DAHAS AND SKEIN THEORY

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Abstract. We give a skein-theoretic realization of the $\mathfrak{gl}_n$ double affine Hecke algebra of Cherednik using braids and tangles in the punctured torus. We use this to provide evidence of a relationship we conjecture between the classical skein algebra of the punctured torus and the elliptic Hall algebra of Burban and Schiffmann.

Contents

1. Introduction

This paper concerns a relation between double affine Hecke algebras and algebras associated to the punctured and unpunctured torus that are defined using skein theory. As partial motivation, we first briefly discuss previous results in the $\mathfrak{sl}_2$ case of the Kauffman bracket, and then go on to discuss the conjectures and results of the present paper.

1.1. **The Kauffman bracket.** The *Kauffman bracket skein algebra* $K_s(F)$ of a surface F is spanned by embedded links in the thickening $F \times [0,1]$, modulo the Kauffman bracket skein relations. These are local relations depending on a parameter $s \in \mathbb{C}^*$, and which are similar to equation (3.13). The product is given by stacking links in the $[0,1]$ direction, and this algebra can be viewed as a quantization of the ring of functions on the SL_2 character variety of F [PS00, BFKB99]. For the torus and punctured torus these algebras have been described explicitly by Frohman, Gelca, and by Bullock, Przytycki, respectively.

The *double affine Hecke algebra* was defined by Cherednik (see, e.g. [Che05]) using explicit generators and relations, and it depends on two parameters, $q, t \in \mathbb{C}^*$. In rank 1, its *spherical subalgebra* $SH_{q,t}$ of the DAHA was described explicitly by Koornwinder and later by Terwilliger.

Combining these explicit descriptions leads to the following theorem.

Theorem 1.1 ([FG00, Ter13, Koo08]). *There is an isomorphism*

$$K_s(T^2) \cong SH_{s,s}$$

between the Kauffman bracket skein algebra of the torus and the $t = q = s$ specialization[1] *of the rank 1 spherical DAHA.*

Combining the same algebraic theorems with the description of the skein algebra of the punctured torus instead, we obtain the following.

Theorem 1.2 ([BP00, Ter13, Koo08]). *There is a surjective map*

$$K_s(T^2 - D^2) \twoheadrightarrow SH_{q=s,t}$$

from the skein algebra of the punctured torus to the spherical rank 1 DAHA.

We note that the source algebra still only depends on one parameter – the second parameter t in the target appears in the relations describing the kernel of the map. One rough way of thinking of these results is that the spherical DAHA can be obtained from the skein algebra of the punctured torus using some kind of "decoration" at the puncture.

Let us also comment briefly on the importance of two parameters. The Macdonald polynomials are symmetric polynomials depending on the parameters q and t which have been studied intensively, and this had led (at the very least) to very interesting combinatorics, geometry, and algebra. In the $t = q$ specialization, the Macdonald polynomials degenerate to Schur polynomials, which are much more well-understood and less subtle. It is therefore desirable to be able to "see" both parameters from topology.

1.2. **The elliptic Hall algebra and Homfly skeins.** This paper deals with the "infinite rank" versions of the algebras in the previous section. The Kauffman bracket skein algebras are replaced with the *Homflypt skein algebra* $\mathrm{Sk}_s(F)$, and the spherical DAHA is replaced by the *elliptic Hall algebra* $\mathcal{E}_{\sigma,\bar{\sigma}}$ defined by Burban and Schiffmann [BS12]. In earlier work we proved the analog of Theorem 1.1:

Theorem ([MS17]). *There is an isomorphism*

$$\mathrm{Sk}_s(T^2) \cong \mathcal{E}_{s,s}$$

between the Homflypt skein algebra of the torus and the $\sigma = \bar{\sigma} = s$ specialization of the elliptic Hall algebra.

We make the following conjecture which is the analog of Theorem 1.2.

Conjecture 1.3. There is a surjective algebra map $\mathrm{Sk}_s(T^2 - D^2) \twoheadrightarrow \mathcal{E}_{\sigma,\bar{\sigma}}$.

The currently available proofs of all three of the previous theorems involve giving explicit presentations of the algebras in the statements, and then using these presentation to construct an algebra map by hand. Giving a presentation of $\mathrm{Sk}_s(T^2 - D^2)$ seems difficult, so instead of doing this, we give some evidence for this conjecture using other techniques, which we describe in the next subsection. These techniques are closely related to some techniques used or mentioned by others – one reason we make precise statements of our own version is that it gives evidence for the conjecture above.

[1]Technically, Frohman and Gelca showed skein algebra is isomorphic to the $t_{DAHA} = 1$, $q_{DAHA} = s_{skein}$ specialization, but the presentations of Koornwinder and Terwilliger show that this is isomorphic to the $t_{DAHA} = q_{DAHA} = s_{skein}$ specialization, which is a nontrivial statement.

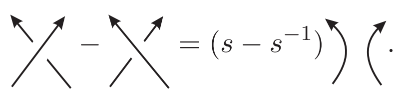
1.3. **DAHAs for $\mathfrak{gl}_n$.** The elliptic Hall algebra $\mathcal{E}_{\sigma,\bar{\sigma}}$ is closely related to the double affine Hecke algebras $\ddot{H}_n$ of type $\mathfrak{gl}_n$, as detailed in the work of Schiffman and Vasserot, [SV11]. We were intrigued by the nature of the presentation of the algebras $\ddot{H}_n$, which involved Homfly type relations and braids in the torus T^2. This led us to speculate on the possibility of constructing some form of skein theoretic model which would incorporate both the algebra $\ddot{H}_n$, in terms of braids, and our original algebra of closed curves in the thickened torus.

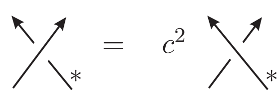
As a start we considered the possibility of a direct skein-based model in terms of n-braids in T^2 for the double affine Hecke algebra $\ddot{H}_n$ with parameters t and q.

The naive approach of considering n-braids in T^2 modulo the Homfly relations gives a model that works for one of the parameters t but only covers the case $q = 1$ for the other parameter. A search of the literature came up with a paper by Burella et al, [BWPV14], suggesting that a model based on framed braids could handle the more general case of $q \neq 1$, where adding a twist to the framing of a braid was reflected in multiplication by q. Their model depends on the product of certain braids with explicit framing resulting in a single twist on the framing of one string. We tried without success to follow the diagrammatic views of this product, which appears to us to have the trivial framing on all strings, and not the desired twist. We worked out a uniform way of specifying a framing on the strings of a torus braid, noted in Theorem 4.1 below, and we came to the conclusion that the use of framing alone would not provide a means of incorporating the second parameter q into a geometric model for $\ddot{H}_n$.

We were still hopeful of making a skein-based geometric model, and we came up instead with one that includes an extra string. Instead of working with n-braids in the torus we use $(n + 1)$-braids in which one distinguished string, called the *base string*, is fixed throughout. Equivalently our geometric elements are n-braids in the once-punctured torus, regarding the fixed base string as determining the puncture. In our model we use linear combinations of these braids. The regular n-string braids are allowed to interact as braids by the Homfly relations and the parameter q is introduced when a regular string is allowed to cross through the base string.

These relations can be summarised in diagrammatic form as





In Section 3 of this paper we set up our skein model starting from $\mathbf{Z}[s^{\pm 1}, c^{\pm 1}]$-linear combinations of n-braids in the punctured torus, up to equivalence. We give a presentation for this algebra as a quotient of the group algebra of the braid group of n-braids in the punctured torus, using an explicit presentation by Bellingeri, [Bel04, Theorem 1.1], for this braid group with generators $\sigma_1, \ldots, \sigma_{n-1}, a, b$. Our emphasis here is on the use of geometric diagrams to represent the elements of the algebra. Such an approach is used elsewhere with oriented framed (banded) curves in a variety of manifolds as the basic ingredients subject to the 3-term linear relations above. A new addition in our current setting is the use of the base string, and the relation introducing the second parameter c^2 when a string is moved across it.

We give diagrammatic illustrations of some useful braids and their interrelations, and show how to interpret Bellingeri's presentation in terms of our braids.

The skein relations can then be included by adding the relations

$$\sigma_i - \sigma_i^{-1} = s - s^{-1}$$

or

$$(\sigma_i - s)(\sigma_i + s^{-1}) = 0$$

in quadratic form, and

$$P = c^2,$$

where P is the braid taking string n once round the puncture and fixing the other strings. There is a simple formula for P in terms of the generators of the punctured braid group, which we provide.

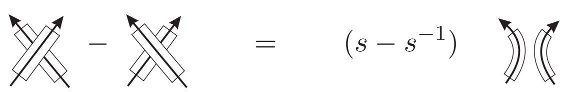
The outcome is a presentation for the algebra of braids in the punctured torus modulo the skein relations, which is our skein-based model, $\mathrm{BSk}_n(T^2, *)$, see Definition 3.1. We establish this presentation in Theorem 3.5, and show how it corresponds exactly to the presentation in [SV11] for the double affine Hecke algebra $\ddot{H}_n$:

Theorem (see Thm. 3.5)**.** *The braid skein algebra* $\mathrm{BSk}_n(T^2, *)$ *is isomorphic to the* $\mathfrak{gl}_n$ *double affine Hecke algebra* $\ddot{H}_{n;q,t}$*, with* $t = s^2$ *and* $q^{-1} = c^2$*.*

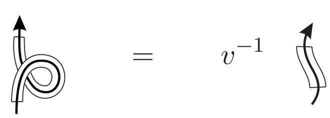
Remark 1.4. While this paper was in preparation, D. Jordan and M. Vazirani proved a very similar statement, that the DAHA is a quotient of the group algebra of the braid group of the punctured torus [JV17, Prop. 4.1]. The difference between their statement and ours is that they impose relations algebraically, while we impose ours topologically, giving a visually appealing interpretation of elements of the algebra and the underlying relations. It is therefore a-priori possible that we impose more relations than them, and the content of this theorem is that their relations imply ours. Let us also mention here that Cherednik states in [Che05] that the $\mathfrak{gl}_n$ DAHA is a deformation of the Hecke quotient of the braid group of the *closed* torus. One of our desires was to remove the word "deformation" from this statement so that the deformation parameter has a direct topological meaning.

Remark 1.5. One puzzling aspect of the comparison between this theorem and Theorem 1.2 is that in the Kauffman bracket case, the quadratic parameter in the DAHA corresponds to the "puncture parameter" on the skein side, and the q parameter in the DAHA corresponds to the quantum group parameter on the skein side. However, in Theorem 3.5, these parameter correspondences are reversed, and the quadratic parameter in the DAHA corresponds to the quantum group parameter on the skein side. We do not know a conceptual explanation for this.

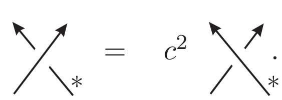
In Section 5 we make use of framed n-tangles in the full framed Homfly skein of the punctured torus, $\mathrm{Sk}_n(T^2, *)$. In this setting an n-tangle consists of n framed oriented arcs in the thickened torus, along with a number of framed oriented closed curves. The arcs are no longer restricted to lying as braids in $T^2 \times I$. We work with linear combinations of framed tangles and impose the local relation



between framed tangles, as well as the change of framing relation



using a second parameter v. In keeping with the first section we include a base string defining the puncture, and allow a framed string to cross through it at the expense of the scalar c^2, with local relation



There is a homomorphism from $\mathrm{BSk}_n(T^2, *)$ to $\mathrm{Sk}_n(T^2, *)$, since the braids in $\mathrm{BSk}_n(T^2, *)$ can be given a consistent framing using a nonvanishing vector field on the torus so that the relations in $\mathrm{BSk}_n(T^2, *)$ continue to hold in the wider tangle skein. It is not clear however whether this homomorphism is injective. One point at issue is that, as well as the extra elements introduced, the additional relations between them might have the effect of collapsing the algebra considerably. Nonetheless, we make the following:

Conjecture 1.6. The map $\mathrm{BSk}_n(T^2,*) \to \mathrm{Sk}_n(T^2,*)$ from the braid skein algebra to the tangle skein algebra is an isomorphism.

Remark 1.7. This may seem surprising, since tangles can contain closed curves, which means that a-priori, the tangle algebra is "much bigger" and the map in question should not be surjective. Indeed, for other surfaces the analogous map is not surjective. However, on the torus, a closed curve "on one side of the puncture" is isotopic to the same curve "on the other side of the puncture," and we use this fact in Theorem 4.1 to show that the map in the conjecture is surjective.

There is an algebra map from the (classical) Homflypt skein module $\mathrm{Sk}(T^2-D^2)$ of closed links in the thickened punctured torus to our skein algebra $\mathrm{Sk}_n(T^2,*)$ of tangles with a base string, given by "filling in the puncture with the identity braid and the base string." If we assume Conjecture 1.6, this gives us an algebra map $\mathrm{Sk}(T^2-D^2) \to \ddot{H}_n$ for any n. We can compose this map with multiplication by the symmetrizer $\mathbf{e}$ in the finite Hecke algebra to obtain a map $\mathrm{Sk}(T^2-D^2) \to \mathbf{e}\ddot{H}_n\mathbf{e}$ to the so-called *spherical subalgebra*.

Schiffmann and Vasserot showed that the elliptic Hall algebra is the $n \to \infty$ limit of the spherical subalgebras (see Theorem 2.3 for a precise statement). Let $\mathrm{Sk}^+(T^2-D^2)$ be the subalgebra generated by "curves lifted from the closed torus which only cross the y-axis positively" (see Definition 5.6 for a precise statement). We then show the following, which we view as evidence for Conjecture 1.3.

Theorem (see Thm. 5.7)**.** *Assuming Conjecture 1.6 holds, there is a surjective algebra map* $\mathrm{Sk}^+(T^2-D^2) \twoheadrightarrow \mathcal{E}^+_{\sigma,\bar{\sigma}}$.

One corollary of this theorem (again assuming Conjecture 1.6) is that the generator $u_{\mathbf{x}}$ of the elliptic Hall algebra has a simple interpretation as a sum $W_{\mathbf{x}}$ of certain closed curves on the torus, with homology class $\mathbf{x} \in \mathbb{Z}^2 = H_1(T^2-D^2)$. In fact, these elements are lifts of the exact same elements in $\mathrm{Sk}(T^2)$ that were used in [MS17]. The subtle point here is that not all the relations between $W_{\mathbf{x}}$ that were proved in [MS17] hold in the punctured torus, since the proofs of some of these relations used global isotopies on T^2 that don't lift to the punctured torus. Roughly, the problem is that some curves "get caught on the puncture." When pushed into our skein algebra of tangles with a base string, these curves can once again be pushed through the puncture, but at the cost of some "lower order terms" involving braids, and these lower order terms contribute to the generating series relations in the elliptic Hall algebra.

This suggests one purely algebraic question of possible interest. The Schiffmann-Vasserot elements $u_{\mathbf{x}}$ are in the spherical DAHA $e_n\ddot{H}_{q,t}e_n$, where e_n is the symmetrizer in the finite Hecke algebra. However, the images of our elements $W_{\mathbf{x}}$ most naturally lie in the centralizer $Z_{\ddot{H}_n}(H_n)$ of the finite Hecke algebra H_n inside the double affine Hecke algebra. This suggests there may be an interesting limit of these centralizers which would include the elliptic Hall algebra a subalgebra.

Let us briefly comment on related or future work. In [JV17], Jordan and Vazirani used factorization homology to construct representations of the braid-skein algebra $\mathrm{BSk}_n(T^2,*)$, and more skein-theoretic techniques to construct representations are being used in work in progress of Vazirani and Walker. We hope that some combination of these approaches could be used to prove Conjecture 1.6, but we don't discuss this in the present paper.

We also note that the so-called $\mathbb{A}_{q,t}$ algebra introduced by Carlsson and Mellit in [CM18] has a relation that looks like a 3-term version of the skein relation involving the base string. Discussions with Jordan and Mellit indicate that more precise versions of this statement are available, but these won't be discussed here either.

A summary of the contents of the paper is as follows. In Section 2 we recall algebraic background involving DAHAs and the elliptic Hall algebra. In Section 3 we define the braid skein algebra and

show it is isomorphic to the DAHA, and in Section 4 we discuss the tangle skein algebra. In Section 5 we compare this and the classical skein algebra of the punctured torus to the elliptic Hall algebra.
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2. Algebraic background

In this section we recall the algebraic definitions and results that we need in the rest of the paper. In particular, we define the elliptic Hall algebra and double affine Hecke algebras (DAHAs), and we recall results of Schiffmann and Vasserot relating the two. In later sections we use these results to relate the skein algebra of the punctured torus to the elliptic Hall algebra.

2.1. The Elliptic Hall algebra.

Let us recall the definition of the elliptic Hall algebra $\mathcal{E} = \mathcal{E}_{\sigma,\bar\sigma}$ of Burban and Schiffmann [BS12], using the conventions of [SV11]. It is an algebra over the ring $\mathbb{Q}(\sigma,\bar\sigma)$, and it is generated by elements $u_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$, subject to relations:

(1) If $\mathbf{x}$ and $\mathbf{x}'$ belong to the same line in $\mathbb{Z}^2$, then $[u_{\mathbf{x}}, u_{\mathbf{x}'}] = 0$.
(2) Assume that $\mathbf{x}$ is primitive and that the triangle with vertices 0, $\mathbf{x}$, and $\mathbf{x}+\mathbf{y}$ has no interior lattice points. Then

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}} \frac{\theta_{\mathbf{x}+\mathbf{y}}}{\alpha_1}$$

where the elements $\theta_{\mathbf{z}}$ with $\mathbf{z} \in \mathbb{Z}^2$ are obtained by the generating series identity

$$\sum_i \theta_{i\mathbf{x}_0} z^i = \exp\left(\sum_{i\geq 1} \alpha_i u_{i\mathbf{x}_0} z^i\right)$$

for $\mathbf{x}_0 \in \mathbb{Z}^2$ primitive.

In the above relations we used the constants $\epsilon_{\mathbf{x},\mathbf{y}} = \operatorname{sign}(\det(\mathbf{x}\,\mathbf{y}))$ and

$$\alpha_i = (1-\sigma^i)(1-\bar\sigma^i)(1-(\sigma\bar\sigma)^{-i})/i$$

We also define the following subsets of $\mathbf{Z} := \mathbb{Z}^2$:

$$\mathbf{Z}^{>} := \{(x,y) \mid x > 0\}, \quad \mathbf{Z}^{+} := \mathbf{Z}^{>} \sqcup \{(0,y) \mid y \geq 0\} \tag{2.1}$$

We use similar notation to define subalgebras of $\mathcal{E}$, for example,

$$\mathcal{E}^+ := \langle u_{\mathbf{x}} \mid \mathbf{x} \in \mathbf{Z}^+\}$$

We will use similar notation for algebras generated by elements indexed by $\mathbf{Z}$. Finally, let $d(\mathbf{x})$ be the greatest common denominator of the entries of $\mathbf{x} \in \mathbb{Z}^2$.

2.2. Limits of DAHAs.

We now recall the definition of the double affine Hecke algebra $\ddot{H}_n$ as given in [SV11]. This is an algebra over $\mathbf{Z}[t^{\pm 1/2}, q^{\pm 1}]$ with generators

$$\{T_i\}, 1 \leq i \leq n-1, \{X_j\}, \{Y_j\}, 1 \leq j \leq n$$

and relations

$$(T_i + t^{1/2})(T_i - t^{-1/2}) = 0 \tag{2.2}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \tag{2.3}$$

$$[T_i, T_j] = 0, |i-j| > 1 \tag{2.4}$$

$$[T_i, X_j] = [T_i, Y_j] = 0, j \neq i, i+1 \tag{2.5}$$

$$[X_i, X_j] = [Y_i, Y_j] = 0 \tag{2.6}$$

$$X_{i+1} = T_iX_iT_i, \tag{2.7}$$

$$Y_{i+1} = T_i^{-1}Y_iT_i^{-1} \tag{2.8}$$

$$X_1^{-1}Y_2 = Y_2X_1^{-1}T_1^{-2} \tag{2.9}$$

$$Y_1X_1 \cdots X_n = qX_1 \cdots X_nY_1 \tag{2.10}$$

Let e_n be the symmetrizing idempotent in the finite Hecke algebra (which is generated by the T_i's), which is characterized by $T_je_n = e_nT_j = t^{1/2}e_n$ for all j. The spherical DAHA is the subalgebra $\mathrm{S}\ddot{H}^n_{q,t} := e_n\ddot{H}^n_{q,t}e_n$ of $\ddot{H}^n_{q,t}$, and it is also $\mathbb{Z}^2$-graded. There is an $\mathrm{SL}_2(\mathbb{Z})$ action on the subalgebra $\mathrm{S}\ddot{H}^n_{q,t}$ (see the paragraph above Lemma 2.1 in [SV11]).

Following [SV11, Sec. 2.2] (except for the notational change $P \to Q$), for $k > 0$ we define elements

$$Q^n_{0,k} = e_n \sum_i Y_i^k e_n$$

Elements $Q^n_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ are defined using the $\mathrm{SL}_2(\mathbb{Z})$ action. We define $\mathrm{S}\ddot{H}^{n,>}_{q,t}$ to be the subalgebra of $\mathrm{S}\ddot{H}^n_{q,t}$ generated by $Q^n_{a,b}$ with $a > 0$.

Let us identify parameters $\sigma = q$ and $\bar{\sigma} = t^{-1}$. Then Schiffmann and Vasserot proved the following theorem relating the elliptic Hall algebra and spherical DAHAs.

Theorem 2.1 ([SV11, Thm. 3.1])**.** *The assignment*

$$u_{\mathbf{x}} \mapsto \frac{1}{q^{d(\mathbf{x})} - 1} Q^n_{\mathbf{x}}$$

extends uniquely to a $\mathbb{Z}^2$-graded $\mathrm{SL}_2(\mathbb{Z})$-equivariant surjective algebra homomorphism

$$\phi^n : \mathcal{E}_{q,t} \twoheadrightarrow \mathrm{S}\ddot{H}^n_{q,t}$$

Given the previous theorem, a natural question is whether there is some type of limit one can take as $n \to \infty$. It turns out that there is, but to describe it Schiffmann and Vasserot first had to prove the following theorem.

Theorem 2.2 ([SV13, Prop. 4.1])**.** *The assignment $Q^n_{\mathbf{x}} \mapsto Q^{n-1}_{\mathbf{x}}$ for each $\mathbf{x} \in \mathbb{Z}^+$ extends to a unique surjective algebra map $\Phi_n : \mathrm{S}\ddot{H}^{n,+}_{q,t} \to \mathrm{S}\ddot{H}^{n-1,+}_{q,t}$.*

This theorem allows us to construct a projective limit $\varprojlim \mathrm{S}\ddot{H}^n_{q,t}$. Also, the generators $Q^n_{\mathbf{x}}$ provide elements in this projective limit, and we let $\mathrm{S}\ddot{H}^{\infty,+}_{q,t}$ be the subalgebra generated by these elements. Theorem 2.1 shows that there is a map from the elliptic Hall algebra to $\mathrm{S}\ddot{H}^{\infty,+}_{q,t}$.

Theorem 2.3 ([SV13, Thm. 4.6])**.** *The induced map $\phi^\infty : \mathcal{E}^+_{q,t} \to \mathrm{S}\ddot{H}^{\infty,+}_{q,t}$ is an isomorphism.*

Summarizing this work of Schiffmann and Vasserot, we obtain the following corollary which we use below.

Corollary 2.4. *Suppose A is an algebra generated by elements $a_{\mathbf{x}}$ for $\mathbf{x} \in S \subset \mathbb{Z}^2$. Suppose there are algebra maps $A \to \mathrm{S}\ddot{H}^{n,+}_{q,t}$ for each n such that $a_{\mathbf{x}} \mapsto Q_{\mathbf{x}}$. Then there is an algebra map $A \to \mathcal{E}^+_{q,t}$ sending $a_{\mathbf{x}} \to (q^{d(\mathbf{x})} - 1)u_{\mathbf{x}}$.*

3. Skeins with a base string

We will describe some skeins which use the framed Homfly relations on oriented framed curves and braids in the thickened torus $T^2 \times I$, together with a single fixed base string $\{*\} \times I \subset T^2 \times I$.

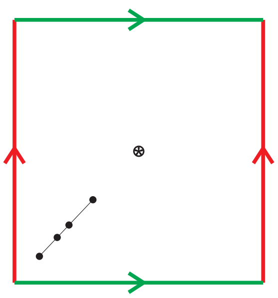
In this section we define the braid skein algebra $\mathrm{BSk}_n(T^2, *)$ in terms of $\mathbf{Z}[s^{\pm 1}, c^{\pm 1}]$-linear combinations of braids, and their composites, and prove that it is isomorphic to the double affine Hecke algebra $\ddot{H}_n$ (following the conventions in [SV11]). (See Theorem 3.7.)

3.1. **Isotopies of braids in the punctured torus.** We start by considering the group of n-braids in the punctured torus $T^2 - D^2$. We will work with the thickened torus $T^2 \times I$ with a single fixed base string $\{*\} \times I$ to determine by the puncture $* \in T^2$. Braids are made up of n strings oriented monotonically from $T^2 \times \{0\}$ to $T^2 \times \{1\}$ which do not intersect each other or the base string. Braids are considered equivalent when the strings are isotopic avoiding the base string.

Composition of braids is defined by placing one on top of the other, using the convention that AB means braid A lying below braid B.

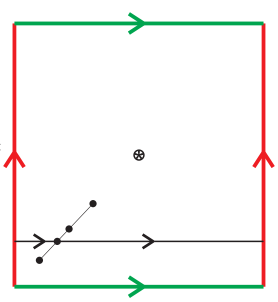
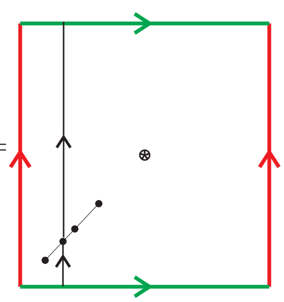
As in [MS17] we shall regard T^2 as given by identifying opposite pairs of sides in the unit square $[0, 1] \times [0, 1]$. Take the base point $*$ to be the centre $(1/2, 1/2)$ of the square. Fix $n > 0$ points in order on the lower part of the diagonal of the square between $(0, 0)$ and $*$ as the end points for n-string braids in $T^2 \times I - \{*\} \times I$.

We can draw the thickened torus in plan view as a square with opposite pairs of edges identified. We show the braid points and the base string position in the figure below.



We can indicate some simple braids where only one or two of the points move by drawing the path of the moving points on the plan view, rather as in the diagrams in [AM98]. In this view the braid product is given by concatenation of the paths.

For example, write x_i for the braid in which point i moves uniformly around the $(1, 0)$ curve in the torus, and y_i where point i moves around the $(0, 1)$ curve, with all other points remaining fixed. These are shown in plan view as

$x_i =$  , $y_i =$ 

and in a side view in figures 2 and 3. Similarly the braid σ_i appears in plan view as in figure 1, concentrating only on the region around the braid points.

A side elevation for x_i viewed in the $(0, 1)$ direction is shown in figure 2, and y_i viewed in the $(-1, 0)$ direction is seen in elevation in figure 3.

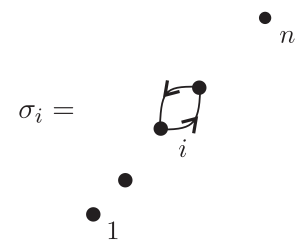


FIGURE 1. Plan view of σ_i

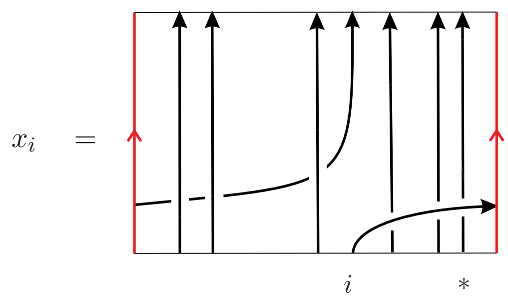


FIGURE 2. Side view of x_i

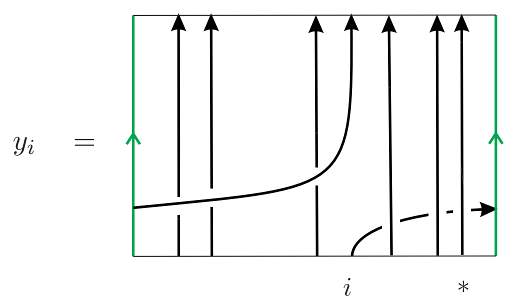


FIGURE 3. Side view of y_i

Using either of these two elevation views the braids σ_i appear in their usual form above, and it is immediate from these views that

$$\sigma_i^{-1} x_i \sigma_i^{-1} = x_{i+1} \tag{3.1}$$
$$\sigma_i y_i \sigma_i = y_{i+1}. \tag{3.2}$$

In a plan view we assume that paths are projections of braid strings which rise monotonically from their initial braid point to their final braid point. The product of two braids corresponds to the concatenation of their paths.

We can see that the braids $\{x_i\}$ commute among themselves, since their paths in the plan view are disjoint. The same applies to the braids $\{y_i\}$, and equally the braids σ_i commute with x_j and y_j when $j \neq i, i+1$.

The relations

$$x_1 x_2 = x_2 x_1, \quad y_1 y_2 = y_2 y_1$$

become

$$x_1 \sigma_1^{-1} x_1 \sigma_1^{-1} = \sigma_1^{-1} x_1 \sigma_1^{-1} x_1, \tag{3.3}$$
$$y_1 \sigma_1 y_1 \sigma_1 = \sigma_1 y_1 \sigma_1 y_1 \tag{3.4}$$

in terms of the generators x_1, y_1.

We can use the plan view for a braid where two paths cross, taking the usual convention of knot crossings to show which strand lies at a higher level. For example in the plan view of x_1y_2 the path of point 1 lies below that of point 2, giving views of x_1y_2 and y_2x_1 in figure 4.

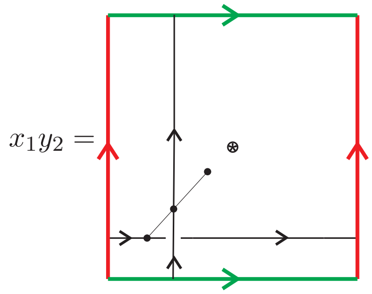


FIGURE 4. Plan views of x_1y_2 and y_2x_1

When two braids are composed there may be a path on the plan view that passes through a braid point at an intermediate stage. The plan can be altered to avoid such intermediate calls, by diverting the path slightly away from the braid point. For example the braid x_1y_1 starts with a plan view in figure 5. When the intermediate visit to braid point 1 is diverted a plan view for x_1y_1 is shown in figure 6 along with a view for y_1x_1.

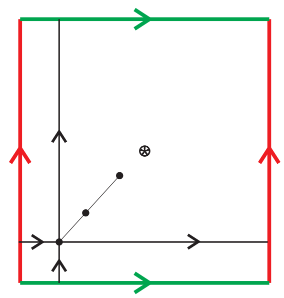


FIGURE 5. Plan view of x_1y_1

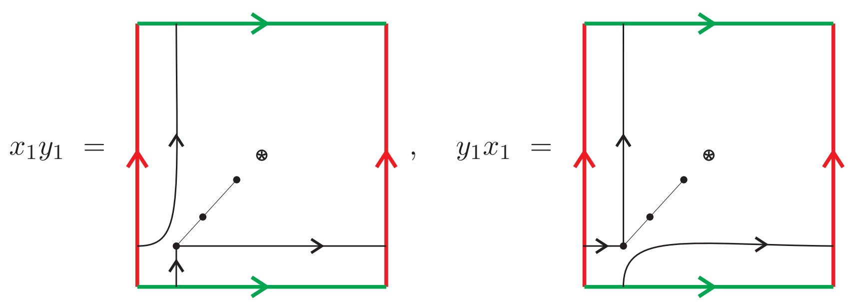


FIGURE 6. Smoothed plan view of x_1y_1 and y_1x_1

With further smoothing we get the plan view of the commutator $x_1y_1x_1^{-1}y_1^{-1}$ as shown in figure 7. From its elevation view in figure 8 we can write it as

$$x_1y_1x_1^{-1}y_1^{-1} = \sigma_1\sigma_2\cdots\sigma_{n-1}P\sigma_{n-1}\cdots\sigma_2\sigma_1.$$

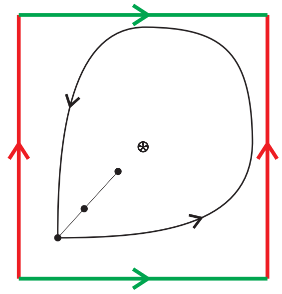
$x_1 y_1 x_1^{-1} y_1^{-1} =$ 

FIGURE 7. Plan view of $x_1 y_1 x_1^{-1} y_1^{-1}$

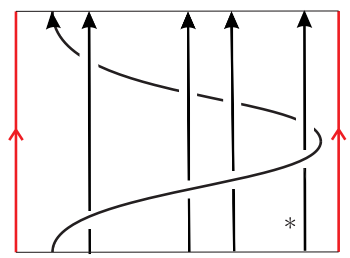
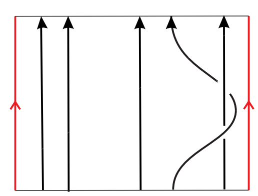
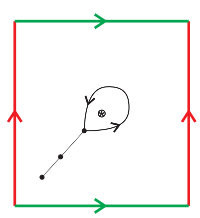


FIGURE 8. Elevation of $x_1 y_1 x_1^{-1} y_1^{-1}$

Here

$P =$ 

is the braid taking string n once round the base string, with plan view



This gives an expression

$$P = \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} x_1 y_1 x_1^{-1} y_1^{-1} \sigma_1^{-1} \cdots \sigma_{n-1}^{-1}.$$

as a braid in the punctured torus, in terms of the generators x_1, y_1, σ_i.

As a further help in using the plan view for paths we can alter the view near the projection of one of the braid points, where a path starts out at the lowest level from the braid point and finishes at the highest level. Then another path crossing nearby (with either orientation) can be moved across the braid point as shown locally in figure 9.

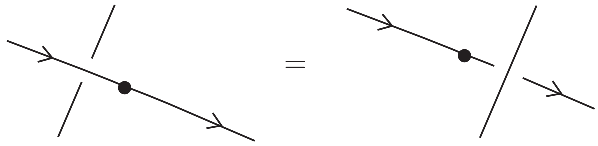
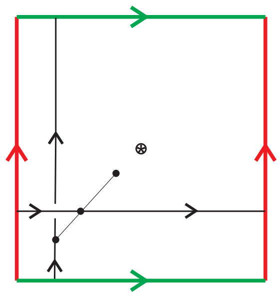
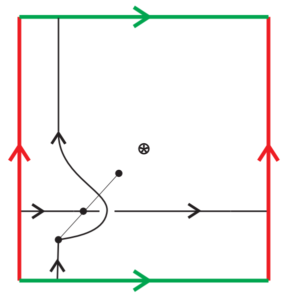
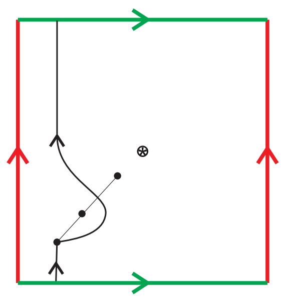


FIGURE 9. Moving an arc past a braidpoint

Apply this to the view of $y_1 x_2$ by moving the path from braid point 1 across braid point 2. This gives

$y_1 x_2 =$  $=$  $= x_2 \alpha_2$

where

$\alpha_2 =$  $= \sigma_1^2 y_1,$

and thus

$$x_2 y_1^{-1} \quad = \quad y_1^{-1} x_2 \sigma_1^2. \tag{3.5}$$

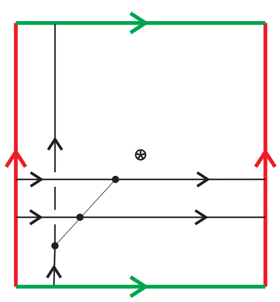
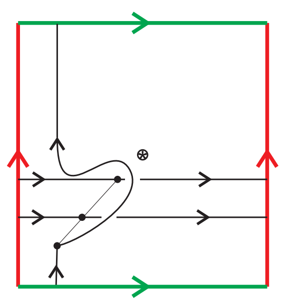
We can rewrite this equation in terms of the generators x_1 and y_1 as

$$\sigma_1^{-1} x_1 \sigma_1^{-1} y_1^{-1} = y_1^{-1} \sigma_1^{-1} x_1 \sigma_1$$

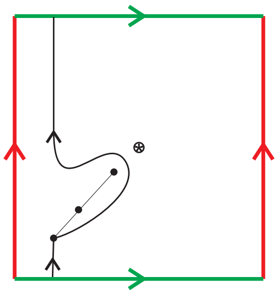
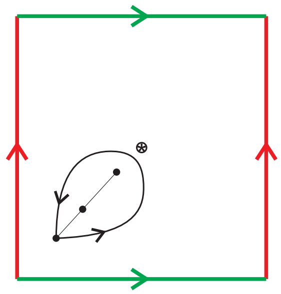
and further

$$\sigma_1^{-2} x_1 y_2^{-1} = y_2^{-1} x_1.$$

A similar argument, moving one path across braid points $2 \dots n$, shows that

$y_1 x_2 x_3 \cdots x_n =$  $=$  $= x_2 x_3 \cdots x_n \alpha_n$

in the punctured braid group, where

$\alpha_n =$  $= \beta_n y_1$, with $\beta_n =$ 

as in figure 10, giving

$$y_1 x_2 \cdots x_n = x_2 \cdots x_n \beta_n y_1.$$

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FIGURE 10. Side view of the braid $\beta_n = \sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1$

Bellingeri [Bel04, theorem 1.1] gives a presentation for the group of n-braids in the punctured torus with generators

$$\sigma_1, \cdots, \sigma_{n-1}, a, b,$$

and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1 \tag{3.6}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \tag{3.7}$$
$$\sigma_i a = a\sigma_i, i > 1 \tag{3.8}$$
$$\sigma_i b = b\sigma_i, i > 1 \tag{3.9}$$
$$a\sigma_1^{-1}a\sigma_1^{-1} = \sigma_1^{-1}a\sigma_1^{-1}a \tag{3.10}$$
$$b\sigma_1^{-1}b\sigma_1^{-1} = \sigma_1^{-1}b\sigma_1^{-1}b \tag{3.11}$$
$$b\sigma_1^{-1}a\sigma_1 = \sigma_1^{-1}a\sigma_1^{-1}b \tag{3.12}$$

In our notation this corresponds to a presentation with generators x_1, y_1, σ_i taking $a = y_1$ and $b = x_1^{-1}$ and σ_i^{-1} in place of σ_i.

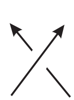
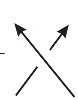
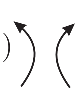
Bellingeri's relations involving a and b correspond to the equations

$$\begin{aligned} x_1x_2 &= x_2x_1 \\ y_1y_2 &= y_2y_1 \\ x_2y_1^{-1} &= y_1^{-1}x_2\sigma_1^2 \end{aligned}$$

when written in terms of the generators x_1, y_1, σ_1.

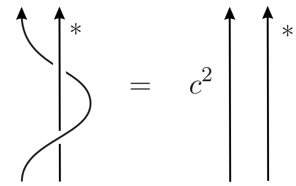
3.2. A presentation for the algebra $\mathrm{BSk}_n(T^2, *)$.

Definition 3.1. The braid skein algebra $\mathrm{BSk}_n(T^2, *)$ is defined to be $\mathbf{Z}[s^{\pm 1}, c^{\pm 1}]$-linear combinations of n-braids in the punctured torus, up to equivalence, subject to the local relations

(3.13)  $-$  $= (s - s^{-1})$ 

and

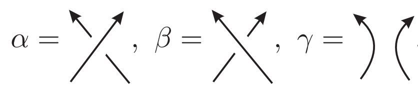
(3.14)



between braids.

By the term *local relation* in this definition we mean that the braids in the relations only differ as shown inside a 3-ball. We would like to find a "small" generating set for the ideal defined by these relations, which we do in the following three theorems. (To simplify exposition, Theorems 3.2 and 3.3 are proved in Subsection 3.3.)

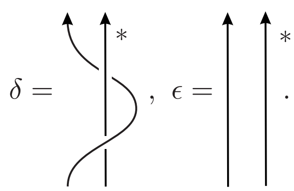
Theorem 3.2. *Suppose that α, β, γ are three n-braids in the punctured torus whose diagrams can be isotoped in $(T^2 - \{*\}) \times I$, fixing the boundary, so that they differ only inside a ball as*



Then there exist braids A and B such that

$$\alpha = A\sigma_1 B, \beta = A\sigma_1^{-1} B, \gamma = AB.$$

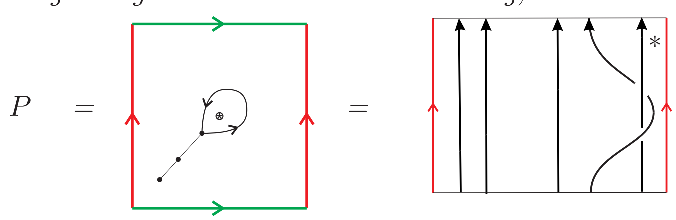
Theorem 3.3. *Suppose that δ, ϵ are two n-braids in the punctured torus whose diagrams can be isotoped in $(T^2 - D^2) \times I$, fixing the boundary, so that they differ only in a ball as*



Then there exist braids A and B such that

$$\delta = APB, \epsilon = AB,$$

where P is the braid taking string n once round the base string, shown here in plan and elevation.

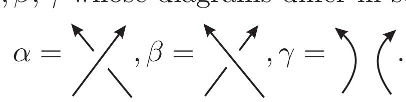


Theorem 3.4. *The ideal generated by* (3.13) *and* (3.14) *is the same as the ideal defined by*

$$\sigma_1 - \sigma_1^{-1} = (s - s^{-1}) \tag{3.15}$$

$$P = c^2. \tag{3.16}$$

Proof. Clearly the equations $\sigma_1 - \sigma_1^{-1} = s - s^{-1}$ and $P = c^2$ are special cases of (3.13) and (3.14) .
Conversely given any braids α, β, γ whose diagrams differ in some ball as



By theorem 3.2 we can write

$$\alpha - \beta = A(\sigma_1 - \sigma_1^{-1})B$$

Then equation (3.15) shows that

$$\alpha - \beta = (s - s^{-1})AB = (s - s^{-1})\gamma,$$

showing that α, β and γ satisfy equation (3.13).

To deduce equation (3.14) for braids δ and ϵ as in theorem 3.3 write

$$\delta = APB$$

and apply equation (3.16) to get

$$\delta = APB = c^2 AB = c^2 \epsilon.$$

□

We can now adjoin these relations to Bellingeri's presentation for the braid group of the punctured torus to give a presentation of the algebra $\mathrm{BSk}_n(T^2, *)$.

Theorem 3.5. *The algebra* $\mathrm{BSk}_n(T^2, *)$ *can be presented by the braids*

$$\sigma_1, \cdots, \sigma_{n-1}, x_1, y_1,$$

with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| > 1 \tag{3.17}$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{3.18}$$
$$\sigma_i x_1 = x_1 \sigma_i, i > 1 \tag{3.19}$$
$$\sigma_i y_1 = y_1 \sigma_i, i > 1 \tag{3.20}$$
$$x_1 \sigma_1^{-1} x_1 \sigma_1^{-1} = \sigma_1^{-1} x_1 \sigma_1^{-1} x_1 \tag{3.21}$$
$$y_1 \sigma_1 y_1 \sigma_1 = \sigma_1 y_1 \sigma_1 y_1 \tag{3.22}$$
$$x_1^{-1} \sigma_1 y_1 \sigma_1^{-1} = \sigma_1 y_1 \sigma_1 x_1^{-1} \tag{3.23}$$
$$(\sigma_1 - s)(\sigma_1 + s^{-1}) = 0 \tag{3.24}$$
$$x_1 y_1 x_1^{-1} y_1^{-1} = c^2 \sigma_1 \sigma_2 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_2 \sigma_1 \tag{3.25}$$

Proof. In our notation Bellingeri's generators are $a = y_1, b = x_1^{-1}$, and our σ_i is Bellingeri's σ_i^{-1}.

Relations (3.17) to (3.23) then present the algebra of n-braids in the punctured torus, by [Bel04]. Relation (3.24) is equivalent to relation (3.15). Relation (3.25) is equivalent to the relation (3.16), $P = c^2$, since

$$x_1 y_1 x_1^{-1} y_1^{-1} = \sigma_1 \sigma_2 \cdots \sigma_{n-1} P \sigma_{n-1} \cdots \sigma_2 \sigma_1.$$

□

Remark 3.6. As confirmation that our conventions are consistent with these relations note that with $x_2 = \sigma_1^{-1} x_1 \sigma_1^{-1}$ and $y_2 = \sigma_1 y_1 \sigma_1$ the relations between the generators x_1 and y_1 become $x_1 x_2 = x_2 x_1, y_1 y_2 = y_2 y_1$ and $y_2 x_1^{-1} = x_1^{-1} y_2 \sigma_1^{-2}$. These relations have already been demonstrated in our illustrations above.

Theorem 3.7. *The skein algebra* $\mathrm{BSk}_n(T^2, *)$ *is isomorphic to the double affine Hecke algebra* $\ddot{H}_n$.

Proof. We construct inverse homomorphisms between the two algebras.

- Define a homomorphism from $\mathrm{BSk}_n(T^2, *)$ to $\ddot{H}_n$ by sending x_1, y_1, σ_i to X_1, Y_1, T_i^{-1} and s^2, c^2 to t, q^{-1}.

  To show that this gives a homomorphism it is enough to check that the relations in the presentation of $\mathrm{BSk}_n(T^2, *)$ hold after the assignment of generators in $\ddot{H}_n$.

  The only relation for which this is not immediately clear is relation (3.25) in $\mathrm{BSk}_n(T^2, *)$. Relation (3.25) can be written

$$x_1 y_1 x_1^{-1} = c^2 \beta_n y_1.$$

We also know that

$$y_1 x_2 \cdots x_n = x_2 \cdots x_n \beta_n y_1.$$

The relation can then be rewritten as

$$c^{-2} x_1 y_1 x_1^{-1} = (x_2 \cdots x_n)^{-1} y_1 x_2 \cdots x_n.$$

In our assignment to $\ddot{H}_n$ we can see that each x_i is sent to X_i. It is then enough to check that

$$q X_1 Y_1 X_1^{-1} = (X_2 \cdots X_n)^{-1} Y_1 X_2 \cdots X_n$$

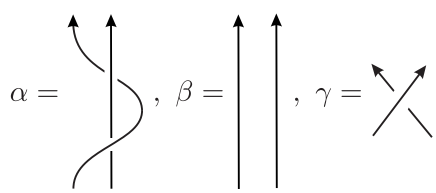
in $\ddot{H}_n$. This follows immediately from the last relation for $\ddot{H}_n$ and the fact that the elements X_i all commute.

- We can define an inverse homomorphism from $\ddot{H}_n$ to $\mathrm{BSk}_n(T^2, *)$ by sending X_i, Y_i, T_i to x_i, y_i, σ_i^{-1} and t, q to s^2, c^{-2}. Our illustrations above confirm that the relations from $\ddot{H}_n$ hold in $\mathrm{BSk}_n(T^2, *)$ after this assignment.

□

3.3. **Isotopies of relations.**

Proof of theorem 3.2. For convenience of drawing we include an extra crossing inside the ball so that



inside the ball and all three agree outside it. The two strings crossing in the ball in the diagram for α must belong to different braid strings, otherwise the diagram of γ would have a closed component, and so could not be a braid. Premultiply all three diagrams by the same braid C so that the two strings in the ball have started at braid points 1 and 2. Then postmultiply all three diagrams by the braid $\beta^{-1}C^{-1}$ to get three diagrams of braids α', β', γ' which again agree except inside a ball.

We can isotope the diagram for the identity braid β' to straighten out all the strings. We keep track of the ball, by adding two small meridian circles joined by a ribbon in the diagram for β'.

After any isotopy this ribbon gives us the means to recover diagrams of α' and γ'. In the case of α' we break string 1 where it comes to the meridian ring, and then follow the edge of the ribbon to the other meridian ring, and go round the meridian ring and then back along the other edge of the ribbon, to rejoin string 1. Similarly we can use the edges of the ribbon, combined with breaks of both strings at the meridian rings, to recover a diagram of γ'.

When we isotope β' to the identity braid with straight strings the ribbon will then join a meridian ring on string 1 to one on string 2. Push these meridian rings down to the bottom of their strings. The ribbon will now lie in $T^2 \times I$ in some potentially complicated way in the complement of the n straight strings. Using the ribbon we then have a diagram for α' in which strings 2 to n, and the base string, are all straight, while string 1 follows one edge of the ribbon, in the complement of the other strings, then passes once around string 2 in the positive sense on the meridian ring and finally returns along the other side of the ribbon to continue straight up along string 1.

At this stage string 1 may not lie monotonically in the direction of I. It does determine an element in the fundamental group of the complement of the straight strings based at point 1. This element can be written $l\sigma_1^2 l^{-1}$, where l is represented by the path from 1 along the ribbon, returning directly to 1 from the meridian ring at 2, while the path from 1 to 2, round the meridian ring, and back to 1 represents σ_1^2, and l^{-1} corresponds to the final return to 1 along the ribbon. Now the classical analysis of pure braids in surfaces [Bir74] shows that a braid with all but string 1 fixed is determined uniquely up to isotopy by the homotopy class of the first string as an element of the fundamental group of the complement of the remaining strings.

Write L for the braid determined in this way by the loop l. Then considering the homotopy class of the first string in the braid α' shows that

$$\alpha' = L\sigma_1^2 L^{-1}.$$

We can similarly write

$$\gamma' = L\sigma_1 L^{-1}.$$

The homotopy techniques above can be applied in a two-stage process to the pure braid $L^{-1}\gamma' L\sigma_1^{-1}$, by firstly removing string 1 and showing that string 2 is trivial, and then showing that string 1 is trivial, so that the pure braid is the identity.

The outcome is that

$$\alpha = C\alpha'\beta^{-1}C^{-1} = (CL\sigma_1)\sigma_1(L^{-1}\beta^{-1}C^{-1})$$

while

$$\beta = C\beta'\beta^{-1}C^{-1} = C\beta^{-1}C^{-1} = (CL\sigma_1)\sigma_1^{-1}(L^{-1}\beta^{-1}C^{-1})$$

and

$$\gamma = C\gamma'\beta^{-1}C^{-1} = (CL\sigma_1)(L^{-1}\beta^{-1}C^{-1}).$$

The result now follows, taking $A = CL\sigma_1$ and $B = L^{-1}\beta^{-1}C^{-1}$. □

Proof of theorem 3.3. Again we can put meridian rings and a ribbon around the two strings in the braid ϵ to keep track of the ball. Premultiply both braids by a braid C so that string n becomes the braid string in the ball, and then postmultiply both braids by $\epsilon^{-1}C^{-1}$ to get δ' and ϵ'. Then ϵ' is the identity braid. After isotoping all strings to be straight and moving the two meridian rings down to the bottom of string n and the base string the ribbon will allow a reconstruction of the braid δ' as a braid in which only string n moves. It represents an element lPl^{-1} as before, where l is represented by the path along one edge of the ribbon to the meridian ring at the base string, and back directly to point n, while the path from n going round the meridian ring at the base string gives the braid P. Write L for the pure braid determined by the loop l, to finish with the description

$$\delta' = LPL^{-1} = C\delta\epsilon^{-1}C^{-1},$$

Then

$$\delta = (C^{-1}L)P(L^{-1}C\epsilon),$$

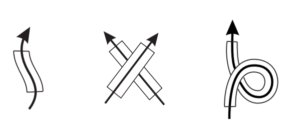
giving the result $\delta = APB, \epsilon = AB$, where $A = C^{-1}L$ and $B = L^{-1}C\epsilon$. □

4. The tangle skein algebra

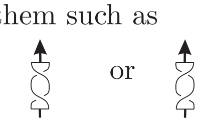
In this section we generalize the definition of the braid skein algebra using framed tangles, and we conjecture that this produces the same algebra as the braid skein algebra. In the next section we use this conjecture to relate the classical skein algebra of the punctured torus to the elliptic Hall algebra.

In Homflypt skein theory we consider oriented *banded* curves in a 3-manifold M, possibly with marked input and output points on its boundary.

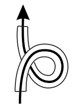
Here are some such pieces



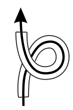
We can think of these as made of flat tape rather than rope. The only difference from rope is that the tapes can have extra twists in them such as



Twists may be dealt with by drawing little kinks in the diagram, replacing

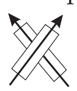
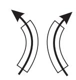
 by 

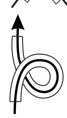
and

 by 

When there are boundary points the curves will include oriented arcs joining input to output points. In addition we can have some closed oriented curves.

The general Homflypt skein $\mathrm{Sk}(M)$ is defined to be $\mathbb{Z}[s^{\pm 1}, v^{\pm 1}]$ -linear combinations of banded links, up to isotopy, with the basic linear relations

(1)  $-$  $=$ $(s - s^{-1})$ 

(2)  $=$ v^{-1} 

between banded links whose diagrams differ only locally as shown.

Special cases of interest to us are where $M = F \times I$ for a surface F, with or without boundary. In such cases we write $\mathrm{Sk}(F)$ for the skein $\mathrm{Sk}(M)$, which has the structure of an algebra, with product induced by stacking curves in the direction of the interval I.

In [MS17] we have looked at the case where $F = T^2$, and given a presentation for $\mathrm{Sk}(T^2)$.

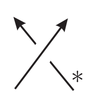
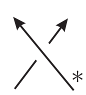
The case $\mathcal{C} = \mathrm{Sk}(A)$, where A is the annulus, is a commutative algebra. It has been widely studied, originally by Turaev [Tur97], and subsequently by Morton and others.

In our present work we will incorporate the skein of the torus with one hole, $\mathrm{Sk}(T^2 - D^2)$, including elements which map to the generators of $\mathrm{Sk}(T^2)$ under the homomorphism induced by the inclusion $T^2 - D^2 \to T^2$.

Again in the case $M = F \times I$ we will consider the case where we fix n input points in $F \times \{0\}$, and take the corresponding n output points in $F \times \{1\}$. Stacking in the I direction will give this skein the structure of an algebra over $\mathbb{Z}[s^{\pm 1}, v^{\pm 1}]$ which we denote by $\mathrm{Sk}_n(F)$.

The simplest case of this, when $F = D^2$, gives the algebra $\mathrm{Sk}_n(D^2)$. This algebra is a version of the Hecke algebra $H_n(z)$ of type A, based on the quadratic relation $\sigma_i^2 = z\sigma_i + 1$, where $z = s - s^{-1}$.

In anticipation of the next section we are led to consider the skein $\mathrm{Sk}_n(T^2 - *)$ of the punctured torus. In order to incorporate our algebra $\mathrm{BSk}_n(T^2, *)$ into this framework we will adjoin the relation

 $=$ c^2 

to allow a string to cross through the fixed string $* \times I$ in $T^2 \times I$ which defines the puncture. With this extra relation in place we use the notation $\mathrm{Sk}_n(T^2, *)$ for the resulting algebra over $\mathbb{Z}[s^{\pm 1}, v^{\pm 1}, c^{\pm 1}]$.

Theorem 4.1. *There is a surjective algebra homomorphism*

$$F : \mathrm{BSk}_n(T^2, *) \cong \ddot{H}_n \twoheadrightarrow \mathrm{Sk}_n(T^2, *).$$

Proof. It is enough to observe that braids in T^2 can be framed by fixing a direction in T^2, say the $(1, 0)$ direction, and taking the band on each braid string in this direction, which is always transverse to the string at all points. Under any braid isotopy the bands will be preserved, and the relations between braids will satisfy the skein relations between banded curves.

Now we show that this map is surjective. Let $\varphi_n : \mathrm{Sk}(T^2 - D^2) \to \mathrm{Sk}_n(T^2, *)$ be the algebra map given by filling in the disk with the identity braid, see equation (5.1). As an intermediate step we take a diagram C in $\mathrm{Sk}(T^2 - D^2)$ and show that there is a (non-commutative) polynomial $K(W_{\mathbf{x}_\alpha})$ such that $\varphi_n(C - K) \in \mathrm{Sk}_n(T^2, *)$ is represented by braids. In other words $\varphi_n(C - K) \in \mathrm{Im}(\mathrm{BSk}_n(T^2, *))$. To show this we first use induction on number of crossings in the diagram C. Using skein relations, we may switch crossings so that all components of the diagram are contained in non-overlapping intervals when projected to $[0, 1]$. Since the map F is clearly an algebra map, we may now assume (by induction) that C has a single component. Using the same induction, we may assume that this component is a totally ascending curve, with basepoint on the boundary of D.

Next, consider the annulus A_α defined by the homology $[C] = \alpha \in \mathbb{Z}^2$ of C, which is the neighborhood of an embedded curve of slope α immediately adjacent to D^2. The kernel of the map $\pi_1(T^2 - D^2)$ is generated by the loop around D^2, and, modulo braids, this loop is contractible. Since the component C is totally ascending, this shows that using skein relations, we may move C to be contained in A_α, modulo braids. (If C weren't totally ascending, the "lower order terms" in this argument would not be braids.)

Finally, the skein of the annulus is generated by the $W_{\mathbf{x}}$ with $\mathbf{x}$ parallel to α. In equation (5.2), it is shown that $W_{(m,0)}$ can be written as a braid, and combining this with the $SL_2(\mathbb{Z})$ action completes the proof.

□

Remark 4.2. It is not clear whether this homomorphism is injective. There can be the question of possible further relations between elements in the image of $\mathrm{BSk}_n(T^2, *) \cong \ddot{H}_n$ coming from the additional closed curves that can be used in $\mathrm{Sk}_n(T^2, *)$.

Despite the previous remark, we conjecture (see Conjecture 1.6 in the introduction) that the algebra map $\ddot{H}_n \to \mathrm{Sk}_n(T^2, *)$ in Theorem 4.1 is an isomorphism.

5. Relations with the elliptic Hall algebra

In [SV11] the authors relate the double affine Hecke algebras $\ddot{H}_n$ to the elliptic Hall algebra. As part of their construction they make use of the sums of powers

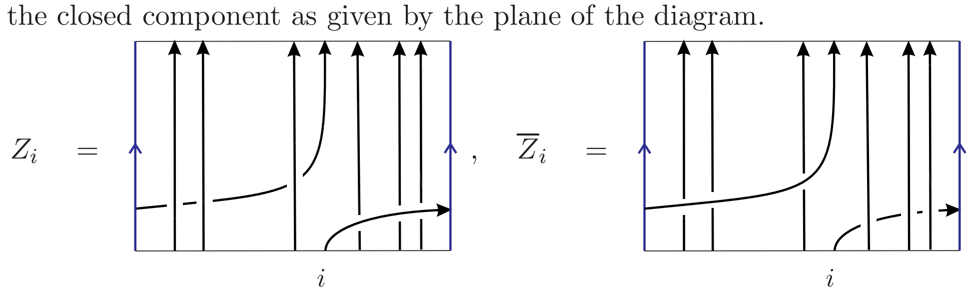
$$\sum_i X_i^l, \sum_i Y_i^l \in \ddot{H}_n$$

which have a very useful skein theoretic description, and which led us to try including closed curves in our skein $\mathrm{BSk}_n(T^2, *)$. We will show that the images of these elements in $\mathrm{Sk}_n(T^2, *)$ agree with the images of certain natural elements in $\mathrm{Sk}(T^2 - D^2)$. In Theorem 5.7, we combine this with results of Schiffmann and Vasserot to show that Conjecture 1.6 implies a weakened version of Conjecture 1.3.

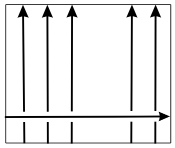
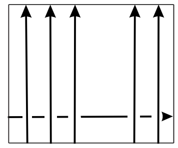
5.1. **Certain closed curves.** For the moment consider the Homfly skein $\mathrm{Sk}_n(A)$ where A is an annulus, using oriented diagrams in the thickened annulus $A \times I$ with n output points on the top $A \times \{1\}$, and n matching input points on $A \times \{0\}$. We also allow closed components in the diagrams.

When restricted to braid diagrams the skein $\mathrm{BSk}_n(A)$ is used by Graham and Lehrer as a model for the affine Hecke algebra $\dot{H}_n$, where composition is again induced by composition of braids.

Write Z_i and $\overline{Z}_i$ for the elements represented in $\mathrm{Sk}_n(A)$ by the diagrams shown here. Take the framing of the closed component as given by the plane of the diagram.



It is readily established that

 $-$  $= (s-s^{-1})\sum Z_i$

$= (s-s^{-1})\sum \overline{Z}_i$

There is also a well-established element P_m for each m in the skein $\mathrm{Sk}(A) = \mathcal{C}$ of the annulus with no boundary points which satisfies the relation

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$= (s^m - s^{-m})$

When we embed A into T^2 around the $(1,0)$ curve, matching the braid points suitably, the induced homomorphism from $\mathrm{Sk}_n(A)$ to $\mathrm{Sk}_n(T^2,*)$ gives the equation

$(s^m - s^{-m})\sum_i x_i^m =$ uxfront.{ps,eps,pdf} not found (or no BBox) $-$ uxback.{ps,eps,pdf} not found (or no BBox)

$= (1-c^{2m})$ uxfront.{ps,eps,pdf} not found (or no BBox)

Similarly, taking A around the $(0,1)$ curve on T^2 we get

$(s^m - s^{-m})\sum_i y_i^m =$ uyfront.{ps,eps,pdf} not found (or no BBox) $-$ uyback.{ps,eps,pdf} not found (or no BBox)

$= (c^{-2m}-1)$ uyback.{ps,eps,pdf} not found (or no BBox)

In $\mathrm{Sk}_n(A)$, taking account of the crossing signs, we also have

backidleftblue.{ps,eps,pdf} not found (or no BBox) $-$ frontidleftblue.{ps,eps,pdf} not found (or no BBox) $= -(s$

$= -(s$

Placing A along the $(1,0)$ curve then gives

$-(s^m - s^{-m})\sum x_i^{-m} =$ uxfrontleft.{ps,eps,pdf} not found (or no BBox) $-$ uxbackleft.{ps,eps,pdf} not found (o

$= (1-c^{-2m})$ uxfrontleft.{ps,eps,pdf} not found (or no BBox) ,

while placing A along the $(0,1)$ curve gives

$-(s^m - s^{-m})\sum_i y_i^{-m} =$ uyfrontdown.{ps,eps,pdf} not found (or no BBox) $-$ uybackdown.{ps,eps,pdf} not found (o

$= (c^{2m}-1)$ uybackdown.{ps,eps,pdf} not found (or no BBox)

In [SV11] there is a description of the elliptic Hall algebra which involves generators $u_{\mathbf{x}}$ for every non-zero $\mathbf{x} \in \mathbb{Z}^2$. These elements satisfy certain commutation relations, and the comparison with the algebras $\ddot{H}_n$ requires the prescription of an image for each $u_{\mathbf{x}}$, and a check on their commutation properties.

We can give a version of this comparison by using the skein $\mathrm{Sk}_n(T^2,*)$, and the skein $\mathrm{Sk}(T^2-D^2)$. Fix a disc D^2 in T^2 which includes the braid points and the base point. A suitable choice for our purposes is a neighbourhood of the diagonal in the square. There is then a homomorphism

$$\varphi_n : \mathrm{Sk}(T^2-D^2) \to \mathrm{Sk}_n(T^2,*) \tag{5.1}$$

defined by taking the banded curves in T^2-D^2 along with the identity n-braid in $\mathrm{Sk}_n(T^2,*)$, consisting of n vertical strings in $D^2\times I$ and the base string.

Now any oriented embedded curve in T^2-D^2 is determined up to isotopy by a primitive element $\mathbf{y}\in\mathbb{Z}^2$, representing the homology class of the curve. This curve, framed by its neighbourhood in T^2 defines an element $W_{\mathbf{y}}\in\mathrm{Sk}(T^2-D^2)$. For any other non-zero $\mathbf{x}\in\mathbb{Z}^2$ write $\mathbf{x}=m\mathbf{y}$ with $m>0$ and $\mathbf{y}$ primitive, and define $W_{\mathbf{x}}$ to be $W_{\mathbf{y}}$ with the closed curve decorated by the element P_m.

We will write $W_{\mathbf{x}}$ also for its image in the skein $\mathrm{Sk}_n(T^2,*)$. We then have plan views of $W_{(\pm m,0)}$ and $W_{(0,\pm m)}$ as

$W_{(m,0)}$ = uxfront.{ps,eps,pdf} not found (or no BBox) , $W_{(-m,0)}$ = uxfrontleft.{ps,eps,pdf} not found (or no BBox) ,

$W_{(0,m)}$ = uyback.{ps,eps,pdf} not found (or no BBox) , $W_{(0,-m)}$ = uybackdown.{ps,eps,pdf} not found (or no BBox) ,

Our equations above show that

$$\begin{aligned}(1-c^{2m})W_{(m,0)} &= (s^m-s^{-m})\sum x_i^m,\\ (c^{-2m}-1)W_{(-m,0)} &= (s^m-s^{-m})\sum x_i^{-m}\\ (c^{-2m}-1)W_{(0,m)} &= (s^m-s^{-m})\sum y_i^m\\ (1-c^{2m})W_{(0,-m)} &= (s^m-s^{-m})\sum y_i^{-m}.\end{aligned} \tag{5.2}$$

5.2. **Comparison with the algebraic approach.** For non-zero $\mathbf{x}\in\mathbb{Z}^2$ Schiffman and Vasserot in [SV11] define elements $Q_{\mathbf{x}}$ in the spherical algebra $S\ddot{H}_n$, where $S\ddot{H}_n$ is defined as $e_n\ddot{H}_ne_n$, with $e_n\in H_n$ being the symmetrizer. They use the elements $Q_{\mathbf{x}}$ in setting up their comparisons with the elliptic Hall algebra.

Using the identification of $\mathrm{BSk}_n(T^2,*)$ with $\ddot{H}_n$, where $q=c^{-2}, s^2=t$, we show now that our elements $W_{\mathbf{x}}$ are closely related to $Q_{\mathbf{x}}\in S\ddot{H}_n$, when mapped into the full skein algebra $\mathrm{Sk}_n(T^2,*)$.

Before doing this we note the construction of the symmetrizer $e_n\in H_n\subset\ddot{H}_n$ in the braid skein setting, as used by Aiston and Morton in [AM98].

We use the model of the Hecke algebra H_n described in [MT90], and further in [AM98]. The symmetrizer is given there as a multiple of the quasi-idempotent $a_n=\sum s^{l(\pi)}\omega_\pi$, where ω_π is the positive permutation braid associated to the permutation π with length $l(\pi)$ in the symmetric group. The symmetrizer is then $e_n=\frac{1}{\alpha_n}a_n$ where α_n is given by the equation $a_na_n=\alpha_na_n$ [Luk05, AM98]. Using the quasi-idempotent $b_n=\sum(-s)^{-l(\pi)}\omega_\pi$ in a similar way gives the antisymmetrizer.

We prefer to avoid the notation S for the symmetrizer, because of conflict with the notation for the symmetric group. In [SV11] the element a_n is denoted by $\tilde{S}$, and the symmetrizer by S.

Theorem 5.1. *For $\mathbf{x}\in\mathbb{Z}^2$ we have the following equality in $\mathrm{Sk}_n(T^2,*)$:*

$$(q^m-1)e_nW_{\mathbf{x}}e_n=(s^m-s^{-m})Q_{\mathbf{x}},$$

where $\mathbf{x}=m\mathbf{y}$ with $\mathbf{y}$ primitive and $m>0$.

Proof. We start from the definition in [SV11] which sets $Q_{(0,m)}:=e_n\sum Y_i^me_n$ for $m>0$.

Our third equation above proves the theorem for $\mathbf{x}=(0,m)$, since

$$(q^m-1)e_nW_{(0,m)}e_n=(s^m-s^{-m})e_n\sum Y_i^me_n=(s^m-s^{-m})Q_{(0,m)}.$$

When $\mathbf{x} = (\pm m, 0), (0, -m)$ the values of $Q_{\mathbf{x}}$ are shown in [SV11, Eq. 2.16-2.18] to be

$$\begin{aligned} Q_{(-m,0)} &= e_n \sum X_i^{-m} e_n \\ Q_{(0,-m)} &= q^m e_n \sum Y_i^{-m} e_n \\ Q_{(m,0)} &= q^m e_n \sum X_i^{m} e_n. \end{aligned}$$

The theorem follows immediately in these cases too from our three other equations, since

$$(q^m - 1) W_{(-m,0)} = (s^m - s^{-m}) \sum x_i^{-m},$$

giving the case $\mathbf{x} = (-m, 0)$, while

$$(q^m - 1) W_{(m,0)} = (s^m - s^{-m}) q^m \sum x_i^{m}$$

and

$$(q^m - 1) W_{(0,-m)} = (s^m - s^{-m}) q^m \sum y_i^{-m},$$

giving the other two cases.

We use automorphisms of $\ddot{H}_n$, and their counterpart in the skein models $\mathrm{BSk}_n(T^2, *)$ and $\mathrm{Sk}_n(T^2, *)$ to establish the proof for general $\mathbf{x}$.

Firstly, in our skein model, a right-hand Dehn twist about the (unoriented) $(1, 0)$ curve in $T^2 - D$ induces an automorphism τ_1 of $\mathrm{Sk}(T^2 - D^2)$, which carries $W_{\mathbf{x}}$ to $W_{\mathbf{y}}$ with

$$\mathbf{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

A left-hand Dehn twist about the unoriented $(0, 1)$ curve in $T^2 - D$ induces an automorphism τ_2 of $\mathrm{Sk}(T^2 - D^2)$, which carries $W_{\mathbf{x}}$ to $W_{\mathbf{y}}$ with

$$\mathbf{y} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

These two automorphisms generate all homeomorphisms of T^2 which fix D, up to isotopy fixing ∂D. This group of automorphisms is isomorphic to the braid group B_3 with τ_1 and τ_2^{-1} playing the roles of the usual Artin generators σ_1, σ_2. The kernel of the map to $SL(2, \mathbb{Z})$ is infinite cyclic, generated by $(\tau_1 \tau_2^{-1} \tau_1)^4$, which is the right-hand Dehn twist about ∂D.

For any $\mathbf{x}$ with $d(\mathbf{x}) = m > 0$ we can find an automorphism γ so that $\mathbf{x} = \gamma((0, m))$.

Now the effect of τ_1 on the generators σ_i, x_i, y_i of $\mathrm{BSk}_n(T^2, *)$ is

$$\begin{aligned} \tau_1(\sigma_i) &= \sigma_i \\ \tau_1(x_i) &= x_i \\ \tau_1(y_i) &= \eta_i \end{aligned}$$

where

$$\eta_i = y_i x_i \delta_i$$

and

$$\delta_i = \sigma_{i-1} \ldots \sigma_1 \sigma_1 \ldots \sigma_{i-1}.$$

The effect of τ_2 is

$$\begin{aligned} \tau_2(\sigma_i) &= \sigma_i \\ \tau_2(x_i) &= \xi_i \\ \tau_2(y_i) &= y_i \end{aligned}$$

where

$$\xi_i = x_i y_i \delta_i^{-1}.$$

The automorphisms ρ_1 and ρ_2 used in [SV11] agree with τ_2 and τ_1, given the correspondence of x_i, y_i, σ_i with X_i, Y_i, T_i^{-1} respectively.

Since $Q_{\mathbf{x}}$ is given from $Q_{(0,m)}$ by applying a suitable product of ρ_1 and ρ_2, the same automorphism will carry $W_{(0,m)}$ to $W_{\mathbf{x}}$ and the theorem will follow. □

5.3. **Without the symmetrizer.** Theorem 5.1, which refers to elements of $\mathrm{Sk}_n(T^2,*)$, suggests that $Q_{\mathbf{x}}$ could be defined unambiguously from an element $\tilde{Q}_{\mathbf{x}}$ in $\ddot{H}_n \cong \mathrm{BSk}(T^2,*)$ before passing to $S\ddot{H}_n$. The kernel of the map from B_3 to $SL(2,\mathbb{Z})$ is generated by $(\tau_1\tau_2^{-1}\tau_1)^4$. In the skein model this is a Dehn twist about the boundary of the disc D, and so in this model we expect the following theorem, which we can prove algebraically.

Proposition 5.2. *For any $Z \in \ddot{H}_n \cong \mathrm{BSk}_n(T^2,*)$ we have*

$$(\tau_1\tau_2^{-1}\tau_1)^4 Z = \Delta^{-2} Z \Delta^2,$$

where Δ^2 is the full twist braid in the finite Hecke algebra H_n.

Proof. It is enough to prove this when $Z = x_1$ and $Z = y_1$, since these elements, along with σ_i, generate $\mathrm{BSk}_n(T^2,*)$. In the case $Z = \sigma_i$ we have $\tau_1(\sigma_i) = \tau_2(\sigma_i) = \sigma_i$, while the full twist Δ^2 commutes with each σ_i.

We also know that

$$\begin{aligned}
\tau_1(x_1) &= x_1\\
\tau_1(y_1) &= y_1x_1\\
\tau_2^{-1}(x_1) &= x_1y_1^{-1}\\
\tau_2^{-1}(y_1) &= y_1
\end{aligned}$$

Writing $\tau_1\tau_2^{-1}\tau_1 = \theta$ we get

$$\theta(x_1) = y_1^{-1}, \theta(y_1) = y_1x_1y_1^{-1}$$

so

$$\theta^2(x_1) = (\theta(y_1))^{-1} = y_1x_1^{-1}y_1^{-1} = (y_1x_1)x_1^{-1}(y_1x_1)^{-1}$$
$$\theta^2(y_1) = \theta(y_1)\theta(x_1)(\theta(y_1))^{-1} = (y_1x_1)y_1^{-1}(y_1x_1)^{-1}$$

Finally

$$\begin{aligned}
\theta^4(x_1) &= \theta^2(y_1x_1)\theta^2(x_1^{-1})(\theta^2(y_1x_1))^{-1} = (y_1x_1)(y_1^{-1}x_1^{-1})x_1(x_1y_1)(y_1x_1)^{-1}\\
&= [x_1,y_1]^{-1}x_1[x_1,y_1]\\
\theta^4(y_1) &= [x_1,y_1]^{-1}x_1[x_1,y_1]
\end{aligned}$$

Now $[x_1,y_1] = c^2\beta_n$ so

$$\begin{aligned}
\theta^4(x_1) &= \beta_n^{-1}x_1\beta_n\\
\theta^4(y_1) &= \beta_n^{-1}y_1\beta_n
\end{aligned}$$

The result now follows since

$$\Delta^2 = w(\sigma_2,\cdots,\sigma_{n-1})\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1 = w\beta_n$$

and the braid w commutes with x_1 and y_1. □

Remark 5.3. Simmental, [Sim, Lemma 2.4.20], in notes which are part of a seminar series at MIT and Northeastern in 2017, makes a similar observation when applied to the spherical algebra $\mathrm{S}\ddot{H}_n$, to demonstrate the construction of the elements $Q_{\mathbf{x}}$.

We can go further and define $\tilde{Q}_{0,m}$ for $m > 0$, by

$$\tilde{Q}_{0,m} = y_1^m + y_2^m + \cdots + y_n^m.$$

Then

$$Q_{0,m} = e_n \tilde{Q}_{0,m} e_n,$$

in [SV11]. We follow the same procedure as in [SV11] to define $\tilde{Q}_{\mathbf{x}}$ from $\tilde{Q}_{0,m}$ by applying an automorphism from $SL(2,\mathbb{Z})$ which takes $(0,m)$ to $\mathbf{x}$.

This gives a well-defined element $\tilde{Q}_{\mathbf{x}}$, provided we can show that $(\tau_1\tau_2^{-1}\tau_1)^4 = \theta^4$ acts trivially on $\tilde{Q}_{0,m} = \sum y_i^m$. So we prove

Lemma 5.4.

$$\Delta^{-2}(y_1^m + \cdots + y_n^m)\Delta^2 = y_1^m + \cdots + y_n^m.$$

Proof. It is enough to show that $y_1^m + \cdots + y_n^m$ commutes with σ_i for all i. Now σ_i commutes with y_j for $j \neq i, i+1$. So we just need to show that σ_i commutes with $y_i^m + y_{i+1}^m$.

This in turn follows once we prove that

$$\begin{aligned} \sigma_i(y_i + y_{i+1}) &= (y_i + y_{i+1})\sigma_i \\ \sigma_i(y_i y_{i+1}) &= (y_i y_{i+1})\sigma_i \end{aligned}$$

Now

$$\begin{aligned} \sigma_i(y_i + y_{i+1}) &= \sigma_i y_i + \sigma_i^2 y_i \sigma_i = \sigma_i y_i + y_i \sigma_i + (s - s^{-1})\sigma_i y_i \sigma_i \\ &= y_i \sigma_i + \sigma_i y_i \sigma_i^2 = (y_i + y_{i+1})\sigma_i, \\ \sigma_i(y_i y_{i+1}) &= \sigma_i y_i \sigma_i y_i \sigma_i = y_{i+1} y_i \sigma_i = (y_i y_{i+1})\sigma_i. \end{aligned}$$

This completes the proof. □

So we have constructed elements $\tilde{Q}_{\mathbf{x}} \in \ddot{H}_n$ with $Q_{\mathbf{x}} = e_n \tilde{Q}_{\mathbf{x}} e_n$, which are related even more directly to the elements $W_{\mathbf{x}}$ in $\mathrm{Sk}(T^2, *)$, in the following enhancement of theorem 5.1.

Theorem 5.5. *For every non-zero* $\mathbf{x} \in \mathbb{Z}^2$ *we have*

$$(q^m - 1)W_{\mathbf{x}} = (s^m - s^{-m})\tilde{Q}_{\mathbf{x}},$$

where $\mathbf{x} = m\mathbf{y}$ *with* $\mathbf{y}$ *primitive and* $m > 0$.

5.4. **The punctured torus and elliptic Hall algebra.** In this subsection, we use the previous results in this section to show that Conjecture 1.6 implies a weakened version of Conjecture 1.3. Recall that $\mathbb{Z}^+ \subset \mathbb{Z}^2$ is defined by

$$\mathbb{Z}^+ := \{(a,b) \in \mathbb{Z}^2 \mid a > 0\} \sqcup \{(0,b) \mid b \geq 0\}$$

Definition 5.6. Let $\mathrm{Sk}^+(T^2 - D^2)$ be the subalgebra of $\mathrm{Sk}(T^2 - D^2)$ generated by $W_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^+$.

Theorem 5.7. *There is a surjective algebra map* $\mathrm{Sk}^+(T^2 - D^2) \twoheadrightarrow \mathcal{E}^+_{\sigma,\bar{\sigma}}$ *sending* $W_{\mathbf{x}}$ *to* $(s^{d(\mathbf{x})} - s^{-d(\mathbf{x})})u_{\mathbf{x}}$.

Proof. By Conjecture 1.6, the map $\ddot{H}_n \to \mathrm{Sk}_n(T^2, *)$ is an isomorphism, and we can compose its inverse with the natural map

$$\varphi_n : \mathrm{Sk}(T^2 - D^2) \to \mathrm{Sk}_n(T^2, *)$$

to obtain a map $\mathrm{Sk}(T^2 - D^2) \to \ddot{H}_n$. By Theorem 5.1, this map satisfies the following equation:

$$W_{\mathbf{x}} \mapsto \frac{s^{d(\mathbf{x})} - s^{-d(\mathbf{x})}}{q^{d(\mathbf{x})} - 1} Q_{\mathbf{x}}$$

By Corollary 2.4, this proves the existence of the algebra map stated in the theorem, and surjectivity follows immediately from the definition of the subalgebra $\mathcal{E}^+_{\sigma,\bar{\sigma}}$. □

Remark 5.8. It would be desirable to extend this map to a much larger subalgebra of $\mathrm{Sk}^+(T^2 - D^2)$, and it seems that the main difficulty is showing compatibility between the Schiffmann-Vasserot projections of spherical DAHAs and the maps $\mathrm{Sk}(T^2 - D^2)$. Ideally this would follow from a topological interpretation of these projections as some kind of partial trace, but it isn't clear if such an interpretation exists.

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