.7.1).

For a constant step size $h > 0$ and all the usual notation, let $\vartheta \in [0,1]$. For the differential equation

$$y'(t) = \mathcal{F}(t, y(t)), y(0) = y_0 \tag{3.7.3}$$

the approximate solution given by the linear $\vartheta - method$ is given by

$$\widetilde{y}_{m+1} = \widetilde{y}_m + h(1-\vartheta)F_m + h\vartheta F_{m+1}, \quad F_\ell := F(t_\ell, \widetilde{y}_\ell). \tag{3.7.4}$$

The corresponding quadrature rule approximates the integral

$$\int_{mh}^{(m+1)h} f(s)ds \approx h(1-\vartheta)f(mh) + h\vartheta f((m+1)h). \tag{3.7.5}$$

To integrate over an interval of length unity ($[0,1]$, say) we may take a step $h = 1/M$, write

$$\int_0^1 f(s)ds = \sum_{k=0}^{M-1} \int_{kh}^{(k+1)h} f(s)ds \tag{3.7.6}$$

and approximate each integral over an interval $[kh, (k+1)h]$ using (3.7.5). The approximation simplifies to read

$$\int_0^1 f(s)ds = h(1-\vartheta)f_0 + h\sum_{k=1}^{M-1} f_k + h\vartheta f_M \quad (h = 1/M). \tag{3.7.7}$$

Example 3.7.1 *We consider examples based on choices of r and q in the equation $y'(t) = \int_{-1}^0 y(t-r(\varsigma))dq(\varsigma)$ given in* (3.7.1). *The simplest case arises*

$$y'(t) = \int_{-1}^0 y(t+\varsigma)d\varsigma \tag{3.7.8}$$

with $r(s) = -s$, $q(s) = s$. Applying a discrete scheme based on an Euler rule ($\vartheta = 0$) for the differential equation and the corresponding forward rectangular rule (the repeated explicit Euler rule – $\vartheta = 0$) for the quadrature we then obtain

$$\widetilde{y}_{m+1} = \widetilde{y}_m + 2h^2 \sum_{j=0}^{M-1} jh\widetilde{y}_{m-j} \quad (h = 1/M), \tag{3.7.9}$$

This is an elementary finite order difference scheme.

For a further example, consider the equation

$$y'(t) = 2\int_{-1}^0 y(t+\varsigma)\varsigma d\varsigma \tag{3.7.10}$$

This is of the form (3.7.1) *with*

$$q(\varsigma) = \varsigma^2, r(\varsigma) = -\varsigma.$$

However the example

$$y'(t) = \int_{-1}^0 y(t-\varsigma^2)dq(\varsigma) \text{ where } q(\varsigma) = \varsigma^2 \tag{3.7.11}$$

(to which we return later) shows how the discretisation of apparently simple equations may become unexpectedly complicated. Indeed, we write $y'(t) =$

$\int_{-1}^{0} y(t-\varsigma^2)dq(\varsigma)$ as $y'(t) = 2\int_{-1}^{0} y(t-\varsigma^2)\varsigma d\varsigma)$ and direct application of a simple discrete scheme based on the explicit Euler formula gives us

$$\widetilde{y}_{m+1} = \widetilde{y}_m + 2h^2 \sum_{j=0}^{M-1} jh\tilde{y}(t_m - j^2 h), \tag{3.7.12}$$

It is easy to see that we need to interpolate the values of $\tilde{y}$ since $t_m - j^2h$ will not always be one of the values $\{t_\ell\}$. However, depending on the function function r, we may be able to avoid this problem (a) by restricting the choice of h or (b) by using a non-uniform grid for discretising the integral (thereby using only values $\tilde{y}_\ell$). Both approaches can be applied for the present example.

3.7.2 Some remarks on quadrature

An early discussion of the definition of Riemann-type integration can be found in [87]. We recall the standard definition of a Riemann integral (see, also, [1], which yields

$$\int_0^1 f(s)ds := \sum_{j=1}^{N} \{\sigma_{N,j} - \sigma_{N,j-1}\} f(\sigma'_{N,j}) = o(1) \text{ as } \max_j \{\sigma_{N,j+1} - \sigma_{N,j}\} \to 0 \tag{3.7.13}$$

where, for each N,

$$0 = \sigma_{N,0} \leq \sigma'_{N,1} \leq \sigma_{N,1} \leq \cdots \leq \sigma'_{N,N} \leq \sigma_{N,N} = 1. \tag{3.7.14}$$

Any particular choice (3.7.14) can be used to give an approximation. Taking $\sigma'_{N,j} = (j-1)h$ (for $j = 1, 2, \cdots, N$ and $h = 1/(N-1)$) we can obtain as particular examples the composite versions of the ϑ rules (compare (3.7.7)).

Remark 3.7.1 *(a) Given low-order smoothness on f (assume Lipschitz-continuity, or a bounded first derivative, for f) the term $o(1)$ in (3.7.13) is actually $\mathcal{O}(h)$. Thus the errors in any repeated ϑ rule (3.7.7)) are $\mathcal{O}(h)$ under the mildest of conditions.*

(b) However, with sufficiently high-order differentiability of f, the quadrature errors are $\mathcal{O}(h^{2r})$ for any r when f is periodic of period unity. This result follows from formulae of Euler-Maclaurin type. The result is of interest when one approximates an integral around a simple closed contour, since the integrand is periodic – with period equal to the length of the closed contour – when the variable of integration is distance along the contour from some fixed point.

We turn, now, to the approximation of a class of Riemann-Stieltjes integrals $\int_0^1 f(s)d\Omega(s)$. The classical analysis literature contains more than one definition of the Riemann-Stieltjes integral, but this need not concern us as we consider a subset of the possible choices of Ω. Moreover, there is little in the classical numerical analysis literature on this topic, and we shall restrict ourselves to elementary approximation formulae.

Lemma 3.7.1 *(a) If f is continuous and Ω is of bounded variation on $[0,1]$, then the Riemann-Stieltjes integral $\int_0^1 f(s)d\Omega(s)$ exists. Furthermore, (b) if Ω has a continuous derivative on $[0,1]$ we have*

$$\int_0^1 f(s)d\Omega(s) = \int_0^1 f(s)\Omega'(s)ds \tag{3.7.15}$$

$$= \sum_{j=1}^{N}\{\Omega(\sigma_{N,j}) - \Omega(\sigma_{N,j-1})\}f(\sigma'_{N,j}) + o(1) \text{ as } \max_j\{\sigma_{N,j+1} - \sigma_{N,j}\} \to 0. \tag{3.7.16}$$

where, for each N,

$$0 = \sigma_{N,0} \leq \sigma'_{N,1} \leq \sigma_{N,1} \leq \cdots \leq \sigma'_{N,N} \leq \sigma_{N,N} = 1. \tag{3.7.17}$$

The second result (b) follows because each term $\{\Omega(\sigma_{N,j}) - \Omega(\sigma_{N,j-1})\}$ in the sum can be expressed as $(\sigma_{N,j} - \sigma_{N,j-1}) \times \Omega'(\widehat{\sigma}_{N,j})$ where $\widetilde{\sigma}_{N,j} \in [\sigma_{N,j-1}, \sigma_{N,j}]$ and we can invoke (3.7.13) with the choice ${\sigma'}_{N,j} = \widetilde{\sigma}_{N,j}$.

3.7.3 Quadrature for functional equations

In the context of the discretisation of functional differential equations that involve integrals, we consider formulae that generate approximations $\widetilde{y}_m \approx y(t_m)\}$ where $t_m = t_0 + mh$. It is appropriate to ask whether quadrature formulae that approximate the integrals can be expressed directly in terms of values $\{\widetilde{y}_m\}$.

To assist the discussion, let us focus on

$$y'(t) = \int_{-1}^{0} y(t - r(\varsigma))dq(\varsigma) \tag{3.7.18}$$

that appeared in (3.7.1). Let us seek an approximation to $\int_{-1}^{0} y(t_m - r(\varsigma))dq(\varsigma)$ that involves only values $\widetilde{y}_m\}$. If we base this approximation on (3.7.16) we will have an approximation

$$\int_{-1}^{0} \widetilde{y}(t_m - r(\varsigma))dq(\varsigma) \approx \sum_{j=1}^{N}\{q(\sigma_{N,j}) - q(\sigma_{N,j-1})\}\widetilde{y}(t_m - r(\sigma'_{N,j})) \tag{3.7.19}$$

in which the $\sigma-$ and σ'-values satisfy (3.7.17), and we are looking for such a choice subject to the condition $t_m - r(\sigma'_{N,j}) \in \{t_\ell\}$; thus, we require

$$r(\sigma'_{N,j}) = \kappa_{m,j} h \text{ for some } \kappa_{m,j} \in \mathbb{Z}_-. \tag{3.7.20}$$

At the same time, it is desirable, on the grounds of consistency, that

$$\max_j \{\sigma_{N,j+1} - \sigma_{N,j}\} \to 0 \text{ as } h \to 0. \tag{3.7.21}$$

Remark 3.7.2 *Finding a suitable approximation may not always be possible, but an alternative procedure in which $\widetilde{y}(t_m - r(\sigma'_{N,j}))$ is expressed (using some interpolation formula) in terms of $\{\widetilde{y}_\ell\}$ is always open to us.*

Returning to (3.7.19), it is not a big step to deduce the related approximations

$$\int_{-1}^{0} \widetilde{y}(t_m - r(\varsigma))dq(\varsigma) \approx \sum_{j=1}^{N} \{\sigma_{N,j}) - \sigma_{N,j-1}\} \widetilde{y}(t_m - r(\sigma'_{N,j}))q(\sigma_{N,j}) \tag{3.7.22}$$

or

$$\int_{-1}^{0} \widetilde{y}(t_m - r(\varsigma))dq(\varsigma) \approx \sum_{j=1}^{N} \{\sigma_{N,j} - \sigma_{N,j-1}\} \widetilde{y}(t_m - r(\sigma'_{N,j}))q(\sigma_{N,j-1}) \tag{3.7.23}$$

Example 3.7.2 *(a) Consider the approximation of $\int_{-1}^{0} y(t_m - s^2)sds$ which corresponds to $r(s) = s^2$ and $q(s) = \frac{1}{2}s^2$ in the discussion above. Pick*

$$\sigma_j = j\sqrt{h} \textit{ where for } j \in \{0, 1, 2, \cdots M\} \textit{ and } h = 1/M^2. \tag{3.7.24}$$

For convenience we shall here set $N = M^2$. *Then* (3.7.22) *combined with the additional approximation $y(t) \approx \widetilde{y}(t)$ provides us with the approximation*

$$\sum_{k=0}^{N-1} \{\sqrt{(k+1)h} - \sqrt{kh}\} \widetilde{y}(t_{m-k^2})\sqrt{kh} = h \sum_{k=0}^{N-1} \{\sqrt{(k+1)k} - k\} \widetilde{y}(t_{m-k^2}). \tag{3.7.25}$$

Observe that $\sigma_{N,j} - \sigma_{N,j-1} \to 0$ as $h \to 0$.

(b) Next, consider the approximation of $\int_{-1}^{0} y(t_m - s^4)s^3 ds$ which corresponds to $r(s) = s^2$ and $q(s) = \frac{1}{3}s^4$ in the discussion above. Pick, in place of (3.7.24),

$$\sigma_j = j\sqrt[4]{h} \textit{ where for } j \in \{0, 1, 2, \cdots M\} \textit{ and } h = 1/M^4 \tag{3.7.26}$$

(again, $\sigma_{N,j} - \sigma_{N,j-1} \to 0$ as $h \to 0$) and by a similar process to the one above we obtain

$$\int_{-1}^{0} y(t_m - s^4)s^3 ds \approx h \sum_{k=0}^{N-1} \{\sqrt[4]{(k+1)k^3} - k\}\widetilde{y}(t_{m-k^4}). \qquad (3.7.27)$$

For convenience we have here set $N = M^4$ and $h = 1/N$.

3.8 IDVEs with oscillatory solutions

In this section we discuss qualitative behaviour of solutions of first order DVEs. These have oscillatory solutions given certain conditions. We generate a first order linear autonomous homogeneous DVE from the delay IDE (1.5.12) and having the generic form

$$\widetilde{y}_{m+1} = \widetilde{y}_m + \sum_{j=0}^{N-1} p_j \widetilde{y}_{m-j} = 0 \qquad (3.8.1)$$

where $\{p_k\}$ and h are constants and $\widetilde{y}_m \in \mathbb{R}^n$. Every solution of the equation (3.8.1) oscillates if and only if the characteristic values of the equation do not include at least one positive real number.

3.8.1 Generating DVEs from delay IDEs using numerical methods

Let us consider the non-oscillatory IDE (2.24), namely

$$y'(t) = 2\int_{-1}^{0} y(t - \theta^2)\theta d\theta$$

Refer back to the discussion of Section 3.7.3. Let us suppose $h = 1/M^2$, and we derive equations for $\widetilde{y}_m \approx y(t_m)$. by applying the explicit Euler rule to discretise the derivative and discretising the integral using (3.7.25). We have (on simplifying)

$$\widetilde{y}_{m+1} - \widetilde{y}_m - 2h^2 \sum_{k=0}^{M^2-1} (\sqrt{(k+1)k} - k)\widetilde{y}_{m-k} = 0. \qquad (3.8.2)$$

Similarly let us consider the non-oscillatory IDE of the form,

$$y'(t) = 4\int_{1}^{0} y(t - \theta^4)\theta^3 d\theta \qquad (3.8.3)$$

If the Euler forward rule is applied to the DDE (3.34), we have, on using the discretisation (3.7.27) and simplifying,

$$\widetilde{y}_{m+1} - \widetilde{y}_m = 4h^2 \sum_{k=0}^{M^4-1} (\sqrt[4]{(k+1)k^3} - k)\widetilde{y}_{m-k} \quad (h = 1/M^4). \tag{3.8.4}$$

In the next chapter the most sophisticated analytical and numerical methods will be studied to develop the analysis for the new numerical and analytical methods.

Chapter 4

Some Sample Differential Equations with Oscillatory Solutions

4.1 Introduction

This Chapter introduces some differential equations with oscillatory solutions and provides examples of the types of equation that will be used later in the thesis. We endeavour to set the equations that we consider in context. The basic constant-coefficient homogeneous equations that we have introduced, that is, the scalar cases

$$y'(t) = ay(t), \quad y'(t) = b\, y(t-\tau),\ y'(t) = c\, y(t+\tau), \tag{4.1.1a}$$

$$y'(t) = a\, y(t) + b\, y(t-\tau),\ y'(t) = a\, y(t) + b\, y(t-\tau) + c\, y(t+\tau), \tag{4.1.1b}$$

or the rather more general systems (with $y(t) \in \mathbb{R}^n$)

$$y'(t) = Ay(t), \quad y'(t) = B\, y(t-\tau),\ y'(t) = C\, y(t+\tau) \tag{4.1.2a}$$

$$y'(t) = Ay(t) + By(t-\tau) + Cy(t+\tau), \tag{4.1.2b}$$

can be generalised in a number of ways. First, we may increase the number of deviating arguments, as in

$$y'(t) = Ay(t) + \sum_j B_j y(t-\tau_j) + \sum_j C_j y(t+\tau_j^\star), \tag{4.1.3}$$

(where each τ_j and each $\tau_j^\star$ is non-negative). Second, we may replace constants by variables, as in

$$y'(t) = A(t)y(t) + \sum_j B_j(t)y(t-\tau_j(t)) + \sum_j C_j(t)y(t+\tau_j^\star(t)). \tag{4.1.4}$$

Then, we may replace sums by Riemann integrals or Riemann-Stieltjes integrals, as in

$$y'(t) =$$

$$A(t)y(t) + \sum_j B_j(t)y(t - \tau_j(t)) + \sum_j C_j(t)y(t + \tau_j^*(t)) + \int_{-1}^{1} [d\eta(s)]y(t + r(s)), \tag{4.1.5}$$

or, in the scalar case

$$y'(t) = a(t)y(t) + \sum_j b_j(t)y(t - \tau_j(t)) + \sum_j c_j(t)y(t + \tau_j^*(t)) + \int_{-1}^{1} y(t + r(s))d\Omega(s). \tag{4.1.6}$$

Remark 4.1.1 *Non-linear versions can sometimes be related to linear ones through a process of linearization, but, for ill-posed equations, this linearization may prove difficult to justify. The step away from equations with instantaneous or retarded arguments to those where arguments may be advanced presents challenges that have been addressed in the literature.*

As part of our orientation, we here quote from Krisztin's paper [90], of 2000. In that paper, an example is given to show that the linear autonomous functional differential equation of mixed type $y'(t) = \int_{-1}^{1}[d\eta(s)]y(t+s)$ may have a non-oscillatory solution in spite of the nonexistence of real roots of its characteristic equation. However, under a regularity condition on η at 1, exponential boundedness is shown for the non-oscillatory solutions. The following quote (taken from [90], edited and adapted to our notation), is instructive. It is assumed that η is an $n \times n$ matrix-valued function of bounded variation.

> *Recall that for linear autonomous FDEs of retarded type of the form $y'(t) = \int_{-1}^{0}[d\eta(s)]y(t+s)$, the Cauchy problem for positive times is well-posed. There is an exponential bound on the growth of the solutions at ∞. Such a bound is related to the fact that there is an upper bound for the real parts of the roots of the characteristic equation* $\det[\lambda I - \int_{-1}^{0} \exp(\lambda s) d\eta(s)]$ *… Furthermore, the existence of a non-oscillatory solution is equivalent to the existence of a real root of the characteristic equation. The situation for $y'(t) = \int_{-1}^{1}[d\eta(s)]y(t+s)$ is, in general, strikingly different. The Cauchy problem is not well posed for both positive and negative times. There is no upper bound for the real parts of the roots of the characteristic equation. As a consequence, there are solutions growing faster than any exponential as $t \to \infty$. Series*

representations for all solutions ... also are not known. ... First we give an example showing that $y'(t) = \int_{-1}^{1}[d\eta(s)]y(t+s)$ may have a non-oscillatory solution in spite of the fact that the corresponding characteristic equation has no real roots. The function grows faster than any exponential as $t \to \infty$. This example provides a counterexample to the conjecture that the oscillation of all solutions of $y'(t) = \int_{-1}^{1}[d\eta(s)]y(t+s)$ can be characterised by the nonexistence of real roots of the characteristic equation $\det[\lambda I - \int_{-1}^{1}\exp(\lambda s)d\eta(s)] = 0$. *For our second result it is assumed that η satisfies a weak non-degeneracy condition at $s = 1$. Then it is proved that, although there is no exponential bound for all solutions, the non-oscillatory solutions are exponentially bounded at ∞ ....*

We return to related issues in §4.4 (in particular Remark 4.5.1), and refer to [90] for additional detail.

In what follows, we restrict ourselves to scalar equations and to a discussion of (4.1.1) supplemented by some particular examples that fit in with the generalisations indicated above. To be specific, in the following sections we consider the examples shown:

$$2y''(t) + 3y'(t) + 4y(t) = 0, \quad t \geq 0.$$

$$y'(t) = 3y(t-1), \quad t \geq 0.$$

$$y'(t) = -y(t-1), \quad t \geq 0.$$

$$y'(t) = 2\int_{-1}^{0} y(t-s^2)s\,ds, \quad t \geq 0.$$

$$y'(t) = \frac{-1}{a}\int_{-1}^{0} y(t - \frac{1}{5} + \theta)d\theta, \quad t \geq 0, a > 0.$$

$$y'(t) + y(t-1) + y(t+1) = 0, \quad t \geq 0.$$

$$y'(t) = \int_{-1}^{1} y(t+s)ds + y(t+1), \quad t \geq 0.$$

etc.

4.1.1 Oscillation

Before we proceed, we recall from Chapter 1, Section 1.1 the definitions 1.1.3 of an oscillatory solution and of an oscillatory equation.

Definition 4.1.1 *If $\mathcal{T}_0 \subseteq [t_0, \infty)$ is a set of real numbers with no finite upper bound, then a real-valued function u of $t \in \mathcal{T}_0$ is said to be oscillatory (or oscillatory about zero) if there does not exists a value $T \in [t_0, \infty)$ such that either*

$$u(t) > 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty) \text{ or } u(t) < 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty). \tag{4.1.7}$$

A function that is not oscillatory is called non-oscillatory: that is, there does exists a value $T \in [t_0, \infty)$ such that either

$$u(t) > 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty) \text{ or } u(t) < 0 \text{ for } t \in \mathcal{T}_0 \cap [T, \infty). \tag{4.1.8}$$

A function is either oscillatory about a value k or non-oscillatory about a value k if the function with values $u(t) - k$ is respectively oscillatory or non-oscillatory.

Now we recall what we will mean by an *oscillatory equation.*

Definition 4.1.2 *A scalar differential equation is said to be an oscillatory equation if and only if all the solutions of the equation are oscillatory functions in the sense of Definition 4.1.1.*

As we have seen, for the cases of certain linear equation, the presence (or otherwise) of non-oscillatory solutions can be studied by considering the characteristic equation. We recall the following result

Lemma 4.1.1 *Let $\mathcal{L}$ be a linear differential operator with constant coefficients and deviating argument such that all the characteristic values (eigenvalues) of $\mathcal{L}$ are complex then the equation $\mathcal{L}y = 0$ is oscillatory. If one or more of the characteristic values is real then the equation is non-oscillatory.*

The proof is immediate, simply by writing the solution as a linear combination of eigenfunctions and setting all but one coefficient to zero.

4.2 ODEs with oscillatory solutions

In Section 1.1 and in Lemma 1.1.5 we discuss qualitative behaviour of first order and second order ODEs. First order scalar linear ODEs with real coefficients have no oscillatory solutions but second-order real ODEs may have oscillatory solutions.

Example 4.2.1 *Consider the second order linear autonomous homogeneous differential equation obtained by setting $a = 2$, $b = 3$ and $c = 4$ in the equation (1.1.17), i.e.,*

$$2y''(t) + 3y'(t) + 4y(t) = 0, \tag{4.2.1}$$

where $y(t) \in \mathbb{R}$ and we have initial values $y(t_0) = y_0$, $y'(t) = y'_0$. According to the Lemma 1.1.6, every solution of the equation oscillates (and we would then conclude that the equation is oscillatory) if and only if the characteristic values of the equation are complex with non-zero imaginary part.

Remark 4.2.1 *For equation (4.2.1), the characteristic equation can be written*

$$2\lambda^2 + 3\lambda + 4 = 0 \tag{4.2.2}$$

Therefore the characteristic values are $\lambda_{1,2} = \dfrac{-3 \pm i\sqrt{23}}{4}$, and have non-zero imaginary part. So, according to Lemma 1.1.7 all solutions of the equation (4.2.1) oscillate.

4.3 DDEs with oscillatory solutions

In Chapter 1, Section 1.2, we discuss the qualitative behaviour of first order DDEs. First order linear autonomous DDEs have oscillatory solutions under conditions that can be determined.

Example 4.3.1 *Let us consider the first order scalar linear autonomous homogeneous delay differential equation obtained on obtained by setting $\mu = 3$, $\tau = 1$ in (1.2.8), namely:*

$$y'(t) = 3y(t-1), \text{ with initial function } y(t) = \phi(t) \text{ for } t_0 - 1 \leq t \leq t_0, \tag{4.3.1}$$

where $y(t) \in \mathbb{R}$. The characteristic function is $\lambda - 3\exp(-\lambda)$. At least one or more solutions of the equation does not oscillate *if and only if one or more of the characteristic values of the equation is* real *and the equation is then a non-oscillatory DDE.*

Example 4.3.2 *Consider the DDE (4.3.1) which has characteristic function $\lambda - 3\exp(-\lambda)$. The related quasi-polynomial*

$$\chi_{aux}(\lambda) := \lambda\exp(\lambda) - 3. \tag{4.3.2}$$

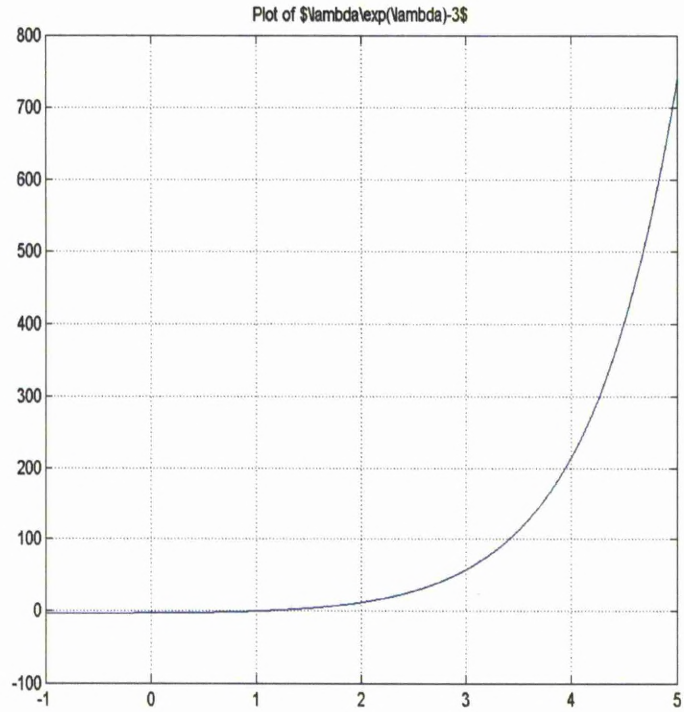

Figure 4.3.1: Results for Example 4.3.1: Plot of the auxiliary characteristic function (for (4.3.1)) defined in (4.3.2)

is an an auxiliary characteristic function, and $\chi_{aux}(1) < 0$, $\chi_{aux}(1.1) > 0$ *so* χ_{aux} *has one real zero. Thus, at least one solution of the DDE (4.3.1) does not oscillate, so that equation (4.3.1) is a non-oscillatory DDE.*

The graph of the auxiliary function (Figure 4.3.1) shows evidence of the real zero which confirms that the DDE (4.3.1) is a non-oscillatory equation.

Example 4.3.3 *Now consider a first order scalar linear autonomous homogeneous delay differential equation obtained by setting* $\mu = -1$, $\tau = 1$ *in (1.2.8), that is,*

$$y'(t) = -y(t-1), \text{ with initial function } y(t) = \phi(t) \text{ for } t_0 - 1 \leq t \leq t_0, \tag{4.3.3}$$

where $y(t) \in \mathbb{R}$*. According to Lemma 1.2.3, every solution of the equation oscillates if and only if none of the characteristic values of the equation are*

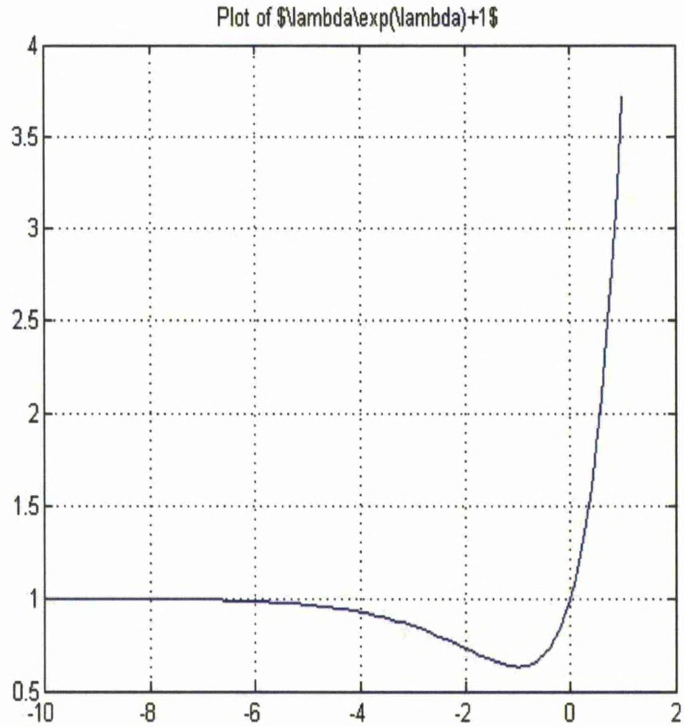

Figure 4.3.2: Results for Example 4.3.3: Plot of auxiliary characteristic function (4.3.4) for (4.3.3)

real and the equation is then said to be an oscillatory DDE.

For the DDE (4.3.3)

$$\chi_{aux}(\lambda) := \lambda \exp(\lambda) + 1 \tag{4.3.4}$$

defines an auxiliary function. The function (4.3.4) satisfies $\chi_{aux}(\lambda) > 0$, where $\lambda \in \mathbb{R}$. So all the solutions of the DDE (4.3.3) oscillate and the equation (4.3.3) is an oscillatory DDE. The graph of the auxiliary characteristic polynomial (4.3.4) shows evidence of the absence of the real zeros, confirming that the DDE (4.3.1) is an oscillatory equation.

4.3.1 Determination of an upper bound on the real part of the characteristic roots

Definition 4.3.1 *The expression 'dominant'characteristic root will in this section refer to the characteristic value or values which has (have) the largest real part.*

Lemma 4.3.1 *Consider the usual prototype DDEs (1.2.6) and (1.2.8), namely*

$$y'(t) = ay(t) + by(t - \tau), \text{ and } y'(t) = \mu y(t - 1). \tag{4.3.5}$$

(a) These two equations are equivalent under an appropriate transformation of variables. (b) For $\mu \in [\frac{-1}{e}, \infty)$ *there is a real characteristic root while for* $\mu \in (-\infty, \frac{-1}{e})$*, there are no real characteristic values.*

Consider the DDE $y'(t) = \mu y(t-1)$ from (1.2.8). The function $y(t) = \exp(\lambda t)$ satisfies the above equation if

$$\lambda \exp(\lambda) = \mu. \tag{4.3.6}$$

The characteristic function is $\chi(\lambda) = \lambda - \mu \exp(-\lambda)$ and the characteristic values for (1.2.8) are solutions of the equation $\lambda e^{\lambda} = \mu$. Writing $\lambda = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$, we obtain from(4.3.6) the two equations

$$\alpha \exp(\alpha) \cos\beta - \beta \exp(\alpha) \sin\beta = \mu \tag{4.3.7}$$

and

$$\alpha \exp(\alpha) \sin\beta + \beta \exp(\alpha) \cos\beta = 0 \tag{4.3.8}$$

which yield

$$\mu = -\frac{\beta \exp(\alpha)}{\sin\beta} \tag{4.3.9}$$

$$\exp(\alpha) = -\frac{\mu \sin\beta}{\beta} \tag{4.3.10}$$

Taking the natural logarithm of both sides of the equation (4.3.10) we have,

$$\alpha = \ln\left(-\frac{\mu \sin\beta}{\beta}\right)$$

From equation (4.3.8), we obtain,

$$\alpha = -\beta \cot\beta \tag{4.3.11}$$

and from (4.3.10), we obtain,

$$\beta = -\mu \exp(-\alpha) \sin\beta. \tag{4.3.12}$$

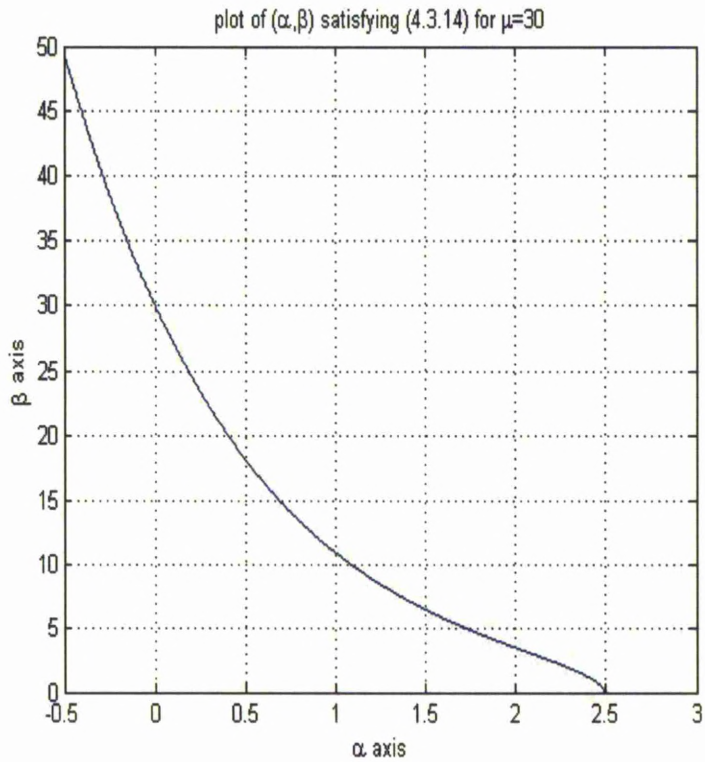

Figure 4.3.3: Results of DDE 1.2.8: Plot of 4.3.14 with positive imaginary part for $\mu = 30$

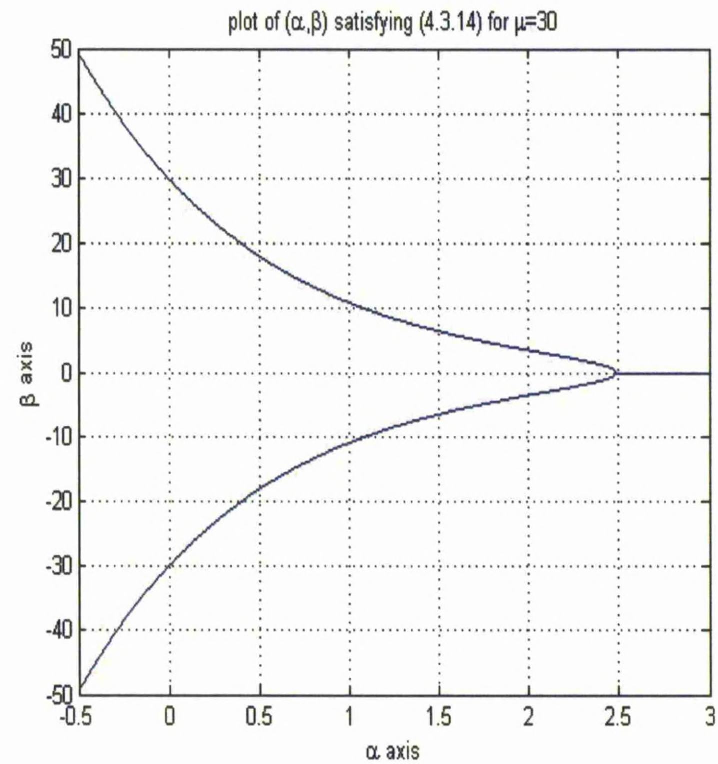

Figure 4.3.4: Results of DDE 1.2.8: Plot of 4.3.14 with positive and negative imaginary parts for $\mu = 30$

Equations (4.3.11) and (4.3.12) can be combined, giving

$$\mu = -\frac{\beta}{\sin(\beta)}\exp(\alpha) = -\frac{\beta}{\sin(\beta)}\exp\big(-\beta\cot(\beta)\big), \tag{4.3.13}$$

where $\sin(\beta) \neq 0$. Equation (4.3.13) can be written

$$\beta = \pm\sqrt{\left(\frac{\mu}{\exp(\alpha)}\right)^2 - \alpha^2} \tag{4.3.14}$$

$$\beta^2 = \left(\frac{\mu}{\exp(\alpha)}\right)^2 - \alpha^2 \tag{4.3.15}$$

Since β must be real, $\beta^2 \geq 0$ and this imposes an upper limit on α, say α^*_μ dependent upon μ. So the growth rate of the eigenfunctions is bounded by $\exp\big(\alpha^*_\mu t\big)$. The growth rate of the eigenfunctions is $\exp\big(\alpha t\big)$ and oscillation rate is governed by β. The graph of β against α satisfying 4.3.14 show that the greatest value of α with $\beta^2 \geq 0$ is approximately 2.5.

4.4 Delay integro-differential equations with oscillatory solutions

We shall consider delay IDEs and mixed IDEs (with delayed and advanced terms). Here we wish to study some examples of oscillatory IDEs. In Chapter 1, Section 1.5.2, we discussed the qualitative behaviour of first order IDEs with delay. First order IDEs with delay term may have oscillatory solutions under appropriate conditions. We shall consider (1.5.12), namely

$$y'(t) = \int_{-1}^{0} y(t - \tau(s))d\Omega(s)$$

Example 4.4.1 *Let us consider the first order linear autonomous homogeneous Delay IDE obtained by setting,* $\tau(s) = s^2$ *and* $\Omega(s) = s^2$ *in the equation (1.5.12),*

$$y'(t) = 2\int_{-1}^{0} y(t - s^2)sds \tag{4.4.1}$$

where $y(t) \in \mathbb{R}$, $r(\varphi)$ *is a positive real continuous function on* $[-1, 0]$ *and* $q(\varphi)$ *is a real function of bounded variation on* $[-1, 0]$.

Lemma 4.4.1 *By way of illustration of the fundamental analytical ideas, let us consider the characteristic polynomial of the equation (4.4.1)*

$$\chi(\lambda) := \lambda - 2\int_{-1}^{0} \exp(-\lambda\varphi^2)\varphi d\varphi \tag{4.4.2}$$

where $r(\varphi)$ *is a real continuous non-negative function on* $[-1, 0]$ *and* $q(\varphi)$ *is a real function of bounded variation on* $[-1, 0]$. *Equation (4.4.1) is oscillatory if and only if* $\chi(\lambda)$ *is not equal to zero for* $\lambda \in \mathbb{R}$ *(see [47]).*

Equations (1.5.12) generalise single and multi-term delay differential equations and various delay integro-differential equations. They are sufficiently complicated to exhibit the range of behaviours we need to study while remaining tractable to analysis. In particular the underlying linear nature of the problem makes the work amenable to analysis using characteristic values (see [47]).

The result of the above Lemma 4.4.1 can be found in the work of Krisztyn [89] (see also [63], [64], [120]) who also develop additional criteria to identify oscillatory equations) and can be derived by the usual approach of searching for solutions of (1.5.12) that take the form $e^{\lambda t}$.

Example 4.4.2 *A broad class of examples takes the form*

$$r(\varsigma) = \varsigma^n, \qquad q(\varsigma) = c\varsigma^m, \text{ for some } c \in \mathbb{R} \text{ and } n, m \in \mathbb{N}. \tag{4.4.3}$$

When n *is even,* r *is non-negative and such problems are found to be non-oscillatory (see [47]). When* n *is odd, the equation is advanced and goes beyond the scope of the present section. Nevertheless, the approach can be extended to this case too and this will be explored in a sequel.*

Example 4.4.3 *If we take a sub-set of the problems given in Example 4.4.2 of the form*

$$r(\varsigma) = \varsigma^n, \text{ for some even } n \in \mathbb{N}, \quad dq(\varsigma) = r'(\varsigma)d\varsigma, \tag{4.4.4}$$

then a family of equations will be obtained which are all non-oscillatory (see [47])

Example 4.4.4 *Pinelas (see [119]) considered the equation*

$$y'(t) = \frac{-1}{a}\int_{-1}^{0} y(t - \frac{1}{5} + s)ds \tag{4.4.5}$$

and showed for some constant $a > 0$, *the equation is oscillatory. We look at it numerically in Chapter 6.*

Lemma 4.4.2 *Oscillatory and non-oscillatory delay IDEs can be expressed by*

$$y'(t) = j\int_{-1}^{0} y(t-\alpha^{2j})\alpha^{j-1}d\alpha \tag{4.4.6}$$

where $y(t) \in \mathbb{R}$, $\alpha \in [-1,0]$, j=1,2,3,...,N. The delay IDE (4.4.6) will be oscillatory if and only if every solution of the equation is an oscillatory function and the equation (4.4.6) will be non-oscillatory if and only if at least one solution is not an oscillatory function.

4.5 Mixed type IDEs or MTFDEs with oscillatory solutions

In Chapter 1, Section 1.5.6 we discussed qualitative behaviour of first order IDEs. First order IDEs with delay and advanced terms may be shown to have oscillatory solutions under suitable conditions.

Let us consider the first order linear autonomous homogeneous mixed IDE (1.5.28), namely

$$y'(t) + \int_{-1}^{1}[d\Omega(s)]y(t+s) = 0, \tag{4.5.1}$$

where $y(t) \in \mathbb{R}^n$ and $\Omega : [-1,1] \to \mathbb{R}^{n\times n}$ are functions of bounded variation.

Remark 4.5.1 *The starting point needs to be a consideration of the conditions under which an equation of the form (1.5.28) has at least one non-oscillatory solution. This is a problem that has been considered in detail in the works of Krisztin [89], [90], [91]. In his work [89], for example, Krisztin gives an example (1.5.28) of a linear mixed type equation with constant coefficients, where the characteristic equation has no real roots, but where the equation has a non-oscillatory solution. Thus the conditions that are so well-known from the theory of ordinary differential equations, and which extend to consideration of delay differential equations and integral equations, are no longer adequate to provide full information about oscillatory behaviour in the case of mixed-type equations. Krisztin gives a non-degeneracy condition which, if satisfied, provides us with the usual insights based on characteristic roots. He calls the non-degeneracy* Condition (H) *(see [90, 91]), and we recall it below.*

Definition 4.5.1 (Kristin's Condition H) *Condition (H) holds for the equation*

$$y'(t) + \int_{-1}^{1}[d\Omega(s)]y(t+s) = 0 \quad (y(t) \in \mathbb{R}^n), \tag{4.5.2}$$

if there exist a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ *and a non-decreasing function* $\beta : [-1, 1] \to \mathbb{R}$ *such that* $\beta(s) < \beta(1)$ *for all* $s \in [-1, 1)$*, and*

$$\frac{\det\{\Omega(s)A - \beta(s)I\} - \det\{\Omega(1)A - \beta(1)I\}}{\beta(s) - \beta(1)} \to 0 \tag{4.5.3}$$

as $s \to 1-$.

Lemma 4.5.1 *[90, p. 335]. Condition (H) holds, provided either*

(a) $\Omega(1) - \Omega(1-)$ *is a nonsingular matrix; or*

(b) there is a nonsingular matrix $\mathbb{B} \in \mathbb{R}^{n\times n}$, $s_0 \in [-1, 1)$*, and a monotone function* $v : [s_0, 1] \to \mathbb{R}$ *so that* $\Omega(s)\mathbb{B} = v(s)I$ *for all* $s \in [s_o, 1]$ *and* $v(s) \neq v(1)$ *for all* $s \in [s_0, 1)$.

Remark 4.5.2 *In 2009, Pinelas [119] used a condition related to Condition (a) in Lemma 4.5.1.*

Let $L(t)\phi = \int_{-1}^{1} \phi(t + s)d\Omega(s)$*. For each* t*,* $L(t)$ *is an operator that maps* ϕ *into* $\mathbb{R}^n$*. Since* Ω *is assumed to have bounded variation, it follows that* L *is continuous for each* t*.*

Definition 4.5.2 *In the scalar case* $(n = 1)$*, the operator* $L(t)$ *is atomic at* θ *in the sense of [83] when the left and right-hand limits of* Ω *at* θ *do not coincide.*

Lemma 4.5.2 *(See Krisztyn [90].) If the operator* L *that maps the function* $\psi \in C([-1, 1], \mathbb{R}^n)$ *to the function* $L\psi$ *where*

$$L\psi(t) = \int_{-1}^{1} [d\Omega(s)]\psi(t + s) \in \mathbb{R}^n \tag{4.5.4}$$

is atomic at 1 in the sense of [83, pp. 52–53], then Condition (H) holds.

Assumption (H) may be satisfied in cases where L is non-atomic at 1 (see [90]). Verduyn Lunel and Hale [83] trace the use of their terminology to Hale and Oliva [82] who themselves refer to Hale's 1971 book [80]. For completeness, we include the definition from [83] that applies for a nonlinear operator:

Remark 4.5.3 *Suppose* $\Omega \subseteq \mathbb{R} \times C$ *is open with elements* (t, ϕ)*. A function* $D : \Omega \to \mathbb{R}^n$ *(not necessarily linear) is said to be atomic at* β *on* Ω *if* D *is continuous together with its first and second Frechet derivatives with respect* *to ϕ; and D_ϕ, the derivatives with respect to ϕ, is atomic at β on Ω. If $D(t, \phi)$ is linear in ϕ and continuous in $(t, \phi) \in \mathbb{R} \times C$,*

$$D(t, \phi) = \int_{-r}^{0} [d_\theta \eta(t, \theta)]\phi(\theta)$$

then $A(t, \phi, \beta) = A(t, \beta)$ is independent of ϕ and $A(t, \beta) = \eta(t, \beta+) - \eta(t, \beta-)$. Thus, $D(t, \phi)$ is atomic at β on $\mathbb{R} \times C$ if det $A(t, \beta) \neq 0$ *for all $t \in \mathbb{R}$*

We now consider the scalar version of (4.5.2).

Lemma 4.5.3 *Consider the scalar case*

$$y'(t) + \int_{-1}^{1} y(t+s)d\Omega(s) = 0 \quad (y(t) \in \mathbb{R}). \tag{4.5.5}$$

Krisztin's Condition (H) for oscillation is satisfied (see [90]) when there exists $A \in \mathbb{R}$ such that

$$\frac{\{\Omega(s)A - \beta(s)\} - \{\Omega(1)A - \beta(1)\}}{\beta(s) - \beta(1)} \to 0 \text{ as } s \to 1-. \tag{4.5.6}$$

Example 4.5.1 *We demonstrate how the preceding result may be applied to the equation*

$$y'(t) + y(t-1) + y(t+1) = 0. \tag{4.5.7}$$

We define

$$\Omega\,(s) = \begin{cases} 0, & \text{if } s = -1\ ; \\ 1, & \text{if } -1 < s < 1; \\ 2, & \text{if } \ s = 1. \end{cases} \tag{4.5.8}$$

and consider the scalar *mixed type IDE or MTFDE of the form*

$$y'(t) + \int_{-1}^{1} y(t+s)d\Omega(s) = 0. \tag{4.5.9}$$

Then, (4.5.9) *reduces to* $y'(t) + y(t-1)(1-0) + 0 + y(t+1)(2-1) = 0$, *which is (4.5.7). Let us now choose $\beta = \Omega$, that is*

$$\beta(s) = \Omega\,(s) = \begin{cases} 0, & \text{if } s = -1\ ; \\ 1, & \text{if } -1 < s < 1; \\ 2, & \text{if } \ s = 1 \end{cases}$$

and β is a nondecreasing function. Let us make the choice $A = 1$, then as $s \to 1-$

$$\frac{\{\Omega(s)-\beta(s)\}-\{\Omega(1)-\beta(1)\}}{\beta(s)-\beta(1)} \longrightarrow \frac{0-0}{2-1}=\frac{0}{1}=0. \tag{4.5.10}$$

Therefore, the mixed type IDE or MTFDE ((4.5.9)) satisfies Condition (H), and the equivalent MTFDE (4.5.7) is oscillatory.

We revisit the example studied here in Chapter 6 when we investigate the problem numerically.

Example 4.5.2 *Let us consider a mixed type IDE or MTFDE of the form*

$$y'(t) = \int_{-1}^{1} y(t+s)ds + y(t+1). \tag{4.5.11}$$

Equivalently,

$$y'(t) = \int_{-1}^{1} y(t+s)d\Omega_1(s) + \int_{-1}^{1} y(t+s)d\Omega_2(s)$$

where, $\Omega_1(s) = s$ *and* $\Omega_2(s) = \left\{ \begin{array}{l} 0, \text{ if } -1 < s < 1; \\ 1, \text{ if } s = 1; \end{array} \right\}$.

Equation (4.5.11) can also be written

$$y'(t) = \int_{-1}^{1} y(t+s)d\Omega(s) \tag{4.5.12}$$

where $\Omega(s) = \Omega_1(s) + \Omega_2(s) = s + \Omega_2(s)$ *and*

$$\Omega(s) = \left\{ \begin{array}{l} s+0, \text{ if } -1 < s < 1; \\ s+1, \text{ if } s = 1 \end{array} \right\}.$$

We seek to apply Condition (H), with $n = 1$,

$$\frac{\det\{\Omega(s)A-\beta(s)I\}-\det\{\Omega(1)A-\beta(1)I\}}{\beta(s)-\beta(1)} \to 0$$

as $s \to 1^-$ (for oscillation). Let us choose,

$$\beta(s) = \Omega(s) = \left\{ \begin{array}{l} s, \text{ if } -1 < s < 1; \\ s+1, \text{ if } s = 1 \end{array} \right. \tag{4.5.13}$$

where β is a nondecreasing function. With $A = 1$, we have

$$\frac{\det\{\Omega(s)-\beta(s)I\}-\det\{\Omega(1)-\beta(1)I\}}{\beta(s)-\beta(1)} \to \frac{0-0}{(s+1)-s}=\frac{0}{1}=0 \text{ as } s \to 1-.$$

Therefore, the MTFDE satisfies Condition (H) and (4.5.11) is oscillatory.

Example 4.5.3 *Again let us consider the linear autonomous mixed type IDE or MTFDE (1.5.26) (see [89], [90], [91]), that is,*

$$y'(t) + \int_{-1}^{1} y(t+s)d\Omega(s) = 0,$$

and apply a different method to check the characteristic equation,

Let us suppose, $\Omega(s) = s, d\Omega(s) = ds$

$$y'(t) + \int_{-1}^{1} y(t+s)ds = 0 \tag{4.5.14}$$

Consider two tests for oscillation:

(1) using the characteristic function,

(2) using Condition (H) ([90]).

For (1), examine

$$\lambda + \frac{e^{\lambda}}{\lambda} - \frac{e^{-\lambda}}{\lambda} = 0$$

This is the characteristic equation of the MTFDE (4.5.14). It can be shown that the characteristic equation has no real roots. Therefore, the MTFDE (4.5.14) is oscillatory;

For (2) we must check (but for the case $n = 1$*) that*

$$\frac{\det\{\Omega(s)A - \beta(s)I\} - \det\{\Omega(1)A - \beta(1)I\}}{\beta(s) - \beta(1)} \to 0$$

as $s \to 1^-$ *holds where* $A \in \mathbb{R}^{n\times n}$ *is some nonsingular matrix and* β *is a nondecreasing function on [-1,1]. Let us take* $A = 1$ *then* $\Omega(s) = \beta(s) = s$*,* $\Omega(s)A - \beta(s)I = s - s = 0$ *on* $[-1, 1]$ *and*

$$\frac{\{\Omega(s)A - \beta(s)I\} - \{\Omega(1)A - \beta(1)I\}}{\beta(s) - \beta(1)} = \frac{0}{\beta(1) - \beta(s)} = 0 \text{ on } [-1, 1],$$

$$\lim_{s\to 1^-} \frac{\{\Omega(s)A - \beta(s)I\} - \{\Omega(1)A - \beta(1)I\}}{\beta(s) - \beta(1)} \to 0.$$

Thus the equation (4.5.14) satisfies Condition (H) [90]) and the original equation (1.5.26) is therefore an oscillatory MTFDE.

Remark 4.5.4 *We revisit the examples studied here in Chapter 6 when we investigate them numerically.*

4.5.1 Further mixed type equations with oscillatory solutions

In Chapter 1, Section 1.5.6, we discussed the qualitative behaviour of first order IDEs. First order IDEs with delay and advanced terms have oscillatory solutions subject to appropriate conditions (see also [48]). In this section we shall study the oscillatory behaviour of some mixed type IDEs.

We consider the mixed type IDE (1.5.26) of the general form:

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta) \tag{4.5.15}$$

where $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalised so that $\nu(-1) = \eta(-1) = 0$, $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. Taking

$$\|\tau\| = \max\{\tau(\theta) : \theta \in [-1, 0]\},$$

the advance $\tau(\theta)$ will be assumed to satisfy

$$\tau(\theta_0) = \|\tau\| > \tau(\theta), \quad \forall\theta \neq \theta_0. \tag{4.5.16}$$

In case of having $\tau(\theta_0) > 0$, the function $\eta(\theta)$ is supposed to be atomic at θ_0, that is, such that

$$\eta(\theta_0^+) - \eta(\theta_0^-) \neq 0. \tag{4.5.17}$$

The equation (4.5.15) represents another wider class of linear functional differential equations of mixed type and is considered by Krisztin (see [90]) as a basis for some mathematical applications appearing in the literature, such as in [34] and [123].

Letting $R = \max\{\|r\|, \|\tau\|\}$, a solution of (1.5.28) will mean any differentiable function $x : [-R, +\infty) \to \mathbb{R}$ which satisfies (4.5.15) for every $t \in [0, +\infty)$.

Remark 4.5.5 *As usual, a solution x of (4.5.15) oscillates if satisfies Definition 1.1.3. In [90] $x(t)$ is called oscillatory if there is no cone, $\mathcal{K}$, such that $x(t) \in \mathcal{K}$, eventually. Notice that for these equations, both definitions coincide.*

By assuming that delays and advances are positive and differentiable on $[-1, 0]$, one can obtain some special criteria for (4.5.15) to be oscillatory. In this section this case will be analyzed, complementing the results in [120] for the case where delays and advances are only continuous. Further theoretical results for delay equations are obtained in [124] and these can be extended in a natural way to the mixed equation.

The two main ingredients in the theory of linear delay equations (see [83]) are the existence of a unique solution, for any given initial condition, and the exponential boundedness on those solutions. As is shown in [122], this is not at all the situation for a differential equation of mixed type such as (4.5.15). However the following Lemma goes some way towards helping with the analysis.

Lemma 4.5.4 *However, under the atomicity assumption (4.5.17), one has that every oscillatory solution is exponentially bounded as $t \to \infty$ ([90], [Proposition 4]).*

This fact enables the oscillatory behaviour of (4.5.15) to be studied through the analysis of the zeros of the generalised characteristic equation

$$\lambda = \int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\nu\left(\theta\right) + \int_{-1}^{0} \exp\left(\lambda \tau\left(\theta\right)\right) d\eta\left(\theta\right). \tag{4.5.18}$$

Theorem 4.5.5 *If*

$$F\left(\lambda\right) = \int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\nu\left(\theta\right) + \int_{-1}^{0} \exp\left(\lambda \tau\left(\theta\right)\right) d\eta\left(\theta\right).$$

Equation (4.5.15) is oscillatory if and only if $F\left(\lambda\right) \neq \lambda$ then the generalised characteristic function is

$$\chi(\lambda) = \lambda - F(\lambda)$$

Equivalently if

$$F\left(\lambda\right) > \lambda \tag{4.5.19}$$

or if

$$F\left(\lambda\right) < \lambda, \quad \forall \lambda \in \mathbb{R} \tag{4.5.20}$$

(where $\chi := \lambda - F$) equation (4.5.15) is oscillatory.

4.5.2 Oscillatory mixed-type equations with differentiable delays and advances

Definition 4.5.3 *Assuming that $-1 \leqslant \theta_1 \leq 0$, let $D^{+}(\theta_1)$ be the family of all positive differentiable functions, which are strictly increasing on $[-1, \theta_1]$ and decreasing on $[\theta_1, 0]$.*

Definition 4.5.4 *The set we denote by D_i^+ consists of all positive increasing differentiable functions on the interval $[-1,0]$. The class D_d^+ denotes all decreasing positive differentiable functions on $[-1,0]$.*

The set of $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$ with θ_0 as in (4.5.16), we define the value for (1.5.28)

$$S_1 = e^{-1}\left(\int_{-1}^{0} \nu(\theta)\, d\ln r(\theta) + \int_{-1}^{0} \eta(\theta)\, d\ln \tau(\theta)\right).$$

Theorem 4.5.6 *Consider (4.5.15) and suppose $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$ and*

$$\begin{aligned} \nu(\theta) &\leqslant 0 \text{ for } \theta \in [-1,\theta_1),\ \nu(\theta) \geqslant 0 \text{ for } \theta \in [\theta_1, 0) && (4.5.21)\\ \eta(\theta) &\leqslant 0 \text{ for } \theta \in [-1,\theta_0),\ \eta(\theta) \geqslant 0 \text{ for } \theta \in [\theta_0, 0), && (4.5.22) \end{aligned}$$

such that $\eta(0) > 0$. If

$$1 + \ln(\tau(0)\,\eta(0)) + \tau(0)\, S_1 > 0 \tag{4.5.23}$$

then the equation (4.5.15) is oscillatory.

Proof: [48] We shall prove that 4.5.19 is satisfied for $\lambda = 0$, we have $F(0) = \nu(0) + \eta(0) > 0$. Let $\lambda \neq 0$. Using integration by parts we obtain

$$\begin{aligned} F(\lambda) &= \exp(-\lambda r(0))\,\nu(0) + \exp(\lambda\tau(0))\,\eta(0)\\ &+ \lambda \int_{-1}^{0} \exp(-\lambda r(\theta))\,\nu(\theta)\, dr(\theta)\\ &- \lambda \int_{-1}^{0} \exp(\lambda\tau(\theta))\,\eta(\theta)\, d\tau(\theta). \end{aligned} \tag{4.5.24}$$

Since $\nu(\theta)\, r'(\theta) \leqslant 0$ and $\eta(\theta)\,\tau'(\theta) \leqslant 0$ for $\theta \in [-1,0]$, and $u\exp(-u) \leqslant 1/e$, for every real u, we have

$$F(\lambda) \geqslant \exp(-\lambda r(0))\,\nu(0) + \exp(\lambda\tau(0))\,\eta(0) + S_1.$$

Therefore

$$\begin{aligned} F(\lambda) - \lambda &\geqslant \exp(-\lambda r(0))\,\nu(0) + \exp(\lambda\tau(0))\,\eta(0) - \lambda + S_1\\ &\geqslant \exp(\lambda\tau(0))\,\eta(0) - \lambda + S_1. \end{aligned} \tag{4.5.25}$$

As $\eta(0) > 0$, the function $f(\lambda) = \exp(\lambda\tau(0))\,\eta(0) - \lambda$ attains an absolute minimum at

$$\lambda_0 = -\frac{\ln(\tau(0)\,\eta(0))}{\tau(0)}$$

and consequently

$$F(\lambda) - \lambda \geqslant \frac{1}{\tau(0)} + \frac{1}{\tau(0)} \ln(\tau(0)\eta(0)) + S_1 > 0.$$

Thus (4.5.19) is satisfied, which completes the proof.

The following examples illustrate the application of Theorem 4.5.6.

Example 4.5.4 *Consider the equation (4.5.15) for*

$$\nu(\theta) = (3\theta + 1)(\theta + 1), \eta(\theta) = (\theta + 1)(2\theta + 1)$$

$$r(\theta) = -\frac{3}{2}\theta^2 - \theta + 1 \text{ and } \tau(\theta) = -\theta^2 - \theta + 2$$

As

$$S_1 = e^{-1} \int_{-1}^{0} (3\theta + 1)(\theta + 1) \frac{-3\theta + 1}{-\frac{3}{2}\theta^2 - \theta + 1} d\theta$$
$$+ e^{-1} \int_{-1}^{0} (\theta + 1)(2\theta + 1) \frac{-2\theta - 1}{-\theta^2 - \theta + 2} d\theta \approx \quad -0.1421,$$

$$1 + \ln(\tau(0)\eta(0)) + \tau(0) S_1 = 1 + \ln 2 + 2S_1 \approx 1.4089,$$

the corresponding equation (4.5.15) is oscillatory.

Example 4.5.5 *Consider the equation (4.5.15) with*

$$\nu(\theta) = \begin{cases} -\theta - 1, & \text{if } \theta \in [-1, 0[\\ 1, & \text{if } \theta = 0 \end{cases},$$
$$\eta(\theta) = \theta + 1, \quad r(\theta) = \theta + 2 \quad \text{and} \quad \tau(\theta) = -\theta + 1.$$

The corresponding equation is oscillatory since

$$S_1 = e^{-1} \int_{-1}^{0} \frac{-\theta - 1}{\theta + 2} d\theta + e^{-1} \int_{-1}^{0} \frac{-(\theta + 1)}{-\theta + 1} d\theta \approx -0.25499$$

and

$$1 + \ln(\tau(0)\eta(0)) + \tau(0) S_1 = 1 + \ln 1 + S_1 \approx 0.74501.$$

To state a further theorem, we define

$$S_2 = \int_{-1}^{0} \nu(\theta)\, dr(\theta) - \int_{-1}^{0} \eta(\theta)\, d\tau(\theta).$$

Theorem 4.5.7 *[48] Let* $r \in D^{+}(\theta_1)$, $\tau \in D^{+}(\theta_0)$. *If (4.5.21)–(4.5.22) hold such that* $\nu(0) + \eta(0) > 0$ *and*

$$1 - e\tau(0)\eta(0) < S_2 < 1 + er(0)\nu(0) \tag{4.5.26}$$

then equation (4.5.15) is oscillatory.

Proof: The case where $\lambda = 0$, follows as in the proof of Theorem 4.5.6. For $\lambda \neq 0$, by (4.5.24) we have

$$\begin{aligned} \frac{F(\lambda)}{\lambda} &= \frac{\exp(-\lambda r(0))}{\lambda}\nu(0) + \frac{\exp(\lambda\tau(0))}{\lambda}\eta(0) \\ &+ \int_{-1}^{0} \exp(-\lambda r(\theta))\,\nu(\theta)\,dr(\theta) \\ &- \int_{-1}^{0} \exp(\lambda\tau(\theta))\,\eta(\theta)\,d\tau(\theta). \end{aligned} \tag{4.5.27}$$

Now let $\lambda > 0$. Since $\exp(-u) < 1$, $\exp u > 1$, $\frac{\exp(-u)}{u} > 0$ and $\frac{\exp u}{u} \geq e$, for $u > 0$, we obtain

$$\frac{F(\lambda)}{\lambda} > e\tau(0)\eta(0) + S_2 > 1$$

and so $F(\lambda) > \lambda$. Finally, for $\lambda < 0$, the same arguments imply that

$$\frac{F(\lambda)}{\lambda} < -er(0)\nu(0) + S_2 < 1$$

and $F(\lambda) > \lambda$. Hence (4.5.19) is again satisfied and (4.5.15) is oscillatory.

The following examples illustrate Theorem 4.5.7.

Example 4.5.6 *Consider the equation (4.5.15) with*

$$\nu(\theta) = (5\theta + 4)(\theta + 1), \quad \eta(\theta) = (10\theta + 9)(\theta + 1),$$

$$r(\theta) = -\frac{5}{2}\theta^2 - 4\theta + 5 \text{ and } \tau(\theta) = -5\theta^2 - 9\theta + 1.$$

We have

$$S_2 = -\int_{-1}^{0} (5\theta + 4)^2 (\theta + 1)\,d\theta + \int_{-1}^{0} (10\theta + 9)^2 (\theta + 1)\,d\theta \approx 15.417$$

and

$$-23.465 \approx 1 - 9e = 1 - e\tau(0)\eta(0) < S_2 < 1 + er(0)\nu(0) = 1 + 20e \approx 55.366.$$

Therefore, the corresponding equation is oscillatory.

Notice that in this case

$$\begin{aligned} S_1 &= e^{-1}\left(\int_{-1}^{0}\frac{-(5\theta+4)^2(\theta+1)}{-\frac{5}{2}\theta^2-4\theta+5}d\theta+\int_{-1}^{0}\frac{-(10\theta+9)^2(\theta+1)}{-5\theta^2-9\theta+1}d\theta\right) \\ &\approx -3.6737 \end{aligned}$$

and

$$1+\ln(\tau(0)\eta(0))+\tau(0)S_1 = 1+\ln 9+S_1 = -0.47648 < 0.$$

so that, in consequence, we cannot apply Theorem 4.5.6.

With respect to condition (4.5.20) the following theorem will be obtained.

Theorem 4.5.8 *[48] Let $r \in D^{+}(\theta_1)$, $\tau \in D^{+}(\theta_0)$ and assume $\nu(0) < 0$ and that*

$$\nu(\theta) \geqslant 0 \text{ for } \theta \in [-1,\theta_1),\ \nu(\theta) \leqslant 0 \text{ for } \theta \in [\theta_1,0], \tag{4.5.28}$$

$$\eta(\theta) \geqslant 0 \text{ for } \theta \in [-1,\theta_0),\ \eta(\theta) \leqslant 0 \text{ for } \theta \in [\theta_0,0] \tag{4.5.29}$$

If

$$1+\ln(r(0)|\nu(0)|)-r(0)S_1 > 0 \tag{4.5.30}$$

then equation (4.5.15) is oscillatory.

Proof: We show that 4.5.20 is satisfied for $\lambda = 0$, we have $F(0) = \nu(0)+\eta(0) < 0 = \lambda$.

Let $\lambda \neq 0$. Applying (4.5.24) and taking into account that now $\nu(\theta)r'(\theta) \geqslant 0$ and $\eta(\theta)\tau'(\theta) \geqslant 0$ for $\theta \in [-1,0]$, and $u\exp(-u) \leqslant 1/e$, for every real u, we have

$$F(\lambda) \leqslant \exp(-\lambda r(0))\nu(0)+\exp(\lambda\tau(0))\eta(0)+S_1.$$

Notice that, in this case, $M(\lambda) \to -\infty$, as $\lambda \to \pm\infty$.

Therefore

$$F(\lambda)-\lambda \leq \exp(-\lambda r(0))\nu(0)-\lambda+S_1. \tag{4.5.31}$$

The function $g(\lambda) = \exp(-\lambda r(0))\nu(0)-\lambda$ has a maximum at

$$\lambda_0 = \frac{\ln(r(0)|\nu(0)|)}{r(0)}$$

and consequently by (4.5.30)

$$F(\lambda)-\lambda \leqslant -\frac{1}{r(0)}-\frac{1}{r(0)}\ln(r(0)|\nu(0)|)+S_1 < 0,$$

for every $\lambda \in \mathbb{R}$.

Thus (4.5.20) is satisfied and (4.5.15) is oscillatory.

Remark 4.5.6 *Notice that conditions (4.5.21) and (4.5.22) of Theorem 4.5.6 by (4.5.25), imply that* $M(\lambda) - \lambda \to +\infty$, *as* $\lambda \to \pm\infty$. *Analogously to (4.5.28) and (4.5.29) of Theorem 4.5.8, by (4.5.31), one has* $M(\lambda) - \lambda \to -\infty$, *as* $\lambda \to \pm\infty$. *This means that in such situations the real roots of the characteristic equation (4.5.18) are bounded.*

Example 4.5.7 *Consider the equation (4.5.15) with*

$$\nu(\theta) = (-\theta - 1)(4\theta + 3), \quad \eta(\theta) = -8\theta - 8,$$

$$r(\theta) = -2\theta^2 - 3\theta + 1, \ \textit{and}\ \tau(\theta) = -\theta + 1.$$

Notice that

$$S_1 = e^{-1}\int_{-1}^{0} \frac{(-\theta - 1)(4\theta + 3)(-4\theta - 3)}{-2\theta^2 - 3\theta + 1} d\theta + e^{-1}\int_{-1}^{0} \frac{8\theta + 8}{-\theta + 1} d\theta \approx 1.6372.$$

and

$$1 + \ln(r(0)|\nu(0)|) - r(0) S_1 = 1 + \ln 3 - S_1 \approx 0.4614.$$

By Theorem 4.5.8, the corresponding equation (1.5.28) is oscillatory.

Example 4.5.8 *Consider*

$$\nu(\theta) = \begin{cases} \theta + 1, & \textit{if}\ \theta \in [-1, 0[\\ -1, & \textit{if}\ \theta = 0 \end{cases},$$

$$\eta(\theta) = -\theta - 1, \quad r(\theta) = -\theta^2 + 2 \quad \textit{and} \quad \tau(\theta) = -\theta + 3.$$

The equation (4.5.15) is oscillatory since

$$S_1 = e^{-1}\left(\int_{-1}^{0} \frac{-2\theta(\theta + 1)}{-\theta^2 + 2} d\theta + \int_{-1}^{0} \frac{\theta + 1}{-\theta + 3} d\theta\right) \approx 0.1291,$$

and

$$1 + \ln(r(0)|\nu(0)|) - r(0) S_1 = 1 + \ln 2 - 2S_1 \approx 1.4349.$$

Theorem 4.5.9 *Let* $r \in D^+(\theta_1)$, $\tau \in D^+(\theta_0)$ *and assume that (4.5.28)–(4.5.29) are satisfied such that* $\nu(0) + \eta(0) < 0$. *If*

$$1 + er(0)\nu(0) < S_2 < 1 - e\tau(0)\eta(0) \tag{4.5.32}$$

then the equation (4.5.15) is oscillatory.

Proof: When $\lambda = 0$, as before one has $F(0) = \nu(0) + \eta(0) < 0$.

Let $\lambda > 0$. Using (4.5.27) and the arguments as in Theorem 4.5.7, we obtain

$$\frac{F(\lambda)}{\lambda} < e\tau(0)\eta(0) + S_2,$$

and by (4.5.32) follows that $M(\lambda) < \lambda$.

For $\lambda < 0$, the same arguments as before enable us to conclude that

$$\frac{F(\lambda)}{\lambda} > er(0)|\nu(0)| + S_2 > 1.$$

So, by (4.5.32) one has also $F(\lambda) < \lambda$, which achieves the proof.

For the case where $\theta_0 = \theta_1 = -1$, the delays and advances are in $D_{d.}^{+}$. When $\theta_0 = \theta_1 = 0$, the delays and advances are in D_i^{+}. The following example illustrates this situation for Theorem 4.5.9.

Example 4.5.9 *Let the equation (4.5.15)*

$$\nu(\theta) = -(5\theta + 1)(\theta + 1), \quad \eta(\theta) = -(6\theta + 1)(\theta + 1),$$

$$r(\theta) = -10\theta^2 - 4\theta + 10,$$

and

$$\tau(\theta) = -3\theta^2 - \theta + 1.$$

We have

$$\begin{aligned} S_2 &= \int_{-1}^{0} (5\theta + 1)(\theta + 1)(20\theta + 4)\, d\theta - \int_{-1}^{0} (6\theta + 1)^2 (\theta + 1)\, d\theta \\ &\approx 2.1667 \end{aligned}$$

$$-26.138 \approx 1 - 10e = 1 + er(0)\nu(0) < S_2 < 1 - e\tau(0)\eta(0) = 1 + e \approx 3.7183,$$

therefore, the corresponding equation is oscillatory.

Remark 4.5.7 *We revisit the examples studied here in Chapter 6 when we investigate them numerically.*

Chapter 5

Rouché's Theorem and the Argument Principle

5.1 Introduction

This Chapter introduces Rouché's theorem and the Argument Principle that will be used later in the thesis to count the number of zeros of a polynomial that lie in a given region or a given interval. (The principles are the same when we consider the zeros of a quasi-polynomial rather than a polynomial.)

5.2 Rouché's and related theorems

In this section we introduces Rouché's theorem and some related results.

In the following theorems, we suppose that the functions f and g are analytic on a simply-connected open domain $D \subset \mathbb{C}$ whose boundary $\gamma = \partial D$ is a simple closed contour (taken to be positively oriented).

Theorem 5.2.1 (Rouché's Theorem) *Suppose that $f(z)$ and $g(z)$ have no zeros or poles for $z \in \gamma$. If $|g(z)| < |f(z)|$ for all $z \in \gamma$ then f and $f+g$ have the same number of zeros inside γ (that is, in D).*

An equivalent result is the following.

Theorem 5.2.2 (Attributed to Estermann [46]) *If $|f(z)-g(z)| < |f(z)|+|g(z)|$ for all $z \in \gamma$ then f and g have the same number of zeros inside γ.*

Suppose now that a function f is analytic in D except at a finite set of poles. We denote the number of zeros of f in D by z_f and the number of poles in D by p_f. If f is a analytic on the closed contour γ then

$$z_f - p_f = \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz.$$

The above result is stated formally in Theorem 5.2.3 and is often called the Principle of the Argument [38]. In the cases we consider, f is often a polynomial (sometimes a quasi-polynomial) and so the integral permits us to calculate the number of zeros (counting multiple zeros according to multiplicity) lying within γ.

Moreover, our aim is to test for real zeros. Typical of our results for oscillation of solutions of a discrete recurrence is the characterisation property that states that *the equation (1.7.1) is oscillatory if and only if none of the zeros of (1.7.2) lie on the non-negative real axis* or its positive part. For many of the analytic problems (ODES, DDEs, etc.) considered here, *the equation is oscillatory if and only if none of the zeros of some characteristic function lie on the whole real axis.* In each case, we do not need to find the zeros of the function, we merely need to count how many lie on the nonnegative real axis (respectively, on the real axis). The Argument Principle provides an ideal tool for this type of investigation. In the next section we demonstrate the use of the Argument Principle in a simple example, showing how to count the zeros of a function in a given region.

We recall that a meromorphic function is a function that is the quotient of two analytic functions, and state the result formally:

Theorem 5.2.3 (The Argument Principle) *Let f be meromorphic assumption on $\mathcal{D} \in Cset$ with poles $p_1, p_2, \ldots, m_p$ and zeros $z_1, z_2, \ldots, m_z$ repeated as necessary according to multiplicity. Let γ be a positively oriented Jordan (simple closed) curve in $\mathcal{D}$ which does not pass through any of the zeros or poles of f. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = z_f - p_f = \begin{array}{c}\textit{the number of zeros of } f \textit{ lying within } \gamma - \\ \textit{the number of poles of } f \textit{ lying within } \gamma\end{array} \tag{5.2.1}$$

5.2.1 Basic application of the Argument Principle

In this section we shall present an example where the location and number of the zeros of a simple function is already known and the purpose is merely to illustrate the principle we shall use later. Our numerical approach depends upon selecting contours γ_M for integer M and using them in place of γ. We start with a problem where we wish to count the zeros on the real axis ($\mathbb{R}$)

and later we deal with a problem where we wish to count the zeros on the positive real axis ($\mathbb{R}^+$).

Example 5.2.1 *The problem here is to find the number and location of zeros of the function f where*

$$f(z) = z^2 - 1, \quad z \in \mathbb{C}. \tag{5.2.2}$$

on the real axis using the Argument Principle. (We know the zeros to be at ± 1. We can verify the known result using numerical techniques based on the Argument Principle!)

Definition 5.2.1 (Rectangular contours Γ_{2M} and γ_M) *(a) The contour γ_M consists of the rectangular contour in the complex plane (specifically, in the right half-plane) with vertices $A = M + \mathrm{i}/M$, $B = \mathrm{i}/M$, $C = -\mathrm{i}/M$, $D = M - \mathrm{i}/M$. (b) The contour Γ_{2M} is defined as the rectangular contour having vertices $A = M + \mathrm{i}/M$, $B = -M + \mathrm{i}/M$, $C = -M - \mathrm{i}/M$, $D = M - \mathrm{i}/M$.*

Refer to Figure 5.2.1, where we display a contour γ_M. This contour is used to find zeros in $\mathbb{R}_+$ (Figure 5.2.1); Γ_{2M} is used to find zeros in $\mathbb{R}$ (Figure 5.2.2). If the length of the rectangular contour $ABCD$ is increased by increasing M then the height of the rectangular will decrease and in the limit as $M \to \infty$ the number of zeros on the real half-axis (respectively, real axis) will be determined.

First, we consider results that follow using γ_M. When $M = 1$ one of the zeros of f lies on γ_M and the Argument Principle cannot be applied. This problem arises with γ_M only for $M = 1$, and we can take any other value of M and apply the principle.

For the given function

$$f(z) = z^2 - 1 \tag{5.2.3}$$

we have

$$f'(z) = 2z \tag{5.2.4}$$

By the Argument Principle. the number and location of zeros inside $\gamma_M = ABCD$ (Figure 5.2.1) is

$$z_f = \frac{1}{2\pi i}\oint_{\gamma_M} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{\gamma_M} \frac{2z}{z^2-1}dz. \tag{5.2.5}$$

Here, $\gamma_M = ABCD$ is a closed curve and

$$z_f = \frac{1}{2\pi i}\left[\int_{DA} + \int_{AB} + \int_{BC} + \int_{CD} \left(\frac{2z}{z^2-1}\right) dz\right]. \tag{5.2.6}$$

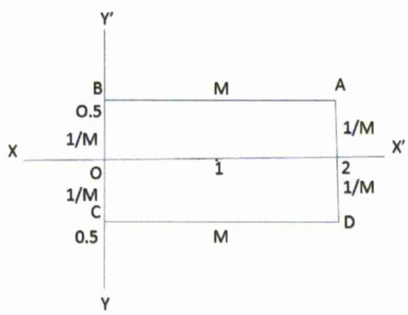

Figure 5.2.1: Results for Example 5.2.1: The rectangle γ_M for $M = 2$

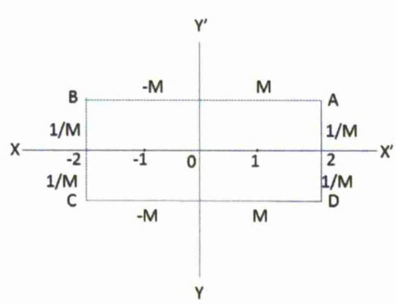

Figure 5.2.2: Result for Example 5.2.1: The rectangle Γ_{2M} for $M = 2$

We propose to discretise each of the four integrals in (5.2.6). *Define a step-size* $h = 1/(LM)$ *where* L *is an integer and define* $\gamma_M(h)$ *to be the set of points lying on* γ_M*, starting at the vertex* A *and at a distance* h *from its neighbour. (The set* $\Gamma_{2M}(h)$ *can be defined in a similar fashion.)*

We obtain

$$z_f \approx \frac{h}{2\pi i} \sum_{z \in \gamma_M(h)} \frac{2z}{z^2 - 1}. \tag{5.2.7}$$

Indeed, since $h = 1/(LM) \to 0$ *as* $L \to \infty$,

$$z_f = \lim_{L, M \to \infty} \frac{h}{2\pi i} \sum_{z \in \gamma_M(h)} \frac{2z}{z^2 - 1}. \tag{5.2.8}$$

and we are aided by the fact that the true value is an integer ($z_f \in \mathbb{N}$) *so that the limit is more readily apparent on taking a small* h *(taking a large* L *with a large* M*). Already, for* $M = L = 2$*, the approximation to* z_f *is 0.56. The number of zeros on real axis is one. Figure 5.2.1 shows the result.*

Now consider the number of real zeros (of any sign) and construct γ_{2M}*. Thus, we consider both sides of the real axis then the number and location of the real zeros will be obtained. The number of the real roots will be 2. There is one positive and one negative root. Figures 5.2.1 and 5.2.2 based on the Argument Principle show clearly the accurate determination of the number of positive and negative zeros for the polynomial. Using* $\Gamma_{2M}(h)$ *gives the number of real zeros; while using* $\Gamma_M(h)$ *gives the number of positive real zeros, in each case for sufficiently small* h *and sufficiently large* M*.*

Remark 5.2.1 *Example 5.2.1 shows that the Argument Principle can be used to locate and count the number of zeros of a polynomial on the real axis. Also the example shows the process is automatic, quicker and accurate for the smaller step-size (see figure 5.2.3).*

The application of the Argument Principle to count zeros of a function will be given in the next section.

5.2.2 The Use of the Argument Principle to Count Zeros

We proceed without formal justification in this section (further theoretical justification will be deferred to Chapter 7).

We let $M > 0$ be fixed and we let γ_M be the rectangle with vertices at $(0, \pm\frac{1}{M})$, $(M, \pm\frac{1}{M})$. We approximate the integral (5.2.1) using numerical integration. That means we approximate, numerically, the integral on the

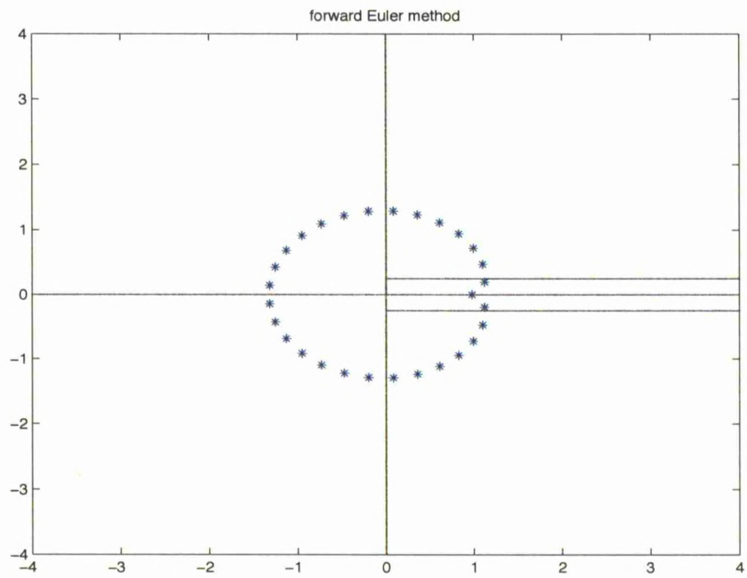

Figure 5.2.3: Results for Example 5.2.1: Each symbol $*$ represents a characteristic value obtained using $h = 0.05$, $M = 10$ and the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$

left hand side of (5.2.1) around γ_M as we allow M to vary through a sequence of increasing values. This enables us to count the number of zeros of f lying on the positive real axis in a very straightforward way. Of course we must be careful to ensure the accuracy of the numerical method used to evaluate the integral as well as ensuring that we choose M sufficiently large to capture all the positive real zeros of f. We give examples illustrating the use of this approach in the next section. Figure 5.2.3 illustrates the method using $M = 4$ and for which the integral will determine three zeros lying inside the rectangle. The zeros of the characteristic polynomial are marked with $*$ and as $M \to \infty$ we can see that the method will count the single real positive zero.

Remark 5.2.2 *Table 5.2.1 records the results of the numerical calculations using different values of M to calculate the number of zeros of a polynomial lying in the rectangle. As M becomes larger, the number of zeros of the polynomial counted by the Argument Principle tends to* 1. *There is one positive real characteristic value for the discrete scheme. The underlying problem is non-oscillatory, as is the discrete scheme. Throughout, the term 'Large' refers to all values of M larger than in the preceding row for which*

Step length h	Length of rectangle M	Number of zeros
0.0001	10	7
0.0001	20	1
0.0001	30	1
0.0001	Large	1

Table 5.2.1: Results for Example 5.2.1: Number of zeros located inside γ_M by the Argument Principle 5.2.1.

experiments were performed.

Chapter 6

Counting characteristic values for continuous and discrete problems using the Argument Principle

6.1 Introduction

The oscillatory behaviour of solutions of DDEs and MTDEs (with delayed and advanced terms) are based on the existence or non-existence of real zeros of the corresponding characteristic equation and also depends on the smoothness of the delay function (see [63], [64]).

6.2 Discretisation of oscillatory DDEs and DIDEs

We want to determining the number of real characteristic values of continuous and discrete DDEs and delay IDEs or MTFDEs. Essentially, we wish to know whether there are no real zeros or one or more real zeros. We use basic numerical methods for discretisation. Actually we are interested to know the fundamental oscillatory properties of an equation and its solutions under a numerical discretisation.

This is particularly important when investigating functional differential equations which often do not admit an exact closed form solution, and the use of numerical approximation methods becomes essential. It is known that numerical schemes may suppress certain properties of an equation, or introduce spurious properties and one needs to be able to select numerical methods that exhibit accurately the important true patterns of behaviour from the original continuous problem (see [56, 60, 61]).

For counting zeros a MATLAB program has been produced based on the Principle of the Argument to count the number of zeros of characteristic equations of DDEs and delay IDEs to investigate whether or not they are oscillatory. Similarly a MATLAB program has been produced based on the Principle of the Argument to count the number of zeros of characteristic equation of its discrete equations to investigate whether they are also oscillatory or not. We demonstrate some examples which will show how to count the number of zeros using the program and it will also show how the numerical methods are more effective and reliable than original analytical methods.

6.2.1 Experiments of real characteristic values of DDEs

In this section we consider some examples of oscillatory and non-oscillatory DDEs. Our approach is to discretise the equations.

Example 6.2.1 *Consider the non-oscillatory problem obtained by setting $\Omega = 3$ in the basic DDE (1.2.8),*

$$y'(t) = 3y(t-1), \quad t \geq 0. \tag{6.2.1}$$

The characteristic function of the above equation can be written $\chi(\lambda) = \lambda - 3\exp(-\lambda)$.

Applying the forward Euler rule (3.3.3), where $Nh = 1$, N is a positive integer and $m \in \{0, 1, 2, \cdots\}$, $t_m = t_0 + mh$, $\widetilde{y}_m \approx y(t_m)$, we obtain

$$\widetilde{y}_{m+1} - \widetilde{y}_m - 3h\widetilde{y}_{m-N} = 0. \tag{6.2.2}$$

The characteristic equation for (6.2.2) can be written

$$\lambda^{N+1} - \lambda^N - 3h = 0 \tag{6.2.3}$$

and the characteristic polynomial can be written

$$\chi^D(\lambda) = \lambda^{N+1} - \lambda^N - 3h.$$

Figure 6.2.2 presents the characteristic values of the discretised case 6.2.2for $h = 0.01$, $N = 100$. Figure 6.2.2 and Table 6.2.1 show that the discrete problem (6.2.2) is non-oscillatory as is the underlying DDE (6.2.1).

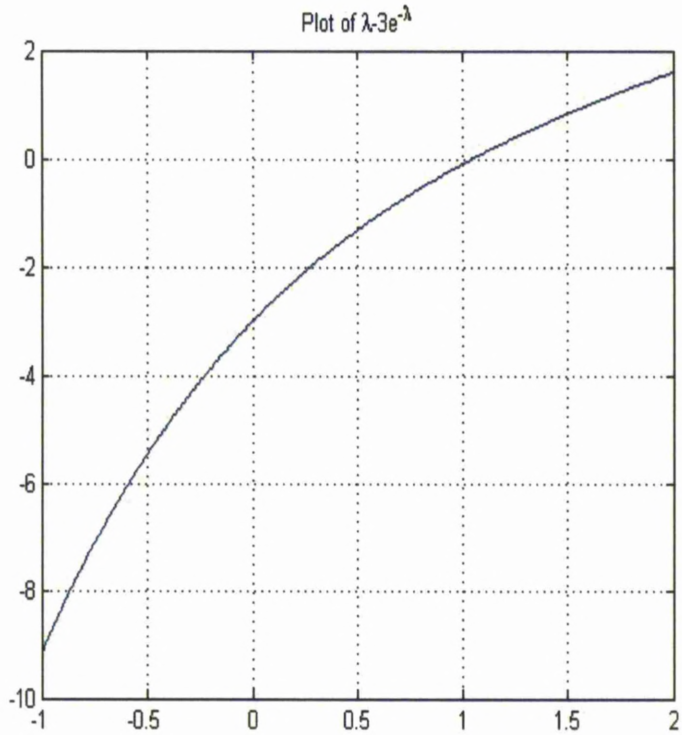

Figure 6.2.1: Results for Example 6.2.1: Plot of the characteristic function for $y'(t) = 3y(t-1)$.

Step length h	Length of rectangle M	Number of zeros
0.01	2	17
0.001	10	3
0.001	20	3
0.001	30	1
0.001	40	1
0.0001	40	1
0.0001	Large	1

Table 6.2.1: Results for Example 6.2.1 : Number of positive real zeros located inside γ_M by the Argument Principle 5.2.1

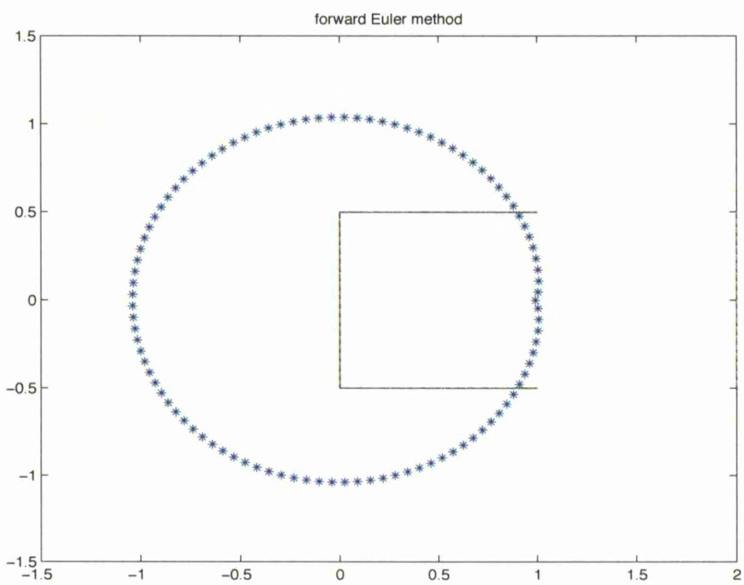

Figure 6.2.2: Results for Example 6.2.1: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.01$, $M = 2$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Example 6.2.2 *Let us consider the oscillatory DDE (4.3.1), i,e.,*

$$y'(t) = -y(t-1), \quad t \geq 0.$$

The characteristic function $\chi(\lambda) = \lambda + \exp(-\lambda)$. *We can choose as an auxiliary characteristic function* $\lambda \exp(\lambda) + 1$.

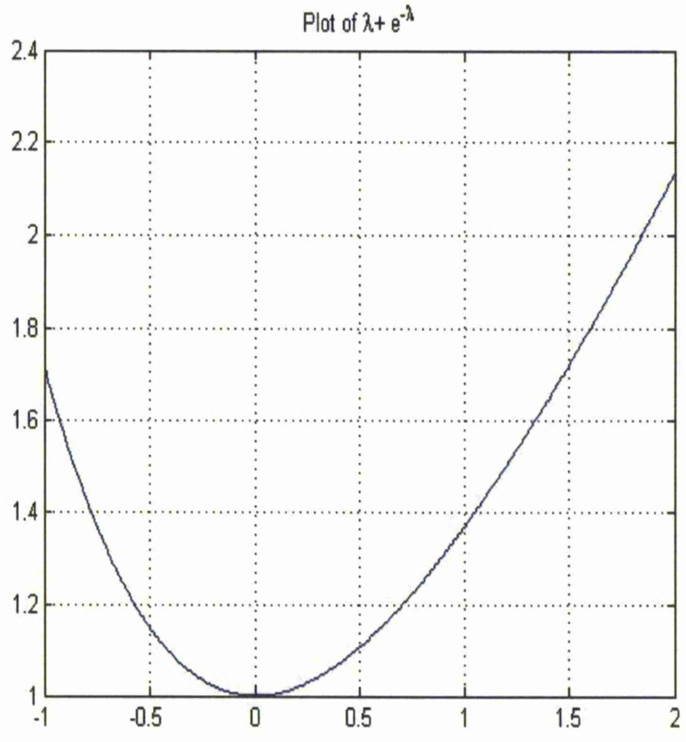

Figure 6.2.3: Results for Example 6.2.2: Plot of the characteristic function for $y'(t) = -y(t-1)$.

Applying the forward Euler rule (3.3.3) with the usual notation, we obtain

$$\widetilde{y}_{m+1} - \widetilde{y}_m + h\widetilde{y}_{m-N} = 0, \quad h = \frac{1}{N}. \tag{6.2.4}$$

The characteristic polynomial can be written $\chi^D(\lambda) = \lambda^{N+1} - \lambda^N + h$ *and the characteristic equation can be written*

$$\lambda^{N+1} - \lambda^N + h = 0 \tag{6.2.5}$$

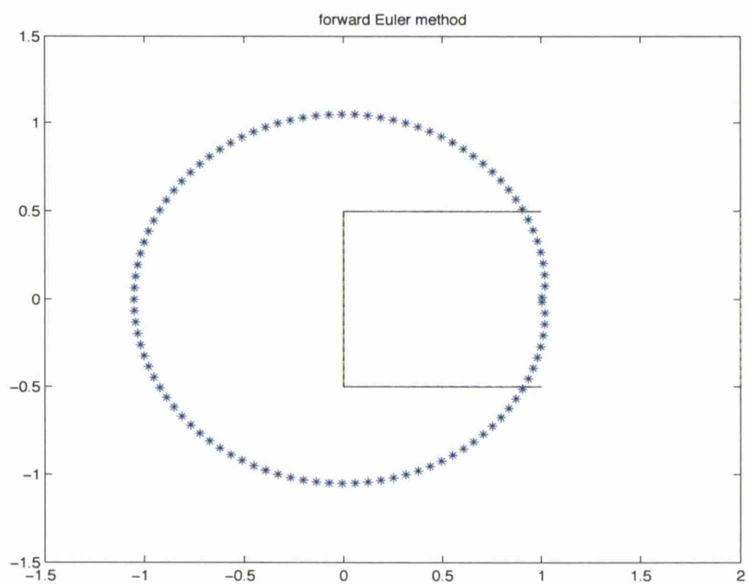

Figure 6.2.4: Results for Example 6.2.2: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.01$, $M = 2$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Step length h	Length of rectangle M	Number of zeros
0.01	2	18.0955
0.001	10	4
0.001	20	2
0.001	30	2
0.001	40	2
0.0001	40	2
0.0001	Large	0

Table 6.2.2: Results for Example 6.2.2: Number of positive real roots located inside γ_M by the Argument Principle 5.2.1

The characteristic values are represented in Figure 6.2.4 and Table 6.2.2 confirming that the problem is oscillatory.

Remark 6.2.1 *Figure 6.2.4 and Table 6.2.2 confirming that the discrete problem (6.2.4) of the problem (4.3.1) is oscillatory.*

6.2.2 Experiments for counting characteristic values of oscillatory and non-oscillatory DIDEs

Example 6.2.3 *Let us consider the non-oscillatory IDE (4.4.1),*

$$y'(t) = 2\int_{-1}^{0} y(t-s^2)sds.$$

The characteristic function of the above equation can be written $\chi(\lambda) = \lambda - \frac{\exp(-\lambda)-1}{\lambda}$ *for* $\lambda \neq 0$ *– with* $\chi(0) = \lim_{\lambda\to 0}\chi(\lambda)$.

Applying the forward Euler rule (3.3.3) for the differential equation, and a specially selected quadrature rule with step length $\sqrt{h}$, *the discrete scheme will be obtained,*

$$\frac{\widetilde{y}_{m+1} - \widetilde{y}_m}{h} = 2h\sum_{k=-N}^{0} w_k \widetilde{y}_{m-k^2}.k, \qquad (6.2.6)$$

where $N = M^2$ *is a positive integer and* $M \in (N = 0, 1, 2, ..., M)$, $h > 0, t = mh, y(mh) \approx \widetilde{y}_m, w_0 = w_{-N} = \frac{1}{2}, w_k = 1, h = \frac{1}{M^2}$.

$$\widetilde{y}_{m+1} - \widetilde{y}_m = 2h^2\sum_{k=-N}^{0} k w_k \widetilde{y}_{m-k^2}$$

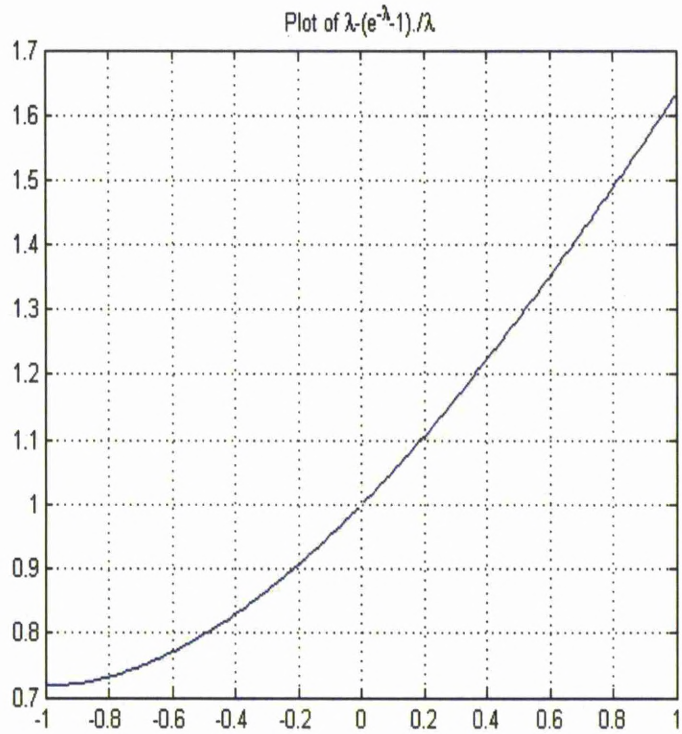

Figure 6.2.5: Results for Example 6.2.3: Plot of the characteristic function for $y'(t) = 2\int_{-1}^{0} y(t-s^2)sds$.

and the characteristic equation is

$$\lambda^{N^2+1} - \lambda^{N^2} + \frac{2}{N^2}w_1\lambda^{N^2-1^2} + \frac{2}{N^2}.2.w_2\lambda^{N^2-2^2} + ... + \frac{2}{N^2}Nw_{-N}\lambda^{N^2-N^2} = 0.$$

The characteristic polynomial can be written

$$\chi^D(\lambda) = \lambda^{N^2+1} - \lambda^{N^2} + \frac{2}{N^2}w_1\lambda^{N^2-1^2} + \frac{2}{N^2}.2.w_2\lambda^{N^2-2^2} + ... + \frac{2}{N^2}Nw_{-N}\lambda^{N^2-N^2}.$$

where $\{w_\ell\} = \frac{1}{2}, 1, \cdots, 1, \frac{1}{2}$.

Remark 6.2.2 *Figure 6.2.6 and Table 6.2.3 confirm that the discrete problem (6.2.6) and the underlying continuous problem are non-oscillatory.*

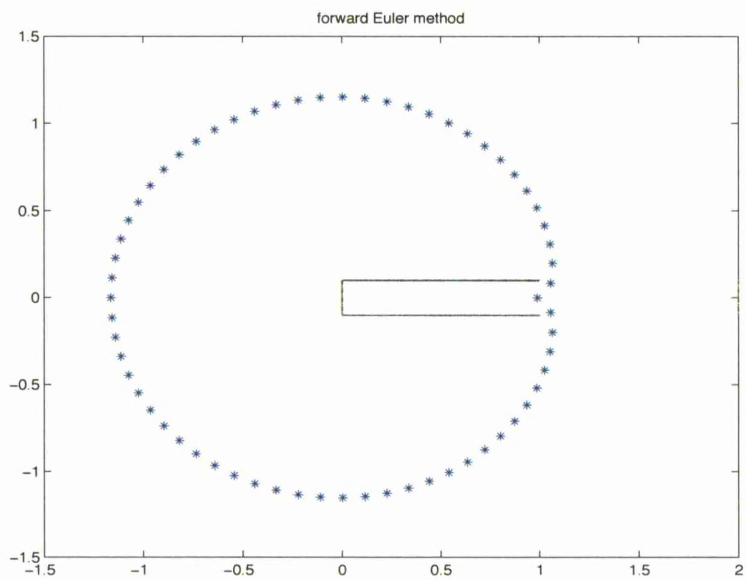

Figure 6.2.6: Results for Example 6.2.3: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.0125$, $M = 10$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Step length h	Length of rectangle M	Number of zeros
0.01	2	13.0482
0.001	10	3.0002
0.001	20	1
0.001	30	1
0.001	40	1
0.0001	40	1
0.0001	Large	1

Table 6.2.3: Results for Example 6.2.3: Number of positive real roots located inside γ_M by the Argument Principle 5.2.1.

Example 6.2.4 *Let us consider the oscillatory equation (4.4.5) of the form*

$$y'(t) = -\frac{1}{a}\int_{-1}^{0} y(t - \frac{1}{5} + s)ds, \quad a > 0 \tag{6.2.7}$$

The characteristic function of the above equation can be written $\chi(\lambda) = \lambda + \frac{\exp(-\lambda/5)(1-exp(-\lambda))}{a\lambda}$ *for* $\lambda \neq 0$.

Applying the Euler forward rule to discretise (6.2.7) *with* $t = mh$, $\widetilde{y}_m \approx y(mh)$, $Nh = \frac{1}{5}$, $0 < a < 2$ *we obtain:*

$$\frac{\widetilde{y}_{m+1} - \widetilde{y}_m}{h} = -\frac{1}{a}\sum_{j=-5N}^{0} \widetilde{y}_{m-N+j}.h. \tag{6.2.8}$$

and the characteristic equation takes the form

$$\frac{h^2}{a}[\lambda^0 + \lambda^1 + \lambda^2 + ... + \lambda^{5N}] - \lambda^{6N} + \lambda^{6N+1} = 0 \tag{6.2.9}$$

And characteristic polynomial can be written

$$\chi^D(\lambda) = \frac{h^2}{a}[\lambda^0 + \lambda^1 + \lambda^2 + ... + \lambda^{5N}] - \lambda^{6N} + \lambda^{6N+1}$$

Remark 6.2.3 *Figure 6.2.8 and Table 6.2.4 confirm that the discrete problem (6.2.8) and the underlying continuous problem (4.4.5) are oscillatory.*

Remark 6.2.4 *We have seen that the numerical approach introduced here does provide a reliable method for determining whether or not linear functional differential equations are oscillatory. The experiments have shown that the technique works also for some non-linear problems, but there is a need for further analytical results in this case.*

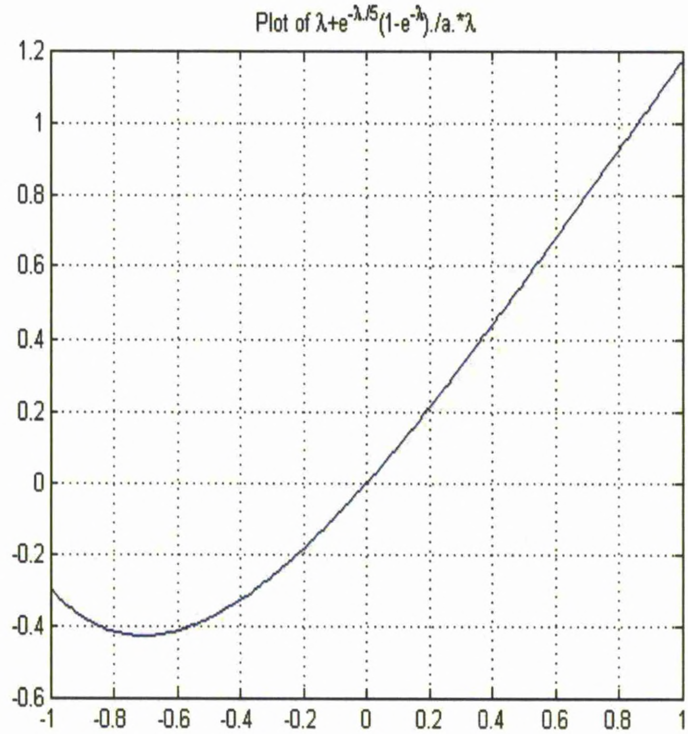

Figure 6.2.7: Results for Example 6.2.4: Plot of the characteristic function for Example 6.2.4, i.e.,$y'(t) = -\frac{1}{a}\int_{-1}^{0} y(t - \frac{1}{5} + s)ds.$

Step length h	Length of rectangle M	Number of zeros
0.01	2	22
0.01	4	10
0.01	10	4
0.01	20	2
0.01	30	2
0.01	40	2
0.01	Large	0

Table 6.2.4: Results for Example 6.2.4: Number of positive real roots located inside γ_M by the Argument Principle 5.2.1.

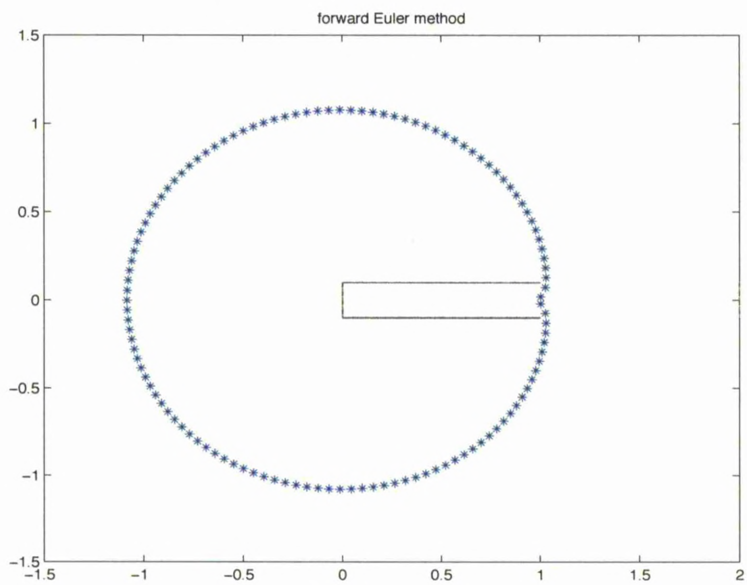

Figure 6.2.8: Results for Example 6.2.4: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.01$, $M = 20$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

6.3 Numerical treatment of oscillatory MT-DEs and mixed type IDEs

This section shows how to count zeros of continuous and discrete mixed type differential equations (MTDEs or MTFDEs) with delayed and advanced terms and continuous and discrete mixed type integro-differential equations (MTIDEs) with differentiable delays and advances.

In this section we are interested to study fundamental properties of a mixed type differential equations (MTDEs) and their solutions under numerical discretisation. The work presented develops new insights and builds upon the previous section where equations without advanced term were studied.

As is well known, mixed-type equations, which have both advanced and delay terms can be particularly difficult to analyse, even in the case of linear equations.

The importance of understanding the dynamics of numerical schemes for solution of mixed-type equations is considerable. Where, as in these cases, the problems under investigation have no closed form exact solution, one needs to know for certain whether numerical approximations reproduce faithfully the true behaviour of the original problem, or whether they introduce new behaviour of their own. The aim is always to select well-behaved numerical methods that exhibit accurately true behaviour from the original continuous problem. In other words, it is important that we should observe oscillatory behaviour in the approximate solutions if and only if it would be found in the exact solutions.

A basic mixed type differential equation has a much wider range of potential dynamical behaviour than a delay differential equation.

A MATLAB program has been produced based on the Principle of the Argument to count the number of zeros of the characteristic equation of a mixed type differential equation and we shall investigate whether it is oscillatory or not. Similarly a MATLAB program has been produced based on the Principle of the Argument to count the number of zeros of the characteristic equation of its discrete equation and we shall investigate whether it is oscillatory or not. We demonstrate some examples which will show how to count the number of zeros for any order of polynomial or a quasi=polynomial and also show how the numerical methods can be more effective and reliable than original analytical methods.

6.3.1 Experiments for counting characteristic values of Mixed Type FDEs

In this section we consider some examples of oscillatory MTDEs with differentiable delays and advances.

Example 6.3.1 *Let us consider the oscillatory MTDE (1.4.8)*

$$y'(t) + y(t+1) + y(t-1) = 0$$

The characteristic function can be written $\chi(\lambda) = \lambda + \exp(\lambda) + \exp(-\lambda)$

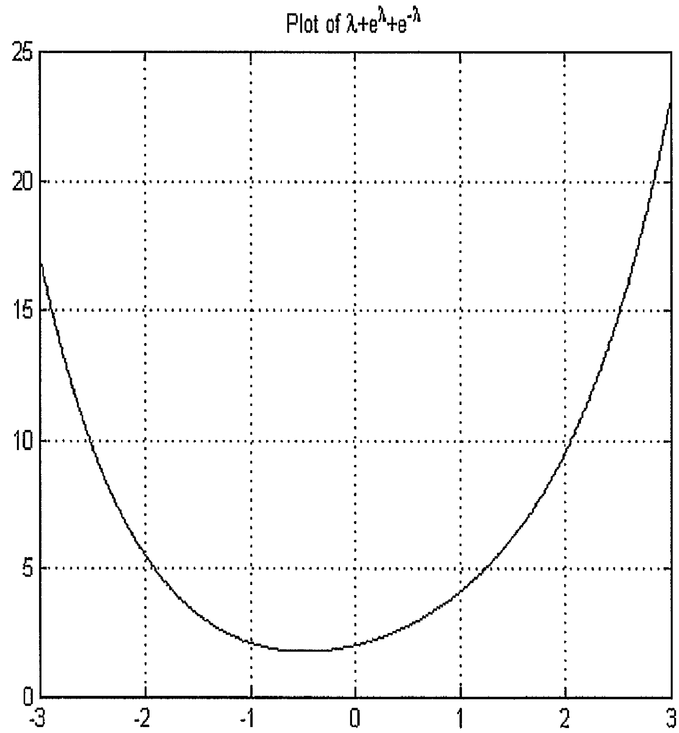

Figure 6.3.1: Results for Example 6.3.1: Plot of the characteristic function for $y'(t) + y(t+1) + y(t-1) = 0$.

Applying the Euler rule, we have,

$$\frac{\widetilde{y}_{m+1} - \widetilde{y}_m}{h} + y(t_m + Nh) + y(t_m - Nh) = 0$$

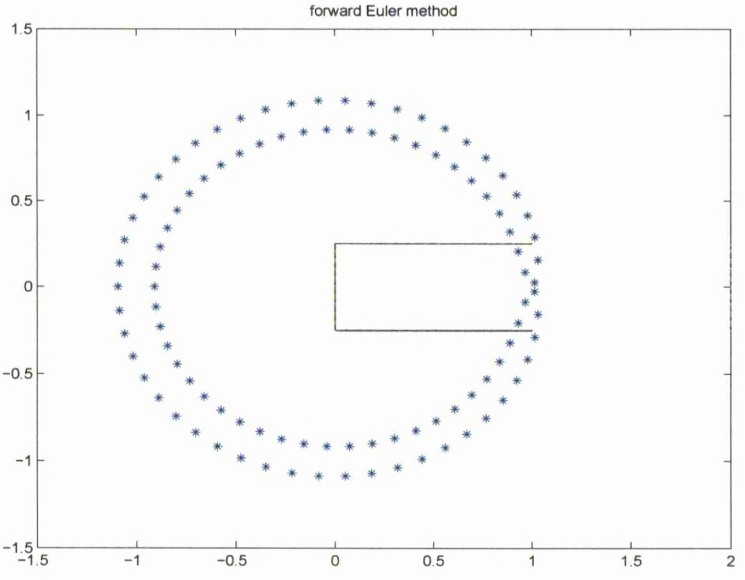

Figure 6.3.2: Results for Example 6.3.1: Each symbol $*$ represents a characteristic value obtained using $h = 0.01$, $M = 20$ and the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$

where, $h > 0$, $Nh = 1$, $t_m = mh$, $y(mh) \approx \widetilde{y}_m$

$$\widetilde{y}_{m+1} - \widetilde{y}_m + hy((m+N)h) + h\widetilde{y}((m-N)h) = 0$$

$$\widetilde{y}_{m+1} - \widetilde{y}_m + h\widetilde{y}_{m+N} + h\widetilde{y}_{m-N} = 0$$

Replace m *by* $m + N$ *to obtain*

$$\widetilde{y}_{m+N+1} - \widetilde{y}_{m+N} + h\widetilde{y}_{m+N+N} + h\widetilde{y}_{m+N-N} = 0$$

The characteristic equation can be written $\lambda^{N+1} - \lambda^N + h\lambda^{N+N} + h\lambda^{N-N} = 0$ *or*

$$h\lambda^{2N} + \lambda^{N+1} - \lambda^N + h\lambda^0 = 0;$$

its roots are the characteristic value. *Using the* MATLAB *program all roots of the characteristic equation are represented by the diagram 6.3.2. Obviously, there is no positive real root. Just one negative real root and the rest are all complex roots with non-zero imaginary part, which satisfy the criteria for oscillation of the discrete equation. Therefore, the behaviour reproduces in the numerical scheme.*

A summary of the results is presented in Table 6.3.1

Step length h	Length of Rectangle M	Number of zeros
0.01	2	18.0938
0.01	4	8
0.001	10	4
0.001	20	2
0.001	30	2
0.001	40	2
0.0001	Large	0

Table 6.3.1: Results for Example 6.3.1: Number of zeros located inside γ_M by the Argument Principle 5.2.1.

Remark 6.3.1 *The above Diagram 6.3.2 and the Table 6.3.1 confirm that the corresponding discrete equation is also oscillatory.*

6.3.2 Experiments for counting zeros of mixed type IDEs

In this section we consider some example of oscillatory mixed type IDEs with differentiable delays and advances.

Example 6.3.2 *Let us consider the oscillatory mixed type IDE*

$$y'(t) = \int_{-1}^{1} y(t+s)ds + y(t+1)$$

The characteristic function can be written $\chi(\lambda) = \lambda - \frac{\exp(\lambda)-\exp(-\lambda)}{\lambda} - \exp(\lambda)$
Applying the explicit Euler rule, we have,

$$\frac{\widetilde{y}_{m+1} - \widetilde{y}_m}{h} = \sum_{k=-N}^{N-1} y(t_m + kh).h + y(t_m + Nh) \tag{6.3.1}$$

where $h > 0$*,* $Nh = 1$*,* $t_m = t_0 + mh$*,* $y(t_m) \approx \widetilde{y}_m$*. Thus,*

$$\widetilde{y}_{m+1} - \widetilde{y}_m = h^2 \sum_{k=-N}^{N-1} \widetilde{y}(t_{(m+k)}) + h\widetilde{y}(t_{(m+N)})$$

and thus

$$\widetilde{y}_{m+1} - \widetilde{y}_m - h^2[(\widetilde{y}_{m-N} + \widetilde{y}_{m-N+1} + \ldots + \widetilde{y}_{m-1} + \widetilde{y}_m) + (\widetilde{y}_{m+1} + \widetilde{y}_{m+2} + \ldots + \widetilde{y}_{m+N-1} + \widetilde{y}_{m+N-1})]$$
$$-h\widetilde{y}_{m+N} = 0.$$

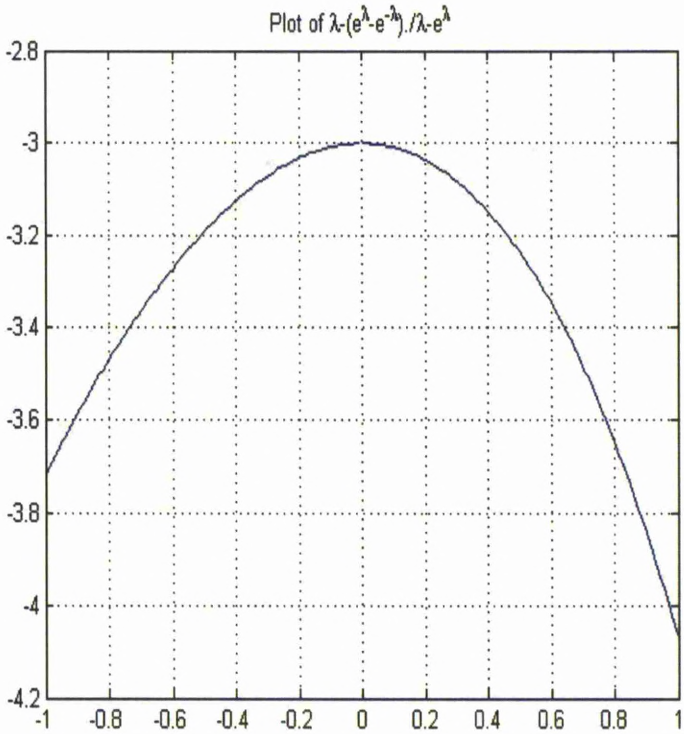

Figure 6.3.3: Results for Example 6.3.2: Plot of the characteristic function for $y'(t) = \int_{-1}^{1} y(t+s)ds + y(t+1)$.

The characteristic polynomial has the form $p(\lambda) = a_1\lambda^0 + a_2\lambda^1 + a_3\lambda^2 + ... + a_{2N+1}\lambda^{2N}$ *– a polynomial of degree* $2N$*.*

For various values of parameters all roots of the polynomial can be determined in order to indicate the behaviour of solutions of the discrete equation. The roots of the polynomial are represented by the above diagram using the Argument Principle. Obviously, there is no positive real root. All are complex roots which satisfy the criteria for oscillation of discrete equation. Therefore, the behaviour reproduces in the numerical scheme.

Summary of the experiment can be represented by the following Table 6.3.2:

Remark 6.3.2 *The above diagram and table confirm that the corresponding discrete equation is also oscillatory.*

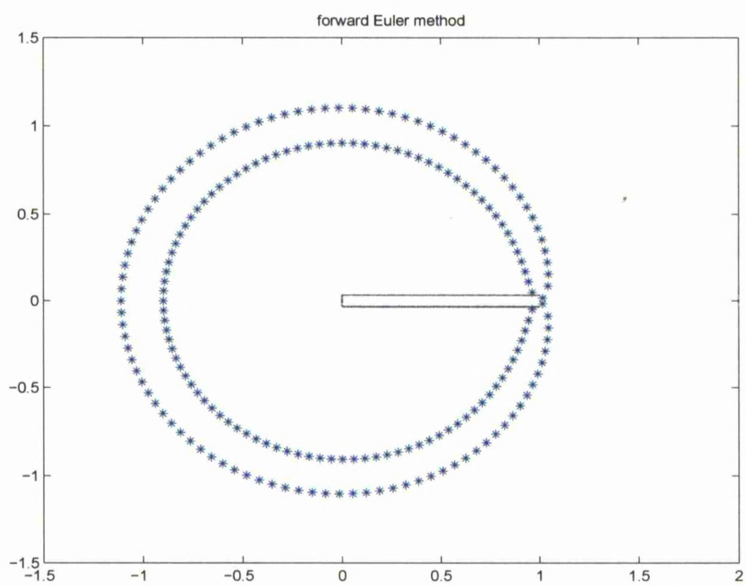

Figure 6.3.4: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.01$, $M = 30$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Step length h	Length of rectangle M	Number of zeros
0.02	2	18
0.02	4	8
0.002	10	4
0.002	20	2
0.0002	30	2
0.0002	Large	0

Table 6.3.2: Results for Example 6.3.2: Number of zeros located inside γ_M by the Argument Principle 5.2.1.

Example 6.3.3 *Let us consider the equation (1.5.28), namely*

$$y'(t) + \int_{-1}^{1} y(t+s)ds = 0,$$

a linear autonomous IDE of mixed type (see [89], [90], [91]). The characteristic function can be written $\chi(\lambda) = \lambda - \frac{\exp(\lambda)-\exp(-\lambda)}{\lambda}$

Applying Euler's explicit (forward) rule, we obtain,

$$\frac{\widetilde{y}_{m+1} - \widetilde{y}_m}{h} + h \sum_{k=-N}^{N-1} \widetilde{y}_{m+k} = 0 \tag{6.3.2}$$

where $y(t_m) \approx \widetilde{y}_m$*,* $Nh = 1$ *(*$h > 0$*,* $N \in \mathbb{N}$*). Thus, we have* $\widetilde{y}_{m+1} - \widetilde{y}_m + h^2 \sum_{K=-N}^{N-1} \widetilde{y}_{m+k} = 0$*, which we can write as*

$$\widetilde{y}_{m+1} - \widetilde{y}_m \tag{6.3.3}$$

$$+h^2[\widetilde{y}_{m-N} + \widetilde{y}_{m-N+1} + \ldots + \widetilde{y}_{m-1} + \widetilde{y}_m) + (\widetilde{y}_{m+1} + \widetilde{y}_{m+2} + \cdots + \widetilde{y}_{m+N-1}] = 0$$

This is a DVE derived from the MTIDE (1.5.28). The characteristic equation can be written

$$h^2 + h^2\lambda + h^2\lambda^2 + \ldots + (h^2-1)\lambda^N + (h^2+1)\lambda^{N+1} + \ldots + h^2\lambda^{N+N-1} + h^2\lambda^{N+N} = 0,$$

(which is satisfied when λ *is a characteristic root). Therefore, the polynomial can be written by*

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a4z^4 + \ldots\ldots + a_{2N} z^{2N}$$

where the coefficients are given by

$$a_1 = h^2, a_2 = h^2, \ldots, a_{N+1} = h^2 - 1, a_{N+2} = h^2 + 1, a_{N+3} = h^2, \ldots, a_{2N+1} = h^2.$$

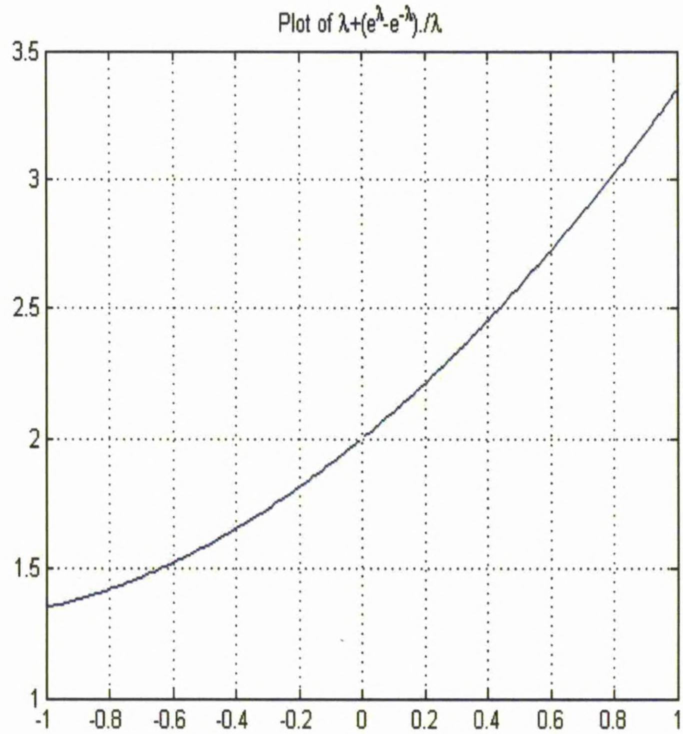

Figure 6.3.5: Results for Example 6.3.3: Plot of the characteristic function for $y'(t) + \int_{-1}^{1} y(t+s)ds = 0$.

By computation based on the Argument Principle for various values of parameters all roots of the polynomial can be determined which indicate the behaviour of the solutions of the discrete equation (3.8.4). The roots are displayed by the diagram 6.3.6. Obviously, there is no positive real root which satisfies the conditions for oscillation of discrete equation (3.8.4). Therefore, the behaviour reproduces in the numerical schemes.

Remark 6.3.3 *Finally the Figure 6.3.6 and 6.3.7 and the Table 6.3.3 illustrate the results and confirming that the discrete problem is also oscillatory.*

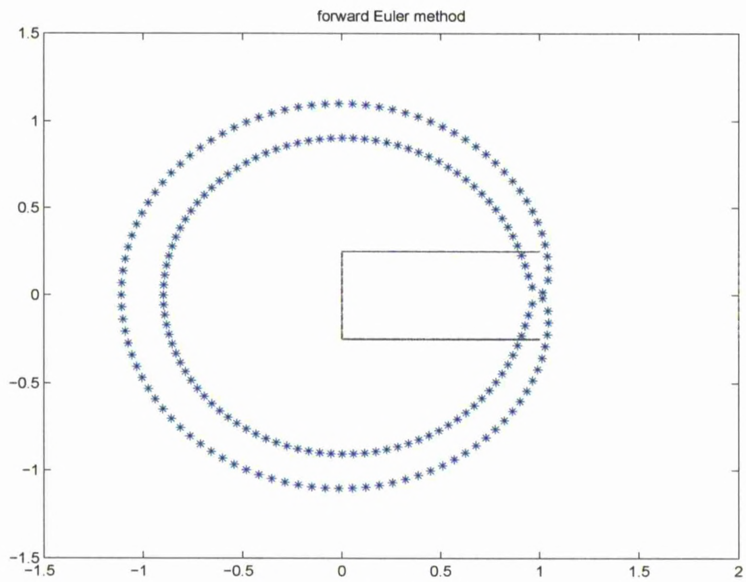

Figure 6.3.6: Results for Example 6.3.3: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.01$, $M = 4$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

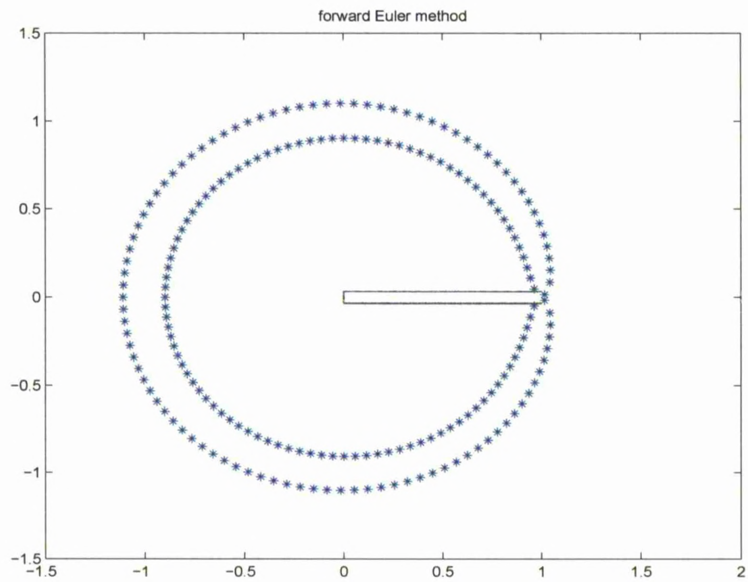

Figure 6.3.7: Results for Example 6.3.3: Each symbol $*$ represents a characteristic value of the discretised scheme obtained using $h = 0.01$, $M = 30$ inside the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Step length h	Length of rectangle M	Number of zeros
0.01	2	32
0.01	4	14
0.01	10	6
0.01	20	2
0.01	30	2
0.01	Large	0

Table 6.3.3: Results for Example 6.3.3: Number of zeros located inside γ_M by the Argument Principle 5.2.1.

6.4 Numerical treatment for further oscillatory mixed type IDEs with delays and advances using the Argument Principle

In this section, we show how numerical approximations can be used to derive information about oscillation or non-oscillation of solutions to a mixed-type equation. To begin, an overview of the approach is given, which builds on that adopted in [47]. More details are given later.

The general approach is to derive a discrete system that approximates the underlying mixed-type equation and to analyse the behaviour of solutions of the discrete scheme. The approach is adopted here is to use a very simple discretisation, based on an Euler rule to approximate the derivative on the left hand side of the equation, and a trapezoidal rule to approximate the integrals on the right hand side. In principle, one could use a more complicated approach, but the results we obtain here are very good and the method is already effective in our view.

As a general principle, a fixed step length $h > 0$ will be used and the resulting system of discrete equation will take the form of difference equations or a recurrence relation. This can be analysed using its characteristic equation and (for no oscillatory solutions) we are looking for the case where there are no non-negative real characteristic roots.

The root counting method is adopted (see [47]) for further discussion) is based as usual on an application of the argument principle and Rouché's Theorem to count zeros of a polynomial function inside a closed path. We choose the same rectangular path with vertices at $0 \pm \frac{1}{M}i$, $M \pm \frac{1}{M}i$ for large positive values of $M \in \mathbb{R}$ and count the zeros inside the rectangle as $M \to \infty$. As is seen in [47], one can show that the characteristic polynomial of the discrete problem has zeros close to the positive real axis only if the characteristic equation of the underlying continuous problem has characteristic values close to the real axis. Further details of the analytical results will be found in [47] (see also [8], [38], [51], [57], [58]).

For the detail, consider the numerical scheme for the equation (1.5.28),

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta) \tag{6.4.1}$$

where, $y(t) \in \mathbb{R}$, and ν and η are real functions of bounded variation on $[-1, 0]$ normalised in manner that $\nu(-1) = \eta(-1) = 0$, while r and τ are non-negative real continuous functions on $[-1, 0]$. The backward Euler method is used to approximate the time derivative and we use the trapezoidal method to approximate the integral. Then the corresponding discrete characteristic polynomial is obtained. Further we use the Argument Principle to find the numbers of real roots of the discrete characteristic polynomial. We observe that the equation (6.4.1) is oscillatory if and only if the characteristic polynomial has no real roots, which is consistent with the theoretic results.

We will describe how to find the discrete characteristic polynomial of (6.4.1). In all the numerical examples, it is assumed that $r(\theta), \nu(\theta), \tau(\theta)$ and $\eta(\theta)$ are quadratic polynomials.

6.4.1 A novel approach

The equation under consideration here is

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) \tag{6.4.2}$$

which is a special case of (6.4.1). Similar ideas can be applied to the integral $\int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta)$ when we consider (6.4.1). For examples see §6.4.2.

In a practical numerical algorithm, it is often necessary to know the maximum deviation that arises in a differential equation with deviating arguments. With this in mind, let us consider a different approach on how to discretise the integral $\int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta)$. The significant detail is that we wish to know the maximum value of $r(\theta)$.

With our new approach it is necessary. first, to find the critical points θ_r of $r(\theta)$ on $[-1, 0]$, i.e., points where $r'(\theta_r) = 0$. Assume for simplicity that $r(\theta)$ attains its maximum value at θ_r, i.e., $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$. It is also assumed that

$$r(-1) = r_{-1} > 0, \qquad r(0) = r_0 > 0.$$

Obviously, in this case $r(\theta_r) = r_c \geq \max\{r_{-1}, r_0\}$. The integral is now written in two parts:

$$\int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) = \int_{-1}^{\theta_r} y(t - r(\theta))\, d\nu(\theta) + \int_{\theta_r}^{0} y(t - r(\theta))\, d\nu(\theta).$$

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < \ldots$ be time points and let $h = t_{j+1} - t_j$ be the time step. The idea in the discretisation of the integral $\int_{-1}^{\theta_r} y(t + r(\theta))\, d\nu(\theta)$ is to find two nonnegative integers $N_1,\ N_2,\ N_1 > N_2$ such that

$$-1 = \theta_{-N_1} < \theta_{-N_1+1} < \cdots < \theta_{-N_2} = \theta_r,$$

is a partition of $[-1, \theta_r]$ and

$$r(\theta_{-N_1}) = r(-1) = r_{-1} = N_1 h, \tag{6.4.3}$$

$$r(\theta_j) = N_1 h + m_r(N_1 + j)h, \quad j = -N_1 + 1, -N_1 + 2, \ldots, -\omega_1 + N_1 - N_2 - 1), \tag{6.4.4}$$

$$r(\theta_{-N_2}) = r(\theta_r) = r_c = N_1 h + m_r(N_1 - N_2)h. \tag{6.4.5}$$

Here m_r is some positive integer which guarantees that $\omega_2 \geq 0$. Such ω_1 and ω_2 can be obtained by (6.4.3) and (6.4.5),

$$N_1 = \frac{r_{-1}}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r}. \tag{6.4.6}$$

Note that $\theta_j,\ j = -N_1, -N_1 + 1, \ldots, -N_1 + (N_1 - \omega_2)$ can be obtained by solving (6.4.3) - (6.4.5) for the given $r(\theta)$.

The idea of the discretisation of the integral $\int_{\theta_r}^{0} y(t + r(\theta))\, d\nu(\theta)$ is to find two nonnegative integers $N_3,\ N_4$ such that

$$\theta_r = \theta_{-N_3} < \theta_{-N_3+1} < \cdots < \theta_{-1} < \theta_0 = 0,$$

is a partition of $[\theta_r, 0]$ and

$$r(\theta_0) = r(0) = r_0 = N_4 h, \tag{6.4.7}$$

$$r(\theta_l) = (N_3 h + N_4 h) - (N_3 + l)h, \quad l = -N_3 + 1, -N_3 + 2, \ldots, -1, \tag{6.4.8}$$

$$r(\theta_{-N_3}) = r(\theta_r) = r_c = N_3 h + N_4 h. \tag{6.4.9}$$

Such N_3 and N_4 can be obtained by (6.4.7) and (6.4.9),

$$N_4 = \frac{r_0}{h}, \qquad N_3 = \frac{r_c}{h} - N_4. \tag{6.4.10}$$

Note that $\theta_l,\ l = -N_3, -N_3+1, \ldots, -1, 0$ can be obtained by solving (6.4.7) - (6.4.9) for the given $r(\theta)$.

Now the integral $\int_{-1}^{0} y(t + r(\theta))\, d\nu(\theta)$ at $t = t_n$ can be discretised. We have

$$\int_{-1}^{0} y(t_n - r(\theta))\, d\nu(\theta) = \int_{-1}^{\theta_r} y(t_n - r(\theta))\, d\nu(\theta) + \int_{\theta_r}^{0} y(t_n - r(\theta))\, d\nu(\theta) \tag{6.4.11}$$

$$\approx \sum_{j=-N_1}^{-N_2-1} y(t_n - r(\theta_j))\Big(\nu(\theta_{j+1}) - \nu(\theta_j)\Big)$$

$$+ \sum_{l=-N_3}^{-1} y(t_n - r(\theta_l))\Big(\nu(\theta_{l+1}) - \nu(\theta_l)\Big)$$

$$= \sum_{j=-N_1}^{-N_2-1} y(nh - [N_1 h + m_r(N_1 + j)h])\Big(\nu(\theta_{j+1}) - \nu(\theta_j)\Big)$$

$$+ \sum_{l=-N_3}^{-1} y(nh - [N_3 h + N_4 h - (N_3 + l)h])\Big(\nu(\theta_{l+1}) - \nu(\theta_l)\Big).$$

Similarly the integral $\int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta)$ can be discretised. Now let us summarise the steps to find the characteristic polynomial of the *discretised* version of (6.4.2).

Step 1. Find the critical point[1] θ_r of $r(\theta)$ on $[-1, 0]$. Without loss of the generality, this is assumed that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} > 0,\ r(0) = r_0 > 0$.

Step 2. Find the nonnegative integers $N_1, N_2,\ N_1 > N_2$ by

$$N_1 = \frac{r_{-1}}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r},$$

where $r_c = r(\theta_r)$ and m_r is some positive integer such that $N_2 \geq 0$.

Find the nonnegative integers N_3 and N_4 by

$$N_4 = \frac{r_0}{h}, \qquad N_3 = \frac{r_c}{h} - N_4.$$

[1]The critical value $r(\theta_r)$ represents the maximum delay over $[-1, 0]$ and this determines the maximum numbers of terms in the history of the solution which will need to be retained by the algorithm.

Step 3. Find the critical point θ_τ of $\tau(\theta)$ on $[-1,0]$. Without loss of the generality, this is assumed that $\tau(\theta)$ is increasing on $[-1,\theta_\tau]$ and decreasing on $[\theta_\tau,0]$ and $\tau(-1)=\tau_{-1}>0$, $\tau(0)=\tau_0>0$.

Step 4. Find the nonnegative integers M_1, M_2, $M_1>M_2$ by

$$M_1=\frac{\tau_{-1}}{h},\qquad M_2=\frac{(m_\tau+1)M_1-\frac{\tau_c}{h}}{m_\tau},$$

where $\tau_c=\tau(\theta_\tau)$ and m_τ is some positive integer such that $M_2\geq 0$.

Find the nonnegative integers M_3 and M_4 by

$$M_4=\frac{\tau_0}{h},\qquad M_3=\frac{\tau_c}{h}-M_4.$$

Step 5. Approximating the time derivative in (6.4.2) by the backward Euler method and approximating the integral in (6.4.2) by the Trapezoidal method, we obtain, at time t_n,

$$\begin{aligned}\frac{y(t_{n+1})-y(t_n)}{h}\approx&\sum_{j=-N_1}^{-N_2-1}y\big(nh-[N_1h+m_r(N_1+j)h]\big)\big(\nu(\theta_{j+1})-\nu(\theta_j)\big)\\&+\sum_{l=-N_3}^{-1}y\big(nh-[N_3h+N_4h-(N_3+l)h]\big)\big(\nu(\theta_{l+1})-\nu(\theta_l)\big)\\&+\sum_{j=-M_1}^{-M_2-1}y\big(nh+[M_1h+m_\tau(M_1+j)h]\big)\big(\eta(\theta'_{j+1})-\eta(\theta'_j)\big)\\&+\sum_{l=-M_3}^{-1}y\big(nh+[M_3h+M_4h-(M_3+l)h]\big)\big(\eta(\theta'_{l+1})-\eta(\theta'_l)\big).\end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j)=N_1h+m_r(N_1+j)h,\quad j=-N_1,-N_1+1,\ldots,-N_2,$$

and

$$r(\theta_l)=(N_3h+N_4h)-(N_3+l)h,\quad l=-N_3,-N_3+1,\ldots,-1,0.$$

Similarly, θ'_j and θ'_l are determined by

$$r(\theta'_j)=M_1h+m_\tau(M_1+j)h,\quad j=-M_1,-M_1+1,\ldots,-M_2,$$

and

$$r(\theta'_l)=(M_3h+M_4h)-(M_3+l)h,\quad l=-M_3,-M_3+1,\ldots,-1,0.$$

Write $y_m \approx y(t_m)$, $n = 0, 1, 2, \ldots$. We have

$$\frac{y_{m+1} - y_m}{h} = \tag{6.4.12}$$

$$\begin{aligned}
&\sum_{j=-N_1}^{-N_2-1} y_{m-[N_1+m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\
&\quad + \sum_{l=-N_3}^{-1} y_{m-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\
&\qquad + \sum_{j=-M_1}^{-M_2-1} y_{m+[M_1+m_r(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\
&\qquad\quad + \sum_{l=-M_3}^{-1} y_{m+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big).
\end{aligned}$$

Write $N = \max\{N_1 + m_r(N_1 - N_2 - 1), N_3 + N_4\}$. The characteristic polynomial of (6.4.12) is obtained as

$$\begin{aligned}
\chi^D(z) = z^{N+1} - z^N - h\Big[&\sum_{j=-N_1}^{-N_2-1} z^{n-[N_1+m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\
&- \sum_{l=-N_3}^{-1} z^{n-[N_3+N_4-N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\
&- \sum_{j=-M_1}^{-M_2-1} z^{n+[M_1+m_r(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\
&- \sum_{l=-M_3}^{-1} z^{n+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big].
\end{aligned}$$

Step 6. Apply the Argument Principle to determine the existence of the positive real roots of the characteristic polynomial $p(z) \equiv \chi^D(z)$.

$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}\,dz = \text{Number of zeros of } p(z) \text{ inside the closed curve } \gamma.$$

In the numerical simulation, the curve γ is chosen as the boundary of a rectangle with vertices $A = +\mathrm{i}\frac{1}{M}$, $B = -\mathrm{i}\frac{1}{M}$, $C = M - \mathrm{i}\frac{1}{M}$ and $D = M + \mathrm{i}\frac{1}{M}$ for some sufficiently large M.

Remark 6.4.1 *A similar idea can be used to work on the case where* $r(-1) = r_{-1} < 0$ *and* $r(0) = r_0 < 0$*, or* $r(-1) \cdot r(0) < 0$.

Remark 6.4.2 *A similar idea also can be used to work on the case where* $r(\theta)$ *(or* $\tau(\theta)$*) is decreasing on* $[-1, \theta_c]$ *(or* $[-1, \theta_\tau]$*) and increasing on* $[\theta_c, 0]$ *(or* $[\theta_\tau, 0]$*).*

6.4.2 Examples

Below, we will consider how to construct the discrete characteristic polynomials for a selection of examples.

Example 6.4.1 *Consider the equation (6.4.1) with the conditions of Example 4.5.4*

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta). \tag{6.4.13}$$

Here

$$\nu(\theta) = (3\theta + 1)(\theta + 1), \qquad \eta(\theta) = (\theta + 1)(2\theta + 1),$$

and

$$r(\theta) = -\frac{3}{2}\theta^2 - \theta + 1, \qquad \tau(\theta) = -\theta^2 - \theta + 2.$$

Let us find the discrete characteristic polynomial of (6.4.13). *First the critical point* θ_r *of* $r(\theta)$ *on* $[-1, 0]$ *is to be found. Let* $r'(\theta) = -3\theta - 1 = 0$*. we get* $\theta_r = -\frac{1}{3}$*. Further it is easy to find that* $r(\theta)$ *is increasing on* $[-1, \theta_r]$ *and decreasing on* $[\theta_r, 0]$ *and* $r(-1) = r_{-1} = \frac{1}{2} > 0$*,* $r(0) = r_0 = 1 > 0$ *and* $r(\theta_r) = r_c = \frac{7}{6}$.

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ *can be determined by*

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{2h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = \frac{1}{6h},$$

where $m_r = 2$ *is chosen which guarantees that* $N_2 > 0$.

The nonnegative integers N_3 *and* N_4 *can be determined by*

$$N_4 = \frac{r_0}{h} = \frac{1}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{1}{6h}.$$

Next the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1, 0]$ *will be found. Let* $\tau'(\theta) = -2\theta - 1 = 0$*. we get* $\theta_\tau = -\frac{1}{2}$*. Further it is easy to find that* $\tau(\theta)$ *is increasing on* $[-1, \theta_\tau]$ *and decreasing on* $[\theta_\tau, 0]$ *and* $\tau(-1) = \tau_{-1} = 2 > 0$*,* $\tau(0) = \tau_0 = 2 > 0$ *and* $\tau(\theta_\tau) = \tau_c = 2.25$.

The nonnegative integers $M_1, M_2,\ M_1 > M_2$ *can be determined by*

$$M_1 = \frac{\tau_{-1}}{h} = \frac{2}{h}, \qquad M_2 = \frac{(m_\tau + 1)M_1 - \frac{\tau_c}{h}}{m_\tau} = \frac{7}{4h},$$

where $m_\tau = 1$ *is chosen which guarantees that* $M_2 \geq 0$.

The nonnegative integers M_3 *and* M_4 *can be determined by*

$$M_4 = \frac{\tau_0}{h} = \frac{2}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{4h}.$$

Finally, $N = \max\{N_1 + 2(N_1 - N_2 - 1), N_3 + N_4\}$ *is denoted. Then the following discrete characteristic equation of* (6.4.13) *is obtained*

$$\begin{aligned} p(z) = -z^{N+1} + z^N + h\Big[& \sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ & + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ & + \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\ & + \sum_{l=-M_3}^{-1} z^{\omega+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned}$$

Here θ_j *and* θ_l *are determined by*

$$r(\theta_j) = -\frac{3}{2}\theta_j^2 - \theta_j + 1 = (3N_1 + 2j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -\frac{3}{2}\theta_l^2 - \theta_l + 1 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{1 + \sqrt{1 + 6\big(1 - (3N_1 + 2j)h\big)}}{2 \times (-3/2)}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$\theta_l = \frac{1 - \sqrt{1 + 6\big(1 - N_4 h + lh\big)}}{2 \times (-3/2)}, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

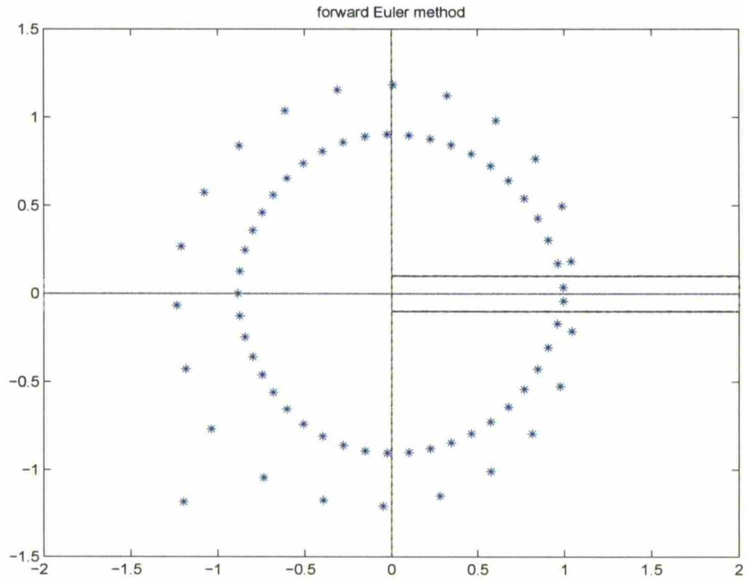

Figure 6.4.1: Results for Example 6.4.1: Each symbol $*$ represents a characteristic value obtained using $h = 0.05$, $M = 10$ and the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Similarly, θ'_j and θ'_l are determined by

$$\theta'_j = \frac{1 + \sqrt{1 + 4\big(2 - (2M_1 + j)h\big)}}{2 \times (-1)}, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$\theta'_l = \frac{1 - \sqrt{1 + 4\big(2 - M_4 h + lh\big)}}{2 \times (-1)}, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Remark 6.4.3 *Applying the Argument Principle, we have discovered that $p(z)$ has no positive real roots and so this satisfies the conditions for discrete equation to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillatory property of the equation* (6.4.13)*. See Figure 6.4.1 and Table 6.4.1.*

Example 6.4.2 *Consider the equation (6.4.1) of Example 4.5.5*

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta). \tag{6.4.14}$$

Step length h	Length of rectangle M	Number of zeros
0.05	2	12
0.05	4	6
0.05	10	2
0.05	20	2
0.05	Large	0

Table 6.4.1: Results for Example 6.4.1: Number of zeros located inside γ_M by the Argument Principle 5.2.1.

Here

$$\nu(\theta) = \begin{cases} -\theta - 1, & -1 \leq \theta < 0, \\ 1, & \theta = 0, \end{cases}$$

and

$$\eta(\theta) = \theta + 1, \qquad r(\theta) = \theta + 2, \qquad \tau(\theta) = -\theta + 1.$$

Now the discrete characteristic polynomial of (6.4.14) *can be found. We first find the critical point* θ_r *of* $r(\theta)$ *on* $[-1, 0]$*. We get* $\theta_r = 0$*. It is easy to see that* $r(\theta)$ *is increasing on* $[-1, \theta_r]$ *and* $r(-1) = r_{-1} = 1 > 0$, $r(0) = r_0 = 2 > 0$ *and* $r(\theta_r) = r_c = 2$.

The nonnegative integers N_1, N_2, $N_1 > N_2$ *can be determined by*

$$\omega_1 = \frac{r_{-1}}{h} = \frac{1}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 0,$$

where $m_r = 1$ *is chosen which guarantees that* $N_2 \geq 0$.

Next the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1, 0]$ *will be found. Then* $\theta_\tau = -1$*. It is easy to see that* $\tau(\theta)$ *is decreasing on* $[\theta_\tau, 0]$ *and* $\tau(-1) = \tau_{-1} = 2 > 0$, $\tau(0) = \tau_0 = 1 > 0$ *and* $\tau(\theta_\tau) = \tau_c = 2$.

The nonnegative integers M_3, M_4 *can be determined by*

$$M_4 = \frac{\tau_0}{h} = \frac{1}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{h}.$$

Finally, $N_1 + (N_1 - N_2 - 1)$ *is denoted by* N*. Then the following discrete characteristic equation of* (6.4.14) *is obtained*

$$p(z) = -z^{N+1} + z^N + h\Big[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big)$$
$$+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \quad (6.4.15)$$

Here θ_j are determined by

$$r(\theta_j) = \theta_j + 2 = N_1 h + (N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

which implies that

$$\theta_j = N_1 h + (N_1 + j)h - 2, \quad j = -N_1, -N_1 + 1, \ldots, -N_2.$$

Similarly, the θ'_l are determined by

$$\tau(\theta'_l) = -\theta'_l + 1 = (M_3 h + M_4 h) - (M_3 + l)h, \quad j = -M_3, -M_3 + 1, \ldots, -1, 0.$$

which implies that

$$\theta'_l = -(M_3 h + M_4 h) + (M_3 + l)h + 1, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Remark 6.4.4 *Note that $\nu(\theta)$ has a jump at $\theta = 0$, therefore we have, in* (6.4.15),

$$\begin{aligned} \nu(\theta_{-N_2}) - \nu(\theta_{-N_2-1}) &= \nu(0) - \nu(N_1 h + N_1 h - N_2 h - h) \\ &= \nu(0) - \nu(2 - h) = 1 - \big(-(2 - h) - 1 \big) \\ &= 4 - h, \end{aligned}$$

Remark 6.4.5 *Applying the Principle of the Argument, we find that $p(z)$ has no positive real roots and therefore satisfies the conditions for the discrete equation to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation property for the equation* (6.4.14). *See Figure 6.4.2.*

Example 6.4.3 *Consider the equation (6.4.1) for the example 4.5.6*

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta). \tag{6.4.16}$$

Here

$$\nu(\theta) = (5\theta + 4)(\theta + 1), \qquad \eta(\theta) = (10\theta + 9)(\theta + 1),$$

and

$$r(\theta) = -\frac{5}{2}\theta^2 - 4\theta + 5, \qquad \tau(\theta) = -5\theta^2 - 9\theta + 1.$$

Now the discrete characteristic polynomial of (6.4.16) *can be found. First the critical point θ_r of $r(\theta)$ on $[-1, 0]$ is found. Let $r'(\theta) = -5\theta - 4 = 0$. Then we can see that $\theta_r = -\frac{4}{5}$. Further it is easy to find that $r(\theta)$ is increasing on*

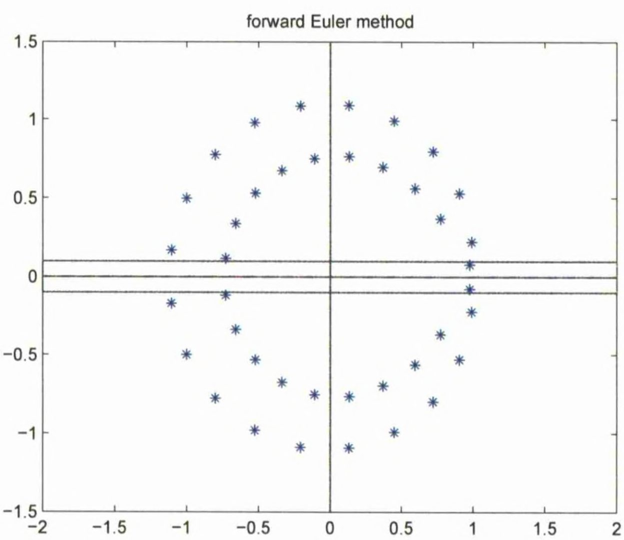

Figure 6.4.2: Results for Example 6.4.2: Each symbol $*$ represents a characteristic value obtained using $h = 0.01$, $M = 8$ and the rectangle Γ_M with corners at $(-M \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$

$[-1,\theta_r]$ *and decreasing on* $[\theta_r,0]$ *and* $r(-1)=r_{-1}=6.5>0,\ r(0)=r_0=5>0$ *and* $r(\theta_r)=r_c=6.6$.

The nonnegative integers $N_1, N_2,\ N_1>N_2$ *can be determined by*

$$N_1=\frac{r_{-1}}{h}=\frac{6.5}{h},\qquad N_2=\frac{(m_r+1)N_1-\frac{r_c}{h}}{m_r}=2N_1-\frac{6.6}{h},$$

where $m_r=1$ *is chosen which guarantees that* $N_2\geq 0$.

The nonnegative integers N_3 *and* N_4 *can be determined by*

$$N_4=\frac{r_0}{h}=\frac{5}{h},\qquad N_3=\frac{r_c}{h}-N_4=\frac{6.6}{h}-N_4.$$

Next the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1,0]$ *will be found. Let* $\tau'(\theta)=-10\theta-9=0$. *we get* $\theta_\tau=-\frac{9}{10}$. *Further it is easy to find that* $\tau(\theta)$ *is increasing on* $[-1,\theta_\tau]$ *and decreasing on* $[\theta_\tau,0]$ *and* $\tau(-1)=\tau_{-1}=5>0,\ \tau(0)=\tau_0=1>0$ *and* $\tau(\theta_\tau)=\tau_c=5.05$.

The nonnegative integers $M_1, M_2,\ M_1>M_2$ *can be determined by*

$$M_1=\frac{\tau_{-1}}{h}=\frac{5}{h},\qquad M_2=\frac{(m_\tau+1)M_1-\frac{\tau_c}{h}}{m_\tau}=2M_1-\frac{5.05}{h},$$

where $m_\tau=1$ *is chosen which guarantees that* $M_2\geq 0$.

The nonnegative integers M_3 *and* M_4 *can be determined by*

$$M_4=\frac{\tau_0}{h}=\frac{2}{h},\qquad M_3=\frac{\tau_c}{h}-M_4=\frac{1}{4h}.$$

Finally, $N=\max\{N_1+2(N_1-N_2-1),N_3+N_4\}$ *is denoted. Then the following discrete characteristic equation of* (6.4.16) *is obtained*

$$\begin{aligned}p(z)=-z^{N+1}+z^N+h\Big[&\sum_{j=-N_1}^{-N_2-1}z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1})-\nu(\theta_j)\big)\\&+\sum_{l=-N_3}^{-1}z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1})-\nu(\theta_l)\big)\\&+\sum_{j=-M_1}^{-M_2-1}z^{N+[M_1+(M_1+j)]}\big(\eta(\theta'_{j+1})-\eta(\theta'_j)\big)\\&+\sum_{l=-M_3}^{-1}z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1})-\eta(\theta'_l)\big)\Big].\end{aligned}$$

Here θ_j *and* θ_l *are determined by*

$$r(\theta_j)=-\frac{5}{2}\theta_j^2-4\theta_j+5=(2N_1+j)h,\quad j=-N_1,-N_1+1,\ldots,-N_2,$$

and

$$r(\theta_l) = -\frac{5}{2}\theta_j^2 - 4\theta_j + 5 = N_4 h - lh, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{4 + \sqrt{16 + 10\big(5 - (2N_1 + j)h\big)}}{2 \times (-5/2)}, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

and

$$\theta_l = \frac{4 - \sqrt{16 + 10\big(5 - N_4 h + lh\big)}}{2 \times (-5/2)}, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

Similarly, θ'_j and θ'_l are determined by

$$\theta'_j = \frac{9 + \sqrt{81 + 20\big(1 - (2M_1 + j)h\big)}}{2 \times (-5)}, \quad j = -M_1, -M_1+1, \ldots, -M_2,$$

and

$$\theta'_l = \frac{9 - \sqrt{81 + 20\big(1 - M_4 h + lh\big)}}{2 \times (-5)}, \quad l = -M_3, -M_3+1, \ldots, -1, 0.$$

Remark 6.4.6 *Applying the Principle of the Argument, we find that $p(z)$ has no positive real roots which means that the equation satisfies the conditions to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation* (6.4.16). *See Figure 6.4.3 and Table 6.4.2.*

Step length h	Length of rectangle M	Number of zeros	
0.05	2	78	
0.05	4	38	
0.05	10	14	
0.05	20	8	
0.05	Large	0	

Table 6.4.2: Results for Example 6.4.3: Number of zeros located inside Γ_M by the Argument Principle 5.2.1.

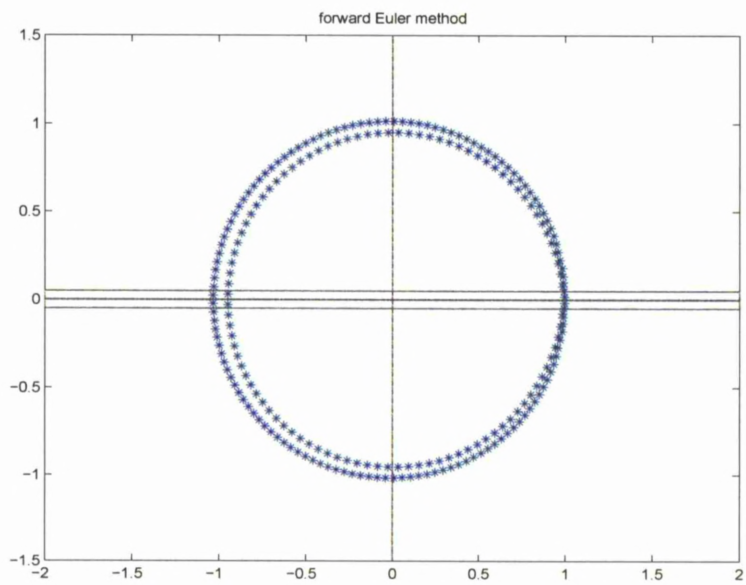

Figure 6.4.3: Results for Example 6.4.3: Each symbol $*$ represents a characteristic value obtained using $h = 0.05$, $M = 20$ and the rectangle γ_M with corners at $(-M \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Example 6.4.4 *Consider the equation (6.4.1) for Example 4.5.7*

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta). \tag{6.4.17}$$

Here

$$\nu(\theta) = (-\theta - 1)(4\theta + 3), \qquad \eta(\theta) = -8\theta - 8,$$

and

$$r(\theta) = -2\theta^2 - 3\theta + 1, \qquad \tau(\theta) = -\theta + 1.$$

Now the discrete characteristic polynomial of (6.4.17) *is found. First the critical point* θ_r *of* $r(\theta)$ *on* $[-1, 0]$ *is found. Let* $r'(\theta) = -4\theta - 3 = 0$. *We get* $\theta_r = -\frac{3}{4}$. *Further it is easy to find that* $r(\theta)$ *is increasing on* $[-1, \theta_r]$ *and decreasing on* $[\theta_r, 0]$ *and* $r(-1) = r_{-1} = 2 > 0$, $r(0) = r_0 = 1 > 0$ *and* $r(\theta_r) = r_c = \frac{17}{8}$.

The nonnegative integers N_1, N_2, $N_1 > N_2$ *can be determined by*

$$N_1 = \frac{r_{-1}}{h} = \frac{2}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{17}{8h},$$

where $m_r = 1$ *is chosen which guarantees that* $N_2 \geq 0$.

The nonnegative integers N_3 *and* N_4 *can be determined by*

$$N_4 = \frac{r_0}{h} = \frac{1}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{17}{8h} - N_4.$$

Next the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1, 0]$ *will be found. We obtain* $\theta_\tau = -1$. *Further it is easy to find that* $\tau(\theta)$ *is decreasing on* $[\theta_\tau, 0]$ *and* $\tau(-1) = \tau_{-1} = 2 > 0$, $\tau(0) = \tau_0 = 1 > 0$ *and* $\tau(\theta_\tau) = \tau_c = 2$.

The nonnegative integers M_3 *and* M_4 *can be determined by*

$$M_4 = \frac{\tau_0}{h} = \frac{1}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{2}{h} - M_4.$$

Finally, $N = \max\{N_1 + (N_1 - N_2 - 1), N_3 + N_4\}$ *is denoted. Then the following discrete characteristic equation for* (6.4.17) *is obtained*

$$\begin{aligned} p(z) = -z^{N+1} + z^N + h\Big[&\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ &+ \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ &+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -2\theta_j^2 - 3\theta_j + 1 = (2N_1 + j)h, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

and

$$r(\theta_l) = -2\theta_l^2 - 3\theta_l + 1 = N_4 h - lh, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{3 + \sqrt{9 + 8\big(1 - (2N_1 + j)h\big)}}{2 \times (-2)}, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

and

$$\theta_l = \frac{3 - \sqrt{9 + 8\big(1 - N_4 h + lh\big)}}{2 \times (-2)}, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

Similarly, θ_l' are determined by

$$\theta_l' = -M_4 h + lh + 1, \quad l = -M_3, -M_3+1, \ldots, -1, 0.$$

Remark 6.4.7 *Applying the Principle of the Argument, we find that $p((z)$ has no positive real roots which means that the equation satisfies the conditions to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation* (6.4.17). *See Figure 6.4.4 and Table 6.4.3.*

Step length h	Length of rectangle M	Number of zeros
0.05	2	6
0.05	8	2
0.05	10	2
0.05	20	2
0.05	30	2
0.05	Large	0

Table 6.4.3: Results for Example 6.4.4 Number of zeros located inside Γ_M by the Argument Principle 5.2.1.

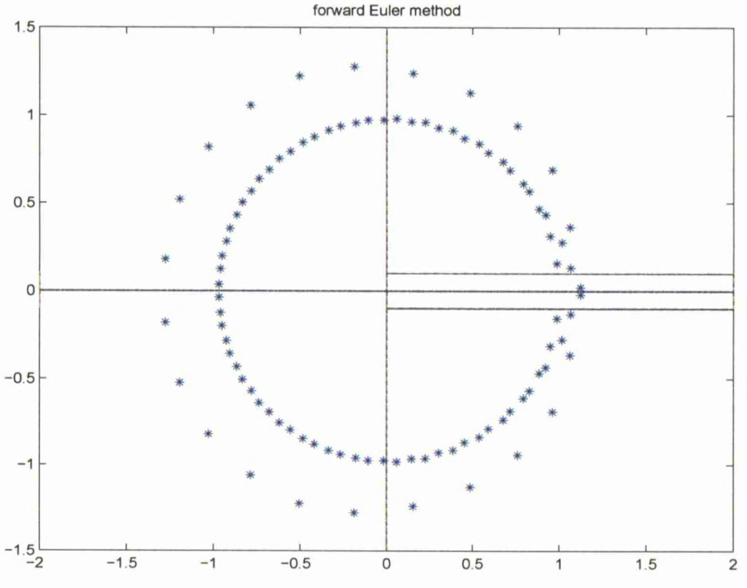

Figure 6.4.4: Results for Example 6.4.4: Each symbol $*$ represents a characteristic value obtained using $h = 0.05$, $M = 10$ and the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Example 6.4.5 *Consider the equation (6.4.1) for the example 4.5.8*

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta). \tag{6.4.18}$$

Here

$$\nu(\theta) = \begin{cases} \theta + 1, & -1 \le \theta < 0, \\ 0, & \theta = 0, \end{cases}$$

and

$$\eta(\theta) = -\theta - 1, \qquad r(\theta) = -\theta^2 + 2, \qquad \tau(\theta) = -\theta + 3.$$

Now the discrete characteristic polynomial of (6.4.18) *is found. First the critical point θ_r of $r(\theta)$ on $[-1, 0]$ is found. Let $r'(\theta) = -2\theta = 0$. Then it is seen that $\theta_r = 0$. Further it is easy to see that $r(\theta)$ is increasing on $[-1, \theta_r]$ and $r(-1) = r_{-1} = 1 > 0$, $r(0) = r_0 = 2 > 0$ and $r(\theta_r) = r_c = 2$.*

The nonnegative integers N_1, N_2, $N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{2}{h},$$

where $m_r = 1$ is chosen which guarantees that $N_2 \ge 0$.

Next the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$ will be found. We get $\theta_\tau = -1$. Further it is easy to see that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 4 > 0$, $\tau(0) = \tau_0 = 3 > 0$ and $\tau(\theta_\tau) = \tau_c = 4$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{3}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{h}.$$

Finally, $N = N_1 + (N_1 - N_2 - 1), N_3 + N_4\}$ is denoted. Then the following discrete characteristic equation of (6.4.18) *is obtained*

$$p(z) = -z^{N+1} + z^N + h\Big[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big)$$
$$+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big].$$

Here θ_j are determined by

$$r(\theta_j) = -\theta_j^2 + 2 = (2N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

which implies that

$$\theta_j = -\sqrt{2 - (2N_1 + j)h}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2.$$

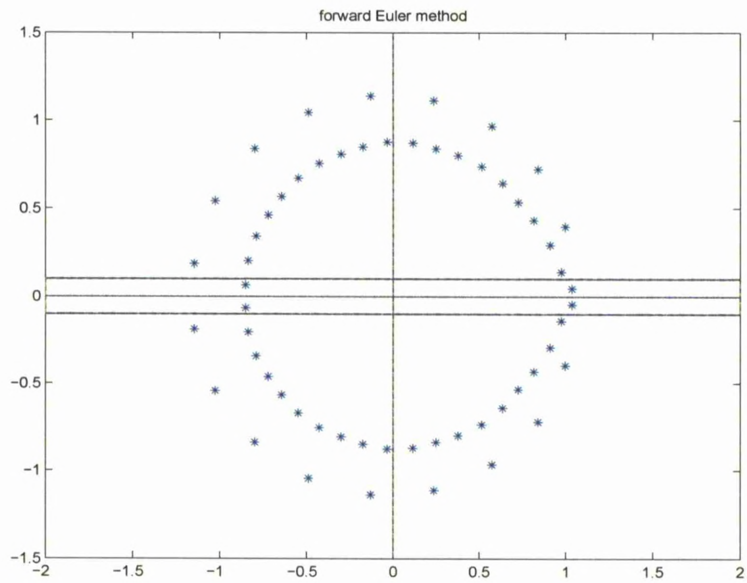

Figure 6.4.5: Results for Example 6.4.5: Each symbol $*$ represents a characteristic value obtained using $h = 0.01$, $M = 10$ and the rectangle Γ_M with corners at $(-M \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

Similarly, θ_l' are determined by

$$\tau(\theta_l') = -\theta_l' - 1 = M_4 h - lh,$$

which implies that

$$\theta_l' = -M_4 h + lh - 1, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Remark 6.4.8 *Applying the Principle of the Argument, we find that $p(z)$ has no positive real roots which mean that the equation satisfies the condition to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation* (6.4.18). *See Figure 6.4.5 and Table 6.4.4.*

Example 6.4.6 *Consider the equation (6.4.1) for Example 4.5.9*

$$y'(t) = \int_{-1}^{0} y(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} y(t + \tau(\theta))\, d\eta(\theta). \qquad (6.4.19)$$

Here

$$\nu(\theta) = -(5\theta + 1)(\theta + 1), \qquad \eta(\theta) = -(6\theta + 1)(\theta + 1),$$

Step length h	Length of rectangle M	Number of zeros
0.01	2	10
0.01	4	4
0.01	8	2
0.01	10	2
0.01	20	2
0.01	Large	0

Table 6.4.4: Results for Example 6.4.5: Number of zeros located inside Γ_M by the Argument Principle 5.2.1.

and

$$r(\theta) = -10\theta^2 - 4\theta + 10, \qquad \tau(\theta) = -3\theta^2 - \theta + 1.$$

Now the discrete characteristic polynomial of (6.4.19) *is found. We First the critical point* θ_r *of* $r(\theta)$ *on* $[-1, 0]$ *is found. Let* $r'(\theta) = -20\theta - 4 = 0$. *Then it is seen that* $\theta_r = -\frac{1}{5}$. *Further it is easy to find that* $r(\theta)$ *is increasing on* $[-1, \theta_r]$ *and decreasing on* $[\theta_r, 0]$ *and* $r(-1) = r_{-1} = 4 > 0$, $r(0) = r_0 = 10 > 0$ *and* $r(\theta_r) = r_c = 10.4$.

The nonnegative integers N_1, N_2, $N_1 > N_2$ *can be determined by*

$$N_1 = \frac{r_{-1}}{h} = \frac{4}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = \frac{3N_1 - \frac{10.4}{h}}{2},$$

where we choose $m_r = 2$ *which guarantees that* $N_2 \geq 0$.

The nonnegative integers N_3 *and* N_4 *can be determined by*

$$N_4 = \frac{r_0}{h} = \frac{10}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{10.4}{h} - N_4.$$

Next the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1, 0]$ *will be found. Then it is seen that* $\theta_\tau = -\frac{1}{6}$. *Further it is easy to find that* $\tau(\theta)$ *is increasing on* $[-1, \theta_\tau]$ *and decreasing on* $[\theta_\tau, 0]$ *and* $\tau(-1) = \tau_{-1} = -1 < 0$, $\tau(0) = \tau_0 = 1 > 0$ *and* $\tau(\theta_\tau) = \tau_c = \frac{13}{12}$.

Note that here $\tau(-1) = \tau_{-1} = -1 < 0$. *Let us discretise the integral* $\int_{-1}^{\theta_\tau} x(t + \tau(\theta))\, d\eta(\theta)$. *It is necessary to find some nonnegative integers* M_1, M_2, $M_1 > M_2$ *such that* $-1 = \theta_{-M_1} < \theta_{-M_1+1} < \cdots < \theta_{-M_2} = \theta_\tau$ *is a partition of* $[-1, \theta_\tau]$ *and*

$$\begin{aligned}
&\tau(\theta_{-M_1}) = \tau(-1) = \tau_{-1} = -M_1 h,\\
&\tau(\theta_j) = -M_1 h + m_\tau(M_1 + j)h, \quad j = -M_1 + 1, -M_1 + 2, \ldots, -M_2 - 1,\\
&\tau(\theta_{-M_2}) = \tau(\theta_\tau) = \tau_c = -M_1 h + m_\tau(M_1 - M_2)h,
\end{aligned}$$

Here m_τ is some positive integer which guarantees that $M_2 \geq 0$. In fact, we can determine $M_1, M_2,\ M_1 > M_2$ by the following:

$$M_1 = -\frac{\tau_{-1}}{h} = \frac{1}{h},$$

and, with $m_\tau = 3$,

$$-M_1 h + m_\tau(M_1 - M_2)h = \frac{13}{12},$$

which implies that $M_2 = \frac{2M_1 - \frac{13}{12h}}{3} > 0$. This is remarkable that the bigger m_r is, the faster $\tau(\theta_j),\ j = -M_1+1, -M_1+2, \ldots, -M_2$ increase. In order to guarantee M_2 is nonnegative, we need to choose $m_r \geq 3$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{21}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{12h}.$$

Finally, $N = \max\{N_1 + m_r(N_1 - N_2 - 1), N_3 + N_4\}$ is denoted. Then the following discrete characteristic equation of (6.4.19) is obtained

$$\begin{aligned} p(z) = -z^{N+1} + z^N + h\Big[&\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ &+ \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ &+ \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\ &+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -10\theta_j^2 - 4\theta_j + 10 = N_1 h + m_r(N_1+j)h, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

and

$$r(\theta_l) = -10\theta_l^2 - 4\theta_l + 10 = N_4 h - lh, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{4 + \sqrt{16 + 40\big(10 - (N_1 h + m_r(N_1+j)h\big)}}{2 \times (-10)}, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

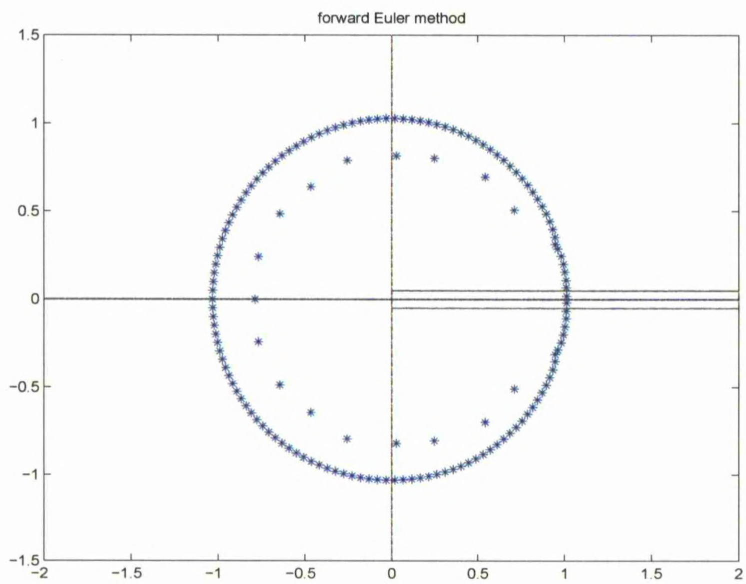

Figure 6.4.6: Results for Example 6.4.6: Each symbol $*$ represents a characteristic value obtained using $h = 0.05$, $M = 32$ and the rectangle γ_M with corners at $(0 \pm \mathrm{i}\frac{1}{M})$, $(M \pm \mathrm{i}\frac{1}{M})$.

and

$$\theta_l = \frac{4 - \sqrt{16 + 40(10 - N_4 h + lh)}}{2 \times (-10)}, \quad l = -N_3, -N_3 + 1, \dots, -1, 0.$$

Similarly, θ'_j and θ'_l are determined by

$$\theta'_j = \frac{1 + \sqrt{1 + 12(1 + M_1 h - m_\tau(M_1 + j)h)}}{2 \times (-3)}, \quad j = -M_1, -M_1{+}1, \dots, -M_2,$$

and

$$\theta'_l = \frac{1 - \sqrt{1 + 12(1 - M_4 h + lh)}}{2 \times (-3)}, \quad l = -M_3, -M_3 + 1, \dots, -1, 0.$$

Remark 6.4.9 *Applying the Principle of the Argument, we find that $p(z)$ has no positive real roots which means that the equation satisfies the condition to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation* (6.4.19). *See Figure 6.4.6 and Table 6.4.5.*

Step length h	Length of rectangle M	Number of zeros
0.05	2	24
0.05	4	12
0.05	8	6
0.05	10	4
0.05	20	2
0.05	30	2
0.05	Large	0

Table 6.4.5: Results for Example 6.4.6: Number of zeros located inside γ_M by the Principle of the Argument 5.2.1.

Remark 6.4.10 *We have seen that the numerical approach introduced here does provide a reliable method for determining whether or not linear mixed functional differential equations are oscillatory. Based on the experiments we have tried, the technique works also for non-linear problems, but there is a need for further analytical results in this case.*

Chapter 7

Theoretical justification of our numerical approaches

7.1 Introduction

The organisation of this Chapter is based on a perspective communicated by Professor Baker. We give results (in particular, a number of the formal statements) that he proposed to indicate the mathematical foundations of our methodology. The material also draws on insight obtainable from related work in the literature, including publications of Professor Ford and his co-authors, and the thesis of Dr Lumb [105] on small solutions.

7.1.1 Basic analysis

Before we recall the properties of auxiliary and characteristic functions and their generalisations, it is useful to remind ourselves of some relevant facts from analysis.

Lemma 7.1.1 (Bolzano-Weirstrass) *If a bounded set $\mathcal{S} \subset \mathbb{R}$ contains infinitely many points then there is at least one $x \in \mathbb{R}$ that is an accumulation point of $\mathcal{S}$.*

According to this well-known theorem (for which see [1, p.43]), the boundedness of $\mathcal{S}$ implies that there exists $\{x_k\}_0^\infty \subset \mathcal{S}$ such that $x = \lim_{k\to\infty} x_k$ exists ($x \in \mathbb{R}$).

We now recall some basic results for analytic functions of a complex variable. If f is a single-valued continuous complex function in a domain $\mathcal{D} \subset \mathbb{C}$ and f is complex-differentiable in $\mathcal{D}$ then f is described as being analytic (or holomorphic) on $\mathcal{D}$. Such an analytic function is infinitely differentiable.

Lemma 7.1.2 (Taylor series) *For z in an open neighbourhood $\mathcal{D}_0$ of z_0 ($\mathcal{D}_0 \subset \mathcal{D}$), $f(z)$ has a series expansion*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \textit{where } z, z_0 \in \mathcal{D}_0, \tag{7.1.1}$$

in which the values $\{a_j\}$ depend on z_0. Indeed, when $C_r(z_0)$ is (for a given r and z_0) a positively oriented circle $|z - z_0| = r$ that lies in $\mathcal{D}_0$,

$$a_j = \frac{f^{(j)}(z_0)}{j!} = \frac{1}{2\pi \mathrm{i}} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{j+1}} dw, \quad j \in \{0, 1, 2, \cdots\}. \tag{7.1.2}$$

Lemma 7.1.3 (Taylor sum with remainder) *As in Lemma 7.1.2, suppose that f is analytic on $\mathcal{D} \subset \mathbb{C}$, and let $C_r(z_0)$ again be the positively-oriented circle $|z - z_0| = r$. Then,*

$$f(z) = \sum_{j=0}^{m} \frac{(z - z_0)^j}{j!} f^j(z_0) + R_m(z_0; z) \textit{ for } z, z_0 \in C_r(z_0), \tag{7.1.3}$$

where

$$R_m(z_0; z) = \sum_{j=m+1}^{\infty} \frac{(z - z_0)^j}{j!} f^j(z_0) = \frac{(z - z_0)^{m+1}}{2\pi \mathrm{i}} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{m+1}(w - z)} dw. \tag{7.1.4}$$

Corollary 7.1.4 *Given that $C_r(z_0)$ is the circle with centre z_0 and radius r,*

$$|R_m(z_0; z)| \le \max_{w \in C_r(z_0)} |f(w)| \frac{\beta^{m+1}}{1 - \beta} \textit{ for } |z - z_0| \le \beta r \textit{ with } 0 \le \beta < 1, \tag{7.1.5}$$

and, hence, the magnitude of $|R_m(z_0; z)|$ depends on the maximum absolute value assumed by f on the circle $C_r(z_0)$:

$$M_r = \max_{w \in r(z_0)} |f(w)| \equiv \max_{0 \le \theta \le 2\pi} |f(z_0 + r \exp(\mathrm{i}\theta))|. \tag{7.1.6}$$

See, for example {`http://en.wikipedia.org/wiki/Taylor's\_theorem`}.

Example 7.1.1 *If we choose $z_0 = 0$, we have*

$$\exp(z) = 1 + z + z^2/2! + z^3/3! + \cdots, \textit{ for all } z \in \mathbb{C};$$

$$\exp(-z) = 1 - z + z^2/2! - z^3/3! \cdots, \textit{ for all } z \in \mathbb{C};$$

and

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots \textit{ provided } |z| < 1;$$

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots \textit{ provided } |z| < 1.$$

Selecting from the previous results, we can deduce that

$$\frac{(1+z)^{\vartheta}}{(1-z)^{1-\vartheta}} = 1 + z + (1-\vartheta)z^2 + \cdots \textit{ for } |z| < 1 \textit{ when } \vartheta \in [0,1]. \quad (7.1.7)$$

It follows that

$$\exp(z) = (1+z)^{\vartheta}/(1-z)^{1-\vartheta} + \mathcal{O}(z^2) \textit{ as } z \to 0, \textit{ for } \vartheta \in [0,1] \quad (7.1.8)$$

and

$$\exp(z) = (1+z)^{\vartheta}/(1-z)^{1-\vartheta} + \mathcal{O}(z^3) \textit{ as } z \to 0 \textit{ for } \vartheta = \tfrac{1}{2}. \quad (7.1.9)$$

Lemma 7.1.5 *Suppose that the function $f \in (\mathbb{C} \to \mathbb{C})$ is analytic on an open region $\mathcal{R} \subset \mathbb{C}$ and $f(z_0) = 0$ for some $z_0 \in \mathcal{R}$. If f does not vanish identically on $\mathcal{R}$ then there is a neighbourhood $N_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| \le \delta\}$ on which f vanishes only at z_0 .*

According to the previous theorem (see [1, p.518]) a zero z_0 of an analytic function is isolated. Apostol (loc. cit.) establishes this well-known property using the Taylor series. (For a zero z_0 of multiplicity precisely m, $f(x) = (z - z_0)^m f_0(z)$ where f_0 does not vanish at z_0 and is analytic and has an infinite Taylor series expansion centered on z_0.)

From the preceding lemmas we deduce:

Theorem 7.1.6 *Given a non-vanishing analytic function $f \in (\mathbb{C} \to \mathbb{C})$, the number of real zeros of f lying in any given bounded interval $[u_0, u_1] \subset \mathbb{R}$ is finite.*

Proof: Denote by $\mathcal{U} \subset [u_0, u_1]$ the set of real zeros of f. If $\mathcal{U}$ contains infinitely many points then $[u_0, u_1] \subset \mathbb{R}$ contains a point of accumulation x with $x = \lim_{k\to\infty} x_k$ where $\{x_k\} \subseteq \mathcal{U}$. By continuity of f, we have $f(x) = 0$. But the zero x must be an isolated zero, so the hypothesis is false and $\mathcal{U}$ contains a finite number of points.

Corollary 7.1.7 *Suppose $u_0 < u_1$ and $v \in \mathbb{R}$ are prescribed. Given a non-vanishing analytic function $f \in (\mathbb{C} \to \mathbb{C})$, the number of zeros of f lying on the bounded line segment $u_0 \le \Re(z) \le u_1$, $\Im(z) = v$ is finite.*

Remark 7.1.1 *We shall be treating functions (denoted f, say) of a complex variable (denoted $z \in \mathbb{C}$, say) whose restrictions to real arguments define real-valued functions. Without ambiguity, the real function will also be denoted f and its values written $f(x)$ (for $x \in \mathbb{R}$).*

7.1.2 Basic auxiliary and characteristic functions

In the classical literature, auxiliary and characteristic functions and their generalisations are defined both for linear autonomous difference equations and for linear autonomous differential equations and delay-differential equations with one retarded argument; their extension in the literature to linear autonomous functional differential equations (including those with multiple constant deviations) then follow very naturally. In particular, they arise in the solution of such difference equations based on $\mathfrak{z}$-transforms and the solution of the functional differential equations based on Laplace transforms (assuming the applicability of these techniques).

Remark 7.1.2 *In the case of Laplace transform techniques, assuming them to be applicable, the (classical) characteristic function, and the characteristic values, assumes a prominent rôle when the theory of Cauchy residues is employed to invert the Laplace transform that defines the solution.*

7.1.3 Oscillation in discrete equations

There exist parallels between the role of characteristic values for FDEs in the search for oscillatory or non-oscillatory solutions and corresponding results for analogous discrete equations. The discrete theory is easier than the continuous theory because there are finitely many degrees of freedom in the definition of the solution (and the auxiliary function is an auxiliary polynomial).

We shall consider a generic homogeneous constant-coefficient finite-term linear recurrence scalar relation

$$\widehat{\gamma}_0\widehat{u}_m + \widehat{\gamma}_1\widehat{u}_{m-1} + \cdots + \widehat{\gamma}_k\widehat{u}_{m-k} = 0 \quad (m \in \mathbb{Z}) \text{ where } \widehat{\gamma}_0\widehat{\gamma}_k \neq 0, \tag{7.1.10}$$

with fixed $k \in \mathbb{N}$, for which the characteristic polynomial is

$$\widehat{\chi}(\zeta) := \sum_{\ell=0}^{k} \widehat{\gamma}_\ell \zeta^{k-\ell}. \tag{7.1.11}$$

In our current context, (7.1.10) depends upon a given FDE and a choice of discretisation parameter $h > 0$. We could normalise so that $\widehat{\gamma}_0 = 1$, but in general

$$h = 1/N \text{ for some } N \in \mathbb{N} \tag{7.1.12}$$

and

$$k \equiv k(N), \ \widehat{\gamma}_\ell = \widehat{\gamma}_\ell(h) \in \mathbb{R} \quad (\ell \in \{0, 1, 2, \cdots, k\}). \tag{7.1.13}$$

Since $\widehat{\chi}(\zeta)$ is a polynomial of degree k it has k zeros (counting according to multiplicity) and if the distinct zeros are denoted $\{\zeta_\ell\}_1^{k_*}$ where ζ_ℓ has a multiplicity μ_ℓ then

$$\widehat{\chi}(\zeta) = \prod_{\ell=1}^{k_*}(\zeta - \zeta_\ell)^{\mu_\ell}. \tag{7.1.14}$$

If $\zeta_* \in \{\zeta_\ell\}_1^{k_*}$ then $\overline{\zeta_*} \in \{\zeta_\ell\}_1^{k_*}$. We recall that the general complex-valued solution of (7.1.10) is expressible as $u_m = \sum_{j=1}^{k_*} p_{\mu_j}(m) r_j^m \exp(\mathrm{i}m\xi_\ell)$ when we write $\zeta_\ell = r_\ell \exp(\mathrm{i}\xi_\ell)$, which for real-valued sequences gives us the form

$$u_m = \Re\{\sum_{j=1}^{k_*} p_{\mu_j}(m)\zeta_j^m\} = \Re\{\sum_{j=1}^{k_*} p_{\mu_j}(m) r_j^m \exp(\mathrm{i}m\xi_j)\} \tag{7.1.15}$$

where the notation p_r denotes a polynomial of degree r with complex coefficients, as usual i denotes $\sqrt{-1}$, and $\Re\{w\}$ denotes the real part of $w \in \mathbb{C}$. It follows that the general form for real valued sequences satisfying the relation (7.1.10) is

$$u_m = \sum_{j=1}^{k_*}\{q^0_{\mu_j}(m) r_j^m \cos(m\xi_j) + q^1_{\mu_j}(m) r_j^m \sin(m\xi_j)\} \tag{7.1.16}$$

where we denote by q_r^0 and by q_r^1 any polynomials of degree r with real coefficients. For some purposes, (7.1.15) is a more helpful expression.

Lemma 7.1.8 *(a) For $j \in \{1, 2, \cdots, k_*\}$, the j-th term in* (7.1.15) *is oscillatory if and only if*

$$\text{either } (i)\ \Im\{\zeta_j\} \neq 0 \text{ or } (ii)\ \Im\{\zeta_j\} = 0 \text{ and } \Re\{\zeta_j\} < 0 \tag{7.1.17}$$

(b) When (7.1.17) *holds for all $j \in \{1, 2, \cdots, k_*\}$ then all solutions of* (7.1.10) *are oscillatory.*

Proof: The elementary formulae for trigonometric functions can be used to show that the sum is oscillatory when each term is oscillatory.

One implication is that there exists a non-oscillatory solution if and only if there exists at least one $j \in \{1, 2, \cdots, k_*\}$ for which (7.1.17) is not satisfied.

Theorem 7.1.9 *Equation* (7.1.10) *is non-oscillatory if and only if there exists a characteristic value that is positive ($\{\zeta_\ell\} \cap (0, \infty)$ is non-empty).*

Remark 7.1.3 *For vector-valued recurrences there are similar equations with $\widehat{\gamma}_\ell$ replaced by matrices $\widehat{\Gamma}_\ell \in \mathbb{R}^{n\times n}$, and $\widehat{\chi}(\zeta) := \det \sum_0^k \widehat{\Gamma}_\ell \zeta^{k-\ell}$, and the corresponding results (including Theorem 7.1.9) expressed in terms of characteristic values are valid.*

7.1.4 Solution expansion based on the characteristic values

We assume, unless noted otherwise, that $n = 1$ and our equations have solutions have values in $\mathbb{R}$ (extensions for solutions with values in $\mathbb{R}^n$ for $1 < n \in \mathbb{N}$ are often straightforward).

In the classical theory of functional differential equations, the characteristic function χ is analytic.

Lemma 7.1.10 *The characteristic values* $\{\lambda_\ell\}$ *constitute a set of isolated complex numbers such that* $\exp(\lambda t)$ *is a solution of the functional differential equation whenever* $\lambda \in \{\lambda_\ell\}$.

Recall that, in the current context, if $\lambda \in \{\lambda_\ell\}$, then $\overline{\lambda} \in \{\lambda_\ell\}$. It follows immediately that if $\Im(\lambda) \neq 0$ for some $\lambda \in \{\lambda_\ell\}$ then there are corresponding *oscillatory* solutions $\sin(\lambda t)$ and $\cos(\lambda t)$. (In the case of a multiple characteristic value we also have oscillatory solutions of the form $t^k \sin(\lambda t)$ and $t^k \cos(\lambda t)$ for certain integer k.) In this classical case, we show that a non-oscillatory solution exists when we find some $\lambda \in \{\lambda_\ell\}$ with $\Im(\lambda) = 0$, and an oscillatory solution exists when we find $\lambda \in \{\lambda_\ell\}$ with $\Im(\lambda) \neq 0$. More thought is required, even if we know all of the characteristic values, if we wish to assert that *no* oscillatory solution exists or that *no* non-oscillatory solution exists. Such assertions will generally rely on being able to prove that all the solutions are expressible in the form

$$y(t) = \sum_{\ell=0}^{\infty} p_\ell(t) \exp(\lambda_\ell t) \tag{7.1.18a}$$

or

$$y(t) \sim \sum_{\ell=0}^{\infty} p_\ell(t) \exp(\lambda_\ell t) \text{ as } t \to \infty \tag{7.1.18b}$$

or (see Remark 7.1.4) some similar claim, where $p_\ell(t)$ is some polynomial in t whose degree does not exceeed the multiplicity of λ_ℓ as a characteristic value.

Observe that (7.1.18) refers to the possible solutions of the FDE and they will in general form a subset – a subset that is defined by the FDE – of the set of continuous functions. The relevant theory for retarded (delay) equations is more complete than that for advanced or mixed-type arguments. The situation in Section 7.1.3 was more straightforward because the result corresponding to (7.1.18) involved a finite sum.

Remark 7.1.4 *We are interested in providing motivation for later sections (§§7.3 onwards) and allow ourselves to forego some rigour, which may be recaptured by consulting the cited literature. It may be necessary to show (for every solution y) not only that*

$$\lim_{N\to\infty} \sup_{t\geq T} |\varepsilon_N(t)| = 0 \tag{7.1.19a}$$

where

$$\varepsilon_N(t) = y(t) - \sum_{\ell=0}^{N} p_\ell(t)\exp(\lambda_\ell t), \tag{7.1.19b}$$

but also that, for given solution y the corresponding function

$$\varepsilon_N \textit{ is non-oscillatory for all } N \geq N_*, \tag{7.1.20}$$

or

$$\varepsilon_N \textit{ is oscillatory for all } N \geq N_*. \tag{7.1.21}$$

Lemma 7.1.11 *For an autonomous homogeneous constant-coefficient linear DDE,*

(a) if $\Im(\lambda) = 0$ for some $\lambda \in \{\lambda_\ell\}$ then the equation possesses a non-oscillatory solution;

(b) every oscillatory solution can be expressed in a suitable form (7.1.18) *where* (7.1.21) *holds, and*

(c) the FDE is oscillatory if $\{\lambda_\ell\} \not\subset \mathbb{R}$ (that is, $\Im(\lambda) \neq 0$ for every $\lambda \in \{\lambda_\ell\}$).

Part (a) applies equally to any autonomous homogeneous constant-coefficient linear FDE (functional differential equation) of the type considered here, and the applicability of (b)–(c) in such cases is sometimes established separately and sometimes assumed.

7.1.5 Small solutions

There is a connection between *small solutions* and the possibility of expressing a solution as a sum of characteristic functions.

Definition 7.1.1 *A* small solution *of a FDE is a solution that decays to zero faster than the exponential of any multiple of the argument t as $t \to \infty$.*

It can be shown that some FDEs possess solutions that do not have the required expansion of the form (7.1.18). In particular, small solutions provide an example in the literature on DDEs (the coefficients of the exponential terms $\exp(\lambda t)$ in in (7.1.18) all vanish). When such small solutions exist, they may, in principle, be oscillatory – for example, $\sin(t)\exp(-t^2)$, or not oscillatory – for example, $\exp(-t^2)$. See Verduyn Lunel, [100] . We draw attention to the results (see [100] and for a shortened form [97]) by Verduyn Lunel and others that indicate that under certain simple conditions all small solutions for delay equations become zero in finite time and would therefore be oscillatory.

Example 7.1.2 *Dr Lumb's thesis [105] contains a summary of some results in the literature on small solutions of retarded equations. We state some results that attracted our interest and may be found in [105] where original sources are given:*

1. *The equation $y'(t) = \mu_\natural y(t-1)$ has no small solutions when $\mu_\natural \in \mathbb{R}$;*
2. *If the equation $y'(t) = \mu_\natural(t)y(t-1)$, with $\mu_\natural(t) \in \mathbb{R}$, has a small solution, then $\mu_\natural$ must change sign;*
3. *The vector-valued equation $y'(t) = B_0(t)y(t) + B_1(t)y(t-1)$ with analytic $B_{0,1}(t) \in \mathbb{R}^{n\times n}$ has no non-trivial small solutions when $|\det(B_1(t)| > 0$ for all $t \geq t_0$.*

7.1.6 Generalised characteristic functions

We offer some remarks as an introduction to for further reading and investigation. We have already referred, earlier in this thesis, to research papers that explore rigorously various links between oscillatory equations and generalised characteristic functions.

Lemma 7.1.10 provides a description of characteristic functions and characteristic values that allows their definition to be extended to non-autonomous FDEs; compare (1.5.14).

Definition 7.1.2 *A generalised characteristic function $\chi_\natural$ of a given FDE is (it it exists) a function of $\lambda \in \mathbb{C}$ that vanishes if and only if $\exp(\lambda t)$ is a solution of the given FDE. If $\chi_\natural(\lambda) = 0$ then λ is called a generalised characteristic value.*

Clearly, if any characteristic value is real, then the FDE is non-oscillatory. For a homogeneous equation the null function is always a solution. However, if zero is a real characteristic value, then any constant solution is necessarily a solution, and (corresponding to zero as a multiple characteristic value) certain polynomials may also be solutions.

Remark 7.1.5 *Instead of seeking solutions of the form* $\exp(\lambda t)$ *one might search for solutions* $\exp(\lambda(t))$ *where* $\lambda(t) \in \mathbb{C}$ *for* $t \geq t_0$. *Since the FDE involves the derivative of the solution, it would be appropriate to suppose that* λ *is differentiable and in consequence seek solutions of the form*

$$y(t) = \exp(\lambda(t)), \quad \lambda(t) := \int_{t_0}^{t} \alpha(s)ds. \tag{7.1.22}$$

(Compare this form with solutions of the ODE $y'(t) = a(t)y(t)$*.) Substitution of* (7.1.22) *into the FDE will give a relation that must be satisfied by* α *(equivalently, a relation that must be satisfied by* λ*) that may prove tractable. Clearly, if* λ *satisfies* (7.1.22) *then so does its complex conjugate and there is an oscillatory solution if* $\Im\lambda(t) \neq 0$ *for all sufficiently large* t *and there is a non-oscillatory solution if* $\Im\lambda(t) = 0$ *for all sufficiently large* t

A somewhat different approach to that suggested by the remarks above follows from a search for solutions that are ultimately positive (or ultimately negative). A solution that is ultimately strictly positive is non-oscillatory, as also is a solution that is ultimately strictly negative. In [73, Chapters 2 – 3], conditions are established for a positive solution of

$$y'(t) + \sum_{k=1}^{N} g_k y(t - \tau_k) = 0 \quad (\tau_k > 0 \text{ for all } k \in \{1, 2. \cdots, N\}) \tag{7.1.23}$$

and the existence of a real zero of χ (where $\chi(\lambda) := \lambda + \sum_{k=1}^{N} g_k \exp(\lambda\tau_k)$). The theory is extended to provide conditions for the existence of a positive solution of

$$y'(t) + \sum_{k=1}^{N} g_k(t) y(t - \tau_k(t)) = 0 \quad (\tau_k(t) > 0 \text{ for all } k \in \{1, 2. \cdots, N\}). \tag{7.1.24}$$

7.2 Further reading

The literature provides further insight, and as a selection for further reading we suggest [2, 7, 24, 40, 47, 51, 58, 65, 70, 73, 96, 97, 98, 99, 100, 83, 89, 138, 48, 90, 119, 118, 43]. Chapter 3 of [73] concerns generalised characteristic equation and existence of positive solution.

7.3 Criteria based on characteristic values

We seek to explain the theoretical background of the numerical methods that we use to count the number of real or positive zeros of an analytic function defined by the characteristic function or the characteristic polynomial. We emphasise that we concentrate, now, on criteria for equations to be oscillatory where these criteria take the form:

1. The *continuous problem* is oscillatory when the zeros of a characteristic function defined by the continuous problem all have non-zero imaginary part, or (equivalently) that for an oscillatory equation *the function* χ *must have no real zeros.*

2. The corresponding *discretised problem* is oscillatory when the zeros of a characteristic polynomial defined by the discretised problem are all non-positive or have non-vanishing imaginary part. Equivalently, for an oscillatory equation *the function* χ^D *must have no positive real zeros.*

It may prove convenient to consider a related auxiliary function in place of the characteristic function/polynomial. (An auxiliary function has precisely the same zeros as the characteristic function.)

7.3.1 Theoretical aspects of the application of the Argument Principle

We consider the basic requirements for the application of the Argument Principle using rectangular contours Γ_{2M} or γ_M. We suppose these are rectangular contours in $\mathbb{C}_- \cup \mathbb{C}_+$ or in $\mathbb{C}_+$, used to determine the location of appropriate zeros of a suitable corresponding characteristic function. Of these,

$$\Gamma_{2M} \text{ is the ("full") rectangle with corners } \pm M \pm \mathrm{i}\frac{1}{M}. \tag{7.3.1}$$

$$\gamma_M \text{ is the ("half") rectangle with corners } M \pm \mathrm{i}\frac{1}{M} \text{ and } 0 \pm \mathrm{i}\frac{1}{M} \tag{7.3.2}$$

is a rectangle in $\mathbb{C}_+$. By 'rectangle' we mean the positively-oriented rectangular contour that bounds the rectangular area! Characteristic polynomials χ^D from discretised equations are treated using γ_M.

Remark 7.3.1 *We may require, in the discretised case (if the properties of* χ^D *are to indicate properties of* χ*), that the discretisation parameter* h *should be sufficiently small – and this means that the degree of the characteristic polynomial may have to be large.*

We take $f(z) = \chi(z)$ for some characteristic function or $f(z) = \chi^D(z)$ for some characteristic polynomial and the properties required are that

(i) the integrand $f'(z)/f(z)$ should be analytic within and on Γ_{2M} (respectively, γ_M) and

(ii) f should not vanish on Γ_{2M} (respectively, γ_M).

The first requirement (i) is automatic because χ is a quasi-polynomial and χ^D is a polynomial. The second requirement (ii) will generally require that M is taken sufficiently large and $h = 1/N$ sufficiently small. *We shall need to show that with such conditions, we capture all the relevant real zeros identified in the opening remarks of § 7.1.*

7.3.2 A general approach to bounding real zeros

The following lemma forms the basis of our approach to bounding the real zeros of a smooth (that is, continuous) function of a real variable. In applications, this will result from restricting a relevant analytic function of a complex variable to the real numbers (this restriction is real-valued).

Lemma 7.3.1 *(a) An upper bound for the real zeros of a continuous function $f \in (\mathbb{R} \to \mathbb{R})$ will exist if*

$\exists\ x_* \in \mathbb{R}$ *such that* $f(x) > 0\ \forall\ x > x_*$ *or* $\exists\ x_*$ *such that* $f(x) < 0\ \forall\ x > x_*$.

(b) Similarly, a lower bound for the real zeros of a continuous function f will exist if there esists a real value x_* such that $f(x) > 0$ for all $x < x_*$ or such that $f(x) < 0$ for all $x < x_*$.

In the cases that we consider we can often show that $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to +\infty$ and this yields the desired result in (a). Indeed it is sufficient to show that $f(x) \sim f^* \neq 0$ as $x \to \infty$ (f_* can be finite or infinite), and this result will be provable if $f(x) = f_0(x) + f_1(x)$ where $f_0(x) \to 0$ while $f_1(x) \sim f_1^* \neq 0$ both as $x \to \infty$. The discussion of lower bounds follows parallel lines to that of upper bounds.

Definition 7.3.1 *Here, the notation $f_0(x) = o(1)f_1(x)$ as $x \to x_*$ signifies that*

$$f_0(x)/f_1(x) \to 0 \text{ as } x \to x_* \tag{7.3.3}$$

where x_* is a finite or infinite real number. With what is a minor abuse of convention, this notation will also be used

1. *in the case that* $f_0(x)$ *is a non-zero constant and* $f_1(x) \to \infty$ *or* $f_1(x) \to -\infty$ *as* $x \to x_*$ *or*

2. *in the case that* $f_0(x)$ *is* zero *constant provided that* $f_1(x)$ *remains either positive or negative for all sufficiently large arguments.*

Lemma 7.3.2 *Suppose that* $x \in \mathbb{R}$ *and* $f(x) = f_0(x) + f_1(x)$ *where* $f_{0,1}$ *are continuous real-valued functions* ($f_{0,1} \in C(\mathbb{R} \to \mathbb{R})$)*. If*

$$f_0(x) = o(1) f_1(x) \text{ as } x \to \infty \tag{7.3.4a}$$

and if

$$f_1(x) \to +\infty \text{ or } f_1(x) \to -\infty \text{ as } x \to \infty \tag{7.3.4b}$$

then there exists an upper bound on the real zeros of f*. Such a bound is provided by any value* $\widehat{x}$ *such that,* for all $x \geq \widehat{x}$,

$$|f_1(x)| > |f_0(x)| \text{ and also } \mathrm{sign} f_1(x) = \mathrm{sign} \lim_{x \to \infty} f(x). \tag{7.3.5}$$

Likewise, if $f_0(x) = o(1) f_1(x)$ *as* $x \to -\infty$ *and if* $f_1(x) \to +\infty$ *or* $f_1(x) \to -\infty$ *as* $x \to -\infty$*, then there exists a lower bound on the real zeros of* f*.*

7.4 Mixed type homogeneous linear equations

In this section, we do not give the detailed analysis for the general case but consider simple examples that indicate the general form of the theoretical justification of the use of the Argument Principle. We shall start by looking at characteristic functions for undiscretised equations.

We gain relatively easy insight if we examine the characteristic function for the simple scalar equation of mixed type introduced in (1.4.8), namely,

$$y'(t) = a\, y(t) + b\, y(t-1) + c\, y(t+1) \tag{7.4.1}$$

where $y \in (\mathbb{R} \to \mathbb{R})$. (The case $y'(t) = \alpha y(t) + \beta y(t-\tau) + \gamma y(t+\tau)$ in (1.4.7) follows immediately from the discussion.) The characteristic function for (7.4.1) reads

$$\chi(z) = z - a - b \exp(-z) - c \exp(z) \quad (z \in \mathbb{C}). \tag{7.4.2}$$

Example 7.4.1 *The value* $z = 0$ *is a zero of* χ *only if* $a + b + c = 0$*. Moreover,* $\chi'(0) = 0$ *only if* $1 + b + c = 0$*. If* $a = 1$*,* $b = -1$ *and* $c = 0$*, then zero is a multiple root of* $\chi(z) = 0$*.*

Suppose $b = c = 0$*, then* $\chi(z) = z - a$ *and* χ *has a zero only at* a*.*

Vanishing parameters	Dominating term as $x \to \infty$	Limiting behaviour as $x \to \infty$	Dominating term as $x \to -\infty$	Limiting behaviour as $x \to -\infty$
a can be 0	$-c\exp(x)$	$\to -\text{sign}(c)\infty$	$-b\exp(-x)$	$\to -\text{sign}(b)\infty$
$b = 0$	$-c\exp(x)$	$\to -\text{sign}(c)\infty$	x	$-\infty$
$c = 0$	x	∞	$-b\exp(-x)$	$\to -\text{sign}(b)\infty$
$b = c = 0$	x	∞	x	$-\infty$

Table 7.4.1: Limiting behaviours

Motivated by Lemma 7.3.4 we examine the dominating terms in $\chi(x) = x - a - b\exp(-x) - c\exp(x)$, as $x \to \infty$ or as $x \to -\infty$. See Table 7.4.1.

The following result holds in general.

Lemma 7.4.1 *(i) There is an upper bound on the real zeros of* (7.4.2)*, and (ii) there is a lower bound on the real zeros of* (7.4.2).

Proof: We shall follow the lines of argument indicated in §7.3.2. Consider $z \in \mathbb{R}$ and, as a reminder, write it as $z = x$. For $c \neq 0$, it is clear that as $x \to \infty$, $\text{sign}(-c)\chi(x) \to +\infty$, and in consequence there exists $\widehat{x} \in [0, \infty)$ such that $\text{sign}\{\chi(x)\} = \text{sign}\{-c\}$ for all $x \in [\widehat{x}, \infty)$. The case $c = 0$ is similar. The totality of possibilities for vanishing coefficients is indicated in Table 7.4.1 and the analogous arguments are straightforward. The result (i) follows and (ii) follows on replacing z by $-z$ to show that as $x \to -\infty$, $\text{sign}(-b)\chi(x) \to +\infty$.

As a corollary of the preceding work, we deduce

Theorem 7.4.2 *For sufficiently large M, χ has no zeros on Γ_{2M}.*

Proof: Take M sufficiently large so that (i) $[-M, M]$ contains all the real zeros and (ii) $1/M < \min|\Im(\lambda)|$ where the minimum is taken over those $\lambda \in \{\lambda_\ell\}$ (finite in number) such that $|\Re(\lambda_\ell) \in [-M, M]$.

Remark 7.4.1 *If characteristic values are to be found on a contour, it is possible to "indent" the rectangle with a semicircle (or a set of semicircles) of sufficiently small radius ϵ_M such that no zeros lie on the indented rectangle. Compensating adjustments can be made to our results.*

7.4.1 Analogues in the discretised case

Recall the equation (7.4.1), namely

$$y'(t) = a\, y(t) + b\, y(t-1) + c\, y(t+1) \tag{7.4.3}$$

and we assume it to be discretised using a stepsize $h = 1/N$ with the ϑ-method to give the scheme

$$\widetilde{y}_{m+1} - \widetilde{y}_m \tag{7.4.4}$$
$$= \vartheta h\{a\,\widetilde{y}_m + b\,\widetilde{y}_{m-N} + c\, y_{m+N})\} + (1-\vartheta)h\{a\,\widetilde{y}_{m+1} + b\,\widetilde{y}_{m+1-N} + c\, y_{m+1+N})\}.$$

Suppose $b \neq 0$; then, for $\vartheta = 0$ the characteristic function is

$$\chi^{D(\vartheta=0)}(\lambda) = \lambda^{N+1} - \lambda^N - h\{a\lambda^{N+1} + b + c\lambda^{2N+1}\} \tag{7.4.5}$$

while for $\vartheta = 1$ the characteristic function is

$$\chi^{D(\vartheta=1)}(\lambda) = \lambda^{N+1} - \lambda^N - h\{a\lambda^N + b + c\lambda^{2N}\}. \tag{7.4.6}$$

If $b = 0$, then spurious factors λ^N should be cancelled throughout to yield

$$\chi^{D(\vartheta=0)}(\lambda) = \lambda - 1 - h\{a\lambda + c\lambda^{N+1}\}, \tag{7.4.7}$$

$$\chi^{D(\vartheta=1)}(\lambda) = \lambda - 1 - h\{a + c\lambda^N\}. \tag{7.4.8}$$

The general case ($0 \leq \vartheta \leq 1$) follows from

$$\chi^{D(\vartheta)}(\lambda) = \vartheta\chi^{D(\vartheta=1)}(\lambda) + (1-\vartheta)\chi^{D(\vartheta=0)}(\lambda). \tag{7.4.9}$$

Lemma 7.4.3 *For $b \neq 0$,*

$$\chi^{D(\vartheta)}(\lambda) = \lambda^{N+1} - \lambda^N - \vartheta h\{a\lambda^N + b + c\lambda^{2N}\} - (1-\vartheta)h\{a\lambda^{N+1} + b + c\lambda^{2N+1}\}. \tag{7.4.10}$$

For $b = 0$, $\chi^{D(\vartheta)}(\lambda) = \lambda - 1 - \vartheta h\{a + c\lambda^N\} - (1-\vartheta)h\{a\lambda + c\lambda^{N+1}\}$.

We see readily that the following Lemma holds.

Lemma 7.4.4 $\chi^{D(\vartheta)}(0) \neq 0$ *for all $h = 1/N$ and all $\vartheta \in [0,1]$.*

Remark 7.4.2 *We gain general insight when, for simplicity, we consider the case $\vartheta = 1$ and determine what may be said about the zeros of $\chi^{D(\vartheta=1)}(\lambda)$,*
$\chi^{D(\vartheta=1)}(\lambda)$ simplifies, when expressed in terms of N rather than h, as

$$\chi^{D(\vartheta=1)}(\lambda) = -\frac{c}{N}\lambda^{2N} + \lambda^{N+1} - (1 + \frac{a}{N})\lambda^N - \frac{b}{N} \text{ if } b \neq 0. \tag{7.4.11}$$

We also have $\chi^{D(\vartheta=0)}(\lambda) = -\frac{c}{N}\lambda^{2N} + \lambda^{N+1} - (1 - \frac{a}{N})\lambda^N + \frac{b}{N}$ *if* $b \neq 0$.

A polynomial of degree K has exactly K zeros and it follows that for fixed N, and $h = 1/N$, the real zeros of $\chi^{D(\vartheta)}(\lambda)$ have an upper bound and a lower bound. This result is insufficient for our purposes. Rather, we need to provide an upper bound that can be quantified and apply uniformly and we shall now achieve this aim. Assuming $c \neq 0$,

$$\operatorname{sign}\{\chi^{D(\vartheta=1)}(\lambda)\} = \operatorname{sign} \lim_{\lambda \to -\infty} \{\chi^{D(\vartheta=1)}(\lambda)\} \tag{7.4.12}$$

when $|\lambda| > 1$ and $|\lambda^N| > 2 + |ah| + |bh|$ and hence, in particular, when

$$|\lambda| > 2 + |a| + |b| \tag{7.4.13}$$

(the condition implies (7.4.12)). In the case $c = 0$ a corresponding bound is $|\lambda| > 1 + |a| + |b|$. The general case $\vartheta \in [0, 1]$ follows in a similar manner, and we can establish the following result.

Lemma 7.4.5 *There is an upper bound on the real zeros of $\chi^{D(\vartheta)}$ that is independent of $N \geq 1$.*

Remark 7.4.3 *We can supplement Lemma 7.4.4 by establishing: $\chi^{D(\vartheta=1)}$ has no purely imaginary zeros.*

If we evaluate $\chi^{D(\vartheta=1)}(\mathrm{i}v)$, the even powers of $\mathrm{i}v$ are real and the odd powers are purely imaginary; both these terms must vanish to ensure that $\chi^{D(\vartheta=1)}(\mathrm{i}v) = 0$ and we see by inspection that this cannot happen.

For $\vartheta \in [0, 1]$ and for general $\chi^{D(\vartheta=1)}$ we can deduce instead the weaker statement that we require, namely:

Lemma 7.4.6 *Whenever $h = 1/N$ is sufficiently small, $\chi^{D(\vartheta=1)}$ has no zeros of the form $\mathrm{i}\frac{\nu}{M}$ (with $-1 \leq \nu \leq 1$) for every sufficiently large M.*

As a corollary of our lemmas, we deduce:

Theorem 7.4.7 *For sufficiently large M, $\chi^{D(\vartheta)}$ has no zeros on γ_M.*

7.4.2 From the continuous to the discrete

It is clear, on the basis of Theorems 7.4.2 and 7.4.7 and the analytic properties of χ and χ^D, that we are justified in our use of the Argument Principle to determine, in each case, the number of zeros of χ (or of χ^D) within the contour $\gamma_2 N$ (the contour γ_N).

It remains, however, to show that (*when the stepsize h is sufficiently small*) results for oscillation based on the Argument Principle for the continuous and the discrete case *are consistent with each other.*

The criteria that should carry across from the continuous case to the discrete case (for any sufficiently small step-size) are the following:

1. the function χ where
$$\chi(z) = z - a - b\exp(-z) - c\exp(z) \quad (z \in \mathbb{C}); \qquad (7.4.14)$$
(when $b = 0$, this is $z - a - c\exp(z)$) has no *real zeros*;

2. χ^D has no *positive* real zeros; where the general form of $\chi^{D(\vartheta)}$ is given by Lemma 7.4.3 and, in particular,
$$\chi^{D(\vartheta=1)}(z) = -chz^{2N} + z^{N+1} - (1+ah)z^N - bh \text{ where } h = 1/N \quad (7.4.15)$$
unless $b = 0$ when
$$\chi^{D(\vartheta=1)}(z) = -chz^N + z - (1 + ah). \qquad (7.4.16)$$
We distinguish this latter case (the case $b = 0$) since the function (7.4.15) has a multiple zero at $z = 0$ which is removed in (7.4.16).

7.4.3 A route to comparison

Consider the characteristic polynomial (7.4.15). To compare its zeros with those of (7.4.14), we can equally consider the auxiliary function $\exp(z)\chi(z)$
$$\chi_{\text{aux}}(z) = z\exp(z) - a\exp(-z) - b - c\exp(2z) \quad (z \in \mathbb{C}), \qquad (7.4.17)$$
defined by (7.4.14).

In (7.4.15) we seek to replace z by $\exp \zeta h$, with $h = 1/N$. To the previous end, we employ the maps
$$\zeta h \in \mathbb{R} \overset{z=\exp(\zeta h)}{\longrightarrow} z \in \mathbb{R}_+; \text{ and } z \in \mathbb{R}_+ \overset{\zeta h = \mathrm{Ln}(z)}{\longrightarrow} \zeta h \in \mathbb{R} \qquad (7.4.18)$$
associated with restrictions of the exponential $w = \exp(z)$ for $z \in \mathbb{C}$ and $z = \mathrm{Ln}(w)$ for $w \in \mathbb{C}$, in which Ln denotes the principal value of the natural logarithm (with $\Im\{\mathrm{Ln}(w)\} \in (-\pi, \pi]$ for all $w \in \mathbb{C}$).

The general procedure is well-illustrated by studying the special case based on the choosing $b = 0$ and a comparison of χ and $\chi^{D(\vartheta=1)}$. For ease of access we restate the functions for comparison as
$$\begin{aligned} \chi(z) &:= \quad z - a - c\exp(z) \quad \text{(the case } b = 0); && (7.4.19)\\ \chi^{D(\vartheta=1)}(z) &:= \quad z - (1 + ah) - chz^N \quad \text{(the case } b = 0). && (7.4.20)\end{aligned}$$
Now, from the fact that $Nh = 1$, we have $z = \exp(\zeta h)$ and $z^N = \exp(\zeta)$) and (7.4.20) when expressed in terms of ζ reads
$$\begin{aligned} \chi^{D(\vartheta=1)}(z) &:= \quad \exp(\zeta h) - (1 + h) - ch\exp(\zeta) && (7.4.21)\\ &= \quad h\Big\{\frac{\exp(\zeta h) - 1}{h} - a - c\exp(\zeta)\Big\} && (7.4.22)\end{aligned}$$

(this is the case $b = 0$). It remains, therefore, to compare the zeros of the analytic functions

$$\begin{aligned} z \in \mathbb{C} \longrightarrow \quad & f(z) := z - a - c\exp(z) \\ \zeta \in \mathbb{C} \longrightarrow \quad & g_h(\zeta) := \frac{\exp(\zeta h) - 1}{h} - a - c\exp(\zeta). \end{aligned} \tag{7.4.23}$$

where we have scaled the second function in (7.4.21) by a factor h to give g_h. We recall from Example 7.1.1 that $\exp(z) = 1 + z + z^2/2! + z^3/3! + \cdots$, so that

$$\frac{\exp(\zeta h) - 1}{h} = \zeta + h\zeta^2/2 + h^2\zeta^3/3! + \cdots. \tag{7.4.24}$$

and it follows that $f(z) - g_h(z) = h\zeta^2/2 + h^2\zeta^3/3! + \cdots$, and (from the remainder in the Taylor sum[1]) we see that for any rectangular region Γ^{2M} we have a bound of the form

$$\sup_{z \in \Gamma^{2M}} |f(z) - g_h(z)| \leq hK_M \sup_{z \in \Gamma^{2M}} |\exp(z)| \tag{7.4.25}$$

for some constant K_M. Taking M sufficiently large so that Γ^{2M} contains all the real zeros of f, (7.4.25) together with the second theorem in Chapter 5 (Theorem 5.2.2) this result establishes that the number of real zeros of f does not exceed the number of real zeros of g_h and by a symmetric argument, the number of real zeros of f equals the number of real zeros of g_h. In the special case $\vartheta = 1$ and $b = 0$ we have established Theorem 7.4.8. The case $\vartheta = 1$ and $b \neq 0$ follows virtually identical lines with

$$\begin{aligned} z \in \mathbb{C} \longrightarrow \quad & f(z) := z - a - b\exp(-z) - c\exp(z) \\ \zeta \in \mathbb{C} \longrightarrow \quad & g_h(\zeta) := \frac{\exp(\zeta h) - 1}{h} - a - b\exp(-\zeta) - c\exp(\zeta). \end{aligned} \tag{7.4.26}$$

so that $f(z) - g_h(z)$ has the same expression as it had in the case $b = 0$. The general case $\vartheta \in [0, 1]$ involves rather more manipulation. The principal result related to this discussion is the following generalisation to the ϑ-method with step h:

Theorem 7.4.8 *For all sufficiently small h the number of real zeros of the characteristic function*

$$z - a - b\exp(-z) - c\exp(z) \quad (z \in \mathbb{C}),$$

is equal to the number of positive zeros of the discretised characteristic polynomial $\chi^{D(\vartheta)}$.

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[1]We use a modified form of Corollary 7.1.4.

Chapter 8

Conclusions and further work

As we have seen, our approach to the analysis of oscillatory equations based upon the use of the Principle of the Argument has been quite effective in the examples we have studied. Indeed, we have found examples of work by other authors in which equations had been wrongly classified using conventional analysis, and where the approach we have introduced here has enabled the analysis to be corrected. However, as we already remarked, much remains to be done.

The work on small solutions for delay equations was mentioned in Chapter 7. Our analysis is heavily dependent upon an understanding that the characteristic values fully characterise the behaviour of the solution set for the given problem. It is known that, in the presence of small solutuions, this may not be correct. However it is also known that, for delay equations at least, small solutions often become zero after finite time, and therefore they have little new to contribute to the long-term dynamics of the solution.

More problemmatic may be the application of the method to mixed-type equations, where the existing analytical results are much more limited. One can surmise the existence of an analogous property to a small solution, but one which decays from right to left, rather than left to right. However analysis of this type of problem lies beyond the scope of the present thesis. Nevertheless, we recognise that not being able to be sure that one can resolve the solution fully in terms of characteristic functions is a significant limitation to our results. For delay equations, small solutions represent a form of degeneracy of the equation, and the same would apply to their reverse analogue in the mixed equation. Therefore the results of this thesis will apply to all problems where this degeneracy does not arise. For delay equations, the degeneracy cannot arise for autonomous problems, and we believe the same would be true for mixed-type problems. Our experimental evidence suggests that the results of our approach are much more widely applicable than can currently be shown analytically, and we believe that a significant piece of further work would be to widen the scope of the analytical understanding of the problem.

Other possible investigations could be based upon applying our methods to known mixed-type equations that arise in mathematical models of the real world, since these equations are not always the ones that are most naturally candidates for analysis.

In conclusion, it is our intention to continue working in this area.

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