Supersymmetry Algebras in Arbitrary Dimension and Signature

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Abstract

In this thesis, a formalism is presented that allows the construction of supersymmetry algebras in arbitrary dimension in such a way that the space-time $\mathrm{SO}(t,s)$ and R-symmetry transformations are disentangled completely (for odd dimensions) or almost completely (in even dimensions). This is done by first taking multiple copies of the underlying spinor representation and defining complex bilinear superbrackets on the resulting space. Real supersymmetry algebras are then obtained by imposing signature-dependent reality conditions. This construction generalises and includes symplectic Majorana spinors. For dimensions up to twelve, we classify all supersymmetry algebras of any space-time signature whose R-symmetry groups are real forms of the R-symmetry group of complex superbrackets based on charge conjugation matrices. While not providing a full classification up to isomorphism, this method allows one to identify cases where more than one supersymmetry algebra exists for a given signature with any number of supercharges. In particular, for Lorentz signature, we find alternative 'type-*' or 'twisted' superalgebras with non-compact R-symmetry groups.

This formalism is then applied to five- and four-dimensional $\mathcal{N} = 2$ supersymmetry algebras and is used to derive vector multiplet theories in any signature. In five dimensions, the physical Lagrangians and supersymmetry representations are found by imposing signature-dependent reality conditions on a holomorphic master Lagrangian and associated supersymmetry variations to obtain signature-dependent theories. Four-dimensional Lagrangians are found through the dimensional reduction of these Lagrangian and supersymmetry representations. In four-dimensional Minkowski signature the existence of a 'twisted' supersymmetry algebra with $\mathrm{U}(1,1)$ R-symmetry is demonstrated and the vector multiplet theory derived from this algebra is shown to necessarily have ghost fields.

Additionally, an alternative classification of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry algebras is performed in five and four dimensions following the method of [1], classifying the possible superalgebras in each signature up to isomorphism. In five dimensions there is a one-parameter family of superalgebras, and in four dimensions the space of superbrackets is found to have the same structure as the associated space-time $\mathbb{R}^{t,s}$.

Publication List

This thesis contains material by the author that has appeared in the following publications:

- L. Gall and T. Mohaupt, *'Five-dimensional Vector Multiplets in Arbitary Signature,'* JHEP 1809 (2018) 053, arXiv:1805.06312, [2].
- V. Cortes, L. Gall and T. Mohaupt, *'Four-dimensional Vector Multiplets in Arbitary Signature,'* arXiv:1907.12067, [3].

and from the forthcoming paper

- L. Gall and T. Mohaupt, *'Supersymmetry Algebras in Arbitrary Signature and Dimension*,' TBA [4].

Contents

1 Introduction

Symmetries are the foundation of most approaches to modern physics. Supersymmetry is a proposal that extends the conventional symmetries of particle physics by allowing fermionic generators, in doing so circumventing the Coleman-Mandula theorem [5, 6]. While there has been no experimental evidence of supersymmetry so far, it is a mathematically fertile area that is interesting in its own right. In particular, supersymmetry is a necessary feature of superstring theory that is a candidate for a 'theory of everything' that unifies particle physics and gravity. In turn, attempts to unify the variety of string theories lead to M-theory [7, 8] and non-perturbative completion of the Type IIB string theories led to F-theory [9].

Supersymmetric theories in non-standard signatures arise in string theory in a variety of situations. For example, string theory with local $\mathcal{N} = 2$ worldsheet supersymmetry has a four-dimensional target space with signature $(2, 2)$ and excitations corresponding to self-dual gravity and self-dual Yang-Mill theory [10, 11]. F-theory can be interpreted as a theory in signature $(2, 10)$ and hidden symmetries of M-theory imply an embedding into a theory of signature $(2, 11)$ [12]. Exceptional field theory also leads to supergravities with non-standard space-time signature [13].

The different formulations of string theory are related by T-duality and S-duality. T-duality relates string theories compactified on a circle of radius R with another compactified on a circle of radius $1/R$, and S-duality relates strongly-coupled theories to weakly-coupled theories. In [14–16] the existence of a web of dualities connecting different types of type-II string theories and M-theories using a chain of T and S-duality was revealed. Allowing T-duality transformations over time-like directions naturally leads to relations between theories of different space-time signature, leading to M-theory in different eleven-dimensional signatures, Type-II* theories in Minkowski signature and a variety of Type II theories in all other ten-dimensional signatures.

Euclidean signature theories are used to study non-perturbative effects in the Euclidean path integral formalism, such as instantons. The time-like reduction of Minkowski theories leads to Euclidean theories. Solutions of these Euclidean theories can then be used to generate stationary solutions in Minkowski signature by dimensional uplifting. In [17] there are examples of dynamically changing space-time signature, so understanding the effects space-time signature has on the physical theories is useful here. Supersymmetric two-time theories have also been studied in [18, 19].

Existing work on this topic has certain shortcomings. Common methods of obtaining Euclidean and exotic signature theories involve doctoring a Minkowski signature theory by flipping signs and inserting factors of i, or dimensional reduction over time of theories in Minkowski signature to obtain a Euclidean theory in one dimension fewer, such as in [20–27]. However, dimensional reduction may not be able to reach all Euclidean theories (for example, see Chapter 5 where it is shown there are four-dimensional supersymmetry algebras that cannot be found from dimensional reduction unless one performs a reparametersation of the supercharges). In most cases, the fermionic terms are omitted, or the reductions are carried out in an on-shell formalism (however [20] and [28] fully include the fermions in an on-shell formulation). In [26] the supersymmetry variations of the fermions were found by analytic continuation of the Killing spinor equations, these were then used to find the bosonic terms of the on-shell Lagrangian of five-dimensional vector multiplets coupled to supergravity.

It is therefore desirable to develop a systematic manner of constructing and relating supersymmetric theories with arbitrary space-time signature. Said systematic construction should start with the supersymmetry algebra, which is then used this to construct a (preferably off-shell, where possible) representation of the algebra on fields and then building a Lagrangian invariant under this representation.

The goal of this thesis is to provide a systematic manner for the construction of supersymmetry algebras and then to apply it to build physical theories. To do this a formalism is introduced that allows one to construct supersymmetry algebras in any space-time dimension and signature, with any number of supersymmetries. This is achieved by first complexifying an arbitrary sum of irreducible spinor modules (spin representations) and then defining a complex superbracket. Then by applying signature-dependent reality conditions on this complexified space, real supersymmetry algebras are obtained in that signature. Symplectic Majorana spinors are a well-known example of such a construction, complexifying and then imposing a reality condition, which this formalism naturally incorporates and generalises.

This construction has the additional benefit of producing manifestly R-symmetric spinors, where the action of R-symmetry and Spin has been separated. When constructing physical theories disentangling R-symmetry from the Lorentz symmetry is highly useful: it makes writing terms in a Lagrangian and supersymmetry representations easier and offers an insight into necessary reality conditions of fields in the Lagrangian.

Starting from a complex supersymmetry algebra and restricting to a real form does not necessarily lead to inequivalent supersymmetry algebras. To aid in this classification, R-symmetry groups are calculated in all signatures for dimensions up to 12. As a guiding principle, having a different R-symmetry allows one to identify non-isomorphic supersymmetry algebras. In the scope of this formalism, some isomorphisms are outlined between supersymmetry algebras, but the complete classification of supersymmetry algebras up to isomorphism is left to further work. Finally, this chapter ends by discussing dimensional reduction and T-duality in terms of this framework to provide demonstrative examples of its usage and its ability to provide interesting insights.

A similar approach, in ten and eleven dimensions, was used in [29,30], however here the supersymmetry algebras were obtained from contractions of orthosymplectic Lie superalgebras. Since it is not known whether all Poincaré Lie superalgebras (in any signature) can be obtained as contractions of a larger algebra, the construction outlined here provides a useful alternative. While the orthosymplectic framework naturally provides BPS-charges (also called polyvector extensions or central-charges). Such BPS-charges can be added to our construction, as in [31], though this is left to future work.

The complete classification of superbrackets on real and complex spinor modules has been carried out in [1], but a classification of super-Poincaré Lie algebras up to isomorphism has not been attempted. Necessary and sufficient condition for two Poincaré Lie superalgebras to be isomorphic were added in [3], which also performs the classification explicitly in $\mathcal{N} = 2$ superalgebras in four dimensions. It is demonstrated that the space of $\mathcal{N} = 2$ superbrackets in all four-dimensional signatures is parameterised by the same vector space of the underlying space-time, $\mathbb{R}^{t,s}$, in all signatures, though this is a chance alignment in four dimensions and is not a general statement. Details on this are also included in this thesis in Section 2.9 and the explicit calculations in four dimensions are included in Chapter 5.

Chapter 3 details this formalism for constructing superalgebras in arbitrary dimension and signature with manifestly R-symmetric spinors, providing all relevant information for a reader to construct supersymmetry algebras in dimensions up to 12. This chapter involves work from [4], that is to appear soon.

Having the ability to construct a supersymmetry algebra in arbitrary signature and dimension allows one to study the effects of supersymmetry in any signature and dimension. In pursuit of this goal, we consider vector multiplets in five and four dimensions. In both cases, we begin with the supersymmetry algebra in an arbitrary signature and construct off-shell Abelian vector multiplet representations of these superalgebras. Descriptions of the supersymmetry algebras are also provided in the formalism of [1], and that outlined in Chapter 3, and explicit isomorphisms are provided, where applicable, in terms of both formalisms.

In five dimensions there is a unique minimal algebra, up to scaling, that in all cases is '$\mathcal{N} = 2$'.[1] In each signature, these minimal superalgebras are found by imposing a signature-dependent reality condition on a complexified supersymmetry algebra. This complexified supersymmetry algebra has a complex rigid off-shell vector multiplet representation and an associated Lagrangian that is invariant under these transformations (referred to as a holomorphic master Lagrangian). The signature-dependent reality properties of the spinor modules induce reality conditions on the complexified representations and Lagrangians, in doing so deriving $\mathcal{N} = 2$ five-dimensional vector multiplet theories in a particular space-time signature.

For the five-dimensional Euclidean theory we find that the scalar and vector kinetic terms always come with a relative sign difference, as predicted indirectly in [25], demonstrating that the supersymmetry algebra mandates this relative sign. In five dimensions, where there is a unique $\mathcal{N} = 2$ superalgebra in each signature, we find the relative sign choices in the Lagrangian to be completely determined by supersymmetry. In all cases, agreement is found with [26], where the bosonic on-shell Lagrangians for five-dimensional vector multiplets coupled to supergravity were obtained for all signatures

[1]Counting in multiples of the number of supercharges in the minimal four-dimensional superalgebra, so that $\mathcal{N} = 2$ implies eight real supercharges.

by analytic continuation of the Killing spinor equations from Minkowski signature.

The work in five dimensions can be found in Chapter 4 and is based on [2].

In even dimensions, the space of potential supersymmetry algebras is larger, so four dimensions provided a good step up that demonstrates more features of supersymmetric theories in an arbitrary signature. In each four-dimensional signature, the space of $\mathcal{N} = 2$ superbrackets (with which we define a superalgebra, as is outlined in Section 2.8) is four-dimensional. In Euclidean and neutral, $(2,2)$, signature it is shown that all these superbrackets lead to isomorphic supersymmetry algebras, leading to a single unique supersymmetry algebra up to isomorphism in these signatures. In Minkowski signature, there are two distinct $\mathcal{N} = 2$ supersymmetry algebras, distinguished by their $\mathrm{U}(2)$ and $\mathrm{U}(1,1)$ R-symmetry. The supersymmetry algebra with $\mathrm{U}(2)$ R-symmetry is the standard and well-known $\mathcal{N} = 2$ supersymmetry algebra, while the $\mathrm{U}(1,1)$ R-symmetric supersymmetry algebra is similar to the 'twisted' or 'type-*' supersymmetry algebras in Type II* supersymmetry as in [14, 15].

An important distinguishing feature of these 'twisted' Minkowski signature theories is that some fields have the 'wrong sign' in front of their kinetic term, meaning all fields cannot have positive-indefinite energy. While this raises concerns about whether these theories are stable, they naturally arise in string theory as described in [14].

Explicit off-shell vector multiplet representations of the four-dimensional $\mathcal{N} = 2$ superalgebras are then found in all signatures. These vector multiplet representations are obtained by the dimensional reduction of the five-dimensional vector multiplet representations found in Chapter 4. There are six five-dimensional signatures $(0,5), ..., (5,0)$ that can be reduced along a time-like or space-like direction resulting in ten different four-dimensional theories, two in each four-dimensional signature $(0,4), ..., (4,0)$. The knowledge obtained in the discussion of supersymmetry algebras – that Euclidean and neutral signature have a unique $\mathcal{N} = 2$ supersymmetry algebra and Minkowski signature has two – is used to determine which theories are isomorphic and provide explicit local field redefinitions that relate them in Euclidean and neutral signature.

The scalar manifold is special Kähler in Lorentz signature and special para-Kähler in Euclidean and neutral signature. For each signature, we obtain two theories, one with and one without a relative sign difference between the scalar and vector kinetic terms (and other conventional differences related to the signature-dependence of the spinor module). The two Minkowski signature theories are inequivalent; the theory with scalar and vector kinetic terms with the same sign corresponds to the regular $\mathcal{N} = 2$ vector multiplet theory and the theory with a relative minus sign arises from the twisted $\mathcal{N} = 2$ superalgebra. In Euclidean and neutral signature, the two Lagrangians are shown to be equivalent, and a local field redefinition that changes this relative sign (and all other conventional differences) is provided. In doing so, we confirm the result that the relative sign between scalar and vector kinetic terms is conventional, as was first outlined in [25]. However, the transformation proposed there is a strong-weak coupling duality, acting non-locally on the vector potential, while we give a local field redefinition defined at the level of the off-shell vector multiplet representation and is induced by an isomorphism of the underlying supersymmetry algebras.

Chapter 5 concerns four-dimensional $\mathcal{N} = 2$ supersymmetry algebras and vector multiplet theories. Based on the work in [3] (currently available in e-print).

To recap, the outline of this thesis is as follows. We begin with Chapter 2 that introduces preliminary mathematics and physics that are used throughout the thesis. This includes general information on the construction of the spinor module in arbitrary signature and the classification of supersymmetry algebras as originally derived in [1]. The formalism for systematically constructing supersymmetry algebras is outlined in Chapter 3, which also includes the calculation of R-symmetry groups in all signatures up to twelve. Chapters 4 and 5 then study $\mathcal{N} = 2$ supersymmetry in five and four dimensions, making use of the formalism developed in Chapter 3 to derive physical theories (supersymmetry representations and Lagrangians) with off-shell rigid vector multiplets.

2 | Background Material

2.1 Conventions and Notation

$\mathbb{K}(n)$ is the algebra of $n \times n$ matrices over $\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H}'\}$. $m\mathbb{K}(n) = \mathbb{K}(n) \oplus ... \oplus \mathbb{K}(n)$ is the m-fold direct sum of algebras $\mathbb{K}(n)$.

$\mathbb{K}(n)$ acts naturally on $\mathbb{K}^n$. $m\mathbb{K}(n)$ has m inequivalent irreducible representations, where one factor acts on $\mathbb{K}^n$ and all others act trivially.

$\langle A, B \rangle_{algebra}$ is used to mean the algebra generated by elements A, B and all possible combinations, for example

$$\langle i, j, k \rangle_{algebra} = \mathbb{H},$$

where the identity of $\mathbb{H}$ is implied because it can be obtained by squaring any element.

$\mathbb{1}_n$ is the $n \times n$ identity matrix.

Spinor Index Conventions

In this thesis the same conventions as [20] will be used for spinor indices. Dirac spinors $\psi \in \mathbb{S}$ have lowered indices, $\psi \equiv \psi_\alpha$. γ-matrices are endomorphisms on the spinor module with index structure $\gamma_\mu \equiv (\gamma_\mu)_\alpha{}^\beta$.

The matrices A and C representing a sesquilinear and bilinear form on $\mathbb{S}$ are $A \equiv A^{\alpha\beta}$ and $C \equiv C^{\alpha\beta}$ so that

$$A(\lambda, \chi) = \lambda^*_\alpha A^{\alpha\beta} \chi_\beta, \quad C(\lambda, \chi) = \lambda_\alpha C^{\alpha\beta} \chi_\beta. \tag{2.1}$$

The inverse matrices are then $A^{-1} = A^{-1}_{\alpha\beta}$ and $C^{-1} \equiv C^{-1}_{\alpha\beta}$.

In some sections we will work with Dirac spinors with a sesquilinear form, A, in these cases indices are raised and lowered using A and A^{-1}, i.e. $\lambda^\alpha = A^{\alpha\beta}\lambda_\beta$ and $\lambda_\alpha = A^{-1}_{\alpha\beta}\lambda^\beta$.

Other parts of the thesis, namely those that use extended spinors (and the sub-case of doubled spinors) have a bilinear form C on $\mathbb{S}$, so this controls raising and lowering the spinorial indices: such that $\lambda^\alpha = C^{\alpha\beta}\lambda_\beta$ and $\lambda_\alpha = C^{-1}_{\alpha\beta}\lambda^\beta$.

Spinor indices are very rarely used explicitly and will mostly be suppressed as a result.

Complex conjugation of spinor bilinear quantities is done without changing order, e.g.

$$(\lambda^T C\chi)^* = \lambda^\dagger C^* \chi^*.$$

Expressions involving γ-matrices, A and C, such as $A = \Pi_\tau \gamma_\tau$ and $B = (CA^{-1})^T$, are relationships between matrices not maps. Whilst the definitions of A, C, γ_μ and all derived quantities are basis dependent, all spinorial quantities appearing in Lagrangians and supersymmetry transformations are covariant with respect to Lorentz transformations because all spinor and other indices are contracted. Therefore the results are independent of the representation of the spinor module.

Bilinear Forms on $\mathbb{C}^N$

On $\mathbb{C}^N$ we will usually use the NW-SE conventions, to match the convention used with symplectic Majorana spinors that have SU(2)-indices. This is done so that notation can be universal. For a bilinear form M we write

$$M(z,w) = z^i w^j M_{ji}, \qquad i,j = 1,...,N.$$

Raising and lowering the indices are done using M_{ij} and its inverse M^{ij} such that

$$z^i = M^{ij} z_j, \quad z_i = z^j M_{ji},$$

Usually, two bilinear forms are considered on $\mathbb{C}^N$, a symmetric bilinear form called δ

and an antisymmetric bilinear form called J, whose Gram matrices are

$$\delta = \mathbb{1}_N, \quad J = \begin{pmatrix} 0 & \mathbb{1}_{\frac{N}{2}} \\ -\mathbb{1}_{\frac{N}{2}} & 0 \end{pmatrix}.$$

These are motivated and used in Chapter 3.

2.2 Super Vector Spaces and Modules

2.2.1 Super Vector Spaces

A super vector space is a $\mathbb{Z}_2$-graded vector space [32], such that V can be decomposed as

$$V = V_0 + V_1. \tag{2.2}$$

A homogenous vector $v \in V_i$ has parity $|v| = i$. Elements with parity 0 are called even elements (or sometimes bosonic when used in a physics circumstance) and those with parity 1 are called odd (or fermionic in a physics perspective).

If V has a finite dimension with V_0 having dimension p and V_1 having dimension q then V is said to have dimension $p|q$.

A superalgebra, A, over $\mathbb{K}$ is a super vector space equipped with a bilinear multiplication $A \times A \to A$ such that $|ab| = |a| + |b|$ for $a, b \in A$. The Clifford algebra is a superalgebra, and will be discussed in more detail later.

A Lie superalgebra (also called a super Lie algebra) is a super vector space with a product called a Lie superbracket (also called a supercommutator) that satisfies two conditions:

$$[x, y] = -(-1)^{|x||y|}[y, x] \tag{2.3}$$

and

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0. \tag{2.4}$$

This is the super Jacobi identity, the generalisation of the Jacobi identity of a regular

Lie algebra. For even elements, the Lie superbracket is the regular commutator, and for odd elements, it is the anticommutator.

2.2.2 Modules and Supermodules

A module over a ring is a generalisation of a vector space over a field, replacing the scalars of the field with elements of a ring R [32]. A left R-module M is made from an Abelian group $(M, +)$ and a ring R with identity 1_R, equipped with an operation $\cdot : R \times M \to M$ such that for all $r, s \in \mathbb{R}$ and $x, y \in M$ we have

$$\begin{aligned} &r \cdot (x + y) = r \cdot x + r \cdot y \\ &(r + s) \cdot x = r \cdot x + r \cdot x \\ &(rs) \cdot x = r \cdot (s \cdot x) \\ &1_R \cdot x = x. \end{aligned}$$

A right R-module is defined analogously but with multiplication happening from the right, replacing $r \cdot x$ with $x \cdot r$, etc.

In particular the real/complex spinor module is a real/complex Clifford module (equipped with a $\mathrm{Spin}_0(t, s)$-invariant bilinear form, as outlined later in Section 2.5) [1, 32].

Supermodules are the generalisation of a super vector space, extending the scalars to include odd variables that are elements of a superalgebra, $A = A_0 + A_1$. A supermodule is a $\mathbb{Z}_2$-graded module over a superalgebra. It is a module M with a decomposition

$$M = M_0 + M_1. \tag{2.5}$$

For a left A-supermodule the multiplication by elements $a \in A$ satisfies the above axioms for a module and additionally

$$|a \cdot x| = |a| + |x|. \tag{2.6}$$

For a right A-supermodule we replace left multiplication with right multiplication. Similarly to a super vector space, elements of A_0 and M_0 are called even and A_1 and M_1 are called odd.

2.2.3 Parity Change Functor

On a super vector space (or module) $V = V_0 + V_1$, we define the parity change morphism Π [33, 34]:

$$\Pi(V_0) = V_1, \quad \Pi(V_1) = V_0, \tag{2.7}$$

e.g. a field $\mathbb{K}$ is replaced by $\Pi(\mathbb{K}) = \mathbb{K}^{(0|1)}$, so for any $\mathbb{K}$-super vector space W we have

$$\Pi(W) = \mathbb{K}^{(0|1)} \otimes_{\mathbb{K}} W. \tag{2.8}$$

Any map between super vector spaces/modules preserves parity, so Π is a functor between the category of supermodules to itself. Avoiding any more category theory, this implies that any results we obtain using commuting elements also apply when we move to purely anticommuting elements. In Chapter 3 we will work with commuting spinors when defining superalgebras (following the conventions of [1]) before moving to anticommuting spinors to write physical Lagrangians and supersymmetric variations in Chapters 4 and 5.

2.3 Para-complex numbers, Quaternions and Para-quaternions

2.3.1 Para-complex and ϵ-complex Numbers

An ϵ-complex number combines complex and para-complex numbers [1].

$$z \in \mathbb{C}_\epsilon \qquad \mathbb{C}_\epsilon = \{z = x + i_\epsilon y \quad | \quad x, y \in \mathbb{R}\} \qquad s.t. \quad i_\epsilon^2 = \epsilon = \pm 1. \tag{2.9}$$

In both cases, we will call x the real part of the ϵ-complex number and y the imaginary part. We see that $\mathbb{C}_{-1}$ corresponds to the regular complex numbers – we may sometimes omit the subscript – and $\mathbb{C}_{+1}$ are the para-complex numbers – omitting the subscript we may refer to these as $\mathbb{C}'$. In this thesis we will usually use $e = i_{+1}$ to refer to the para-complex unit and $i = i_{-1}$ to refer to the complex unit.

We define the ϵ-complex conjugate of z, usually written z^* or $\bar{z}$ as

$$z^* = x - i_\epsilon y. \tag{2.10}$$

[1] Para-complex numbers are sometimes called split-complex numbers, double numbers or hyperbolic numbers

We use this to define the modulus of an ϵ-complex number:

$$|z| = \sqrt{zz^*} = \sqrt{x^2 - \epsilon y^2}. \tag{2.11}$$

We see that $\mathbb{C}_{+1}$ contains zero divisors and it is not a division algebra or a field. The group of invertible elements is isomorphic to $\mathrm{SO}(1,1)$; it is not connected and the four components correspond to $z = \pm exp(et)$ and $\pm eexp(et)$.

Alternatively, we can describe para-complex numbers in terms of a pair of real numbers (and analogously describe para-complex manifolds in terms of entirely real coordinates called adapted coordinates). Defining $z_\pm = x \pm y$, we see that

$$|z| = \sqrt{z_+ z_-} = \sqrt{x^2 - y^2}. \tag{2.12}$$

This makes the isomorphism $\mathbb{C}' \cong \mathbb{R} \oplus \mathbb{R}$ explicit, with the extra ingredient of conjugation (a real structure) that takes the pair (z_+, z_-) to (z_-, z_+).

2.3.2 Quaternions

The quaternions $q \in \mathbb{H}$ are defined by

$$\mathbb{H} = \{q_0 + q_1 i + q_2 j + q_3 k | q_i \in \mathbb{R}\}, \tag{2.13}$$

where

$$i^2 = j^2 = k^2 = -1, \qquad ijk = -1, \qquad ij = k, \qquad ki = j, \qquad jk = i. \tag{2.14}$$

$\mathbb{H} \cong Cl_{(2,0)}$ as real associative algebras:

$$Cl_{(2,0)} = \langle \gamma_1, \gamma_2 \rangle_{\text{algebra}}. \tag{2.15}$$

Where $\gamma_1^2 = \gamma_2^2 = -1$. These generate the element $\gamma_{12} = \gamma_1 \gamma_2$ which necessarily squares to -1. Mapping $\gamma_1 \to i, \gamma_2 \to j$ and $\gamma_{12} \to k$ gives an explicit isomorphism.

The quaternions are a real four-dimensional associative algebra. The conjugate of a quaternion is

$$q^* = q_0 - q_1 i - q_2 j - q_3 k \tag{2.16}$$

with which we can define the norm

$$N(q) = qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2. \tag{2.17}$$

As this is positive definite, it is common to use the modulus of a quaternion as the norm instead. This is

$$|q| = (N(q))^{\frac{1}{2}}. \tag{2.18}$$

Moreover, as they have non-negative norm they do not have zero-divisors so are a (non-commutative) division ring.

We can write a quaternion as a pair of complex numbers:

$$q = (q_0 + iq_1) + (q_2 + iq_3)j = u + vj. \tag{2.19}$$

$\mathbb{H}$ is also equivalent to the matrix algebra

$$\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \qquad u, v \in \mathbb{C}, \tag{2.20}$$

where u and v are the same u and v as in (2.19). One can show

$$det(M(q)) = N(q) = uu^* + vv^*. \tag{2.21}$$

Matrices of this form with unit determinant belong to the group SU(2) which is therefore isomorphic to the group of unit quaternions $\mathrm{U}(1, \mathbb{H})$.

Any matrix $M(q)$ with non-zero determinant can be written as the product of a real number and a matrix with unit determinant. The group of invertible quaternions is therefore

$$\mathbb{H}^* = \{q \in \mathbb{H} | N(q) \neq 0\} \cong \mathbb{R}^{>0} \times \mathrm{SU}(2). \tag{2.22}$$

(2.20) can be generalised to quaternionic matrices. Given a quaternionic matrix

$$Q = Q_0 + iQ_1 + jQ_2 + kQ_3, \qquad Q \in \mathbb{H}(n), \quad Q_i \in \mathbb{R}(n), \tag{2.23}$$

defining

$$Q = U + jV, \quad U = Q_0 + iQ_1, \quad V = Q_2 + iQ_3, \quad U, V \in \mathbb{C}(n), \tag{2.24}$$

this can then be written as a $2n \times 2n$ complex matrix:

$$\tilde{M}(Q) = \begin{pmatrix} U & V \\ -V^* & U^* \end{pmatrix} \in \mathbb{C}(2n), \tag{2.25}$$

with $det(\tilde{M}(Q)) = det(Q)$.

On $\mathbb{H}^n$ we can define the Hermitian form

$$\langle q^i, p^i \rangle = -(q^1)^* p^1 - ... - (q^t)^* p^t (q^{t+1})^* p^{t+1} + ... + (q^n)^* p^n, \tag{2.26}$$

which is invariant under the group $\mathrm{U}(t, s, \mathbb{H})$.

$\mathrm{U}(n, \mathbb{H})$ is isomorphic to $\mathrm{Sp}(p, q)$. A general element of the Lie algebra $\mathfrak{u}(p, q, \mathbb{H})$ has the following form

$$u = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \in \mathfrak{u}(p, q, \mathbb{H}), \qquad X^\dagger = -X, \quad Z^\dagger = -Z. \tag{2.27}$$

X is a $t \times t$, Y a $t \times s$ and Z a $s \times s$ quaternionic matrix. Writing each of X, Y and Z using (2.20) we obtain a generic element of the Lie algebra $\mathfrak{sp}(p, q)$. As the groups are connected this means the two associated Lie groups are isomorphic too. In particular, $\mathrm{U}(2, \mathbb{H}) \cong \mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ and $\mathrm{U}(1, 1, \mathbb{H}) \cong \mathrm{Sp}(1, 1) \cong \mathrm{Spin}(1, 4)$ will be used.

2.3.3 Para-quaternions

The para-quaternions $q \in \mathbb{H}'$ are defined by

$$\mathbb{H}' = \{q_0 + q_1 i + q_2 j + q_3 k | q_i \in \mathbb{R}\}, \tag{2.28}$$

where

$$i^2 = -1, \qquad j^2 = k^2 = +1, \qquad ijk = +1, \qquad ij = k, \qquad ki = j, \qquad jk = -i. \tag{2.29}$$

$\mathbb{H}' \cong Cl_{(1,1)} \cong Cl_{(0,2)}$ as real associative algebras:

$$Cl_{(0,2)} = \langle\gamma_1, \gamma_2\rangle_{\text{algebra}}, \tag{2.30}$$

where $\gamma_1^2 = \gamma_2^2 = +1$. These generate the element $\gamma_{12} = \gamma_1\gamma_2$ which necessarily squares to -1. Mapping $\gamma_1 \to j, \gamma_2 \to k$ and $\gamma_{12} \to i$ gives the explicit isomorphism. Similarly,

$$Cl_{(1,1)} = \langle\gamma_1, \gamma_2\rangle_{\text{algebra}}. \tag{2.31}$$

Where now $\gamma_1^2 = -1$. This change means $\gamma_{12}^2 = +1$, so we can obtain the para-quaternions by setting $\gamma_1 \to i$ and $\gamma_{12} \to k$ instead.

They are a real four-dimensional associative algebra. The conjugate of a para-quaternion is

$$q^* = q_0 - q_1 i - q_2 j - q_3 k, \tag{2.32}$$

which then is used to define the norm

$$N(q) = qq^* = q_0^2 + q_1^2 - q_2^2 - q_3^2. \tag{2.33}$$

$N(q)$ can be zero when $q_i \neq 0$, meaning the para-quaternions permit zero-divisors so $\mathbb{H}'$ is not a division ring unlike the quaternions.

$\mathbb{H}'$ is equivalent to $\mathbb{R}(2)$ as a normed algebra, with the norm provided by the determinant. We can map $\mathbb{H}' \to \mathbb{R}(2)$ by

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.34}$$

So that

$$q \to M(q) = \begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_2 - q_1 & q_0 - q_3 \end{pmatrix}. \tag{2.35}$$

It is easy to show $det(M(q)) = N(q)$. The group of invertible para-quaternions,

$$(\mathbb{H}')^* = \{q \in \mathbb{H}' | N(q) \neq 0\}, \tag{2.36}$$

is therefore isomorphic to $\mathrm{GL}(2,\mathbb{R})$, and the subgroup of unit quaternions

$$\mathrm{U}(1,\mathbb{H}') = \{q \in \mathbb{H}' | N(q) = 1\} \cong \mathrm{SL}(2,\mathbb{R}). \tag{2.37}$$

(2.35) can be extended to a mapping of para-quaternionic $n \times n$ matrices, $Q \in \mathbb{H}'(n)$, to $2n \times 2n$ real matrices. A para-quaternionic matrix can be decomposed into four real matrices according to

$$Q = Q_0 + Q_1 i + Q_2 j + Q_3 k, \qquad Q_i \in \mathbb{R}(n). \tag{2.38}$$

This can then be mapped to a $2n \times 2n$ real matrix in an analogous manner

$$M(Q) = \begin{pmatrix} Q_0 + Q_3 & Q_1 + Q_2 \\ Q_2 - Q_1 & Q_0 - Q_3 \end{pmatrix} \in \mathbb{R}(2n). \tag{2.39}$$

A para-quaternion can also be considered a pair of complex numbers, writing

$$q = (q_0 + q_1 i) + (q_2 + q_3 i)j = u + vj, \quad u, v \in \mathbb{C}. \tag{2.40}$$

$\mathbb{H}'$ can also be viewed as a normed algebra with the norm given by the determinant of 2×2 complex matrices of the form

$$q \to \tilde{M}(q) = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}, \tag{2.41}$$

with the same u and v from (2.40). Matrices of this form with unit norm are the standard form of $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{U}(1,\mathbb{H}')$. Similarly to before, $det(\tilde{M}(q)) = N(q)$.

(2.41) too can be generalised to para-quaternionic matrices

$$Q = U + jV, \quad U = Q_0 + iQ_1, \quad V = Q_2 + iQ_3, \quad U, V \in \mathbb{C}(n). \tag{2.42}$$

This can then be written as a $2n \times 2n$ complex matrix:

$$\tilde{M}(Q) = \begin{pmatrix} U & V \\ V^* & U^* \end{pmatrix} \in \mathbb{C}(2n), \tag{2.43}$$

On $\mathbb{H}'^n$ we can define the definite Hermitian form

$$\langle q^i, p^i \rangle = (q^1)^* p^1 + \ldots + (q^n)^* p^n, \tag{2.44}$$

which is invariant under the group $\mathrm{U}(n, \mathbb{H}')$.

$\mathrm{U}(n, \mathbb{H}')$ is isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$. A general element of the Lie algebra $\mathfrak{u}(N, \mathbb{H}')$ is a anti-Hermitian matrix. Being anti-Hermitian means we can decompose $u \in \mathfrak{u}(N, \mathbb{H}')$ as

$$u = u_0 + iu_1 + ju_2 + ku_3, \quad u_0^T = -u_0, \quad u_i^T = u_i \quad i = 1, 2, 3, \quad u_0, u_i \in \mathbb{R}(n). \tag{2.45}$$

Using (2.39) we can then write this as a real matrix

$$M(u) = \begin{pmatrix} u_0 + u_3 & u_1 + u_2 \\ u_2 - u_1 & u_0 - u_3 \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{R}). \tag{2.46}$$

As the groups are connected, this means the two associated Lie groups are isomorphic.

2.3.4 ϵ-quaternions

Similarly to the definition of ϵ-complex numbers we can combine quaternions and para-quaternions into an ϵ-quaternion. An ϵ-quaternion, with $\epsilon = \pm 1$, is

$$\mathbb{H}_\epsilon = \{q_0 + iq_1 + jq_2 + kq_3 | q_i \in \mathbb{R}\} \tag{2.47}$$

such that i, j, k obey

$$i^2 = -1, j^2 = k^2 = \epsilon, \quad ijk = \epsilon, \quad ij = k, \quad jk = i, \quad ki = \epsilon j. \tag{2.48}$$

$\mathbb{H}_{+1}$ is the para-quaternions, and $\mathbb{H}_{-1}$ is the quaternions. $\mathbb{H}$ without a subscript always refers to the regular quaternions. This notation will be primarily used for convenience.

2.4 Clifford Algebras

The tensor algebra $T(V)$ of a vector space, V, over a field $\mathbb{K}$, is the algebra of tensors on V with multiplication given by the tensor product [35]. A rank N tensor is an element of $T^N(V)$, where

$$T^N(V) = V^{\otimes N} = V \otimes \ldots \otimes V \tag{2.49}$$

$T^0(V)$ is defined to be the base field of the tensor product, $\mathbb{K}$. $T(V)$ is the sum of all $T^N(V)$

$$T(V) = \bigoplus_{N=0}^{\infty} T^N(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots \tag{2.50}$$

Given a vector space equipped with a quadratic form Q, the Clifford algebra generated by V, $Cl(V)$, is the quotient algebra of the tensor algebra with the ideal generated by $v \otimes v - Q(v)Id$ for all $v \in V$ [36–38]:

$$Cl(V) = T(V)/(v \otimes v - Q(V)Id). \tag{2.51}$$

When $V \cong \mathbb{R}^{t,s}$ we call the associated real Clifford algebra $Cl_{t,s}$, for brevity we write $Cl_{m,0} \equiv Cl_m$ or $Cl_{0,m} \equiv Cl_m$. All real Clifford algebras, $Cl_{t,s}$, are isomorphic to $m\mathbb{K}(n)$, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ for $m = \{1, 2\}$.

The real Clifford algebras obey the following isomorphisms,

$$Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2}, \quad Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}, \quad Cl_{s,t} \otimes Cl_{1,1} \cong Cl_{s+1,t+1}. \tag{2.52}$$

These lead to the Bott periodicities

$$Cl_{n+8,0} \cong Cl_{n,0} \otimes \mathbb{R}(16), \quad Cl_{0,n+8} \cong Cl_{0,n} \otimes \mathbb{R}(16), \quad Cl_{s+4,t+4} \cong Cl_{s,t} \otimes \mathbb{R}(16). \tag{2.53}$$

Using these isomorphisms, and $Cl_{1.1}^{\otimes 4} \cong \mathbb{R}(16)$ and $Cl_{2,0}^{\otimes 2} \otimes Cl_{0,2}^{\otimes 2} \cong \mathbb{R}(16)$,
we can classify all real Clifford algebras, as seen in Table 2.1

$s - t$ mod 8	$Cl_{t,s}$
$0, 6$	$\mathbb{R}(2^{\frac{d}{2}})$
7	$\mathbb{R}(2^{\frac{d-1}{2}}) \otimes \mathbb{R}(2^{\frac{d-1}{2}})$
$1, 5$	$\mathbb{C}(2^{\frac{d-1}{2}})$
$2, 4$	$\mathbb{H}(2^{\frac{d-2}{2}})$
3	$\mathbb{H}(2^{\frac{d-3}{2}}) \otimes \mathbb{H}(2^{\frac{d-3}{2}})$

Table 2.1: Classification of real Clifford algebras, $Cl_{t,s}$, with $d = t + s$.

If instead $V = \mathbb{C}^d$ we obtain the complex Clifford algebra, $\mathbb{C}l_d$. All Clifford algebras considered in this thesis are real or complex Clifford algebras. All complex Clifford algebras, $\mathbb{C}l_d$, are isomorphic to $m\mathbb{C}(n)$, for $m = \{1, 2\}$.

The complex Clifford algebras obey

$$\mathbb{C}l_{d+2} \cong \mathbb{C}l_d \otimes_{\mathbb{C}} \mathbb{C}(2). \tag{2.54}$$

Applying this to $\mathbb{C}l_0 \cong \mathbb{C}$ and $\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}$ we find

$$\mathbb{C}l_d \cong \begin{cases} \mathbb{C}(2^{\frac{d}{2}}), & d \text{ even}, \\ \mathbb{C}(2^{\frac{d-1}{2}}) \oplus \mathbb{C}(2^{\frac{d-1}{2}}), & d \text{ odd}. \end{cases} \tag{2.55}$$

Clifford algebras have a natural $\mathbb{Z}_2$-grading. Defining the grade involution

$$\alpha : v \to -v \tag{2.56}$$

we see that $\alpha^2 = 1$, so that we can decompose $Cl(V)$ into two eigenspaces

$$Cl(V) = Cl^0(V) + Cl^1(V), \tag{2.57}$$

such that $\alpha(x) = x$ for $x \in Cl^0(V)$ and $\alpha(y) = -u$ for $y \in Cl^1(V)$.

$Cl^0(V)$ is called the even subalgebra, and $Cl^1(V)$ is the odd subalgebra. They are

$$Cl^0(V) = \bigoplus_{N=0}^{\infty} T^{2n}(V)/v \otimes v - Q(v)Id, \qquad Cl^1(V) = \bigoplus_{N=0}^{\infty} T^{2n+1}(V)/v \otimes v - Q(v)Id. \tag{2.58}$$

All real even Clifford algebras, $Cl^0_{t,s}$, are of the form $m\mathbb{K}(n)$, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ for $m = \{1, 2\}$ with $Cl^0_{t,s} \subset Cl_{t,s}$. Indeed one can show that

$$Cl^0_{t,s} \cong \begin{cases} Cl_{t,s-1}, & s > 0, \\ Cl_{s,t-1}, & t > 0. \end{cases} \tag{2.59}$$

Similarly all complex even Clifford algebras are of the form $m\mathbb{C}(n)$ with $m \in \{1, 2\}$.

The complex Clifford algebras and their even subalgebra are related by

$$\mathbb{C}l_d^0 \cong \mathbb{C}l_{d-1} \tag{2.60}$$

On a Clifford algebra $Cl(V)$ we define the spinor norm q

$$q(x) = x^t \otimes x, \quad x \in Cl(V). \tag{2.61}$$

The operation ^{t} reverses the order of the tensor product:

$$x^t = x_n \otimes \ldots \otimes x_1. \tag{2.62}$$

The Pin group, $\mathrm{Pin}(V)$, is the subgroup of the Clifford algebra (regarded as a group with product given by multiplication) with unit spinor norm, i.e., they can be written as products of unit vectors (with respect to the quadratic form Q). The Pin group is sensitive to the signature of the quadratic form Q. In particular for $V = \mathbb{R}^{p,q}$ and $V = \mathbb{R}^{q,p}$, $\mathrm{Pin}(p,q) \ncong \mathrm{Pin}(q,p)$.

There exists a map from $\mathrm{Pin}(V)$ to $\mathrm{O}(V)$:

$$a \otimes v \otimes a^t = \rho(a)v, \qquad a \in \mathrm{Pin}(V), \quad \rho(a) \in \mathrm{O}(V). \tag{2.63}$$

We see that a and $-a$ are both mapped to $\rho(a)$, so that $\mathrm{Pin}(V)$ double covers $\mathrm{O}(V)$. The action $\mathrm{Pin}(V)$ corresponds to a collection of reflections in some hyperplane dependent on the form of a.

$\mathrm{Spin}(V) \subset \mathrm{Pin}(V)$ is the subgroup of even elements (with respect to α in $\mathrm{Pin}(V)$:

$$\mathrm{Spin}(V) \cong \mathrm{Pin}(V) \cap Cl^0(V) \tag{2.64}$$

$\mathrm{Pin}(V)$ is the double-cover of $\mathrm{O}(V)$, and $\mathrm{Spin}(V)$ is the double-cover of $\mathrm{SO}(V)$ under the same map

$$b \otimes v \otimes b^t = \rho(b)v, \qquad b \in \mathrm{Spin}(V), \quad \rho(b) \in \mathrm{SO}(V). \tag{2.65}$$

An element of $\mathrm{Spin}(V)$ corresponds to an even number of reflections, which always maintain orientation so that $\mathrm{Spin}(V)$ double covers $\mathrm{SO}(V)$. Note that unlike *Pin*, $\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$.

All elements of Spin(V) can be generated by quadratic elements of the Clifford algebra (elements of the form $v \otimes w$). This defines $\mathfrak{spin}(V)$, the Lie algebra associated to Spin(V)

$$\mathfrak{spin}(V) = \{b \in Cl^0(V) \quad | \quad b = v \otimes w, \quad |v| = |w| = 1\}. \tag{2.66}$$

2.5 Spinor Modules

2.5.1 Clifford Modules

Clifford algebras are a unital associative algebra, which is a ring that also has a scalar multiplication. The Clifford module is a R-module where the ring is the Clifford algebra [38]. In this thesis we consider real and complex Clifford modules where $R = Cl_{t,s}$ and $R = \mathbb{C}l_{t,s}$ respectively.

The algebra $m\mathbb{K}(n)$ has m inequivalent irreducible representations, with one factor $\mathbb{K}(n)$ acting on $\mathbb{K}^n$ and the rest acting trivially. All Clifford algebras and even Clifford algebras are of the form $m\mathbb{K}(n)$. Therefore Clifford algebras (and even Clifford algebras) have a unique irreducible module Σ when $m = 1$, or precisely two irreducible modules $\Sigma_1 \neq \Sigma_2$ when $m = 2$. The most general $Cl_{t,s}$-module (or $Cl^0_{t,s}$-module) is of the form $S = p\Sigma$ or $S = p_1\Sigma_1 \oplus p_2\Sigma_2$. This is also true for complex Clifford modules.

For a real or complex Clifford module, this has $2^{[\frac{D}{2}]}$ real or complex dimension. Upon the Clifford module we can act by elements of the Clifford algebra, producing further elements of the Clifford module. We will use this ability to multiply by Clifford algebra elements to convert bilinears of spinors into space-time SO(p,q) scalars, vectors, and tensors.

2.5.2 Spinor Modules

A real/complex spinor module is a Clifford module equipped with a Spin$_0(t,s)$-equivariant bilinear form. The construction of Spin$_0(t,s)$-equivariant bilinear forms is dealt with in Section 2.8. As all Clifford algebras on $V = \mathbb{R}^{t,s}$ and $\mathbb{C}^d$ are of the form $m\mathbb{K}(n)$, the spinor module is isomorphic to $\mathbb{K}^{mn}$.

The complex spinor module, usually referred to as $\mathbb{S}$ in this thesis, is complex-irreducible in odd dimensions. Elements of $\mathbb{S}$ are called Dirac spinors in physics and, as a result, it may be referred to as the Dirac spinor module. In even dimensions, $\mathbb{S}$ can be decomposed into two inequivalent irreducible semi-spinor modules, $\mathbb{S}_\pm$, such that

$$\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-, \tag{2.67}$$

where elements of the complex semi-spinor modules, $\mathbb{S}_\pm$, are called Weyl spinors by physicists (and thus they will sometimes be referred to as the Weyl spinor modules).

$\mathbb{S}$ always has the natural $\mathrm{Spin}_0(t,s)$-invariant complex structure, I, that is multiplication by the complex unit i:

$$I : \lambda \to i\lambda, \qquad \lambda \in \mathbb{S}. \tag{2.68}$$

Consider the complexification of $\mathbb{S}$,

$$\mathbb{S} \otimes_R \mathbb{C} = \mathbb{S} \otimes_R \bar{\mathbb{S}}. \tag{2.69}$$

In Section 2.8, it will be shown there always exists an additional $\mathrm{Spin}_0(t,s)$-invariant real or quaternionic structure (or both) on $\mathbb{S}$. The presence of a $\mathrm{Spin}_0(t,s)$-invariant real or quaternionic structure(s) implies that $\bar{\mathbb{S}} = \mathbb{S}$, so that

$$\mathbb{S} \otimes_R \mathbb{C} = \mathbb{S} \oplus \mathbb{S}. \tag{2.70}$$

Real spinor modules, usually called $S_{\mathbb{R}}$ or S when the context is clear, can be irreducible or reducible regardless of dimension. If it is reducible, S can be decomposed into two real semi-spinor modules, such that

$$S = S_+ + S_-. \tag{2.71}$$

These real semi-spinor modules may be equivalent or inequivalent. They are equivalent when the even Clifford algebra is simple. If they are inequivalent then the even Clifford algebra is of the form $2\mathbb{H}(n)$ or $2\mathbb{R}(n)$ (the only non-simple possibilities). When S is reducible but $S_+ \cong S_-$ it follows that S is the complexification of either $S_\pm$:

$$S = S_+ \oplus S_- \cong S_\pm \oplus S_\pm \cong S_\pm \otimes \mathbb{C}. \tag{2.72}$$

S and $\mathbb{S}$ are not necessarily distinct, i.e. for some signatures $\mathbb{S} \cong S$ when $\mathbb{S}$ carries a $\mathrm{Spin}_0(t,s)$-invariant real structure. If S and $\mathbb{S}$ are inequivalent then we find that

$S_+ \ncong S_-$ always.

To summarise we have the following possibilities

1. $\mathbb{S} \ncong S$
   (a) and S is irreducible. Elements of S are called Majorana spinors.
   (b) and $S = S_+ + S_-$, with $S_+ \ncong S_-$. Elements of $S_\pm$ are Majorana-Weyl spinors.
2. $\mathbb{S} \cong S$
   (a) and S is irreducible. The irreducible spinors are Dirac spinors.
   (b) and $S = S_+ + S_-$, with $S_+ \ncong S_-$. This implies $\mathbb{S}_\pm \cong S_\pm$ and the irreducible spinors are Weyl spinors.
   (c) and $S = S_+ + S_-$, with $S_+ \cong S_-$. This implies $S = S_\pm \otimes \mathbb{C}$, the irreducible spinors, elements of $S_\pm$, are Majorana spinors.

2.5.3 Schur Algebra and Group

In this thesis, we make frequent use of the so-called Schur algebra. The Schur is the algebra of endomorphisms of a spinor module, S, that commute with $\mathrm{Spin}_0(V)$ [1]:

$$\mathcal{C}(S) = Z_{End(S)}(\mathfrak{spin}(V)) = End_{Cl^0_{t,s}}(S). \tag{2.73}$$

Here S can be the real or complex spinor module. In particular, the Schur algebra of the complex spinor module in any signature is always isomorphic to $\mathbb{H}_\epsilon$ in odd dimensions, and $2\mathbb{H}_\epsilon$ or $\mathbb{C}(2)$ in even dimensions. For shorthand, abusing language we may refer to 'the' complex/real Schur algebra which corresponds to the Schur algebra of the complex/real spinor module, $\mathcal{C}(\mathbb{S})/\mathcal{C}(S)$.

The invertible elements of $\mathcal{C}(S)$ form the Schur group $\mathcal{C}(S)^*$, which are are elements of the general linear transformations of S, $\mathrm{GL}(S)$, that commute with $\mathrm{Spin}_0(V)$:

$$\mathcal{C}(S)^* = Z_{\mathrm{GL}(S)}(\mathfrak{spin}(V)). \tag{2.74}$$

The Schur group of the complex spinor module is, therefore, the group of invertible quaternions or para-quaternions, $\mathbb{H}^*_\epsilon$ in odd dimensions. In even dimensions it can be $2\mathbb{H}^*_\epsilon$ or $\mathrm{GL}(2,\mathbb{C})$.

2.6 Physics Reformulation

In physics it is common to work with a representation of $\mathbb{C}l_{t,s} = \mathbb{C}l_d$ in terms of γ-matrices. The γ-matrices are elements of $\mathbb{C}(\frac{d}{2})$ that obey the Clifford algebra [20, 37]

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \qquad \mu,\nu = 1,...,t+s, \tag{2.75}$$

where $\eta^{\mu\nu}$ is the inverse of the space-time metric $\eta_{\mu\nu}$; it is a diagonal matrix with t entries of -1 followed by s entries of $+1$. The γ-matrices act on complex spinors that are elements of $\mathbb{S} \cong \mathbb{C}^{\frac{d}{2}}$.

The first t γ-matrices are chosen to be anti-Hermitian, and the following s γ-matrices are chosen to be Hermitian.

All other elements of the Clifford algebra are then products of these γ-matrices:

$$Id, \gamma^\mu, \gamma^{\mu\nu}, \gamma^{\mu\nu\rho}, etc. \in Cl_{t,s},$$

where we have used the notation $\gamma^{\mu_1\cdots\gamma_{\mu_n}} = \gamma^{[\mu_1,\ldots,\gamma_{\mu_n}]}$. The shorthand $\gamma^{(n)} \equiv \gamma^{[\mu_1,\ldots,\gamma_{\mu_n}]}$ will also be used.

Given a spinor λ we can construct quantities with spacetime indices by multiplication with γ-matrices, e.g. $\gamma^\mu\lambda$, $\gamma^{\mu\nu}\lambda$.

On the complex spinor module we can define a $\mathrm{Spin}_0(t,s)$-equivariant sesquilinear or bilinear form. These are complex valued; real quantities can then obtained by taking the real or imaginary parts of the sesquilinear or bilinear forms. $\mathfrak{spin}(t,s) \subset Cl_{t,s}$ is generated by Clifford algebra elements of the form $\gamma^{\mu\nu}$, so a sesquilinear/bilinear form, β, is Spin_0-equivariant if it satisfies (with no sum)

$$\beta(\gamma^{\mu\nu}\cdot, \gamma^{\mu\nu}\cdot) = \beta(\cdot,\cdot). \tag{2.76}$$

For a general sesquilinear form, $A(\cdot,\cdot)$, defined by

$$\begin{aligned} &A : \mathbb{S}\times\mathbb{S} \to \mathbb{C}, \\ &A(\lambda,\chi) = \lambda^\dagger A\chi = \lambda^*_\alpha A^{\alpha\beta}\chi_\beta, \end{aligned} \tag{2.77}$$

$\mathrm{Spin}_0(t,s)$-invariance implies the Gram matrix A must satisfy

$$(\gamma^\mu)^\dagger = (-1)^t A\gamma^\mu A^{-1}. \tag{2.78}$$

This is solved by

$$A = \Pi_\tau \gamma_\tau, \tau = 1, ..., t. \tag{2.79}$$

This generalises the γ_0 found in spinor bilinears in Minkowski-signature theories. We will refer to this as the Dirac sesquilinear form (though it will not be used often), and $\lambda^\dagger A$ is the Dirac conjugate of λ.

We will call the $\mathrm{Spin}_0(t,s)$-equivariant bilinear form C, and define it by

$$\begin{aligned} &C : \mathbb{S} \times \mathbb{S} \to \mathbb{C}, \\ &C(\lambda,\chi) = \lambda^T C\chi = \lambda_\alpha C^{\alpha\beta}\chi_\beta. \end{aligned} \tag{2.80}$$

This will be referred to as the Majorana bilinear form; the Majorana conjugate of λ is $\bar{\lambda} = \lambda^T C$. The Gram matrix of this bilinear form, also called C, is commonly known as the charge conjugation matrix. To be $\mathrm{Spin}_0(t,s)$-invariant, C must satisfy

$$(\gamma^\mu)^T = \tau C\gamma^\mu C^{-1}, \quad \tau^2 = 1. \tag{2.81}$$

In odd dimensions, there is a unique choice of C, and in even dimensions there are two, one with each value of τ. They are conventionally known as $C_{-\tau}$[2]. It is always possible to choose a basis where $C = C^\dagger = C^{-1}$ (for both Cs simultaneously in even dimensions).

The symmetry of the bilinear form is equal to the symmetry of the Gram matrix C:

$$C(\lambda,\chi) = \sigma C(\chi,\lambda) \iff C^T = \sigma C, \quad \sigma = \pm 1. \tag{2.82}$$

The symmetry of $C_\pm$ will be called $\sigma_\pm$.

In even dimensions, we can define another matrix that anticommutes with the other

[2] In the conventions of [37] these were defined by an invariant $\eta = -\tau$, explaining this name.

γ-matrices

$$\gamma_* = (-i)^{t+\frac{D}{2}}\gamma_1 ... \gamma_D \tag{2.83}$$

γ_* commutes with the generators of Spin(t,s) and satisfies $(\gamma_*)^2 = 1$, so we can use it define chiral spinor modules, $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_-$, which are the ± 1 eigenspaces of γ_*

$$\gamma_* \lambda_\pm = \pm\lambda_\pm, \quad \lambda_\pm \in \mathbb{S}_\pm. \tag{2.84}$$

From the matrices A and C we can define a third matrix $B = (CA^{-1})^T$ that allows one to define a one-parameter family of real or quaternionic Spin$_0(t,s)$-invariant structure on $\mathbb{S}$:

$$J^{(\epsilon)(\alpha)} : \lambda \to \alpha^* B^* \lambda^*, \qquad |\alpha| = 1. \tag{2.85}$$

A real or quaternionic structure is an anti-linear involution that squares to $\epsilon = +1$ or -1 respectively. $J^{(+1)(\alpha)}$ is a real structure, and $J^{(-1)(\alpha)}$ is a quaternionic structure. Observe that

$$(J^{(\epsilon)(\alpha)})^2(\lambda) = B^* B \lambda \tag{2.86}$$

We see the value of ϵ is controlled by the product $B^*B = \epsilon$, such that a $J^{(\epsilon)(\alpha)}$ is a real structure when $B^*B = +1$ and a quaternionic structure when $B^*B = -1$. The form and properties of B, and therefore $J^{(\epsilon)(\alpha)}$, are signature-dependent because A is signature-dependent. It can be shown that

$$B^* B = \sigma(-\tau)^t (-1)^{t(t+1)/2}. \tag{2.87}$$

When $J^{(\epsilon)(\alpha)} = J^{(+1)(\alpha)}$, i.e. it is a real structure, one can define Majorana spinors that are invariant under J, such that $J(\lambda) = \lambda$. $J^{(\epsilon)(\alpha)}$ also links the Dirac and Majorana conjugate

$$J^{(\epsilon)(\alpha)}(\lambda^\dagger A) \propto \lambda^T C. \tag{2.88}$$

If γ_* commutes with B then $J^{(\epsilon)(\alpha)}$ is a real or quaternionic structure on $\mathbb{S}_\pm$:

$$\gamma_* B = B\gamma_* \implies J^{(\epsilon)(\alpha)}(\mathbb{S}_\pm) \in \mathbb{S}_\pm. \tag{2.89}$$

If γ_* anticommutes with $J^{(\epsilon)(\alpha)}$ changes the chirality of a spinor

$$\gamma_* B = -B\gamma_* \implies J^{(\epsilon)(\alpha)}(\mathbb{S}_\pm) \in \mathbb{S}_\mp. \tag{2.90}$$

2.6.1 Commuting vs. Anticommuting Spinors

In this thesis, we work with commuting spinors when defining superalgebras. When constructing physical theories we will then work with anticommuting spinors. A commuting spinor is an element of the complex or real spinor module, say S, and to define anticommuting spinors we replace this with ΠS where Π is the parity change functor discussed earlier. ΠS is a purely odd super vector space with dimension $(0|d_S)$, where d_S is the dimension of S. Because parity change is a functor, any results found with commuting spinors translate to anticommuting with minimal changes: we need only to invert all symmetry statements. For example, if we need a symmetric bilinear form with commuting spinors, we need an antisymmetric bilinear form with anticommuting spinors.

However, a physical theory involves spinor fields that depend on space-time. A commuting spinor field is a section of the spinor bundle, $S(\mathbb{R}^{t,s}) = \mathbb{R}^{t,s} \times S \to \mathbb{R}^{t,s}$, the trivial bundle over $\mathbb{R}^{t,s}$ with fibres S [20]. We cannot just replace S with $\Pi(S)$ here, as the super vector bundle $\mathbb{R}^{t,s} \times \Pi S \to \mathbb{R}^{t,s}$ has no non-zero sections. This is because the local components of a section must be purely odd superfunctions, which requires the base of the bundle to have a non-trivial odd part. Therefore we also have to replace $\mathbb{R}^{t,s}$ with $\mathbb{R}^{t,s|d_S} = \mathbb{R}^{t,s} \times M$ where M is an internal, purely odd parameter space of dimension d_S. $\mathbb{R}^{t,s|d_S} \times \Pi S \to \mathbb{R}^{t,s}$ has non-trivial sections and can therefore be the anticommuting spinor bundle. An anticommuting spinor field is a section of this bundle.

2.7 Super-Poincaré Algebras

A super-Poincaré algebras, $\mathfrak{g}$, is a $\mathbb{Z}_2$-graded algebras of the form [3]

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_0 + \mathfrak{g}_1, \\ \mathfrak{g}_0 &= \mathfrak{so}(V) + V. \end{aligned} \tag{2.91}$$

$\mathfrak{g}_0$ is the regular Poincaré algebra whose Lie bracket is defined by

$$[A, B] = AB - BA, \qquad [A, v] = Av, \qquad [v_1, v_2] = 0, \tag{2.92}$$

where $A, B \in \mathfrak{so}(V)$ and $v, v_1, v_2 \in V$.

The odd subalgebra $\mathfrak{g}_1$ is an arbitrary sum of irreducible spinor modules, associated to $V = \mathbb{R}^{t,s}$. $\mathfrak{so}(V)$ has a spinor representation, ρ_S.

$$[A, s] = \rho_s(A)s, \qquad [s_1, s_2] = \Pi(s_1, s_2) \in V. \tag{2.93}$$

For $s, s_1, s_2 \in \mathfrak{g}_1$. The definition of $\Pi(\cdot,\cdot)$ is discussed in Section 2.8.

In a physical context, we usually write P_μ for the generators of V, $M_{\mu\nu}$ for the generators of $\mathrm{SO}(V)$ and Q for the supersymmetry generators that are usually called supercharges. The bosonic generators obey [39]

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\ &[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \end{aligned} \tag{2.94}$$

where $\eta_{\mu\nu}$ is the space-time metric of $\mathbb{R}^{t,s}$.

Supersymmetric field theories are then made of multiplets, which are representations of a super-Poincaré algebra. Each field in a multiplet is a representation of the Poincaré algebra alone, and are transformed into one another by supersymmetry.

2.8 Classification of N-extended Super-Poincaré Algebras

This section follows [1] heavily and outlines important foundational concepts for the rest of the thesis. The original paper also contains information on $\mathbb{Z}_2$-graded Lie algebras, though, as they are not used here this is omitted. In addition, some additional proofs and remarks are omitted where deemed appropriate.

Definition – An N-extended Poincaré algebra (also called an N-extended super-Poincaré algebra) of $V = \mathbb{R}^{p,q}$ is a super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ where

- $\mathfrak{g}_0 \cong \mathfrak{p}(V) = V + \mathfrak{so}(V)$.
- $\mathfrak{g}_1$ is the sum of N irreducible spinor or semi spinor modules, S, of $\mathfrak{p}(V)$ with trivial action on V.
- The superbracket, $\{S, S\} \subset V$ (also referred to as Π_β as outlined shortly).

Note that the (super-)Jacobi identities are automatically satisfied as $[[x,y],z] = 0$ for $x,y,z \in \mathfrak{g}_1$.

To define on $\mathfrak{g}$ the structure of a super Lie algebra we need a superbracket, $j^* : \bigvee^2 S \to V$. Classifying such superbrackets is equivalent to classifying $\mathfrak{so}(V)$-equivariant mappings, $j : V^* \to \bigvee^2 S^*$. The space of $\mathfrak{so}(V)$-equivariant mappings is called $\mathcal{J}$.

$\mathcal{B}$ is the space of $\mathrm{SO}(V)$-equivariant bilinear forms on S. Clifford multiplication, $\gamma : V \otimes S \cong V^* \otimes S \to S$ (this is the restriction of the natural Clifford module multiplication on S to just V) provides the isomorphism between $\mathcal{J}$ and $\mathcal{B}$. Given $\beta \in \mathcal{B}$ we define

$$j_\gamma(\beta) : v^* \in V^* \to \beta \cdot \gamma(v^*) = \beta(\gamma(v^*)\cdot, \cdot) \in S^* \otimes S^* \tag{2.95}$$

We see that $j_\gamma : \mathcal{B} \to \mathcal{J}$. We can therefore determine all possible superbrackets by finding all bilinear forms. A superbracket is be built from a bilinear form according to

$$\langle \Pi_\beta(\cdot,\cdot), v\rangle = \beta(\gamma(v)\cdot, \cdot), \tag{2.96}$$

where Π_β is used in place of $\{\cdot,\cdot\}$ to refer to the superbracket built from a particular β. The classification of $\mathfrak{so}(V)$-equivariant bilinear forms on S is equivalent to describing the Schur algebra, $\mathcal{C}(S)$.

Definition – the Schur algebra $\mathcal{C}(S)$ is the algebra of $\mathfrak{so}(V)$ (and hence $\mathfrak{spin}(V)$)-equivariant endomorphisms of S. $\mathcal{C}(S)$ depends only on the spacetime signature and is isomorphic to $\mathbb{K}$, $\mathbb{K}(2)$ or $2\mathbb{K}$ for $\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Before describing $\mathfrak{so}(V)$-equivariant bilinear forms and the Schur algebra it is first useful to define an admissible bilinear form.

Definition – an admissible $\mathfrak{so}(V)$-equivariant bilinear form β on S has the following properties

- β has a definite symmetry, i.e. it is either symmetric or antisymmetric. We encode this in the symmetry $\sigma(\beta)$ such that $\beta(s,t) = \sigma(\beta)\beta(t,s)$.
- Clifford multiplication is β-symmetric or β-antisymmetric. This is called the type $\tau(\beta)$ such that $\beta(\gamma(v)s,t) = \tau(\beta)\beta(s,\gamma(v)t)$.

- If the spinor module is reducible, i.e. $S = S_+ + S_-$, then $S_\pm$ are mutually orthogonal or isotropic. This is the isotropy $\iota(\beta)$, which is +1 when β is orthogonal and −1 when β is isotropic.

Note sometimes the more compact σ_β and τ_β will be used, when the context is clear, instead of $\sigma(\beta)$ and $\tau(\beta)$.

A superbracket is a symmetric $\mathrm{Spin}_0(t,s)$-equivariant vector-valued bilinear form:

$$\beta(\gamma(v)\cdot,\cdot) \in \vee^2 S^* \implies \beta(\gamma(v)s,t) = \beta(\gamma(v)t,s), \quad s,t \in S. \tag{2.97}$$

$\beta(\gamma(v)t,s) = \sigma_\beta\tau_\beta\beta(\gamma(v)s,t)$ so only bilinear forms with $\sigma\tau = +1$ define a non-vanishing superbracket.

Definition – β is a super-admissible bilinear form when $\sigma_\beta\tau_\beta = +1$, so-called because they naturally define a superbracket.

Having a type implies a bilinear form is $\mathfrak{so}(V)$-equivariant, so all admissible bilinear forms are $\mathfrak{so}(V)$-equivariant. $\mathfrak{spin}(V) \cong \mathfrak{so}(V)$ is a subalgebra of the Clifford algebra composed of elements of the form $\gamma(v)\gamma(w)$, inserting this into both arguments of the bilinear form:

$$\beta(\gamma(v)\gamma(w)s, \gamma(v)\gamma(w)t) = \tau^2\beta(s,t). \tag{2.98}$$

As $\tau = \pm1$, $\tau^2 = 1$ so β is $\mathfrak{spin}(V) \cong \mathfrak{so}(V)$ equivariant.

Definition – given an admissible $\beta \in \mathcal{B}$, an endomorphism $A \in \mathcal{C}$ is called β-admissible if

- A is β-symmetric or β-antisymmetric. The β-symmetry of A is $\sigma_\beta(A)$.
- Clifford multiplication commutes or anticommutes with A. This is the type of A, $\tau(A)$. $\tau(A) = +1$ if it commutes and $\tau(A) = -1$ if it anticommutes.
- When S is reducible, $AS_\pm \subset S_\pm$ or $AS_\pm \subset S_\mp$. In the first case the isotropy of A is $\iota(A) = +1$ and in the second $\iota(A) = -1$.

Having a definite type means that A is an $\mathfrak{so}(V)$-equivariant endomorphism. Therefore given an admissible bilinear form β and a β-admissible A, $\beta \cdot A = \beta(A\cdot,\cdot) \in \mathcal{B}$, is admissible and the invariants are multiplicative:

$$\sigma(\beta \cdot A) = \sigma(\beta)\tau(A), \quad \tau(\beta \cdot A) = \tau(\beta)\tau(A), \quad \iota(\beta \cdot A) = \iota(\beta)\tau(A). \tag{2.99}$$

If A is not β-admissible then $\beta \cdot A$ is in general not admissible. It will be demonstrated that there exists a canonical bilinear form h from which we can construct a basis for all admissible bilinear forms using the Schur algebra.

It is always possible to decompose $\mathbb{R}^{p,q}$ as

$$\mathbb{R}^{p,q} \cong \mathbb{R}^{q,q} + \mathbb{R}^{p-q,0}, \quad p > q \quad \text{or} \quad \mathbb{R}^{p,q} \cong \mathbb{R}^{p,p} + \mathbb{R}^{0,q-p}, \quad p < q. \tag{2.100}$$

For $V = V_1 + V_2$, as above, there exists a canonical isomorphism of $\mathbb{Z}_2$-graded algebras

$$Cl(V) \cong Cl(V_1)\hat{\otimes}Cl(V_2), \tag{2.101}$$

where $\hat{\otimes}$ is the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded algebras. We can then build the spinor module as the tensor product of the spinor module built from $Cl(V_1)$ and $Cl(V_2)$:

$$S = S_1 \otimes S_2 \tag{2.102}$$

We then build bilinear forms (and thus superbrackets) from tensor products of bilinear forms on this product space. Therefore for a complete classification we need only consider 3 cases, spacetimes of signature (m,m), $(0,k)$ and $(k,0)$. In each signature, we will derive the canonical bilinear form and the Schur algebra, in doing so providing a basis for all bilinear forms. We will then provide the invariants of said bilinear forms. Following this, we discuss how to combine these cases to describe the general signatures (p,q).

(m,m)

Let U and U^* be two complementary isotropic subspaces of $V = \mathbb{R}^{m,m} = U + U^*$. Using the standard scalar product, $\langle \cdot, \cdot \rangle$, on V we identify U^* with the dual space of U:

$$\mathrm{U}^*(u) = 2\langle u, u^* \rangle, \quad u \in U, \quad u^* \in U^*. \tag{2.103}$$

The spinor module can be realised as $S = \bigwedge U$ that has decomposition $S = \bigwedge^{even} U +$

$\bigwedge^{odd} U = S_+ + S_-$, where S_+ and S_- are inequivalent irreducible $\mathfrak{so}(m,m)$-submodules.

We define an irreducible $Cl_{(m,m)}$-module on $S = \bigwedge U$ with Clifford multiplication γ:

$$\gamma(u)s = u \wedge s, \qquad \gamma(u^*)s = -u^* \lrcorner s, \quad u \in U, \quad u^* \in U^* \tag{2.104}$$
$$\implies \gamma(u)\gamma(u^*) + \gamma(u^*)\gamma(u) = -2\langle u, u^* \rangle Id, \quad \gamma(u)^2 = \gamma(u^*)^2 = 0. \tag{2.105}$$

where $\lrcorner$ is the interior product. The second line is the familiar equations defining a Clifford algebra.

We define the nilpotent endomorphisms of S ϵ_a and ι_α, for $a \in \bigwedge U$ and $\alpha \in \bigwedge U^*$

$$\epsilon_a = a \wedge s, \quad \iota_\alpha = \alpha \lrcorner s. \tag{2.106}$$

The Lie algebra $\mathfrak{so}(m,m)$ has the graded decomposition

$$\mathfrak{so}(m,m) = \mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2 = \iota_{\wedge^2 U} + \mathfrak{sl}(U) + \epsilon_{\wedge^2 U}. \tag{2.107}$$

such that $\mathfrak{sl}(U) \cong [\iota_{U^*}, \epsilon_U]$, $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ with $\mathfrak{g}^{i+j} = 0$ if $|i+j| > 2$. $\iota_{\wedge^2 U}$ and $\epsilon_{\wedge^2 U}$ are Abelian subalgebras.

An $\mathfrak{so}(m,m)$-equivariant endomorphism E of S is

$$Es_\pm = \pm s_\pm, \quad s_\pm \in S_\pm. \tag{2.108}$$

This corresponds to the chiral projection matrix in physics, that is often called 'γ_5' in four dimensions or 'γ_*' more generally. In this case, this is the only endomorphism that exists on S, meaning the Schur algebra is $\mathcal{C}(S) \cong \mathbb{R} \oplus \mathbb{R}$ with a basis given by Id, E.

We can give an admissible bilinear form f on S by fixing a volume form, $vol \in \wedge^m U$ on U^* and defining

$$f(\wedge^i U, \wedge^j U) = 0 \quad \text{if} \quad i + j \neq m, \tag{2.109}$$
$$f(s,t) vol = (-1)^{\frac{i(i+1)}{2}} s \wedge t, \quad s \in \wedge^i U, \quad t \in \wedge^{m-i} U. \tag{2.110}$$

The space $\mathcal{B}$ of $\mathfrak{so}(m,m)$-equivariant bilinear forms is two dimensional, with basis f

and $f_E = f(E\cdot,\cdot)$. Note that E is both f- and f_E-admissible. It has invariants

$$\sigma(E) = \sigma_{f_E}(E) = (-1)^m, \quad \tau(E) = -1, \quad \iota(E) = +1. \tag{2.111}$$

m	f	f_E
0	$+-+$	$+++$
1	$---$	$++-$
2	$--+$	$-++$
3	$+--$	$-++$

Table 2.2: Invariants (σ, τ, ι) of admissible bilinear forms in signatures (m, m).

$(k, 0)$

First we work in even dimensions, setting $k = 2m$. We can decompose $\mathbb{R}^{2m} = \mathbb{R}^m + \tilde{\mathbb{R}}^m$ for some isometry $\tilde{}: \mathbb{R}^m \to \tilde{\mathbb{R}}^m$.

On S we can define a Clifford structure for $m = 0$ or 3 mod 4.

$$\gamma(v)s = vs, \quad \gamma(\tilde{v})s = \begin{cases} \omega s v & m = 0 \quad mod \quad 4, \\ \omega \alpha(s) v & m = 3 \quad mod \quad 4. \end{cases} \tag{2.112}$$

For $v \in \mathbb{R}^m$, $\tilde{v} \in \tilde{\mathbb{R}}^m$. ω is the volume element of $Cl_m = e_1 ... e_m$. α is the grading automorphism from Section 2.4, that, in particular, takes $v \to -v$, $v \in \mathbb{R}^m$.

If $m = 1, 2$ mod 4 we instead use

$$\gamma(v)s = vs, \quad \gamma(\tilde{v})s = i\alpha(s)v. \tag{2.113}$$

Using these definitions, one finds, for all m, that

$$\gamma(v)^2 = -\langle v, v\rangle Id, \quad \gamma(\tilde{v})^2 = -\langle \tilde{v}, \tilde{v}\rangle Id, \quad \gamma(v)\gamma(\tilde{v}) + \gamma(\tilde{v})\gamma(v) = 0. \tag{2.114}$$

Which correctly gives a Clifford algebra structure on S. Note that m-even implies

$\{\omega, v\} = 0$ and m-odd means $[\omega, v] = 0$.

$$\omega^2 = \begin{cases} +1 & m = 1, 3, \\ -1 & m = 2, 4. \end{cases} \tag{2.115}$$

We seek a canonical bilinear form, which we shall call h. The form of h varies depending on the value of m mod 4 again. For $m = 0, 2$ the S is reducible to two inequivalent Weyl spinor modules, for $m = 1$ it is reducible but the two Weyl spinor modules are equivalent and for $m = 3$ it is irreducible.

We can identify $\bigwedge \mathbb{R}^m$ and Cl_m by identifying $e_{i_1} \wedge ... \wedge e_{i_k} \to e_{i_1}...e_{i_k}$. Clifford multiplication of $v \in \mathbb{R}^m$ and $\phi \in Cl_m$ is given by

$$v\phi = v \wedge \phi - v \lrcorner \phi, \quad \phi v = v \wedge \alpha(\phi) + x \lrcorner \alpha(\phi). \tag{2.116}$$

The standard scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^m$ induces a standard scalar product on $\bigwedge \mathbb{R}^m$ (that we will call the same name). It is invariant under exterior and interior multiplication by unit vectors $v \in \mathbb{R}^m$. It is also invariant under left and right multiplication by unit vectors $v \in \mathbb{R}^m$, so that it is $\mathrm{Pin}(2m)$-invariant.

For $m = 0$ or 3 mod 4 then $h = \langle \cdot, \cdot \rangle$ is the admissible $\mathrm{Pin}(2m)$-invariant scalar product on S. If $m = 1$ or 2 we must extend $S = \bigwedge \mathbb{R}^m$ to $\bigwedge \mathbb{C}^m$ so that $\langle \cdot, \cdot \rangle$ is a symmetric complex bilinear form on S. $h = Re\langle c\cdot, \cdot \rangle$ is then a symmetric real bilinear form where c is the complex conjugation operator.

We now seek an h-admissible basis for the Schur algebra to find all bilinear forms. The basis depends on m, and we will deal with each in turn.

If $m = 0$ mod 4 then $\mathcal{C}_{2m,0} \cong \mathbb{R} \oplus \mathbb{R}$, with an h-admissible basis given by Id and $E = \alpha$ (the grade automorphism from above). E has invariants $\tau(E) = -1$, $\sigma(E) = \sigma_h(E) = +1$ and $\iota(E) = +1$. The space of admissible bilinear forms is then $span\{h, h_E\}$ and they have invariants $(\sigma, \tau, \iota)(h) = (+1, -1, +1)$ and $(\sigma, \tau, \iota)(h_E) = (+1, +1, +1)$.

For $m = 3$ mod 4, $\mathcal{C}_{2m,0} \cong \mathbb{C}$ with h-admissible basis given by Id and $J = L_\omega \cdot \alpha$. L_ω is left multiplication by the volume element of Cl_m, $\omega = e_1...e_m$. This has invariants $\tau(J) = -1$ and $\sigma(J) = -1$. The associated bilinear forms have invariants $(\sigma, \tau)(h) = (+1, -1)$ and

$(\sigma, \tau)(h_J) = (-1, +1)$.

For odd $k = 2m + 1$ we consider the decomposition $\mathbb{R}^k = \mathbb{R}e_0 + \mathbb{R}^{2m}$, where e_0 is a unit vector. First we remark the following

$$S_{2m+1,0} = \begin{cases} S_{2m,0} \otimes \mathbb{C} = \mathbb{S}_{2m} & m = 0 \mod 4, \\ S_{2m,0} & m = 1, 2, 3 \mod 4. \end{cases} \tag{2.117}$$

We can deal with $m = 1$ or $2 \mod 4$ in tandem. The Schur algebra has an h-admissible basis given by

$$Id, \quad I : s \to is, \quad J = L_\omega \cdot c, \quad K = IJ, \quad E = \alpha, \quad EI, \quad EJ, \quad EK, \tag{2.118}$$

where once again L_w is left multiplication by the volume element of Cl_m. One can show that these operators obey the following relations:

$$\begin{aligned} & I^2 = J^2 = -1, \quad E^2 = +1, \\ & \{I, J\} = [I, E] = 0 \quad \implies K^2 = -1, \quad (EI)^2 = -1, \\ & \{J, E\} = 0 \quad \text{if} \quad m = 1 \text{ mod } 4 \implies (EJ)^2 = +1, \\ & [J, E] = 0 \quad \text{if} \quad m = 2 \text{ mod } 4 \implies (EJ)^2 = -1. \end{aligned} \tag{2.119}$$

For $m = 1 \mod 4$ we find $\mathcal{C} \cong \mathbb{C}(2)$ and when $m = 2$ we find $\mathcal{C} \cong \mathbb{H} \oplus \mathbb{H}$.

The invariants of the endomorphisms and the resulting bilinear forms can be found in the following table

m	Id	I	J	K	E	EI	EJ	EK
1	$+++$	$-++$	$-++$	$-++$	$+-+$	$--+$	$+-+$	$+--$
2	$+++$	$-++$	$--+$	$--+$	$+-+$	$--+$	$-++$	$-++$
m	h	h_I	h_J	h_K	h_E	h_{EI}	h_{EJ}	h_{EK}
1	$+-+$	$--+$	$--+$	$---$	$+++$	$-++$	$++-$	$++-$
2	$+-+$	$--+$	$-++$	$-++$	$+++$	$-++$	$--+$	$--+$

Table 2.3: Invariants (σ, τ, ι) Schur algebra basis elements and associated bilinear forms in signatures $(2m, 0)$ with $m = 1, 2 \mod 4$.

For odd $k = 2m + 1$ we consider the decomposition $\mathbb{R}^k = \mathbb{R}e_0 + \mathbb{R}^{2m}$, where e_0 is a unit vector. First we remark the following

$$S_{2m+1,0} = \begin{cases} S_{2m,0} \otimes \mathbb{C} = \mathbb{S}_{2m} & m = 0 \mod 4, \\ S_{2m,0} & m = 1,2,3 \mod 4. \end{cases} \tag{2.120}$$

For $m = 1,2$ we have the Cl_{2m}-invariant complex structure I defined previously. I is also a Cl_{2m}-invariant complex structure on $S_{2m,0} \otimes \mathbb{C}$ when $m = 0$.

Given a representation γ of Cl_{2m} on $S_{2m,0}$ we can extend this to a representation $\tilde{\gamma}$ of $Cl_{2m+1,0}$ on $S_{2m+1,0}$:

$$\tilde{\gamma}(\mathbb{R}^{2m}) = \gamma(\mathbb{R}^{2m}), \tag{2.121}$$

$$\tilde{\gamma}(e_0) = \begin{cases} \gamma(\omega_{2m}) & \text{if} \quad m = 1 \text{ or } 3 \mod 4 \\ I \cdot \gamma(\omega_{2m}) & \text{if} \quad m = 0 \text{ or } 2 \mod 4 \end{cases} \tag{2.122}$$

One can show that $\tilde{\gamma}(e_0)^2 = -1$ and $\{\tilde{\gamma}(e_0), \tilde{\gamma}(v)\} = 0$ for $v \in \mathbb{R}^{2m}$.

(2.120) implies that we can the same canonical bilinear form as before for $m = 1,2,3 \mod 4$, which was $\langle \cdot, \cdot \rangle$ for $m = 3$ and $Re\langle c\cdot, \cdot \rangle$ for $m = 1,2$. In $m = 0 \mod 4$ we must now use $h = Re\langle c\cdot, \cdot \rangle$. These bilinear forms are $\mathrm{Pin}(2m+1)$-invariant; by Schur's lemma for $m = 1,2,3$ and by remarking that h is invariant under $\tilde{\gamma}(e_0)$ explicitly for $m = 0$.

For $m = 0,1,2$ the Schur algebra is four dimensional, given by Id, $I : s \to is$,

$$J = \begin{cases} L_\omega \cdot c & m = 1,2 \mod 4, \\ \alpha \cdot c & m = 0 \mod 4. \end{cases} \tag{2.123}$$

and $K = IJ$. For $m = 1,2$ this is isomorphic to $\mathbb{H}$, and for $m = 0$ this is $\mathbb{H}' \cong \mathbb{R}(2)$. The resulting invariants are the same as the corresponding entries in Table 2.3 for $m = 1,2$ and as detailed in the relevant paragraph for $m = 0$.

For $m = 3$ the Schur algebra is one dimensional, $\mathcal{C} \cong \mathbb{R}$, and the space of admissible bilinear forms is similarly one dimensional, $\mathcal{B} = \mathbb{R}h$.

$(0, k)$

$(0, k)$ works very similarly to $(k, 0)$. In this section I will outline any key differences, which are mostly just changes in behaviour of m. We begin with even, $k = 2m$ and we use the decomposition $\mathbb{R}^{0,2m} = \mathbb{R}^{0,m} + \tilde{\mathbb{R}}^{0,m}$ for some isometry $\tilde{}: \mathbb{R}^{0,m} \to \mathbb{R}^{0,m}$.

The volume element of Cl_m, ω, obeys

$$\omega^2 = \begin{cases} +1 & m = 1, 3, \\ -1 & m = 2, 4. \end{cases} \tag{2.124}$$

ω (anti)commutes for m-even (m-odd).

We define the Clifford algebra representation on $S_{0,2m}$ with $m = 0, 1$ according to

$$\gamma(v)s = vs, \quad \gamma(\tilde{v})s = \begin{cases} \omega sv & m = 0 \quad mod \quad 4, \\ \omega\alpha(s)v & m = 1 \quad mod \quad 4. \end{cases} \tag{2.125}$$

For $v \in \mathbb{R}^{0,m}$, $s \in S$. The spinor module is reducible for $m = 0$ and irreducible for $m = 1$.

When $m = 2, 3$ the representation is instead

$$\gamma(v)s = vs, \quad \gamma(\tilde{v})s = i\alpha(s)v. \tag{2.126}$$

The Weyl spinor modules are (in)equivalent for $m = 2$ ($m = 3$).

For $(k, 0)$ we used that $\bigwedge \mathbb{R}^m = Cl_m$ and used the standard scalar product on $\bigwedge \mathbb{R}^m$. In this case we consider

$$\mathbb{R}^{0,m} = i\mathbb{R}^{m,0} \subset \mathbb{C}l_m = Cl_m \otimes \mathbb{C}. \tag{2.127}$$

Following the same logic

$$Cl_{0,m} = Cl^0_{0,m} + Cl^1_{0,m} = Cl^0_m + iCl^1_m. \tag{2.128}$$

We define the isomorphism

$$\begin{aligned} \varphi : Cl_m &\to Cl_{0,m}, \\ \varphi(a) &= i^{deg(a)} a. \end{aligned} \tag{2.129}$$

For an element $a \in Cl_m$ with pure degree. We can then define a scalar product $\langle \cdot, \cdot \rangle$ on $Cl_{0,m}$ by the condition that φ is an isometry for the standard scalar product on $\bigwedge \mathbb{R}^m = Cl_m$

For $m = 0, 1$ $\langle \cdot, \cdot \rangle$ is the canonical admissible $\mathrm{Pin}(0, 2m)$-invariant scalar product on S. For $m = 2, 3$ we extend the scalar product to the symmetric complex bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $S = \bigwedge C^m$. The canonical bilinear form on S is then $h = Re\langle c\cdot, \cdot \rangle$.

The Schur algebra for $m = 0$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$, with basis elements Id and $E = \alpha$. They have invariants $\sigma(E) = \sigma_h(E) = +1$, $\tau(E) = -1$ and $\iota(E) = +1$, and the associated bilinear forms have invariants $(\sigma, \tau, \iota)(h) = (+1, +1, +1)$ and $(\sigma, \tau, \iota)(h_E) = (+1, -1, +1)$.

When $m = 1$ the Schur algebra is $\mathcal{C} \cong \mathbb{C}$, an admissible basis is given by Id and $J = L_\omega \cdot c$. Note that ω is the volume element of $Cl_{0,m}$. The admissible bilinear forms then have a basis h and h_J with invariants $(\sigma\tau)(h) = (+1, +1)$ and $(\sigma, \tau)(h_J) = (-1, -1)$.

We can deal with $m = 2, 3$ in tandem. The Schur algebra has an admissible basis given by

$$Id, \quad I : s \to is, \quad J = L_\omega \cdot c, \quad K = IJ, \quad E = \alpha, \quad EI, \quad EJ, \quad EK. \tag{2.130}$$

These basis elements obey the following relations

$$\begin{aligned} &I^2 = J^2 = -1, \quad E^2 = +1, \\ &\{I, J\} = [I, E] = 0 \quad \Longrightarrow K^2 = -1, \quad (EI)^2 = -1, \\ &\{J, E\} = 0 \quad \text{if} \quad m = 2 \bmod 4 \Longrightarrow (EJ)^2 = +1, \\ &[J, E] = 0 \quad \text{if} \quad m = 3 \bmod 4 \Longrightarrow (EJ)^2 = -1. \end{aligned} \tag{2.131}$$

For $m = 2 \bmod 4$ we find $\mathcal{C} \cong \mathbb{C}(2)$ and when $m = 3$ we find $\mathcal{C} \cong \mathbb{H} \oplus \mathbb{H}$.

The invariants of the endomorphisms and the resulting bilinear forms can be found in the following table

m	Id	I	J	K	E	EI	EJ	EK
1	$+++$	$-++$	$--+$	$--+$	$+-+$	$--+$	$-++$	$-+-$
2	$+++$	$-++$	$-+-$	$-+-$	$+-+$	$--+$	$+--$	$+--$
m	h	h_I	h_J	h_K	h_E	h_{EI}	h_{EJ}	h_{EK}
1	$+++$	$-++$	$--+$	$--+$	$+-+$	$--+$	$-++$	$-++$
2	$+++$	$-++$	$-+-$	$-+-$	$+-+$	$--+$	$+--$	$+--$

Table 2.4: Invariants (σ, τ, ι) Schur algebra basis elements and associated bilinear forms in signatures $(0, 2m)$ with $m = 2, 3 \mod 4$.

Bilinear forms in odd dimensions are then built similarly as in $(k, 0)$. Analogously, we write $\mathbb{R}^{0,2m+1} = \mathbb{R}e_0 + \mathbb{R}^{0,2m}$ with $\langle e_0, e_0 \rangle = -1$. The spinor module of $(0, 2m+1)$ relates to that of $(0, 2m)$:

$$S_{0,2m+1} = \begin{cases} S_{0,2m} \otimes \mathbb{C} = \mathbb{S}_{2m} & m = 1 \mod 4, \\ S_{0,2m} & m = 0, 2, 3 \mod 4. \end{cases} \tag{2.132}$$

On $S_{0,2m}$ we defined a representation γ of $Cl_{0,2m}$. We extend this to a representation $\tilde{\gamma}$ of $Cl_{0,2m+1}$ on $S_{0,2m+1}$ according to

$$\tilde{\gamma}(\mathbb{R}^{0,2m}) = \gamma(\mathbb{R}^{0,2m}), \tag{2.133}$$

$$\tilde{\gamma}(e_0) = \begin{cases} \rho(\omega_{2m}) & \text{if} \quad m = 0 \text{ or } 2 \mod 4, \\ I \cdot \rho(\omega_{2m}) & \text{if} \quad m = 1 \text{ or } 3 \mod 4. \end{cases} \tag{2.134}$$

When $m = 0, 2, 3$ we can use the same h as the original even dimension. For $m = 1$ we must use the complex bilinear extension of h, setting $h = Re\langle c\cdot, \cdot \rangle$ analogous to the $m = 0$ case in signatures $(k, 0)$.

We now describe the Schur algebra for each m. For $m = 0$ the Schur algebra $\mathcal{C} = \mathbb{R}Id$ and the space of admissible bilinear forms is one dimensional. The bilinear form has the invariants $(\sigma\tau) = (++)$.

For $m \neq 0$ we define

$$\tilde{J} = L_\omega \cdot \alpha \cdot c, \tag{2.135}$$

where we have again used grade automorphism α and complex conjugation operator c. $\omega = \omega_{0,m}$, the volume element of $Cl_{0,m}$.

Using I as before and $\tilde{J}$ we can generate the basis of the Schur algebra. We have

$$Id, \quad I, \quad \tilde{J}, \quad \tilde{K} = I\tilde{J} \tag{2.136}$$

In all cases $I^2 = -1$ and $\{I, \tilde{J}\} = 0$. For $m = 1, 2$, $\tilde{J}^2 = -Id$ (which implies $\tilde{K}^2 = -1$) and $\mathcal{C} \cong \mathbb{H}$. Finally for $m = 3$, $\tilde{J}^2 = +Id$ (so $\tilde{K}^2 = +Id$) and we see that $\mathcal{C} \cong \mathbb{H}' \cong \mathbb{R}(2)$.

m	Id	I	J	K
1	$++$	$-+$	$--$	$--$
2	$++$	$-+$	$-+$	$-+$
3	$+++$	$-+-$	$+-+$	$+--$
m	h	h_I	h_J	h_K
1	$++$	$-+$	$--$	$--$
2	$++$	$-+$	$-+$	$-+$
3	$+++$	$-+-$	$+-+$	$+--$

Table 2.5: Invariants (σ, τ) ((σ, τ, ι) for $m = 3$) Schur algebra basis elements and associated bilinear forms in signatures $(0, 2m + 1)$ with $m = 1, 2, 3 \bmod 4$.

Combinations

We now discuss how to calculate signatures (p, q). Setting

$$V = \mathbb{R}^{p,q} = V_1 + V_2, \quad \text{s.t.} \quad V_1 = \mathbb{R}^{m,m}, \quad V_2 = \mathbb{R}^{k,0} \quad \text{or} \quad \mathbb{R}^{0,k}. \tag{2.137}$$

The associated spinor module to V_i will be called S_i. S_i is a spinor module of $\mathfrak{so}(V_i)$ so the product space, $\mathbb{S}_1 \otimes \mathbb{S}_2$ is a spinorial $\mathfrak{so}(V_1 + V_2)$-module.

Given an admissible bilinear form β_2 on S_2 there is a unique (up to scaling) admissible bilinear form β_1 on S_1 such that $\tau(\beta_2) = \iota(\beta_1)\tau(\beta_1)$. Correspondingly, given a β_2-admissible endomorphism A_2 on S_2 there is a unique β_1-admissible endomorphisms A_1 on S_1 such that $\tau(A_2) = \iota(A_1)\tau(A_1)$. This means the Schur algebra of $S = S_1 \otimes S_2$ and S_2 are equivalent; $\mathcal{C}(S) = \mathcal{C}(S_2)$ [3].

[3]This is expected, because S and S_2 have the same signature

We can calculate the invariants of the products $A = A_1 \otimes A_2$ and $\beta{=}\beta_1 \otimes \beta_2$ by the following rules

$$\begin{aligned}&\tau(\beta) = \tau(\beta_1) = \iota(\beta_1)\tau(\beta_2), \quad \sigma(\beta) = \sigma(\beta_1)\sigma(\beta_2), \quad \iota(\beta) = \iota(\beta_1)\iota(\beta_2), \\ &\tau(A) = \tau(A_1) = \iota(A_1)\tau(A_2), \quad \sigma_\beta(A) = \sigma_{\beta_1}(A_1)\sigma_{\beta_2}(A_2), \quad \iota(A) = \iota(A_1)\iota(A_2).\end{aligned} \tag{2.138}$$

We have shown on pseudo-Euclidean space, $V_2 = \mathbb{R}^{k,0}$ or $\mathbb{R}^{0,k}$ there exists a canonical $\mathrm{Pin}(V_2)$-invariant bilinear form, h_2. This has invariants $\sigma(h_2) = +1$ and $\tau(h_2) = -1$ for $V_2 = \mathbb{R}^{k,0}$ or $\tau(h_2) = +1$ for $V_2 = \mathbb{R}^{0,k}$. If S_2 is reducible then $\iota(h_2) = +1$.

Therefore there exists a canonical bilinear form $h = h_1 \otimes h_2$ from which we can then construct endomorphisms from the product of two endomorphisms on each factor. In doing so, we can repeat the construction outlined above for any spacetime $\mathbb{R}^{p,q}$.

2.9 Supersymmetry Algebra Isomorphisms

While [1] focused on the construction of superalgebras, it did not consider whether the resulting superalgebras are unique (up to isomorphism). At the beginning of [3], we extended this research to include and solve this problem.

Theorem – In all signatures (t,s) except $(1,1)$, two Poincaré Lie superalgebras, $(\mathfrak{g},[\cdot,\cdot])$ and $(\mathfrak{g},[\cdot,\cdot]')$, are isomorphic if and only if there exists $\psi = \psi' \cdot a \in \mathrm{Pin}(V)\cdot\mathcal{C}(S)^*$, where $\psi' \in \mathrm{Pin}(V)$ and $a \in \mathcal{C}(S)^*$, such that

$$\Pi'(\psi s_1, \psi s_2) = \pm\varphi(\Pi(s_1,s_2)), \quad \forall s_1, s_2 \in S \tag{2.139}$$

φ is the imagine of ψ' under the homomorphism $Ad : \mathrm{Pin}(V) \to \mathrm{O}(V)$. $\mathrm{Pin}(V)\cdot\mathcal{C}(S)*$ is the subgroup of $\mathrm{GL}(S)$ that is generated by $\mathrm{Pin}(V)$ and $\mathcal{C}(S)^*$. Note that $\mathrm{Pin}(V)$ normalises $\mathcal{C}(S)^*$.

Proof – Every isomorphism $\phi : (\mathfrak{g},[\cdot,\cdot])$ and $(\mathfrak{g},[\cdot,\cdot]')$ maintains the grading, mapping $\mathfrak{g}_i \to \mathfrak{g}_0$. More than this, it also maps $V \subset \mathfrak{g}_i$ to V because V is the kernel of the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ that is induced by the adjoint representation of $\mathfrak{g}$ with either bracket (that on the bosonic and fermionic generators).

Let us define

$$\varphi = \phi|_V \in \mathrm{GL}(V), \quad \psi = \phi|_S \in \mathrm{GL}(S) \tag{2.140}$$

ϕ also induces an automorphism ξ of $\mathfrak{so}(V) = \mathfrak{g}_0/V = (\mathfrak{so}(V) + V)/V$. The subalgebra $\phi(\mathfrak{so}(V)) \subset \mathfrak{so}(V) + V$ is conjugate to $\mathfrak{so}(V)$ by a translation. Up to composition of ϕ by an inner automorphism of $(\mathfrak{g}, [\cdot,\cdot]')$ we can assume $\phi(\mathfrak{so}(V) = \mathfrak{so}(V)$, so we can identify $\xi = \phi|_{\mathfrak{so}(V)} \in Aut(\mathfrak{so}(V)$. This implies that ϕ is an isomorphism between the two superalgebras if and only if the derived automorphisms ξ, φ, ψ satisfy

$$\xi(A)\varphi(v) = \varphi(Av) \tag{2.141}$$

$$\xi(A)\psi(s) = \psi(As) \tag{2.142}$$

for all $A \in \mathfrak{so}(V)$, $v \in V$ and $s \in S$. (2.141) means that $\xi = C_\varphi$, with $C_\varphi : A \to \varphi \cdot A \cdot \varphi^{-1}$ denoting conjugation by ϕ. We can therefore write (2.141) as a condition on φ alone:

$$\varphi \in N_{\mathrm{GL}(V)}(\mathfrak{so}(V)) = \{A \in \mathrm{GL}(V) | A^*\langle\cdot,\cdot\rangle = \lambda\langle\cdot,\cdot\rangle, \lambda \neq 0\} \tag{2.143}$$

Here is where the signature dependence enters. A linear transformation that normalises the Lie algebra $\mathfrak{so}(V)$ preserves the standard scalar product up to a factor for all signatures $(t,s) \neq (1,1)$. Further if $t \neq s$ there are no anti-isometries and λ is necessarily positive. This means

$$\varphi \in \mathrm{CO}(V) = \{A \in \mathrm{GL}(V) | A^*\langle\cdot,\cdot\rangle = \lambda\langle\cdot,\cdot\rangle, \quad \lambda > 0\} = \mathbb{R} \cdot \mathrm{O}(V), \tag{2.144}$$

$\mathrm{CO}(V)$ is the linear conformal group.

(2.142) goes one further than this and shows that $\varphi \in \mathrm{CO}(V)$ for all signatures $(t,s) \neq (1,1)$ too. Assume $t = s \geq 2$ and $\varepsilon \in \mathrm{GL}(V)$ is an anti-isometry. We will prove there is no $\psi \in \mathrm{GL}(S)$ normalising the image of $\mathfrak{spin}(V)$ in End$S$ that acts on $\mathfrak{spin}(V) \cong \mathfrak{so}(V)$ as ξ.

Proof – The homomorphism $Ad : \mathrm{Pin}(V) \to \mathrm{O}(V)$ is surjective, so we can assume without loss of generality that φ exchanges space-like and time-like vectors:

$$\varphi(e_i) = e_i', \quad \varphi(e_i') = e_i, \tag{2.145}$$

where $(e_1, ...e_t, e_1', ..., e_t')$ is an orthonormal basis. e_i are the time-like vectors and e_i' the space-like. ξ must then interchange e_ie_j with $-e'_ie'_j$ $(i \neq j)$ and $e_ie'_j$ with $-e'_ie_j = e_je'_i$ $(i, j$ arbitrary in this notation).

Let us first consider the case $t = s = 2$, [4] and derive the rest by induction.

In $(2,2)$ the real Clifford algebra is $Cl_{2,2} \cong \mathbb{R}(4)$ with even subalgebra $Cl^0_{2,2} \cong 2\mathbb{R}(2)$. We can therefore consider $S = \mathbb{R}^2 \otimes \mathbb{R}^2$ and give a Clifford representation as

$$\gamma_{e_1} = J \otimes I, \quad \gamma_{e_2} = K \otimes I, \quad \gamma_{e'_1} = Id \otimes J, \quad \gamma_{e'_2} = Id \otimes K. \tag{2.146}$$

Where $I, J, K = IJ$ are pairwise anticommuting operators on $\mathbb{R}^2$ that obey the para-quaternion algebra, such that $I^2 = -1$ and $J^2 = K^2 = +1$. This is done explicitly in Chapter 5 but the details are not strictly necessary here.

$\mathfrak{so}(V)$ is generated by pairwise multiplication of the Clifford generators, these pairs are

$$\begin{aligned} \gamma_{e_1}\gamma_{e_2} = -I \otimes Id, \quad \gamma_{e_1}\gamma_{e'_1} = J \otimes K, \quad \gamma_{e_1}\gamma_{e'_2} = -J \otimes J \\ \gamma_{e_2}\gamma_{e'_1} = K \otimes K, \quad \gamma_{e_2}\gamma_{e'_2} = -K \otimes J, \quad \gamma_{e'_1}\gamma_{e'_2} = Id \otimes I. \end{aligned} \tag{2.147}$$

Allowing us to read the effects ξ: it preserves the elements $J \otimes K$ and $K \otimes J$ and interchanges $Id \otimes I \leftrightarrow -I \otimes Id$ and $J \otimes J \leftrightarrow K \otimes K$.

A generic element of $\psi \in End(S)$ has the form

$$\psi = Id \otimes A_0 + I \otimes A_1 + J \otimes A_2 + K \otimes A_3, \tag{2.148}$$

with $A_a \in End(\mathbb{R}^2)$. (2.142) gives the following equations

$$\psi \cdot (J \otimes K) = (J \otimes K) \cdot \psi, \quad \psi \cdot (K \otimes J) = (K \otimes J) \cdot \psi, \tag{2.149}$$

$$\psi \cdot (Id \otimes J) = -(Id \otimes I) \cdot \psi, \quad \psi \cdot (K \otimes K) = -(J \otimes J) \cdot \psi. \tag{2.150}$$

This has no solution except $\psi = 0$ (and thus $A_a = 0$) confirming the proposition above.

The irreducible Clifford module in $(t+1, t+1)$ is $S = \mathbb{R}^2 \otimes (\mathbb{R}^2)^{\otimes t}$ with the Clifford

[4]This applies readily to research contained later in this thesis about four-dimensional vector multiplets, see Chapter 5.

algebra representation given by

$$\gamma_{e_i} = J \otimes \tilde{\gamma}_i, \quad \gamma_{e_i'} = J \otimes \tilde{\gamma}_i', \tag{2.151}$$

$$\gamma_{e_{t+1}} = Id \otimes I, \quad \gamma_{e_{t+1}} = K \otimes Id, \tag{2.152}$$

where $\tilde{\gamma}_i, \tilde{\gamma}_i'$ are the Clifford generators of signature (t, t).

ψ has the same form as in (2.148) with the caveat that $A_a \in End((\mathbb{R}^2)^{\otimes n})$ instead. (2.142) results in the equations

$$A_a \tilde{\gamma}_i \tilde{\gamma}_j = -\tilde{\gamma}_i' \tilde{\gamma}_j' A_a \quad i \neq j, \quad A_a \tilde{\gamma}_i \tilde{\gamma}_j' = \tilde{\gamma}_j \tilde{\gamma}_i' A_a. \tag{2.153}$$

These equations imply, by induction, that $A_0 = 0$ and therefore $\psi = 0$.

A homothety with factor μ on S and a simultaneous homothety with factor μ^2 on V defines an automorphism of any super Poincaré Lie algebra, so we will consider $\varphi \in \mathrm{O}(V)$ instead of $\mathrm{CO}(V)$. There exists $\psi_1 \in \mathrm{Pin}(V)$ such that $Ad(\psi_1) = \varphi$ and/or $\psi_2 \in \mathrm{Pin}(V)$ such that $Ad(\psi_2) = -\varphi$ for any choice of V. All solutions take these form and any solution solves (2.142).

ψ therefore corresponds, up to an element of the Schur group $\mathcal{C}(S)^*$, to the pre-image ψ_1 of φ or ψ_2 of $-\varphi$ under the adjoint map. Such a ψ satisfies (2.139).

Any solution (ψ, φ) defines an isomorphism from $(\mathfrak{g}, [\cdot,\cdot]_\Pi])$ to $(\mathfrak{g}, [\cdot,\cdot]_{\Pi'}])$ or $(\mathfrak{g}, [\cdot,\cdot]_{-}\Pi])$ to $(\mathfrak{g}, [\cdot,\cdot]_{\Pi'}])$. Further $(\mathfrak{g}, [\cdot,\cdot]_\Pi])$ and $(\mathfrak{g}, [\cdot,\cdot]_{-\Pi}])$ are isomorphic [5], so we have proven the theorem.

The classification of Poincaré Lie superalgebras up to isomorphism is reduced to the classification of the orbits

$$\mathcal{O}_\Pi = \mathcal{C}(S)^* \cdot \mathrm{Pin}(V) \cdot \Pi \tag{2.154}$$

of the group $\frac{\mathcal{C}(S)^* \cdot \mathrm{Pin}(V)}{\mathrm{Spin}_0(V)}$ on $(Sym^2 S^* \otimes V)^{\mathrm{Spin}_0(V)}$. $\mathrm{Pin}(V)/\mathrm{Spin}_0(V) \cong \mathrm{O}(V)/SO_0(V) \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ depending on the signature (t, s).

[5] This can be realised by simply replacing (A, v, s) with $(A, -v, s)$ for $A \in \mathfrak{so}(V), v \in V, s \in S$.

From now we restrict to when $t+s=\dim V$ is even because in odd dimensions there is only a single super-admissible bilinear form. In even dimensions

$$\frac{\mathrm{Pin}(V)}{\mathrm{Spin}_0(V)}=\begin{cases}\{[1],[e_1],[\omega],[\omega e_1]\} & V \text{ is indefinite}, \quad t,s \quad \text{odd},\\ \{[1],[e_1],[e_{t+s}],[e_{t+s}e_1]\} & V \text{ is indefinite}, \quad t,s \quad \text{even},\\ \{[1],[e_1]\} & V \text{ is definite}.\end{cases} \tag{2.155}$$

$e_1,...,e_{t+s}$ is an orthonormal basis for V and $\omega=e_1...e_{t+s}$. $\omega\in\gamma(\mathrm{Pin}(V))\cap\mathcal{C}(S)^*$ so we have two cases. First if V is indefinite and t,s are both even then

$$\begin{aligned}\mathcal{C}(S)^*\cdot\gamma(\mathrm{Pin}(V))=&\mathcal{C}(S)^*\cdot\gamma(\mathrm{Spin}_0(V))\cup\mathcal{C}(S)^*\cdot\gamma(\mathrm{Spin}_0(V)e_1) \\ &\cup\mathcal{C}(S)^*\cdot\gamma(\mathrm{Spin}_0(V)e_{t+s})\cup\mathcal{C}(S)^*\cdot\gamma(\mathrm{Spin}_0(V)e_1e_{t+s})\end{aligned} \tag{2.156}$$

And second when V is definite or V is indefinite and t,s are odd

$$\mathcal{C}(S)^*\cdot\gamma(\mathrm{Pin}(V))=\mathcal{C}(S)^*\cdot\gamma(\mathrm{Spin}_0(V))\cup\mathcal{C}(S)^*\cdot\gamma(\mathrm{Spin}_0(V)e_1). \tag{2.157}$$

Therefore the orbit $\mathcal{O}_\Pi$ is given by

$$\mathcal{O}_\Pi=\mathcal{C}(S)^*\cdot\Pi\cup\mathcal{C}(S)^*\cdot\gamma_{e_{t+s}}\cdot\Pi\cup\mathcal{C}(S)^*\gamma_{e_1}\cdot\Pi\cup\mathcal{C}(S)^*\gamma_{e_1e_{t+s}}\cdot\Pi \tag{2.158}$$

when V is indefinite and t,s both even and

$$\mathcal{O}_\Pi=\mathcal{C}(S)^*\cdot\Pi\cup\mathcal{C}(S)^*\cdot\gamma_{e_1}\cdot\Pi. \tag{2.159}$$

when V is definite or V is indefinite and t,s are odd.

Therefore when V is indefinite and t,s are even, two super-Poincaré algebras $(\mathfrak{g},[\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}',[\cdot,\cdot]'_\Pi)$ are isomorphic if and only if $\pm\Pi$, $\pm e_1\Pi$, $\pm e_{t+s}\Pi$ or $\pm e_1e_{t+s}\Pi$ is related to Π' by an element of the Schur group.

And when V is definite or V is indefinite and t,s are odd, then two super-Poincaré algebras $(\mathfrak{g},[\cdot,\cdot]_\Pi)$ and $(\mathfrak{g}',[\cdot,\cdot]'_\Pi)$ are isomorphic if and only if $\pm\Pi$ or $\pm e_1\Pi$ is related to Π' by an element of the Schur group.

2.9.1 Schur Group Action on Vector-valued Bilinear Forms

An element $a \in \mathcal{C}(\mathbb{S})^*$ acts on a symmetric vector-valued bilinear form, $\Pi_\beta \in (Sym^2\mathbb{S}^* \otimes \mathbb{R}^{t,s})^{\mathrm{Spin}_0(t,s)}$ in the contragredient/dual representation:

$$(a, \Pi_\beta) \to \Pi'_\beta = a \cdot \Pi_\beta = \Pi_\beta(a^{-1}\cdot, a^{-1}\cdot). \tag{2.160}$$

Consider a one-parameter subgroup $a(u) = exp(uA)$ with $A \in \mathcal{C}(\mathbb{S})$ (such that a is as element of the Schur algebra regarded as a Lie algebra). This gives the corresponding infinitesimal action

$$(A, \Pi_\beta) \to a \cdot \Pi_\beta = -\Pi_\beta(A\cdot, \cdot) - \Pi_\beta(\cdot, A\cdot). \tag{2.161}$$

If β is admissible and A is β-admissible then $\beta(A\cdot, \cdot)$ defines a new admissible bilinear form. Recall that $\langle \Pi_\beta, v\rangle = \beta(\gamma_v\cdot, \cdot)$ by definition, so that

$$\begin{aligned}
&\langle \Pi_\beta(A\cdot, \cdot) + \Pi_\beta(\cdot, A\cdot), v\rangle = \beta(\gamma_v\cdot, \cdot) + \beta(\gamma_v\cdot, \cdot) \\
=&(\tau(A) + \sigma_\beta(A))\beta(A\gamma_v\cdot, \cdot) = (\tau(A) + \sigma_\beta(A))\beta_A(\gamma_v\cdot, \cdot) \\
=&(\tau(A) + \sigma_\beta(A))\langle \Pi_{\beta_A}(A\cdot, \cdot)\rangle.
\end{aligned} \tag{2.162}$$

Therefore the infinitesimal action of a Schur algebra A on a superbracket Π_β is

$$A \cdot \Pi_\beta = -(\tau(A) + \sigma_\beta(A))\Pi_{\beta_A} = \begin{cases} -2\tau(A)\Pi_{\beta_A}, & \tau(A)\sigma_\beta(A) = 1, \\ 0 & \tau(A)\sigma_\beta(A) = -1. \end{cases} \tag{2.163}$$

If β is a super-admissible bilinear form we see that a β-admissible Schur algebra element $A \in \mathcal{C}(\mathbb{S}$ only acts non-trivially if β_A is super-admissible. The connected component of the stabiliser group of Π_β is generated by elements A that satisfy $\tau(A)\sigma_\beta(A) = -1$. Writing $a = exp(uA)$ again, the stabiliser group is

$$Stab_{\mathcal{C}(\mathbb{S}^*)}(\Pi_\beta) = \{a \in \mathcal{C}(\mathbb{S})^* | \beta(\gamma_v a\cdot, a\cdot) = \beta(\gamma_v\cdot, \cdot)\}, \tag{2.164}$$

in physics, this is called the R-symmetry group of the supersymmetry algebra (defined using the superbracket Π_β). Up to conjugation the stabiliser only depends on the $\mathcal{C}(\mathbb{S})^*$-orbit of Π_β and is therefore isomorphic for all superbrackets that define isomorphic super-Poincaré algebras. This makes R-symmetry a useful classification tool for supersymmetry algebras and is one that will be employed in Chapter 3.

2.10 Real Forms

A real form of a complex Lie algebra $\mathfrak{g}$, is a real Lie algebra, $\mathfrak{g}_0$, that satisfies [40]

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}. \tag{2.165}$$

Similarly, a real Lie group, G, is a real form of a complex Lie group, $G_{\mathbb{C}}$, if $G_{\mathbb{C}} = G \otimes \mathbb{C}$.

In general, a complex Lie algebra can have many real forms. There are two readily accessible unique real forms of a complex semisimple Lie algebra; the split real form and the compact real form. A real form $\mathfrak{g}_0$ of a complex semisimple Lie algebra $\mathfrak{g}$ is split if in each Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, the space $\mathfrak{p}_0$ contains a maximal Abelian subalgebra of $\mathfrak{g}_0$ (which is its Cartan subalgebra). The compact real form is compact, as the name suggests, and can be obtained from the split real form by a 'Weyl unitary trick' taking

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \rightarrow \mathfrak{g}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0. \tag{2.166}$$

Given a compact real form and an involutive automorphism T we can find all real forms by a similar method. An involutive automorphism, T, of $\mathfrak{g}$ satisfies $T\mathfrak{g}T^{-1} = \mathfrak{g}$ and $T^2 = 1$. All possible involutive automorphisms have a basis given by complex conjugation and

$$T = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = I_{p,q} \quad \text{or} \quad T = \begin{pmatrix} 0 & \mathbb{1}_p \\ -\mathbb{1}_p & 0 \end{pmatrix} = J_p. \tag{2.167}$$

Given such an automorphism we can decompose $\mathfrak{g}_0$ into the ± 1 eigenspaces of T:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \tag{2.168}$$

where $Tk = k$ for $k \in \mathfrak{k}_0$ and $Tp = -p$ for $p \in \mathfrak{p}_0$. $\mathfrak{k}_0$ is a subalgebra of $\mathfrak{g}_0$ and $\mathfrak{p}_0$ is its orthogonal complementary subspace.

If $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}$ all other real forms are obtained by decomposing $\mathfrak{g}_0$ into the ± 1-eigenspaces of T and performing the Weyl unitary trick

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \rightarrow \mathfrak{g}_0^* = \mathfrak{g}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0. \tag{2.169}$$

$\mathfrak{g}_0^*$ is a different real form of $\mathfrak{g}$ that is not necessarily different for each possible T. Therefore finding the real forms of a simple complex Lie algebra, $\mathfrak{g}$, is equivalent to finding all involutive automorphisms of the compact real form $\mathfrak{g}_0$.

Chapter 3 provides a manner to derive supersymmetry algebras whose R-symmetry groups are real forms of $\mathrm{O}(N,\mathbb{C})$, $\mathrm{Sp}(2N,\mathbb{C})$ and $\mathrm{GL}(N,\mathbb{C})$, or the products $\mathrm{O}(N,\mathbb{C}) \times \mathrm{O}(M,\mathbb{C})$ and $\mathrm{Sp}(2N,\mathbb{C}) \times \mathrm{Sp}(2M,\mathbb{C})$. Note that $\mathfrak{o}(N,\mathbb{C})$ is a real form of $\mathfrak{o}(N,\mathbb{C}) \oplus \mathfrak{o}(N,\mathbb{C})$, and similarly $\mathfrak{sp}(2N,\mathbb{C})$ is a real form of $\mathfrak{sp}(2N,\mathbb{C}) \oplus \mathfrak{sp}(2N,\mathbb{C})$.

2.11 $\mathcal{N} = 2$ Vector Multiplets

In this thesis $\mathcal{N} = 2$ rigid vector multiplet theories are used as examples of physical theories derived using the supersymmetry algebras formalism in Chapter 3. This section will provide a brief overview of the conventional manner of defining a vector multiplet for the reader's convenience. This mostly follows [20], though the notation has been altered to follow the conventions in this thesis. We will focus on those with Abelian gauge groups only, for simplicity, as this was done in our papers [2] and [3] and in the respective thesis chapters, Chapter 4 and 5.

We will begin first by discussing the conventional definition of a Minkowski signature $\mathcal{N} = 2$ vector multiplet theory in five dimensions before moving onto four dimensions, deriving the Lagrangian through dimensional reduction as we will also do in this thesis in Chapter 5. In this section space-time indices in five-dimensions will be $\mu, \nu, ... = 0, ..., 4$ and in four-dimensions they will be $m, n, ... = 0, .., 3$ in Minkowski signature and $m, n, ... = 1, ..., 4$ in Euclidean signature.

Five Dimensions

A five-dimensional $\mathcal{N} = 2$ off-shell vector multiplet contains the vector field A^μ, a symplectic Majorana fermion λ^i with $i = 1, 2$, a scalar field σ and a triplet of auxiliary fields packaged as a real, symmetric SU(2) tensor Y^{ij}. An SU(2) tensor obeys $(Y^{ij})^* = \varepsilon_{ik}\varepsilon_{kl}Y^{kl} = Y_{ij}$. This is induced by the reality condition of the fermions, that obey

$$(\lambda^i)^* = -B\lambda^j \varepsilon_{ji}. \tag{2.170}$$

This involves the matrix $B = (CA)^T = B = -(C\gamma_0)^T$ as was outlined in Section 2.6.

Reality conditions like this and their generalisation are studied in Chapter 3. The i, j indices of λ^i and Y^{ij} are so-called SU(2) indices that are raised and lowered with ε_{ij}, which is the Levi-Civita symbol in two dimensions:

$$\lambda^i = \varepsilon^{ij}\lambda_j, \quad \lambda_i = \lambda^j \varepsilon_{ji}, \tag{2.171}$$

using the NW-SE conventions as is standard. Contracted fermion terms use the Majorana conjugate, $\bar{\lambda}^i = (\lambda^i)^T C$, with the single charge conjugate matrix in five-dimensions, more details on this are available in Chapter 3. Spinor bilinears are then written

$$\bar{\lambda}\gamma^{\mu_1\dots\mu_r}\chi = (\lambda^i)^T C\gamma^{\mu_1\dots\mu_r}\chi^j\varepsilon_{ji}. \tag{2.172}$$

The five-dimensional Poincaré superalgebra involves a superbracket given by

$$\{Q_{i\alpha}, Q_{j\beta}\} = -\frac{1}{2}(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu\varepsilon_{ij}, \tag{2.173}$$

with has an off-shell representation:

$$\begin{aligned} \delta A^\mu &= \frac{1}{2}\bar{\epsilon}\gamma^\mu\lambda, \qquad \delta\sigma = \frac{1}{2}\bar{\epsilon}\lambda, \qquad \delta Y^{ij} = -\frac{i}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)}, \\ \delta\lambda^i &= -\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}\epsilon^i - \frac{i}{2}\not{\partial}\sigma\epsilon^i - Y^{ij}\epsilon_j. \end{aligned} \tag{2.174}$$

The supersymmetry parameter, ϵ^i, is also a symplectic Majorana spinor obeying the same reality condition as λ^i. As this is an off-shell representation, additional terms can be added to the Lagrangian and these variations will not change. The superbracket is invariant under SU(2) transformations that act entirely on the i, j indices.

Generalising to n_V multiplets, we require the supersymmetry variations to hold for each vector multiplet individually, so that

$$\begin{aligned} \delta A^{\mu I} &= \frac{1}{2}\bar{\epsilon}\gamma^\mu\lambda^I, \qquad \delta\sigma^I = \frac{1}{2}\bar{\epsilon}\lambda^I, \qquad \delta Y^{ijI} = -\frac{i}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I}, \\ \delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i - \frac{i}{2}\not{\partial}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j, \quad I = 1, \dots, n_V. \end{aligned} \tag{2.175}$$

The following Lagrangian is invariant under the variations in (2.175)

$$L_{(1,4)} = \left(-\frac{1}{4}F^{I}_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I\partial\!\!\!/\lambda^J + Y^I_{ij}Y^{ijJ}\right)\mathcal{F}_{IJ}(\sigma) \\ + \left(-\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} - \frac{i}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{i}{2}\bar{\lambda}^{iI}\lambda^{jJ}Y^K_{ij}\right)\mathcal{F}_{IJK}(\sigma). \tag{2.176}$$

$F^{\mu\nu}$ is the field strength of A^μ. The coupling coefficients are the derivatives of the pre-potential $\mathcal{F}(\sigma)$:

$$\mathcal{F}_{IJ}(\sigma) = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\mathcal{F}(\sigma), \qquad \mathcal{F}_{IJK}(\sigma) = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\frac{\partial}{\partial\sigma^K}\mathcal{F}(\sigma). \tag{2.177}$$

The pre-potential $\mathcal{F}(\sigma)$ is an arbitrary cubic polynomial in σ, i.e. $\mathcal{F}_{IJKL} = 0$. This is a necessary condition for the interaction terms to be invariant under both gauge and supersymmetry. In contrast, if we were to work with a superconformal vector multiplet that is a representation of the five-dimensional Minkowski signature superconformal algebra then $\mathcal{F}(\sigma)$ must be a homogenous polynomial of degree 3.

We can interpret σ^I as a map from space-time, $\mathbb{R}^{1,4}$, to a n_V-dimensional Riemannian manifold $\mathcal{M}$ with metric given by $\mathcal{F}_{IJ}$. The metric $\mathcal{F}_{IJ}$ is a Hessian metric derived from $\mathcal{F}(\sigma)$, which is a polynomial of degree at most 3. The resulting manifold is called an affine special real manifold [20, 39].

The Lagrangian is not invariant under general coordinate transformations of $\mathcal{M}$, only affine transformations, $\sigma^I \to R^I_J\sigma^J + a^I$, with constant and invertible R^I_J and constant a^I. Hence σ^I are affine coordinates, in analogy to $(1,3)$ signature theories they are often also called special coordinates.

Additional references for five-dimensional vector multiplet theories can be found in [41–43], where they were studied using string theory, and [44–47] which use superconformal vector multiplets. Additional terms also arise in the supersymmetry variations due to the special supersymmetry transformations in the superconformal algebra.

Four Dimensions

Four-dimensional vector multiplet Lagrangians can be found by dimensional reduction, usually from five or six dimensions [20, 39]. This section will summarise [20], of which the physical aspects in this thesis were based upon, where a Minkowski-signature five-dimensional theory was reduced to Minkowski and Euclidean signature four-dimensional theories. The explicit details will be omitted here, though in Chapter 5 the dimensional reductions from any five-dimensional signature to any four-dimensional signature are demonstrated in detail.

Dimensional reduction is assumed to be along the 0 direction for a time-like reduction and the 4 direction for a space-like reduction. For shorthand, this will be referred to as #, which is 0 or 4 depending on the context. In doing so, the conventions for the four-dimensional space-time indices m, n agree with those given at the start of this section.

The field content of n_V four-dimensional $\mathcal{N} = 2$ off-shell vector multiplet involves complex or para-complex scalar fields, X^I, symplectic Majorana spinors, λ^{iI} [6], the eponymous vector fields A^{mI} and auxiliary fields Y^{ijI}. The scalar fields are complex in Minkowski signature and para-complex in Euclidean signature.[7] Using the language of ϵ-complex numbers we can call this an ϵ-complex scalar field and treat the two cases in tandem.

Compared to five dimensions the vector has lost one degree of freedom and the scalar has gained one (going from a real scalar field to an ϵ-complex scalar field). This is because, upon dimensional reduction, the components of the vectors along the removed dimension become the extra scalar fields. Reducing along a space-like direction, and setting $A^I_{\#} = b^I$ the kinetic term of the vector field becomes

$$-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} \to \frac{1}{4}F^I_{mn}F^{Jmn} + (-1)^t\frac{1}{2}\partial_m b^I \partial^m b^J. \tag{2.178}$$

Combining these new scalar fields b^I with the scalar fields σ^I such that $X^I = \sigma^I + i_\epsilon b^I$, with $\epsilon = (-1)^t$, we can write

$$-\frac{1}{2}\partial_m \sigma^I \partial^m \sigma^J + (-1)^t\frac{1}{2}\partial_m b^I \partial^m b^J = -\frac{1}{2}\partial_m X^I \partial^m \bar{X}^J, \tag{2.179}$$

where we have used the ϵ-complex conjugate of X^I, $\bar{X}^I = \sigma^I - i_\epsilon b^I$ and t is the number of time-like dimensions of the daughter theory, e.g. in Minkowski signature $t = 1$ and for Euclidean signature $t = 0$. Note the sign difference of the b^I kinetic term in Euclidean

[6] An equivalent formulation in terms of a pair of Majorana spinors is possible and well-known, which is also discussed in Chapter 5.

[7] Alternatively one could write these theories using a pair of related real coordinates, called ‘adapted coordinates’ but this is not used in this thesis.

and Minkowski signature was necessary to allow the writing in terms of an ϵ-complex scalar field in both cases.

In five dimensions the pre-potential was a real function $\mathcal{F}(\sigma)$ which is now extended to an ϵ-holomorphic function $\mathcal{F}(X)$ by simply substituting σ^I with X^I. The coupling matrices become

$$\mathcal{F}_{IJ}(X) = \frac{\partial^2 \mathcal{F}(X)}{\partial X^I \partial X^J}, \quad \bar{\mathcal{F}}_{IJ}(\bar{X}) = \frac{\partial^2 \bar{\mathcal{F}}(\bar{X})}{\partial \bar{X}^I \partial \bar{X}^J}, \tag{2.180}$$

with $\bar{\mathcal{F}}(\bar{X}) = (\mathcal{F}(X))^*$, the ϵ-complex conjugate of $\mathcal{F}(X)$. Because $\mathcal{F}(\sigma)$ is a cubic polynomial, we can write $\mathcal{F}_{IJ}(X)$ and its conjugate in terms of the real fields σ^I and b^I:

$$\mathcal{F}_{IJ}(X) = \mathcal{F}_{IJ}(\sigma) + i_\epsilon b^K \mathcal{F}_{IJK}, \quad \bar{\mathcal{F}}_{IJ}(\bar{X}) = \mathcal{F}_{IJ}(\sigma) - i_\epsilon b^K \mathcal{F}_{IJK}. \tag{2.181}$$

Therefore we can write

$$\mathcal{F}_{IJ}(\sigma) = \frac{1}{2}\big(\mathcal{F}(X)_{IJ} + \bar{\mathcal{F}}(\bar{X})_{IJ}\big) = N_{IJ}(X, \bar{X}). \tag{2.182}$$

$N_{IJ}(X, \bar{X})$ is an ϵ-Kähler metric with potential

$$K(X, \bar{X}) = \frac{1}{2}\big(\mathcal{F}_I \bar{X}^I + \bar{\mathcal{F}}_I X^I\big), \tag{2.183}$$

such that $N_{IJ} = \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J}$. This is not a generic ϵ-Kähler potential, as it can be expressed in terms of a ϵ-holomorphic prepotential $\mathcal{F}(X)$. This is the defining feature of an affine special ϵ-Kähler manifold [20, 39].

In four dimensions we can split the field strength of the vector field into self-dual and anti-self-dual field strengths, given $F_{mn} = \partial_m A_n - \partial_n A_m$ and its dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{mnpq}F^{pq}$ we define

$$F_{\pm mn} = \frac{1}{2}\Big(F_{mn} + \frac{1}{i_\epsilon}\tilde{F}_{mn}\Big) \tag{2.184}$$

such that $\tilde{F}_{\pm mn} = \frac{1}{2}\epsilon_{mnpq}F_{\pm}^{pq} = \pm F_{\pm mn}$. The dimensionally reduction of the Chern-Simons term in the five-dimensional Lagrangian, see (2.176), combines with the Maxwell term to allow one to write the Lagrangian in terms of self-dual and anti-self-dual field strengths.

The dimension of the complex spinor module in five and four dimensions is the same, so dimensional reduction does not affect the symplectic Majorana spinor, or the associated matrices B and C. Spinor bilinears and the reality condition are the same.

However in even dimensions we can define γ_*, which anti-commutes with all γ-matrices, and use it to construct the projectors $\Gamma_\pm = \frac{1}{2}(1 \pm \gamma_*)$. For the dimensionally reduced theory γ_* is proportional to the removed Clifford algebra generator, this is

$$\gamma_* = -i i_\epsilon \gamma_\#. \tag{2.185}$$

The factor of i_ϵ arises to allow one to write entirely ϵ-holomorphic terms involving the spinors; this is shown in greater detail in Chapter 5. With this the symplectic Majorana spinor can be decomposed $\lambda^i = \lambda^i_+ + \lambda^i_-$. In Minkowski signature the chiral pieces, $\lambda^i_\pm$, are not symplectic Majorana-Weyl spinors, one can show

$$(\lambda^i_\pm)^* = -B^{1,3}_- \lambda^j_\mp \varepsilon_{ji}, \tag{2.186}$$

however, in Euclidean signature, one can define symplectic Majorana spinors that obey

$$(\lambda^i_\pm)^* = -i B^{0,4}_+ \lambda^j_\pm \varepsilon_{ji}, \tag{2.187}$$

where $B^{t,s}_\pm$ arises from relating the five-dimensional B to the four-dimensional B matrices in each signature (similar calculations are found in Chapter 3). Here $*$ refers only to complex conjugate, not ϵ-complex conjugation as the para-complex elements are associated to the scalar manifold, not to the spinor module.

The dimensional reduction of the superbracket gives

$$\{Q_{i\alpha}, Q_{j\beta}\} = -\frac{1}{2}(\gamma^m C^{-1})_{\alpha\beta} P_m \varepsilon_{ij}, \tag{2.188}$$

where we have chosen to ignore a central-charge like term arising from the dimensional reduction. This is invariant under $\mathrm{U}(2)$ for Minkowski signature and $\mathrm{SO}(1,1) \times \mathrm{U}(2) \cong \mathrm{U}^*(2)$ for Euclidean signature, due to the presence of Weyl spinors in four dimensions allowing additional transformations the R-symmetry groups are larger than in five dimensions.

The full dimensional reduction of the Lagrangian follows the standard procedure, see [20] or Chapter 5 in this thesis for details. The resulting Lagrangian is true for both signatures:

$$
\begin{aligned}
L = &-\frac{1}{4}\big(F^I_{-mn}F^{Jmn}_{-}\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_{+}\bar{\mathcal{F}}_{IJ}(\bar{X})\big)\\
&-\frac{1}{2}\partial_m X^I\partial^m\bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij}Y^J_{ij}N_{IJ}(X,\bar{X})\\
&-\frac{1}{2}\big(\bar{\lambda}^I_+\not{\partial}\lambda^J_- + \bar{\lambda}^I_-\not{\partial}\lambda^J_+\big)N_{IJ}(X,\bar{X})\\
&-\frac{1}{4}\big(\bar{\lambda}^I_-\not{\partial}\mathcal{F}_{IJ}(X)\lambda^J_+ + \bar{\lambda}^I_+\not{\partial}\bar{\mathcal{F}}_{IJ}(\bar{X})\lambda^J_-\big)\\
&-\frac{i}{8}\big(\bar{\lambda}^I_+\gamma^{mn}F^J_{-mn}\lambda^K_+\mathcal{F}_{IJK} + \bar{\lambda}^I_-\gamma^{mn}F^J_{+mn}\lambda^K_-\bar{\mathcal{F}}_{IJK}\big)\\
&-\frac{i}{2}\big(\bar{\lambda}^{Ii}_+\lambda^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} + \bar{\lambda}^{Ii}_-\lambda^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\big).
\end{aligned}
\tag{2.189}
$$

This is invariant under the following signature-independent supersymmetric variations, where the $\bar{\cdot}$ is understood to be the ϵ-complex conjugate for the scalar fields X^I and the Majorana conjugate for λ^i:

$$
\begin{aligned}
\delta X^I &= i\bar{\epsilon}_+\lambda^I_+, \qquad \delta\bar{X}^I = i\bar{\epsilon}_-\lambda^I_-,\\
\delta A^I_m &= \frac{1}{2}\big(\bar{\epsilon}_+\gamma_m\lambda^I_- + \bar{\epsilon}_-\gamma_m\lambda^I_+\big),\\
\delta Y^I_{ij} &= -\frac{1}{2}\big(\bar{\epsilon}_{+(i}\not{\partial}\lambda^I_{-j)} + \bar{\epsilon}_{-(i}\not{\partial}\lambda^I_{+j)}\big),\\
\delta\lambda^{Ii}_+ &= -\frac{1}{4}\gamma^{mn}F^I_{-mn}\epsilon^i_+ - \frac{i}{2}\not{\partial}X^I\epsilon^i_- - Y^{Iij}\epsilon_{+j},\\
\delta\lambda^{Ii}_- &= -\frac{1}{4}\gamma^{mn}F^I_{+mn}\epsilon^i_- - \frac{i}{2}\not{\partial}\bar{X}^I\epsilon^i_+ - Y^{Iij}\epsilon_{-j}.
\end{aligned}
\tag{2.190}
$$

Note that, in Minkowski signature, this Lagrangian is in terms of the so-called old conventions that are related to the new conventions, established in [48], by setting

$$
\mathcal{F}^{(new)}(X) = \frac{1}{2i}\mathcal{F}^{(old)}(X). \tag{2.191}
$$

An analogous rescaling is possible in Euclidean signature, replacing i with e.

These two examples, of five-dimensional and four-dimensional vector multiplets, provided one of the key motivations for this thesis. The space-time signature controls the signs of the kinetic term for the b^I scalar fields in the Lagrangian, which in turn affects the scalar manifold by forcing the usage of complex or para-complex scalar fields.

One of the primary goals of this thesis was to assess the extent to which this sign difference between the scalar and vector field kinetic terms is mandated by supersymmetry and see if alternate pathways can avoid it. For example, reducing a five-dimensional Euclidean theory to four dimensions may result in a different theory. This is explored in Chapter 4 and 5, that also include any signature theories in both dimensions. The formalism that was developed to deal with the Lagrangians and supersymmetry variations, initially called 'doubled spinors' in [2] and [3], was then generalised to allow similar constructions in any signature and dimension and can be found in Chapter 3. In Euclidean signature, it was shown the sign difference can be removed (though the scalar field remains para-complex). In Minkowski signature an alternative $\mathcal{N} = 2$ theory is found that has a sign difference that cannot be removed, therefore necessarily having ghost fields.

2.12 Supersymmetry in Ten Dimensions

2.12.1 Type IIA and Type IIB

In ten dimensions we can define two types of $\mathcal{N} = 2$ superalgebra, one where the supercharges are of opposite chiralities (a.k.a. $\mathcal{N} = (1,1)$) and one where the supercharges are both the same chirality (a.k.a. $\mathcal{N} = (2,0)$). The first kind of superalgebra arise from a string theory called Type IIA (and also the less common Type IIA*) and the second gives Type IIB (and similarly Type IIB* and IIB'). We will give more details on the alternative theories in Section 2.12.3.

The low energy limit of string theory is described by ten-dimensional supergravity. The features we wish to study (that are present in our four-dimensional theories too) are visible at this level so we will exclusively discuss supergravity. As they have a different superalgebras, Type IIA and Type IIB supergravity have different field contents. Note that the starred theories have the same field content as the non-starred versions.

The Type IIA supergravity multiplet contains the graviton, $g_{\mu\nu}$, a pair of chiral gravitino, $\psi_{+\mu}$ and $\psi_{-\mu}$, the Kalb-Ramond two form, $B_{\mu\nu}$, odd-dimensional Ramond-Ramond gauge fields, A_μ and $C_{\mu\nu\rho}$, the dilaton, ϕ, and the dilatino.

The gravitini can be combined into a single Majorana vector-spinor $\psi_\mu = \psi_{+\mu} + \psi_{-\mu}$.

The chirality matrix is $\Gamma_{11} = \Gamma_1 ... \Gamma_{10}$, such that $\Gamma_{11}\psi_\pm = \pm\psi_\pm$. In [14] pseudo-Majorana spinors are also used, but are omitted here and the distinction between pseudo- and regular Majorana is ignored for brevity.

These fields form a representation of the Type IIA superalgebra, which is (with spinor indices suppressed)

$$\{Q, Q\} = (\Gamma^\mu C)^{-1} P_\mu \tag{2.192}$$

for a Majorana supercharge, Q, that can be split into Majorana-Weyl supercharges, $Q_\pm$, such that $Q = Q_+ + Q_-$.

The bosonic Lagrangian is, with conventional normalisation and omitting higher-order terms,

$$L_{IIA} = \int d^{10}x\sqrt{-g}\big(e^{-2\phi}(R + 4\partial^\mu\phi\partial_\mu\phi - H^2) - G_2^2 - G_4^2\big) + \int \frac{4}{3} G_4 \wedge G_4 \wedge B_2 + ... \tag{2.193}$$

where G_2 is the 2-form field strength of the 1-form gauge potential A_μ and G_4 is the field strength of $C_{\mu\nu\rho}$.

The gravitini both obey the same reality condition,

$$(\Psi^\mu_\pm)^* = B\Psi^\mu_\pm, \tag{2.194}$$

so they can be combined into a single Majorana gravitino that satisfies the same reality condition, $(\Psi^\mu)^* = B\Psi^\mu$.

The kinetic term for the gravitini is of the form

$$\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = \bar{\psi}_{+\mu}\Gamma^{\mu\nu\rho}\partial_\nu\psi_{+\rho} + \bar{\psi}_{-\mu}\Gamma^{\mu\nu\rho}\partial_\nu\psi_{-\rho}. \tag{2.195}$$

As our formalism involves the definition of spinors and superalgebras, it is this term that is particularly important to us.

Type IIB supergravity contains the same field content, except the gauge potentials have even dimensions, which are a 0-form axion χ, two form $\tilde{B}_{\mu\nu}$ and a self-dual 4-form

$D_{\mu\nu\rho\sigma}$, and the two gravitini (with the same chirality), $\psi^i_\mu = \psi^i_{+\mu}$.

The Type IIB superalgebra is

$$\{Q_{+i}, Q_{+j}\} = (\Gamma^\mu C)^{-1} P_\mu \delta_{ij}. \tag{2.196}$$

The supercharges are Majorana-Weyl spinors of a single chirality, Q_{+i}, that was arbitrarily chosen to be +.

The bosonic Lagrangian density is, once again with conventional normalisation and omitting higher-order terms,

$$L_{IIB} = \int d^{10}x\sqrt{-g}\big(e^{-2\phi}(R + 4\partial^\mu\phi\partial_\mu\phi - H^2) - G_1^2 - G_3^2 - G_5^2\big) + ... \tag{2.197}$$

Similarly G_1, G_3 and G_5 are the field strengths of the axion, 2-form and 4-form respectively.

The kinetic term for the gravitini is proportional to

$$\bar{\psi}^i_\mu \Gamma^{\mu\nu\rho}\partial_\nu \psi^j_\rho \delta_{ij}. \tag{2.198}$$

2.12.2 T-Duality

Conventional (space-like) T-duality links type IIA and Type IIB, but if we allow the compactified dimension to be time-like we can reach other ten-dimensional supergravities that correspond to different types of string theory, called IIA*, IIB* and IIB'.

Type IIA* is obtained from Type IIB following a time-like T-duality, and Type IIB* is similarly obtained from Type IIA using a time-like T-duality. Alternatively one could allow a 'mixed' T-duality, where one theory is compactified on a space-like circle is T-dual to one compactified on a time-like circle. This means the signature of the two theories must differ by one, so this way we can reach other space-time signatures. The starred theories have different signs in the Lagrangian but identical field content to the non-starred versions.

This section will focus on the effects T-duality has on the fermionic pieces, especially the supersymmetry algebra, as this is what we will be concerned with later. The author used [49] was used for general information on T-duality. On the bosonic pieces, T-duality exchanges the even/odd R-R tensor fields with the odd/even R-R tensor fields so that the Type IIA Lagrangian is mapped to the Type IIB Lagrangian.

In the following a space-like T-duality will be considered to act on the X^9 direction, flipping the sign of the right-moving components of X^9:

$$X^9_L \to X^9_L, \qquad X^9_R \to -X^9_R. \tag{2.199}$$

This must also be true for the superpartners of $X^9_{L/R}$, the world sheet fermions $\psi^9_{L/R}$

$$\psi^9_L \to \tilde{\psi}^9_L = \psi^9_L, \qquad \psi^9_R \to \tilde{\psi}^9_R = -\psi^9_R. \tag{2.200}$$

Time-like T-duality will act on the X^0 direction, such that $X^0_R \to -X^0_R$, thereby inducing

$$\psi^0_L \to \tilde{\psi}^0_L = \psi^0_L, \qquad \psi^0_R \to \tilde{\psi}^0_R = -\psi^0_R. \tag{2.201}$$

The zero-modes of the world-sheet fermions (with periodic boundary conditions) satisfy the Clifford algebra, up to normalisation,

$$\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}, \tag{2.202}$$

so that one can associate the zero-mode of the right-moving fermions with the Γ-matrices

$$\psi^\mu_{R,0} \propto \Gamma^\mu. \tag{2.203}$$

Therefore we can interpret T-duality can be interpreted as a transformation on the Γ-matrices that swaps the sign on Γ^0 or Γ^9. One can show that the following implements the necessary change

$$\tilde{\Gamma}^\mu = T^\dagger \Gamma^\mu T, \quad T = \beta \Gamma_* \Gamma^{0/9}, \quad |\beta| = 1. \tag{2.204}$$

Where $\Gamma^{0/9}$ is the Γ-matrix associated with the direction the T-duality is performed and $\Gamma_* = (-i)^t \Gamma_0 ... \Gamma_9$ is the (1,9) signature chirality matrix. One can show that this correctly implements the T-duality transformation on the Γ-matrices, such that

$$\tilde{\Gamma}^{0/9} = -\Gamma^{0/9}, \qquad \tilde{\Gamma}^m = \Gamma^m, \quad m \neq 0/9, \tag{2.205}$$

For a space-like T-duality this is always unitary transformation, but for a time-like Γ^0 only when $\beta = \pm 1$ is this a unitary transformation, for $\beta = \pm i$ it is real but not unitary. Under this transformation $A = \Pi_\tau \gamma_\tau$ becomes $\tilde{A} = T^\dagger A T$.

Alternatively, we may wish to leave the Γ-matrices invariant and implement the transformation onto the spinorial states. Considering the Ramond sector ground state, that is obtained by applying a spin operator S_L and S_R onto the NS-groundstate $|0\rangle$, giving $|S_L\rangle = S_L|0\rangle$ and $|S_R\rangle = S_R|0\rangle$. In Type IIA $|S_L\rangle$ and $|S_R\rangle$ are Majorana-Weyl fermions with opposite chirality, and in Type IIB they are Majorana-Weyl fermions with the same chirality.

Recasting T-duality as a transformation acting on the left and right-moving spin operators we set

$$S_L \to \tilde{S}_L = S_L, \quad S_R \to \tilde{S}_R = T S_R. \tag{2.206}$$

This changes the chirality of S_R: we observe that $\Gamma_* T = -\Gamma_* T$ so that

$$\Gamma_* S_R = \pm S_R \implies \Gamma_* \tilde{S}_R = \Gamma_* T S_R = -T \Gamma_* S_R = \mp \tilde{S}_R. \tag{2.207}$$

The space-time supercharges are the integral of the spin operators at zero momentum [50], so the transformation is passed on to the supersymmetry algebra.

Say Q_R is the supercharge associated to S_R, such that $Q_R \to T Q_R$ under T-duality. One can show that

$$\begin{aligned} \{Q_R, Q_R\} &= (\Gamma^\mu C)^{-1} P_\mu \\ \implies \{T Q_R, T Q_R\} &= \frac{1}{\beta^2}\{Q_R, Q_R\}. \end{aligned} \tag{2.208}$$

Choosing $\beta = \pm i$ gives the conventional sign on the superbracket according to [14]. More details about this choice can be found in Chapter 3 Section 3.11.3.

2.12.3 Type IIA* and IIB*

The conventions in this section follow the original formulation in [14]. This is slightly different from the conventions that are used in Chapter 3, any differences will be described in that section.

The Type IIA* and IIB* field content is the same as IIA and IIB respectively, the differences between the theories are in the definition of the spinorial aspects, that induce various sign differences in the Lagrangian.

The conventional writing of the Type IIA* superbracket is

$$\{Q_{\pm i}, Q_{\pm j}\} = \pm(\Gamma^{\mu} C)^{-1} P_{\mu} \tag{2.209}$$

such that we can no longer combine this into a superbracket for a Majorana supercharge $Q = Q_{+} + Q_{-}$.

The bosonic Lagrangian is

$$L_{IIA^{*}} = \int d^{10}x\sqrt{-g}\big(e^{-2\phi}(R + 4\partial^{\mu}\phi\partial_{\mu}\phi - H^{2}) + G_{2}^{2} + G_{4}^{2}\big) - \int \frac{4}{3} G_{4} \wedge G_{4} \wedge B_{2} + ... \tag{2.210}$$

The kinetic terms of the Majorana-Weyl gravitini have a different sign,

$$\bar{\psi}_{+\mu}\Gamma^{\mu\nu\rho}\partial_{\nu}\psi_{+\rho} - \bar{\psi}_{-\mu}\Gamma^{\mu\nu\rho}\partial_{\nu}\psi_{-\rho}, \tag{2.211}$$

however, this can be compensated for using the chirality matrix, Γ_{*},

$$\begin{aligned} &\bar{\psi}_{+\mu}\Gamma^{\mu\nu\rho}\partial_{\nu}\psi_{+\rho} - \bar{\psi}_{-\mu}\Gamma^{\mu\nu\rho}\partial_{\nu}\psi_{-\rho} \\ &= \bar{\psi}_{\mu}\Gamma^{\mu\nu\rho}\Gamma_{*}\partial_{\nu}\psi_{\rho}. \end{aligned} \tag{2.212}$$

The Type IIB* superbracket is similarly twisted, putting the 'wrong sign' on the $i = j = 2$ component:

$$\{Q_{+i}, Q_{+j}\} = (\Gamma^{\mu} C)^{-1} P_{\mu} \eta_{ij}. \tag{2.213}$$

The reality condition on the supercharges is the standard SO(2)-Majorana reality condition:

$$(Q_{+}^{i})^{*} = B Q_{+}^{i}. \tag{2.214}$$

The twist in (2.213) can be compensated for by taking $Q_{2} \to iQ_{2}$, such that the superbracket becomes the same as in Type IIB

$$\{Q_{+i}, Q_{+j}\} = (\Gamma^\mu C)^{-1} P_\mu \delta_{ij}. \tag{2.215}$$

However, this modifies the reality condition to

$$(Q_+^i)^* = B Q_+^j \eta_{ji}. \tag{2.216}$$

We see that one can move around the relative sign between the kinetic term signs or the reality condition of the $i = 1$ and $i = 2$ components, but it cannot be eliminated.

The Type IIB* bosonic Lagrangian is

$$L_{IIB^*} = \int d^{10}x\sqrt{-g}\big(e^{-2\phi}(R + 4\partial^\mu\phi\partial_\mu\phi - H^2) + G_1^2 + G_3^2 + G_5^2\big) + ... \tag{2.217}$$

There is a sign difference between the terms involving the p-forms in the starred theories. This feature is shared by the vector multiplet theories obtained from the 'twisted' or 'type-*' four-dimensional superalgebras with $\mathrm{U}(1,1)$ R-symmetry group. The standard $\mathcal{N} = 2$ vector multiplet has the same sign for the scalar and vector kinetic terms, but the twisted version necessarily has a different sign. These are first hinted at in Chapter 3 and are discussed in detail in Chapter 5.

Additionally, there is the Type IIB' theory, that is the S-dual of the Type IIB* theory. S-duality is a duality between coupling limits of the same theory, so they have the same supersymmetry algebra but different bosonic Lagrangians:

$$L_{IIB'} = \int d^{10}x\sqrt{-g}\big(e^{-2\phi}(R + 4\partial^\mu\phi\partial_\mu\phi - H^2) + G_1^2 - G_3^2 + G_5^2\big) + ... \tag{2.218}$$

We see the sign of the kinetic term of B_2 and G_3 have changed. The analysis of Chapter 3 focuses on the superalgebras so the differences between Type IIB and Type IIB' will not be discussed.

2.12.4 Exotic Signature Theories in String Theory

The mixed T-dualities are dualities between Type II string theories in different ten-dimensional signatures. The types of spinor one can define varies with signature. More details can be found in Chapter 3.

A Majorana condition can be defined in (0,10), (2,8) and (4,6) (and the mirror signatures) but one cannot have Majorana-Weyl spinors. As a result, one can only define Type IIA style superalgebras with Majorana supercharges. Type IIA* theories cannot be defined because to do so we need Majorana-Weyl spinors. The superalgebra is the standard Type IIA algebra in (2.192).

In (3,7) and (7,3) signature there is no Majorana condition, so one works with symplectic Majorana-Weyl spinors, mandating Type IIB superalgebras with the superbracket

$$\{Q_{+i}, Q_{+j}\} = (\Gamma^{\mu} C)^{-1} P_{\mu} \varepsilon_{ij}. \tag{2.219}$$

with chiral supercharges that satisfy the symplectic Majorana reality condition

$$(Q_+^i)^* = B Q_+^j \varepsilon_{ji}. \tag{2.220}$$

Moving to different signatures changes the sign of the kinetic terms in the Lagrangian. These are intimately related to the definition of the superalgebra, which is induced in-turn by the signature-specific features of the spinor module. Additionally, there may be more than one possible superalgebra (for a given $\mathcal{N}$), such as in (1,9) where one can define IIA, IIA*, IIB and IIB*, which also affect the signs in the Lagrangian.

Each Type IIA and Type IIA* bosonic Lagrangian is of the form

$$L_{IIA} = \int d^{10}x \sqrt{-g} \big(e^{-2\phi} (R + 4\partial^{\mu}\phi \partial_{\mu}\phi - s_H H^2) + s_2 G_2^2 + s_4 G_4^2 \big) + ... \tag{2.221}$$

with signature-dependent signs $s_H, s_i = \pm 1$ for $i = 2, 4$.

Similarly, every type IIB, IIB* and IIB' Lagrangian has the following form

$$L_{IIB^*} = \int d^{10}x \sqrt{-g} \big(e^{-2\phi} (R + 4\partial^{\mu}\phi \partial_{\mu}\phi + s_H H^2) + s_1 G_1^2 + s_3 G_3^2 + s_5 G_5^2 \big) + ... \tag{2.222}$$

which again has signature-dependent signs $s_H, s_i = \pm 1$ for $i = 1, 3, 5$. The signs are collected in the following table.

Signature	Theory	s_H	s_2	s_4
$(0,10)$	IIA	$-$	$+$	$+$
$(1,9)$	IIA	$+$	$-$	$+$
	IIA*	$-$	$-$	$-$
$(2,8)$	IIA	$+$	$+$	$-$
$(4,6)$	IIA	$-$	$+$	$+$
$(5,5)$	IIA	$-$	$-$	$-$
	IIA*	$+$	$-$	$+$
$(6,4)$	IIA	$+$	$+$	$-$
$(8,2)$	IIA	$-$	$+$	$+$
$(9,1)$	IIA	$-$	$-$	$-$
	IIA*	$+$	$-$	$+$
$(10,0)$	IIA	$+$	$+$	$-$

Signature	Theory	s_H	s_1	s_3	s_5
$(1,9)$	IIB	$-$	$-$	$-$	$-$
	IIB*	$+$	$-$	$+$	$+$
	IIB'	$+$	$+$	$-$	$+$
$(3,7)$	IIB	$-$	$+$	$+$	$-$
$(5,5)$	IIB	$-$	$-$	$-$	$-$
	IIB*	$+$	$-$	$+$	$+$
	IIB'	$+$	$+$	$-$	$+$
$(7,3)$	IIB	$-$	$+$	$+$	$-$
$(9,1)$	IIB	$-$	$-$	$-$	$-$
	IIB*	$+$	$-$	$+$	$+$
	IIB'	$+$	$+$	$-$	$+$

Table 2.6: Signs in bosonic Lagrangian of Type II theories.

A similar paradigm is explored in Chapter 4 and 5. In five dimensions each signature has a single unique minimal superalgebra, so the signs in a supersymmetric Lagrangian are determined by the signature. One can realise more than one minimal superalgebra in some four-dimensional signatures; this affects the signs in the Lagrangian along with the signature-dependent aspects of the spinor module. This is paralleled in the ten-dimensional signatures, where the possibility of defining different types of superalgebra and signature-dependence of the spinor module affect the signs in the Lagrangian. This means one can have different theories with different sign attributions in the same signature. Instead of using T-duality, the theories in this thesis are constructed ab initio by imposing signature-dependent reality conditions on a complexified holomorphic Lagrangian.

3 Extended Supersymmetry Algebras

3.1 Introduction

This chapter details a method of constructing supersymmetry algebras in any signature and dimension with supercharges that are elements of an arbitrary number of copies of irreducible spinor modules. All relevant details are included in all signatures in up to 12 dimensions in an entirely self-contained manner that allows a reader to construct a superalgebra in any of these scenarios from first principles.

We begin by outlining the complexification of these extended spinor modules and the bilinear forms upon them. Then we define a signature-dependent real structure on the complexified spaces to obtain a real supersymmetry algebra that can then be used to define a physical theory. In doing this, we are generalising the Majorana and symplectic Majorana constructions and expand on the doubled spinor formalism in our previous work (also with Vicente Cortes) [2] and [3], and work by others such as [14].

In odd dimensions, this process disentangles the R-symmetry group from the Lorentz group – such that R-symmetry transformations act only an internal space – and almost entirely in even dimensions, where R-symmetry transformations may act on each Weyl spinor module with a different sign. R-symmetry transformations then act entirely on the internal index that enumerates the spinor modules. When constructing physical theories, this is highly useful; it makes writing terms in a Lagrangian and supersymmetry representations easier and offers an insight into necessary reality conditions of fields in the Lagrangian. In addition, the reality condition of the spinors are related to the scalar target geometry and can induce different target space geometries than the standard cases.

Next, we investigate the uniqueness of the resulting superalgebras, which are not necessarily unique. Maps between some families of isomorphic families are included. To do this, R-symmetry is used to identify non-isomorphic superalgebras, thereby providing a classification tool that is constrained by signature and dimension. All R-symmetry groups captured by our formalism are calculated for each signature in up to 12 dimensions. While not providing a full classification up to isomorphism, it provides all known supersymmetry algebras and more. Additionally, this method allows one to identify cases where more than one supersymmetry algebra exists for a given signature and dimension of the spinor representation. In particular, for Lorentz signature, we find 'type-*' algebras with non-compact R-symmetry groups.

Finally, we introduce some physical examples, detailing how they arise in this formalism. In particular, we look at the dimensional reduction of superalgebras in various scenarios and T-duality including exotic signatures (like in [14, 15]. As this formalism disentangles R-symmetry from the Lorentz index, dimensional reduction is straightforward. Later chapters in this thesis use this construction to derive five-dimensional and four-dimensional vector multiplet theories.

Commonly used notation

- D – dimension of space-time, with signature (t, s), with indices μ, ν etc.
- $\mathbb{S}$ – the complex spinor module, indices α, β etc. but often suppressed.
- S – the real spinor module.
- $d_{\mathbb{S}} = 2^{[\frac{D}{2}]}$ – dimension of the spinor module (number of components of real/Dirac spinor).
- $\mathbb{S} \otimes \mathbb{C}^K$ – 'K-extended (complex) spinor module'.
- N – number of copies of the real spinor module, which is then complexified.
- $K = N \quad \text{or} \quad 2N$ – resulting number of copies of $\mathbb{C}$ in the complexification. Value depends on whether $\mathbb{S}$ has spin invariant real structure.
- Indices on $\mathbb{C}^K$ are i, j etc. When/if we 'double again' for Weyl spinors the indices will be I, J etc. which run from $1, ..., N_+ + N_-$. Spinor indices are α, β.

Note that in this chapter, we work exclusively with commuting spinors, following the previous mathematical work this is based on. Changing to anti-commuting (Grassmannian) variables is a perfectly understood functor, as outlined previously, and all-important structures transfer, see Section 2.2.3. Working with Grassmannian variables effectively inverts all symmetry statements (exchanging symmetric and antisymmetric where it arises) and does not have any effect on the conclusions. As physical theories are written in terms of anti-commuting variables, this is a distinction that is worth keeping in mind.

3.2 Useful Formulae

The following formula are used extensively throughout this chapter and are provided here for ease-of-reference. These include $D = 2, 6, 10$, $\sigma_+ = -\sigma_-$ and in $D = 4, 8, 12$, $\sigma_+ = \sigma_-$, as motivated in the text.

$$C_\pm \gamma_* = \begin{cases} \pm i C_\mp & D = 2,6,10 \\ C_\mp & D = 4,8,12 \end{cases} \tag{3.1}$$

$$\gamma_* C_\pm = \begin{cases} \pm i C_\mp & D = 2,6,10 \\ -C_\mp & D = 4,8,12 \end{cases} \tag{3.2}$$

$$B_\pm \gamma_* = \begin{cases} \pm i \sigma_+ \sigma_- B_\mp = \mp i B_\mp & D = 2,6,10 \\ \sigma_+ \sigma_- B_\mp = B_\mp & D = 4,8,12 \end{cases} \tag{3.3}$$

$$\gamma_* B_\pm = \begin{cases} (-1)^t \mp i \sigma_+ \sigma_- B_\mp = \pm (-1)^t i B_\mp & D = 2,6,10 \\ (-1)^t \sigma_+ \sigma_- B_\mp = (-1)^t B_\mp & D = 4,8,12 \end{cases} \tag{3.4}$$

$$B^*_\pm \gamma_* = \begin{cases} \pm i B^*_\mp & D = 2,6,10 \\ B^*_\mp & D = 4,8,12 \end{cases} \tag{3.5}$$

3.3 Complexified Spinor Modules

A generic supersymmetric theory involves $\mathcal{N}$ supercharges, which are the spinorial generators of a super-Poincaré algebra, $\mathfrak{g} = \mathfrak{so}(t,s) + \mathbb{R}^{t,s} + \mathfrak{s}$ as outlined in Section 2.7, where $\mathfrak{s}$ is an arbitrary sum of irreducible real spinor modules. This section describes the complexification of this sum of real spinor modules, which is dependent on the space-time dimension and signature. Eventually, we will use these complexified spinor modules to define a real supersymmetry algebra with manifest R-symmetry.

First, we begin in odd dimensions. As outlined in Section 2.5, recall that in odd dimensions, the real spinor module can be reducible or irreducible as a real module. However, in odd dimensions, one cannot define chiral spinors, and one finds that when the real spinor module is reducible $\mathbb{S} \cong S$ so we do not need to consider the reducible case separately.

The real spinor module is either equivalent to the complex spinor module, $\mathbb{S} \cong S$, or inequivalent, $\mathbb{S} \ncong S$. When the real and complex spinor module are equivalent, one cannot define a $\mathrm{Spin}(t,s)$-invariant real structure on $\mathbb{S}$. In signatures without a real structure, we will find we can always define a $\mathrm{Spin}(t,s)$-invariant quaternionic structure instead, this was motivated briefly in Section 2.6 and is detailed in further details in this chapter, Section 3.5.

Complexifying the real spinor module means taking

$$S \to S \otimes_{\mathbb{R}} \mathbb{C}. \tag{3.6}$$

For our two possibilities, either $\mathbb{S} \ncong S$ so that there is a real structure on $\mathbb{S}$ or $\mathbb{S} \cong S$ with no real structure on $\mathbb{S}$, the complexification of the spinor module is

$$S \otimes_{\mathbb{R}} \mathbb{C} \cong \begin{cases} \mathbb{S} & \text{Real structure exists,} \\ \mathbb{S} \otimes_{\mathbb{R}} \mathbb{C} & \text{No real structure.} \end{cases} \tag{3.7}$$

Where the second line is the complexification of the complex spinor module. Therefore when we complexify N copies of the real spinor module we obtain

$$S^{\oplus N} \to (S \otimes_{\mathbb{R}} \mathbb{C})^{\oplus N} \cong (S \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}^N \cong \begin{cases} \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^N & \text{Real structures exist,} \\ \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^{2N} & \text{No real structures.} \end{cases} \tag{3.8}$$

We will call the objects on the right the K-extended spinor modules, and refer to it in shorthand as $\mathbb{S} \otimes \mathbb{C}^K$ where it is understood that $K = N$ when $S \ncong \mathbb{S}$ and $K = 2N$ when $S \cong \mathbb{S}$.

In even dimensions the complex spinor module is reducible, i.e., Dirac spinors can be decomposed into Weyl spinors. The complex semi-spinor modules $\mathbb{S}_{\pm}$, also called the

Weyl spinor modules, are inequivalent and complex irreducible. The real spinor module can be irreducible or reducible, able to be decomposed into two real semi-spinor modules $S = S_+ + S_-$. These real semi-spinor modules may be equivalent, $S_+ \cong S_-$ or inequivalent, $S_+ \ncong S_-$.[1] Finally $\mathbb{S} \cong S$ or $\mathbb{S} \ncong S$, determined by whether one can define a $\mathrm{Spin}(t,s)$-invariant real structure on $\mathbb{S}$.

Therefore in even dimensions, we need to consider the complexification of

$$S^{\oplus N} \quad \text{and} \quad S_+^{\oplus N_+} \oplus S_-^{\oplus N_-},$$

the former when S is irreducible and the latter when S is reducible. The complexification of the first is the same as in odd dimensions:

$$S^{\oplus N} \to (S \otimes_{\mathbb{R}} \mathbb{C})^{\oplus N} \cong (S \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}^N \cong \begin{cases} \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^N & \text{Real structures exist,} \\ \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^{2N} & \text{No real structures.} \end{cases} \tag{3.9}$$

When S is reducible and the two real semi-spinor modules are isomorphic one finds that $S \cong S_\pm \otimes_{\mathbb{R}} \mathbb{C}$ and there never exists a real structure on $\mathbb{S}$ so that $S \cong \mathbb{S}$. Therefore, in this case, an arbitrary sum of irreducible spinor modules is

$$S_+^{\oplus N_+} \oplus S_-^{\oplus N_-} \cong S_+^{\oplus N_+ + N_-} \cong S_+^{\oplus N}. \tag{3.10}$$

In the last equation we have defined $N = N_+ + N_-$. The complexification of this is

$$S_+^{\oplus N} \to (S_+ \otimes_{\mathbb{R}} \mathbb{C})^{\oplus N} \cong S^{\oplus N} \cong \mathbb{S}^{\oplus N} \cong \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^K. \tag{3.11}$$

This is again of the form $\mathbb{S} \otimes \mathbb{C}^K$, like in odd dimensions.

When the two semi-spinor modules are inequivalent, there are two possibilities, either a real structure exists on $\mathbb{S}_\pm$ or not. If a real structure exists then $\mathbb{S}_\pm \cong S_\pm \otimes \mathbb{C}$, and if not $\mathbb{S}_\pm \cong S_\pm$. Therefore, $S_+^{\oplus N_+} \oplus S_-^{\oplus N_-}$ has the complexification

$$S_+^{\oplus N_+} \oplus S_-^{\oplus N_-} \to \begin{cases} \mathbb{S}_+ \otimes \mathbb{C}^{N_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{N_-} & \text{Real structures exist,} \\ \mathbb{S}_+ \otimes \mathbb{C}^{2N_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{2N_-} & \text{No real structures.} \end{cases} \tag{3.12}$$

Similarly to before we will abbreviate this to $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$, where $K_\pm = N_\pm$ when

[1] Recall, as in Chapter 2 Section 2.5 this is when the even Clifford algebra is simple.

there is a $\mathrm{Spin}_0(t,s)$-invariant structure on $\mathbb{S}$ and $K_\pm = 2N_\pm$ when there is not.

We see that there are two distinct types of K-extended spinor module: in odd dimensions, or even dimensions when the real spinor module is irreducible, or when the real spinor module is reducible, and the real semi-spinor modules are equivalent, the complex supercharges are elements of $\mathbb{S}\otimes\mathbb{C}^K$. In the remaining case, in even dimensions when the real semi spinor is reducible, and the real semi-spinor modules are inequivalent, the complex supercharges live on $\mathbb{S}_+\otimes\mathbb{C}^{K_+}\oplus\mathbb{S}_-\otimes\mathbb{C}^{K_-}$. To construct a superalgebra, we need a vector-valued bilinear form, so we now need to define a complex bilinear form on these complex spaces. From now on the $\mathbb{C}$ subscript will be omitted from the tensor product.

The quantity N is not necessarily what is often called $\mathcal{N}$, the 'number of supersymmetries.' In odd dimensions, we will pick the convention that $\mathcal{N} = K$; that is $\mathcal{N} = N$ when we have a spin-invariant real structure and $\mathcal{N} = 2N$ when we have no such real structure. In even dimensions we use a similar convention, classifying algebras as $(\mathcal{N}_+,\mathcal{N}_-)$, where $\mathcal{N}_\pm$ are the number of copies of each chiral spinor module, where $\mathcal{N}_\pm = K_\pm$ in the same way, or simply $\mathcal{N} = K_+ = K_-$ when we have equal numbers of both chiralities.

For example, in (1,4) signature the real and complex spinor module are isomorphic. Therefore the complexification of a single 'real' spinor module gives us $\mathbb{S}\otimes\mathbb{C}^2$ such that spinors come in SU(2) doublets. As a result, both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ are used. We prefer to use $\mathcal{N} = 2$ as two copies of the spinor module are used to define a superalgebra and they have the same number of supercharges as 4D $\mathcal{N} = 2$ theories. In (2,3), one can define a real structure, so the real and complex spinor module are distinct. The equivalent supersymmetry algebra would therefore only be called $\mathcal{N} = 2$ (this turns out to be the smallest possible supersymmetry algebra, with all superbrackets on a single copy of the real spinor module vanishing).

3.4 Bilinear Forms

To summarise the previous section; we work with complexified extended spinors that are elements of the spaces

$$\mathbb{S}\otimes\mathbb{C}^K \qquad \text{or} \qquad \mathbb{S}_+\otimes\mathbb{C}^{K_+}\oplus\mathbb{S}_-\otimes\mathbb{C}^{K_-}. \tag{3.13}$$

On these spaces, we require a $\mathrm{Spin}_0(t,s)$-invariant product. For this we will focus on bilinear forms (on complex spaces we could also use a sesquilinear form) as we are primarily interested in the complexification of real spinor modules taking a complex bilinear form on is the natural complex extension of a bilinear form on the real space.

We construct a bilinear form on these product spaces from tensor products of bilinear forms on each factor. The following section details our choices and conventions for bilinear forms on each factor individually and then specialises to those that can define a superbracket, and therefore a superalgebra.

3.4.1 Bilinear forms on the complex spinor module, $\mathbb{S}$

For a sesquilinear or bilinear form, β, on the complex spinor module, we define two invariants: the symmetry σ and the type τ

$$\beta(\lambda,\chi) = \sigma\beta(\chi,\lambda), \tag{3.14}$$

$$\beta(\gamma^\mu\lambda,\chi) = \tau\beta(\lambda,\gamma^\mu\chi). \tag{3.15}$$

An admissible bilinear form has $\sigma,\tau \in \{\pm 1\}$. Having a definite $\tau = \pm 1$ implies Spin_0 invariance: for a given γ-matrix (so that the following does not imply a sum over n)

$$\beta(\gamma^n\lambda,\gamma_n\chi) = \tau\beta(\gamma^n\gamma_n\lambda,\chi) = \tau\beta(\lambda,\chi). \tag{3.16}$$

We see that a single γ-matrix is an infinitesimal isometry or anti-isometry depending on the value of τ. Therefore for a spin generator (once again no sum over m,n)

$$\beta(\gamma^{mn}\lambda,\gamma^{mn}\chi) = \tau^2\beta(\gamma^{mn}\gamma^{nm}\lambda,\chi) = \beta(\lambda,\chi) \tag{3.17}$$

All admissible bilinear forms on the real and complex spinor module were defined in [1].

Recall in Section 2.6 we detailed the construction of a sesquilinear and bilinear form on the spinor module. The sesquilinear form, A, was given by

$$\begin{aligned} &A : \mathbb{S}\times\mathbb{S} \to \mathbb{C} \\ &A(\lambda,\chi) = \lambda^\dagger A\chi = \lambda^*_\alpha A^{\alpha\beta}\chi_\beta \end{aligned} \tag{3.18}$$

We will refer to this as the Dirac sesquilinear form (though it will not be used often).

The complex bilinear form was defined by

$$\begin{aligned} &C : \mathbb{S} \times \mathbb{S} \to \mathbb{C} \\ &C(\lambda, \chi) = \lambda^T C \chi = \lambda_\alpha C^{\alpha\beta} \chi_\beta . \end{aligned} \tag{3.19}$$

This will be referred to as the Majorana bilinear form. The Gram matrix of this bilinear form, also called C, is commonly known as the charge conjugation matrix. The symmetry of the bilinear form is equal to the symmetry of the Gram matrix C, such that $C^T = \sigma C$.

In odd dimensions, there is a unique choice of C (up to equivalence) with a definite σ and τ, and in even dimensions, we have two, one with each value of τ. They are conventionally known as $C_{-\tau}$. The symmetry of $C_\pm$ will be called $\sigma_\pm$. Knowing that γ_* anti-commutes with all γ-matrices one finds

$$C_\pm \gamma_* \propto C_\mp . \tag{3.20}$$

It is always possible to choose a basis where $C = C^\dagger = C^{-1}$, for both C's simultaneously. This, along with our previous choice of $\gamma_* = (-i)^{\frac{D}{2}+t} \gamma_1 ... \gamma_D$ implies we can assume

$$C_\pm \gamma_* = \begin{cases} \pm i C_\mp & \text{D = 2, 6, 10} \\ C_\mp & \text{D = 4, 8, 12} \end{cases} \tag{3.21}$$

Locking in a basis is not strictly required, but it is useful for the explicit formulas given later for examples of isomorphisms between superalgebras.

We stress that at this point this is a complex-valued bilinear form, and will only be real after the imposition of a reality condition.

Given a bilinear/sesquilinear form, say $\beta(\cdot,\cdot)$, we can insert elements of the Clifford algebra in the first argument to obtain tensorial quantities

$$\begin{aligned} \beta^p : \mathbb{S} \times \mathbb{S} \to T^p, \quad \beta^p(\cdot,\cdot) = \beta(\gamma^{(p)} \cdot, \cdot), \\ \text{e.g.} \quad \beta^1 : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^D \otimes \mathbb{C} \cong \mathbb{C}^D, \quad \beta^1(\cdot,\cdot) = \beta(\gamma^\mu \cdot, \cdot). \end{aligned} \tag{3.22}$$

Where $T^p = (\mathbb{C}^D)^{\otimes p}$ are complex-valued tensors of rank p. The second quantity, the vector-valued bilinear/sesquilinear form, is of particular relevance as it will be used to define the Lie bracket between the supercharges, and thus a super-Poincaré algebra. This is dealt with in Section 3.6.

The symmetry of the rank-p tensor-valued bilinear forms are given by

$$\beta(\gamma^{(p)}\lambda,\chi) = \sigma\beta(\chi,\gamma^{(p)}\lambda) = (-1)^{\frac{p(p-1)}{2}}\sigma\tau^{p}\beta(\gamma^{(p)}\chi,\lambda) \tag{3.23}$$

Where the $(-1)^{\frac{p(p-1)}{2}}$ factor has come from rearranging the indices of $\gamma^{(p)}$. For the vector-valued bilinear form the coefficient reduces to simply $\sigma\tau$; vector-valued bilinear forms with $\sigma\tau = 1$ are symmetric.

We will restrict to using the Majorana bilinear forms exclusively, making the description as 'Majorana-like' as possible. As outlined in greater detail later Section 3.10, when using complex-valued bilinear forms on the complex spinor module, the choice of bilinear form is mostly irrelevant. In odd dimensions, Schur's lemma tells us invariance group is the trivial group because the spinor module is complex-irreducible. When we take multiple copies of the spinor module, we end up with spaces of the type $\mathbb{S}\otimes\mathbb{C}^K$, and any transformations are therefore restricted to the $\mathbb{C}^K$ factor. Our choice of bilinear form on $\mathbb{S}$ is therefore almost trivial in odd dimensions. In even dimensions, the spinor module is not complex-irreducible, but this leads to groups that act infinitesimally as only Id or γ_* on the $\mathbb{S}$ factor.

In this chapter, the extended complex spinor modules arise from the complexification of an arbitrary sum of real spinor modules, so working with a complex bilinear form is the natural complex extension of this. Additionally, a bilinear form is easier to use when working with a reality condition, as we do not have to undo the complex conjugation in the sesquilinear form. The form of the Majorana bilinear form is only dimension-dependent and does not change with signature unlike the Dirac sesquilinear form A. The 'Majorana flip properties' are well known, and this makes dealing with spinor bilinears easy in Lagrangian descriptions of the theory. Also, the use of C makes calculations involving the reality conditions easier as these involve B, which has various easily calculable relations with C and γ_*.

3.4.2 Bilinear forms on the Weyl Spinor modules, $\mathbb{S}_\pm$

As we have settled on using the Majorana bilinear forms on $\mathbb{S}$, we will now detail their behaviour on the complex semi-spinor modules, also known as the Weyl spinor modules. As we are working with Weyl spinors, we are in even dimensions where we have access to two potential Majorana bilinear forms.

Equivalence of C_+ and C_-

As was mentioned in the Section 3.4.1, the two charge conjugation matrices are proportional, obeying (3.1). Recalling that $\gamma_*\lambda_\pm = \pm\lambda_\pm$, for $\lambda_\pm \in \mathbb{S}_\pm$, we can extend this proportionality to the Majorana bilinear forms acting on $\mathbb{S}_\pm$

$$C_{(\pm)}(\cdot, \mathbb{S}_\pm) = C_{(\pm)}(\cdot, \pm\gamma_*\mathbb{S}_\pm) \propto C_{(\mp)}(\cdot, \mathbb{S}_\pm). \tag{3.24}$$

Here the bracketed signs are not linked to the unbracketed signs. The first argument can be either Weyl spinor module, $\mathbb{S}_+$ or $\mathbb{S}_-$,[2] though we will find in a given spacetime dimension the bilinear form will vanish with one of these. As the first argument is arbitrary, this proportionality holds for the tensor-valued bilinear forms that add an element of the Clifford algebra into the first argument. Using (3.1) we find the proportionality to be

$$C_{(\pm)}(\cdot, \mathbb{S}_\pm) = \begin{cases} iC_{(\mp)}(\cdot, \mathbb{S}_\pm) & \mathrm{D} = 2,\ 6,\ 10, \\ \pm C_{(\mp)}(\cdot, \mathbb{S}_\pm) & \mathrm{D} = 4,\ 8,\ 12. \end{cases} \tag{3.25}$$

Once again, these are basis dependent, but we have chosen to work with the basis outlined earlier that provide as helpful proportionality. We will use these relations to demonstrate features of Weyl spinors and later for isomorphisms between superalgebras.

Restriction of bilinear form to Weyl spinor Modules

$C_\pm$ is a bilinear form on $\mathbb{S}$ and we wish to see how it works under the decomposition $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$. We have four restrictions of the bilinear form (and all those derived from it through the insertion of Clifford algebra elements) to consider

$$C(\mathbb{S}_+, \mathbb{S}_+), \qquad C(\mathbb{S}_+, \mathbb{S}_-), \qquad C(\mathbb{S}_-, \mathbb{S}_+), \qquad C(\mathbb{S}_-, \mathbb{S}_-).$$

[2]Note that the complex semi-spinor modules are always inequivalent so that the complex bilinear forms on the K-extended complexified spinor module can be expressed in terms of $\lambda_\pm \in \mathbb{S}_\pm$ without redundancy regardless of signature.

For a particular dimension, as the properties of C do not depend on the signature, either the homogeneous or mixed terms will be zero for both bilinear forms. This is to be expected as the Majorana bilinear forms are proportional when restricted to Weyl spinors.

In Section 3.6 we will define a superbracket using the vector-valued bilinear form – and in doing so a super-Poincaré algebra– so we will detail this calculation first. Consider the orthogonal vector-valued bilinear forms

$$\begin{aligned} & C_+(\gamma^\mu \lambda_\pm, \chi_\pm) = kC_-(\gamma^\mu \lambda_\pm, \chi_\pm) \\ \implies & \sigma_+\tau_+C_+(\gamma^\mu \chi_\pm, \lambda_\pm) = k\sigma_-\tau_-C_-(\gamma^\mu \chi_\pm, \lambda_\pm) \\ \implies & \sigma_+\tau_+C_+(\gamma^\mu \chi_\pm, \lambda_\pm) = \sigma_-\tau_-C_+(\gamma^\mu \chi_\pm, \lambda_\pm) \end{aligned} \tag{3.26}$$

Where $k = 1$ or $k = i$ dependent on dimension according to (3.1). By definition $\tau_+ = -\tau_- = -1$ so we obtain

$$\sigma_+C_+(\gamma^\mu \chi_\pm, \lambda_\pm) = -\sigma_-C_+(\gamma^\mu \chi_\pm, \lambda_\pm) \tag{3.27}$$

We see that only when $\sigma_+ = -\sigma_-$ are the vector-valued Majorana bilinear forms entirely orthogonal, i.e. non-zero on $\mathbb{S}_+$ or $\mathbb{S}_-$ alone.

On the mixed chirality vector-valued bilinear forms we obtain the opposite sign

$$\begin{aligned} & C_+(\gamma^\mu \lambda_\pm, \chi_\mp) = kC_-(\gamma^\mu \lambda_\pm, \chi_\mp) \\ \implies & \sigma_+\tau_+C_+(\gamma^\mu \chi_\mp, \lambda_\pm) = \sigma_-\tau_-kC_-(\gamma^\mu \chi_\mp, \lambda_\pm) \\ \implies & \sigma_+C_+(\gamma^\mu \chi_\mp, \lambda_\pm) = \sigma_-C_+(\gamma^\mu \chi_\mp, \lambda_\pm) \end{aligned} \tag{3.28}$$

We can see that only when $\sigma_+ = \sigma_-$ can the final line hold. Both conditions cannot be satisfied simultaneously, so we see that the restriction of the vector-valued bilinear form is either orthogonal (homogeneous terms only, mixed terms are zero) or isotropic (the opposite). This is the 'isotropy' of the vector-valued bilinear form. In $D = 2, 6, 10$ we have $\sigma_+ = -\sigma_-$ so that these signatures are always orthogonal (allowing the definition of Type IIA and IIB string theory, for example) and $D = 4, 8, 12$ have $\sigma_+ = \sigma_-$ and thus are always isotropic.

The isotropy of the vector-valued bilinear form does not hold universally for all tensor-valued bilinear forms (those with first argument $\gamma^{(p)}\lambda$, for a general element of the

Clifford algebra with p indices). Consider the orthogonal scalar-valued bilinear form,

$$\begin{aligned} C_+(\lambda_\pm, \chi_\pm) &= kC_-(\lambda_\pm, \chi_\pm) \\ \implies \sigma_+ C_+(\chi_\pm, \lambda_\pm) &= \sigma_- C_+(\chi_\pm, \lambda_\pm) \end{aligned} \tag{3.29}$$

Opposite to the vector-valued bilinear form, the orthogonal scalar-valued terms are only non-zero if $\sigma_+ = \sigma_-$ (the mixed terms are non-zero when $\sigma_+ = -\sigma_-$).

Generalising this to the insertion of any element of the Clifford algebra in the first argument

$$\begin{aligned} C_+(\gamma^{(p)}\lambda_\pm, \chi_\pm) &= kC_-(\gamma^{(p)}\lambda_\pm, \chi_\pm) \\ \implies \sigma_+\tau_+^p C_+(\gamma^{(p)}\chi_\pm, \lambda_\pm) &= \sigma_-\tau_-^p C_-(\gamma^{(p)}\chi_\pm, \lambda_\pm) \\ \implies (-1)^p\sigma_+ C_+(\gamma^{(p)}\chi_\pm, \lambda_\pm) &= \sigma_- C_+(\gamma^{(p)}\chi_\pm, \lambda_\pm) \end{aligned} \tag{3.30}$$

And we obtain the opposite sign for the mixed-chirality terms.

We can therefore define the isotropy ι_p of the rank-p tensor-valued Majorana bilinear form [3]

$$\iota_p = (-1)^p \sigma_+ \sigma_- \tag{3.31}$$

We see that isotropy of the rank-p tensor-valued Majorana bilinear form alternates. ι_0 is the same as ι in [1], which is the isotropy of the scalar-valued bilinear forms. Because isotropy alternates, supersymmetric theories with chiral superalgebras cannot have chiral mass terms.

Differing slightly from their conventions, we will call Majorana bilinear forms with $\iota_1 = +1$ orthogonal bilinear forms and those with $\iota_1 = -1$ isotropic bilinear forms, though strictly it is the vector-valued bilinear form that is orthogonal or isotropic. We are primarily concerned with defining superalgebras, and this depends only on the properties of the vector-valued bilinear form, so this naming convention is convenient. As the properties of the Majorana bilinear forms are dimension dependent, we will sometimes call the dimensions with an orthogonal/isotropic (vector-valued) bilinear form 'orthogonal/isotropic dimensions' too.

[3] previous equation leads to $\frac{\sigma_\pm}{\sigma_-}$ but recall that $\sigma = \pm 1$ s.t. $\sigma^{-1} = \sigma$

ι_p provides a quick guide for permissible terms in a physical theory. Provided the terms are not already zero according to (3.23), theories with $\iota_p = 1$ can have entirely chiral terms proportional to

$$\bar{\lambda}_{\pm}\gamma^{(p)}\chi_{\pm} \neq 0, \quad \bar{\lambda}_{\pm}\gamma^{(p)}\chi_{\mp} = 0 \quad \text{iff} \quad \iota_p = +1 \tag{3.32}$$

For example if $\iota_0 = 1$ one can have chiral mass terms provided the scalar-valued bilinear form is symmetric (else $\bar{\lambda}\lambda = 0$). Those with $\iota_1 = 1$ can be used to define a chiral superalgebra (provided the superbracket is non-vanishing, which will be discussed in 3.6) and have entirely chiral kinetic terms.

If $\iota_p = -1$ then only mixed-chirality terms are possible,

$$\bar{\lambda}_{\pm}\gamma^{(p)}\chi_{\pm} = 0, \quad \bar{\lambda}_{\pm}\gamma^{(p)}\chi_{\mp} \neq 0, \quad \text{iff} \quad \iota_p = -1. \tag{3.33}$$

For our purposes the most important case is when $\iota_1 = -1$, then we require both chiralities in equal number to define a superalgebra otherwise the superbracket vanishes.

3.4.3 Bilinear forms on $\mathbb{C}^K$

We now wish to define bilinear forms on the other factor of the extended spinor module $\mathbb{S} \otimes \mathbb{C}^K$. Recall that $K = N$ or $2N$ depending on whether we have access to a real structure on the complex spinor module, as outlined in Section 3.3. Once again we only wish to consider bilinear forms as they are most compatible with implementing a real structure on $\mathbb{S} \otimes \mathbb{C}^K$.

Our model complex bilinear form will be called M, and our index conventions are as follows

$$\begin{aligned} &M : \mathbb{C}^K \times \mathbb{C}^K \to \mathbb{C} \\ &M(w, z) = w^i z^j M_{ji} \qquad i, j = 1, ..., M \end{aligned} \tag{3.34}$$

This index convention is chosen to replicate the usual $NW-SE$ convention for symplectic Majorana spinors. We will only consider bilinear forms with a definite symmetry, and this symmetry is encoded as σ_M,

$$M(w,z) = \sigma_M M(z,w), \qquad M_{ij} = \sigma_M M_{ji}, \qquad \sigma_M = \pm 1. \tag{3.35}$$

We therefore have two cases, symmetric and antisymmetric bilinear forms. Given a symmetric bilinear form on $\mathbb{C}^K$ it is always possible to reparameterise $\mathbb{C}^K$ such that $M_{ij} = \delta_{ij}$, the $K \times K$ identity matrix. This bilinear form will be called $\delta(\cdot,\cdot)$ or just δ, referencing the Gram matrix δ_{ij}.

Similarly given an antisymmetric bilinear form it is always possible to reparameterise $\mathbb{C}^K$ such that $M_{ij} = (J_K)_{ij}$ given by

$$(J_K)_{ij} = \begin{cases} \begin{pmatrix} 0 & \mathbb{1}_k \\ -\mathbb{1}_k & 0 \end{pmatrix} & K = 2k, \\ \begin{pmatrix} 0 & \mathbb{1}_k & 0 \\ -\mathbb{1}_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & K = 2k+1. \end{cases} \tag{3.36}$$

$(J_2)_{ij}$ is the Levi-Civita symbol ε_{ij}. Often the K subscript will be omitted when the context is clear. When K is odd, the resulting bilinear form is degenerate, effectively removing one factor of $\mathbb{C}$. Therefore when working with an antisymmetric bilinear form on $\mathbb{C}^K$ we will only consider even values of K. The bilinear form represented by the Gram matrix $(J_K)_{ij}$ will be called $J(\cdot,\cdot)$ or just J.

Invariance Group

The group acting on $\mathbb{C}^K$ under which these bilinear forms are invariant will be called $G_{\mathbb{C}^K}$. For the two choices of M these are, by definition

$$G_{\mathbb{C}^K} = \begin{cases} \mathrm{O}(K,\mathbb{C}) & M = \delta, \\ \mathrm{Sp}(K,\mathbb{C}) & M = J. \end{cases} \tag{3.37}$$

These will be used to calculate the R-symmetry group, which will be subgroups of these two groups (or products of subgroups). In many cases, the R-symmetry group will be a real form of these groups with different signatures able to realise different real forms.

3.4.4 Bilinear forms on K-extended Spinor Module

Odd dimensions

In odd dimensions, the complexified spinor module is always of the form $\mathbb{S} \otimes \mathbb{C}^K$. A bilinear form on $\mathbb{S} \otimes \mathbb{C}^K$ can be made from the tensor product of a bilinear form on each factor. As a shorthand, this will often be called $\beta = C \otimes M$ and is defined by

$$\begin{aligned} &\beta : (\mathbb{S} \otimes \mathbb{C}^K) \times (\mathbb{S} \otimes \mathbb{C}^K) \to \mathbb{C}, \\ &\beta(\lambda^i, \chi^i) = (\lambda^i)^T C \chi^j M_{ji}. \end{aligned} \tag{3.38}$$

The symmetry of the tensor product is a product of the symmetries of each bilinear form. The type is inherited from the bilinear form C on $\mathbb{S}$ as the γ-matrices do not touch the $\mathbb{C}^K$ factor.

$$\beta(\lambda^i, \chi^i) = \sigma_C \sigma_M \beta(\chi^i, \lambda^i) \quad \sigma_\beta = \sigma_C \sigma_M, \tag{3.39}$$

$$\beta(\gamma^\mu \lambda^i, \chi^i) = \tau_C \beta(\lambda^i, \gamma^\mu \chi^i) \quad \implies \tau_\beta = \tau_C. \tag{3.40}$$

From this, we realise that regardless of the spacetime dimension (which mandates the value of σ_C) we can have a bilinear form on $\mathbb{S} \otimes \mathbb{C}^K$ with either symmetry value by selecting M. This will be necessary for the following section where we will define superalgebras using symmetric vector-valued bilinear form.

We can then use this to build rank-p tensor-valued bilinear forms β^p:

$$\begin{aligned} &\beta^p : (\mathbb{S} \otimes \mathbb{C}^K) \times (\mathbb{S} \otimes \mathbb{C}^K) \to T^p, \\ &\beta(\gamma^{(p)} \lambda^i, \chi^i) = (\gamma^{(p)} \lambda^i)^T C \chi^j M_{ji}. \end{aligned} \tag{3.41}$$

To build a superbracket we only need the vector-valued bilinear form, β^1, and will focus on this from now on.

Even Dimensions

In the cases where the real spinor module is irreducible, we work with spinors that live on $\mathbb{S} \otimes \mathbb{C}^K$ that work identically to odd dimensions.

In signatures with a reducible real spinor module we work with spinors that are elements of $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$. As we have shown the two (vector-valued) Majorana bilinear forms are orthogonal or isotropic on $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$, determined by ι_1. This quantity is unaffected by the $\mathbb{C}^K$ factor.

With an orthogonal tensor-valued bilinear form we can have $K_+ \neq K_-$. In these cases, the bilinear form on $\mathbb{C}^{K_\pm}$ will be called $M_\pm$. We can define the bilinear forms on each Weyl spinor module individually, we write

$$\beta_+^p : (\mathbb{S}_+ \otimes \mathbb{C}^{K_+}) \times (\mathbb{S}_+ \otimes \mathbb{C}^{K_+}) \to T^p, \tag{3.42}$$
$$\beta_+(\gamma^{(p)}\lambda_+^i, \chi_+^i) = (\gamma^{(p)}\lambda_+^i)^T C \chi_+^j M_{+ji},$$
$$\beta_-^p : (\mathbb{S}_- \otimes \mathbb{C}^{K_-}) \times (\mathbb{S}_- \otimes \mathbb{C}_-^K) \to T^p, \tag{3.43}$$
$$\beta_-(\gamma^{(p)}\lambda_-^i, \chi_-^i) = (\gamma^{(p)}\lambda_-^i)^T C \chi_-^j M_{-ji}.$$

For notational clarity we omitted the subscript from C, though it could be either choice, C_+ or C_-, available in even dimensions [4]. The total rank-p tensor-valued bilinear form on $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$ is then $\beta_+^p \oplus \beta_-^p$.

For an isotropic tensor-valued Majorana bilinear form we necessarily need the same number of each chirality. The rank-p tensor-valued bilinear form is

$$\beta^p : (\mathbb{S}_\pm \otimes \mathbb{C}^K) \times (\mathbb{S}_\mp \otimes \mathbb{C}^K) \to T^p, \tag{3.44}$$
$$\beta(\gamma^{(p)}\lambda_\pm^i, \chi_\mp^i) = (\gamma^{(p)}\lambda_\pm^i)^T C \chi_\mp^j M_{ji}.$$

The complex spinors are elements of $\mathbb{S}_+ \otimes \mathbb{C}^K \oplus \mathbb{S}_+ \otimes \mathbb{C}^K$. It is natural to combine the Weyl spinors into Dirac spinors, $\lambda^i = \lambda_+^i + \lambda_-^i$, so that one works with $\mathbb{S} \otimes \mathbb{C}^K$. For even dimensions with isotropic bilinear forms ($D = 4, 8, 12, ...$) we will therefore construct superalgebras with supercharges that are elements of the K-extended spinor modules $\mathbb{S} \otimes \mathbb{C}^K$ regardless of whether the real spinor module is reducible or irreducible.

3.5 ϵ-quaternionic structures

To define a physical theory, we need a real supersymmetry algebra, but up to this point, we have only defined a complex bilinear form that would produce a complex-valued su-

[4]Note, we could have a different Majorana bilinear form on each Weyl spinor module though from (3.25) we can see they are proportional up to a factor of i which can be removed (as shown in 3.10.2) so the choice of C is irrelevant.

peralgebra. Therefore we need a real structure on the complexified (extended) spinor module. Taking the real or imaginary part of the bilinear forms defined above is an option, though it is one that does not lead to manifestly R-symmetric spinors. Instead we use a reality condition on $\mathbb{S}\otimes\mathbb{C}^K$ or $\mathbb{S}_+\otimes\mathbb{C}^{K_+}\oplus\mathbb{S}_-\otimes\mathbb{C}^{K_-}$; only considering the elements that are invariant under said real structure. A widely-used example of this is Majorana and symplectic Majorana spinors, the following section generalise these concepts and gives a prescription on how to define real structures on the extended spinor module in any signature and dimension.

Real structures on $\mathbb{S}\otimes\mathbb{C}^K$ and $\mathbb{S}_+\otimes\mathbb{C}^{K_+}\oplus\mathbb{S}_-\otimes\mathbb{C}^{K_-}$ will be made from the product of two real or two quaternionic structures on each factor, so this section follows a similar plan to the previous where we define the necessary parts on each factor then on the total product space.

3.5.1 ϵ-quaternionic structures on the complex spinor module, $\mathbb{S}$

From the matrices A and C in Section 3.4 we define a new matrix B

$$B = (CA^{-1})^T. \tag{3.45}$$

In a given odd-dimensional signature this is a unique choice, as there is a single A and C. There are two possible choices for C in even dimensions and hence two possible choices for B, which will called $B_{-\tau} = (C_{-\tau}A^{-1})^T$.

With B we can define a one-parameter family of $\mathrm{Spin}_0(t,s)$-invariant real or quaternionic structures on $\mathbb{S}$:

$$J^{(\varepsilon)(\alpha)} : \lambda \to \alpha^* B^* \lambda^*, \qquad |\alpha| = 1. \tag{3.46}$$

A real or quaternionic structure is an anti-linear involution that squares to $\epsilon = +1$ or -1 respectively. $J^{(+1)(\alpha)}$ is a real structure, and $J^{(-1)(\alpha)}$ is a quaternionic structure. The value of ϵ is controlled by the product $B^*B = \epsilon$, such that a $J^{(\epsilon)(\alpha)}$ is a real structure when $B^*B = +1$ and a quaternionic structure when $B^*B = -1$. The form and properties of B are signature dependent, it can be shown that

$$B^*B = \sigma(-\tau)^t(-1)^{t(t+1)/2}. \tag{3.47}$$

To make the text more legible, the (α) superscript will sometimes be omitted when the phase is unimportant.

Along with the natural complex structure on $\mathbb{S}$ given by multiplication by i, that we will call I, $J^{(-1)}$ defines another complex structure that anticommutes with I:

$$IJ(\lambda) = i\alpha^* B^* \lambda^*, \quad JI(\lambda) = J(i\lambda) = -i\alpha^* B^* \lambda^*. \tag{3.48}$$

I and J are $\mathrm{Spin}_0(t,s)$-invariant endomorphisms and are therefore in the Schur algebra of the complex spinor module, $\mathcal{C}(\mathbb{S})$. Defining $K = IJ^{(-1)} \in \mathcal{C}(\mathbb{S})$ we see that $I, J^{(-1)}, K$ pair-wise anti-commute and $K^2 = -1$ such that together I and $J^{(-1)}$ generate an algebra isomorphic to $\mathbb{H}$.

Similarly, in signatures with I and $J^{(+1)}$ we define $K = IJ^{(+1)}$ and obtain an algebra isomorphic to $\mathbb{H}' \cong \mathbb{R}(2)$. $I, J^{(+1)}, K$ are all $\mathrm{Spin}_0(t,s)$-invariant and thus are contained in the Schur algebra. Indeed from Section 2.8, we saw the Schur algebra in odd dimensions can only be $\mathbb{H}$ or $\mathbb{H}'$ so in this section we have derived all possibilities, having derived the form of the Schur algebra elements in a language more familiar to physicists.

As I always exists, we refer to $J^{(\varepsilon)}$ as an ϵ-quaternionic structure rather than just a complex and real structure, as together they generate an algebra $\mathbb{H}_\epsilon$. Recall a -1-quaternion is a regular quaternion, and a $+1$-quaternion is a para-quaternion.

In a physical theory, the phase of the reality condition, α, is not free. It is chosen so that the vector-valued bilinear form is real. For more details, see Section 3.6.

3.5.2 ϵ-quaternionic structures in even dimensions

In even dimensions, we have two possible charge conjugation matrices and two corresponding $\mathrm{Spin}_0(t,s)$ invariant ϵ-quaternionic structures

$$J_\pm^{(\epsilon)(\alpha)} : \lambda \to \alpha^* B_\pm^* \lambda^*, \qquad |\alpha| = 1. \tag{3.49}$$

The subscript on $J_\pm^{(\epsilon)(\alpha)}$ refers to $B_\pm$ being used to define the structure. Later, we will use different numbers of each Weyl spinor module, when this is done the particular α on each chirality will possess a subscript $\alpha_\pm$ as we do not mandate the structure to act the same on each Weyl spinor module.

Along with I, $J_+^{(\epsilon)}$ and $J_-^{(\epsilon)}$ generate a larger Schur algebra in even dimensions. If both J_+ and J_- have the same value of ϵ then this is isomorphic to $2\mathbb{H}_\epsilon$. If they have different values of ϵ, one obtains a Schur algebra isomorphic to $\mathbb{C}(2)$ (which contains both the quaternions and para-quaternions as sub-algebras but as they overlap on $\mathbb{C} \subset \mathbb{C}(2)$ it is not $\mathbb{H} + \mathbb{H}'$). This exhausts all possibilities for $\mathcal{C}(\mathbb{S})$ in even dimensions.

In the following table the type of structures given by $J^{(\epsilon)}$ are described in physically relevant signatures. There is a natural $(t,s) \leftrightarrow (s,t)$ symmetry, though in even dimensions one must also replace $J_{-\tau}^{(\epsilon)}$ with $J_\tau^{(\epsilon)}$. For example, if we have (t,s) $J_-^{(-1)}$ in signature (s,t) there will be a quaternionic structure $J_+^{(-1)}$. See Appendix 3.13.2.

D	$(0,D)$	$(1,D-1)$	$(2,D-2)$	$(3,D-3)$	$(4,D-4)$	$(5,D-5)$	$(6,D-6)$
1	$+1$	$+1$					
2	$-1_+,+1_-$	$+1_+ + 1_-$	$+1_+,-1_-$				
3	-1	$+1$	$+1$	-1			
4	$-1_+,-1_-$	$+1_+,-1_-$	$+1_+,+1_-$	$-1_+,+1_-$	$-1_+,-1_-$		
5	-1	-1	$+1$	$+1$	-1	-1	
6	$+1_+,-1_-$	$-1_+,-1_-$	$-1_+,+1_-$	$+1_+,+1_-$	$+1_+,-1_-$	$-1_+,-1_-$	$-1_+,+1_-$
7	$+1$	-1	-1	$+1$	$+1$	-1	-1
8	$+1_+,+1_-$	$-1_+,+1_-$	$-1_+,-1_-$	$+1_+,-1_-$	$+1_+,+1_-$	$-1_+,+1_-$	$-1_+,-1_-$
9	$+1$	$+1$	-1	-1	$+1$	$+1$	-1
10	$-1_+,+1_-$	$+1_+,+1_-$	$+1_+,-1_-$	$-1_+,-1_-$	$-1_+,+1_-$	$+1_+,+1_-$	$+1_+,-1_-$
11	-1	$+1$	$+1$	-1	-1	$+1$	$+1$
12	$-1_+,-1_-$	$+1_+,-1_-$	$+1_+,+1_-$	$-1_+,+1_-$	$-1_+,-1_-$	$+1_+,-1_-$	$+1_+,+1_-$

Table 3.1: Entries are the values of ϵ for each signature-dependent $J^{(\epsilon)}$. In even dimensions the sign subscript corresponds to the ϵ-value of $J_\pm^{(\epsilon)}$.

If $J_\pm^{(\epsilon)}$ is a real or quaternionic structures on $\mathbb{S}$ it is not necessarily a real or quaternionic structure on the Weyl spinor modules $\mathbb{S}_\pm$ alone. As $(J_\pm^{(\epsilon)})^2(\lambda) = \pm\lambda$, for $\lambda \in \mathbb{S}$, we have two possibilities

$$J_\pm^{(\epsilon)}(\mathbb{S}_{(\pm)}) = \mathbb{S}_{(\pm)}, \quad J_\pm^{(\epsilon)}(\mathbb{S}_{(\pm)}) = \mathbb{S}_{(\mp)}. \tag{3.50}$$

Only the bracketed signs on $\mathbb{S}_\pm$ are linked, with the non-bracketed signs on $J_\pm^{(\epsilon)}$ being unrelated. In the first case, $J_\pm^{(\epsilon)}$ will be called a Weyl-compatible structure and the second case it will be called Weyl-incompatible, and similarly we will refer to the signatures themselves as Weyl-compatible and Weyl-incompatible as we will show it is a signature-dependent quality. In Weyl-compatible signatures the Weyl spinor modules are self conjugate, i.e $(\mathbb{S}_\pm)^* = \mathbb{S}_\pm$.

In Weyl compatible signatures the real spinor module is reducible and the real semi-spinor modules are inequivalent. In Weyl incompatible signatures the real spinor is either irreducible, or it is reducible and the real-semi spinors are equivalent. Therefore Weyl-compatibility is required to work with entirely chiral theories or theories with an arbitrary number of each chirality. If the signature is Weyl-incompatible the complexified spinors are elements of $\mathbb{S} \otimes \mathbb{C}^K$.

In 3.4.2 it was shown that only orthogonal bilinear forms can be defined on $\mathbb{S}_\pm \otimes \mathbb{C}^K$ alone, therefore to construct a chiral superalgebra we require both an orthogonal bilinear form and a Weyl-compatible ϵ-quaternionic structure.

Isotropic bilinear forms require an equal number of spinors of both chiralities that can naturally be combined into Dirac spinors. With a Weyl-compatible reality condition, one could define a different reality condition on $\mathbb{S}_+$ and $\mathbb{S}_-$, so that they cannot be combined into a single reality condition on $\mathbb{S}$. However, in this thesis, when using an isotropic bilinear form, we will only require the reality condition to be the same on both chiralities, allowing one to work with Dirac supercharges and fermions. The complex supercharges are then elements of $\mathbb{S} \otimes \mathbb{C}^K$.

If $J_\pm^{(\epsilon)}$ is an ϵ-quaternionic structure then $B_\pm^* B_\pm = \epsilon$. The properties of B are signature and dimension dependent, and we find that in signature (t,s), with $D = t+s$,

$$B_+^* B_+ = \begin{cases} iB_-^*\gamma_* B_+ & = (-1)^{t+1} B_-^* B_- \qquad D = 2,6,10, \\ B_-^*\gamma_* B_+ & = (-1)^t B_-^* B_- \qquad D = 4,8,12. \end{cases} \tag{3.51}$$

This can be found using (3.4) and (3.5) from the Useful Formulae found at the start of this chapter. We see that in the orthogonal dimensions, $D = 2,6,10$, $J_\pm^{(\epsilon)}$ are both real or quaternionic structures, i.e. $(J_+^{(\epsilon)})^2 = (J_-^{(\epsilon)})^2 = \epsilon$, when the number of time-like direction, t, is odd (for example in Minkowski signature in 10 dimensions), and in isotropic dimensions, $D = 4,8,12$, they are the same type of structure when t is even.

Further, if they are both real/quaternionic structures, they are necessarily Weyl-compatible. To see this we remark that Weyl-compatibility means that $B_\pm$ commutes with γ_*, and this implies

$$B_+\gamma_* = \gamma_* B_+ \implies \begin{cases} B_- = (-1)^{t+1}B_- & D = 2,6,10, \\ B_- = (-1)^t B_- & D = 4,8,12. \end{cases} \tag{3.52}$$

Here we used (3.3) and (3.4) from the Useful Formulae. By inspection the equation for B_- only makes sense when t is odd in the orthogonal dimensions t is even in the isotropic dimensions – these two criteria correspond exactly with the requirement that both $J_\pm^{(\epsilon)}$ are real or quaternionic.

Weyl-compatibility is a property of the spacetime signature and it alternates as we increment t. As we change signatures the matrix A gains or loses time-like γ-matrices and so does $B = (CA^{-1})^T$. Therefore changing from t to $t \pm 1$ means there is one extra/fewer γ-matrix in B, changing whether γ_* (anti)commutes with B, causing the structures $J_\pm^{(\epsilon)}$ to alternate between being Weyl-compatible and incompatible. Further, Weyl-compatibility(-incompatibility) implies that both $J_+^{(\epsilon)}$ and $J_-^{(\epsilon)}$ on $\mathbb{S}$ have the same (opposite) value for ϵ.

Finally, $J_+^{(\epsilon)}$ and $J_-^{(\epsilon)}$ are proportional on the Weyl spinor modules

$$J_+^{(\epsilon)(\alpha)}(\lambda_\pm) = \begin{cases} \alpha^* B_+^* \lambda_\pm^* = i\alpha^* B_-^* \lambda_\pm^* = J_-^{(\epsilon)(i\alpha)}(\lambda_\pm) & D = 2,6,10 \\ \alpha^* B_+^* \lambda_\mp^* = \mp\alpha^* B_-^* \lambda_\pm^* = J_-^{(\epsilon)(\mp\alpha)}(\lambda_\pm) & D = 4,8,12 \end{cases} \tag{3.53}$$

To obtain this, we used (3.5). We see the two ϵ-quaternionic structures are proportional, with modified phases depending on the dimension.

3.5.3 ϵ-quaternionic structures on $\mathbb{C}^K$

Next, we need to define an ϵ-quaternionic structure on $\mathbb{C}^K$. Our conventions will be the following, once again tailored to the $NW - SE$ convention,

$$j^{(\epsilon)} : z^i \to (z^j)^* L_{ji} \tag{3.54}$$

If $L^2 = 1$ then $j^{(\epsilon)}$ is a real structure, if $L^2 = -1$ then it is a quaternionic structure.

Strictly, L can be any matrix that squares to ± 1. However, we will restrict the form of L to matrices are involutive automorphisms of the Lie algebras $\mathfrak{o}(K, \mathbb{C})$ and $\mathfrak{sp}(K, \mathbb{C})$

under conjugation, i.e.

$$L \cdot \mathfrak{g} \cdot L^{-1} = \mathfrak{g}, \qquad \mathfrak{g} = \mathfrak{o}(K,\mathbb{C}) \quad \text{or} \quad \mathfrak{sp}(K,\mathbb{C}). \tag{3.55}$$

This results in the real R-symmetry group being a real form of the complexified R-symmetry group in the cleanest possible manner. In the Appendix provide an example of a real R-symmetry group when L is not an involutive automorphism of the complexified R-symmetry Lie algebra.

Given an involution of a complex Lie group/algebra, there is a corresponding real form. By finding all involutions we exhaust all possible real forms, though each involution does not necessarily produce a different real form. The involutive automorphisms of $\mathfrak{o}(K,\mathbb{C})$ and $\mathfrak{sp}(K,\mathbb{C})$ are different, so we will consider both separately. Real forms and their relations to involutive automorphisms were discussed in Section 2.10.

$\mathfrak{o}(K,\mathbb{C})$

Using complex conjugation and the following matrices,

$$\mathbb{1}_K, \qquad I_{p,q} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}, \qquad J_K = \begin{pmatrix} 0 & \mathbb{1}_k \\ -\mathbb{1}_k & 0 \end{pmatrix}, \qquad K = p + q = 2k, \tag{3.56}$$

we can construct all involutive automorphisms of $\mathfrak{o}(K,\mathbb{C})$. We can only use J_K when K is even.

$L = \mathbb{1}_K, I_{p,q}$ define $j^{(\epsilon)} = j^{(+1)}$, a real structure and $L = J_K$ define $j^{(\epsilon)} = j^{(-1)}$, a quaternionic structure.

Recall that $G^K_{\mathbb{C}} = \mathrm{O}(K,\mathbb{C})$ when the bilinear form on the $\mathbb{C}^K$ factor is $M = \delta$. When we work with such a bilinear form, we will only consider L's of the forms outlined in (3.56), whether in odd or even dimensions. These forms of L will often be called the canonical choices for L.

$\mathfrak{sp}(K,\mathbb{C})$

Working with $\mathfrak{sp}(K,\mathbb{C})$ means K is even, we set $K = 2k$, for a real structure we have the following possibilities for L

$$\mathbb{1}_K, \qquad J_K = \begin{pmatrix} 0 & \mathbb{1}_k \\ -\mathbb{1}_k & 0 \end{pmatrix}, \qquad I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{only when } K = 2, \tag{3.57}$$

$$\tilde{I}_{2r,2s} = \begin{pmatrix} \mathbb{1}_r & 0 & 0 & 0 \\ 0 & -\mathbb{1}_s & 0 & 0 \\ 0 & 0 & \mathbb{1}_r & 0 \\ 0 & 0 & 0 & -\mathbb{1}_s \end{pmatrix} = \begin{pmatrix} I_{r,s} & 0 \\ 0 & I_{r,s} \end{pmatrix}, \quad k = r + s.$$

Once again, along with complex conjugation these form a basis for all involutive automorphisms of the algebra $\mathfrak{sp}(K,\mathbb{C})$. Note that $\tilde{I}_{2r,2s}$ cannot be used when $K = 2$, $I_{1,1}$ takes it place, but generally conjugation with $I_{p,q}$ is not an involutive automorphism for $\mathfrak{sp}(K,\mathbb{C})$.

$L = \mathbb{1}_K, I_{1,1}, \tilde{I}_{2r,2s}$ make $j^{(\epsilon)} = j^{(+1)}$, a real structure and $L = J_K$ make $j^{(\epsilon)} = j^{(-1)}$, a quaternionic structure.

$G^K_{\mathbb{C}} = \mathrm{Sp}(K,\mathbb{C})$ when the bilinear form on the $\mathbb{C}^K$ factor is $M = J$. The choices for L in (3.58) are the canonical choices for the reality condition when we choose to work with $M = J$.

3.5.4 ϵ-quaternionic structures on $\mathbb{S} \otimes \mathbb{C}^K$ and $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$.

Odd dimensions

To construct a real structure on the $\mathbb{S} \otimes \mathbb{C}^K$ we have two choices – the product of two real or two quaternionic structures. Recall that the type of structure is dependent on the possible $J^{(\epsilon)}$ available in each signature. Therefore in odd dimensions, we have only one type of structure on $\mathbb{S}$, and in even we have two (which may both be real or quaternionic). So once again the choice we make on the $\mathbb{C}^K$ factor is determined by the behaviour of $\mathbb{S}$.

The real structure on $\mathbb{S} \otimes \mathbb{C}^K$ is

$$\rho = J^{(\epsilon)} \otimes j^{(\epsilon)} : \lambda^i \to \alpha B^* (\lambda^j)^* L_{ji}. \tag{3.58}$$

Elements of $\mathbb{S}\otimes\mathbb{C}^K$ that are invariant under this real structure are the real spinors of the theory. The restriction of all tensor-valued bilinear forms to these elements is entirely real or imaginary (and therefore real after multiplication by i), we choose the phase α so that the vector-valued bilinear form is real.

When K is odd, we cannot define quaternionic structures on $\mathbb{C}^K$ because a quaternionic structure requires an even number of dimensions. This does not impede defining a real structure on $\mathbb{S}\otimes\mathbb{C}^K$ – K is odd only if $\mathbb{S}$ has a real structure, so being limited to real structures on $\mathbb{C}^K$ never prevents the definition of a real structure on the product space, indeed it is the only choice.

If we only have access to a quaternionic structure on $\mathbb{S}$ the extended spinor modules are always of the form $\mathbb{S}\otimes\mathbb{C}^{2k}$ so once again we can always define a real structure on the product space because there is no impediment to defining a quaternionic structure on $\mathbb{C}^{2k}$. A corollary is that in signatures without a real structure on $\mathbb{S}$ we cannot have theories with an odd number of supersymmetries within this framework.

Even Dimensions

Recapping, in even dimensions we have Weyl-compatible and -incompatible ϵ-quaternionic structures. Weyl-compatible ϵ-quaternionic structures are maps from $\mathbb{S}_\pm$ to $\mathbb{S}_\pm$, and both have the same ϵ. Weyl-incompatible structures are maps from $\mathbb{S}_\pm$ to $\mathbb{S}_\mp$, with one having $\epsilon = 1$ and the other $\epsilon = -1$.

Weyl-compatible ϵ-quaternionic structures work exactly the same as in odd dimensions, replacing the Dirac spinor module with either Weyl-spinor module. Both $J_\pm^{(\epsilon)}$ have the same value of ϵ so restrict the form of $j^{(\epsilon)}$ in the same way. Real structures can be defined on each Weyl spinor module individually:

$$\rho_\pm(\lambda_\pm^i) = \alpha B^*_{(\pm)}(\lambda_\pm^i)^* L_{ji}. \tag{3.59}$$

These are a real structure provided $B^*B = L^2 = \epsilon$. The canonical choices of L that we will consider are those listed above. The total real structure is then $\rho = \rho_+ + \rho_-$. For superalgebras with both chiralities present, we can have different real structures defined on each chirality.

Weyl-incompatible signatures link the two Weyl spinor modules. In terms of Weyl spinors, the model real structure is of the form

$$\rho(\lambda^i_\pm) = \alpha B^*_{(\pm)}(\lambda^i_\mp)^* L_{ji}. \tag{3.60}$$

Here the choice of B is meaningful, as $B^*_+ B_+ = -B^*_- B_-$, so the form of L we would choose depends on the choice of B. This reality condition can be written as a reality condition on $\mathbb{S} \otimes \mathbb{C}^K$:

$$\rho(\lambda^i) = \rho(\lambda^i_+) + \rho(\lambda^i_-) = \alpha B^*_{(\pm)}(\lambda^i_-)^* L_{ji} + \alpha B^*_{(\pm)}(\lambda^i_+)^* L_{ji} = \alpha B^*_{(\pm)}(\lambda^i)^*. \tag{3.61}$$

3.6 Defining a Real Superalgebra

We now turn to defining a super-Poincaré algebra whose supercharges are elements of the complex extended spinor modules outlined in the previous sections. The form of the complex extended spinor modules is dependent on the signature. They are of the form $\mathbb{S} \otimes \mathbb{C}^K$ for odd dimensions and even-dimensional signatures with isotropic bilinear forms and/or Weyl-incompatible reality conditions. In even-dimensional signatures with orthogonal vector-valued bilinear forms and Weyl-compatible reality conditions the supercharges are elements of $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$.

To define a Poincaré superalgebra we need a superbracket for the supercharges, Q^i_α, that is proportional to the spacetime translation generators, $P_\mu \in \mathbb{R}^{t,s}$, in other words a superbracket is a real-valued vector-valued bilinear form [5]. Fortunately, we have already outlined the pieces we need to obtain a real vector-valued bilinear form on the complexified spinor modules (that will be isomorphic to an arbitrary sum of irreducible real spinor modules after imposing a reality condition).

This section will first discuss reality conditions imposed on the complex extended spinor modules before describing how this relates to defining a superbracket.

3.6.1 Reality Conditions

The K-extended spinor module, either $\mathbb{S} \otimes \mathbb{C}^K$ or $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$, is equipped with a bilinear form, β or $\beta = \beta_+ \oplus \beta_-$, that up to this point is complex-valued, as are all derived rank-p tensor-valued bilinear forms. On these spinor modules we have shown

[5] the Lie bracket on the bosonic sector $\mathfrak{so}(t,s) + \mathbb{R}^{t,s}$ is the standard Lie bracket as outlined in Chapter 2 Section 2.7.

how to define a real structure, ρ or $\rho_+ \oplus \rho_-$. Real K-extended spinors are the elements of $\mathbb{S} \otimes \mathbb{C}^K$ that are invariant under ρ, i.e. those that satisfy

$$\rho(\lambda^i) = \lambda^i \implies (\lambda^i)^* = \alpha B \lambda^j L_{ji}. \tag{3.62}$$

The form on the right is how we will usually specify a reality condition. The subspace invariant under this reality condition is isomorphic to

$$(\mathbb{S} \otimes \mathbb{C}^K)^\rho \cong S^{\oplus N}, \quad (\mathbb{S}_+ \otimes \mathbb{C}^{K_+})^{\rho_+} \oplus (\mathbb{S}_- \otimes \mathbb{C}^{K_-})^{\rho_-} \cong S_+^{\oplus N_+} \oplus S_-^{\oplus N_-}, \tag{3.63}$$

where K and N are related as outlined in Section 3.3. While we have ended up with a module isomorphic to an arbitrary sum of irreducible real spinor modules (where we started), in doing so, we will find we have disentangled Lorentz and R-symmetry transformations, which act on the internal $\mathbb{C}^K$ factor as is shown in Section 3.8.

On this invariant subspace we need a real vector-valued bilinear form, i.e. one that satisfies

$$([C \otimes M](\gamma^\mu \lambda, \chi))^* = [C \otimes M](\gamma^\mu \lambda, \chi). \tag{3.64}$$

From (3.62) we see the reality properties of the spinors is determined up to a phase α, we choose to fix the value of α such that the vector-valued bilinear form is real. This is done to avoid factors of i in the Lagrangian and superalgebra but is otherwise entirely conventional. In an alternative formulation, one could permit any value for α and compensate by multiplication of M by the necessary factor to guarantee the reality of the vector-valued bilinear form, as is seen in [14]. Relating the two is not difficult, an example is contained in Section 3.11.3.

To fix α we calculate

$$\begin{aligned}
([C \otimes M](\gamma^\mu \lambda, \chi))^* &= (\gamma^\mu \lambda^i)^T C \chi^j M_{ji}, \\
&= \alpha^2 (\gamma^\mu B \lambda_+^k L_{ki})^T C^* B \chi^l L_{lj} M_{ji}, \\
&= \alpha^2 \tau_B (-1)^t (\gamma^\mu \lambda_\pm^i)^T B^T C^* B \chi^j (L^T M L)_{ji}.
\end{aligned} \tag{3.65}$$

τ_B means the τ associated to the choice of $B_\pm = B_{-\tau}$ as this is not necessarily the same as the choice of sign for C, in odd dimensions this can be ignored. Going further requires fixing the signature, L and M (and the choice of B and C in even dimensions). For all possible choices, $B^T C^* B = \pm C$ and $L^T M L = \pm M$ [6] so that $\alpha^2 = \pm 1$ is the only necessary compensating factor. This fixes α to a sign, so one can choose between $\alpha = \pm 1$ or $\alpha = \pm i$.

In odd dimensions the sign is conventional, though if one wishes to make the Majorana vector-valued bilinear form proportional to the vector-valued Dirac sesquilinear form, such as in our five-dimensional paper [2] and the five-dimensional work in Chapter 4, a particular sign choice is required.

On $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$ each summand has a complex bilinear form, β_+ and β_- and real structure, ρ_+ and ρ_-. The phase of ρ_+ and ρ_- are chosen such that the vector-valued bilinear forms β_+^1 and β_-^1 are real, respectively. The relative sign choice of $\alpha_\pm$ affects physical theories. For example in (1,9) signature the difference between the Type IIA and Type IIA* theories can be expressed as to whether the two Weyl spinor modules have the same or different phase in the reality condition. More details on these can be found in Section 3.11.3.

By considering the subspace $(\mathbb{S} \otimes \mathbb{C}^K)^\rho \cong S^{\oplus N}$ or $(\mathbb{S}_+ \otimes \mathbb{C}^{K_+})^{\rho_+} \oplus (\mathbb{S}_- \otimes \mathbb{C}^{K_-})^{\rho_-}$ and selecting the phase of the real structures ρ or $\rho_\pm$, we have constructed an extended spinor modules with a real-valued $\mathrm{Spin}_0(t,s)$-invariant vector-valued bilinear form, with which we can define a superbracket, and in doing so a super-Poincaré algebra.

3.6.2 ...on $\mathbb{S} \otimes \mathbb{C}^K$

The supercharges are $Q \in (\mathbb{S} \otimes \mathbb{C}^K)^\rho$, the ρ-invariant subspace of $\mathbb{S} \otimes \mathbb{C}^K$. A superbracket requires a map, K from the spinor module to $\mathbb{R}^{p,q}$ defined by an anticommutator,

$$K : \mathrm{Sym}(\mathbb{S} \otimes \mathbb{C}^K \times \mathbb{S} \otimes \mathbb{C}^K) \to \mathbb{R}^{t,s}. \tag{3.66}$$

With explicit indices this is

$$\{Q^i_\alpha, Q^j_\beta\} = (K^\mu)^{ij}_{\alpha\beta} P_\mu. \tag{3.67}$$

K is a symmetric $\mathrm{Spin}_0(t,s)$-invariant vector-valued bilinear form, and $(K^\mu)^{ij}_{\alpha\beta}$ is its Gram matrix. On $(\mathbb{S} \otimes \mathbb{C}^K)^\rho$ we defined a real symmetric $\mathrm{Spin}_0(t,s)$-invariant vector-valued bilinear form, $\beta|_\rho = C \otimes M$. This vector-valued bilinear form, β, is symmetric if

[6] This only holds for when L is one of the canonical choices outlined in Section 3.5.3.

$\sigma_\beta \tau_\beta = +1$:

$$\beta(\gamma^\mu \lambda, \chi) = \beta(\gamma^\mu \chi, \lambda). \tag{3.68}$$

Recall for $\beta = C \otimes M$, $\sigma_M = \sigma_C \sigma_\beta$ and $\tau_\beta = \tau_C$. We call bilinear forms with $\sigma_\beta \tau_\beta = 1$ super-admissible, because it can be used to define a superbracket, as will now be shown.

Given any Majorana bilinear form on $\mathbb{S}$ we can create a super-admissible bilinear form on $\mathbb{S} \otimes \mathbb{C}^K$ by choosing M such that $\sigma_\beta \tau = \sigma_C \sigma_M \tau = 1$. We choose M to be symmetric when $\sigma_C \tau_C = +1$ so the resulting combination is super-admissible. And when $\sigma_C \tau_C = -1$ we choose an antisymmetric M to obtain a super-admissible bilinear form. Our canonical forms for a symmetric and antisymmetric bilinear form on $\mathbb{C}^K$ were called δ and J respectively. In summary:

$$\begin{aligned} \sigma_C \tau_C = +1 &\to M = \delta, \\ \sigma_C \tau_C = -1 &\to M = J. \end{aligned} \tag{3.69}$$

Assuming $\beta = C \otimes M$ is super-admissible, K is set to be proportional $\beta(\gamma^\mu \cdot, \cdot)$:

$$\beta(\gamma^\mu \lambda, \chi) = (\gamma^\mu \lambda^i)^T C \chi^j M_{ji} = \lambda^i_\alpha (k K^\mu)^{\alpha\beta}_{ji} \chi^j_\beta,$$

such that the superbracket is given by

$$\{Q^i_\alpha, Q^j_\beta\} = k M^{ij} (\gamma^\mu C)^{-1}_{\alpha\beta} P_\mu, \tag{3.70}$$

with supercharges Q^i that satisfy the reality condition $\rho(Q^i) = Q^i$, i.e.

$$(Q^i)^* = \alpha B Q^j L_{ji}. \tag{3.71}$$

In odd dimensions this construction leads to unique superbracket, as there is a single choice for C and this mandates the choice of M. A corollary of this is when the Majorana bilinear form is not super-admissible it is not possible to define an odd-K superbracket, for $\mathbb{C}^K$ with K-odd the bilinear form J is degenerate and thus equivalent to the even superbracket with $K-1$ supercharges. Recall that K can only be odd when we have a spin-invariant real structure, such that $S \ncong \mathbb{S}$. Combining these two facts, we see that we can only define theories with an odd number of supersymmetric generators when we have a spin-invariant real structure on $\mathbb{S}$ and a super-admissible Majorana bilinear form. This prevents '$\mathcal{N} = 1$' theories in signatures where one may expect them due to the existence of Majorana spinors, such as (2,3).

There is no impediment to defining a superbracket when K is even. We can always use a Majorana bilinear form and a compensating bilinear form on $\mathbb{C}^K$.

It was shown in Section 3.4.2 that orthogonal signatures have $\iota_1 = 1$ which means that $\sigma_+\tau_+ = \sigma_-\tau_-$, so both are either super-admissible or not. In isotropic signatures we have $\sigma_+\tau_+ = -\sigma_-\tau_-$ so they always have one super-admissible bilinear form and one not.

For an orthogonal vector-valued bilinear form in even dimensions, one can have either C but both have the same choice of M (as it was shown they are both super-admissible or they are both not). One can write the superbracket in terms of chiral supercharges if desired:

$$\{Q^i_\alpha, Q^j_\beta\} = \{Q^i_{+\alpha}, Q^j_{+\beta}\} + \{Q^i_{-\alpha}, Q^j_{-\beta}\}. \tag{3.72}$$

In isotropic dimensions one Majorana bilinear form is super-admissible, and one is anti-super-admissible. Call the super-admissible bilinear form C, and the anti-super-admissible bilinear form C' (either could be $C_\pm$ depending on dimension), we then have two possible superbrackets

$$\{Q^i_\alpha, Q^j_\beta\} = k\delta^{ij}(\gamma^\mu C)^{-1}_{\alpha\beta}P_\mu \tag{3.73}$$

$$\{Q^i_\alpha, Q^j_\beta\} = kJ^{ij}(\gamma^\mu C')^{-1}_{\alpha\beta}P_\mu \tag{3.74}$$

Later we will show these two bilinear forms (regardless of reality condition) are isomorphic for Weyl-compatible reality conditions. In Weyl-incompatible signatures, these two superbrackets can be shown to be equivalent but doing so modifies the reality condition.

If desired these can be written in terms of chiral supercharges

$$\{Q^i_\alpha, Q^j_\beta\} = \{Q^i_{+\alpha}, Q^j_{-\beta}\} + \{Q^i_{-\alpha}, Q^j_{+\beta}\} = 2\{Q^i_{+\alpha}, Q^j_{-\beta}\}. \tag{3.75}$$

The constant k is a free choice, in our previous work we have chosen it to be $-\frac{1}{2}$ to match the standard choice for symplectic Majorana spinors (it is also in this guise that one can show it is equal to the sesquilinear form on a single Dirac spinor, but this is not needed for this chapter, see [20]).

3.6.3 ...on $\mathbb{S}_+ \otimes \mathbb{C}^{K_+} \oplus \mathbb{S}_- \otimes \mathbb{C}^{K_-}$

In (and only in) orthogonal Weyl-compatible signatures do we work with supercharges that are elements of $(\mathbb{S}_+ \otimes \mathbb{C}^{K_+})^{\rho_+} \oplus (\mathbb{S}_- \otimes \mathbb{C}^{K_-})^{\rho_-}$. In these signatures both Majorana bilinear forms are super-admissible or anti-super-admissible.

On each Weyl spinor module alone we define a superbracket independently:

$$\{Q^i_{\pm\alpha}, Q^j_{\pm\beta}\} = (K^\mu_\pm)^{ij}_{\alpha\beta} P_\mu. \tag{3.76}$$

Where we again interpret $(K^\mu_\pm)^{ij}_{\alpha\beta}$ as the structure constants of a real symmetric $\mathrm{Spin}_0(t,s)$-invariant vector-valued bilinear form on $\mathbb{S}_\pm$. We set $K^\mu_\pm$ to be proportional to a super-admissible $C \otimes M$ so that the superbracket is given by

$$\{Q^i_{\pm\alpha}, Q^j_{\pm\beta}\} = kM^{ij}(\gamma^\mu C)^{-1}_{\alpha\beta} P_\mu. \tag{3.77}$$

If the Majorana bilinear forms are super-admissible, we can then define a superalgebra using only elements of a single Weyl spinor module, obtaining a $(1,0)$ or $(0,1)$ superalgebra. If they are not super-admissible, then the minimal superalgebra in that dimension involves two Weyl spinor modules.

To summarise, in this formalism a superalgebra is completely specified by a pair of a complex $\mathrm{Spin}_0(t,s)$-invariant vector-valued bilinear form and reality condition, though said pair is not necessarily unique. Pictorially,

$$\text{Superalgebra} \iff (\text{Bilinear form, Reality Condition}). \tag{3.78}$$

3.7 Matrix Notation for Weyl Spinors

In this section, we will introduce a notation that makes calculations easier when dealing with Weyl spinors in even dimensions. Essentially we are 'doubling' the spinor module once more to incorporate the two Weyl spinors modules. We also employ matrix methods when dealing with chiral spinors we can simplify some calculations and calculate symmetry algebras and groups.

Using the natural embedding $\mathbb{S}_\pm \subset \mathbb{S}$ we combine the two Weyl spinor modules into a single 'doubled-again' spinor module

$$\left(\lambda_+^i, \lambda_-^{\hat{i}}\right) = \left(\lambda_+^1, ..., \lambda_+^{K_+}, \lambda_-^1, ..., \lambda_-^{K_-}\right) \in \mathbb{S}_+^{\oplus K_+} \oplus \mathbb{S}_-^{\oplus K_-} \subset \mathbb{S}^{\oplus K_+ + K_-} = \mathbb{S} \otimes \mathbb{C}^{K_+ + K_-} \tag{3.79}$$

Later these are used again in Chapter 5 where λ^I, with $I = 1, ..., K_+ + K_-$, is used as shorthand for $\left(\lambda_+^i, \lambda_-^i\right)$. In doing so we can write the calculations in a shorter manner, though it is not necessary in this chapter.

Orthogonal and isotropic bilinear forms in this notation can then be written

$$\left(\bar{\lambda}_+^i, \bar{\lambda}_-^{\hat{i}}\right) \begin{pmatrix} M_{ji} & 0 \\ 0 & M'_{\hat{j}\hat{i}} \end{pmatrix} \begin{pmatrix} \chi_+^j \\ \chi_-^{\hat{j}} \end{pmatrix} \qquad \text{Orthogonal,} \tag{3.80}$$

$$\left(\bar{\lambda}_+^i, \bar{\lambda}_-^i\right) \begin{pmatrix} 0 & M_{ji} \\ M_{ji} & 0 \end{pmatrix} \begin{pmatrix} \chi_+^j \\ \chi_-^j \end{pmatrix} \qquad \text{Isotropic.} \tag{3.81}$$

Where M is the bilinear form chosen on the C^{K_+} factor, M' on the C^{K_-} factor, $i, j = 1, ..., K_+$ and $\hat{i}, \hat{j} = 1, ..., K_-$. For isotropic signatures, necessarily $M = M'$ and $K_+ = K_-$.

In addition, we have a real structure, ρ, which is either Weyl-compatible or incompatible. For a Weyl-compatible reality condition, we write this as

$$\rho(\lambda_+^i) = \alpha^* B^* (\lambda_+^j)^* L_{ji}, \qquad \rho(\lambda_-^{\hat{i}}) = \beta^* B'^* (\lambda_-^{\hat{j}})^* L'_{\hat{j}\hat{i}} \tag{3.82}$$

$$\to \rho \begin{pmatrix} \lambda_+^i \\ \lambda_-^{\hat{i}} \end{pmatrix} = \begin{pmatrix} \alpha^* B^* L_{ji} & 0 \\ 0 & \beta^* B'^* L'_{\hat{j}\hat{i}} \end{pmatrix} \begin{pmatrix} \lambda_+^j \\ \lambda_-^{\hat{j}} \end{pmatrix}^* . \tag{3.83}$$

Here B and B' can refer to either $B_\pm$.

Weyl-incompatible reality conditions are written as

$$\rho(\lambda_\pm^i) = \alpha^* B^* (\lambda_\mp^j)^* L_{ji} \to \rho \begin{pmatrix} \lambda_+^i \\ \lambda_-^i \end{pmatrix} = \alpha^* B^* \begin{pmatrix} 0 & L_{ji} \\ L_{ji} & 0 \end{pmatrix} \begin{pmatrix} \lambda_+^j \\ \lambda_-^j \end{pmatrix}^* . \tag{3.84}$$

Often when writing expressions we will suppress the i, j indices and write them as a vector-of-vectors and block matrices

$$\left(\underline{\bar{\lambda}}_+, \underline{\bar{\lambda}}_-\right)\begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}\begin{pmatrix} \underline{\chi}_+ \\ \underline{\chi}_- \end{pmatrix} \qquad \text{Orthogonal,} \tag{3.85}$$

$$\left(\underline{\bar{\lambda}}_+, \underline{\bar{\lambda}}_-\right)\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}\begin{pmatrix} \underline{\chi}_+ \\ \underline{\chi}_- \end{pmatrix} \qquad \text{Isotropic.} \tag{3.86}$$

When we do this, we will change the indices such that normal matrix multiplication makes sense for the resulting expressions. This means that that J alone represents J_{ij}, so the above with $M = J$ translates to

$$\left(\bar{\lambda}^i_+, \bar{\lambda}^i_-\right)\begin{pmatrix} 0 & J_{ji} \\ J_{ji} & 0 \end{pmatrix}\begin{pmatrix} \chi^j_+ \\ \chi^j_- \end{pmatrix} = \left(\underline{\bar{\lambda}}_+, \underline{\bar{\lambda}}_-\right)\begin{pmatrix} 0 & -J \\ -J & 0 \end{pmatrix}\begin{pmatrix} \underline{\chi}_+ \\ \underline{\chi}_- \end{pmatrix}. \tag{3.87}$$

For a final example, we would write (3.83) as

$$\rho\begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = B^*\begin{pmatrix} \alpha^* L & 0 \\ 0 & \beta^* L' \end{pmatrix}\begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix}. \tag{3.88}$$

Using this notation we only need to consider linear transformations which act on the expanded internal space $\mathbb{C}^{K_+ + K_-}$, with no transformations acting on the internal space at all. This disentangling of spinor and internal indices with respect to the action of the Schur group is the main advantage of this notation. It reflects that while the Schur group only acts on internal space in odd dimensions, it can act differently on the chiral components in even dimensions. By doubling the auxiliary space, any chiral effects are encoded in the larger matrix acting on this doubled space. Afterwards, this effects on each Weyl spinor module can be reconstructed and rewritten in terms of Id and γ_* acting on the original spinor module.

3.8 R-Symmetry Group

3.8.1 Complexified R-Symmetry Group

The R-symmetry group is the invariance group of the vector-valued bilinear form (and the associated superbracket) that commutes with $\mathrm{Spin}_0(t, s)$. Initially, we define a complex vector-valued bilinear form, before imposing a reality condition. The invariance group of the complex vector-valued bilinear form will be called the complexified R-symmetry group, $G_R^{\mathbb{C}}$, i.e.

$$\beta(\gamma^\mu R\cdot, R\cdot) = \beta(\gamma^\mu\cdot,\cdot) \implies R \in G_R^{\mathbb{C}}. \tag{3.89}$$

G_R is the real R-symmetry group, it is the subgroup of the invariance group of the bilinear form that commutes with the reality condition, and will be a real form of $G_R^{\mathbb{C}}$.

Before enforcing a reality condition, we have three cases, odd-dimensional superalgebras and even dimensions with orthogonal and isotropic bilinear forms. We will treat each in turn. The complex R-symmetry group depends only on the dimension, not on the signature. Signature dependence enters through the reality condition via the matrix B, whose definition is signature-dependent).

Odd Dimensions

When $\mathbb{S}$ is complex irreducible, Schur's lemma implies that the complexified R-symmetry group acts trivially on the $\mathbb{S}$ factor, acting entirely on $\mathbb{C}^K$. The invariance group on $\mathbb{C}^K$ we called $G_{\mathbb{C}^K}$ and this is therefore also the complexified R-symmetry group, $G_R^{\mathbb{C}}$. This group is $\mathrm{O}(K,\mathbb{C})$ or $\mathrm{Sp}(K,\mathbb{C})$ if we chose a symmetric or antisymmetric bilinear form on $\mathbb{C}^K$ respectively.

Therefore in odd dimensions an R-symmetry transformation is given by

$$\lambda^i \to R^i{}_j\lambda^j, \tag{3.90}$$

where $R^i{}_j$ does not act upon the spinor indices. The corresponding R-symmetry Lie algebra element is written $r^i{}_j$ such that $R^i{}_j = exp(r^i{}_j)$.

The need for a symmetric or antisymmetric bilinear form on $\mathbb{C}$ depends on the dimension (because the symmetry of the Majorana bilinear form depends only on the dimension). We find that the complex R-symmetry group in odd dimensions is

$$G_R^{\mathbb{C}} = \begin{cases} \mathrm{O}(K,\mathbb{C}) & D = 1,3,9,11, \\ \mathrm{Sp}(K,\mathbb{C}) & D = 5,7. \end{cases} \tag{3.91}$$

Even Dimensions

Orthogonal Bilinear Form

In orthogonal dimensions, our R-symmetry ansatz will be

$$\begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} \to R \begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = \begin{pmatrix} A^i{}_j & 0 \\ 0 & B^{\hat{i}}{}_{\hat{j}} \end{pmatrix} \begin{pmatrix} \lambda_+^j \\ \lambda_-^{\hat{j}} \end{pmatrix}. \tag{3.92}$$

With $i = 1, ..., K_+$ and $\hat{i} = 1, ..., K_-$. The matrices $A^i{}_j$ and $B^{\hat{i}}{}_{\hat{j}}$ act only on the internal spaces $\mathbb{C}^{K_+}$ and $\mathbb{C}^{K_-}$ because due to Schur's lemma R-symmetry transformations are inert on the spinor indices.

Invariance of the vector-valued bilinear form implies

$$R^T \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} R = \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}, \tag{3.93}$$

which, after inserting the components of R, becomes

$$\begin{pmatrix} A^T M A & 0 \\ 0 & B^T M' B \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}. \tag{3.94}$$

Recall that both M and M' are of the same form (either identity or J) but of potentially different size (M is $K_+ \times K_+$ and M' is $K_- \times K_-$). These equations for A and B define $\mathrm{O}(K_+, \mathbb{C})$ and $\mathrm{O}(K_-, \mathbb{C})$ if the two Majorana bilinear forms are super-admissible or $\mathrm{Sp}(K_+, \mathbb{C})$ and $\mathrm{Sp}(K_-, \mathbb{C})$ if the Majorana bilinear forms are anti-super-admissible. Therefore we obtain the complexified R-symmetry groups as

$$G_R^{\mathbb{C}} = \begin{cases} \mathrm{O}(K_+, \mathbb{C}) \times \mathrm{O}(K_-, \mathbb{C}) & D = 2, 10, \\ \mathrm{Sp}(K_+, \mathbb{C}) \times \mathrm{Sp}(K_-, \mathbb{C}) & D = 6. \end{cases} \tag{3.95}$$

Isotropic Bilinear Form

In isotropic signatures, we make a similar ansatz to orthogonal signatures with different A and B, but necessarily with $K_+ = K_- = K$,

$$\begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} \to R \begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = \begin{pmatrix} A^i{}_j & 0 \\ 0 & B^i{}_j \end{pmatrix} \begin{pmatrix} \lambda_+^j \\ \lambda_-^j \end{pmatrix}. \tag{3.96}$$

To preserve the vector-valued bilinear form implies

$$R^T \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} R = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}. \tag{3.97}$$

This leads to

$$\begin{pmatrix} 0 & A^T M B \\ B^T M A & 0 \end{pmatrix} = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}. \tag{3.98}$$

This is solved by $B = M^{-1}(A^T)^{-1}M$ and therefore

$$R = \begin{pmatrix} A & 0 \\ 0 & M^{-1}(A^T)^{-1}M \end{pmatrix}. \tag{3.99}$$

A must be invertible for this to make sense but is otherwise unconstrained, i.e. $A \in \mathrm{GL}(K, \mathbb{C})$. We observe

$$\big(M^{-1}(A^T)^{-1}M\big)\big(M^{-1}(A'^T)^{-1}M\big) = M^{-1}((AA')^T)^{-1}M. \tag{3.100}$$

A is in the fundamental representation and $M^{-1}(A^T)^{-1}M$ is the contragredient/dual representation (that also has undergone a change of basis given by M) of $\mathrm{GL}(K, \mathbb{C})$. R is the direct sum of two representations of $\mathrm{GL}(K, \mathbb{C})$, which is a (reducible) representation of $\mathrm{GL}(K, \mathbb{C})$ in its own right. In isotropic dimensions the complex R-symmetry group is therefore

$$G_R^{\mathbb{C}} = \mathrm{GL}(K, \mathbb{C}) \qquad D = 4, 8, 12. \tag{3.101}$$

Summary Table

We, therefore, have a few possibilities for the complexified R-symmetry group, and they are presented below. In addition, we list the available charge conjugation matrix and its invariants in each signature. This table is a useful reference for following sections.

Invariants are from [37]. Here invariants were given as $\epsilon = -\sigma$ and $\eta = -\tau$. The table is mod 8 but it was included here for physically relevant dimensions.

D		σ	τ	ι	M	$G_{\mathbb{C}^K}$	$G_R^{\mathbb{C}}$
1		+1	+1	N/A	δ	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K,\mathbb{C})$
2	C_+	−1	−1	+1	J	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K_+,\mathbb{C}) \times \mathrm{O}(K_-,\mathbb{C})$
	C_-	+1	+1	+1	δ	$\mathrm{O}(K,\mathbb{C})$	
3		−1	−1	N/A	δ	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K,\mathbb{C})$
4	C_+	−1	−1	−1	δ	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{GL}(K,\mathbb{C})$
	C_-	−1	+1	−1	J	$\mathrm{Sp}(K,\mathbb{C})$	
5		−1	+ 1	N/A	J	$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{Sp}(K,\mathbb{C})$
6	C_+	+1	−1	+1	J	$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{Sp}(K_+,\mathbb{C}) \times \mathrm{Sp}(K_-,\mathbb{C})$
	C_-	−1	+1	+1	J	$\mathrm{Sp}(K,\mathbb{C})$	
7		+1	−1	N/A	J	$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{Sp}(K,\mathbb{C})$
8	C_+	+1	−1	−1	J	$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{GL}(K,\mathbb{C})$
	C_-	+1	+1	−1	δ	$\mathrm{O}(K,\mathbb{C})$	
9		+1	+1	N/A	δ	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K,\mathbb{C})$
10	C_+	−1	−1	+1	J	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K_+,\mathbb{C}) \times \mathrm{O}(K_-,\mathbb{C})$
	C_-	+1	+1	+1	δ	$\mathrm{O}(K,\mathbb{C})$	
11		−1	−1	N/A	δ	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K,\mathbb{C})$
12	C_+	−1	−1	−1	δ	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{GL}(K,\mathbb{C})$
	C_-	−1	+1	−1	J	$\mathrm{Sp}(K,\mathbb{C})$	

Table 3.2: Table summarising the previous sections

3.8.2 Real R-symmetry Group

We recall that (in our prescription) a superalgebra is defined by a complex vector-valued bilinear form on the complexified spinor module and a reality condition. Given the vector-valued bilinear form obtained from a bilinear form β with a reality condition given by ρ, we then seek the transformations that leave the bilinear form invariant and commute with the reality condition. This is the real R-symmetry group. An element $R \in G_R$ obeys

$$\beta(\gamma^\mu R\cdot, R\cdot) = \beta(\gamma^\mu\cdot,\cdot) \qquad \rho(R\cdot) = R\rho(\cdot). \tag{3.102}$$

Different definitions of the R-symmetry group exist: some authors define the R-symmetry group to be the invariance group of the superalgebra and Lagrangian. In other cases the theory may enjoy accidental conformal symmetry and generators of the R-symmetry algebra are associated with superconformal generators. For example with $\mathcal{N} = 4$ super Yang-Mills the R-symmetry group is often written as SU(4) and not U(4) as a U(1) factor falls out of the R-symmetry group as it is attributed to the superconformal algebra.

In addition, it is common to give the R-symmetry groups as the connected components, for example as SO(2) in Type IIB string theories. In our table we will give the groups defined by our conventions, which will usually be slightly larger disconnected groups, using the same example our prescription gives the R-symmetry for Type IIB string theories as O(2).

The R-symmetry groups for all signatures with dimension ≤ 12 will be calculated in the following section. As we are now considering reality conditions too, we now have five scenarios – odd dimensions and the four cases in even dimensions: Weyl-compatible orthogonal, Weyl-compatible isotropic, Weyl-incompatible orthogonal and Weyl-incompatible isotropic spinor modules.

Finally, we present a table summarising this information. The R-symmetry group is then used in further sections to determine those superalgebras that are isomorphic.

Odd Dimensions

In odd dimensions we work with complex-irreducible Dirac spinors, with a real structure defined on $\mathbb{S}\otimes\mathbb{C}^K$. Due to Schur's lemma we found that $G_R^{\mathbb{C}} = G_{\mathbb{C}^K}$ which was $\mathrm{O}(K,\mathbb{C})$ or $\mathrm{Sp}(K,\mathbb{C})$.

A generic reality condition is

$$(\lambda^i)^* = \alpha B \lambda^j L_{ji}. \tag{3.103}$$

Invariance under an R-symmetry transformation leads to

$$(R^i{}_j)^* L_{kj} = R^j{}_k L_{ji} \qquad \to R^* L^T = L^T R. \tag{3.104}$$

All canonical choices for L have a definite symmetry, i.e. $L^T = \pm L$, such that the above equation can be written

$$R^* L = LR. \tag{3.105}$$

Lie algebra elements ${r^i}_j$ obey the same equation

$$r^* L = L r \implies r = L^{-1} r^* L. \tag{3.106}$$

This equation demonstrates that the R-symmetry Lie algebra is a real form of the complexified Lie algebra, $\mathfrak{g}_R^{\mathbb{C}}$, see Section 2.10 for more details. For each involution on a Lie algebra we have an associated real form (that is not necessarily different). As described in 3.5.3 the matrix L in real and quaternionic structures defined on $\mathbb{C}^K$ were limited to those that are involutive automorphisms of the complex Lie algebras we obtained from complex R-symmetry Lie algebra.

In odd dimensions, we only need to focus on the real forms of $\mathrm{O}(N,\mathbb{C})$ and $\mathrm{Sp}(N,\mathbb{C})$. The possible real forms we can access from our two complex groups are dependent on the signature, as this controls the value of ϵ in the ϵ-quaternionic structures on $\mathbb{S}$ and therefore the type of structure chosen on $\mathbb{C}^K$, which were defined by choice of the representative matrix L.

In Table 3.3 we list the real forms and the corresponding involution, and the type of structure needed on the spinor module needed to realise this involution

$G_R^{\mathbb{C}}$	G_R	ϵ	L
$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(K)$	+1	δ
	$\mathrm{O}(p,q)$	+1	$I_{p,q}$
	$\mathrm{SO}^*(K)$	−1	J_K
$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{Sp}(K,\mathbb{R})$	+1	δ
	$\mathrm{USp}(2r,2s)$	+1	$\tilde{I}_{2r,2s}$
	$\mathrm{USp}(K)$	−1	J_K

Table 3.3: Real forms of $\mathrm{O}(N,\mathbb{C})$ and $\mathrm{Sp}(N,\mathbb{C})$ and the corresponding structure on $\mathbb{S}$ needed to realise them. $p+q=K$ and $2r+2s=K$

We can see that there is a broader possibility of R-symmetry groups in odd-dimensional signatures that have a real structure on $\mathbb{S}$ allowing multiple Majorana spinors or ‘twisted’ Majorana spinors with different reality conditions on each Majorana spinor. We can choose the ‘signature’ of the R-symmetry group (we can have definite or indefinite orthogonal groups), or we can realise $\mathrm{Sp}(K,\mathbb{R})$ or $\mathrm{USp}(2r,2s)$. Those with a quaternionic structure are restricted to either having $\mathrm{SO}^*(K)$ if the Majorana bilinear form is super-admissible or $\mathrm{USp}(K)$ if it is not.

Next we summarise this for each odd-dimensional signature up to $D = 11$:

D	$(0,D)$	$(1,D-1)$	$(2,D-2)$	$(3,D-3)$	$(4,D-4)$	$(5,D-5)$	$(6,D-6)$
1	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$					
3	$\mathrm{SO}^*(K)$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(K)$			
5	$\mathrm{USp}(K)$	$\mathrm{USp}(K)$	$\mathrm{Sp}(K,\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{Sp}(K,\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{USp}(K)$	$\mathrm{USp}(K)$	
7	$\mathrm{Sp}(K,\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{USp}(K)$	$\mathrm{USp}(K)$	$\mathrm{Sp}(K,\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{Sp}(K,\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{USp}(K)$	$\mathrm{USp}(K)$
9	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(K)$	$\mathrm{SO}^*(K)$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(K)$
11	$\mathrm{SO}^*(K)$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(K)$	$\mathrm{SO}^*(K)$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$

Table 3.4: R-symmetry groups possible in each odd dimension in any signature, $p+q = K$.

From this table provides some useful insights. When $K = 1$ only the group $\mathrm{O}(1) \cong \mathbb{Z}_2$ is defined. Therefore only signatures with $\mathrm{O}(p,q)$ R-symmetry can have a '$\mathcal{N} = 1$' algebra. The signatures with $\mathrm{SO}^*(K)$ R-symmetry group only quaternionic structures on $\mathbb{S}$ so we cannot have K-odd theories. This means that in 11 dimensions $\mathcal{N} = K = 1$ algebras can only be defined in (1,10), (2,9) and (5,6); these give M, M* and M' theories respectively [14]. For $K = 2$ the R-symmetry groups are commonly known by different names – $\mathrm{SO}^*(2) \cong \mathrm{SO}(2)$, $\mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$ and $\mathrm{USp}(2) \cong \mathrm{SU}(2)$.

Orthogonal Weyl-compatible

In orthogonal Weyl-compatible signatures we found the two Weyl spinor modules can be used independently, each working like in odd dimensions, so we do not need any additional calculations. The complex R-symmetry groups were

$$G_R^{\mathbb{C}} = \begin{cases} \mathrm{O}(K_+,\mathbb{C}) \times \mathrm{O}(K_-,\mathbb{C}) & D = 2, 10, \\ \mathrm{Sp}(K_+,\mathbb{C}) \times \mathrm{Sp}(K_-,\mathbb{C}) & D = 6. \end{cases} \tag{3.107}$$

We can define different reality conditions on both chiralities if desired (defined with different L) and in doing so, obtain a different real form in each factor of the product. However, we recall that Weyl-compatibility implies that both $J_\pm^{(\epsilon)}$ have the same ϵ (i.e. they are both real or quaternionic structures on the Weyl spinor modules). Therefore we cannot realise drastically different groups on each factor.

In the following table, we have listed the possible R-symmetry groups obtainable in orthogonal Weyl-compatible signatures

$G_R^{\mathbb{C}}$	G_R	ϵ
$\mathrm{O}(K_+,\mathbb{C}) \times \mathrm{O}(K_-,\mathbb{C})$	$\mathrm{O}(p_+,q_+) \times \mathrm{O}(p_-,q_-)$	$+1$
	$\mathrm{SO}^*(K_+) \times \mathrm{SO}^*(K_-)$	-1
$\mathrm{Sp}(K_+,\mathbb{C}) \times \mathrm{Sp}(K_-,\mathbb{C})$	$(\mathrm{Sp}(K_+,\mathbb{R})$ or $\mathrm{USp}(2r_+,2s_+)) \times (\mathrm{Sp}(K_-,\mathbb{R})$ or $\mathrm{USp}(2r_-,2s_-))$	$+1$
	$\mathrm{USp}(K_+) \times \mathrm{USp}(K_-)$	-1

Table 3.5: Real forms of $\mathrm{O}(N,\mathbb{C})$ and $\mathrm{Sp}(N,\mathbb{C})$ and the corresponding structure on $\mathbb{S}$ needed to realise them. $p+q=K$ and $2r+2s=K$

Orthogonal Weyl-incompatible

As they are Weyl-incompatible the spinor module is of the form $\mathbb{S}\otimes\mathbb{C}^K$, however, R-symmetry transformations can act differently on the Weyl spinor modules. As a result, in this signature, it is useful to use the matrix notation.

Weyl-incompatible ϵ-quaternionic structures link the two chiralities, transformations must be compatible with the reality condition. In matrix notation, this means that R satisfies

$$\rho\left(R\begin{pmatrix}\lambda_+^j\\ \lambda_-^j\end{pmatrix}\right)=R\rho\begin{pmatrix}\lambda_+^j\\ \lambda_-^j\end{pmatrix}. \tag{3.108}$$

Where for a Weyl-incompatible signature this implies

$$R^*\begin{pmatrix}0 & L\\ L & 0\end{pmatrix}=\begin{pmatrix}0 & L\\ L & 0\end{pmatrix}R. \tag{3.109}$$

For our previous ansatz, we obtain

$$\begin{pmatrix}0 & AL\\ BL & 0\end{pmatrix}=\begin{pmatrix}0 & LB^*\\ LA^* & 0\end{pmatrix}. \tag{3.110}$$

Which leads to

$$B=LA^*L^{-1}. \tag{3.111}$$

$A \in \mathrm{O}(K, \mathbb{C})$ or $\mathrm{Sp}(K, \mathbb{C})$, and we see that

$$R = \begin{pmatrix} A & 0 \\ 0 & LA^* L^{-1} \end{pmatrix}. \tag{3.112}$$

This is a reducible representation of $\mathrm{O}(K, \mathbb{C})$ or $\mathrm{Sp}(K, \mathbb{C})$; R is the direct sum of the fundamental representation and the conjugate representation that has also undergone a basis transformation given by L. The R-symmetry group of orthogonal Weyl-incompatible signatures is, therefore, $\mathrm{O}(K, \mathbb{C})$ if the Majorana bilinear forms are super-admissible and $\mathrm{Sp}(K, \mathbb{C})$ if they are not.

Isotropic Weyl-compatible

Previously we found that for isotropic signatures an R-symmetry transformation is given by

$$R = \begin{pmatrix} A & 0 \\ 0 & M^{-1}(A^T)^{-1} M \end{pmatrix}. \tag{3.113}$$

For the following calculations we find that working with the associated Lie algebra element is easier, this is

$$r = \begin{pmatrix} a & 0 \\ 0 & -M^{-1} a^T M \end{pmatrix}. \tag{3.114}$$

Where we define a such that $A = e^a$, and therefore $R = e^r$. As described earlier, R is a representation of the same group as A, and therefore r is a representation of the same Lie algebra as a.

In isotropic signatures, $M = \delta$ or $M = J$ can be realised, as the two Majorana bilinear forms have opposite superadmissibility. However, we will see that this choice of M is irrelevant. Additionally, we will also find that the exact form of the reality condition is irrelevant too (not dependent on the form of L chosen); the only thing that will matter if we have a real or quaternionic structure on $\mathbb{C}^K$. In Section 3.10 we show there is a map between the two choices of Majorana bilinear form and maps between possible reality condition choices, demonstrating this further.

Specialising to Weyl-compatible signatures, to commute with the reality condition means that r must satisfy

$$r^* \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} r. \tag{3.115}$$

For r given in (3.114) this implies

$$a = L^{-1} a^* L, \qquad M^{-1} a^T M = L^{-1} (M^{-1} a^T M)^* L. \tag{3.116}$$

Rearranging the final equation in (3.116) we get

$$a = (MLM^{-1})^{-1} a^* (MLM^{-1}). \tag{3.117}$$

However for $M = \delta$ obviously $MLM^{-1} = L$ and we find that even for $M = J$ we see that $MLM^{-1} = L$ when we choose one of our canonical choices for L ($L = Id$, $L = \tilde{I}_{2r,2s}$ or $L = J$). Therefore we only seek an a that solves

$$a = L^{-1} a^* L. \tag{3.118}$$

As the signature is Weyl-compatible, we only have access to two real or two quaternionic structures on $\mathbb{S}$ depending on the signature, restricting possible values of L.

Real Structures on $\mathbb{S}$

We shall first deal with real structures, such that L is given by either Id or $I_{p,q}$ if $M = \delta$ or L is Id or $\tilde{I}_{2r,2s}$ if $M = J$.

For $L = \delta$ equation (3.118) means that $a \in \mathfrak{gl}(K, \mathbb{R})$, though we also find this to be true for any L that defines a real structure with either bilinear form on $\mathbb{S}$.

For $L = I_{p,q}$ we see that

$$a = I_{p,q}^{-1} a^* I_{p,q} \implies a = \begin{pmatrix} w & ix \\ iy & z \end{pmatrix}. \tag{3.119}$$

Where w is a $p \times p$ real matrix, z is $q \times q$, x is $p \times q$ and y is $q \times p$. We see that a is a $K \times K$ matrix with K^2 real numbers. On dimensional grounds, this implies that $a \in \mathfrak{gl}(K, \mathbb{R})$, though with an unconventional representation, as it is the only possible Lie group it could be.

If $M = J$ we only consider $L = \delta$ and $L = \tilde{I}_{2r,2s}$ for our canonical forms of L. Obviously for $L = \delta$ we obtain $a \in \mathfrak{gl}(K, \mathbb{R})$ again. For $L = \tilde{I}_{2r,2s}$ we find that a must have the form

$$a = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}. \tag{3.120}$$

Where W, X, Y, Z are $\frac{K}{2} \times \frac{K}{2}$ matrices that also obey $V = I_{r,s} V^* I_{r,s}$ for $V = W, X, Y, Z$. They are of the form

$$V = \begin{pmatrix} V_1 & iV_2 \\ iV_3 & V_4 \end{pmatrix}. \tag{3.121}$$

With the same reasoning for the previous case, a is a representation of $\mathfrak{gl}(K, \mathbb{R})$. This means that $r \in \mathfrak{gl}(K, \mathbb{R})$ in isotropic Weyl-compatible signatures with real structures, regardless of the choice of M and L (this is explained later when we show the choices to be isomorphic). The R-symmetry group is then given by $\mathrm{GL}(K, \mathbb{R})$.

Quaternionic Structures on $\mathbb{S}$

If the signature has quaternionic structures we can only realise one form for $L = J$. A matrix $a \in \mathfrak{gl}(K, \mathbb{C})$ that satisfies (3.118) defines the Lie algebra $\mathfrak{u}^*(K) = \mathfrak{gl}(\frac{K}{2}, \mathbb{H})$. a has the form

$$a = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \qquad x, y \in M_{\frac{K}{2}}(\mathbb{C}). \tag{3.122}$$

Upon exponentiation this retains the same form

$$A = e^a = \begin{pmatrix} X & Y \\ -Y^* & X^* \end{pmatrix}, \qquad X, Y \in \mathrm{GL}(\frac{K}{2}, \mathbb{C}). \tag{3.123}$$

r is also a representation of $\mathfrak{u}^*(K)$ and the R-symmetry group is $\mathrm{U}^*(K)$.

Isotropic Weyl-incompatible

Once again Dirac spinors are the building blocks as the bilinear form is isotropic and the reality condition is Weyl-incompatible.

For this calculation it is easier to use the R-symmetry group element. Commuting with the reality condition means R obeys

$$R^* \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix} = \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix} R. \tag{3.124}$$

For the form given in (3.113) this equation is

$$\begin{pmatrix} 0 & A^* L \\ M^{-1}(A^\dagger)^{-1} M L & \end{pmatrix} = \begin{pmatrix} 0 & L M^{-1}(A^T)^{-1} M \\ LA & 0 \end{pmatrix}. \tag{3.125}$$

This is two copies of the equation

$$A^\dagger (ML) A = (ML). \tag{3.126}$$

This defines the unitary group $\mathrm{U}(p,q)$ where (p,q) is the signature of the matrix ML. For some choices of M and L their product ML will not be diagonal, giving an A that is an unconventional representation of the unitary group, but a representation all the same.

For $M = \delta$ the signature depends entirely on L and is $(K,0)$ for $L = \delta$, (p,q) for $L = I_{p,q}$ and (k,k) for $L = J$ (where $K = 2k$). Setting $M = J$ we obtain signature (k,k) for $L = \delta$, $(2r,2s)$ when $L = \tilde{I}_{2r,2s}$ and $(K,0)$ when $L = J$.

The following table summarises the isotropic signatures

	M	L	G_R		M	L	G_R
WC	δ	δ	$\mathrm{GL}(K,\mathbb{R})$	WI	δ	δ	$\mathrm{U}(K)$
	δ	$I_{p,q}$	$\mathrm{GL}(K,\mathbb{R})$		δ	$I_{p,q}$	$\mathrm{U}(p,q)$
	δ	J	$\mathrm{U}^*(K)$		δ	J	$\mathrm{U}(k,k)$
	J	δ	$\mathrm{GL}(K,\mathbb{R})$		J	δ	$\mathrm{U}(k,k)$
	J	$\tilde{I}_{2r,2s}$	$\mathrm{GL}(K,\mathbb{R})$		J	$\tilde{I}_{2r,2s}$	$\mathrm{U}(2r,2s)$
	J	J	$\mathrm{U}^*(K)$		J	J	$\mathrm{U}(K)$

Table 3.6: Real forms of $\mathrm{GL}(K,\mathbb{C})$ obtained in our description and corresponding structure on $\mathbb{S}\otimes\mathbb{C}^K$ needed to realise them. $K = 2k = 2r+2s = p+q$. WC means the signature is Weyl compatible and WI means the signature is Weyl incompatible.

3.8.3 Tables of R-symmetry Groups in Even Dimensions

We now present the results for each signature in dimensions up to twelve. We begin by first outlining the possible $(1,0)$ or $(0,1)$ algebras, theories in which the supercharges form a single Majorana-Weyl spinor. Following this, we define $(1,1)$, $(2,0)$, $(0,2)$ and $\mathcal{N}=1$ superalgebras in even dimensions, where the supercharges are either two Majorana-Weyl, a symplectic Majorana spinor or a Dirac spinor. Finally, a table for general (K_+,K_-) supersymmetry algebras is presented. Though technically the general table is all that is required, the specific cases are presented because they are used very often in physics.

$(1,0)$ or $(0,1)$ algebras

First, we will discuss those theories with minimal superalgebras that have a single Majorana-Weyl spinor, therefore having $d_{\mathbb{S}}/2$ real supercharges.

This is only possible in signatures which are Weyl-compatible with orthogonal bilinear form. They are just a single Majorana-Weyl spinor. They, therefore, have a $\mathbb{Z}_2$ symmetry in the same way that a single Majorana spinor in odd dimensions does.

D	$(0,D)$	$(1,D-1)$	$(2,D-2)$	$(3,D-3)$	$(4,D-4)$	$(5,D-5)$	$(6,D-6)$
2	–	$\mathbb{Z}_2$	–				
4	–	–	–	–	–		
6	–	–	–	–	–	–	–
8	–	–	–	–	–	–	–
10	–	$\mathbb{Z}_2$	–	–	–	$\mathbb{Z}_2$	–
12	–	–	–	–	–	–	–

Table 3.7: R-symmetry groups possible in even signatures for '$\mathcal{N}=\frac{1}{2}$' algebras with $d_{\mathbb{S}}/2$ supercharges. A dash means no such algebra can be defined.

We remark that F-theory does not come with a super-Poincaré algebra in 12 dimensions [51], instead the supercharges obey

$$\{Q_+,Q_+\}=C\Gamma^{MN}Z_{MN}+C\Gamma^{MNPQRS}Z_{MNPQRS}. \tag{3.127}$$

Where Z_{MN} and Z_{MNPQRS} are BPS-charges. These are currently outside the scope of this construction, but the author wishes to pursue this in the future.

For dimensional reasons, F-theory requires a spinor with 32 real components, which is possible in (2,10), as the Weyl condition and Majorana condition take the 2^6 complex numbers (128 real numbers) and halve it twice. Majorana-Weyl spinors can only be defined in (2,10), hence the ambiguous statement that F-theory has two time-like dimensions. However, our analysis is only interested in super-admissible bilinear forms leading to a super-Poincaré algebra which is not possible here with only a single Majorana-Weyl spinor as the bilinear form is isotropic. We have not considered extensions to the algebra, so this is not covered in our analysis.

In 10 dimensions the only signatures that allow a Type I string theory are (1,9) and (5,5); we see this in the table above where they are represented by their $\mathbb{Z}_2$ R-symmetry group.

$(0,2),(1,1),(2,0)$ or $\mathcal{N}=1$ algebras

More signatures have a minimal superalgebra that has $d_{\mathbb{S}}$ supercharges, which are often called $(2,0)$ or $(1,1)$ algebras in orthogonal dimensions (like 6 and 10) and $\mathcal{N}=1$ superalgebras in isotropic dimensions (like 4 and 8). This is because in isotropic signatures we necessarily need equal copies of both, so we cannot have '$(K_+,0)$' algebras, necessarily need algebras of the form (K,K) which are then called $\mathcal{N}=K$ algebras with the chiral spinors combined into a single Dirac spinor.

In orthogonal dimensions, both Majorana bilinear forms are super-admissible (in 2 and 10 dimensions) or anti-super-admissible (in 6 dimensions). Weyl-compatible signatures have both $J_\pm^{(\epsilon)}$ defining a real or quaternionic structure on the Weyl spinor modules. For orthogonal Weyl-compatible signatures with a super-admissible Majorana bilinear forms and $J_\pm^{(\epsilon)}$ both giving real structures we can define a $(1,1)$ superalgebra with $\mathbb{Z}_2\times\mathbb{Z}_2$ or a $(2,0)$ superalgebra with R-symmetry group $\mathrm{O}(1,1)$ or $\mathrm{O}(2)$. If however we have super-admissible Majorana bilinear forms, but $J^{(\epsilon)}$ are quaternionic structures on $\mathbb{S}_\pm$ we can only define a $(2,0)$ superalgebra with $\mathrm{SO}(2)$ R-symmetry. When the Majorana bilinear forms are anti-super-admissible we can only define a $(2,0)$ algebra with R-symmetry group given by $\mathrm{SU}(2)$ if $J^{(\epsilon)_\pm}$ are quaternionic structures or $\mathrm{SU}(1,1)$ if they are real structures.

Orthogonal Weyl-incompatible can only have a $(1,1)$ superalgebra which therefore needs a super-admissible Majorana bilinear form. The result is a $\mathbb{Z}_2$ R-symmetry group because both chiralities are needed to define a reality condition, which then links the R-symmetry transformation on the Weyl spinor modules.

For isotropic signatures, we have the $\mathcal{N} = 1$ algebras. In Weyl-compatible signatures, we have an SO(1, 1) R-symmetry group, and in isotropic Weyl-incompatible signatures we have a U(1) R-symmetry group.

D	$(0,D)$	$(1,D-1)$	$(2,D-2)$	$(3,D-3)$	$(4,D-4)$	$(5,D-5)$	$(6,D-6)$
2	$\mathbb{Z}_2$	$\mathrm{O}(1,1),\mathrm{O}(2),\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$				
4	–	$\mathrm{U}(1)$	$\mathrm{SO}(1,1)$	$\mathrm{U}(1)$	–		
6	–	$\mathrm{SU}(2)$	–	$\mathrm{SU}(1,1)$	–	$\mathrm{SU}(2)$	–
8	$\mathrm{SO}(1,1)$	$\mathrm{U}(1)$	–	$\mathrm{U}(1)$	$\mathrm{SO}(1,1)$	$\mathrm{U}(1)$	–
10	$\mathbb{Z}_2$	$\mathrm{O}(1,1),\mathrm{O}(2),\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathrm{SO}(2)$	$\mathbb{Z}_2$	$\mathrm{O}(1,1),\mathrm{O}(2),\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$
12	–	$\mathrm{U}(1)$	$\mathrm{SO}(1,1)$	$\mathrm{U}(1)$	–	$\mathrm{U}(1)$	$\mathrm{SO}(1,1)$

Table 3.8: R-symmetry groups possible in even signatures with $d_{\mathbb{S}}$ supercharges

In (1,9) the superalgebra with $\mathbb{Z}_2 \times \mathbb{Z}_2$ R-symmetry is that of Type IIA or A* theories (this is discussed further in Section 3.11.3). The superalgebra with O(2) R-symmetry gives a Type IIB theory and that with O(1, 1) R-symmetry gives Type IIB*.

Any signatures without an entry in this table have a minimal superalgebra with $2d_{\mathbb{S}}$ real supercharges – in orthogonal signatures these would be of the (2,2) superalgebras, and in isotropic signatures they would be $\mathcal{N} = 2$ algebras (for example in (0,4) the minimal superalgebra is $\mathcal{N} = 2$).

General (K_+, K_-)

Below we present the even-dimensional R-symmetry groups for any signature in even dimensions, followed by a table of all signatures, combining the general table for odd and even dimensions.

D	$(0,D)$	$(1,D-1)$	$(2,D-2)$	$(3,D-3)$	$(4,D-4)$	$(5,D-5)$	$(6,D-6)$
2	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(p_+,q_+)\times\mathrm{O}(p_-,q_-)$	$\mathrm{O}(K,\mathbb{C})$				
4	$\mathrm{U}^*(K)$	$\mathrm{U}(p,q)$	$\mathrm{GL}(K,\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(K)$		
6	$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{USp}(K_+)\times\mathrm{USp}(K_-)$	$\mathrm{Sp}(K,\mathbb{C})$	X	$\mathrm{Sp}(K,\mathbb{C})$	$\mathrm{USp}(K_+)\times\mathrm{USp}(K_-)$	$\mathrm{Sp}(K,\mathbb{C})$
8	$\mathrm{GL}(K,\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(K)$	$\mathrm{U}(p,q)$	$\mathrm{GL}(K,\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(K)$
10	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(p_+,q_+)\times\mathrm{O}(p_-,q_-)$	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{SO}^*(K_+)\times\mathrm{SO}^*(K_-)$	$\mathrm{O}(K,\mathbb{C})$	$\mathrm{O}(p_+,q_+)\times\mathrm{O}(p_-,q_-)$	$\mathrm{O}(K,\mathbb{C})$
12	$\mathrm{U}^*(K)$	$\mathrm{U}(p,q)$	$\mathrm{GL}(K,\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(K)$	$\mathrm{U}(p,q)$	$\mathrm{GL}(K,\mathbb{R})$

Table 3.9: R-symmetry groups possible in each even dimension in any signature. $p_+ + q_+ = K_+$, $p_- + q_- = K_-$ and $K_+ + K_- = 2K$. X = $(\mathrm{Sp}(K_+,\mathbb{R})$ or $\mathrm{USp}(2r_+,2s_+)) \times (\mathrm{Sp}(K_-,\mathbb{R})$ or $\mathrm{USp}(2r_-,2s_-))$

Table of R-symmetry groups in all signatures up to $D = 12$

D	$(0,D)$	$(1,D-1)$	$(2,D-2)$	$(3,D-3)$	$(4,D-4)$	$(5,D-5)$	$(6,D-6)$
1	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$					
2	$\mathrm{O}(\mathcal{N},\mathbb{C})$	$\mathrm{O}(p_+,q_+)\times\mathrm{O}(p_-,q_-)$	$\mathrm{O}(\mathcal{N},\mathbb{C})$				
3	$\mathrm{SO}^*(\mathcal{N})$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(\mathcal{N})$			
4	$\mathrm{U}^*(\mathcal{N})$	$\mathrm{U}(p,q)$	$\mathrm{GL}(\mathcal{N},\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(\mathcal{N})$		
5	$\mathrm{USp}(\mathcal{N})$	$\mathrm{USp}(\mathcal{N})$	$\mathrm{Sp}(\mathcal{N},\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{Sp}(\mathcal{N},\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{USp}(\mathcal{N})$	$\mathrm{USp}(N)$	
6	$\mathrm{Sp}(\mathcal{N},\mathbb{C})$	$\mathrm{USp}(\mathcal{N}_+)\times\mathrm{USp}(\mathcal{N}_-)$	$\mathrm{Sp}(\mathcal{N},\mathbb{C})$	X	$\mathrm{Sp}(\mathcal{N},\mathbb{C})$	$\mathrm{USp}(\mathcal{N}_+)\times\mathrm{USp}(\mathcal{N}_-)$	$\mathrm{Sp}(\mathcal{N},\mathbb{C})$
7	$\mathrm{Sp}(\mathcal{N},\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{USp}(\mathcal{N})$	$\mathrm{USp}(N)$	$\mathrm{Sp}(\mathcal{N},\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{Sp}(N,\mathbb{R})$, $\mathrm{USp}(2r,2s)$	$\mathrm{USp}(\mathcal{N})$	$\mathrm{USp}(\mathcal{N})$
8	$\mathrm{GL}(\mathcal{N},\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(\mathcal{N})$	$\mathrm{U}(p,q)$	$\mathrm{GL}(\mathcal{N},\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(\mathcal{N})$
9	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(N)$	$\mathrm{SO}^*(\mathcal{N})$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(\mathcal{N})$
10	$\mathrm{O}(N,\mathbb{C})$	$\mathrm{O}(p_+,q_+)\times\mathrm{O}(p_-,q_-)$	$\mathrm{O}(\mathcal{N},\mathbb{C})$	$\mathrm{SO}^*(\mathcal{N}_+)\times\mathrm{SO}^*(\mathcal{N}_-)$	$\mathrm{O}(\mathcal{N},\mathbb{C})$	$\mathrm{O}(p_+,q_+)\times\mathrm{O}(p_-,q_-)$	$\mathrm{O}(\mathcal{N},\mathbb{C})$
11	$\mathrm{SO}^*(\mathcal{N})$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(\mathcal{N})$	$\mathrm{SO}^*(\mathcal{N})$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$
12	$\mathrm{U}^*(\mathcal{N})$	$\mathrm{U}(p,q)$	$\mathrm{GL}(\mathcal{N},\mathbb{R})$	$\mathrm{U}(p,q)$	$\mathrm{U}^*(\mathcal{N})$	$\mathrm{U}(p,q)$	$\mathrm{GL}(\mathcal{N},\mathbb{R})$

Table 3.10: $p+q=\mathcal{N}$, $2r+2s=\mathcal{N}$. $p_++q_+=\mathcal{N}_+$, $p_-+q_-=\mathcal{N}_-$. $\mathrm{X}=(\mathrm{Sp}(K_+,\mathbb{R})$ or $\mathrm{USp}(2r_+,2s_+))\times(\mathrm{Sp}(K_-,\mathbb{R})$ or $\mathrm{USp}(2r_-,2s_-))$. Note – In odd dimensions $\mathcal{N}$ is the number of copies of the complex spinor module involved in the construction. In even dimensions $\mathcal{N}$ is the number of Dirac spinor modules in all cases, except when we have Weyl-compatible signatures with orthogonal bilinear forms and these are described using $\mathcal{N}_+$ and $\mathcal{N}_-$, the number of positive and negative spinor modules.

3.8.4 R-Symmetry transformations on $\mathbb{S}$

In the following section, we will describe briefly how the previously calculated R-symmetry transformations (in our matrix notation) look when acting on a Dirac spinor in a more conventional notation. In even dimensions we can define γ_*, and this can be involved in R-symmetry transformations because it anticommutes with all γ-matrices and hence commutes with $\mathrm{Spin}(t,s)$. We will show that a generic Lie algebra element acts on $\mathbb{S}$ either trivially or up to a factor of γ_*. This is because the R-symmetry group has to necessarily commute with $\mathrm{Spin}(t,s)$ so it cannot be another γ-matrix or combination thereof.

γ_* acts on the Weyl spinors $\lambda_\pm \in \mathbb{S}_\pm$ as $\gamma_*\lambda_\pm = \pm\lambda_\pm$. We can always choose a basis where

$$\gamma_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.128}$$

the 1's in this matrix are to be understood as identity matrices acting on the spinor indices. In matrix notation, $\gamma_*\lambda^i_\pm = \pm\lambda^i_\pm$ acts on $(\lambda^i_+, \lambda^i_-)$ as the matrix

$$\begin{pmatrix} \mathbb{1}_K & 0 \\ 0 & -\mathbb{1}_K \end{pmatrix} = \gamma_* \otimes \mathbb{1}_K \tag{3.129}$$

Orthogonal Bilinear Forms

In orthogonal Weyl-compatible signatures the R-symmetry transformations act independently on each Weyl spinor module and act entirely on the internal $\mathbb{C}^{K_\pm}$ factor. This is because the Weyl spinor modules are complex irreducible modules (so we can apply Schur's lemma exactly like odd dimensions) and the reality condition is defined on a Weyl spinor module alone. If $K_+ \neq K_-$ the only manner to consider an R-symmetry transformation is on each spinor module independently.

When $K_+ = K_-$, we can combine the Weyl spinors into Dirac spinors. It is possible to recast a generic infinitesimal R-symmetry transformation to act in terms of γ_*. Given a generic R-symmetry Lie algebra element

$$r = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{3.130}$$

we rewrite this in terms of $c = \frac{1}{2}(a+b)$ and $d = \frac{1}{2}(a-b)$ so that

$$r = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} = (Id \otimes c) + (\gamma_* \otimes d). \tag{3.131}$$

We can see that, at most, R-symmetry generators act as identity or γ_* on the $\mathbb{S}$ factor. In this case, it is somewhat artificial and can be expressed in a manner where the transformations act independently on each chiral spinor module. In the remaining cases, we will do something similar, but the two successive transformations will be inseparable and dependent on one another.

For a Weyl-incompatible orthogonal signature, the reality condition links the two chiralities, and indeed we find the R-symmetry transformations on the two Weyl spinor modules are linked. We found

$$r = \begin{pmatrix} a & 0 \\ 0 & La^* L^{-1} \end{pmatrix}. \tag{3.132}$$

a acts entirely on $\mathbb{S}_+$ and La^*L^{-1} is the compensating infinitesimal transformation on $\mathbb{S}_-$ to maintain the reality condition. This can be recast into transformations that act on the entire spinor module $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$. Using that conjugation by L and complex conjugation are involutions, we decompose a into eigen-matrices under their composition

$$a_\pm = \frac{1}{2}(a \pm La^* L^{-1}). \tag{3.133}$$

So that we can write

$$r = \begin{pmatrix} a_+ & 0 \\ 0 & a_+ \end{pmatrix} + \begin{pmatrix} a_- & 0 \\ 0 & -a_- \end{pmatrix} = (\mathbb{1} \otimes a_+) + (\gamma_* \otimes a_-). \tag{3.134}$$

This is slightly different from the previous case with Weyl-compatible signatures because a_+ and a_- are functions of a alone. However we see that similarly the generators of the R-symmetry group can be written in a way where they act either as Id or γ_* on $\mathbb{S}$. The construction outlined in this chapter is 'manifestly R-symmetric' up to this level for orthogonal Weyl-incompatible signatures.

Isotropic Bilinear Forms

On an isotropic vector-valued bilinear form, γ_* generates a potential R-symmetry transformation. Under

$$\lambda^i \to e^{\omega\gamma_*}\lambda^i = e^{\omega}\lambda^i_+ + e^{-\omega}\lambda^i_-. \tag{3.135}$$

We see a general vector-valued bilinear form transforms as

$$\begin{aligned}\beta(\gamma^\mu\lambda,\chi) =&(\gamma^\mu\lambda^i_+)^T C\chi^j_- M_{ji} + (\gamma^\mu\lambda^i_-)^T C\chi^j_+ M_{ji} &(3.136)\\ \to&(\gamma^\mu e^{\omega}\lambda^i_+)^T Ce^{-\omega}\chi^j_- M_{ji} + (\gamma^\mu e^{\omega}\lambda^i_-)^T Ce^{-\omega}\chi^j_+ M_{ji} = \beta(\gamma^\mu\lambda,\chi). &(3.137)\end{aligned}$$

In matrix notation the transformation in (3.135) is

$$\begin{pmatrix}\lambda_+\\ \lambda_-\end{pmatrix} \to exp(\omega\gamma_*\otimes \mathbb{1}_K)\begin{pmatrix}\lambda_+\\ \lambda_-\end{pmatrix}. \tag{3.138}$$

Commuting with the reality condition forces ω to be real in Weyl-compatible signatures and ω to be imaginary in Weyl-incompatible signatures. This gives the $\mathrm{SO}(1,1)$ or $\mathrm{U}(1)$ subgroup of the R-symmetry group that often appears following dimensional reduction from odd to even dimensions. Additionally we could have R-symmetry group elements that act as γ_* on $\mathbb{S}$ and simultaneously act non-trivially on the $\mathbb{C}^K$ factor.

Given our generic form of an R-symmetry transformation in isotropic signatures,

$$r = \begin{pmatrix} a & 0\\ 0 & -M^{-1}a^T M\end{pmatrix}, \tag{3.139}$$

we notice, again, that conjugation by M and transposition are both involutions so that we can split a into eigen-matrices of the combination of these two operations

$$a = a_+ + a_-, \qquad \text{with} \qquad a_\pm = \frac{1}{2}(a \pm M^{-1}a^T M). \tag{3.140}$$

Therefore we can write

$$r = \begin{pmatrix} a_+ & 0\\ 0 & -a_+\end{pmatrix} + \begin{pmatrix} a_- & 0\\ 0 & a_-\end{pmatrix} = (\gamma_*\otimes a_+) + (\mathbb{1}\otimes a_-). \tag{3.141}$$

From this, we conclude that in isotropic signatures R-symmetry generators act up to a factor of γ_* on $\mathbb{S}$.

3.9 Recap of Construction

The previous section allows us to classify superalgebras according to the choice of bilinear form and reality condition (which are dimension- and signature-dependent quantities respectively);

$$\text{Superalgebra} \iff (\text{Bilinear Form}, \text{Reality Condition}). \tag{3.142}$$

To do so, we do the following:

- Find charge conjugation matrix details for the particular signature, contained in Table 3.2.
- Pick a C and use the correct form of M for the invariants.
- Select a reality condition. In even dimensions, it is easier to use the corresponding $B_\pm$ that goes with the choice of $C_\pm$, though this is not strictly necessary.
- Calculate α so that the vector-valued bilinear form is real with chosen reality condition.

Worked example

By using the presented tables, one can reconstruct how to define a manifestly R-symmetric theory in each signature. For example – in (2,7) the complex spinor module is equivalent to the real spinor module, so for N copies of the irreducible spinor module, we use complexified spinors that are elements of $\mathbb{S} \otimes \mathbb{C}^{2N}$.

From the Table 3.2 we see that $\sigma_C \tau_C = +1$ in 9 dimensions, therefore we use a symmetric bilinear form on $\mathbb{C}^{2N}$ to obtain a super-admissible bilinear form,

$$[C \otimes \delta](\lambda, \chi) = (\lambda^i)^T C \chi^j \delta_{ij}, \tag{3.143}$$

with $i, j = 1, ..., 2N$. Using Table 3.1 we see that B defines a quaternionic structure on $\mathbb{S}$ in signature (2,7). Therefore we define real spinors as those invariant under a real structure that combines a quaternionic structure on $\mathbb{C}^K$ with the quaternionic structure on $\mathbb{S}$:

$$(\lambda^i)^* = \alpha B \lambda^j J_{ji}. \tag{3.144}$$

For an R-symmetry transformation $\lambda^i \to R^i_j \lambda^j$. The bilinear form chosen on $\mathbb{C}^K$ means the complex R-symmetry group is $G_R^{\mathbb{C}} = \mathrm{O}(2N, \mathbb{C})$. The requirement that the R-symmetry transformation must commute with the reality condition implies

$$R = -JR^*J \tag{3.145}$$

The real form of $\mathrm{O}(2N, \mathbb{C})$ that satisfies this equation is $\mathrm{SO}^*(2N)$, under which λ^i transforms in the fundamental representation. This is also verified by Table 3.10.

3.10 Superalgebra Isomorphisms

In even dimensions, this construction can produce many possible superalgebras that are not necessarily unique. Namely, these choices are: the choice of Majorana bilinear form in even dimensions (which mandates the choice of M on $\mathbb{C}^K$), the choice of $B_\pm$ to define the reality condition and the choice of L in the reality condition (which is constricted by the choice of $B_\pm$). In most cases some or all of these choices are irrelevant, and in the following section we detail explicit relationships between said choices where applicable.

If two superalgebras are isomorphic, they necessarily have the same R-symmetry group (see Section 2.9). The converse is not necessarily true, for example, in orthogonal Weyl-compatible signatures one can define the superbracket and reality condition on each Weyl spinor module alone so one can have a Type IIA and a Type IIA*-like algebra that have the same R-symmetry group but are not isomorphic.

In orthogonal dimensions, we find the choices of B and C do not affect the R-symmetry group, and we demonstrate relations between the two choices. For orthogonal vector-valued bilinear forms with a Weyl-compatible reality condition, the chiral spinor modules function like in odd dimensions, so transformations are not necessary. An orthogonal vector-valued bilinear form along with a Weyl-incompatible reality conditions results in an R-symmetry group of $\mathrm{O}(K, \mathbb{C})$ or $\mathrm{Sp}(K, \mathbb{C})$ always and all choices are equivalent.

Isotropic supersymmetry algebras vary depending on the reality condition. For isotropic Weyl-compatible signatures, the R-symmetry group is always $\mathrm{GL}(K, \mathbb{R})$ or $\mathrm{U}^*(K)$ and any choices are irrelevant. In isotropic Weyl-incompatible signatures, the choice of B and C do have an effect, as the resulting choice of M and L defines the R-symmetry group, for two choices of M and L that give the same R-symmetry group we will give a map between the two descriptions. To maintain the signature of ML, which defines the R-symmetry group (as shown previously in Section 3.8.2), both will be changed by any map between the two isomorphic superalgebras.

In the following section, we will detail some isomorphisms between superalgebras. In some cases, the resulting reality condition is not in a canonical form and may need another reparameterisation of the spinor module to place it in these forms. These are transformations between superalgebras only, and more transformations may be needed to link Lagrangian theories that arise from them. For a worked example of implementing these transformations in a Lagrangian, see Chapter 5.

3.10.1 Isotropic Bilinear Form Map, S

First, we will discuss a very useful map, that for orthogonal dimensions changes the reality condition (leaving the bilinear form invariant) and for isotropic dimensions changes the bilinear form (leaving the reality condition invariant when it is Weyl-compatible, and changing it when it is Weyl-incompatible). Namely, it is

$$\lambda_+^i \to \lambda_+^i, \quad \lambda_-^i \to \lambda_-^j J_{ji}. \tag{3.146}$$

And in matrix notation this is

$$\begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} \to S \begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix}. \tag{3.147}$$

This is derived in the case of isotropic bilinear forms as a motivational example.

With this relatively simple transformation, we can make critical changes. For example, it is used heavily in Chapter 5, where it is used to relate the superalgebras obtained in (0,4) and (2,2), and in (1,3) it relates the U(2) R-symmetric $\mathcal{N} = 2$ algebra expressed in Majorana and symplectic Majorana spinors.

This map has varying effects on the reality condition depending on whether we are in Weyl-compatible or Weyl-incompatible signatures. We will discuss this after first discussing its effect on isotropic and orthogonal bilinear forms.

Isotropic Bilinear Forms

Recall that for an isotropic bilinear form if $C_\pm \otimes \delta$ is super-admissible, so is $C_\mp \otimes J$. The isotropic vector-valued bilinear form $[C_\pm \otimes J]$ is explicitly

$$(\gamma^\mu \lambda^i_+) C_\pm \chi^j_- J_{ji} + (\gamma^\mu \lambda^i_-)^T C_\pm \chi^j_+ J_{ji}. \tag{3.148}$$

Using (3.1) we can write this in terms of the other charge conjugation matrix

$$(\gamma^\mu \lambda^i_+)^T C_\pm \chi^j_- J_{ji} + (\gamma^\mu \lambda^i_-)^T C_\pm \chi^j_+ J_{ji} = -(\gamma^\mu \lambda^i_+)^T C_\mp \chi^j_- J_{ji} + (\gamma^\mu \lambda^i_-)^T C_\mp \chi^j_+ J_{ji}. \tag{3.149}$$

In our matrix notation, this equation is

$$\left((\gamma^\mu \lambda^i_+)^T, (\gamma^\mu \lambda^i_-)^T\right) C_\pm \begin{pmatrix} 0 & J_{ji} \\ J_{ji} & 0 \end{pmatrix} \begin{pmatrix} \chi^j_+ \\ \chi^j_- \end{pmatrix} = \left((\gamma^\mu \lambda^i_+)^T, (\gamma^\mu \lambda^i_-)^T\right) C_\mp \begin{pmatrix} 0 & -J_{ji} \\ J_{ji} & 0 \end{pmatrix} \begin{pmatrix} \chi^j_+ \\ \chi^j_- \end{pmatrix}. \tag{3.150}$$

And for $[C_\mp \otimes \delta]$ we have

$$(\gamma^\mu \Psi^i_+)^T C_\mp \Omega^j_- \delta_{ij} + (\gamma^\mu \Psi^i_-)^T C_\mp \Omega^j_+ \delta_{ij} \to \left((\gamma^\mu \Psi^i_+)^T, (\gamma^\mu \Psi^i_-)^T\right) C_\mp \begin{pmatrix} 0 & \delta_{ji} \\ \delta_{ji} & 0 \end{pmatrix} \begin{pmatrix} \Omega^j_+ \\ \Omega^j_- \end{pmatrix}. \tag{3.151}$$

Seeking a transformation that links (3.150) and (3.151) we set

$$\begin{pmatrix} \underline{\lambda}_+ \\ \underline{\lambda}_- \end{pmatrix} = S \begin{pmatrix} \underline{\Psi}_+ \\ \underline{\Psi}_- \end{pmatrix}. \tag{3.152}$$

this means that S satisfies

$$S^T \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} S = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.153}$$

so that the two vector-valued bilinear forms are equal, we find

$$S = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -J \end{pmatrix}. \tag{3.154}$$

In doing so, we have quickly motivated the map given at the start of this section. To summarise we have shown an equivalency of (complex, due to no reality condition) superalgebras defined by an extended spinor module with bilinear form $C_\pm \otimes J$ and $C_\mp \otimes \delta$ in isotropic signatures.

Orthogonal Bilinear Forms

Orthogonal bilinear forms are entirely chiral, and obviously, the map leaves the vector-valued bilinear form $[C \otimes M](\gamma^\mu \lambda_+, \chi_+)$ invariant. On the negative chirality spinors

$$(\gamma^\mu \lambda_-^i)^T C \chi_-^j M_{ji} = (\gamma^\mu \Psi_-^k)^T C \Omega_-^l M_{ji} J_{ki} J_{lj}. \tag{3.155}$$

For both our choices of $M = \{\delta, J\}$ we find $M_{ji} J_{ki} J_{lj} = M_{ji}$ so that this transformation is entirely inert on orthogonal vector-valued bilinear forms.

Recall that in dimensions with orthogonal Majorana bilinear forms they both necessarily have the same superadmissibility, meaning there is a single choice of M, so it is useful that the map does not change the bilinear form on $\mathbb{C}^K$. It does, however, change the reality condition.

Weyl-compatible signatures

Beginning with spinors $\lambda_\pm^i$ that obey a generic reality condition (the bracketed signs are not linked to the non-bracketed signs)

$$(\lambda_\pm^i)^* = \alpha B_{(\pm)} \lambda_\pm^j L_{ji}. \tag{3.156}$$

This map is not needed for orthogonal Weyl-compatible bilinear forms, so this is not considered here (these theories act like two independent copies of an odd-dimensional theory with unique superalgebras on each chirality). This means that L is given by one of our canonical choices $L = \{\delta, J, I_{1,1}, \tilde{I}_{2r,2s}\}$ [7]. We define the transformed spinors $\Psi_\pm^i$ using the map given above

$$\Psi_+^i = \lambda_+^i, \quad \Psi_-^i = -\lambda_-^j J_{ji}. \tag{3.157}$$

Obviously Ψ_+^i obeys the same reality condition as λ_+^i

$$(\Psi_+^i)^* = \alpha B_{(\pm)} \Psi_+^j L_{ji}. \tag{3.158}$$

[7] For isotropic Weyl-compatible signatures, where relate one bilinear form with $M = \delta$ and one with $M = J$ so that we only consider reality conditions using $L = \delta, I_{1,1}$ or $\tilde{I}_{2r,2s}$

Calculating the reality condition for Ψ^i_- is not so trivial:

$$\begin{aligned}(\Psi^i_-)^* &= -(\alpha B_{(\pm)}\lambda^k_- L_{kj})J_{ji} \\ &= -\alpha B_{(\pm)}\Psi^l_- J_{lk}L_{kj}J_{ji}.\end{aligned} \tag{3.159}$$

For the different choices of L we find

$$J_{lk}L_{kj}J_{ji} = \begin{cases} -\delta_{li} & L_{ij} = \delta_{ij}, \\ -J_{li} & L_{ij} = J_{ij}, \\ -(\tilde{I}_{2r,2s})_{li} & L_{ij} = (\tilde{I}_{2r,2s})_{ij}, \\ +(I_{1,1})_{ij} & L_{ij} = (I_{(1,1)})_{ij}. \end{cases} \tag{3.160}$$

For all choices but $L = I_{1,1}$ we see that Ψ^i_- obeys the same reality condition as λ^i_-. When $L = I_{1,1}$ we have an erroneous minus sign on $(\Psi^i_-)^*$. So far we have

$$(\Psi^i_+)^* = \alpha B_\pm \Psi^j_+ (I_{1,1})_{ji}, \qquad (\Psi^i_-)^* = -\alpha B_\pm \Psi^j_- (I_{1,1})_{ji}. \tag{3.161}$$

In this form, the two chiralities have different reality conditions and therefore cannot be combined into a Dirac spinor. This is fixed in different ways depending on the isotropy of the vector-valued bilinear form and is discussed case-by-case, detailed later.

Weyl-incompatible signatures

Now we will test the transformation on Weyl-incompatible reality conditions. A generic Weyl-incompatible reality condition is

$$(\lambda^i_\pm)^* = \alpha B_{(\pm)}\lambda^j_\mp L_{ji}. \tag{3.162}$$

The transformed spinors, $\Psi^i_\pm$, then obey

$$(\Psi^i_+)^* = (\lambda^i_+)^* = \alpha B_{(\pm)}\lambda^j_- L_{ji} = \alpha B_{(\pm)}\Psi^k_- J_{kj}L_{ji}, \tag{3.163}$$

$$(\Psi^i_-)^* = -(\lambda^i_-)^* J_{ji} = -\alpha B_{(\pm)}\lambda^j_+ L_{kj}J_{ji} = -\alpha B_{(\pm)}\Psi^k_+ L_{kj}J_{ji}. \tag{3.164}$$

We now have a sign difference between the two Weyl spinors. How we deal with this changes based on whether the Majorana bilinear forms are orthogonal or isotropic.

To evaluate the two products in these equations we consider $L = \delta$, $L = I_{1,1}$ and $L = \tilde{I}_{2r,2s}$

for real structures and $L = J$ for quaternionic, once again ignoring $L = I_{1,1}$ as the map only has limited use with orthogonal bilinear forms:

$$
\begin{aligned}
&J_{kj}\delta_{ji} = J_{ki}, \qquad \delta_{kj}J_{ji} = J_{ki}, \\
&J_{kj}J_{ji} = -\delta_{ki}, \qquad J_{kj}J_{ji} = -\delta_{ki}, \\
&J_{kj}(\tilde{I}_{2r,2s})_{ji} = \begin{pmatrix} 0 & I_{r,s}, \\ -I_{r,s} & 0 \end{pmatrix}_{ki}, \qquad (\tilde{I}_{2r,2s})_{kj}J_{ji} = \begin{pmatrix} 0 & I_{r,s}, \\ -I_{r,s} & 0 \end{pmatrix}_{ki}, \\
&(J_2)_{kj}(I_{1,1})_{ji} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{ki}, \qquad (I_{1,1})_{kj}(J_2)_{ji} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ki}.
\end{aligned}
\tag{3.165}
$$

Many of these are non-canonical reality conditions, and how we deal with them depends on the isotropy of the vector-valued bilinear form. It can always be solved by a redefinition of the spinors and examples are provided in the following section.

We have outlined the primary effects of the map on orthogonal and isotropic bilinear forms, and Weyl compatible and incompatible reality conditions. We will now combine these to outline the effects in the four resulting combination.

Orthogonal Weyl-compatible

Here the map has no affect, having an orthogonal bilinear form implies that either Majorana bilinear form mandates the same M and being Weyl-compatible means that the choice of L defines the R-symmetry group entirely. The choice of reality condition can be made independently on each semi-spinor module. Like in odd dimensions, choosing L selects the R-symmetry group entirely, and we cannot map between the choices.

Recall that the phase, α, in the reality condition is fixed by the requirement the vector-valued bilinear form is real up to a sign only. Therefore we can define non-isomorphic algebras with the same R-symmetry group in these signatures. This is discussed further in the 3.11.3.

Orthogonal Weyl-incompatible

In these signatures, we have a fixed M but we have already shown the map is inert on orthogonal vector-valued bilinear forms anyway. Weyl-incompatibility means that all possible L choices (that are canonical for our choice of M) are accessible, however, and the R-symmetry group (which is $\mathrm{O}(K,\mathbb{C})$ or $\mathrm{Sp}(K,\mathbb{C})$) is not affected by choice of the reality condition. From the equations in (3.166) we see that we can change L in the reality condition but may also need to reparameterise the spinors to obtain a canonical bilinear form.

In these signatures (3.3) implies that $B_{(\pm)}\lambda_\pm = -iB_{(\mp)}\lambda_\pm$. We therefore cannot remove the sign difference in (3.163) and (3.164) when $L \neq I_{1,1}$ so a further transformation is required. This sign difference can be dealt with by the next transformation R in Section 3.10.2 proceeding this section.

Isotropic Weyl-compatible

The map transforms the two bilinear forms between each other, and the effect on the reality conditions was demonstrated previously in (3.160), explicitly for all choices but $L = I_{1,1}$ we found that $\Psi^i_\pm$ obeys the same reality condition as $\lambda^i_\pm$, so we are already done. When $L = I_{1,1}$ we have an erroneous minus sign on $(\Psi^i_-)^*$:

$$\begin{aligned}(\Psi^i_+)^* &= \alpha B_\pm \Psi^j_+ (I_{1,1})_{ji},\\ (\Psi^i_-)^* &= -\alpha B_\pm \Psi^j_- (I_{1,1})_{ji}.\end{aligned} \tag{3.166}$$

In this form, the two chiralities have different reality conditions and therefore cannot be combined into a Dirac spinor. However, (3.3) implies that in isotropic signatures $B_{(\pm)}\Psi_\pm = \pm B_{(\mp)}\Psi_\pm$ so that we can fix this sign difference:

$$\begin{aligned}(\Psi^i_+)^* &= \alpha B_\mp \Psi^j_+ (I_{1,1})_{ji},\\ (\Psi^i_-)^* &= \alpha B_\mp \Psi^j_- (I_{1,1})_{ji}.\end{aligned} \tag{3.167}$$

In this case, we have changed from $B_\pm$ to $B_\mp$, while all other cases have maintained the original choice of $B_\pm$.

Isotropic Weyl-incompatible

In isotropic Weyl-incompatible signatures, the R-symmetry group was determined by the product ML. This map changes M, so we should expect this map to have a compensating change in the reality condition (such that the signature of ML is maintained) because isomorphic superalgebras have the same R-symmetry group.

For Weyl-incompatible signatures we have

$$
\begin{aligned}
(\Psi_+^i)^* &= -\alpha B_{(\mp)} \Psi_-^k J_{kj} L_{ji}, \\
(\Psi_-^i)^* &= -(\lambda_-^i)^* J_{ji} = -\alpha B_{(\pm)} \lambda_+^j L_{kj} J_{ji} \\
&= -\alpha B_{(\pm)} \Psi_+^k L_{kj} J_{ji} = +\alpha B_{(\mp)} \Psi_+^k L_{kj} J_{ji}.
\end{aligned}
\tag{3.168}
$$

Where we have again used (3.3) to obtain the final expression. We see that similar to the orthogonal Weyl-incompatible case it may be more natural to change $B_\pm$ to $B_\mp$ so that they have the same sign, depending on the product of matrices in the two equations.

We see that for all but $L = I_{1,1}$ the two equations are identical so that it is more natural to take the final line of (3.163) and (3.164) and find the new reality conditions matches. With $I_{1,1}$ the minus sign cancels out the original so that we do not need to change the chosen B matrix. Alternatively, these changes must be made as we recall that one B defines a real structure, and the other a quaternionic structure, and we must change which to match the structure defined by L.

When $L = \tilde{I}_{2r,2s}$ or $I_{1,1}$ we are no longer in one of our conventional choices for L. For a starting $L = \tilde{I}_{2r,2s}$ we can correct this using a further reparameterisation of the spinors. If

$$
(\Psi^i)^* = \alpha B_\pm \Psi^j \begin{pmatrix} 0 & I_{r,s} \\ -I_{r,s} & 0 \end{pmatrix}_{ji}. \tag{3.169}
$$

We see that the righthand side is antisymmetric, so we wish to rotate to the canonical form with the matrix J while preserving the bilinear form (which is $[C \otimes \delta]$). To do this, we define

$$
\psi^i = -I_{r,s} \Psi^i, \qquad \text{for} \quad 0 < i \leq k \quad \text{or} \quad k < i \leq K. \tag{3.170}
$$

Doing this, one finds that

$$
(\psi^i)^* = \alpha B_\pm \psi^j J_{ji}. \tag{3.171}
$$

For $L = I_{1,1}$ we see it is still a real structure with signature (1,1), so we map to $I_{1,1}$:

$$\psi^1 = \frac{1}{\sqrt{2}}(\Psi^1 + \Psi^2), \qquad \psi^2 = \frac{1}{\sqrt{2}}(\Psi^1 - \Psi^2). \tag{3.172}$$

To summarise, for Weyl-incompatible signatures we have shown $[C_\pm \otimes J](\lambda, \chi) = [C_\mp \otimes \delta](\Psi, \Omega)$ such that

$$(\lambda^i)^* = \alpha B_\pm \lambda^i \to (\Psi^i)^* = -\alpha B_\mp \Psi^j J_{ji}, \tag{3.173}$$

$$(\lambda^i)^* = \alpha B_\pm \lambda^j J_{ji} \to (\Psi^i)^* = \alpha B_\mp \Psi^i, \tag{3.174}$$

$$(\lambda^i)^* = \alpha B_\pm \lambda^j (\tilde{I}_{2r,2s})_{ji} \to (\Psi^i)^* = \alpha B_\mp \Psi^j J_{ji}, \tag{3.175}$$

$$(\lambda^i)^* = \alpha B_\pm \lambda^j (I_{1,1})_{ji} \to (\Psi^i)^* = \alpha B_\pm \Psi^j (I_{1,1})_{ji}. \tag{3.176}$$

The first two shows how the map works transforming Majorana to symplectic Majorana and back again. The third and fourth equations need the additional change of basis so that the reality condition is in the canonical forms presented. In each case, one finds the signature of ML is maintained, such that the R-symmetry group is the same.

3.10.2 Reality Condition Map, R

Next, we introduce another useful isomorphism that can be used for manipulating reality conditions for both orthogonal and isotropic dimensions. It exchanges B_+ and B_- in the reality condition. For an orthogonal vector-valued bilinear form it changes the bilinear form from $C_\pm \otimes M$ to $C_\mp \otimes M$, in addition to changing the sign of the reality condition. In isotropic dimensions it does this without changing the bilinear form or choice of L, demonstrating that the choice of B is irrelevant in isotropic dimensions.

The transformation is that we set

$$\lambda^i = \frac{1}{\sqrt{2}}(1 - i\gamma_*)\Psi^i \implies \Psi^i = \frac{1}{\sqrt{2}}(1 + i\gamma_*)\lambda^i. \tag{3.177}$$

Orthogonal Dimensions, $D = 2, 6, 10$

In this section we assume that λ and Ψ in equation (3.177) obey

$$(\lambda^i_\pm)^* = \alpha_\pm B_- \lambda^j_\pm L_{ji}, \qquad (\Psi^i)^* = \beta_\pm B_+ \Psi^j_\pm L_{ji}. \tag{3.178}$$

For theories with orthogonal vector-valued bilinear forms, it is natural to consider this map acting on the chiral spinors; we see this works as

$$\Psi^i_\pm = \frac{1}{\sqrt{2}}(1 + i\gamma_*)\lambda^i_\pm = \frac{1}{\sqrt{2}}(1 \pm i)\lambda^i_\pm = \sqrt{\pm i}\lambda^i_\pm. \tag{3.179}$$

Note that this map maintains the chirality of the spinors (as γ_* commutes with $(1+i\gamma_*)$). This gives us the reality condition on the projected spinors as

$$(\Psi^i_\pm)^* = \sqrt{\mp i}(\lambda^i_\pm)^* = i\sqrt{\pm i}\alpha_\pm B_- \lambda^j_\pm L_{ji}, \tag{3.180}$$

$$= i\alpha_\pm B_- \Psi^j_\pm L_{ji} = \mp\alpha_\pm B_+ \Psi^j_\pm L_{ji}. \tag{3.181}$$

This sign between the two chiralities is what is needed to sort out the sign problem mentioned in the previous summary about the effects of S on orthogonal Weyl-incompatible structures.

Recall that for an orthogonal bilinear form, if $[C_+ \otimes M]$ is super-admissible then so is $[C_- \otimes M]$. Using (3.1) and (3.3) we find these two super-admissible bilinear forms are related by

$$[C_+ \otimes M](\gamma^\mu \underline{\lambda}_\pm, \underline{\chi}_\pm) = \pm i[C_- \otimes M](\gamma^\mu \underline{\lambda}_\pm, \underline{\chi}_\pm). \tag{3.182}$$

The map above already removes this factor of $\pm i$:

$$[C_+ \otimes M](\gamma^\mu \underline{\lambda}_\pm, \underline{\chi}_\pm) = [C_- \otimes M](\gamma^\mu \underline{\Psi}_\pm, \underline{\Omega}_\pm). \tag{3.183}$$

From (3.3) we know that in orthogonal Weyl-compatible signatures a superalgebra with a $[C_+ \otimes M]$ bilinear form with chiral spinors $\lambda^i_\pm$ that obey $(\lambda^i_\pm)^* = \alpha_\pm B_- \lambda^j_\pm L_{ji}$ is equivalent to a theory with reality condition given by $(\lambda^i)^* = \pm i\alpha_\pm B_+ \lambda^j L_{ji}$. Further we have just shown it is also equivalent to a superalgebra defined using a C_- bilinear form with spinors defined by $\Psi^i_\pm = \sqrt{\mp i}\lambda^i_\pm$ whose reality condition is given by $(\Psi^i_\pm)^* = i\alpha_\pm B_- \psi^j_\pm L_{ji}$ or $(\Psi^i_\pm)^* = \mp\alpha_\pm B_+ \psi^j_\pm L_{ji}$.

From this, in orthogonal Weyl-compatible signatures, we conclude that the choice of C and B in the bilinear form and reality condition is unimportant, as we can always rewrite the spinors to compensate. The only distinguishing feature of these superalgebras is the choice of $\alpha_\pm$. They are fixed up to a sign so we have two choices, $\alpha_+ = \alpha_-$ and $\alpha_+ = -\alpha_-$. Later we will show that Type IIA superalgebras are an example of the former, and the Type IIA* superalgebra is an example of the latter.

Isotropic Dimensions, $D = 4, 8, 12$

In isotropic dimensions it is natural and convenient to consider only Dirac spinors λ^i and Ψ^i. Choosing $(\lambda^i)^* = \alpha B \lambda^j L_{ji}$ we see that

$$(\lambda^i)^* = \alpha B_- \lambda^j L_{ji} \implies (\Psi^i)^* = -i\alpha B_+ \Psi^j L_{ji}. \tag{3.184}$$

To calculate this (3.3) and (3.4) were used. We see that we have effectively swapped B_- and B_+, defining a new reality condition with phase $-i\alpha$.

Using that in isotropic dimensions $\gamma_* C_\pm = C_\pm \gamma_*$, we see an isotropic vector-valued bilinear form is unchanged by this transformation

$$\begin{aligned}
&(\gamma^m \lambda^i)^T C_\pm \chi^j M_{ji} \\
\rightarrow &\frac{1}{2}(\gamma^m (1 + i\gamma_*)\Psi^i)^T C_\pm (1 + i\gamma_*)\Omega^j M_{ji} \\
= &\frac{1}{2}(\Psi^i)^T \Big((\gamma^m)^T C_\pm - \gamma_* (\gamma^m)^T C_\pm \gamma_* + i\gamma_* (\gamma^m)^T C_\pm + i(\gamma^m)^T C_\pm \gamma_* \Big) \Omega^j M_{ji} \\
= &\frac{1}{2}(\Psi^i)^T \Big((\gamma^m)^T C_\pm + (\gamma^m)^T C_\pm (\gamma_*)^2 - i(\gamma^m)^T C_\pm \gamma_* + i(\gamma^m)^T C_\pm \gamma_* \Big) \Omega^j M_{ji} \\
= &(\gamma^m \Psi^i)^T C_\pm \Omega^j M_{ji}.
\end{aligned} \tag{3.185}$$

Therefore we have shown the choice of B_+ or B_- in the reality condition is unimportant in isotropic dimensions (up to compensatory factors of i).

In the isotropic Weyl-incompatible signatures the R-symmetry group is always $\mathrm{GL}(K, \mathbb{R})$ when $J_\pm^{(\epsilon)}$ are real structures and $\mathrm{U}^*(K)$ when they are quaternionic. If they are quaternionic there is one choice for L, and when they are real all choices lead to the same group, so the particular choice of L does not matter either [8]. This map shows that the choice of B is irrelevant, and the S map shows the choice of bilinear form is irrelevant; all supersymmetry algebra definitions in this formalism are equivalent in this case.

[8] See Chapter 5 for an example of how to relate reality conditions with $L = \delta$ and with $L = I_{p,q}$.

3.11 Some Applications in Physics

To aid in using the formalism in physical theories, this section will demonstrate some applications of the formalism and give explicit examples of its usage.

The first examples will concern dimensional reduction, where there are two primary avenues to explore for this formalism. First, we can justify the reduced theories R-symmetry groups in terms of the parent theories groups, and motivate their origin. Second, we can demonstrate how the spinors in the parent theory decompose into those in the daughter theory.

Next, we specialise to ten dimensions, demonstrating how the Lorentzian signature Type IIA, Type IIA*, Type IIB and Type IIB* theories arise in this formalism and then expanding the scope to include exotic theories in alternative signatures. The discussion will be presented in terms of our formalism and how this relates to the other common descriptions will be described.

We will only explicitly deal with reduction by one dimension, along a time-like and space-like direction. For more than one step the following can be composed together, or it can be done all in one go (as is common say from ten dimensions to four) using a slightly different methodology. This will not be discussed here, as we will instead use group theory to derive the daughter theories of these dimensional reductions.

3.11.1 Dimensional Reduction

Dimensional reduction is a commonly used technique in physics to derive lower dimensional theories from higher dimensional theories. The formalism presented in this chapter allows one to perform dimensional reduction quite smoothly, and we can also use this to inform ourselves about the supersymmetry algebras in nine and ten dimensions which provides some insights into T-duality. This section will provide some details on performing dimensional reduction, including common physical examples.

Odd to even dimensions

The space-time indices of the higher dimensional theory are $M = 0, ..., D$. Here we use the conventions that when doing a space-like direction we remove the final direction (going from $(D+1)$ to D dimensions this is the Dth direction) and when doing a time-like reduction we remove the $0th$ direction. Therefore the lower dimensional space-time indices are are $\mu = 1, ..., D$ for a time-like reduction or $\mu = 0, ..., D-1$ for a space-like reduction.

When we reduce from an odd to even dimensions the dimension of the Dirac spinor module does not decrease, making this step simpler than the previous. We equate the higher dimensional spinors and γ-matrices to the lower ones:

$$\lambda^i_{(D+1)} = \lambda^i_{(D)}, \quad \Gamma_M = \begin{cases} \{\gamma_\mu, \gamma_{(D+1)} = \gamma_*\} & \text{Space-like reduction,} \\ \{\gamma_0 = i\gamma_*, \gamma_\mu\} & \text{Time-like reduction.} \end{cases} \tag{3.186}$$

The removed γ-matrix is proportional to the projection operator, γ_*, a representation can always be chosen such that for a space-like reduction $\Gamma_{(D+1)} = \gamma_*$ and for a time-like reduction $\Gamma_0 = i\gamma_*$.

The charge conjugation matrix of the $D+1$-dimensional theory is equal to one of the two charge conjugation matrices the even D dimensions. This can be inferred from Table 3.2, if the lower-dimensional theory is orthogonal it is a C_+ and if it is isotropic it is a C_-. The bilinear form is then $C_+ \otimes M$ or $C_- \otimes M$ with the M inherited from the parent theory.

The reality condition is inherited from the higher dimensional theory, though one will need to rewrite the B matrix in terms of the lower dimensional B matrices. When going from odd- to even-dimensions the dimensionally reduced B matrices satisfy, first for orthogonal dimensions

$$\begin{aligned} B^{(t,s)} &= (C(A^{(t,s)})^{-1})^T = (C_-(A^{(t,s-1)})^{-1})^T = B_+^{(t,s-1)}, \\ B^{(t,s)} &= (-1)^t(-iC_-(A^{(t-1,s)})^{-1})^T = (-1)^t B_-^{(t-1,s)}, \end{aligned} \tag{3.187}$$

and for isotropic dimensions this is

$$\begin{aligned} B^{(t,s)} &= (C(A^{(t,s)})^{-1})^T = (C_-(A^{(t,s-1)})^{-1})^T = B_-^{(t,s-1)} m, \\ B^{(t,s)} &= (-1)^t(-iC_+(A^{(t-1,s)})^{-1})^T = (-1)^{t+1} i B_+^{(t-1,s)}. \end{aligned} \tag{3.188}$$

At least one of the ϵ-quaternionic structures in the reduced signature has the same ϵ as the ϵ-quaternionic structure in the parent signature when going from odd dimensions to even dimensions, so that the daughter theories can have the same L.

A detailed version of this is in [3] and this thesis in Chapter 5.

Even to odd dimensions

Here the dimensionality of Dirac spinors halves as we dimensionally reduce, but we can (very roughly) equate the Weyl spinors of the parent theory with the Dirac spinors of the daughter theory. The charge conjugation matrices and γ-matrices must halve. In dimension too. We must be careful how we do the embedding, though fortunately there are not too many possibilities.

In the odd-dimensional daughter signature, we only have a single C, but it will always be related to one of the two even-dimensional parents ones. When the parent it is orthogonal the daughter theory has the same invariants as C_{-}; when the parent theory is isotropic the daughter theory's charge conjugation matrix has the same invariants as C_{+}. We embed it according to

$$C_{-}^{(d+1)} = C^{(d)} \otimes \sigma_1, \quad \mathrm{d} = 5,\, 9 \qquad \text{or} \qquad C_{+}^{(d+1)} = C^{(d)} \otimes 1, \quad \mathrm{d} = 3,\, 7,\, 11. \tag{3.189}$$

The bilinear form on the extended spinor module of the parent theory is then assumed to be $C^{(d+1)} \otimes M$ with whatever $C^{(d+1)}$ is in the above formula and $M = \{\delta, J\}$ is the correct choice to make the bilinear form super-admissible. If the parent theory bilinear form is different, one can use the maps contained in Section 3.10 to obtain a formulation in the correct form.

Finally we have to choose an embedding of the γ-matrices, those in the parent theory will be called Γ_M, with $M = 1, ..., d+1$ if we are reducing along a space-like direction and $M = 0, ..., d$ if we are reducing along a time-like direction. The γ-matrices of the daughter theory are γ_μ, with $\mu = 1, ..., d$ always. We embed the γ-matrices as follows

$$\Gamma_\mu = \gamma_\mu \otimes \sigma_1, \quad \Gamma_{(d+1)} = 1 \otimes \sigma_2 \quad \text{or} \quad \Gamma_0 = i1 \otimes \sigma_2. \tag{3.190}$$

We only have (and will remove) either $\Gamma_{(d+1)}$ or Γ_0 if we wish to reduce along a space-like or time-like direction. We define Γ_* according to the conventional

$$\Gamma_* = (-i)^{d/2+t} \prod_\mu \Gamma_\mu \tag{3.191}$$

and choose the γ_μ such that

$$\Gamma_* = 1 \otimes \sigma_3. \tag{3.192}$$

Note that it is always possible, as the daughter theory is in odd dimensions, there are two inequivalent representations of the Clifford algebra that vary up to a sign on $\gamma_{(d)}$, so here we assume that this was chosen correctly so that the above holds.

For completeness we then have the other charge conjugation matrix, from (3.1) given as

$$C_+^{(d+1)} = C^{(d)} \otimes \sigma_2 \quad \mathrm{d} = 5,\, 9 \qquad \text{or} \qquad C_-^{(d+1)} = C^{(d)} \otimes \sigma_3 \quad \mathrm{d} = 3,\, 7,\, 11. \tag{3.193}$$

We can therefore decompose the $d+1$ dimensional spinors into d dimensional spinors according to

$$\lambda_+^i = \psi^i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_-^{\hat{i}} = \psi^{i+K_+} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.194}$$

where λ_+^i and $\lambda_-^{\hat{i}}$ are the spinors in $d+1$ dimensions, of which we have K_+ and K_- respectively, and ψ^i the spinors in d dimensions, of which we now have $K_+ + K_-$. We may need to transform the ψ^i quantities to put the bilinear form and reality condition into canonical forms.

We are now able to dimensionally reduce the vector-valued bilinear form. We have two cases, namely orthogonal and isotropic vector-valued bilinear forms. We begin with an orthogonal vector-valued bilinear form with K_+ positive and K_- negative chirality spinors

$$(\Gamma^M\lambda_+^i)^T C^{(d+1)}\chi_+^j M_{ji} + (\Gamma^M\lambda_-^{\hat{i}})^T C^{(d+1)}\chi_-^{\hat{j}} M'_{\hat{j}\hat{i}} \tag{3.195}$$

$$=(\gamma^\mu\psi^i)^T C^{(d)}\phi^j M_{ji} \otimes (\sigma_1\begin{pmatrix}1\\0\end{pmatrix})^T\sigma_1\begin{pmatrix}1\\0\end{pmatrix} + (\gamma^\mu\psi^{\tilde{i}})^T C^{(d)}\phi^{\tilde{j}} M'_{\tilde{j}\tilde{i}} \otimes (\sigma_1\begin{pmatrix}0\\1\end{pmatrix})^T\sigma_1\begin{pmatrix}0\\1\end{pmatrix} \tag{3.196}$$

$$=\left((\gamma^\mu\psi^i)^T C^{(d)}\phi^j M_{ji} + (\gamma^\mu\psi^{\tilde{i}})^T C^{(d)}\phi^{\tilde{j}} M'_{\tilde{j}\tilde{i}}\right)\otimes 1 \tag{3.197}$$

$$=(\gamma^\mu\psi^i)C^{(d)}\phi^j\begin{pmatrix}M & 0\\ 0 & M'\end{pmatrix}_{ji}\otimes 1. \tag{3.198}$$

Where $i,j = 1,...,K$ and $\tilde{i},\tilde{j} = K_+ + 1,...,K_+ + K_-$ until the final line where we have combined the indices so that $i,j = 1,...,K_+ + K_-$. M and M' will be of the same form, either δ or J, but are $K_+ \times K_+$ and $K_- \times K_-$ matrices respectively.

Note if $M = \delta$ this is already correctly lined up so the d dimensional theory has vector-valued bilinear form

$$(\gamma^\mu\psi^i)C^{(d)}\phi^j\delta_{ji}, \qquad i = 1,...,K_+ + K_-. \tag{3.199}$$

However, if $M = J$ we are not in the canonical, in that

$$\begin{pmatrix}J_{K_+} & 0\\ 0 & J_{K_-}\end{pmatrix} \neq J_{K_+ + K_-}. \tag{3.200}$$

We then need a change of basis for ψ^i to realign the spinors into a canonical form (this will also affect the reality condition).

And for isotropic dimensions, remembering that $K_+ = K_-$ and $M = M'$ necessarily in these dimensions, we find the following

$$(\Gamma^M\lambda_+^i)^T C^{(d+1)}\chi_-^j M_{ji} + (\Gamma^M\lambda_-^i)^T C^{(d+1)}\chi_+^j M_{ji} \tag{3.201}$$

$$=(\gamma^\mu\psi^i)^T C^{(d)}\phi^{\tilde{j}} M_{\tilde{j}i} \otimes (\sigma_1\begin{pmatrix}1\\0\end{pmatrix})^T\begin{pmatrix}1\\0\end{pmatrix} + (\gamma^\mu\psi^{\tilde{i}})^T C^{(d)}\phi^j M_{j\tilde{i}} \otimes (\sigma_1\begin{pmatrix}0\\1\end{pmatrix})^T\begin{pmatrix}0\\1\end{pmatrix} \tag{3.202}$$

$$=(\gamma^\mu\psi^i)^T C^{(d)}\phi^j\begin{pmatrix}0 & M\\ M & 0\end{pmatrix}\otimes 1. \tag{3.203}$$

Where $i, j = 1, ..., K$ and $\tilde{i}, \tilde{j} = K + 1, ..., 2K$ until the final line where we have combined the indices so that $i, j = 1, ..., 2K$. In the final expression M represent the original $K \times K$ Gram matrices inherited from the parent theory. We then will want a transformation to take this into our canonical form too.

We then need to consider how the reality condition reduces. Due to our different embedding for C we have different factorisations of the B matrices depending on whether the parent is orthogonal or isotropic. First dimensionally reducing along a space-like dimension, from $(t, s+1)$ to (t, s) we find

$$B_+^{(t,s+1)} = \begin{cases} -B^{(t,s)} \otimes \sigma_1^t \sigma_2 & \text{Orthogonal,} \\ B^{(t,s)} \otimes \sigma_1^t & \text{Isotropic,} \end{cases} \tag{3.204}$$

$$B_-^{(t,s+1)} = \begin{cases} B^{(t,s)} \otimes \sigma_1^{t+1} & \text{Orthogonal,} \\ B^{(t,s)} \otimes \sigma_1^t \sigma_3 & \text{Isotropic.} \end{cases} \tag{3.205}$$

And along a time-like direction, from $(t+1, s)$ to (t, s) we find

$$B_+^{(t+1,s)} = \begin{cases} (-1)^{t+1} i B^{(t,s)} \otimes \sigma_1^t & \text{Orthogonal,} \\ i B^{(t,s)} \otimes \sigma_2 \sigma_1^t & \text{Isotropic,} \end{cases} \tag{3.206}$$

$$B_-^{(t+1,s)} = \begin{cases} i B^{(t,s)} \otimes \sigma_2 \sigma_1^{t+1} & \text{Orthogonal,} \\ (-1)^{t+1} B^{(t,s)} \otimes \sigma_1^{t+1} & \text{Isotropic.} \end{cases} \tag{3.207}$$

Some of these will be useful for explicitly working out the dimensional reductions in the following section, though all were included for completeness.

3.11.2 Dimensional Reduction Examples

In the following section, we will use group theory to demonstrate reductions from 6D to 3D and 4D, in doing so motivating some families of theories that are used in the literature and show some new possible reductions.

We will dimensionally reduce a 10D theory to 9 dimensions and explore T-duality in exotic signatures while doing so. We will also use group theory to discuss the reduction to 4D, but not do this explicitly as this is not particularly enlightening.

6D to 4D

6D $\mathcal{N} = 1$ superalgebras (also called $\mathcal{N} = 2$ in 4D units) become 4D $\mathcal{N} = 2$ theories upon dimensional reduction. In 6D we have the following collection of $\mathcal{N} = 1$ theories (defined by their R-symmetry group).

D	$(0,6)$	$(1,5)$	$(2,4)$	$(3,3)$	$(4,2)$	$(5,1)$	$(6,0)$
6	No	SU(2)	No	SU(1,1)	No	SU(2)	No

and in 4D we have the following $\mathcal{N} = 2$ theories

D	$(0,4)$	$(1,3)$	$(2,2)$	$(3,1)$	$(4,0)$
4	$\mathrm{U}^*(2)$	U(2) or U(1,1)	$\mathrm{GL}(2,\mathbb{R})$	U(2) or U(1,1)	$\mathrm{U}^*(2)$

We first remark that we have three possible signatures for the internal space to obtain a 4D theory from a 6D theory – the compact manifold we reduce upon can have two space-like, two time-like or one space-like and one time-like direction. Therefore we expect to either obtain an SO(1,1) or SO(2) subgroup in our 4D R-symmetry groups which can be attributed to the holonomy of this internal space.

The only 6D theory that can reach $(0,4)$ is $(1,5)$ reduced along a time-like and space-like direction, taking the original SU(2) R-symmetry and adding an SO(1,1) factor to obtain $\mathrm{U}^*(2) \cong \mathrm{SO}(1,1) \times \mathrm{SU}(2)$. Similarly, to reach $(2,2)$ we can only start from $(3,3)$ and reduce along a space-like and time-like direction – therefore we see the original SU(1,1) R-symmetry is supplemented by a SO(1,1) factor to obtain $\mathrm{GL}(2,\mathbb{R}) \approx \mathrm{SO}(1,1) \cdot \mathrm{SU}(1,1)$

In Minkowski signature we have two 4D superalgebras, one with a U(2) R-symmetry group and one with U(1,1). We also have two possible parent theories, a $(1,5)$ theory with SU(2) R-symmetry (which is then reduced along two space-like directions) and a $(3,3)$ with SU(1,1) R-symmetry (which is then reduced along two time-like directions). Either path gives us an R-symmetry subgroup attributed to the internal space of $\mathrm{SO}(2) \cong \mathrm{U}(1)$. Using the local isomorphisms

$$\mathrm{U}(2) \approx \mathrm{U}(1) \cdot \mathrm{SU}(2), \qquad \mathrm{U}(1,1) \approx \mathrm{U}(1) \cdot \mathrm{SU}(1,1), \tag{3.208}$$

we see the $(1,3)$ theory with U(2) R-symmetry would be obtained from the reduction of the $(1,5)$ theory, and the one with U(1,1) R-symmetry is derived from the $(3,3)$ theory. This is in agreement with [3] and Chapter 5 where we obtained the U(1,1) theory from a (2,3) 5D theory, and the U(2) from the reduction of a (1,4) theory, whose only possibles parents are the (3,3) and (1,5) 6D theories respectively.

6D to 3D

We could also reduce the 6D theories to 3D. Similarly, these can be motivated through group theory alone. In three dimensions we have the following $\mathcal{N} = 4$ theories

D	$(0,3)$	$(1,2)$	$(2,1)$	$(3,0)$
3	$\mathrm{SO}^*(4)$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{SO}^*(4)$

First, we note the isomorphism $\mathrm{SO}^*(4) \cong \mathrm{SU}(1,1) \times \mathrm{SU}(2)$ R-symmetry group. We can identify two potential parents for this theory. Interpreting the $\mathrm{SU}(1,1) \cong \mathrm{SO}(1,2)$ factor as the holonomy group of the internal space, we see it matches the reduction of a (1,5) parent with SU(2) R-symmetry. Alternatively, we could say the $\mathrm{SU}(2) \cong \mathrm{SO}(3)$ factor comes from the reduction of a (3,3) theory with $\mathrm{SU}(1,1)$ R-symmetry. Both possible paths to the (0,3) theory produce an identical R-symmetry group as expected as the R-symmetry group only has one possibility in this signature.

In (1,2) signature we have three possibilities: O(4), $\mathrm{O}(1,3)$ and $\mathrm{O}(2,2)$. Remarking that $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, we could get to the theory with O(4) R-symmetry from the entirely space-like reduction from (1,5) with one $\mathrm{SU}(2) \cong \mathrm{SO}(3)$ factor arising from reduction along three space-like directions and the other being inherited from the R-symmetry of the (1,5) theory. Similarly for the theory with $\mathrm{O}(2,2)$ R-symmetry, knowing $\mathrm{SO}(2,2) \cong \mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$ this would be obtained from the reduction of a (3,3) theory with one $\mathrm{SU}(1,1) \approx \mathrm{SO}(1,2)$ factor corresponding to the reduction being along a space with signature (1,2) and the other factor coming from the $\mathrm{SU}(1,1)$ R-symmetry of the (3,3) theory. As $\mathrm{SO}(1,3) \approx \mathrm{SL}(2,\mathbb{C})$ this has no 'geometric' parent theory and cannot be obtained from dimensional reduction.

10D to 4D

$N = 4$ theories in 4 dimensions can be obtained through the compactification of 10-dimensional theories. We will now explore how this arises in our formalism. The only superalgebras with the correct number of supercharges, 16, are in (1,9), (5,5) and (9,1), corresponding to Type I theories (this is ignoring other effects that may reduce the total number of supersymmetries through compactification).

For the Euclidean signature, (0,4), the R-symmetry group is $\mathrm{U}^*(4)$ that contains $S\mathrm{U}^*(4) \cong \mathrm{SO}(1,5)$. We can see that this could be a reduction of a (1,9) theory where the (1,5) compactified dimensions giving rise to an internal $\mathrm{SO}(1,5)$ R-symmetry, or it could be the (5,5) in much the same way.

Similarly, the standard (1,3) theory with $\mathrm{U}(4)$ R-symmetry has a well-known (1,9) signature origin, due to $\mathrm{SU}(4) \cong \mathrm{SO}(6)$. This provides a natural geometric interpretation for the $\mathrm{SO}(6)$ subgroup. For the alternative (1,3) theory with $\mathrm{U}(2,2)$ R-symmetry, we remark that $\mathrm{SU}(2,2) \cong \mathrm{SO}(2,4)$, demonstrating that this should arise from the compactification of a (5,5)-signature parent theory. There is also a possible $\mathrm{U}(1,3)$ theory, but this has no geometric justification because $\mathrm{SU}(1,3) \cong \mathrm{SO}^*(6)$.

Finally the $N = 4$ (2,2) has $\mathrm{GL}(4,\mathbb{R})$ R-symmetry. This contains $\mathrm{SL}(4,\mathbb{R}) \cong \mathrm{SO}(3,3)$. Only the (5,5) theories could be reduced to (2,2), correctly giving once again the $\mathrm{SO}(3,3)$ subgroup we see in the R-symmetry group.

3.11.3 T-Duality

In ten dimensions there exists Type IIA and Type IIB string theories, and derivatives of these, often called Type IIA*, Type IIB* and Type IIB', as outlined in Section 2.12. There is a web of dualities found in [14, 15] that links them (and M-theory in eleven dimensions). T-duality is an example of a map between superalgebras that is not included in our discussion above, in that T-duality relates two theories with different R-symmetry, which our basic isomorphisms outlined above could never do. In the following section, we outline how the existence of T-dualities can be inferred from the available nine-dimensional superalgebras.

T-duality maps Type IIA theories, which have one of each chirality, to Type IIB theories which have 2 of the same chirality. By using space- and time-like T-dualities, many different string theories can be reached, which were called IIA, IIA*, IIB, IIB* and IIB' [14]. In the following section, we will detail how this is captured in this formalism, as there are conventional differences which can be clarified.

Basic T-duality links theories that have been compactified upon a circle, on the nine free dimensions we expect to have a regular nine-dimensional theory and all that comes along with it, including a superalgebra. Given the reduction of ten-dimensional superalgebras (which in this formalism is given by the pair of bilinear form and reality condition) we can see the T-dualities as maps between supersymmetry algebras that have the same contraction to nine dimensions. Given two ten-dimensional superalgebras that dimensionally reduce to the same superalgebra in nine dimensions, there exists a T-duality between them. If they reached the nine-dimensional superalgebra after a both undergoing a space/time-like reduction they are related by space/time-like T-duality if one was space-like and one was time-like reduced we have the mixed T-dualities. In the second part of this section, we will define the nine-dimensional superalgebras in our conventions and how this relates to T-duality, performing the dimensional reduction explicitly.

Space-like T-duality

We will implement a space-like T-duality on a Type IIA supersymmetry algebra and obtain a Type IIB algebra. A type IIA theory involves two Majorana-Weyl supercharges with opposite chirality. Let us work with $\lambda_\pm \in \mathbb{S}_\pm$ that obey:

$$\begin{aligned} &(\lambda_+)^* = \alpha B_+ \lambda_+, \qquad && (\lambda_-)^* = \alpha B_+ \lambda_-, \\ &\Gamma_* \lambda_+ = +\lambda_+, \qquad && \Gamma_* \lambda_- = -\lambda_-. \end{aligned} \tag{3.209}$$

Here the choice was made to use B_+ in the reality condition without loss of generality because either choice of B matrix can be mapped to one another. Similarly, the bilinear form is $C_+ \otimes \delta$ can be chosen as we have shown that a bilinear form with C_- is equivalent in ten dimensions. The vector-valued bilinear form is then

$$(\Gamma^\mu \lambda)^T C_+ \chi = (\Gamma^\mu \lambda_+)^T C_+ \chi_+ + (\Gamma^\mu \lambda_-)^T C_+ \chi_- \tag{3.210}$$

A T-duality transformation acts trivially on $\lambda_+ = \tilde{\lambda}_+$ but on λ_- it acts as

$$\lambda_- \to \tilde{\lambda}_- = T\lambda_-, \quad T = \beta \Gamma_* \Gamma^{0/9}, \quad |\beta| = 1. \tag{3.211}$$

Equation (2.207) in Chapter 2 tells us that $\tilde{\lambda}_-$ is now a positive chirality state, such that $T\mathbb{S}_- \cong \mathbb{S}_+$. To avoid ambiguity we shall relabel them $\tilde{\lambda}^1_+ \equiv \tilde{\lambda}_+$ and $\tilde{\lambda}^2_+ \equiv \tilde{\lambda}_-$.

T-duality here is a unitary transformation acting on $\mathbb{S}_-$, so we must also transform all matrices acting on it, this includes A, B_+ and C_+. The associated matrices that act on (the original) $\mathbb{S}_+$ do not change. The vector-valued bilinear form on $\mathbb{S}_-$ becomes

$$(\Gamma^\mu \lambda_-)^T C_+ \chi_- \to (\Gamma^\mu \tilde{\lambda}^2_+)^T T^T \tilde{C}_+ T \tilde{\chi}^2_+ \tag{3.212}$$

To retain the same form of the vector-valued bilinear form on $\mathbb{S}_-$ we require

$$\tilde{C}_+ = (T^T)^{-1} C_+ T^{-1} = \frac{1}{\beta^2} C_+. \tag{3.213}$$

This places it into our conventional forms and allows one to write

$$\begin{aligned}&(\Gamma^\mu \tilde{\lambda}^1_+)^T T^T C_+ T \tilde{\chi}^1_+ + (\Gamma^\mu \tilde{\lambda}^2_+)^T T^T \tilde{C}_+ T \tilde{\chi}^2_+ \\ =&(\Gamma^\mu \tilde{\lambda}^i_+)^T C_+ \tilde{\chi}^j_+ M_{ji}, \qquad M_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & +\frac{1}{\beta^2} \end{pmatrix}.\end{aligned} \tag{3.214}$$

Note that in (2.212) we effectively used a bilinear form with Gram matrix $C\Gamma_*$, which is not a choice considered in this formalism.

Similarly A must transform as $\tilde{A} = TAT^\dagger = A$ for a space-like reduction. This affects the matrix $B_+ = (C_+ A^{-1})^T$, and one can show this transforms as

$$B_+ \to \tilde{B}_+ = \frac{1}{\beta^2} B_+. \tag{3.215}$$

Therefore the reality condition for $\tilde{\lambda}^i_+$ is

$$(\lambda^i_+)^* = \alpha B_+ \lambda^j L_{ji}, \qquad L_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\beta^2} \end{pmatrix}. \tag{3.216}$$

Choosing $\beta = \pm 1$ so that $\beta^2 = 1$ puts us in the conventional description of type IIB theories, which aligns with our conventions for writing vector-valued bilinear forms (such that a symmetric complex bilinear forms always having a Gram matrix $M_{ij} = \delta_{ij}$) and reality condition defined by $L_{ij} = \delta_{ij}$. These choices give us an SO(2) R-symmetry as expected for Type IIB.

Time-like T-duality

First we will consider Type IIA → Type IIB* under a time-like T-duality. For a time-like T-duality $\tilde{A} = -A$. The reality condition is therefore

$$(\lambda^i_+)^* = \alpha B_+ \lambda^j L_{ji}, \qquad L_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\beta^2} \end{pmatrix}. \tag{3.217}$$

C_+ is not signature-dependent, so it does not depend on the type of T-duality, and thus the vector-valued bilinear form is the same

$$(\Gamma^\mu \tilde{\lambda}^i_+)^T C_+ \tilde{\chi}^j_+ M_{ji}, \qquad M_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & +\frac{1}{\beta^2} \end{pmatrix}. \tag{3.218}$$

Let us consider elegant choices for β such that $\beta^2 = \pm 1 \implies \beta = \pm 1, \pm i$. We immediately see that the sign in the reality condition and bilinear form are necessarily opposite. Adhering to our prescription of symmetric complex bilinear forms always having a Gram matrix $M_{ij} = \delta_{ij}$ we choose $\beta = \pm 1$, so that we see the spinor definitions of Type IIB* manifests in our formalism as choosing $L_{ij} = \eta_{ij}$ for the reality condition and $M_{ij} = \delta_{ij}$ for the bilinear form (giving us an SO(1,1) R-symmetry). In [14] the choice with $\beta = \pm i$ was preferred, this swaps L and M in our formalism.

To summarise, the difference between Type IIB and Type IIB* manifests in our formalism as the choice of L in the reality condition. Choosing $L = \delta$ gives the Type IIB superalgebra and choosing $L = \eta$ gives the Type IIB* superalgebra that has an SO(1,1) R-symmetry group.

Now let us consider going from Type IIB to Type IIA*. The difference between Type IIA and Type IIA* is more subtle than for the B and B* theories.

We begin with λ^1_+ and λ^2_+, which we can combine together such that

$$(\lambda^i_+)^* = \alpha B_+ \lambda^i_+, \qquad \Gamma_* \lambda^i_+ = +\lambda^i_\pm. \tag{3.219}$$

The vector-valued bilinear form is then

$$(\Gamma^\mu \lambda^i_+)^T C_+ \chi^j_+ \delta_{ji}. \tag{3.220}$$

time-like T-duality induces

$$\lambda^1_+ \to \tilde{\lambda}_+ = \lambda^1_+, \qquad \lambda^2_+ \to \tilde{\lambda}_- = T\lambda^2_+, \qquad T = \beta\Gamma_*\Gamma^0. \tag{3.221}$$

What could have been called $\tilde{\lambda}^2_+$ has been named $\tilde{\lambda}_-$ for clarity (as it now has negative chirality) and we have discarded the superfluous superscript on $\tilde{\lambda}^1_+$

$$(\tilde{\lambda}_-)^* = -\frac{\alpha}{\beta^2} B_+\tilde{\lambda}_- \tag{3.222}$$

and the vector-valued bilinear form on the negative chirality states is similarly scaled by $\frac{1}{\beta^2}$.

$$(\Gamma^\mu\lambda^2_-)^T C_+\chi^2_- = \frac{1}{\beta^2}(\Gamma^\mu\tilde{\lambda}_-)^T C_+\tilde{\chi}_-. \tag{3.223}$$

Once again we could choose $\beta = \pm 1$ such that $\beta^2 = 1$, to maintain our conventions that the vector-valued bilinear form is $+C_+$ on $\mathbb{S}_\pm$. This means we have a different reality condition on both chiralities:

$$(\gamma^\mu\tilde{\lambda}_+)^* = \alpha B_+\tilde{\lambda}_+, \qquad (\tilde{\lambda}_-)^* = -\alpha B_+\tilde{\lambda}_-. \tag{3.224}$$

If instead we chose $\beta = \pm i$ such that $\beta^2 = -1$ we have the same reality condition on each part, but necessarily different signs on the vector-valued bilinear forms of either chirality (and thus kinetic terms in Lagrangian). It is this way that Type IIA* is given in [14].

Any of these statements separate Type IIA and Type IIA*. To summarise, Type IIA has the same reality condition on both chiralities and the same vector-valued bilinear form on both chiralities. Type IIA* has a sign difference between the reality conditions OR vector-valued bilinear forms on $\mathbb{S}_+$ and $\mathbb{S}_-$. We can always change between the two equivalent Type IIA* descriptions by taking $\tilde{\lambda}_- \to i\tilde{\lambda}_-$. In our formalism, it arises with a sign difference in the reality condition.

These provide a cautionary tale when using this formalism. The relative choices of α (which is only determined up to a sign) in the reality condition on $\mathbb{S}_+$ and $\mathbb{S}_-$ will have changes at a Lagrangian level and should be made with care. Particular physical theories correspond to particular choices of reality condition and bilinear form – as seen here Type IIA and Type IIA* differ by a reality condition which informs the Lagrangian through supersymmetry.

3.11.4 $\mathcal{N} = 2$ superalgebras in Nine Dimensions

In nine dimensions the bilinear form on $\mathbb{S}$ is super-admissible, so we use a symmetric bilinear form on $\mathbb{C}^K$, thus leading to R-symmetry groups that are real forms of $\mathrm{O}(K,\mathbb{C})$. We will denote the nine-dimensional bilinear form as $C^9 \otimes \delta$, constructed from the nine-dimensional Majorana bilinear form C^9 (the two in ten dimensions will be called $C^{10}_{\pm}$). C^9 has invariants $(\sigma,\tau) = (+1,+1)$, $C^{10}_{\pm}$ have invariants $(\sigma,\tau) = (\mp1,\mp1)$. In 10D we will always use C^{10}_{-} as it is mathematically convenient, and we have seen previously the choice is irrelevant.

In nine dimensions we obtain the following R-symmetry groups which have an associated unique superalgebra (as we are in an odd dimension).

$(0,9)$	$(1,8)$	$(2,7)$	$(3,6)$	$(4,5)$
O(1,1) or O(2)	O(1,1) or O(2)	SO(2)	SO(2)	O(1,1) or O(2)

Table 3.11: $\mathcal{N} = 2$ R-symmetry groups in nine dimensions, the R-symmetry group for (s,t) is the same as (t,s).

Where we have an SO(2) R-symmetry group we know we only have a quaternionic structure on $\mathbb{S}$ (effectively the complex structure picks an orientation, locking the group to SO(2) instead of O(2)). To summarise we deal with reality conditions of the form

$$(\lambda^i)^* = \alpha B^{(t,s)} \lambda^i \implies G_R = \mathrm{O}(2), \tag{3.225}$$

$$(\lambda^i)^* = \alpha B^{(t,s)} \lambda^j \eta_{ji} \implies G_R = \mathrm{O}(1,1), \tag{3.226}$$

$$(\lambda^i)^* = \alpha B^{(t,s)} \lambda^j \varepsilon_{ji} \implies G_R = \mathrm{SO}(2). \tag{3.227}$$

3.11.5 10D to 9D Dimensional Reduction

The basis for the ten-dimensional Clifford algebra is Γ_μ within which we embed the nine-dimensional gamma matrices, γ_μ, according to

$$\Gamma_\mu = \gamma_\mu \otimes \sigma_1, \quad \Gamma_{10} = Id \otimes \sigma_2 \quad \text{or} \quad \Gamma_0 = iId \otimes \sigma_2, \tag{3.228}$$

using the same basis for dimensional reduction as was described earlier.

A is the product of the first t gamma matrices. For a time-like reduction, i.e. signature

$(t+1, s)$ is reduced to (t, s) we will define A as

$$A^{(t+1,s)} = \Gamma_0 ... \Gamma_t. \tag{3.229}$$

For a space-like reduction, where $A^{(t,s+1)}$ is reduced to $A^{(t,s)}$, $A^{(t,s+1)} = \Gamma_1 ... \Gamma_t$ as standard. In this basis we find

$$C_-^{10} = C^9 \otimes \sigma_1. \tag{3.230}$$

And the 10D chiral projection matrix $\Gamma_* = (-i)^{t+\frac{D}{2}} \Gamma_1 ... \Gamma_{10}$ is

$$\Gamma_* = 1 \otimes \sigma_3. \tag{3.231}$$

In Type IIA we have two spinors of opposite chirality, say $\lambda_\pm$. Using the standard form of σ_3 into nine-dimensional spinors, ψ^1 and ψ^2 as

$$\lambda_+ = \psi^1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.232}$$

$$\lambda_- = \psi^2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.233}$$

Alternatively, given two spinors of the same chirality, say λ_+^1 and λ_+^2, like in a Type IIB theory, we will decompose them as

$$\lambda_+^1 = \psi^1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.234}$$

$$\lambda_+^2 = \psi^2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.235}$$

Reality Condition Reduction

Here we will only deal with $B_-^{(p,q)}$ (with $p + q = 10$) for ease of use, this is without loss of generality as we recall that we can map between theories with $B_+^{(p,q)}$ and $B_-^{(p,q)}$ used in the reality condition anyway. A space-like reduction gives

$$B_-^{(t,s+1)} = B^{(t,s)} \otimes \sigma_1^{t+1} = \begin{cases} B^{(t,s)} \otimes \sigma_1 & t \quad \text{even}, \\ B^{(t,s)} \otimes 1 & t \quad \text{odd}. \end{cases} \tag{3.236}$$

And a time-like reduction gives

$$B_{-}^{(t+1,s)} = B^{(t,s)} \otimes i\sigma_3\sigma_1^t = \begin{cases} iB^{(t,s)} \otimes \sigma_3 & t \quad \text{even,} \\ -B^{(t,s)} \otimes \sigma_2 & t \quad \text{odd.} \end{cases} \tag{3.237}$$

The second factor in the tensor products does not affect the reality of the 9D spinor bilinears (they would come in pairs and therefore cancel). Indeed they capture the Weyl-compatibility of the ten-dimensional signature; we observe that when the ten-dimensional signature has an even number of time-like directions we have a σ_1 or σ_2 factor that exchange chiralities, as in these signatures the bilinear form is Weyl-incompatible. When the ten-dimensional theory has an odd number of time-like directions the reality condition is Weyl-compatible, so we get Id or σ_3 which do not mix the two chiralities.

For an example allow us to reduce the $(0,10)$ Type IIA algebra (could also be called $\mathcal{N} = (1,1)$ or $\mathcal{N} = 1$) superalgebra to $(0,9)$. $(0,10)$ involves a single Majorana spinor that can be written in terms of a Weyl-incompatible reality condition as

$$(\lambda_{\pm})^* = \alpha B_{-}^{(0,10)} \lambda_{\mp}. \tag{3.238}$$

Decomposing into nine-dimensional quantities, we see this reads

$$(\psi^1)^* \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha(B^{(0,9)} \otimes \sigma_1)(\psi^2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \alpha B^{(0,9)} \psi^2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.239}$$

$$(\psi^2)^* \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha(B^{(0,9)} \otimes \sigma_1)(\psi^1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \alpha B^{(0,9)} \psi^1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.240}$$

Ignoring the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ vector we write

$$(\psi^i)^* = \alpha B^{(0,9)} \psi^j \eta_{ji}. \tag{3.241}$$

This leads to a $(0,9)$ $\mathcal{N} = 2$ superalgebra with an $\mathrm{O}(1,1)$ R-symmetry group.

Reduction of Vector-valued Bilinear Form

Next, we need to reduce the vector-valued bilinear form. This has a different form in the Type IIA and Type IIB theories.

The type IIA vector-valued bilinear form reduces as

$$(\Gamma^\mu\lambda_+)^T C_-^{10}\chi_+ + (\Gamma^\mu\lambda_-)^T C_-^{10}\chi_- = \begin{pmatrix} (\gamma^\mu\psi^i)^T C^9\phi^j\delta_{ji}\otimes 1 \\ -i(\psi^i)^T C^9\phi^j\eta_{ji}\otimes 1 \end{pmatrix}. \tag{3.242}$$

The final component creates a central charge in the lower dimensional super-Poincare algebra, the vector-valued bilinear form reduces into the nine-dimensional one, and thus so do the superalgebras. We then need to assess how the reality condition embeds, which is determined by the parent and daughter signature (as B is signature dependent) as described above.

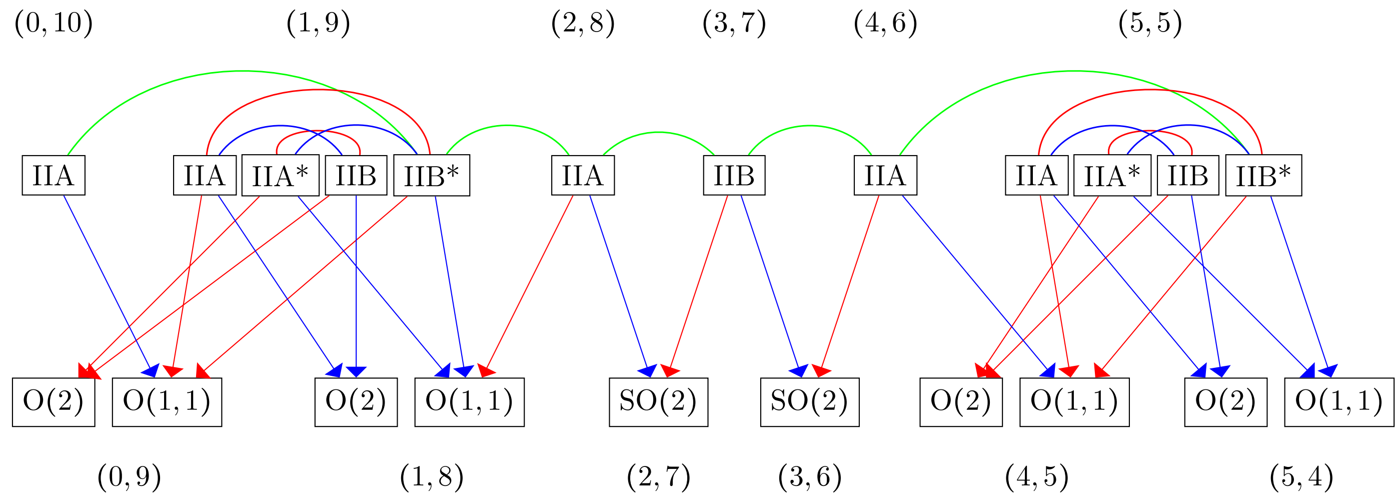
The type IIB vector-valued bilinear form gives

$$(\Gamma^\mu\lambda_+^i)^T C_-^{10}\chi_+^j\delta_{ji} = \begin{pmatrix} (\gamma^\mu\psi^i)^T C^9\phi^j\delta_{ji}\otimes 1 \\ -i(\psi^i)^T C^9\phi^j\eta_{ji}\otimes 1 \end{pmatrix}. \tag{3.243}$$

Summary

Without explicitly performing all reductions, the following diagram summarises all reductions to nine dimensions and then provides the type of T-duality (space-like, time-like or mixed) linking the ten-dimensional superalgebras.

The signature of the compactified dimensions gives the type of T-duality. For example, we have a time-like T-duality when both ten-dimensional superalgebras have the same time-like reduction (and therefore the same starting signature) and we get a mixed T-duality when one ten-dimensional superalgebra was reduced along a time-like direction has the same reduction as one reduced over a space-like direction (so that the starting signatures differ).



A schematic of the T-dualities and their relation to the dimensional reduction of ten-dimensional superalgebras to nine dimensions. A blue arrow corresponds to a space-like reduction and red arrow corresponds to a time-like reduction. A blue curved line means they are related by a space-like T-duality, a red curved line means the two theories are related by a time-like T-duality and a green line implies a mixed T-duality.

The ten-dimensional superalgebras are given their conventional names; the nine-dimensional superalgebras are named after their R-symmetry.

3.12 Conclusion and Outlook

This chapter presented a formalism for defining supersymmetry algebras with manifestly R-symmetric spinors in any signature and dimension with an arbitrary number of supersymmetries.

Next, for physically relevant dimensions (up to 12), the R-symmetry group was calculated in all signatures. This is a useful result in its own right (the dimensional reduction section included some usage of the R-symmetry group) and also provides a guide to which supersymmetry algebras are expected to be isomorphic. The construction does not necessarily lead to unique supersymmetry algebras, this was investigated and these choices were classified up to the scope of this construction.

After this, some physical examples were given using this formalism, namely dimensional reduction and T-duality. The most detailed examples of this formalism are in Chapters 4 and 5 also heavily use this formalism to give Lagrangians and supersymmetry variations. For example, this chapter predicts four-dimensional Lorentzian signature supersymmetry algebras with $\mathrm{U}(1,1)$ R-symmetry, which are justified in these chapters. The supersymmetry algebras and R-symmetry groups calculated here give a guiding hand to defining physical theories, though the full effects this construction can have on the Lagrangian description of theories is a significant undertaking and could be pursued further.

The original inspiration for this work was to provide a physics-tailored version of that found in [1], which was done for the case of $\mathcal{N} = 2$ supersymmetry algebras in [20]. This was then expanded to include a formalism for an arbitrary number of supercharges. Reformulating this construction in the terminology and methodology found in the original paper is a possible avenue for future work, such as calculating the Schur algebra for the extended spinor modules in any case.

Additionally, the original paper [1] was itself expanded in [31] to include polyvector extensions that generalise central charges and [3] that provides a manner of determining the isomorphism classes of superalgebras but does not provide a full classification. This is another potential area for future development; including polyvector extensions/central charges in this framework and expanding the classification to ensure that all possible supersymmetry algebras are contained within it.

3.13 Appendix

3.13.1 Superadmissibility Implies Dynamical Spinor Fields (and Vector-Spinor)

This chapter used commuting spinors; a super-admissible bilinear form has a symmetric vector-valued bilinear form with commuting spinors. Physical theories are written in terms of anticommuting (Grassmann-valued) spinors. A super-admissible bilinear form on *anticommuting* spinors is *antisymmetric*. A kinetic term is proportional to the vector-valued bilinear form

$$\beta(\gamma^\mu\lambda, \partial_\mu\lambda) \propto \bar{\lambda}\gamma^\mu\partial_\mu\lambda \tag{3.244}$$

Using our invariants, σ and τ, the kinetic term can be rewritten

$$\bar{\lambda}\gamma^\mu\partial_\mu\lambda = -\sigma\tau\partial_\mu\bar{\lambda}\gamma^\mu\lambda \tag{3.245}$$

Where we have gained an additional minus sign due to the Grassmannian variables. This means the total derivative is equal to

$$\partial_\mu(\bar{\lambda}\gamma^\mu\lambda) = \bar{\lambda}\gamma^\mu\partial_\mu\lambda + \partial_\mu\bar{\lambda}\gamma^\mu\lambda = (1-\sigma\tau)\bar{\lambda}\gamma^\mu\partial_\mu\lambda \tag{3.246}$$

We can see that if $\sigma\tau = -1$ then the kinetic term is proportional to a total derivative, which we do not want if we require dynamical spinor fields. super-admissible bilinear forms have $\sigma\tau = 1$ so the kinetic term is not a total derivative.

Supergravity theories with fermions also have vector-spinor fields too. From (3.23) we see that the symmetry of the rank-3 tensor-valued bilinear form, which is equal to

$$(-1)^{\frac{3(3-1)}{2}}\sigma\tau^3 = -\sigma\tau \tag{3.247}$$

is opposite that of the vector-valued bilinear form. This means a super-admissible bilinear form gives a symmetric rank-3 tensor-valued bilinear form (with anticommuting spinors).

The standard way of writing the kinetic term of a vector-spinor is proportional to the rank-3 tensor-valued bilinear form

$$\beta(\gamma^{\mu\nu\rho}\psi_\mu, \partial_\nu\psi_\rho) \propto \bar{\psi}_\mu\gamma^{\mu\nu\rho}(\partial_\nu\psi_\rho) \tag{3.248}$$

And once again one can show

$$\partial_\nu\big(\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi_\rho\big) = (1-\sigma\tau)\bar{\psi}_\mu\gamma^{\mu\nu\rho}(\partial_\nu\psi_\rho) \tag{3.249}$$

Where the sign difference from $\sigma_1 = -\sigma_3$ is compensated by relabelling indices. Once again a super-admissible bilinear form means the vector-spinor kinetic term is not a total derivative.

So given a super-admissible bilinear form, we can always define dynamical spinor and vector-spinor fields. We also know from (3.31) the isotropy alternates, so that the vector-valued and rank-3 tensor-valued bilinear form have the same isotropy, so this argument applies to chiral theories too. A super-admissible bilinear form and a reality condition define a (Poincaré) superalgebra which naturally permits multiplets whose fermion fields are always dynamical.

3.13.2 Proof of Signature Flip $\implies B_+ \leftrightarrow B_-$.

Consider the (t,s) signature γ-matrices, which obey (not a sum)

$$(\gamma_i)^2 = \begin{cases} -1 & i \leq t \\ +1 & i > t \end{cases} \tag{3.250}$$

And we define the (s,t) signature γ-matrices as $\gamma'_m = i\gamma_{(D-m+1)}$ (where $D = t+s$) such that they correctly obey

$$(\gamma'_i)^2 = \begin{cases} -1 & i \leq s \\ +1 & i > s \end{cases} \tag{3.251}$$

Both theories have the same charge conjugation matrices, C_+ and C_- and an A given by

$$\begin{aligned} A^{(t,s)} &= \gamma_1 ... \gamma_t \\ A^{(s,t)} &= \gamma'_1 ... \gamma'_s = i^s \gamma_D ... \gamma_{t+1} \end{aligned} \tag{3.252}$$

We then see that, using $C_+ = kC_- \gamma_*$ (k is the constant from (3.1))

$$\begin{aligned} B_+^{(t,s)} &= (C_+ (A^{(t,s)})^{-1})^T \\ &= (kC_- \gamma_* (A^{(t,s)})^{-1})^T \end{aligned} \tag{3.253}$$

Using our definitions for $A^{(t,s)}$ we find

$$\begin{aligned} \gamma_* (A^{(t,s)})^{-1} &= (-i)^t \gamma_1 ... \gamma_D (-1)^t \gamma_t ... \gamma_1 \\ &= (-1)^{st} (-i)^t \gamma_{t+1} ... \gamma_D \\ &= (-1)^{st} (-i)^t (-i)^s \gamma'_s ... \gamma'_1 \\ &= (-1)^{st} (-i)^D (A^{(s,t)})^{-1} = (-1)^{st + \frac{D}{2}} (A^{(s,t)})^{-1} \end{aligned} \tag{3.254}$$

Such that

$$\begin{aligned} & B_+^{(t,s)} = (kC_- (-1)^{st+\frac{D}{2}} (A^{(s,t)})^{-1}) = k(-1)^{st+\frac{D}{2}} B_-^{(s,t)} \\ \Longrightarrow \; & (B_+^{(t,s)})^* B_+^{(t,s)} = (B_-^{(s,t)})^* B_-^{(s,t)} \end{aligned} \tag{3.255}$$

3.13.3 Non-canonical Reality Condition

Allow us to consider a bilinear form $C \otimes J$ on $\mathbb{S} \otimes \mathbb{C}^K$ and a reality condition

$$(\lambda^i)^* = \alpha B \lambda^j (I_{p,q})_{ji}, \qquad p + q = K. \tag{3.256}$$

As $r \in \mathfrak{sp}(K, \mathbb{C})$ means r can be written

$$r = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}, \quad b^T = b, c^T = c. \tag{3.257}$$

In the presented formalism this is a non-canonical reality condition, as it does not lead naturally to a real form. For this example, we set $p > q$ so that we can write

$$I_{p,q} = \begin{pmatrix} \mathbb{1}_{\frac{K}{2}} & 0 \\ 0 & I_{d,q} \end{pmatrix}, \qquad d = p - \frac{K}{2}. \tag{3.258}$$

To be invariant under the reality condition an R-symmetry Lie algebra element r obeys

$$r = L^{-1} r^* L. \tag{3.259}$$

After some calculation finds that r has the form

$$r = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & ib^2 \\ c_1 & 0 & -a_1^T & 0 \\ 0 & ic_2 & 0 & -a_2^T \end{pmatrix}, \tag{3.260}$$

where a_1, b_1, c_1 are $d \times d$ complex matrices and a_2, b_2, c_2 are $q \times q$ complex matrices, b_i and c_i are symmetric. This is a generic element of $\mathfrak{sp}(2d, \mathbb{R}) + \mathfrak{sp}(2q, \mathbb{R})$, to see this we take

$$r \to \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1^T \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2^T \end{pmatrix} \right) \in \mathfrak{sp}(2d, \mathbb{R}) + \mathfrak{sp}(2q, \mathbb{R}). \tag{3.261}$$

While this calculation is for one particular case; we see the algebra obtained is still a subalgebra of the complexified R-symmetry Lie algebra. One expects the associated superalgebra to be like the sum of two supersymmetry algebras, one with $\mathrm{Sp}(2d, \mathbb{R})$ R-symmetry and one with $\mathrm{Sp}(2q, \mathbb{R})$. The study of other such reality conditions will be left to further work.

4 Five-dimensional Superalgebras and Vector Multiplets

4.1 Introduction

The minimal supersymmetry algebra in all five-dimensional signatures is '$\mathcal{N} = 2$' in that they can be written in terms of a single Dirac spinor (which has twice the dimension a real spinor would have if they could be defined) or in terms of doubled spinors (two Dirac spinors equipped with a reality condition, which are symplectic Majorana or a pair of (twisted) Majorana spinors).

First, we will derive these supersymmetry algebras in terms of Dirac spinors, using natural quaternionic and para-quaternionic models for the Clifford algebras and spinor modules. Following this, we will reformulate this in terms of doubled spinors (the $\mathcal{N} = 2$ case of the $\mathcal{N}$-extended spinors formulated in Chapter 3). Five dimensions is an appealing testing ground for the formalism because there is a single Majorana bilinear form and a single ϵ-quaternionic structure on $\mathbb{S}$, therefore in each signature there is a unique supersymmetry algebra in terms of doubled spinors. We will define the possible superalgebras in each five-dimensional signature, derive off-shell representations of the superalgebra and then construct a Lagrangian invariant under these transformations for n_V interacting vector multiplets. Said Lagrangians are found by deriving a holomorphic master Lagrangian that is restricted by signature-dependent reality conditions induced by the doubled spinor module's reality properties.

Applying the formalism to the relatively straightforward case of five-dimensional $\mathcal{N} = 2$ vector multiplets allows a controlled study of the features of supersymmetric theories in arbitrary signatures. As there exists a unique minimal superalgebra in each signature, it allows a controlled setting for the study of which features of supersymmetric field theory are mandated by supersymmetry and what are signature-dependent.

Section 2.11 recapped the original construction for five-dimensional vector multiplets, following the work of [20]. This paper's methodology formed the basis for this line of work and is adapted and used here, though the language is moderately different to align with conventions introduced in the papers [2], [3] and yet to be released work that forms Chapter 3 in this thesis.

This chapter is based on [2].

4.2 Conventions

Most of the conventions in this section follow the universal conventions used in this thesis, see Section 2.1, but there are some additional definitions that this chapter adheres to that are listed here.

Gamma matrices will be labelled, in all signatures with $\mu = 1, 2, 3, 4, 5$. For signature (t, s), the first t gamma-matrices square to -1 and the remaining s matrices square to $+1$.

Odd dimensions permit two choices of for the gamma matrices that differ up to a sign on γ_5. In this chapter the representation is chosen such that in signatures where t is even

$$\gamma_{\mu\nu\rho\sigma\tau} = \varepsilon_{\mu\nu\rho\sigma\tau} \tag{4.1}$$

and in signatures with t-odd the γ-matrices satisfy

$$\gamma_{\mu\nu\rho\sigma\tau} = -i\varepsilon_{\mu\nu\rho\sigma\tau}. \tag{4.2}$$

The opposite sign choice in each condition is possible, but making these allow a unified writing in five dimensions that is useful after reducing to four dimensions. Further details on this can be found later.

We will interpret $min(t, s)$ as the number of time-like directions. Therefore (0,5) and (5,0) are both considered Euclidean with a different metric convention (mostly positive vs mostly negative), we have two Minkowski theories (1,4) and (4,1), and two exotic two-time theories (2,3) and (3,2). Generally, the Euclidean, Minkowski and exotic theories differ up to factors of $\pm i$ due to the different definitions of various properties of the spinors and other signature-dependent quantities induced by the change of metric convention.

4.3 Five-dimensional Clifford Algebras and Spinor Modules

Five dimensional Clifford algebras, Schur algebras and spinors are intimately related to the quaternions and para-quaternions. We will use these relations to construct elegant models for the Clifford algebras and spinor modules. In Table 4.1 various facts about the Clifford algebras, Schur algebras and spinor modules are collected for each of the five-dimensional signatures.

Signature	$Cl_{t,s}$	$Cl^0_{t,s}$	$\mathcal{C}(\mathbb{S})$	$\mathcal{C}(S_\mathbb{R})$	G_R	$\mathbb{S}$
$(0,5)$	$2\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_\mathbb{R}$
$(1,4)$	$\mathbb{C}(4)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_\mathbb{R}$
$(2,3)$	$2\mathbb{R}(4)$	$\mathbb{R}(4)$	$\mathbb{H}'$	$\mathbb{R}$	SU(1,1)	$S_\mathbb{R}\otimes\mathbb{C}$
$(3,2)$	$\mathbb{C}(4)$	$\mathbb{R}(4)$	$\mathbb{H}'$	$\mathbb{H}'$	SU(1,1)	$S_\mathbb{R} = S_{\mathbb{R}\pm}\otimes\mathbb{C}$
$(4,1)$	$2\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_\mathbb{R}$
$(5,0)$	$\mathbb{C}(4)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_\mathbb{R}$

Table 4.1: Summary of five-dimensional signatures, including the Clifford algebras and their even sub-algebras, the Schur group of the complex and real spinor modules, the R-symmetry group

From Table 4.1 we see that the even Clifford algebra is the same for the (t,s) and (s,t), so that the spinor modules in (t,s) and (s,t) are equivalent. Therefore we can just consider three cases, $(0,5)$, $(1,4)$ and $(2,3)$ with the results in (t,s) being applicable to (s,t).

Additionally the five dimensional spin groups are isomorphic to quaternionic or para-quaternionic groups. In particular, $\mathrm{Spin}(5)\cong\mathrm{Sp}(2)\cong\mathrm{U}(2,\mathbb{H})$, $\mathrm{Spin}(1,4)\cong\mathrm{Sp}(1,1)\cong\mathrm{U}(1,1,\mathbb{H})$ and $\mathrm{Spin}(2,3)\cong\mathrm{Sp}(4,\mathbb{R})\cong\mathrm{U}(2,\mathbb{H}')$. Since $\mathrm{Spin}(p,q)\cong\mathrm{Spin}(q,p)$ so we need only look at these three. Details of these isomorphism can be found in Section 2.3.2 and 2.3.3 where the properties of quaternions, para-quaternions and groups using them were discussed.

This section outlines (para-)quaternionic models for the Clifford algebras in 5 dimensions and use these to build the spinor module and extract the necessary data about the bilinear forms.

To define the five-dimensional Clifford algebras acting on $\mathbb{H}^2_\epsilon$ we will use

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

D and E are anticommuting involutions (i.e. they square to identity).

4.3.1 Euclidean Signature

Using D and E along with left and right multiplication by a quaternion q, L_q and R_q respectively, we can define a representation of $Cl_{(0,5)}$ in terms of quaternionic 2×2 matrices acting on $\mathbb{H}^2$:

$$\gamma^1 = D \qquad \gamma^2 = DEL_i \qquad \gamma^3 = DEL_j \qquad \gamma^4 = DEL_k \qquad \gamma^5 = -E. \tag{4.4}$$

In this representation, a spinor is then a pair of quaternions, q^i, with a Spin(5)-invariant bilinear form. By definition the standard Hermitian form is $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ invariant:

$$\langle q^i, p^i \rangle = \bar{q}^1 p^1 + \bar{q}^2 p^2. \tag{4.5}$$

To obtain a real-valued Spin(5)-invariant bilinear form we simply take the real part. This bilinear form is admissible, with symmetry $\sigma = +1$ and type $\tau = +1$. The Clifford generators are isometries of the scalar product and are involutions, so they are symmetric with respect to the scalar product (which corresponds to $\tau = +1$).

We can generate more Spin(5)-invariant bilinear forms using the Schur algebra, the algebra of endomorphisms that commute with Spin(5). Spin(5) is generated by the following elements

$$\begin{aligned} &\gamma^1\gamma^2 = EL_i, \quad \gamma^1\gamma^3 = EL_j, \quad \gamma^1\gamma^4 = EL_k, \quad \gamma^1\gamma^5 = -DE, \quad \gamma^2\gamma^3 = -L_k \\ &\gamma^2\gamma^4 = L_j, \quad \gamma^2\gamma^5 = -DL_i, \quad \gamma^3\gamma^4 = -L_i, \quad \gamma^3\gamma^5 = -DL_j, \quad \gamma^4\gamma^5 = -DL_k. \end{aligned} \tag{4.6}$$

This algebra is isomorphic to $\mathfrak{sp}(2)$. By inspection, we see that none involve right multiplication (which commutes with D, E and L_q) so the Schur algebra is

$$\mathcal{C}(S_{(0,5)}) \cong \langle Id, I = R_i, J = R_j, K = -R_k \rangle_{\text{algebra}} \cong \mathbb{H}. \tag{4.7}$$

The Schur group is the invertible elements of this, which is $C(\mathbb{S}_{(0,5)})^* = \mathbb{H}^*$, the group of invertible quaternions.

Taking the real part of the standard hermitian form, $Re\langle\cdot,\cdot\rangle$, we construct additional bilinear forms by inserting I, J and K into the first argument, which selects the i, j and k-imaginary parts respectively. Table 4.2 gives the invariants of each bilinear form:

β_i	σ	τ
$\beta_0 = Re\langle\cdot,\cdot\rangle$	$+$	$+$
$\beta_1 = Re\langle I\cdot,\cdot\rangle$	$-$	$+$
$\beta_3 = Re\langle J\cdot,\cdot\rangle$	$-$	$+$
$\beta_4 = Re\langle K\cdot,\cdot\rangle$	$-$	$+$

Table 4.2: Symmetry, σ, and type, τ, of bilinear forms in (0,5).

Only $\beta_0 = Re\langle\cdot,\cdot\rangle$ is super-admissible. Next, we calculate the action of the Schur algebra on the space of bilinear forms. To do this, we remark that L_q and R_q are isometries of the standard scalar product that square to -1, so they must be g-skew. D and E are isometries but are involutions so they must be g-symmetric. L_q and R_q commute with all operators, while D and E anti-commute.

A	$\tau(A)$	σ_{β_0}	σ_{β_1}	σ_{β_2}	σ_{β_3}
Id	$+$	$+$	$+$	$+$	$+$
I	$+$	$-$	$-$	$+$	$+$
J	$+$	$-$	$+$	$-$	$+$
K	$+$	$-$	$+$	$+$	$-$

Table 4.3: Type, τ, and β_i-symmetry, σ_{β_i}, of the Schur algebra basis elements, A, in (0,5).

Recall that elements with $\tau(A)\sigma_{\beta_i}(A) = -1$ leave the superbracket Π_{β_i} invariant, such an A is then a generator of the R-symmetry group. For β_0 we see

$$Stab(\Pi_{\beta_0}) = \langle I, J, K\rangle_{\text{algebra}} \cong \mathfrak{su}(2) \tag{4.8}$$

so that the R-symmetry group of an $\mathcal{N} = 2$ theory in Euclidean signature in five dimensions is SU(2).

Alternatively, one can show that for a generic element of the Schur group

$$Z = aId + bI + cJ + dK \in C(\mathbb{S}_{(0,5)})^{*}. \tag{4.9}$$

We obtain

$$\Pi_{\beta_0}(Z\cdot, Z\cdot) = (a^2 + b^2 + c^2 + d^2)\Pi_{\beta_0}(\cdot,\cdot). \tag{4.10}$$

So that Z is equivalent to a unit quaternion which isomorphic (as a multiplicative group) to SU(2).

4.3.2 Minkowski Signature

For our $(1,4)$ model we use the same space as in $(0,5)$, $\mathbb{H}^2$, upon which we define the $Cl_{1,4}$ representation

$$\gamma^1 = -R_iE \qquad \gamma^2 = R_iDL_i \qquad \gamma^3 = R_iDL_j \qquad \gamma^4 = R_iDL_k \qquad \gamma^5 = R_iDE. \tag{4.11}$$

All products of two distinct γ-matrices are

$$\begin{aligned} &\gamma^1\gamma^2 = EDL_i, \quad \gamma^1\gamma^3 = EDL_j \quad \gamma^1\gamma^4 = EDL_k, \quad \gamma^1\gamma^5 = -D, \quad \gamma^2\gamma^3 = -L_k \\ &\gamma^2\gamma^4 = L_j, \quad \gamma^2\gamma^5 = EL_i, \quad \gamma^3\gamma^4 = -L_i, \quad \gamma^3\gamma^5 = EL_j, \quad \gamma^4\gamma^5 = EL_k. \end{aligned} \tag{4.12}$$

This is isomorphic to $\mathfrak{sp}(1,1)$. Again, none of the even elements involve right multiplication by unit quaternions; therefore the following are in the Schur algebra

$$\mathcal{C}(S_{(1,4)}) \cong \langle Id, I = R_i, J = R_j, K = -R_k\rangle_{\text{algebra}} \cong \mathbb{H}. \tag{4.13}$$

For the original bilinear form, we choose

$$\langle q^i, p^i\rangle = \bar{q}^1p^1 - \bar{q}^2p^2. \tag{4.14}$$

This is manifestly $\mathrm{Sp}(1,1) \cong \mathrm{Spin}(1,4)$ invariant. We follow the regular construction for bilinear forms, and find the following set of invariants

β_i	σ	τ
$\beta_0 = Re\langle\cdot,\cdot\rangle$	+	−
$\beta_1 = Re\langle I\cdot,\cdot\rangle$	−	−
$\beta_3 = Re\langle J\cdot,\cdot\rangle$	−	+
$\beta_4 = Re\langle K\cdot,\cdot\rangle$	−	+

Table 4.4: Symmetry, σ, and type, τ, of bilinear forms in (1,4).

$\beta_1 = Re\langle I\cdot,\cdot\rangle$ is the only super-admissible bilinear form. The type and β_i-symmetry of each Schur algebra basis element is

A	$\tau(A)$	σ_{β_0}	σ_{β_1}	σ_{β_2}	σ_{β_3}
Id	+	+	+	+	+
I	+	−	−	−	−
J	−	−	+	−	+
K	−	−	+	+	−

Table 4.5: Type, τ, and β_i-symmetry, σ_{β_i}, of the Schur algebra basis elements, A, in (1,4).

From this we extract the elements such that $\tau(A)\beta_1(A) = -1$ and obtain

$$Stab(\Pi_{\beta_1}) = \langle I, J, K\rangle_{\text{algebra}} \cong \mathfrak{su}(2). \tag{4.15}$$

Inserting a generic element of the Schur group, $Z = aId + bI + cJ + dK \cong C(\mathbb{S}_{(1,4)})^*$, into the superbracket obtained from β_1 we obtain

$$\Pi_{\beta_1}(Z\cdot, Z\cdot) = (a^2 + b^2 + c^2 + d^2)\Pi_{\beta_1}(\cdot,\cdot). \tag{4.16}$$

This again tells us $Z \in \mathbb{H}^*$, the group of unit quaternions, giving the expected SU(2) R-symmetry group.

4.3.3 Exotic Signature

Here we can take a basis identical to that in $(0,5)$ but replace the quaternions with para-quaternions. The order of the γ-matrices is also rearranged, so the first two correspond to the time-like directions as is convention.

$$\gamma^1 = DEL_j \qquad \gamma^2 = DEL_k \qquad \gamma^3 = D \qquad \gamma^4 = DEL_i \qquad \gamma^5 = -E \tag{4.17}$$

is a representation of $Cl_{(2,3)}$ that acts on pairs of para-quaternions, $q^i \in \mathbb{H}'^2$.

The standard canonical Hermitian form on $\mathbb{H}'^2$

$$\langle q^i, p^i\rangle = \bar{q}^1 p^1 + \bar{q}^2 p^2 \qquad q^i, p^i \in \mathbb{H}' \tag{4.18}$$

is invariant under is $\mathrm{U}(2,\mathbb{H}') \cong \mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{Spin}(2,3)$ as outlined in Section 2.3.3. This is the spin group, and it is once again generated by the unit even elements of the Clifford algebra. All products of two γ-matrices are then

$$\begin{aligned}&\gamma^1\gamma^2 = -L_i, \quad \gamma^1\gamma^3 = -EL_j, \quad \gamma^1\gamma^4 = L_k, \quad \gamma^1\gamma^5 = -DL_j, \quad \gamma^2\gamma^3 = -EL_k \\ &\gamma^2\gamma^4 = -L_j, \quad \gamma^2\gamma^5 = -DL_k, \quad \gamma^3\gamma^4 = EL_i, \quad \gamma^3\gamma^5 = -DE, \quad \gamma^4\gamma^5 = -DL_i.\end{aligned} \tag{4.19}$$

This is a basis of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. This basis is the same as Sp(2) in the $(0,5)$ example with quaternions replaced with para-quaternions. Similarly, the Schur algebra is then right multiplication by para-quaternions, as the elements contained in $\mathrm{Spin}(2,3)$ are in terms of left-multiplication by para-quaternions only

$$\mathcal{C}(\mathbb{S}_{(2,3)}) \cong \langle Id, I = R_i, J = R_j, K = -R_k\rangle_{\text{algebra}} \cong \mathbb{H}'. \tag{4.20}$$

As $I^2 = -Id$ and $J^2 = K^2 = Id$, the Schur algebra is isomorphic to the para-quaternions $\mathbb{H}'$. The Schur group is then $C(\mathbb{S}_{(2,3)})^* \cong \mathbb{H}'^*$, the group of invertible para-quaternions.

We once again consider the standard Hermitian bilinear form on pairs of para-quaternions, $\langle\cdot,\cdot\rangle$, take the real part and insert Schur algebra elements to obtain additional $\mathrm{Spin}(2,3)$-invariant bilinear forms. We obtain the following collection of bilinear forms and invariants

β_i	σ	τ
$\beta_0 = Re\langle\cdot,\cdot\rangle$	+	+
$\beta_1 = Re\langle I\cdot,\cdot\rangle$	−	+
$\beta_3 = Re\langle J\cdot,\cdot\rangle$	−	+
$\beta_4 = Re\langle K\cdot,\cdot\rangle$	−	+

Table 4.6: Symmetry, σ, and type, τ, of bilinear forms in (2,3).

$\beta_0 = Re\langle\cdot,\cdot\rangle$ is the super-admissable bilinear form. We once again calculate the interaction with the Schur algebra basis elements and bilinear forms:

A	$\tau(A)$	σ_{β_0}	σ_{β_1}	σ_{β_2}	σ_{β_3}
Id	+	+	+	+	+
I	+	−	−	+	+
J	+	−	+	−	+
K	+	−	+	+	−

Table 4.7: Type, τ, and β_i-symmetry, σ_{β_i}, of the Schur algebra basis elements, A, in (2,3).

And we find

$$Stab(\Pi_{\beta_0}) = \langle I, J, K\rangle_{\text{algebra}} \cong \mathfrak{su}(1,1). \tag{4.21}$$

Multiplying both arguments by a generic Schur group element $Z = aId + bI + cJ + dK$ we obtain

$$Re\langle Z\cdot, Z\cdot\rangle = (a^2 + b^2 - c^2 - d^2)Re\langle\cdot,\cdot\rangle. \tag{4.22}$$

Note – these calculations were identical to (0,5) as we have just replaced quaternions with para-quaternions.

Similarly we could find, using $Z = aId + bI + cJ + dK \in C(\mathbb{S}_{(2,3)})^*$

$$\Pi_{\beta_0}(Z\cdot, Z\cdot) = (a^2 + b^2 - c^2 - d^2)\Pi_{\beta_0}, \tag{4.23}$$

confirming the R-symmetry group as $\mathrm{SU}(1,1)$, which is isomorphic to the group of unit para-quaternions.

4.3.4 Physics-style Reformulation

As models presented above are not in the conventional style used in physics, we will now translate them to the standard language. As usual, we will follow the conventions of [1,20] and use commuting spinors in the mathematical analysis before using anticommuting (Grassmannian-valued) spinors for the Lagrangians. As described previously in Section 2.2.3 this introduces no additional problems, effectively just inverting all symmetry statements.

From $\mathrm{Spin}_0(t,s)$-invariant complex sesquilinear form on $\mathbb{S}$,

$$A(\lambda,\chi) = \lambda^\dagger A\chi, \tag{4.24}$$

we can obtain two bilinear forms from this by taking the real and imaginary parts. The invariants of each depend on the properties of the Gram matrix A. It is straightforward to show

$$A^\dagger = (-1)^{\frac{t(t+1)}{2}}A. \tag{4.25}$$

$Re[A]$ is symmetric when A is Hermitian and antisymmetric when A is anti-Hermitian, and vice versa for $Im[A]$. γ^μ commutes with A when t is even and anticommutes when t is odd, this is unaffected by taking real or imaginary parts so both $Re[A]$ and $Im[A]$ have the same type. Table 4.8 lists the invariants of $Re[A]$ and $Im[A]$ in each signature.

	$Re[A]$	$Im[A]$
$(0,5)$	$(+,+)$	$(-,+)$
$(1,4)$	$(-,-)$	$(+,-)$
$(2,3)$	$(-,+)$	$(+,+)$
$(3,2)$	$(+,-)$	$(-,-)$
$(4,1)$	$(+,+)$	$(-,+)$
$(5,0)$	$(-,-)$	$(+,-)$

Table 4.8: Invariants, $(\sigma = \pm 1, \tau = \pm 1)$, of bilinear forms derived from Dirac sesquilinear form for $t + s = 5$.

In (0,5), (1,4), (4,1) and (5,0) $Re[A]$ is super-admissible and in (2,3) and (3,2) $Im[A]$ is super-admissible.

We can obtain two further bilinear forms from the complex $\mathrm{Spin}_0(t,s)$-invariant bilinear form,

$$C(\lambda, \chi) = \lambda^T C \chi, \tag{4.26}$$

once again by taking the real and imaginary part. In five dimensions the charge conjugation matrix is a 'C_-' with invariants $\sigma = +1$ and $\tau = -1$. To summarise

	$Re[C]$	$Im[C]$
All sigs.	$(-,+)$	$(-,+)$

Table 4.9: Invariants, $(\sigma = \pm 1, \tau = \pm 1)$, of the real and imaginary parts of the Majorana bilinear form for $t + s = 5$.

Neither of these are super-admissible; in each five-dimensional signature we have a single super-admissible bilinear form on $\mathbb{S}$. The only possible real-valued superbracket with Dirac supercharges is therefore

$$\{Q_\alpha, Q_\beta\} = \begin{cases} Re[\gamma^\mu A^{-1}]_{\alpha\beta} P_\mu & t = 0,1,4,5 \\ Im[\gamma^\mu A^{-1}]_{\alpha\beta} P_\mu & t = 2,3 \end{cases} . \tag{4.27}$$

This will be called the superbracket on Dirac spinors, with the associated bilinear form called the bilinear form on Dirac spinors.

Let m be the standard sesquilinear form on $\mathbb{C}^4$, $m(\lambda, \chi) = \lambda^\dagger \chi$. The Dirac sesquilinear form $A_{(t,s)}(\cdot,\cdot)$ is then equivalent to $m(\cdot, A_{(t,s)}\cdot)$. Writing $q^i, p^i \in \mathbb{H}^2$ as $q^i = u^i + v^i j$ and $p^i = w^i + z^i j$, with $u^i, v^i, w^i, z^i \in \mathbb{C}$ we see that

$$g = Re\langle q, p\rangle_{(0,5)} = Re[\bar{q}^1 p^1 + \bar{q}^2 p^2] \equiv Re[m(Z, W)] = Re[\bar{Z}^I W^I] \tag{4.28}$$

with $Z^I = (u^1, v^1, u^2, v^2) \in \mathbb{C}^4$ and $W^I = (w^1, z^1, w^2, z^2) \in \mathbb{C}^4$. The $(0,5)$ subscript was added to specify this was the choice of hermitian form on $\mathbb{H}^2$ in the $(0,5)$ model only.

$A_{(0,5)} = Id$ so that $Re[A]$ corresponds directly to the super-admissible bilinear form $\beta_0 = Re\langle\cdot,\cdot\rangle_{(0,5)}$.

For $(1,4)$ the super-admissible bilinear form is $\beta_1 = Re\langle I\cdot,\cdot\rangle_{(1,4)}$ where $I = R_i$ and $\langle\cdot,\cdot\rangle_{(1,4)}$ is

$$Re[\langle q, p\rangle_{(1,4)}] = Re[\bar{q}^1 p^1 - \bar{q}^2 p^2]. \tag{4.29}$$

$A_{(1,4)} = \gamma_1 = -R_i E$ in our model so that

$$Re[A_{(1,4)}(\cdot,\cdot)] = Re[m(\cdot, A_{(1,4)}\cdot)] \equiv Re[g(\cdot, -R_i E)] = Re[g(R_i E\cdot,\cdot)]. \tag{4.30}$$

Now $Eq^1 = q^1$ and $Eq^2 = -q^2$ so that $Re[g(E\cdot,\cdot) = \langle\cdot,\cdot\rangle_{(1,4)}]$. $Re[m(\cdot, A_{(1,4)}\cdot)]$ is therefore equivalent to $\beta_1 = Re\langle I\cdot,\cdot\rangle_{(1,4)}$.

Similarly writing $q^i, p^i \in \mathbb{H}'^2$ as $q^i = u^i + v^i j$ and $p^i = w^i + z^i j$, with $u^i, v^i, w^i, z^i \in \mathbb{C}$ one can show that for the standard hermitian form on $\mathbb{H}'^2$

$$Im_i[\langle\cdot, L_i\cdot\rangle_{(2,3)}] \equiv Re[m(Z, W)] \tag{4.31}$$

where Im_i is the i-th imaginary component and m, Z and W are defined as above. The bilinear form on $\mathbb{C}^4$ is

$$Im[A_{(2,3)}(\cdot,\cdot)] = Im[m(\cdot, A_{(2,3)}\cdot)] = Re[m(\cdot, -iA_{(2,3)}\cdot)]. \tag{4.32}$$

In the para-quaternionic model $A_{(2,3)} = \gamma_1\gamma_2 = -L_i$ so that

$$Re[m(\cdot, -iA_{(2,3)}\cdot)] \equiv Im_i[\langle\cdot, (L_i)^3\cdot\rangle_{(2,3)}] = Re[\langle\cdot,\cdot\rangle_{(2,3)}]. \tag{4.33}$$

Two L_i factors come from $iA_{(2,3)}$ and the other is from (4.31). We see that the super-admissible bilinear form $\beta_0 = Re[\langle\cdot,\cdot\rangle_{(2,3)}]$ corresponds to $Im[A_{(2,3)}(\cdot,\cdot)]$ so that in all cases we see the analysis from the (para-)quaternionic models and the common formulation in physics agree.

4.4 Doubled Spinor Formulation

Equation (4.27) is not the usual way a (1,4) theory is written; usually, it is written in terms of symplectic Majorana spinors. In (0,5), (1,4), (4,1) and (5,0) we can define and use symplectic Majorana spinors, though in (2,3) and (3,2) we will need a twisted Majorana condition. This section specialises the $\mathcal{N}$-extended spinor construction to the case of five-dimensional $\mathcal{N} = 2$ theories, thereby providing a self-contained example of the construction and its uses, beginning from defining the extended spinor module and the superbracket and through to writing down a Lagrangian theory.

(0,4), (1,4), (4,1) and (0,5) do not have a $\mathrm{Spin}_0(t,s)$-invariant real structure on $\mathbb{S}$, in other words $B^*B = -1$ so that $J^{(\epsilon)}$ is a quaternionic structure (also a complex structure). This means the complexification of the real spinor module is $\mathbb{S}\otimes\mathbb{C}^2$ which is isomorphic to $\mathbb{S}\oplus\mathbb{S}$. It is usually called $\mathcal{N} = 2$ as we work with two copies of the complex spinor module (with a reality condition).

In the signatures with a real structure, (2,3) and (3,2), the complex spinor module and real spinor module is distinct $\mathbb{S} \ncong S$, but the Majorana bilinear form is not super-admissible, so we cannot define a $\mathcal{N} = 1$ algebra (this can not be circumvented by using another bilinear form, as a Majorana reality condition necessarily equates the Dirac sesquilinear form and the Majorana bilinear form). Therefore the minimal algebra is defined on $S \oplus S$, and we work with its complexification $\mathbb{S}\oplus\mathbb{S} \cong \mathbb{S}\otimes\mathbb{C}^2$, so the minimal superalgebra in all five-dimensional signatures is $\mathcal{N} = 2$ and involves spinors that are elements of $(\mathbb{S}\otimes\mathbb{C}^2)^\rho$.

Recall that the construction disentangles $\mathrm{Spin}(t,s)$ and R-symmetry transformations, with the latter being moved entirely onto the internal index, i.e. acting upon the $\mathbb{C}^2$

factor.

In five dimensions the Majorana bilinear form is anti-super-admissible, so on $\mathbb{S} \otimes \mathbb{C}^2$ a super-admissible complex bilinear form is

$$b = C \otimes J_2, \quad b(\lambda^i, \chi^i) = (\lambda^i)^T C \chi^j \varepsilon_{ji}. \tag{4.34}$$

Where the distinction between the previously outlined bilinear form on $\mathbb{S}$ alone is needed, this may also be called the doubled spinor bilinear form. Here we have renamed J_2 to ε_{ji}, as it is the Levi-Civita symbol in 2 dimensions and this is how it is conventionally written. As we usually name bilinear forms with the same name as their Gram matrix, and to align with the literature, we will now call it $\varepsilon(\cdot,\cdot)$. For completeness, $\varepsilon(\cdot,\cdot)$ is an antisymmetric bilinear form on $\mathbb{C}^2$ defined by

$$\varepsilon(z,w) = z^i w^j \varepsilon_{ji}, \quad \varepsilon = \varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.35}$$

Later when writing Lagrangians and supersymmetric variations, when writing terms involving doubled spinors with closed indices, the internal i, j indices will be omitted, e.g.

$$\bar{\lambda}\gamma^{\mu\nu}\chi = \bar{\lambda}^i \gamma^{\mu\nu} \chi^j \varepsilon_{ji} \tag{4.36}$$

Following the construction, we define the complex superbracket from b. This gives

$$\{Q_{i\alpha}, Q_{j\beta}\} = k(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \varepsilon_{ij}. \tag{4.37}$$

As stated previously, conventionally $k = -\frac{1}{2}$, this choice will be motivated in the next section. Where the distinction is needed this will be called the doubled spinor superbracket.

Reality Conditions in Five Dimensions

Whether $J^{(\epsilon)(\alpha)}$ defines a quaternionic or para-quaternionic structure depends on B^*B, in each five-dimensional signature this is

$$B^*B = \begin{cases} -1 & (0,5),(1,4),(4,1) \text{ and } (5,0), \\ +1 & (2,3) \text{ and } (3,2). \end{cases} \tag{4.38}$$

$J^{(\epsilon)}$ is a real structure in (2,3) and (3,2), and a quaternionic structure in (0,5) and (1,4). We therefore use symplectic Majorana spinors in (0,5), (1,4), (4,1) and (5,0) and Majorana or twisted Majorana spinors in (2,3). From the Table 3.10, we expect an R-symmetry of SU(2) in (0,5) and (1,4) and SU(1,1) in (2,3).

We can then build a signature-dependent real structure, ρ, on $\mathbb{S} \otimes \mathbb{C}^2$ [1]:

$$\rho(\lambda^i) = \alpha^* B^* (\lambda^j)^* L_{ji}, \quad L_{ij} = \begin{cases} \varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (0,5),\ (1,4), \\ \eta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & (2,3). \end{cases} \tag{4.39}$$

In (2,3) we have two choices for L that give real structures, $L = \{\delta, \eta\}$, though either gives the same R-symmetry group. As they result in the same R-symmetry group, there should exist an isomorphism between them which is detailed in Section 4.4.

To ensure the vector-valued bilinear form is real $\alpha = \pm 1$ in (0,5) and (1,4) and $\alpha = \pm i$ in (2,3).

The spinors invariant under ρ obey the reality condition

$$(\lambda^i)^* = \alpha B \lambda^j L_{ji}. \tag{4.40}$$

Under this reality condition the complex spinor and the doubled spinors vector-valued bilinear forms (and therefore superbrackets) are related. For example, with the symplectic Majorana condition

$$\begin{aligned} [C_- \otimes \varepsilon](\gamma^\mu \lambda, \chi) &= (\gamma^\mu \lambda^2)^T C \chi^1 - (\gamma^\mu \lambda^1)^T C \chi^2 \\ &= (\alpha^* B^* (\lambda^1)^*)^T (\gamma^\mu)^T C \chi^1 - (\gamma^\mu \lambda^1)^T C (\alpha^* B^* (\chi^1)^*) \\ &= -\alpha^* \big(\tau(A) A(\gamma^\mu \lambda, \chi) + (A(\gamma^\mu \lambda, \chi))^*\big) \\ &= -(-1)^t 2\alpha^* Re[A(\gamma^\mu \lambda, \chi)]. \end{aligned} \tag{4.41}$$

Where we have identified λ^1, χ^1 with $\lambda, \chi \in \mathbb{S}$, the spinors of the complex spinor version of the superalgebra.

[1]Note that we have chosen to use η in place of $I_{1,1}$, which is the diagonalisation of η. The two descriptions are equivalent, as shown later.

In (1,4) the conventional choice is $\alpha = -1$, as in [20]. This gives a coefficient of -2, which corresponds to the conventional $k = -\frac{1}{2}$ in (4.37). In each signature the reality condition will be chosen so that the superalgebra on the doubled spinor module is given by (4.37) with $k = -\frac{1}{2}$. This amounts to selecting one of the particular signs for α in (4.40). For (0,5) this means that $\alpha = +1$, and in (4,1) and (5,0) we get $\alpha = +1$ and -1 respectively.

For the twisted Majorana reality conditions, this is slightly different; the Dirac superbracket is constructed with $Im[A]$:

$$\begin{aligned}[C \otimes \varepsilon](\gamma^\mu, \chi) &= (\gamma^\mu \lambda^2)^T C \chi^1 - (\gamma^\mu \lambda^1)^T C \chi^2 \\ &= (\alpha B \lambda)^\dagger (\gamma^\mu)^T C \chi - (\gamma^\mu \lambda^1)^T C (\alpha B \chi)^* \\ &= \alpha^* \big((\lambda^\dagger A \gamma^\mu \chi - (\gamma^\mu \lambda)^T A^* \chi^* \big) \\ &= 2(-1)^t \alpha^* i Im[A(\gamma^\mu \lambda, \chi)].\end{aligned} \tag{4.42}$$

Keeping the same normalisation, this means that in (2,3) we $\alpha = +i$ and in (3,2) $\alpha = -i$.

A superbracket is defined from the vector-valued bilinear form, so this extends to the proportionality of the complex spinor superbracket and the doubled spinor superbracket:

$$\implies \{Q_\alpha, Q_\beta\} \propto \{Q^i_\alpha, Q^j_\beta\}|_\rho. \tag{4.43}$$

Indeed, it is stronger than that: all tensor-valued bilinear forms are proportional, so that we can say the complex spinor module and the ρ-invariant submodule of the doubled spinor module are isomorphic, i.e.

$$\mathbb{S} \simeq (\mathbb{S} \otimes \mathbb{C}^2)^\rho. \tag{4.44}$$

Note that this only confirms what we already know, that $\mathbb{S} \otimes \mathbb{C}^2$ is the complexification of the complex spinor module $\mathbb{S}$ and is included here as an easy verification of this fact.

Equivalence of Majorana and twisted Majorana Reality Condition

We defined the reality condition with an off-diagonal reality condition to demonstrate the relation to a single Dirac spinor (with no reality condition) clearer. However, we know that regardless of the choice of real condition $\mathcal{N} = 2$ superalgebras in (2,3) and

(3,2) have $\mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$ R-symmetry; therefore each reality condition is equivalent.

We have three choices of the matrix L in the reality condition, namely

$$L = \left\{ \delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{4.45}$$

As the previous sections have given the reality condition with η, we shall relate this to both δ and η'. We begin with spinors, λ^i, that satisfy

$$(\lambda^i)^* = \alpha B \lambda^j \eta_{ji} \tag{4.46}$$

and wish to relate these to spinors with reality condition

$$(\Psi^i)^* = \beta B \Psi^j \eta'_{ji}, \tag{4.47}$$

$$(\phi^i)^* = \gamma B \phi^i = \gamma B \lambda^j \delta_{ji}. \tag{4.48}$$

In each case the vector-valued bilinear form should remain as $b(\gamma^\mu \cdot, \cdot) = [C \otimes \varepsilon](\gamma^\mu \cdot, \cdot)$, as this is the only super-admissible choice. Those with $L = \eta'$ are a pair of $\mathrm{O}(1,1)$ Majorana spinors in the language of [14], and those with $L = \delta$ have an $\mathrm{O}(2)$ Majorana condition.

Changing the reality condition so that $L = \eta'$ is just rotating λ^i to an eigenbasis of the matrix η

$$\Psi^1 = \lambda^1 + \lambda^2, \quad \Psi^2 = \lambda^1 - \lambda^2. \tag{4.49}$$

Ψ^i obey a diagonalised reality condition

$$(\Psi^i)^* = \alpha B \Psi^j \eta'_{ij}. \tag{4.50}$$

Such that $\beta = \alpha$. Under this transformation the vector-valued bilinear form on the doubled spinor module is unaffected

$$(\gamma^\mu \lambda^i)^T C \chi^j \varepsilon_{ji} = (\gamma^\mu \Psi^i)^T C \Omega^j \varepsilon_{ji}. \tag{4.51}$$

Where Ω^i is related to χ^i analogously to how Ψ^i came from λ^i.

Further, this is equivalent to a reality condition with $L = \delta$. We set

$$\phi^1 = \frac{1+i}{\sqrt{2}}(\lambda^1 - \lambda^2), \quad \phi^2 = \frac{1-i}{\sqrt{2}}(\lambda^1 + \lambda^2). \tag{4.52}$$

ϕ^1 and ϕ^2 are a pair of independent Majorana spinors; it is easy to show

$$(\phi^i)^* = i\alpha B\phi^i. \tag{4.53}$$

This too leaves the vector-valued bilinear form invariant:

$$(\gamma^\mu\lambda^i)^T C\chi^j\varepsilon_{ji} = (\gamma^\mu\phi^i)^T C\zeta^j\varepsilon_{ji}. \tag{4.54}$$

Where once again ζ^i is defined from χ^i like ϕ^i is from λ^i. Two equal vector-valued bilinear forms lead to two equal superbrackets and two equivalent superalgebras.

However, the rewriting in (4.49) and (4.52) makes the isomorphism with the vector-valued bilinear form on a Dirac spinor, $Im[A(\gamma^\mu\lambda,\chi)]$, less obvious. As a result, we prefer to give the reality condition in the off-diagonal form.

4.4.1 R-Symmetry

An R-symmetry transformation must commute with $\mathrm{Spin}(t,s)$, leave the vector-valued bilinear form/superbracket invariant and commute with the reality condition. There are two scenarios to consider for the latter due to the different reality conditions used. These calculations were performed in Section 3.8, but here we will give a slightly different derivation.

The bilinear form on the doubled spinor module is the same in all signatures, giving us a complex R-symmetry group of $\mathrm{Sp}(2,\mathbb{C})$ in each signature. The different reality conditions reduce this to a different real form. The real form corresponding to the automorphism J_2 is $\mathrm{USp}(2) \cong \mathrm{SU}(2)$, and the real form from δ, η or η' is $\mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. We will prefer the special unitary groups to highlight the similarities and differences between the signatures.

Using a slightly different approach than in Section 3.8, we can instead derive this by remarking that the symplectic Majorana constraint is invariant under transformations of the form

$$\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}. \tag{4.55}$$

This group is isomorphic to $\mathrm{GL}(1,\mathbb{H})$ using the map between complex matrices and quaternionic matrices given in Section 2.3.2.

The R-symmetry group in signatures $(0,5),(1,4),(4,1)$ and $(5,0)$ is then

$$\mathrm{GL}(1,\mathbb{H}) \cap \mathrm{Sp}(2,\mathbb{C}) = \mathrm{SU}(2). \tag{4.56}$$

The group that commutes with the twisted Majorana constraint is $\mathrm{GL}(1,\mathbb{H}')$, which can be represented as 2×2 complex matrices of the form

$$\begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}. \tag{4.57}$$

The total R-symmetry group for the $(2,3)$ and $(3,2)$ theories is therefore

$$\mathrm{GL}(1,\mathbb{H}') \cap \mathrm{Sp}(2,\mathbb{C}) = \mathrm{SU}(1,1). \tag{4.58}$$

We recall that due to Schur's lemma these R-symmetry transformations act entirely on the internal $\mathbb{C}^2$ factor, hence the name $\mathrm{SU}(2)$ or $\mathrm{SU}(1,1)$ indices for the associated i,j indices of this $\mathbb{C}^2$ space.

4.5 Summary of Doubled Spinor Formulations

The following Lagrangians and supersymmetry representation are in terms doubled spinors, which are elements of $\mathbb{S}\otimes\mathbb{C}^2$ equipped with a complex bilinear and a reality condition.

The bilinear form on $\mathbb{S}\otimes\mathbb{C}^2$ is

$$b(\lambda,\chi) = [C\otimes\varepsilon](\lambda,\chi) = (\lambda^i)^T C\chi^j\varepsilon_{ji}. \tag{4.59}$$

This is used to define the superbracket, which is

$$\{Q_{i\alpha}, Q_{j\beta}\} = -\frac{1}{2}(\gamma^{\mu}C^{-1})_{\alpha\beta}P_{\mu}\varepsilon_{ij}. \tag{4.60}$$

The choice of reality condition in each signature is chosen so that the doubled spinor superbrackets are proportional to the complex spinor superbracket in each signature. They are as follows:

	Reality Condition
$(0,5)$	$(\lambda^i)^* = B\lambda^j\varepsilon_{ji}$
$(1,4)$	$(\lambda^i)^* = -B\lambda^j\varepsilon_{ji}$
$(2,3)$	$(\lambda^i)^* = iB\lambda^j\eta_{ij}$
$(3,2)$	$(\lambda^i)^* = -iB\lambda^j\eta_{ij}$
$(4,1)$	$(\lambda^i)^* = B\lambda^j\varepsilon_{ji}$
$(5,0)$	$(\lambda^i)^* = -B\lambda^j\varepsilon_{ji}$

Table 4.10: Reality Condition in each signature, $B = (CA^{-1})^T$ is signature dependent.

The signatures with symplectic Majorana spinors ((0,5), (1,4), (4,1) and (5,0)) have SU(2) R-symmetry and those with twisted Majorana spinors ((2,3) and (3,2)) have SU$(1,1)$ R-symmetry.

4.6 Field Content

It is well known that $\mathcal{N} = 2$ supersymmetry in five dimensions permits a vector multiplet representation [20, 39]

$$(A^{\mu}, \lambda^i, \sigma, Y^{ij}) \qquad \mu = 1,2,3,4,5 \qquad i = 1,2. \tag{4.61}$$

With the eponymous vector field, A^{μ}, a pair of spinors, λ^i, subject to the signature-dependent reality conditions outlined in the previous section, a scalar field, σ, and a triplet of auxiliary fields packaged as a real, symmetric SU(2) or SU$(1,1)$ tensor, Y^{ij}.

For these fields to be a representation of the complex supersymmetry algebra they transform according to

$$\begin{aligned} &\delta A^{\mu} = \alpha\bar{\epsilon}\gamma^{\mu}\lambda, \qquad \delta\sigma = a\bar{\epsilon}\lambda, \qquad \delta Y^{ij} = \upsilon\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)}, \\ &\delta\lambda^i = \beta\gamma\cdot F\epsilon^i + b\not{\partial}\sigma\epsilon^i + yY^{ij}\epsilon_j. \end{aligned} \tag{4.62}$$

This is an off-shell representation, so is independent of any field equations/Lagrangian.

If (4.62) are to be a representation of the superalgebra (4.60) then the coefficients, which are arbitrary complex numbers, have to obey

$$\begin{aligned}
-\frac{1}{2} &= -2ab = 4\alpha\beta = -uy = 2\alpha\beta - \frac{ab}{4} - \frac{3uy}{8} \\
\alpha\beta &+ \frac{ab}{4} + \frac{uy}{8} = 0 \qquad .
\end{aligned} \tag{4.63}$$

The reality of the coefficients is signature dependent, as the corresponding spinors bilinears reality varies due to the different reality conditions. The reality properties of each coefficient is collected in the following table

Parameter	Real	Imaginary
α	$t = 0, ..., 5$	Never
a	$t = 0, 2, 4$	$t = 1, 3, 5$
β	$t = 0, ..., 5$	Never
b	$t = 0, 2, 4$	$t = 1, 3, 5$
u	$t = 0, 1, 4, 5$	$t = 2, 3$
y	$t = 0, 1, 4, 5$	$t = 2, 3$

Table 4.11: The reality properties of the coefficients in the supersymmetry transformations.

These were found by requiring $\delta\sigma$ and δA^μ to be real, and that $\delta\lambda^i$ and δY^{ij} to obey the same reality conditions as λ^i and Y^{ij} respectively. These are related to the reality properties of the associated spinor bilinear, for example, a is real when $\bar{\epsilon}\lambda$ is real.

The complex conjugate of a general spinor bilinear of arbitrary rank $\bar{\lambda}\gamma^{\mu_1\cdots\mu_r}\chi$ is

$$(\bar{\lambda}\gamma^{\mu_1\cdots\mu_r}\chi)^* = ((\lambda^i)^T C\gamma^{\mu_1\cdots\mu_r}\chi^j\varepsilon_{ji})^* \tag{4.64}$$

$$=\alpha^2(\lambda^k)^T B^\dagger C^* B\chi^l L_{ki}L_{lj}\varepsilon_{ji}. \tag{4.65}$$

One can show $B^\dagger C^* B = (-1)^t C$ and the combination $L_{ki}L_{lj}\varepsilon_{ji}$ has $L = \varepsilon$ for $t = 0, 1, 4, 5$ and $L = \eta$ for $t = 2, 3$. In these two cases

$$\varepsilon_{ki}\varepsilon_{lj}\varepsilon_{ji} = \varepsilon_{lk}, \quad \eta_{ki}\eta_{lj}\varepsilon_{ji} = -\varepsilon_{lk}. \tag{4.66}$$

Note that the α in this equation is that from the reality condition, and is not to be confused with that in (4.62).

As $(\gamma^\mu)^* = B\gamma^\mu B^{-1}$ it follows that

$$(\gamma^{\mu_1\cdots\mu_r})^* = (-1)^{rt} B\gamma^{\mu_1\cdots\mu_r}B^{-1} \tag{4.67}$$

so that

$$(\bar{\lambda}\gamma^{\mu_1\cdots\mu_r}\chi)^* = (-1)^{t(r+1)}\bar{\lambda}\gamma^{\mu_1\cdots\mu_r}\chi. \tag{4.68}$$

Combining all these we see that in five-dimensions using the conventions for the spinor module outlined in this section

$$(\bar{\lambda}\gamma^{\mu_1\cdots\mu_r}\chi)^* = (-1)^{t(r+1)}\bar{\lambda}\gamma^{\mu_1\cdots\mu_r}\chi. \tag{4.69}$$

The reality of the coefficients immediately follows. We see that in all signature the vector-valued bilinear form ($r = 1$) is real, as it should be because this is used to define the real supersymmetry algebra.

SU(2) **and** SU(1,1) **Tensors**

As outlined in Section 2.11 the $\mathcal{N} = 2$ vector multiplet in (1,4) vector multiplets involved Y^{ij}, a real, symmetric SU(2) tensor whose reality properties are induced by the reality condition of the spinors. In (2,3), however, the spinors have a twisted Majorana reality condition that makes the R-symmetry group SU(1,1), so the auxiliary fields are be modified to be a real, symmetric SU(1,1) tensor. We will recap the SU(2) tensors then define SU(1,1) tensors in comparison.

For a generic member of SU(2) the following holds

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in \mathrm{SU}(2), \qquad U^* = \varepsilon U \varepsilon^T \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.70}$$

This demonstrates the equivalence of [2] and $[\bar{2}]$ as real representations.

Under SU(2), $Y^{ij} \in Sym(2,2,\mathbb{C})$ transforms as

$$\begin{aligned} Y^{ij} &\to U^i{}_k U^j{}_l Y^{kl} = (Y')^{ij} \\ Y &\to UYU^T = Y' \end{aligned} \tag{4.71}$$

An SU(2)-invariant reality condition is imposed on Y, reducing the degrees of freedom from 3 complex variables to 3 real variables

$$\begin{aligned} &(Y^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}Y^{kl} \\ &Y \to Y' \implies (Y')^* = \varepsilon Y' \varepsilon^T. \end{aligned} \tag{4.72}$$

The raising and lowering of i, j indices are done using ε_{ij}, as this is involved in the bilinear form for the spinor terms. This means that for an SU(2) tensor $(Y^{ij})^* = Y_{ij}$.

In $(2,3)$ and $(3,2)$ signature the R-symmetry group is $\mathrm{SU}(1,1)$. Therefore Y^{ij} must be an $\mathrm{SU}(1,1)$ tensor, which conjugate differently to SU(2) tensors.

A generic member of $\mathrm{SU}(1,1)$ satisfies the following

$$U = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \in \mathrm{SU}(1,1) \qquad U^* = \eta U \eta \qquad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.73}$$

Similarly this demonstrates the equivalence of $[2]$ and $[\bar{2}]$ as real $\mathrm{SU}(1,1)$ modules.

Analogously $Y^{ij} \in Sym(2,2,\mathbb{C})$ transforms under $\mathrm{SU}(1,1)$ according to

$$Y \to UYU^T = Y'.$$

We once again seek a reality condition on Y^{ij} that is invariant under $\mathrm{SU}(1,1)$. This is given by

$$(Y^{ij})^* = \eta_{ik}\eta_{jl}Y^{kl}. \tag{4.74}$$

This reality condition is correctly $\mathrm{SU}(1,1)$ invariant, that is $(Y')^* = \eta Y' \eta$.

Care should be taken with raising and lowering indices in (2,3) and (3,2), as raising and lowering indices is no longer equivalent to complex conjugation. The bilinear form on $\mathbb{C}^2$ is still ε_{ij} so this is used to raise and lower the i, j indices but η_{ij} is used when complex conjugating. As a result $(Y^{ij})^* \neq Y_{ij}$ in $(2,3)$ and $(3,2)$.

4.7 Lagrangian Description of Theories

Naturally after a multiplet is obtained, a Lagrangian description of the theory is desired. Taking the original Lagrangian calculated for Minkowski signature and removing knowledge of the coefficients/signs for each term we obtain

$$\begin{aligned} L = &\left(\frac{s_F}{4} F^I_{\mu\nu} F^{J\mu\nu} + \frac{s_\sigma}{2} \partial_\mu \sigma^I \partial^\mu \sigma^J + \frac{s_\lambda}{2} \bar{\lambda}^I \not{\partial} \lambda^J + s_Y Y^I_{ij} Y^{Jij}\right) \mathcal{F}_{IJ}(\sigma) \\ &+ \left(\theta_1 \varepsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau} + \theta_2 \bar{\lambda}^I \gamma^{\mu\nu} F^J_{\mu\nu} \lambda^K + \theta_3 \bar{\lambda}^{Ii} \lambda^{Jj} Y^K_{ij}\right) \mathcal{F}_{IJK}(\sigma). \end{aligned} \tag{4.75}$$

This is a 'holomorphic master Lagrangian' that encodes knowledge of the real forms – these real forms correspond to five-dimensional theories in different signatures. We will implement signature-dependent reality conditions on this to obtain the vector multiplet theory in the signature.

The $s_\sigma = s_F = s_\lambda = s_Y = \pm 1$ are signs, so that the conventional normalisations of the kinetic terms are maintained. Once again, $I, J = 1, ..., N$ enumerate the vector multiplets with the coupling coefficients determined by a function $\mathcal{F}(\sigma)$

$$\mathcal{F}_{IJ}(\sigma) = \frac{\partial}{\partial \sigma^I} \frac{\partial}{\partial \sigma^J} \mathcal{F}(\sigma), \qquad \mathcal{F}_{IJK}(\sigma) = \frac{\partial}{\partial \sigma^I} \frac{\partial}{\partial \sigma^J} \frac{\partial}{\partial \sigma^K} \mathcal{F}(\sigma). \tag{4.76}$$

To maintain gauge and supersymmetric invariance the prepotential, $F(\sigma)$, is a polynomial of degree no more than 3. As usual for five-dimensional theories, the scalar kinetic term describes a non-linear sigma model with metric $\mathcal{F}_{IJ}$. $\mathcal{F}(\sigma)$ is often called the prepotential. Before imposing a reality condition, it is a holomorphic Hesse potential of a complex Riemannian manifold. After forcing σ to be real we obtain the expected very special real geometry expected, see Section 2.11.

As noted in [20] the spinor term is written as a partial derivative, not a covariant derivative with respect to the Levi-Civita connection of $\mathcal{F}_{IJ}$; the term containing the connection is identically zero as $\bar{\lambda}^{i(I|} \lambda^{j|J)} \varepsilon_{ji} = 0$.

After varying the above Lagrangian using the transformations (4.62) the following requirements on the coefficients are obtained [2], where t is the number of time-like directions.

$$\begin{aligned} s_F\alpha &= -2s_\lambda\beta & s_\sigma a &= -s_\lambda b & 2s_Y u &= -s_\lambda y \\ 3\theta_1\alpha &= \pm 2i^t\theta_2\beta^* & 4\theta_2\alpha &= -\theta_3 u & \theta_2 y &= \theta_3\beta \\ \alpha s_F &= 8b\theta_2 & a s_Y &= y\theta_3 & a s_\lambda &= 8\alpha\theta_2. \end{aligned} \tag{4.77}$$

* - the sign ambiguity in this equation is explained in the next section. Note that α here is not the α in the reality condition.

Which combined with (4.63) gives the following

$$-\frac{1}{2} = -2\frac{s_F}{s_\lambda}\alpha^2 = 2\frac{s_\sigma}{s_\lambda}a^2 = 2\frac{s_Y}{s_\lambda}u^2. \tag{4.78}$$

Coefficient of Chern-Simons term

As noted, there is an ambiguity concerning the sign of the Chern-Simons term

$$\pm\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau}. \tag{4.79}$$

Here we have chosen to keep the definition of $\varepsilon_{\mu\nu\rho\sigma\tau}$ the same in all signatures, such that $\varepsilon_{12345} = 1$. However, the sign of this term can be changed with a different choice of the γ-matrices.

The coefficient of the Chern-Simons term depends on the chosen representation of the Clifford algebra. In odd dimensions, we have the freedom to choose the volume element of the Clifford algebra to vary by an overall sign. This sign enters the calculation when we look at the invariance of the Lagrangian under supersymmetry, due to the presence of terms containing $\varepsilon^{\mu\nu\rho\sigma\tau}\gamma_\mu$ in the variation of the Lagrangian.

Recall we made the following choices for the Clifford algebra representation

$$\gamma_{\mu\nu\rho\sigma\tau} = \begin{cases} \varepsilon_{\mu\nu\rho\sigma\tau} & t\text{-even} \\ -i\varepsilon_{\mu\nu\rho\sigma\tau} & t\text{-odd} \end{cases}. \tag{4.80}$$

[2] Whilst I have performed these calculations as part of the work, they are relatively standard, repeating those in [20] with arbitrary coefficients and have been omitted due to space considerations.

This makes the coefficient of the Chern-Simons term positive in all signatures

$$+\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau}. \tag{4.81}$$

Primarily this makes the dimensional reduction of this term straightforward, without having to have the five- and four-dimensional Levi-Civita symbols normalised differently, i.e. we have $\varepsilon_{12345} = +1$ and $\varepsilon_{1234} = +1$, whilst in [20] $\varepsilon_{12345} = +1$ and $\varepsilon_{1234} = -1$ was needed to align the four-dimensional terms correctly.

4.8 Lagrangians and Supersymmetric Variations

In this section, the Lagrangians and supersymmetric variations are presented for each signature, after imposing the reality conditions outlined in the previous sections.

For the correct physical interpretation, the Minkowski theory must have a positive-definite Lagrangian, in our conventions, this means the kinetic terms have a negative sign. In the other signatures, the overall sign of the Lagrangian has no natural choice. Other signatures will have their overall sign chosen to look like the Minkowski theory as much as possible.

4.8.1 Minkowski Signature

$(1,4)$

In (1,4) signature Table 4.11 and (4.78) implies the following sign choices

$$s_\sigma = s_\lambda = s_F = -s_Y. \tag{4.82}$$

We can choose all physical fields to have the correct negative sign for their kinetic terms (the sign of the auxiliary field Y is irrelevant).

$$\begin{aligned} L =&\left(-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I\partial\!\!\!/\lambda^J + Y^I_{ij}Y^{ijJ}\right)\mathcal{F}_{IJ} \\ &+\left(\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} - \frac{i}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{i}{2}\bar{\lambda}^{Ii}\lambda^{Jj}Y^K_{ij}\right)\mathcal{F}_{IJK} \end{aligned} \tag{4.83}$$

$$\begin{aligned}
\delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^I \qquad \delta\sigma^I = \frac{i}{2}\bar{\epsilon}\lambda^I \qquad \delta Y^{ijI} = -\frac{1}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I}, \\
\delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i - \frac{i}{2}\not{\partial}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j.
\end{aligned} \tag{4.84}$$

This agrees with [20] except for the sign of the Chern-Simons 'AFF' term, this is the conventional choice explained previously in 4.7

$(4,1)$

We interpret $(4,1)$ to be a Minkowski signature theory with a mostly-negative convention. Table 4.11 and (4.78) tells us the signs obey

$$-s_\sigma = s_\lambda = s_F = -s_Y \tag{4.85}$$

This has created a difference between the scalar and vector kinetic terms, but this is necessary to line up with the mostly-negative convention. In a mostly-negative convention, a positive-definite scalar kinetic term has a plus sign, with the rest still requiring negative, which is a choice we can make here

$$-s_\sigma = s_\lambda = s_F = -s_Y = -1. \tag{4.86}$$

We then obtain

$$\begin{aligned}
L =& \left(-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I\not{\partial}\lambda^J + Y^I_{ij}Y^{ijJ}\right)\mathcal{F}_{IJ} \\
&+ \left(\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} - \frac{1}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{1}{2}\bar{\lambda}^{Ii}\lambda^{Jj}Y^K_{ij}\right)\mathcal{F}_{IJK}
\end{aligned} \tag{4.87}$$

$$\begin{aligned}
\delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^I \qquad \delta\sigma^I = \frac{1}{2}\bar{\epsilon}\lambda^I \qquad \delta Y^{ijI} = -\frac{1}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I}, \\
\delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i + \frac{1}{2}\not{\partial}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j.
\end{aligned} \tag{4.88}$$

The remaining differences amount to factors of $\pm i$ on terms involving the fermions. These are induced from the signature-dependent reality conditions imposed on the spinor module. Considering $min(t,s)$ as 'time' the two Minkowski Lagrangians agree, up to conventional differences of the kinetic term signs and fermionic term's reality properties.

4.8.2 Euclidean Signature

$(0,5)$

For (0,5) we get the following sign attribution

$$-s_\sigma = s_\lambda = s_F = -s_Y = -1. \tag{4.89}$$

If we were using the OS framework for Euclidean theories, we would require that the action is bounded from below. This cannot be the case because changing the overall sign of the Lagrangian always leaves at least one kinetic term with the wrong sign. One of the vector or scalar kinetic terms will be negative-definite and the other positive-definite. This sign attribution has the scalar field flipped from the canonical choice in the associated (1,4) Minkowski signature theory.

This sign difference was predicted in [26], using Killing spinor equations to derive bosonic Lagrangians. The ab initio derivation in this chapter shows there is no choice, and this particular relative sign attribution is mandated by supersymmetry.

$$\begin{aligned} L =& \left(-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I\partial\!\!\!/\lambda^J + Y^I_{ij}Y^{ijJ}\right)\mathcal{F}_{IJ} \\ &+ \left(\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} - \frac{1}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{1}{2}\bar{\lambda}^{Ii}\lambda^{Jj}Y^K_{ij}\right)\mathcal{F}_{IJK} \end{aligned} \tag{4.90}$$

$$\begin{aligned} \delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^I \qquad \delta\sigma^I = \frac{1}{2}\bar{\epsilon}\lambda^I \qquad \delta Y^{ijI} = -\frac{1}{2}\bar{\epsilon}^{(i}\partial\!\!\!/\lambda^{j)I}, \\ \delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i + \frac{1}{2}\partial\!\!\!/\sigma^I\epsilon^i - Y^{ijI}\epsilon_j. \end{aligned} \tag{4.91}$$

$(5,0)$

Here we get

$$s_\sigma = s_\lambda = s_F = -s_Y = -1. \tag{4.92}$$

Once again, this action is indefinite. We again observe a sign flip from the associated Minkowski action ((4,1) in this case).

$$
\begin{aligned}
L =& \Bigg(-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I \not{\partial}\lambda^J + Y^I_{ij}Y^{ijJ}\Bigg)\mathcal{F}_{IJ} \\
&+ \Bigg(\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} - \frac{i}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{i}{2}\bar{\lambda}^{Ii}\lambda^{Jj}Y^K_{ij}\Bigg)\mathcal{F}_{IJK}
\end{aligned} \tag{4.93}
$$

$$
\begin{aligned}
\delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^I \qquad \delta\sigma^I = \frac{i}{2}\bar{\epsilon}\lambda^I \qquad \delta Y^{ijI} = -\frac{1}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I}, \\
\delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i - \frac{i}{2}\not{\partial}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j.
\end{aligned} \tag{4.94}
$$

The factors of $\pm i$ on the fermionic terms, due to the difference in the exact details of the spinor module.

4.8.3 Exotic Signatures

$(2,3)$

We get the following sign attributions

$$
-s_\sigma = s_\lambda = s_F = s_Y = -1. \tag{4.95}
$$

Giving us the Lagrangian, with arbitrary overall sign,

$$
\begin{aligned}
L =& \Bigg(-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I \not{\partial}\lambda^J - Y^I_{ij}Y^{ijJ}\Bigg)\mathcal{F}_{IJ} \\
&+ \Bigg(\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} - \frac{1}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K + \frac{i}{2}\bar{\lambda}^{Ii}\lambda^{Jj}Y^K_{ij}\Bigg)\mathcal{F}_{IJK}
\end{aligned} \tag{4.96}
$$

$$
\begin{aligned}
\delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^I \qquad \delta\sigma^I = \frac{1}{2}\bar{\epsilon}\lambda^I \qquad \delta Y^{ijI} = -\frac{i}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I}, \\
\delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i + \frac{1}{2}\not{\partial}\sigma^I\epsilon^i + iY^{ijI}\epsilon_j.
\end{aligned} \tag{4.97}
$$

$(3,2)$

In this signature, we get the signs as

$$
s_\sigma = s_\lambda = s_F = s_Y = -1. \tag{4.98}
$$

Which results in a Lagrangian as follows

$$\begin{aligned} L =& \left(-\frac{1}{4} F^I_{\mu\nu} F^{J\mu\nu} - \frac{1}{2}\partial_\mu \sigma^I \partial^\mu \sigma^J - \frac{1}{2}\bar{\lambda}^I \not{\partial} \lambda^J - Y^I_{ij} Y^{ijJ} \right) \mathcal{F}_{IJ} \\ &+ \left(\frac{1}{24} \varepsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau} - \frac{i}{8}\bar{\lambda}^I \gamma^{\mu\nu} F^J_{\mu\nu} \lambda^K - \frac{1}{2}\bar{\lambda}^{Ii} \lambda^{Jj} Y^K_{ij} \right) \mathcal{F}_{IJK} \end{aligned} \tag{4.99}$$

$$\begin{aligned} \delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu \lambda^I \qquad \delta\sigma^I = \frac{i}{2}\bar{\epsilon}\lambda^I \qquad \delta Y^{ijI} = -\frac{i}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I}, \\ \delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu} F^I_{\mu\nu}\epsilon^i - \frac{i}{2}\not{\partial}\sigma^I \epsilon^i + iY^{ijI}\epsilon_j. \end{aligned} \tag{4.100}$$

Similarly, in (2,3) and (3,2) signature the kinetic term for σ shows a similar effect, transitioning from mostly-positive to mostly-negative changes the sign of the scalar kinetic term. In addition, we obtain conventional factors of i on the fermionic terms.

4.9 Conclusion

In this chapter we derived two descriptions for the minimal superalgebras in each five-dimensional signature, first in terms of Dirac spinors and then applying the formalism of Chapter 3 to define the doubled spinor constructions (that generalises the conventional manner of using symplectic Majorana spinors). Following this, the doubled spinor formalism was used to derive physical Lagrangians and supersymmetry variations in each signature by imposing signature-dependent reality conditions on a complexified holomorphic master Lagrangian. In doing so, we found signs and coefficients in the Lagrangian are controlled by supersymmetry. These Lagrangians and supersymmetry representations will be used to derive four-dimensional theories by dimensional reduction.

From the obtained Lagrangians, we see the attribution of signs and coefficients in the Lagrangian are fixed up to an overall sign by supersymmetry. In five dimensions, there exists a single one-parameter family of superalgebras up to isomorphism in all cases, so that the relative signs are entirely determined. In other dimensions, as is demonstrated in four dimensions, there may be multiple possible superalgebras in multiple families and therefore distinct theories with different relative signs between the kinetic terms.

Additionally the alternating sign (as the number of time dimensions increase) is necessary to ensure that the reduction from $(p+1,q)$ and $(p,q+1)$ to (p,q), with $p+q=4$, produce a theory with the same scalar geometry (either Kähler or para-Kähler).

In [26] the same relative sign between the vector and scalar kinetic term in theories with even number of time-like dimensions. This what found using Killing spinor equations for the bosonic terms in a supergravity theory, and thus provides verification of this work. In contrast, we included the fermionic terms and were working with rigid supersymmetry. The bosonic terms and fermion supersymmetry variations also agree with [52].

5 Four-dimensional Superalgebras and Vector Multiplets

5.1 Introduction

This chapter focuses on four dimensions, applying the same methodology that we applied in five dimensions. First, we derive and classify the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superalgebras in all four-dimensional signatures. This is done by using models for the spinor module in each signature and studying the space of bilinear forms on the spinor modules. Following this, we apply the N-extended spinor formalism (for the case of doubled spinors again) to derive four-dimensional $\mathcal{N} = 2$ superalgebras in arbitrary signature with manifestly R-symmetric spinors. Finally, $\mathcal{N} = 2$ off-shell vector multiplet theories, including Lagrangians and supersymmetry representations, are derived via the dimensional reduction of the five-dimensional Lagrangians and representations found in Chapter 4.

The first section deals with defining the superalgebras in terms of Dirac spinors (complex spinors) using signature-dependent models that use the natural symmetries of the respective Clifford algebras and spinor modules in each signature. In five dimensions there was a unique super-admissible bilinear form in each signature, but in four dimensions there are four linearly independent super-admissible bilinear forms. Applying the results of Section 3.10 we determine whether these lead to genuinely unique $\mathcal{N} = 2$ superalgebras in each signature, finding in Euclidean and the exotic two-time signature (which is also called neutral signature) there is a unique superalgebra up to isomorphism and in Minkowski signature there is two. In each signature, the space of super-admissible bilinear forms is parameterised by the same vector space of the underlying space-time, $\mathbb{R}^{t,s}$. We then study which superalgebras are obtained by dimensional reduction from five dimensions.

Next, the $\mathcal{N} = 2$ four-dimensional supersymmetry algebras are reformulated in terms of the N-extended spinor formalism, often referred to as doubled spinors in this chapter.

In comparison to five dimensions, four-dimensional supersymmetry algebras are much richer. As it is an even dimension, we have access to two Majorana bilinear forms and two potential $\mathrm{Spin}(t,s)$-invariant ϵ-quaternionic structures, in addition to Weyl spinors. Due to the many possible combinations of bilinear forms and reality conditions, there are multiple possible doubled spinor superalgebras, at least superficially.

In this chapter, supersymmetry representations and Lagrangians are found by dimensional reduction, as opposed to restricting a master Lagrangian as was done in five dimensions. Each four-dimensional signature can have two possible five-dimensional origins (one reduced over a time-like direction, one reduced over a space-like direction) and we find them to differ by signs and coefficients in the representations and Lagrangians.

For example, we obtain, in both Euclidean and neutral signature, two superficially different theories, one from a time-like reduction and one from a space-like reduction. However there is only a single superalgebra up to isomorphism in $(0,4)$ and $(2,2)$, so they should provide isomorphic Lagrangians and associated supersymmetry variations. We will find this to be true and provide explicit local transformations needed to relate the two Lagrangians and supersymmetry variations. In Minkowski signature we find two families of superalgebras, distinguished by their R-symmetry group (which is $\mathrm{U}(2)$ or $\mathrm{U}(1,1)$) and provide maps between the members of each family. The $\mathrm{U}(1,1)$ R-symmetric theory, obtained from dimensional reduction from $(2,3)$, has ghost fields with negative-definite energy and are similar to twisted or type-$*$ theories found in [14,26,53].

5.2 Four-dimensional Clifford Algebra and Spinor Modules

As we did in five dimensions, it is useful to define explicit Clifford algebra models for each signature and calculate the bilinear forms and resulting space of superbrackets. Using the classification of the Clifford algebra and spinor modules we can choose natural models that readily and easily describe the spinor module and the Clifford algebra representation.

We primarily focus on the case of $\mathcal{N}=2$ superalgebras. [1] Therefore we specialise to the case where the spinor module is $\mathbb{S}$, the complex spinor module. Following Section 2.8 we need to derive a basis for the space of bilinear forms and the Schur algebra in each

[1]However we find some things to say about $\mathcal{N}=1$ superalgebras which are detailed as we go along.

signature, using these to define the space of super-admissible bilinear forms. Then, to classify which of the resulting super-admissible bilinear forms lead to unique superalgebras, we study the orbits of the group $\frac{\mathcal{C}(S)^* \cdot \mathrm{Pin}(V)}{\mathrm{Spin}_0(V)}$ on $(Sym^2\mathbb{S}^* \otimes V)^{\mathrm{Spin}_0(V)}$ – however in four-dimensions we find it is sufficient to study the effects of $\mathcal{C}(S)^*$ alone.

Table 5.1 and 5.2 summarise useful information about the Clifford algebras, Schur algebras and spinor modules in four dimensions. The models for the spinor module will use this information to form models for the spinor module and bilinear forms upon it that exploit the natural symmetries in each signature.

$\mathbb{C}l_4$	$\mathbb{C}l_4^0$	$\mathbb{S}$	$\mathbb{S}_\pm$	$\mathcal{C}(\mathbb{S})$	$\mathcal{C}(\mathbb{S}_\pm)$
$\mathbb{C}(4)$	$\mathbb{C}(2)$	$\mathbb{C}^4$	$\mathbb{C}^2$	$2\mathbb{C}$	$\mathbb{C}$

Table 5.1: The complex Clifford Algebra $\mathbb{C}l_4$, the even subalgebra $\mathbb{C}l_4^0$, the complex spinor module, $\mathbb{S}$, the complex semi-spinor modules $\mathbb{S}_\pm$ and associated Schur algebras $\mathcal{C}(\mathbb{S})$ and $\mathcal{C}(\mathbb{S}_\pm)$ in four dimensions.

Signature	$Cl_{t,s}$	$Cl_{t,s}^0$	$\mathbb{S}$	$\mathbb{S}_\pm$	$\mathcal{C}(\mathbb{S})$	$\mathcal{C}(\mathbb{S}_\pm)$
$(0,4),(4,0)$	$\mathbb{H}(2)$	$2\mathbb{H}$	$S_\mathbb{R}$	$S_{\mathbb{R}\pm}$	$2\mathbb{H}$	$2\mathbb{H}$
$(1,3)$	$\mathbb{R}(4)$	$\mathbb{C}(2)$	$S_\mathbb{R} \otimes \mathbb{C}$	$S_\mathbb{R}$	$\mathbb{C}(2)$	$\mathbb{C}$
$(2,2)$	$\mathbb{R}(4)$	$2\mathbb{R}(2)$	$S_\mathbb{R} \otimes \mathbb{C}$	$S_{\mathbb{R}\pm} \otimes \mathbb{C}$	$2\mathbb{R}(2)$	$2\mathbb{R}$
$(3,1)$	$\mathbb{H}(2)$	$\mathbb{C}(2)$	$S_\mathbb{R} = S_{\mathbb{R}\pm} \otimes \mathbb{C}$	$S_{\mathbb{R}\pm}$	$\mathbb{C}(2)$	$\mathbb{C}(2)$

Table 5.2: Classification of the real Clifford Algebras $\mathbb{C}l_4$, the even subalgebra $\mathbb{C}l_4^0$, the relationship between the complex and real complex spinor module, $S_\mathbb{R}$ and $\mathbb{S}$, and the complex and real semi-spinor modules, $S_{\mathbb{R}\pm}$ and $\mathbb{S}_\pm$ and associated Schur algebras $\mathcal{C}(\mathbb{S})$ and $\mathcal{C}(\mathbb{S}_\pm)$ in each four dimensional signature.

The following is heavily based on the paper [3], though the notation and conventions vary slightly. This was done to unify the notation for all three models. Mainly the notation used for the Minkowski and neutral signature models are unified, and thus different from [3] and the Clifford algebra representation in Euclidean signature is slightly different.

5.2.1 Minkowski Signature – $(1,3)$ and $(3,1)$.

Though $Cl_{1,3} \cong \mathbb{R}(4)$ and $Cl_{3,1} \cong \mathbb{H}(2)$, the even Clifford algebras are the same $Cl_{1,3}^0 = Cl_{3,1}^0 \cong \mathbb{C}(2)$ so the $\mathrm{Spin}_0(1,3)$ and $\mathrm{Spin}_0(3,1)$ representations are equivalent. Additionally the Schur algebras are the same, both $\mathcal{C}(\mathbb{S}) \cong \mathbb{C}(2)$ so results obtained for $(1,3)$ are applicable to $(3,1)$ too. We see that one can define Majorana spinors, as the real and complex spinor modules are distinct.

We will now outline a natural model for the Clifford algebra and $\mathbb{S}$ to explicitly calculate the invariants of the bilinear forms and Schur algebra. First, we remark that

$$Cl_{1,3} \cong Cl_{0,2} \otimes Cl_{1,1} \cong \mathbb{R}(2) \otimes \mathbb{R}(2). \tag{5.1}$$

A real spinor is then an element of $S_{\mathbb{R}} = \mathbb{R}^4 \cong \mathbb{R}^2 \otimes \mathbb{R}^2$. First, we will begin by defining a Clifford representation that acts on $S_{\mathbb{R}} \cong \mathbb{R}^2 \otimes \mathbb{R}^2$ before extending this to the complex spinor module $\mathbb{S}$ because ultimately concerned with $\mathcal{N} = 2$ theories in terms of Dirac spinors that live on $\mathbb{S}$.

We can identify $\mathbb{C}$ as $\mathbb{R}^2$ equipped with a complex structure so that

$$\mathbb{S} = S_{\mathbb{R}} \otimes \mathbb{C} \cong S_{\mathbb{R}} \otimes \mathbb{R}^2 \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2. \tag{5.2}$$

Additionally, $S_{\mathbb{R}} \otimes \mathbb{R}^2 \cong S_{\mathbb{R}} \oplus S_{\mathbb{R}}$, such that this is equivalent to the usual $\mathcal{N} = 2$ superalgebra in terms of a pair of real spinors.

We choose the following basis for $\mathbb{R}(2)$:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = IJ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3}$$

I, J, K satisfy

$$\{I, J\} = 0, \quad I^2 = J^2 = Id \implies K^2 = -Id, \quad \{K, I\} = \{K, J\} = 0. \tag{5.4}$$

This basis makes the isomorphism $\mathbb{R}(2) \cong \mathbb{H}'$ explicit, though we will not use the para-quaternions explicitly in this model.

I is a complex structure with which $\mathbb{R}^2$ can be identified with $\mathbb{C} \cong (\mathbb{R}^2, I)$.

We can define the $Cl_{1,3}$ representation on $S_{\mathbb{R}} \cong \mathbb{R}^2 \otimes \mathbb{R}^2$ as

$$\begin{aligned} \gamma_0 &= I \otimes K, \quad \gamma_1 = K \otimes \mathbb{1}, \\ \gamma_2 &= J \otimes 1, \quad \gamma_3 = I \otimes I. \end{aligned} \tag{5.5}$$

The even Clifford algebra is

$$Cl^0_{1,3} = Cl_{0,3} =< \gamma_0\gamma_i | i = 1,2,3 >_{algebra} =< K \otimes K, J \otimes K, 1 \otimes J >_{algebra} \tag{5.6}$$

By inspection we notice that the elements $Id \otimes Id$ and $I \otimes J$ commute with the even Clifford algebra, and therefore $\mathfrak{spin}(1,3)$. $(I \otimes J)^2 = -\mathbb{1}$ so that the Schur algebra is

$$\mathcal{C}(S_{\mathbb{R}}) =< \mathbb{1} \otimes \mathbb{1}, I \otimes J >_{algebra} \cong \mathbb{C}. \tag{5.7}$$

To obtain a Clifford representation acting on $\mathbb{S} \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ we can take (5.17) and trivally extend it via $\gamma_\mu \to \gamma_\mu \otimes \mathbb{1}$. Therefore the Schur algebra is larger as we have no restrictions on the transformations on the third factor:

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) \otimes \mathbb{R}(2) \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2). \tag{5.8}$$

The Schur group is therefore $\mathcal{C}(\mathbb{S})^* \cong \mathrm{GL}(2,\mathbb{C})$. Id, I, J, K are a basis for $\mathbb{R}(2)$ so the following is a basis for $\mathcal{C}(\mathbb{S}) \cong \mathbb{C}(2)$

$$\begin{aligned}
Id &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, & \mathcal{I}_1 &= \mathbb{1} \otimes \mathbb{1} \otimes I, \\
\mathcal{I}_2 &= \mathbb{1} \otimes \mathbb{1} \otimes J, & \mathcal{I}_3 &= \mathbb{1} \otimes \mathbb{1} \otimes K, \\
E &= I \otimes J \otimes \mathbb{1}, & E\mathcal{I}_1 &= I \otimes J \otimes I, \\
E\mathcal{I}_2 &= I \otimes J \otimes J, & E\mathcal{I}_3 &= I \otimes J \otimes K.
\end{aligned} \tag{5.9}$$

Previously we have defined $\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3$ using our conventions in Chapter 3. In this model, the complex structure on the third $\mathbb{R}^2$ is multiplication by I, inducing the complex structure $\mathbb{1} \otimes \mathbb{1} \otimes I$ on $\mathbb{S}$. This takes the place of multiplication by 'i' so that

$$\gamma_* = (Id \otimes Id \otimes I)\gamma_0\gamma_1\gamma_2\gamma_3 = I \otimes J \otimes I = -E. \tag{5.10}$$

The eigenspaces of γ_* are the complex semi-spinor modules $\mathbb{S}_\pm$, also known as the Weyl spinor modules.

In comparison to $\mathcal{C}(\mathbb{S}_{(0,4)}) = 2\mathbb{H}$ and $\mathcal{C}(\mathbb{S}_{(2,2)}) = 2\mathbb{H}'$, $\mathcal{C}(\mathbb{S}_{(1,3)})$ contains both the para-quaternions, generated by

$$\{Id, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\} \subset \mathcal{C}(\mathbb{S}_{(1,3)}) \tag{5.11}$$

and the quaternions, generated by

$$\{Id, \mathcal{I}_1, E\mathcal{I}_2, E\mathcal{I}_3\} \subset \mathcal{C}(\mathbb{S}_{(1,3)}). \tag{5.12}$$

However the subalgebras do not commute, and they intersect on the subalgebra $< Id, \mathcal{I}_1 > \cong \mathbb{C}$ so the Schur algebra is not $\mathbb{H} \oplus \mathbb{H}'$. Further the Schur algebra contains the Lie subalgebra $\mathfrak{sl}(2,\mathbb{R})$, generated by $\mathcal{I}_a$ and $\mathfrak{sl}(2,\mathbb{C})$ that is generated $\{\mathcal{I}_a, E\mathcal{I}_a\}$.

We now wish to define bilinear forms on $\mathbb{S}$. These are products of bilinear forms on each factor. From g, the standard positive-definite symmetric bilinear form on $\mathbb{R}^2$ we define the following basis of bilinear forms on $\mathbb{R}^2$

$$g(\cdot,\cdot), \quad g_I(\cdot,\cdot) = g(I\cdot,\cdot), \quad g_J(\cdot,\cdot) = g(J\cdot,\cdot), \quad g_K(\cdot,\cdot) = g(K\cdot,\cdot). \tag{5.13}$$

g_J and g_K are split-signature symmetric bilinear forms and g_I is the Kähler form constructed from g and I.

The symmetry of the endomorphisms Id, I, J, K with each of these bilinear forms is

b	$\sigma(b)$	$\sigma_b(I)$	$\sigma_b(J)$	$\sigma_b(K)$
g	$+$	$-$	$+$	$+$
g_I	$-$	$-$	$-$	$-$
g_J	$+$	$+$	$+$	$-$
g_K	$+$	$+$	$-$	$+$

Table 5.3: The symmetries and b-symmetries of the endomorphisms on $\mathbb{R}^2$ for each bilinear form, b.

On $S_{\mathbb{R}} \cong \mathbb{R}^2 \otimes \mathbb{R}^2$ we can then make 16 bilinear forms by taking all possible tensor products of $\{g, g_I, g_J, g_K\}$. Of these combinations, only two are admissible (they all have definite symmetry, but only two have a definite type). Symmetry and type can be calculated using Table 5.3.

	σ	τ
$g \otimes g_I$	$-$	$+$
$g_I \otimes g_K$	$-$	$-$

Table 5.4: A basis for the admissible bilinear forms on $S_{\mathbb{R}}$ and their invariants.

From these we can then make eight admissible bilinear forms on $\mathbb{S}$ (the third factor in each γ-matrix is the identity, so the third factors does not affect the type when moving to $\mathbb{S}$). From Section 2.8, we expected the space of admissible bilinear forms to be eight-dimensional, and this is verified here.

From now on we will only focus on the super-admissible bilinear forms, those with $\sigma\tau = +1$. These will be used to define supersymmetry algebras, and we wish to learn which super-admissible bilinear forms lead to isomorphic superalgebras. The super-admissible bilinear forms and their invariants are found in Table 5.5.

β_i	σ	τ	ι
$\beta_0 = g_I \otimes g_K \otimes g$	$-$	$-$	$-$
$\beta_1 = g_I \otimes g_K \otimes g_J$	$-$	$-$	$+$
$\beta_2 = g_I \otimes g_K \otimes g_K$	$-$	$-$	$+$
$\beta_3 = g \otimes g_I \otimes g_I$	$+$	$+$	$-$

Table 5.5: A basis of super-admissible bilinear forms on $\mathbb{S}$ and their invariants.

To check these have been derived correctly, remark that the basis in (5.9) is admissible with respect to β_0, and the other bilinear forms can be found to be

$$\beta_1(\cdot,\cdot) = \beta_0(\mathcal{I}_2\cdot,\cdot), \quad \beta_2(\cdot,\cdot) = \beta_0(\mathcal{I}_3\cdot,\cdot), \quad \beta_3(\cdot,\cdot) = \beta_0(E\cdot,\cdot). \tag{5.14}$$

so they could have been derived by insertion of Schur algebra elements into β_0, though here it was more convenient to find them in an alternative manner.

We now seek the β_i-symmetry and type of each Schur algebra basis element to determine the structure of the space of super-admissible bilinear forms. This is found in Table 5.6.

A	$\tau(A)$	$\sigma_{\beta_0}(A)$	$\sigma_{\beta_1}(A)$	$\sigma_{\beta_2}(A)$	$\sigma_{\beta_3}(A)$
Id	+	+	+	+	+
$\mathcal{I}_1$	+	+	−	+	−
$\mathcal{I}_2$	+	−	+	+	−
$\mathcal{I}_3$	+	+	+	−	−
E	−	+	+	+	+
$E\mathcal{I}_1$	−	+	−	+	−
$E\mathcal{I}_2$	−	−	+	+	−
$E\mathcal{I}_3$	−	+	+	−	−

Table 5.6: Type $\tau(A)$ and β_i-symmetry $\sigma_{\beta_i}(A)$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

An element A with $\sigma_{\beta_i}(A)\tau(A) = +1$ map the superbracket Π_{β_i} to another superbracket and those with $\sigma_{\beta_i}(A)\tau(A) = -1$ leave Π_{β_i} invariant. Calculated in the following table are all values of $\sigma_{\beta_i}(A)\tau(A)$:

A	$\sigma_{\beta_0}(A)\tau(A)$	$\sigma_{\beta_1}(A)\tau(A)$	$\sigma_{\beta_2}(A)\tau(A)$	$\sigma_{\beta_3}(A)\tau(A)$
Id	+	+	+	+
$\mathcal{I}_1$	−	+	+	−
$\mathcal{I}_2$	+	+	−	−
$\mathcal{I}_3$	+	−	+	−
E	−	−	−	−
$E\mathcal{I}_1$	+	−	−	+
$E\mathcal{I}_2$	−	−	+	+
$E\mathcal{I}_3$	−	+	−	+

Table 5.7: $\sigma_{\beta_i}(A)\tau(A) = +1$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

Which gives the following stabiliser algebras

Π_{β_i}	Stabiliser
Π_{β_0}	$<\mathcal{I}_1, E, E\mathcal{I}_2, E\mathcal{I}_3> \cong \mathfrak{u}(1) \oplus \mathfrak{su}(2)$
Π_{β_1}	$<\mathcal{I}_3, E, E\mathcal{I}_1, E\mathcal{I}_2> \cong \mathfrak{u}(1) \oplus \mathfrak{su}(1,1)$
Π_{β_2}	$<\mathcal{I}_2, E, E\mathcal{I}_1, E\mathcal{I}_3> \cong \mathfrak{u}(1) \oplus \mathfrak{su}(1,1)$
Π_{β_3}	$<\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, E> \cong \mathfrak{u}(1) \oplus \mathfrak{su}(1,1)$

Table 5.8: Stabiliser Lie algebras of the basis of superadmissible bilinear forms Π_{β_i}

The stabiliser algebras of the superbrackets are the R-symmetry algebras, so we see that there are two families of $\mathcal{N} = 2$ superalgebras, as expected from Chapter 3, those with U(2) R-symmetry and those with U(1, 1) R-symmetry. Here we derive the stabiliser

Lie algebra, and at the Lie algebra level $\mathfrak{u}(p,q) \cong \mathfrak{u} \oplus \mathfrak{su}(p,q)$.

Notice that E stabilises all bilinear forms so that the bottom half of the table is obtained from the top half of the table by flipping signs. E generates a $\mathrm{U}(1)$ subgroup that acts trivially on all superbrackets. Removing E, the Schur group $\mathcal{C}(\mathbb{S})^*$ effectively acts as $\mathbb{R}^{>0} \times \mathrm{SL}(2,\mathbb{C}) \cong \mathbb{R}^{>0} \times SO_0(1,3) = CSO_0(1,3)$, where $C\mathrm{SO}(t,s)$ is the linear conformal group. The space of superbrackets can then be identified with Minkowski space $\mathbb{R}^{1,3}$ with scalar product given by $(\beta_i, \beta_j) = \eta_{ij}$.

Generally, superbrackets that vary by a scale factor are considered to be the same so that we can focus on the $\mathrm{SL}(2,\mathbb{C})$ subgroup of the Schur group $\mathrm{GL}(2,\mathbb{C})$. This will act on the four-dimensional space of super-admissible bilinear forms. $\mathrm{SL}(2,\mathbb{C})$ is the universal cover of $SO_0(1,3)$ which has two real inequivalent four-dimensional representations – the vector representation and Weyl spinor representation. The vector representation has five open orbits (future and past directed time-like, space-like and null vectors) and the spinorial representation only has 1. We see immediately we have at least two open orbits, because Π_{β_0} has a compact stabiliser group and Π_{β_i} have non-compact stabiliser groups, and will motivate the third. Therefore it acts upon the space of superbrackets in the vector representation.

$\mathrm{SO}(1,3)$ has six orbits: space-like, future- and past-pointing time-like and future- and past-pointing null vectors and the origin. The origin corresponds to the degenerate superbracket that produces the trivial supersymmetry algebra (pictorially those with $Q = 0$). However, the superbrackets $\Pi_{+\beta}$ and $\Pi_{-\beta}$ define isomorphic supersymmetry algebras, so there are only four distinct non-isomorphic types of Lie superalgebras, as future- and past-pointing time-like and null vectors are in the same family of superalgebras.

The time-like orbits of $\Pi_{\pm\beta_0}$ give rise to one family of superalgebras, that is the standard formulation of $\mathcal{N} = 2$ superalgebras in Minkowski signature, with $\mathrm{U}(2)$ R-symmetry. Time-like orbits have the stabiliser group $\mathrm{SO}(3) \cong \mathrm{SU}(2)$, which is the non-abelian part of the R-symmetry $\mathrm{U}(2) \cong \mathrm{U}(1) \cdot \mathrm{SU}(2)$.

The space-like directions correspond to Π_{β_i} with $i = 1,2,3$. Space-like orbits have the stabiliser group $\mathrm{SO}(1,2) \cong \mathrm{SU}(1,1)$ which is the non-abelian part of the R-symmetry group $\mathrm{U}(1,1) \cong \mathrm{U}(1) \cdot \mathrm{SU}(2)$. These give isomorphic non-standard 'twisted' supersymmetry algebras, similar to those found in [14].

Null vectors correspond to partially degenerate superbrackets. Consider the superbracket $\Pi_{\frac{1}{2}(\beta_0+\beta_1)}$ [2]; $\beta_1(\cdot,\cdot) = \beta_0(\mathcal{I}_2\cdot,\cdot)$ such that $\frac{1}{2}(\beta_0+\beta_1) = \beta_0(\frac{1}{2}(\mathbb{1}+\mathcal{I}_2)\cdot,\cdot)$. As $\mathcal{I}_2^2 = +1$ we can define the projection operator

$$P_\pm^{\mathcal{I}_2} = \frac{1}{2}(\mathbb{1} \pm \mathcal{I}_2). \tag{5.15}$$

$P_\pm^{\mathcal{I}_2}$ projects onto the ±1 eigenspaces of $\mathcal{I}_2$. The supercharges live on the four-dimensional submodule $P_+^{\mathcal{I}_2}\mathbb{S} \cong S_\mathbb{R}$. Spinors in $P_-^{\mathcal{I}_2}\mathbb{S}$ are in the kernel of $\Pi_{\frac{1}{2}(\beta_0+\beta_1)}$. $\mathcal{I}_2$ acts as the identity on $P_+^{\mathcal{I}_2}\mathbb{S}$ so that E and $E\mathcal{I}_2$ are equivalent. None of the other elements except Id have the same invariants with β_0 and β_1 so the Schur algebra has basis Id and E, giving an algebra isomorphic to $\mathbb{C}$.

The stabiliser group of $\Pi_{\frac{1}{2}(\beta_0+\beta_1)}$ is the one-dimensional group generated by E, which is isomorphic to U(1). This is precisely the R-symmetry group of an $\mathcal{N} = 1$ superalgebra. $\mathrm{SO}(2) \cong \mathrm{U}(1)$ is the expected little group for null vectors on $\mathbb{R}^{(1,3)}$.

5.2.2 Neutral Signature

From Table 5.2 we see $Cl_{2,2} \cong \mathbb{R}(4)$ and the real spinor module is $S_\mathbb{R} = \mathbb{R}^4 \cong \mathbb{R}^2 \otimes \mathbb{R}^2$ so we can once again use a model like the above, with $\mathbb{S} = \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ again. The even Clifford algebra is $Cl^0_{2,2} \cong 2\mathbb{R}(2)$ which implies we can decompose the spinor module into two inequivalent real semi-spinors $S_\mathbb{R} = S_{\mathbb{R}+} + S_{\mathbb{R}-}$, this means that in this signature there are Majorana-Weyl spinors. However, though one can define Majorana-Weyl spinors there is no $\mathcal{N} = \frac{1}{2}$ algebra whose supercharges are a single Majorana-Weyl spinor because the Majorana bilinear forms are isotropic.

For easy reference, we recall the Schur algebras of the various spinor modules are

$$\mathcal{C}(\mathbb{S}) = 2\mathbb{R}(2) = 2\mathbb{H}', \quad \mathcal{C}(\mathbb{S}_\pm) = \mathbb{R}(2) = \mathbb{H}', \quad \mathcal{C}(S_\mathbb{R}) = 2\mathbb{R}, \quad \mathcal{C}(S_{\mathbb{R}\pm}) = \mathbb{R}. \tag{5.16}$$

$\mathbb{R}(2) = \mathbb{H}'$ is used to draw parallels to the Euclidean signatures (where the Schur algebra of the complex spinor module is $2\mathbb{H}$ instead) and to highlight the presence of two invariant real structures on $\mathbb{S}$. The real (semi-)spinor module(s) are the complexification

[2] As we have shown they are isomorphic, we could have chosen any of the space-like bilinear forms β_i instead of β_1 without loss of generality.

of the complex (semi-)spinor module(s). The semi-spinor modules are self-conjugate as complex $Cl^0_{2,2}$ modules.

Following the same prescription as the previous model, we define a $Cl_{2,2}$ representation acting on $S_{\mathbb{R}} \cong \mathbb{R}^2 \otimes \mathbb{R}^2$ as

$$\begin{aligned}\gamma_1 &= J \otimes I, \quad \gamma_2 = K \otimes I, \\ \gamma_3 &= \mathbb{1} \otimes J, \quad \gamma_4 = \mathbb{1} \otimes K.\end{aligned} \tag{5.17}$$

$\mathfrak{spin}(2,2)$ is generated by

$$\begin{aligned}&\gamma^1\gamma^2 = -I \otimes \mathbb{1}, \quad \gamma^1\gamma^3 = J \otimes K, \quad \gamma^1\gamma^4 = -J \otimes J, \\ &\gamma^2\gamma^3 = K \otimes K \quad \gamma^2\gamma^4 = -K \otimes J, \quad \gamma^3\gamma^4 = -\mathbb{1} \otimes I.\end{aligned} \tag{5.18}$$

By inspection we see that $\mathbb{1} \otimes \mathbb{1}$ and $I \otimes I$ commute with all spin generators. $I \otimes I$ is an involution so that the Schur algebra of the real spinor module is

$$\mathcal{C}(S_{\mathbb{R}}) =< \mathbb{1} \otimes \mathbb{1}, I \otimes I >\cong 2\mathbb{R}. \tag{5.19}$$

Bilinear forms are built with tensor products of $\{g, g_I, g_J, g_K\}$ again. The Clifford algebra has changed, so the elements with definite type have changed, the admissible bilinear forms are:

	σ	τ
$g \otimes g_I$	$-$	$-$
$g_I \otimes g$	$-$	$+$

Table 5.9: A basis for the admissible bilinear forms on $S_{\mathbb{R}}$ and their invariants.

Again we extend the Clifford generators by adding a third factor of $\mathbb{1}$ to each tensor product, $\gamma_\mu \to \gamma_\mu \otimes \mathbb{1}$ to obtain a representation on $\mathbb{S}$. On $\mathbb{S}$ we then have the expected basis of 8 admissible bilinear forms by tensoring one of $\{g, g_I, g_J, g_K\}$ with the two admissible bilinear forms on $S_{\mathbb{R}}$. Of these four are superadmissible, their invariants are contained in the following table

β_i	σ	τ	ι
$\beta_1 = g \otimes g_I \otimes g$	$-$	$-$	$+$
$\beta_2 = g \otimes g_I \otimes g_J$	$-$	$-$	$+$
$\beta_3 = g \otimes g_I \otimes g_K$	$-$	$-$	$+$
$\beta_4 = g_I \otimes g \otimes g_I$	$+$	$+$	$+$

Table 5.10: A basis of super-admissible bilinear forms on $\mathbb{S}$ and their invariants.

The spin generators act trivially on the third $\mathbb{R}^2$ factor, meaning the Schur algebra on the complex spinor module is

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) \otimes \mathbb{R}(2) \cong 2\mathbb{R} \otimes \mathbb{R}(2) = 2\mathbb{R}(2) \cong 2\mathbb{H}'. \tag{5.20}$$

An admissible basis for the Schur algebra of the complex spinor module is therefore obtained by adding I, J, K to the basis elements of the real Schur algebra:

$$\begin{aligned}
&Id = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \quad \mathcal{I}_1 = \mathbb{1} \otimes \mathbb{1} \otimes I, \\
&\mathcal{I}_2 = \mathbb{1} \otimes \mathbb{1} \otimes J, \quad \mathcal{I}_3 = \mathbb{1} \otimes \mathbb{1} \otimes K, \\
&E = I \otimes I \otimes \mathbb{1}, \quad E\mathcal{I}_1 = I \otimes I \otimes I, \\
&E\mathcal{I}_2 = I \otimes I \otimes J, \quad E\mathcal{I}_3 = I \otimes I \otimes K.
\end{aligned} \tag{5.21}$$

E is again proportional to γ_*, in our conventions

$$\gamma_* = \gamma_1\gamma_2\gamma_3\gamma_4 = -I \otimes I \otimes 1 = -E. \tag{5.22}$$

Using the basis (5.22) we can see the super-admissible bilinear forms are related by

$$\beta_2(\cdot,\cdot) = \beta_1(\mathcal{I}_2\cdot,\cdot), \quad \beta_3(\cdot,\cdot) = \beta_1(\mathcal{I}_3\cdot,\cdot), \quad \beta_4(\cdot,\cdot) = \beta_1(E\mathcal{I}_1\cdot,\cdot). \tag{5.23}$$

The Schur algebra can be written as the direct sum $\mathcal{C}(\mathbb{S}) = \mathcal{C}(\mathbb{S})_+ \oplus \mathcal{C}(\mathbb{S})_- \cong \mathbb{H}' \oplus \mathbb{H}'$ with the projectors

$$P_\pm = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \pm I \otimes I \otimes \mathbb{1}) \tag{5.24}$$

Such that $P_\pm \mathcal{C}(\mathbb{S}) = \mathcal{C}(\mathbb{S})_\pm$. The individual $\mathbb{H}'$ factors are then spanned by the operators

$$\begin{aligned}
&1_\pm = P_\pm(\mathbb{1} \otimes \mathbb{1} \otimes 1), \quad I_\pm = P_\pm(\mathbb{1} \otimes \mathbb{1} \otimes I), \\
&J_\pm = P_\pm(\mathbb{1} \otimes \mathbb{1} \otimes J), \quad K_\pm = P_\pm(\mathbb{1} \otimes \mathbb{1} \otimes K).
\end{aligned} \tag{5.25}$$

This provides an alternative basis for the Schur algebra but it does not consist of β_1-admissible elements. This is because $\sigma(g) = +1$ and $\sigma_g(I) = -1$ so that $I_\pm$, $J_\pm$ and $K_\pm$ do not have a definite β_1-symmetry, as the first factor of each is $(\mathbb{1} \pm I)$.

Using the admissible basis, we find the following collection of invariants for each Schur algebra basis element

A	$\tau(A)$	$\sigma_{\beta_1}(A)$	$\sigma_{\beta_2}(A)$	$\sigma_{\beta_3}(A)$	$\sigma_{\beta_4}(A)$
Id	+	+	+	+	+
$\mathcal{I}_1$	+	−	+	+	−
$\mathcal{I}_2$	+	+	+	−	−
$\mathcal{I}_3$	+	+	−	+	−
E	−	+	+	+	+
$E\mathcal{I}_1$	−	−	+	+	−
$E\mathcal{I}_2$	−	+	+	−	−
$E\mathcal{I}_3$	−	+	−	+	−

Table 5.11: Type $\tau(A)$ and β_i-symmetry $\sigma_{\beta_i}(A)$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

Again we calculate $\sigma_{\beta_i}(A)\tau(A)$ to determine the effects of the Schur algebra on the space of superbrackets, in doing so classifying the space of superbrackets.

A	$\sigma_{\beta_1}(A)\tau(A)$	$\sigma_{\beta_2}(A)\tau(A)$	$\sigma_{\beta_3}(A)\tau(A)$	$\sigma_{\beta_4}(A)\tau(A)$
Id	+	+	+	+
$\mathcal{I}_1$	−	+	+	−
$\mathcal{I}_2$	+	+	−	−
$\mathcal{I}_3$	+	−	+	−
E	−	−	−	−
$E\mathcal{I}_1$	+	−	−	+
$E\mathcal{I}_2$	−	−	+	+
$E\mathcal{I}_3$	−	+	−	+

Table 5.12: $\sigma_{\beta_i}(A)\tau(A) = +1$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

It is worth noting that the contents of the tables of invariants in Minkowski and neutral signature are superficially the same, this is a coincidental occurrence and not reflective of any underlying properties. Indeed the explicit form of the Schur algebra basis and the resulting bilinear forms are different. The tables are repeated here to make this section self-contained.

Π_{β_i}	Stabiliser
Π_{β_1}	$<\mathcal{I}_1, E, E\mathcal{I}_2, E\mathcal{I}_3> \cong \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$
Π_{β_2}	$<\mathcal{I}_3, E, E\mathcal{I}_1, E\mathcal{I}_2> \cong \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$
Π_{β_3}	$<\mathcal{I}_2, E, E\mathcal{I}_1, E\mathcal{I}_3> \cong \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$
Π_{β_4}	$<\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, E> \cong \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$

Table 5.13: Stabiliser Lie algebras of the basis of superadmissible bilinear forms Π_{β_i}

Though the details are identical, the resulting algebras are different as now $E^2 = +1$ so that it generates a subgroup isomorphic to $\mathbb{R}^{>0}$. The other generators produce a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(1,2)$. The full R-symmetry group according to Table 3.10 is $\mathrm{GL}(2,\mathbb{R})$, which is validated here as $\mathfrak{gl}(2,\mathbb{R}) \cong \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$.

E acts trivially on all superbrackets so if one only considers isometries the effective action of the Schur group to be

$$CO_0(2,2) \cong \mathbb{R}^{>0} \times SO_0(2,2) \subset \mathcal{C}(\mathbb{S}) = \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}). \tag{5.26}$$

However, neutral signature spacetimes allow anti-isometries that exchange the space-like and time-like direction. We can realise anti-isometries in this model as: $\xi_1 = P_+ + J_-$, that interchanges $\Pi_{\beta_1} \leftrightarrow \Pi_{\beta_2}$ and $\Pi_{\beta_3} \leftrightarrow \Pi_{\beta_4}$, and $\xi_2 = P_- + J_+$, that interchanges $\Pi_{\beta_1} \leftrightarrow \Pi_{\beta_2}$ and $\Pi_{\beta_3} \leftrightarrow -\Pi_{\beta_4}$. These are in the Schur group, so that the total effective action of the Schur group is

$$CO_0(2,2) \cup \xi_1 CO_0(2,2). \tag{5.27}$$

Once again, we see the Schur group acts as the linear conformal group associated with the spacetime signature. Identifying the space of superbrackets with $\mathbb{R}^{2,2}$ one finds that Π_{β_1} and Π_{β_4} are time-like directions and Π_{β_2}, Π_{β_3} are space-like.

$SO_0(2,2)$ has four orbits: the open orbits of space-like and time-like vectors, the three-dimensional orbit of non-zero null vectors and the origin (which once again corresponds to the trivial supersymmetry algebra). The stabiliser of space- and time-like orbits is $SO_0(1,2) \cong \mathrm{SL}(2,\mathbb{R}$, which contained within the R-symmetry group $\mathrm{GL}(2,\mathbb{R})$. ξ_1 exchanges space-like and time-like vectors so that under the Schur group there are only three orbits.

The orbit of null vectors can be explored in the same manner as in Minkowski signature. Without loss of generality we consider the superbracket $\Pi_{\frac{1}{2}(\beta_1+\beta_2)}$. The bilinear form can be written

$$\frac{1}{2}(\beta_1 + \beta_2) = \frac{1}{2}(g \otimes g_I \otimes (g + g_J)) = g \otimes g_I \otimes g\left(\frac{1}{2}(\mathbb{1} + J)\cdot, \cdot\right) \tag{5.28}$$

$J^2 = +1$ so we can construct the projectors

$$P^J_\pm = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \otimes (\mathbb{1} \pm J)) \tag{5.29}$$

that project onto the ± 1 eigenspaces of $(\mathbb{1} \otimes \mathbb{1} \otimes J)$. The four-dimensional projected space $P^J_-\mathbb{S}$ is in the kernel of $\Pi_{\frac{1}{2}(\beta_1+\beta_2)}$, and supercharges therefore live on $P^J_+\mathbb{S} \cong S_\mathbb{R}$. This is the unique $\mathcal{N} = 1$ superalgebra in terms of Majorana spinors in signature $(2, 2)$. The stabiliser group of $\Pi_{\frac{1}{2}(\beta_1+\beta_2)}$ is generated by E. $E^2 = +1$ so the connected component of this group is isomorphic to $SO_0(1, 1)$. Again this aligns with Chapter 3 where we calculate the R-symmetry group to be $\mathrm{SO}(1, 1)$.

Once again we remark that the R-symmetry group of the supersymmetry algebras is associated to the stabiliser of the corresponding orbit, which is $\mathrm{SO}(1, 1)$ for the $\mathcal{N} = 1$ algebra and $\mathrm{SO}(1, 2) \cong \mathrm{SL}(2, \mathbb{R})$ for the $\mathcal{N} = 2$ algebra. Similarly to Minkowski signature, the full R-symmetry group contains an additional Abelian factor, but the non-Abelian factor is the stabiliser of the associated orbit.

Finally, there is no orbit associated with a possible $\mathcal{N} = 1/2$ algebra whose supercharges are Majorana-Weyl spinors. This agrees with the fact the vector-valued bilinear forms are orthogonal, with $\iota(\beta_i) = +1$. Therefore one cannot define a superbracket with a single Majorana-Weyl spinor.

5.2.3 Euclidean Signature

In both Euclidean signatures (using our conventions where $min(t, s)$ is time, we interpret both $(0, 4)$ and $(4, 0)$ as being Euclidean) the Clifford algebra is $Cl_{0,4} \cong Cl_{4,0} \cong \mathbb{H}(2)$ and the real spinor module, which is equal to the complex spinor module due to lack of real structure, is $S_\mathbb{R} = \mathbb{S} = \mathbb{H}^2$. The real spinor module decomposes into two inequivalent semi-spinor modules, $S_\mathbb{R} = S_{\mathbb{R}+} + S_{\mathbb{R}-}$, as does the complex spinor module and the corresponding semi-spinor modules are equal $S_{\mathbb{R}\pm} = \mathbb{S}_\pm$. This can be seen from the even Clifford algebra, $Cl^0_{0,4} = 2\mathbb{H}$, which further implies the existence of two quaternionic structures on $\mathbb{S}$ that are also a quaternionic structure on $\mathbb{S}_\pm$ alone. Under these structures the Weyl spinor modules are self-conjugate, i.e. $\bar{\mathbb{S}}_\pm = \mathbb{S}_\pm$ (in the language of Chapter 3 this means the structure is Weyl compatible). Therefore one can have symplectic Majorana-Weyl spinors. The complex spinor module is therefore also self-conjugate. There is no real structure so one cannot define Majorana spinors. The Schur algebra of the real and complex spinor module is

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) = 2\mathbb{H}, \tag{5.30}$$

and the Schur group is therefore $C(\mathbb{S})^* = 2\mathbb{H}^*$.

Therefore it is natural to work with a spinor module that is a pair of quaternions acted upon by quaternionic matrices. Recall that in Chapter 4 we used a similar representation of $Cl_{0,5}$ in terms of quaternionic matrices acting on spinors that were elements of $\mathbb{S}_{(0,5)} \cong \mathbb{H}^2$. Indeed, the two are related directly by dimensional reduction, and we can then use the γ^i for $i = 1, 2, 3, 4$ from before [3]:

$$\gamma^1 = D \qquad \gamma^2 = DEL_i \qquad \gamma^3 = DEL_j \qquad \gamma^4 = DEL_k. \tag{5.31}$$

Where D and E were anticommuting involutions acting on $\mathbb{H}^2$ given by

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.32}$$

γ^5 has been removed and is proportional to the chirality matrix

$$\gamma_* = \gamma_1\gamma_2\gamma_3\gamma_4 = -E. \tag{5.33}$$

We see that E acts proportional to the identity on the summands $\mathbb{S} = \mathbb{H} + \mathbb{H}$, which are identified with the semi-spinor modules $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_- \implies \mathbb{S}_\pm = \mathbb{H}$.

$\mathfrak{spin}(4)$ is generated by

$$\begin{aligned} \gamma^1\gamma^2 &= EL_i, \quad \gamma^1\gamma^3 = EL_j, \quad \gamma^1\gamma^4 = EL_k, \\ \gamma^2\gamma^3 &= -L_k, \quad \gamma^2\gamma^4 = L_j, \quad \gamma^3\gamma^4 = -L_i. \end{aligned} \tag{5.34}$$

[3] Note that this representation differs from the representation in [3], where we used the dimensional reduction of a $Cl_{1,4}$ algebra, while in this thesis here we used the dimensional reduction of a $Cl_{0,5}$.

We see that this acts exclusively by left multiplication and does not mix the semi-spinor modules. This has fewer elements than $\mathfrak{spin}(5)$ signature, so naturally, we expect the Schur algebra to be bigger. Indeed we now notice that as before in $(0,5)$ the operators

$$\mathcal{I}_a = \{I, J, K\}, \quad a = 1, 2, 3, \tag{5.35}$$

commute with $\mathfrak{spin}(4)$ but so do $E\mathcal{I}_a$ due to the removal of any elements containing the matrix D. The Schur algebra is therefore

$$\mathcal{C}(\mathbb{S}) = \mathcal{C}(S_{\mathbb{R}}) =< \mathcal{I}_a, E\mathcal{I}_a >_{algebra} \cong 2\mathbb{H}. \tag{5.36}$$

Previously we used the Spin(5)-invariant bilinear form

$$< q, p >= \bar{q}^1 p^1 + \bar{q}^2 p^2. \tag{5.37}$$

This is restricted to be $\mathrm{Spin}(4) \subset \mathrm{Spin}(5)$-invariant by interpreting it as the direct sum of the bilinear form on each factor $\mathbb{H}$ individually, forbidding mixing of $q^1 \in \mathbb{S}_+$ and $q^2 \in \mathbb{S}_-$ (and p^1 and p^2). As we have already seen $\mathfrak{spin}(4)$ does not contain any elements that mix the semi-spinors. We write

$$\begin{aligned} &h : \mathbb{H} \otimes \mathbb{H} \to \mathbb{R} \\ &h(q, p) = Re < q, p >= Re(\bar{q}p), \qquad q, p \in \mathbb{H} \end{aligned} \tag{5.38}$$

So that our canonical admissible bilinear form $g = h \oplus h$. This is symmetric and has definite type $\tau(g) = +1$ (for the same reasons as the (0,5) Clifford algebra), so it is an admissible bilinear form. As $\sigma(g)\tau(g) = +1$ it is super-admissible and can be used to define a superbracket.

The larger Schur algebra means the space of admissible bilinear forms is also larger, both now being 8-dimensional. L_q and R_q are isometries of the standard scalar product that square to -1, so they must be g-skew. D and E are isometries of g but are involutions so they are g-symmetric. L_q and R_q commute with all operators, while D and E anti-commute. Using these facts we can calculate all the invariants listed in the table below for each basis element of the Schur algebra

A	$\tau(A)$	$\sigma_g(A)$	$\sigma_g(A)\tau(A)$
Id	+	+	+
$\mathcal{I}_1$	−	+	−
$\mathcal{I}_{2,3}$	−	−	+
E	+	−	+
$E\mathcal{I}_1$	−	−	+
$E\mathcal{I}_{2,3}$	−	+	−

Table 5.14: Type $\tau(A)$ and β_i-symmetry $\sigma_{\beta_i}(A)$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

We see that all eight bilinear forms are admissible (so they form a basis of all admissible bilinear forms) and four are super-admissible. They are

$$\{\beta_i | i = 1,2,3,4\} = \{g(\cdot,\cdot), \quad g(\mathcal{I}_2\cdot,\cdot), \quad g(\mathcal{I}_3\cdot,\cdot), \quad g(E\mathcal{I}_1\cdot,\cdot)\}. \tag{5.39}$$

The associated superbrackets, Π_{β_i} are a basis for the space of symmetric Spin(4)-invariant vector-valued bilinear forms on $\mathbb{S}$, which is equivalent to the space of Poincaré Lie superalgebra structures. Once again we wish to study the action of the Schur algebra on this space to determine the orbit structure and in doing so the number of distinct superalgebras.

To calculate the invariants we used

$$\sigma_{g(B\cdot,\cdot)}(A) = \begin{cases} +\sigma_g(A) & \text{if} \quad [A,B] = 0 \\ -\sigma_g(A) & \text{if} \quad \{A,B\} = 0 \end{cases} . \tag{5.40}$$

A	$\tau(A)$	$\sigma_{\beta_0}(A)$	$\sigma_{\beta_1}(A)$	$\sigma_{\beta_2}(A)$	$\sigma_{\beta_3}(A)$
Id	+	+	+	+	+
I_1	+	−	+	+	−
I_2	−	−	−	+	+
I_3	−	−	+	−	+
E	−	+	+	+	+
EI_1	−	−	+	+	−
EI_2	+	−	−	+	+
EI_3	+	−	+	−	+

Table 5.15: Type $\tau(A)$ and β_i-symmetry $\sigma_{\beta_i}(A)$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

Calculated in the following table are all values of $\sigma_{\beta_i}(A)\tau(A)$:

A	$\sigma_{\beta_0}(A)\tau(A)$	$\sigma_{\beta_1}(A)\tau(A)$	$\sigma_{\beta_2}(A)\tau(A)$	$\sigma_{\beta_3}(A)\tau(A)$
Id	$+$	$+$	$+$	$+$
$\mathcal{I}_1$	$-$	$+$	$+$	$-$
$\mathcal{I}_2$	$+$	$+$	$-$	$-$
$\mathcal{I}_3$	$+$	$-$	$+$	$-$
E	$-$	$-$	$-$	$-$
$E\mathcal{I}_1$	$+$	$-$	$-$	$+$
$E\mathcal{I}_2$	$-$	$-$	$+$	$+$
$E\mathcal{I}_3$	$-$	$+$	$-$	$+$

Table 5.16: $\sigma_{\beta_i}(A)\tau(A) = +1$ of the Schur algebra $\mathcal{C}(\mathbb{S})$ basis elements A.

This gives us the stabiliser algebra for each superbracket:

Π_{β_i}	Stabiliser
Π_{β_1}	$<\mathcal{I}_1, E, E\mathcal{I}_2, E\mathcal{I}_3> \cong \mathbb{R} \oplus \mathfrak{su}(2)$
Π_{β_2}	$<\mathcal{I}_3, E, E\mathcal{I}_1, E\mathcal{I}_2> \cong \mathbb{R} \oplus \mathfrak{su}(2)$
Π_{β_3}	$<\mathcal{I}_2, E, E\mathcal{I}_1, E\mathcal{I}_3> \cong \mathbb{R} \oplus \mathfrak{su}(2)$
Π_{β_4}	$<\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, E> \cong \mathbb{R} \oplus \mathfrak{su}(2)$

Table 5.17: Stabiliser Lie algebras of the basis of superadmissible bilinear forms Π_{β_i}

Id rescales the bilinear forms. E generates the one-dimensional kernel of the representation, acting trivially on all brackets. Factorising this from the Schur algebra we obtain

$$< Id, \mathcal{I}_a, E\mathcal{I}_a > \cong \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathbb{R} \oplus \mathfrak{so}(4), \quad \alpha = 1, 2, 3. \tag{5.41}$$

The SO(4) group acts in a four-dimensional irreducible representation. Each $\mathfrak{su}(2)$ factor acts non-trivially, so this is the vector representation. The Schur group then effectively acts as the linear conformal group $C\mathrm{SO}(4) = \mathbb{R}^{>0} \times \mathrm{SO}(4)$. Once again we see the space of superbrackets can be identified with the spacetime $\mathbb{R}^{t,s} = \mathbb{R}^{0,4}$ in this case.

SO(4) has two orbits, the open orbit of non-zero vectors and the origin. The origin gives the trivial superalgebra.

Each of the superbrackets on $\mathbb{S}$ has a stabiliser Lie algebra of $\mathbb{R} + \mathfrak{su}(2) \cong \mathbb{R} \oplus \mathfrak{so}(3) \cong \mathfrak{u}^*(2)$, as expected in $(0, 4)$ where the R-symmetry group is $\mathrm{U}^*(2)$. The orbit of non-zero vectors has stabiliser group SO(3). We see once again the non-abelian factor of the R-symmetry group is the stabiliser group of the corresponding orbit. Therefore there is a single unique $\mathcal{N} = 2$ superalgebra in Euclidean signature up to isomorphism, corresponding to the orbit of non-zero vectors.

In summary, we see the space of $\mathcal{N} = 2$ superbrackets in all four-dimensional signatures is parameterised by the same vector space of the underlying space-time, $\mathbb{R}^{t,s}$, in all signatures. The types and number of distinct superalgebras correspond to the orbit structures on the group $\mathrm{SO}(t,s)$, with the R-symmetry being the stabiliser group of these orbits (times an additional factor). This is not a general feature of supersymmetry algebras, just a coincidence in four dimensions. For example in $(1,7)$ the superbrackets exhibit the same $\mathbb{R}^{1,3}$ structure (we can see this because again in $(1,7)$ we have $\mathrm{U}(2)$ and $\mathrm{U}(1,1)$ R-symmetry once more).

5.2.4 Dimensional Reduction

The unique super-admissible bilinear form on $\mathbb{S}_{(t,s)}$ was given by $Re[A]$ for $t = 0,1,4,5$ and $Im[A]$ for $t = 2,3$ where $A_{(t,s)}$ is the $\mathrm{Spin}_0(t,s)$-invariant sesquilinear form defined by its Gram matrix, also called $A_{(t,s)}$

$$A_{(t,s)}(\lambda,\chi) = \lambda^\dagger A_{(t,s)}\chi. \tag{5.42}$$

This sesquilinear form is also $\mathrm{Spin}_0(t',s')$-invariant for $t' + s' = 4$, provided $t' \leq t$ and $s' \leq t$ such that $\mathrm{Spin}_0(t',s') \subset \mathrm{Spin}_0(t,s)$.

We can relate the five-dimensional $A_{(t,s)}$ matrix to the four-dimensional $A_{(t',s')}$ matrix and write the five-dimensional sesquilinear form as a $\mathrm{Spin}_0(t',s')$-invariant bilinear form on $\mathbb{S}_{(t',s')} \cong \mathbb{C}^4$. Taking the real or imaginary part of this will give super-admissible bilinear form on $\mathbb{S}_{(t',s')}$ that we can relate to the previous chapter.

Let m be the standard sesquilinear form on $\mathbb{C}^4$, $m(\lambda,\chi) = \lambda^\dagger\chi$. We see that $A_{(t,s)}(\cdot,\cdot) = m(\cdot, A_{(t,s)}\cdot)$. For each signature $m(\cdot,\cdot)$ can be related to the bilinear form given in the particular model (which varies depending on signature) and by writing $A_{(t,s)}$ in terms of four-dimensional quantities one can relate the bilinear form to a super-admissible bilinear form obtained in these models.

The γ-matrices generating $Cl_{0,5}$ will be called $\Gamma_1,...,\Gamma_5$. Our conventional choice is that $\Gamma_1...\Gamma_5 = 1$. The other five-dimensional Clifford algebras, $Cl_{t,s}$, will then be the first t generators replaced with $\Gamma_i' = -i\Gamma_i$. For example the generators of $Cl_{2,3}$ are $\Gamma_1', \Gamma_2', \Gamma_3, \Gamma_4, \Gamma_5$.

Four-dimensional γ-matrices will be written with lower-case γ. They will be related to the five-dimensional Γ's in a case-by-case basis.

Time-like dimensional reductions are always performed over the 1-direction and space-like reductions always remove the 5-direction.

Reduction $(0,5) \to (0,4)$**.**

We relate the Clifford generators as

$$\Gamma_\mu = \gamma_\mu, \quad \mu = 1,2,3,4, \tag{5.43}$$

where we have removed the 5-direction as we have reduced over a space-like direction. The removed $\Gamma_5 = \Gamma_1 ... \Gamma_4$ is then equal to the chirality matrix $\gamma_* = \Gamma_5 = -E$ in terms of quantities in the Section 5.2.3.

$A_{(0,5)} = A_{(0,4)} = 1$ so that

$$Re[A_{(0,5)}(\cdot,\cdot)] = Re[m(\cdot,\cdot)]. \tag{5.44}$$

The model in Section 5.2.3 used spinors that are elements of $\mathbb{H}^2$. Knowing that $\mathbb{H}^2 \cong \mathbb{C}^4$, we can express the bilinear forms on $\mathbb{H}^2$ as bilinear forms on $\mathbb{C}^4$. Writing $q^i = u^i + v^i j$ and $p^i = w^i + z^i j$, with $u^i, v^i, w^i, z^i \in \mathbb{C}$ we see that

$$g(q,p) = Re[\bar{q}^1 p^1 + \bar{q}^2 p^2] \equiv Re[\bar{Z}^I W^I] = Re[m(Z,W)], \tag{5.45}$$

with $Z^I = (u^1, v^1, u^2, v^2) \in \mathbb{C}^4$ and $W^I = (w^1, z^1, w^2, z^2) \in \mathbb{C}^4$. Therefore $(0,4)$ algebra obtained from the dimensional reduction of our $(0,5)$ superalgebra corresponds to that defined using β_1 in the previous model.

Reduction $(1,4) \to (0,4)$

The γ-matrices are

$$\gamma_\mu = \Gamma_{\mu+1}, \quad \mu = 1,2,3,4. \tag{5.46}$$

As we reduce along a time-like direction the A matrices are different:

$$A_{(1,4)} = \Gamma'_1 = -i\Gamma_1 = -i\gamma_1 ... \gamma_4. \tag{5.47}$$

In the quaternionic model E operates as $-\gamma_1...\gamma_4$ and multiplication by i on $\mathbb{C}^4$ corresponds to right-multiplication $R_i = I = \mathcal{I}_1$ on $\mathbb{H}^2 \cong \mathbb{C}^4$, so that $A_{(1,4)}$ corresponds to the operator $E\mathcal{I}_1$. One can show

$$Re[A_{(1,4)}(\cdot,\cdot)] = Re[m(\cdot, A_{(1,4)}\cdot)] = Re[m(\cdot, E\mathcal{I}_1\cdot)] \equiv -g(\cdot, -E\mathcal{I}_1) = \beta_4, \tag{5.48}$$

where g and m are related as above. β_4 is another basis element in the space of super-admissible forms, but it is in the same orbit as β_1 (all super-admissible bilinear forms are in the same orbit in $(0,4)$) and we can use $A = \pm\frac{1}{\sqrt{2}}(Id - E\mathcal{I}_1)$ to map β_1 to β_4.

For time-like reductions the chirality operator is

$$\gamma_* = i\Gamma'_1 = \Gamma_2...\Gamma_5 = \gamma_1...\gamma_4, \tag{5.49}$$

so that E once again corresponds to $-\gamma_*$. To link to [20] we can write $\Gamma'_1 = \gamma_0$ so that $\gamma_* = i\gamma_0$ as before.

Reduction $(1,4) \to (1,3)$

As is convention in $(1,3)$ signature we will call the time-like direction 0 and the space-like directions $1,2,3$ so that

$$\Gamma'_1 = \gamma_0, \quad \Gamma_{i+1} = \gamma_i, \quad i = 1,2,3. \tag{5.50}$$

A space-like reduction keeps the A matrix the same, so that $A_{(1,4)} = A_{(1,3)} = \gamma_0$. γ_0 acts as $I \otimes K \otimes 1$ in Section 5.2.1.

We can write $z^i \in \mathbb{C}^4$ as $z^i = x^i + y^i$ and $w^i = u^i + iv^i$ with $u^i, v^i, x^i, y^i \in \mathbb{R}^4$ such that

$$Re[m(z,w)] = x^i u^i + y^i v^i. \tag{5.51}$$

The right-hand side is equivalent to the standard symmetric bilinear for on $\mathbb{R}^8$. The model in Section 5.2.1 is in terms of $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^8$ and $g \otimes g \otimes g$ is the standard bilinear form on $\mathbb{R}^8$. The real part of m is therefore equivalent to $g \otimes g \otimes g$.

The real part of the Dirac sesquilinear form is then

$$Re[A_{(1,4)}(\cdot,\cdot)] = Re[m(\cdot, A_{(1,4)}\cdot)] \equiv g(\cdot, I\cdot) \otimes g(\cdot, K\cdot) \otimes g(\cdot,\cdot) = -\beta_0. \tag{5.52}$$

β_0 belongs to the time-like orbits under the Schur group $\mathbb{R}^{>0} \cdot SO_0(1,3)$ which have R-symmetry group U(2).

In a space-like reduction the chirality operator is

$$\gamma_* = \Gamma_5 = i\Gamma_1'\Gamma_2\Gamma_3\Gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5, \tag{5.53}$$

where we have defined $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ to match up with the definitions found in [20].

Reduction $(2,3) \to (1,3)$

After removing the 1 direction, we set

$$\gamma_0 = \Gamma_2', \quad \gamma_i = \Gamma_{i+2}, \quad i = 1,2,3. \tag{5.54}$$

The two A-matrices are $A_{(2,3)} = \Gamma_1'\Gamma_2'$ and $A_{(1,3)} = \gamma_0$. In this representation $\Gamma_1' = \Gamma_2'\Gamma_3\Gamma_4\Gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. The conventional choice for $\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$ so we can write

$$A_{(2,3)} = \Gamma_1'\Gamma_2' = -i\gamma_*\gamma_0 = i\gamma_0.. \tag{5.55}$$

$\gamma_* = -E$ in Section 5.2.1, where $E = I \otimes J \otimes \mathbb{1}$, and $\gamma_0 = I \otimes K \otimes \mathbb{1}$ so in total $\gamma_*\gamma_0 = -I \otimes K \otimes \mathbb{1}$.

$$Im[A_{(2,3)}(\cdot,\cdot)] = Im[m(\cdot, A_{(2,3)}\cdot)] = Im[m(\cdot, -i\gamma_*\gamma_0\cdot)] = Re[m(\cdot, \gamma_*\gamma_0\cdot)]. \tag{5.56}$$

$Re[m]$ is equivalent to $g \otimes g \otimes g$ and therefore

$$Im[A_{(2,3)}(\cdot,\cdot)] \equiv g \otimes g(\cdot, I\cdot) \otimes g(\cdot, I\cdot) = \beta_3. \tag{5.57}$$

β_3 is in the space-like orbits (with stabiliser group $SO_0(1,2)$) that have a U(1,1) R-symmetry group. Therefore the two predicted (1,3) superalgebras with U(2) and U(1,1) R-symmetry groups can be realised as the reduction of a (1,4) and (2,3) superalgebra. This confirms the analysis of Chapter 3, and follows logically because the

R-symmetry group of $(2,3)$ $\mathcal{N}=2$ theories is $\mathrm{SU}(1,1) \notin \mathrm{U}(2)$.

Reduction $(2,3) \to (2,2)$

We now work with the convention that the four-dimensional γ_1, γ_2 are time-like and γ_3, γ_4 are space-like; we set

$$\gamma_1 = \Gamma_1', \quad \gamma_2 = \Gamma_2', \quad \gamma_3 = \Gamma_3, \quad \gamma_4 = \Gamma_4. \tag{5.58}$$

The reduction is space-like reduction $A_{(2,3)} = A_{(2,2)} = \gamma_1\gamma_2$. We see that

$$Im[A_{(2,3)}(\cdot,\cdot)] = Im[m(\cdot, A_{(2,3)}\cdot)] = Re[m(\cdot, -i\gamma_1\gamma_2\cdot)]. \tag{5.59}$$

The model Section 5.2.2 $\gamma_1\gamma_2 = I \otimes \mathbb{1} \otimes \mathbb{1}$ and multiplication by i corresponds to $\mathbb{1} \otimes \mathbb{1} \otimes I$. Again $Re[m] = g \otimes g \otimes g$ and we obtain

$$Im[A_{(2,3)}(\cdot,\cdot)] = -g(\cdot, I\cdot) \otimes g(\cdot,\cdot) \otimes g(\cdot, I\cdot) = -\beta_4. \tag{5.60}$$

Due to the presence of anti-isometries $(2,2)$ signature superalgebras only have a single $\mathcal{N}=2$ orbit (that of time-like and space-like vectors) with R-symmetry group $\mathrm{GL}(2,\mathbb{R})$, which we obtained here.

The chirality operator is

$$\gamma_* = \Gamma_5 = \Gamma_1 ... \Gamma_4 = -\gamma_1 ... \gamma_4. \tag{5.61}$$

Reduction $(3,2) \to (2,2)$

The γ-matrices are related by

$$\gamma_1 = \Gamma_2', \quad \gamma_2 = \Gamma_3', \quad \gamma_3 = \Gamma_4, \quad \gamma_4 = \Gamma_5. \tag{5.62}$$

The volume element is

$$\gamma_* = -\gamma_1\gamma_2\gamma_3\gamma_4 = -\Gamma_2'\Gamma_3'\Gamma_4\Gamma_5 = i\Gamma_1'. \tag{5.63}$$

So that $A_{(2,3)} = \Gamma_1'\Gamma_2'\Gamma_3' = -i\gamma_*\gamma_1\gamma_2$. Therefore

$$Im[A_{(2,3)}(\cdot,\cdot)] = Im[h(\cdot, \Gamma_1'\Gamma_2'\Gamma_3'\cdot)] = Re[h(\cdot, i\Gamma_1'\Gamma_2'\Gamma_3'\cdot)] = Re[h(\cdot, -\gamma_*\gamma_1\gamma_2\cdot)]. \tag{5.64}$$

In the model in Section 5.2.2 $\gamma_* = -E = -I \otimes I \otimes \mathbb{1}$ and $\gamma_1\gamma_2 = -I \otimes \mathbb{1} \otimes \mathbb{1}$. $Re[m]$ is equivalent to $g \otimes g \otimes g$ and we get

$$Im[A_{(2,3)}(\cdot,\cdot)] = -g(\cdot,\cdot) \otimes g(\cdot, I\cdot) \otimes g = \beta_1. \tag{5.65}$$

β_1 is in the single orbit of space- and time-like vectors with R-symmetry group $\mathrm{GL}(2,\mathbb{R})$. As we already knew, dimensionally reducing a $(3,2)$ or $(2,3)$ superalgebra to $(2,2)$ results in isomorphic superalgebras as there is only one family of $\mathcal{N} = 2$ superalgebras in $(2,2)$ up to isomorphism.

5.3 Doubled Spinor Formulation

Now we turn to describing the possibilities for $\mathcal{N} = 2$ four-dimensional supersymmetry algebras in the formalism of Chapter 3. All information can be found in Chapter 3 but is repeated here to make this chapter self-contained. Doing so leads to physical theories with manifestly R-symmetric spinors and provides another working example of the formalism. Compared to five dimensions there are many more possibilities in four dimensions, providing a natural step-up.

Now that we are in even dimensions there are two charge conjugation matrices with two associated bilinear forms. The four-dimensional charge conjugation matrices are C_- with invariants $(\sigma_- = -1, \tau_- = +1)$ and C_+ with invariants $(\sigma_+ = -1, \tau_+ = -1)$. $\sigma_-\tau_- = -1$ so that a super-admissible bilinear form on $\mathbb{S}\otimes\mathbb{C}^2$ is given by $C_-\otimes J_2 = C_-\otimes\varepsilon$. The other Majorana bilinear form is super-admissible, $\sigma_+\tau_+ = +1$, so another super-admissible bilinear form on $\mathbb{S}\otimes\mathbb{C}^2$ is given by $C_+ \otimes \delta$.

In even dimensions there are two corresponding B matrices that each define a $\mathrm{Spin}_0(t,s)$-invariant ϵ-quaternionic structure on $\mathbb{S}$

$$J_\pm^{(\epsilon)(\alpha)}(\lambda) = \alpha^* B_\pm^* \lambda^*. \tag{5.66}$$

Both $J_+^{(\epsilon)(\alpha)}$ and $J_-^{(\epsilon)(\alpha)}$ are Weyl-compatible, i.e. $J^{(\epsilon)(\alpha)}(\mathbb{S}_\pm) \in \mathbb{S}_\pm$, or Weyl-incompatible, $J^{(\epsilon)(\alpha)}(\mathbb{S}_\pm) \in \mathbb{S}_\mp$. Recall signatures in which $J_+^{(\epsilon)(\alpha)}$ and $J_-^{(\epsilon)(\alpha)}$ have the same ϵ value are Weyl-compatible (both are quaternionic or para-quaternionic structures), and when their ϵ values are different the signature is Weyl-incompatible (one is a quaternionic structure and the other a paraquaternionic structure).

$(0,4)$ and $(4,0)$ are Weyl-compatible signatures with $B_\pm^* B_\pm = -1$. Therefore only the standard symplectic Majorana reality condition is possible, with either choice of B matrix:

$$(\lambda^i)^* = \alpha B_\pm \lambda^j \varepsilon_{ji}. \tag{5.67}$$

In the Minkowski signatures $(1,3)$ and $(3,1)$ are Weyl-incompatible and have $B_+^* B_+ = +1$ and $B_-^* B_- = -1$. Therefore the following reality conditions on $\mathbb{S} \otimes \mathbb{C}^2$ are possible:

$$(\lambda^i)^* = \alpha B_- \lambda^j \varepsilon_{ji}, \quad (\lambda^i)^* = \alpha B_+ \lambda^i = \alpha B_+ \lambda^j \delta_{ji}, \quad (\lambda^i)^* = \alpha B_+ \lambda^j (I_{1,1})_{ji}. \tag{5.68}$$

Finally in $(2,2)$ $J_\pm^{(\epsilon)(\alpha)}$ are Weyl-compatible and both have $\epsilon = +1$. We can therefore have reality conditions of the form

$$(\lambda^i)^* = \alpha B_\pm \lambda^i = \alpha B_+ \lambda^j \delta_{ji}, \quad (\lambda^i)^* = \alpha B_\pm \lambda^j (I_{1,1})_{ji}. \tag{5.69}$$

As we did in five dimensions, when writing reality conditions we will use the off-diagonal matrix

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.70}$$

instead of $I_{1,1}$ as this is the form found in the papers [2] and [3]. They are related by the map found in Section 4.4.

In $(0,4)$ and $(2,2)$ the choice of bilinear form and reality condition is irrelevant as all choices of bilinear form and reality condition lead to isomorphic supersymmetry algebras with the same R-symmetry group. In $(1,3)$ both the reality condition and bilinear form together determine the R-symmetry group and we do not always end up with isomorphic superalgebras (there are two families, one with $\mathrm{U}(2)$ and one with $\mathrm{U}(1,1)$ R-symmetry). Isomorphisms are described later in Section 5.7.

Not all these supersymmetry algebras (combinations of bilinear form and reality condition) are realised by dimensional reduction of five-dimensional supersymmetry algebras. This is similar to the previous section where different reduced theories ended up with different four-dimensional supersymmetry algebras based on different bilinear forms, though now we are framing this analysis in the doubled spinor framework.

5.3.1 Supersymmetry Algebras Obtained by Dimensional Reduction

To derive four-dimensional theories from the five-dimensional theories we first need to discuss the dimensional reduction of the constituent matrices, A, B, C, involved in the definition of the bilinear forms and reality conditions.

Note that the conventions in this section are different from those in the previous Dimensional Reduction part in Section 5.2.4. Here we use a convention where the γ-matrices for the 5D theories are always γ_0 to γ_4, removing the 0 or 4 direction when we perform a time-like or space-like reduction respectively. The four-dimensional γ matrices are then the remaining $\gamma_1, ..., \gamma_4$ or $\gamma_0, ..., \gamma_3$.

5.3.2 Dimensional Reduction of A, B, C.

In the following always $t + s = 5$. In 5 dimensions the charge conjugation matrix corresponds to the C_- in 4D

$$C^5 = C_-^4 \tag{5.71}$$

When we reduce across a space-like dimension, the reduced direction will be assumed to be γ_4, and for time-like, it will be γ_0. The projection matrix can be found to be $\gamma_* = \gamma_5$ for a space-like reduction or $\gamma_* = i\gamma_0$ for a time-like reduction[4]. Where it is clear the 4 superscript will be omitted, especially as only four-dimensional charge conjugation matrices will be written with $\pm$ subscripts.

As a result of this, all four-dimensional supersymmetry algebras obtained in this form have a superbracket defined using the bilinear form $C_-^4 \otimes \varepsilon$. Where applicable later we will give isomorphisms to equivalent supersymmetry algebras with superbracket derived from $C_+^4 \otimes \delta$.

The five-dimensional A matrix in signature (t, s) contains both the $(t, s-1)$ and $(t-1, s)$ A-matrices (provided $t - 1 < 0$ or $s - 1 < 0$) :

$$A^{(t,s)} = \Pi_\tau \gamma_\tau = A^{(t,s-1)} = \gamma_0 A^{(t-1,s)}. \tag{5.72}$$

[4]Chosen to match with [20]

Similarly, the inverse contains the two corresponding inverses

$$(A^{(t,s)})^{-1} = (-1)^t\gamma_t...\gamma_0 = (A^{(t,s-1)})^{-1} = -(A^{(t-1,s)})^{-1}\gamma_0 = (-1)^t\gamma_0(A^{(t-1,s)})^{-1}. \quad (5.73)$$

This means that the B matrices reduce for a space-like reduction as

$$B^{(t,s)} = (C^5(A^{(t,s)})^{-1})^T = (C^4_-(A^{(t,s-1)})^{-1})^T = B_-^{(t,s-1)}. \quad (5.74)$$

And for a time-like reduction, we obtain

$$B^{(t,s)} = (C^5(A^{(t,s)})^{-1})^T = (-1)^t(C^4_-\gamma_0(A^{(t-1,s)})^{-1})^T. \quad (5.75)$$

After a time-like reduction $\gamma_* = i\gamma_0$ and $C^4_-\gamma_* = C^4_+$, so this can instead be written

$$\implies B^{(t,s)} = (-1)^t(-iC^4_+(A^{(t-1,s)})^{-1})^T = (-1)^{t+1}iB_+^{(t-1,s)} \quad (5.76)$$

5.3.3 Five-Dimensional Supersymmetry Algebras

The five-dimensional supersymmetry algebras were defined in Section 4.4 In five dimensions we have a single Majorana bilinear form, which is a 'C_-', i.e. $\tau(C) = +1$. $\sigma(C) = -1$ so that on $\mathbb{S}\otimes\mathbb{C}^2$ one has to use the super-admissible bilinear form $C\otimes\varepsilon$ and there is no other possibility. The reality condition is signature-dependent, and is contained in the following table (which is repeated from Section 4.5)

	Reality Condition
$(0,5)$	$(\lambda^i)^* = B\lambda^j\varepsilon_{ji}$
$(1,4)$	$(\lambda^i)^* = -B\lambda^j\varepsilon_{ji}$
$(2,3)$	$(\lambda^i)^* = iB\lambda^j\eta_{ij}$
$(3,2)$	$(\lambda^i)^* = -iB\lambda^j\eta_{ij}$
$(4,1)$	$(\lambda^i)^* = B\lambda^j\varepsilon_{ji}$
$(5,0)$	$(\lambda^i)^* = -B\lambda^j\varepsilon_{ji}$

Table 5.18: Reality condition in each five-dimensional signature, $B = (CA^{-1})^T$ is signature dependent.

Four-dimensional Reality Conditions

Using the above we obtain the following reality conditions in the four-dimensional signatures. Here I have introduced the notation $(\not{t},s)$ that signifies the theory is a $(t-1,s)$ signature theory obtained from reduction from (t,s) and similarly $(t,\not{s})$ which signifies the spacetime of the theory has $(t, s-1)$ signature and was derived from the dimensional reduction of a (t, s) theory.

	Reality Condition
$(0,\cancel{5})$	$(\lambda^i)^* = B_-\lambda^j\varepsilon_{ji}$
$(\cancel{1},4)$	$(\lambda^i)^* = -iB_+\lambda^j\varepsilon_{ji}$
$(1,\cancel{4})$	$(\lambda^i)^* = -B_-\lambda^j\varepsilon_{ji}$
$(\cancel{2},3)$	$(\lambda^i)^* = B_+\lambda^j\eta_{ij}$
$(2,\cancel{3})$	$(\lambda^i)^* = iB_-\lambda^j\eta_{ij}$
$(\cancel{3},2)$	$(\lambda^i)^* = B_+\lambda^j\eta_{ij}$
$(3,\cancel{2})$	$(\lambda^i)^* = -iB_-\lambda^j\eta_{ij}$
$(\cancel{4},1)$	$(\lambda^i)^* = -iB_+\lambda^j\varepsilon_{ji}$
$(4,\cancel{1})$	$(\lambda^i)^* = B_-\lambda^j\varepsilon_{ji}$
$(\cancel{5},0)$	$(\lambda^i)^* = -iB_+\lambda^j\varepsilon_{ji}$

Table 5.19: Reality Condition in each four-dimensional signature, $B = (CA^{-1})^T$ is signature dependent.

In all four-dimensional theories obtained from dimensional reduction the bilinear form on $\mathbb{S} \otimes \mathbb{C}^2$ is always $C_-^4 \otimes \varepsilon$. All doubled spinor constructions (the pair (Bilinear Form, Reality Condition)) in $(0,4)$ and $(2,2)$ are isomorphic, with $\mathrm{U}^*(2)$ and $\mathrm{GL}(2,\mathbb{R})$ R-symmetry respectively. The two $(1,3)$ theories are not, with that coming from $(1,4)$ being in the $\mathrm{U}(2)$ R-symmetry group family and that from $(2,3)$ has $\mathrm{U}(1,1)$ R-symmetry. The following section will catalogue all possible doubled spinor constructions in each signature and where possible give maps between them.

5.3.4 (0,4)

In all (0,4) $\mathcal{N} = 2$ theories, no matter the choice of reality condition and bilinear form, the R-symmetry group is $\mathrm{U}^*(2) \cong \mathbb{R}^{>0} \times \mathrm{SU}(2)$. All choices are isomorphic and maps between them are provided.

There are two choices for C (with C_+ forcing $M = \delta$ and C_- forcing $M = J$) and two for B that both lead to quaternionic structures, which means we can only have reality conditions with $L = \varepsilon$. They are collected in the table below [5]:

$$G_R = \mathrm{U}^*(2) \;\Longleftarrow\; \begin{cases} C_- \otimes \varepsilon, & (\lambda^i)^* = \alpha B_- \lambda^j \varepsilon_{ji} \leftarrow (0, \cancel{5}) \\ C_- \otimes \varepsilon, & (\psi^i)^* = \beta B_+ \psi^j \varepsilon_{ji} \leftarrow (\cancel{1}, 4) \\ C_+ \otimes \delta, & (\phi^i)^* = \gamma B_- \phi^j \varepsilon_{ji} \\ C_+ \otimes \delta, & (\xi^i)^* = \delta B_+ \xi^j \varepsilon_{ji} \end{cases} \tag{5.77}$$

We can see that to map all we need two types of transformation: one that interchanges B_- and B_+ in the reality condition, while leaving the vector-valued bilinear form alone, and one that changes the vector-valued bilinear form from $C_- \otimes \varepsilon$ to $C_+ \otimes \delta$, leaving the reality condition alone. By composing these maps, we can then relate all four real superbrackets possible within the doubled spinor construction.

Reality Condition Map, R

This is an example of the R map from Chapter 3 applied to the case of an isotropic Weyl-compatible supersymmetry algebra, more details can be found in this chapter in Section 3.10.2.

We want an isomorphism that changes B_- with B_+ in the reality condition, and preserve the bilinear form, i.e. mapping between the doubled spinors λ^i and ψ^i in (5.77).

Given a doubled spinor λ^i with reality condition

$$(\lambda^i)^* = \alpha B_- \lambda^j L_{ji} \tag{5.78}$$

and a second doubled spinor, ψ^i, that obeys

$$(\psi^i)^* = \beta B_+ \psi^j L_{ji}. \tag{5.79}$$

We wish to find a linear transformation $\lambda^i \to \psi^i$. We make the ansatz

$$\lambda^i = \frac{1}{\sqrt{2}}(a\mathbb{1} + b\gamma_*)\psi^i \tag{5.80}$$

[5]The names of each spinor have no specific meaning, they are just for keeping track in the following section.

Assuming $a \neq \pm b$, so that $a1 + b\gamma_*$ is invertible

$$\psi^i = \frac{1}{\sqrt{2}}(a^*1 + b^*\gamma_*)\lambda^i, \tag{5.81}$$

with $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 2$ and $ab^* + a^*b = 0$.

With this ansatz we find

$$\begin{aligned}(\psi^i)^* &= \frac{1}{\sqrt{2}}(a1 + b\gamma_*)(\lambda^i)^* = \frac{1}{\sqrt{2}}(a1 + b\gamma_*)\alpha B_-\lambda^j L_{ji} \\ &= \frac{1}{\sqrt{2}}\alpha B_-(a1 + b\gamma_*)\lambda^j L_{ji} = \frac{1}{\sqrt{2}}\alpha B_+(a\gamma_* + b1)\lambda^j L_{ji}.\end{aligned} \tag{5.82}$$

This calculation used that in $(0,4)$ signature $\gamma_* B_- = B_-\gamma_* = B_+$. Comparing this to

$$(\psi^i)^* = \beta B_+\psi^j L_{ji} = \frac{1}{\sqrt{2}}\beta B_+(a1 + b\gamma_*)\lambda^j L_{ji} \tag{5.83}$$

we see that a, b, α, β must obey

$$\alpha b = a^*\beta, \quad \alpha a = b^*\beta. \tag{5.84}$$

This is implies that $|a| = |b|$. This equation can be solved by requiring $a = 1$, $b = \frac{\beta}{\alpha}$. Additionally we required that $|a|^2 + |b|^2 = 2$ and $ab^* + a^*b = 0$. From Table 5.19 we see the phases of the two reality conditions obtained from dimensional reduction satisfy $\beta = -i\alpha$, and as they are phases $|\frac{\beta}{\alpha}| = 1$ so both of these equations hold. Therefore $a = 1$ and $b = -i$ so that

$$\lambda^i = \frac{1}{\sqrt{2}}(1 - i\gamma_*)\psi^i \iff \psi^i = \frac{1}{\sqrt{2}}(1 + i\gamma_*)\lambda^i. \tag{5.85}$$

One can easily show that under this change the transformed quantities have the same chirality as the previous spinors

$$\gamma_*\psi_\pm = \gamma_*\frac{1}{\sqrt{2}}(1 + i\gamma_*)\lambda^i_\pm = \frac{1}{\sqrt{2}}(1 + i\gamma_*)\gamma_*\lambda^i_\pm = \pm\psi^i_\pm \tag{5.86}$$

Either vector-valued bilinear form, regardless of choice of C or M, is unchanged under this transformation:

$$
\begin{aligned}
&(\gamma^m\lambda^i)^T C_\pm \chi^j M_{ji} = \frac{1}{2}(\gamma^m(1+i\gamma_*)\psi^i)^T C_\pm (1+i\gamma_*)\Omega^j M_{ji}\\
=&\frac{1}{2}(\psi^i)^T\bigg((\gamma^m)^T C_\pm - \gamma_*(\gamma^m)^T C_\pm\gamma_* + i\gamma_*(\gamma^m)^T C_\pm + i(\gamma^m)^T C_\pm\gamma_*\bigg)\Omega^j M_{ji}\\
=&(\gamma^m\psi^i)^T C_\pm \Omega^j M_{ji}.
\end{aligned}
\tag{5.87}
$$

Where we have used that $\gamma_* C_\pm = C_\pm\gamma_*$ in $(0,4)$ and defined Ω from χ analogously to how ψ is defined from λ. The vector-valued bilinear form is the same in terms of λ^i and ψ^i, so this transformation only exchanges the reality conditions.

Caution is needed, however, as this does not mean the regular scalar-valued bilinear form is invariant. This transforms as

$$
\begin{aligned}
&(\lambda^i)^T C_\pm \chi^j M_{ji} = \frac{1}{2}((1+i\gamma_*)\psi^i)^T C_\pm (1+i\gamma_*)\Omega^j M_{ji}\\
=&\frac{1}{2}(\psi^i)^T\bigg(C_\pm - \gamma_* C_\pm\gamma_* + i\gamma_* C_\pm + iC_\pm\gamma_*\bigg)\Omega^j M_{ji}\\
=&i(\psi^i)^T C_\pm\gamma_*\Omega^j M_{ji}
\end{aligned}
\tag{5.88}
$$

This occurs in the Lagrangians of the two $(0,4)$ theories resulting from dimensional reduction, detailed in Section 5.6. This is dealt with through further reparameterisations of the field content. See this section for more details.

Writing this transformation in terms of Weyl spinors

$$
\lambda^i = \frac{1}{\sqrt{2}}(1-i\gamma_*)\psi^i, \quad \Longrightarrow \lambda^i_\pm = \frac{1}{\sqrt{2}}(1\mp i)\psi^i_\pm
\tag{5.89}
$$

this can be translated into matrix notation:

$$
\lambda^I = R^I{}_J\psi^J, \qquad R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 & 0 & 0\\ 0 & 1-i & 0 & 0\\ 0 & 0 & 1+i & 0\\ 0 & 0 & 0 & 1+i \end{pmatrix}.
\tag{5.90}
$$

Bilinear Form Map, S

Note – this is map S found in Chapter 3 applied to the case where $K = 2$.

If we wish to map the complex vector-valued bilinear forms $[C_- \otimes \varepsilon]$ to $[C_+ \otimes \delta]$ together whilst retaining the reality condition. In (5.77) this corresponds to transforming λ^i into ϕ^i or ψ^i into ξ^i. As we have shown that λ^i can be mapped to ψ^i we can just map λ^i to ϕ^i without loss of generality.

To do this, it is useful to use matrix notation. Recall the vector-valued bilinear form given by $[C_- \otimes \varepsilon]$ is written

$$(\gamma^\mu \lambda_+^i)^T C_- \chi_-^j \varepsilon_{ji} + (\gamma^\mu \lambda_-^i)^T C_- \chi_+^j \varepsilon_{ji} \tag{5.91}$$
$$\to \Big((\gamma^\mu \lambda_+^i)^T, (\gamma^\mu \lambda_+^i)_-^T\Big) C_- \begin{pmatrix} 0 & \varepsilon_{ji} \\ \varepsilon_{ji} & 0 \end{pmatrix} \begin{pmatrix} \chi_+^j \\ \chi_-^j \end{pmatrix} = \Big(\gamma^\mu \underline{\lambda}_+, \gamma^\mu \underline{\lambda}_-\Big) C_- \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \underline{\chi}_+ \\ \underline{\chi}_- \end{pmatrix}$$

Using $C_- \lambda_\pm = C_-(\pm\gamma_* \lambda_\pm) = \pm C_+ \lambda_\pm$, this can be recast into a vector-valued bilinear form using C_+:

$$\Big(\gamma^\mu \underline{\lambda}_+, \gamma^\mu \underline{\lambda}_-\Big) C_- \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \underline{\chi}_+ \\ \underline{\chi}_- \end{pmatrix} = \Big(\gamma^\mu \underline{\lambda}_+, \gamma^\mu \underline{\lambda}_-\Big) C_+ \begin{pmatrix} 0 & +\varepsilon \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \underline{\chi}_+ \\ \underline{\chi}_- \end{pmatrix} \tag{5.92}$$

The vector-valued bilinear form $[C_+ \otimes \delta]$ in matrix notation is

$$\Big(\gamma^\mu \underline{\phi}_+, \gamma^\mu \underline{\phi}_-\Big) C_+ \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \underline{\Omega}_+ \\ \underline{\Omega}_- \end{pmatrix}. \tag{5.93}$$

We seek a linear transformation that relates λ^I and ϕ^I. We find

$$\lambda^I = S^I{}_J \phi^J \implies S^T \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.94}$$

This is solved by

$$S = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\varepsilon \end{pmatrix} \tag{5.95}$$

Explicitly in terms of the components, this is

$$\begin{aligned} \lambda^i_+ &= \phi^i_+ \\ \lambda^i_- &= \phi^j_- \varepsilon_{ji} \end{aligned} \tag{5.96}$$

S is block-diagonal, so it manifestly preserves chirality. Up to this point, the map can be used regardless of signature, though it does not necessarily preserve the reality condition (we will use it in $(1,3)$ signature too, where the reality condition will also change). In $(0,4)$ the reality condition is Weyl compatible, and we see that ϕ^i_+ has the same reality condition as λ^i_+, provided $\alpha = \gamma$. ϕ^i_- satisfies

$$(\phi^i_-)^* = -(\alpha B_\pm \lambda^k_- \varepsilon_{kj})\varepsilon_{ji} = \alpha B_\pm \lambda^i_- = \alpha B_\pm \phi^j_- \varepsilon_{ji}. \tag{5.97}$$

Which once again requires $\alpha = \gamma$.

One can show this behaves identically for ψ^i and ξ^i provided $\beta = \delta$, as expected as the transformation does not act on the spinor indices so that B_+ or B_- in the reality condition is irrelevant. Therefore this mapping preserves the reality condition for either choice of B matrix, interchanging $C_+ \otimes \delta$ and $C_- \otimes \varepsilon$.

Summary for $(0,4)$

The following diagram summarises the situation, where S is the map that exchanges vector-valued bilinear form, named after the matrix representing the linear transformation in the matrix notation, and similarly R is the reality condition exchanging map.

$$\begin{tikzcd}
(C_- \otimes \varepsilon, (\lambda^i)^* = \alpha B_- \lambda^j \varepsilon_{ji}) \arrow[r, leftrightarrow, "R"] & (C_- \otimes \varepsilon, (\psi^i)^* = \beta B_+ \psi^j \varepsilon_{ji}) \\
(C_+ \otimes \delta, (\phi^i)^* = \gamma B_- \phi^j \varepsilon_{ji}) \arrow[r, leftrightarrow, "R"'] \arrow[u, "S"] & (C_+ \otimes \delta, (\xi^i)^* = \delta B_+ \xi^j \varepsilon_{ji}) \arrow[u, "S"']
\end{tikzcd}$$

5.3.5 (1,3)

$(1,3)$ is an isotropic and Weyl-incompatible signature, with J_+ a para-quaternionic structure and J_- a quaternionic structure. The existence of a real structure allows an $\mathcal{N} = 1$ algebra made from Majorana spinors, but as it is isotropic no '$\mathcal{N} = 1/2$' algebra whose supercharges are Majorana-Weyl spinors of a single chirality.

Isotropic, Weyl-incompatible signatures have $\mathrm{U}(p,q)$ R-symmetry group, so for $\mathcal{N}=2$ algebras we can have $\mathrm{U}(2)$ or $\mathrm{U}(1,1)$ R-symmetry. As shown previously, a supersymmetry algebra with $\mathrm{U}(2)$ R-symmetry is obtained from reduction from $(1,4)$ and the reduction from $(2,3)$ results in supersymmetry algebra with $\mathrm{U}(1,1)$ R-symmetry.

We have two choices for bilinear form, as in all four-dimensional signatures they are $C_+\otimes\delta$ and $C_-\otimes\varepsilon$, and three choices for reality condition when $\mathcal{N}=2$, with $L=\{\delta, I_{1,1}, \varepsilon\}$. In the following section we will use the diagonalised form of $I_{1,1}$, η which is the matrix defined in (5.2.3).

Standard $\mathcal{N}=2$ superalgebra, $G_R=\mathrm{U}(2)$.

Two combinations result in a $\mathrm{U}(2)$ R-symmetry group, namely the usual writing in term of symplectic Majorana spinors and Majorana spinors. To summarise:

$$G_R=\mathrm{U}(2) \Longleftarrow \begin{cases} C_-\otimes\varepsilon, & (\lambda^i)^*=\alpha B_-\lambda^j\varepsilon_{ji} \leftarrow (1,\cancel{4}) \\ C_+\otimes\delta, & (\Psi^i)^*=\beta B_+\Psi^i. \end{cases} \tag{5.98}$$

In this chapter, we use the symplectic Majorana description to write the Lagrangians in Section 5.6. Derivations in terms of Majorana spinors can be found in [39].

These two descriptions are therefore isomorphic, and maps between them are already known, though using S from the previous section we can give a marginally more elegant map.

The original manner of relating symplectic Majorana and Majorana spinors, found in the appendix of [20], is to set

$$\lambda^1=\frac{1}{\sqrt{2}}(\Psi^1-i\Psi^2), \tag{5.99}$$

$$\lambda^2=\frac{\beta}{\sqrt{2}\alpha}B_-^*B_+(\Psi^1+i\Psi^2). \tag{5.100}$$

One can show that Ψ^i obeys the correct reality condition. One finds the vector-valued bilinear forms are related by

$$[C_- \otimes \varepsilon](\gamma^\mu \lambda, \chi) = \frac{\beta}{\alpha}[C_+ \otimes \delta](\gamma^\mu \Psi, \Omega) \tag{5.101}$$

In our prescription, the vector-valued bilinear form restricted to the real points of their respective reality condition is real, so $\frac{\beta}{\alpha}$ should be real, though obviously setting $\alpha = \beta$ allows this to two vector-valued bilinear forms to be totally equal.

It was already demonstrated that S exchanges the vector-valued bilinear forms $C_+ \otimes \delta$ and $C_- \otimes \varepsilon$, but it interacts differently with a Weyl-incompatible reality condition:

$$(\Psi^i_+)^* = (\lambda^i_+)^* = \alpha B_- \lambda^j_- \varepsilon_{ji} = -\alpha B_- \Psi^i_- = \alpha B_+ \Psi^i_-, \tag{5.102}$$

$$(\Psi^i_-)^* = -(\lambda^j_-)^* \varepsilon_{ji} = -\alpha B_- \lambda^k_+ \varepsilon_{kj} \varepsilon_{ji} = \alpha B_- \lambda^i_+ = \alpha B_- \Psi^i_+ = \alpha B_+ \Psi^i_+. \tag{5.103}$$

Here we used that $B_\pm \gamma_* = B_\mp$, which is true in all four-dimensional signatures. We see correctly that the map S exchanges the symplectic Majorana and Majorana reality conditions, with phases $\alpha = \beta$, while simultaneously exchanging the bilinear forms.

Twisted $\mathcal{N} = 2$ Superalgebra, $G_R = \mathrm{U}(1,1)$

The remaining four choices all have a $\mathrm{U}(1,1)$ R-symmetry group. Each no longer has the same matrix for the bilinear form and reality condition, M and L. They are:

$$G_R = \mathrm{U}(1,1) \;\Longleftarrow\; \begin{cases} C_- \otimes \varepsilon, & (\lambda^i)^* = \alpha B_+ \lambda^i \\ C_- \otimes \varepsilon, & (\psi^i)^* = \beta B_+ \psi^j \eta_{ji} \leftarrow (\cancel{2}, 3) \\ C_+ \otimes \delta, & (\phi^i)^* = \gamma B_+ \phi^j \eta_{ji} \\ C_+ \otimes \delta, & (\xi^i)^* = \delta B_- \xi^j \varepsilon_{ji} \end{cases} \tag{5.104}$$

S and T are useful again here. S exchanges the bilinear form but does not leave the reality condition invariant. Applying S to λ^i obtains the supersymmetry algebra whose spinors are ξ in (5.104):

$$\begin{aligned}(\xi^i_+)^* = (\lambda^i_+)^* &= \alpha B_+ \lambda^i_- \\ &= \alpha B_+ \xi^j_- \varepsilon_{ji} = -\alpha B_- \xi^j_- \varepsilon_{ji},\end{aligned} \tag{5.105}$$

and the negative chirality spinors obey

$$\begin{aligned}(\xi^i_-)^* &= -(\lambda^j_-)^* \varepsilon_{ji} = -\alpha B_+ \lambda^j_+ \varepsilon_{ji} \\ &= -\alpha B_+ \xi^j_+ \varepsilon_{ji} = -\alpha B_- \xi^j_- \varepsilon_{ji}.\end{aligned} \tag{5.106}$$

where we have used $B_\pm \gamma_* = B_\mp$. We can see that S maps the two supersymmetry algebras whose spinors are λ^i to that with ξ^i with phases that obey $\delta = -\alpha$. Using the matrix notation this is that $\lambda^I = S^I{}_J \xi^J$.

S can also be used to map the superalgebras with ψ^i to ϕ^i, though we will need an additional transformation on top. Setting $\psi^I = S^I{}_J \Phi^J$, we find that

$$\begin{aligned}(\Phi^i_+)^* &= (\psi^i_+)^* = \alpha B_+ \psi^j_- \eta_{ji} \\ &= \alpha B_+ \Phi^k_- \varepsilon_{kj} \eta_{ji}, \\ (\Phi^i_-)^* &= -(\psi^j_-)^* \varepsilon_{ji} = -\alpha B_+ \psi^k_+ \eta_{kj} \varepsilon_{ji} \\ &= -\alpha B_+ \Phi^k_+ \eta_{kj} \varepsilon_{ji}.\end{aligned} \tag{5.107}$$

The two products involved are

$$\eta_{ij}\varepsilon_{jk} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{ik} \equiv -\eta'_{ik} \tag{5.108}$$

$$\varepsilon_{ij}\eta_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ik} \equiv \eta'_{ik}. \tag{5.109}$$

Therefore the reality conditions for Φ^i are

$$(\Phi^i_+)^* = \alpha B_+ \Phi^j_- \eta'_{ji}, \quad (\Phi^i_-)^* = \alpha B_+ \Phi^j_+ \eta'_{ji}. \tag{5.110}$$

Φ^i obeys the diagonalised form of the reality condition given for ϕ^i. This can be undone by implementing

$$\phi^1 = \frac{1}{\sqrt{2}}(\Phi^1 + \Phi^2), \qquad \phi^2 = \frac{1}{\sqrt{2}}(\Phi^1 - \Phi^2). \tag{5.111}$$

The new parameterisation obeys the reality conditions:

$$(\phi^1)^* = \alpha B_+ \frac{1}{\sqrt{2}}(\Phi^1 - \Phi^2) = \alpha B_+ \phi^2, \tag{5.112}$$
$$(\phi^2)^* = \alpha B_+ \frac{1}{\sqrt{2}}(\Phi^1 + \Phi^2) = \alpha B_+ \phi^1.$$
$$\implies (\phi^i)^* = \alpha B_+ \phi^j \eta_{ji}. \tag{5.113}$$

So the new description has the same phase for ψ^i and ϕ^i: $\gamma = \alpha$. This additional transformation does not change any of the bilinear forms:

$$\begin{aligned}
&(\gamma^{(p)}\Phi^1)^T C_+ \Upsilon^1 + (\gamma^{(p)}\Phi^2)^T C_+ \Upsilon^2 \\
=&\frac{1}{2}(\gamma^{(p)}(\phi^1 - \phi^2))^T C_+ (\upsilon^1 - \upsilon^2) + \frac{1}{2}(\gamma^{(p)}(\phi^1 + \phi^2))^T C_+ (\upsilon^1 + \upsilon^2) \\
=&(\gamma^{(p)}\phi^1)^T C_+ \upsilon^1 + (\gamma^{(p)}\phi^2)^T C_+ \upsilon^2.
\end{aligned} \tag{5.114}$$

Where υ^i are related to Υ^i in the same manner ϕ^i and Φ^i are related. We can compose these two maps to obtain a new map:

$$\psi^1_+ = \frac{1}{\sqrt{2}}(\phi^1_+ + \phi^2_+), \quad \psi^2_+ = \frac{1}{\sqrt{2}}(\phi^1_+ - \phi^2_+), \tag{5.115}$$
$$\psi^1_- = \frac{1}{-\sqrt{2}}(\phi^1_- + \phi^2_-), \quad \psi^2_- = \frac{1}{\sqrt{2}}(\phi^1_- + \phi^2_-).$$

In the matrix notation Φ^i and ϕ are related by F

$$\phi^I = F^I{}_J \Phi^J, \qquad F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{5.116}$$

We can then represent the map that takes $\psi^i \to \phi^i$ as the linear transformation $T = SF^{-1}$

$$\psi^I = T^I{}_J \phi^J, \qquad T = SF^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{5.117}$$

Finally, we seek a map that maintains the bilinear form $C_+ \otimes \delta$ and exchanges the reality condition for ϕ and ξ, so that any remaining maps can be formed by composition. We can immediately assume the transformation is block-diagonal, without loss of generality, as we wish to preserve chirality. Defining U such that

$$\phi^I = U^I{}_J \xi^J, \tag{5.118}$$

preserving the bilinear form $C_+ \otimes \delta$ in matrix notation leads to

$$U = \begin{pmatrix} u & 0 \\ 0 & (u^{-1})^T \end{pmatrix}. \tag{5.119}$$

U is a 4×4 matrix acting upon ξ^I so that u is a 2×2 matrix acting on ξ^i_+, with a corresponding complimentary transformation $(u^{-1})^T$ on ξ^i_-.

The reality condition change implies that

$$u^* \eta = -\frac{\beta}{\alpha} \varepsilon (u^{-1})^T. \tag{5.120}$$

This can be solved using

$$u = \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}. \tag{5.121}$$

For definiteness, we will choose

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{5.122}$$

In components this reads

$$\begin{aligned} \phi^1_+ &= \xi^1_+, \qquad & \phi^2_+ &= i\xi^2_+, \\ \phi^1_- &= \xi^1_-, \qquad & \phi^2_- &= -i\xi^2_-. \end{aligned} \tag{5.123}$$

Testing the reality conditions, one finds their phases are related by $\delta = -i\gamma$.

Finally, we can then relate λ^i and ψ^i by composing the maps already obtained.

$$\begin{aligned}\psi^I &= T^I{}_J\,\phi^J = T^I{}_J\,U^J{}_K\,\xi^K = T^I{}_J\,U^J{}_K\,(S^{-1})^K{}_L\lambda^L \\ &= V^I{}_J\,\lambda^J, \quad V = TUS^{-1}.\end{aligned} \tag{5.124}$$

Therefore

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & i & i \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{5.125}$$

This induces the reality condition

$$\begin{aligned}(\lambda_+^1)^* &= \frac{1}{\sqrt{2}}\beta B_+(\psi_-^1 + i\psi_-^2) \\ &= i\beta B_+\frac{1}{\sqrt{2}}(-i\psi_-^1 + \psi_-^2) \\ &= i\beta B_+\lambda_-^1.\end{aligned} \tag{5.126}$$

A similar calculation gives $(\lambda_-^2)^* = i\beta B_+\lambda_+^2$, and the same for all other combinations of $i = 1, 2$ with the two chiralities, so that the phases are related by $\alpha = i\beta$.

Summary for $(1, 3)$

The following commuting diagram summarises the previous section, detailing the relations between the six real superbrackets in $(1, 3)$. There are two distinct isomorphic superalgebras with different R-symmetry that can be realised in a few different ways using the doubled spinor formalism.

$$\begin{tikzcd}
(C_- \otimes \varepsilon, (\lambda^i)^* = \alpha B_-\lambda^j\varepsilon_{ji}) & (C_- \otimes \varepsilon, (\lambda^i)^* = \beta B_+\lambda^i) \arrow[r, "V"] & (C_- \otimes \varepsilon, (\psi^i)^* = \alpha B_+\psi^j\eta_{ji}) \\
(C_+ \otimes \delta, (\Psi^i)^* = \beta B_+\Psi^i) \arrow[u, "S"] & (C_+ \otimes \delta, (\xi^i)^* = \delta B_-\xi^j\varepsilon_{ji}) \arrow[u, "S"'] \arrow[r, "U"] & (C_+ \otimes \delta, (\phi^i)^* = \gamma B_+\phi^j\eta_{ji}) \arrow[u, "T"']
\end{tikzcd}$$

U(2) family U(1,1) family

5.3.6 (2,2)

In (2,2) we have access to two real structures defined by B_+ and B_-, and therefore two choices for $L = \delta, I_{1,1}$. As in all 4D signatures we have two Majorana bilinear forms on $\mathbb{S}$ that lead to the super-admissible bilinear forms $C_+ \otimes \delta$ and $C_- \otimes \varepsilon$ on $\mathbb{S} \otimes \mathbb{C}^2$. All choices lead to the same $\mathrm{GL}(2,\mathbb{R})$ R-symmetry group.

$$G_R = \mathrm{GL}(2,\mathbb{R}) \Longleftarrow \begin{cases} C_- \otimes \varepsilon, & (\lambda^i)^* = \alpha_1 B_- \lambda^j \eta_{ji} \leftarrow (2, \cancel{3}) \\ C_- \otimes \varepsilon, & (\psi^i)^* = \beta_1 B_+ \psi^j \eta_{ji} \leftarrow (\cancel{3}, 2) \\ C_+ \otimes \delta, & (\phi^i)^* = \gamma_1 B_- \phi^j \eta_{ji} \\ C_+ \otimes \delta, & (\xi^i)^* = \delta_1 B_+ \xi^j \eta_{ji} \\ C_+ \otimes \delta, & (\Lambda^i)^* = \alpha_2 B_- \Lambda^i \\ C_+ \otimes \delta, & (\Psi^i)^* = \beta_2 B_+ \Psi^i \\ C_- \otimes \varepsilon, & (\Phi^i)^* = \gamma_2 B_- \Phi^i \\ C_- \otimes \varepsilon, & (\Xi^i)^* = \delta_2 B_+ \Xi^i \end{cases} \tag{5.127}$$

Once again the names of the spinors and the phases have no meaning, just for book-keeping purposes to make the following discussion of isomorphisms legible. We see that we need three transformations, one that relates theories with different vector-valued bilinear forms but maintains reality conditions, one that exchanges $B_+ \leftrightarrow B_-$ while preserving the bilinear form and one that exchanges $\delta_{ij} \leftrightarrow \eta_{ij}$ in the reality condition.

R and S are candidates for the first two, as $(0,4)$ and $(2,2)$ are both Weyl-compatible, so the maps work in a similar manner and we will derive the third. S works fine when the reality condition involves $L = \delta$ but for $L = \eta$ it also exchanges B_+ and B_-.

For $L = \delta$ one can show, repeating the same steps as before, $\Lambda^i = S^I{}_J \Phi^J$ and $\Psi^I = S^I{}_J \Xi^J$ (which both have a reality condition involving $L = \delta$.

Now we test when $L = \eta$, anticipating the answer we set $\lambda^I = S^I{}_J \xi^J$ and we see

$$\begin{aligned} (\xi^i_+)^* &= (\lambda^i_+)^* = \alpha_1 B_- \lambda^j_+ \eta_{ji} = \alpha_1 B_- \xi^j_+ \eta_{ji}, \\ (\xi^i_-)^* &= (-\lambda^i_- \varepsilon_{ji})^* = -\alpha_1 B_- \lambda^k_- \eta_{kj} \varepsilon_{ji} = -\alpha_1 B_- \xi^j_-, \eta_{ji} \end{aligned} \tag{5.128}$$

where the second lines arises because $\varepsilon_{lk}\eta_{kj}\varepsilon_{ji} = \eta_{li}$. We observe there is a sign difference between the reality condition of the chiral pieces. This is fixed using $B_\pm \gamma_* = B_\mp$ so that $B_+ \xi^i_\pm = \pm B_- \xi^i_\mp$ and

$$\Longrightarrow (\xi^i_+)^* = \alpha_1 B_- \xi^j_+ \eta_{ji}, \qquad (5.129)$$
$$(\xi^i_-)^* = \alpha_1 B_- \xi^j_- \eta_{ji}.$$

So we see we have obtained the correct description for ξ^i with the correct reality condition (setting $\delta_1 = \alpha_1$) and bilinear form. Using the R map we can map this to ϕ^i.

R behaves the same as it did in $(0,4)$. This will be demonstrated using λ^i and ψ^i that satisfy

$$(\lambda^i)^* = \alpha_1 B_- \lambda^j L_{ji}, \qquad (5.130)$$
$$(\psi^i)^* = \beta_1 B_+ \psi^j L_{ji}, \qquad (5.131)$$

though it also holds for any two reality conditions that differ by swapping B_+ and B_-.

Setting

$$\lambda^i = \frac{1}{\sqrt{2}}(1 - i\gamma_*)\psi^i, \qquad (5.132)$$

we find this implies again that

$$(\Psi^i)^* = -i\alpha B_+ \Psi^j \eta_{ji}. \qquad (5.133)$$

Where this time we observe that $A = \gamma_0\gamma_1$ means that similarly to $(0,4)$, $\gamma_* B_- = B_- \gamma_* = B_+$.

We seek a transformation that only changes $\delta \leftrightarrow \eta$ in the reality condition. Beginning with Ψ^i which, in matrix notation, has the reality condition

$$\begin{pmatrix} \Psi^i_+ \\ \Psi^i_- \end{pmatrix}^* = \alpha_2 B_+ \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} \Psi^j_+ \\ \Psi^j_- \end{pmatrix}. \qquad (5.134)$$

Using $B_\pm\gamma_* = B_\mp$ we can rewrite this as

$$\begin{pmatrix}\Psi^i_+\\ \Psi^i_-\end{pmatrix}^* = \alpha_2 B_- \begin{pmatrix}\delta_{ij} & 0\\ 0 & -\delta_{ij}\end{pmatrix}\begin{pmatrix}\Psi^i_+\\ \Psi^i_-\end{pmatrix}. \tag{5.135}$$

Anticipating the final result again, we write

$$\Psi^I = Q^I{}_J\lambda^J, \qquad Q = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{5.136}$$

We find λ^i has the reality condition (written in matrix notation)

$$\begin{pmatrix}\lambda^i_+\\ \lambda^i_-\end{pmatrix}^* = \alpha_2 B_- \begin{pmatrix}\eta_{ij} & 0\\ 0 & \eta_{ij}\end{pmatrix}\begin{pmatrix}\lambda^i_+\\ \lambda^i_-\end{pmatrix}. \tag{5.137}$$

This is equivalent to that of λ^i in the table provided $\alpha_1 = \alpha_2$.

This transformation maintains the $C_- \otimes \varepsilon$ bilinear form,

$$\begin{aligned}(\gamma^\mu\Psi^i)C_-\Omega^j\varepsilon_{ji} &= (\gamma^\mu\Psi^2_+)C_-\Omega^1_- - (\gamma^\mu\Psi^1_+)C_-\Omega^2_- + (\gamma^\mu\Psi^2_-)C_-\Omega^1_+ - (\gamma^\mu\Psi^1_-)C_-X^2_+\\ &= (\gamma^\mu(-i\lambda^2_+))C_-(i\chi^1_-) - (\gamma^\mu\lambda^1_+)C_-\chi^2_-\\ &\quad + (\gamma^\mu\lambda^2_-)C_-\chi^1_+ - (\gamma^\mu(i\lambda^1_-))C_-(-i\chi^2_+)\\ &= (\gamma^\mu\lambda^i)C_-\chi^j\varepsilon_{ji}.\end{aligned} \tag{5.138}$$

This also applies to the vector-valued bilinear form as there are no transformations on the spinor indices. One can also show that $\Lambda^I = Q^I{}_J\psi^J$.

We have now derived a sufficient number of linear transformations to get between all superalgebras, any remaining can be found by composition, e.g. $\Xi^I = (SQ^{-1}S^{-1})^I{}_J\xi^J$.

Summary for $(2,2)$

The above is summarised in the following commutative diagram; we see all potential real superbrackets are isomorphic with all resulting in an R-symmetry group of $\mathrm{GL}(2,\mathbb{R})$.

$$\begin{tikzcd}
(C_+ \otimes \delta, (\Xi^i)^* = \delta_2 B_+ \Xi^i) \arrow[r, leftrightarrow, "R"] \arrow[d, "S"'] & (C_+ \otimes \delta, (\Phi^i)^* = \gamma_2 B_- \Phi^i) \arrow[d, "S"] \\
(C_- \otimes \varepsilon, (\Psi^i)^* = \beta_2 B_+ \Psi^i) \arrow[r, "R"] & (C_- \otimes \varepsilon, (\Lambda^i)^* = \alpha_2 B_- \Lambda^i) \\
(C_- \otimes \varepsilon, (\psi^i)^* = \beta_1 B_+ \psi^j \eta_{ji}) \arrow[r, leftrightarrow, "R"'] \arrow[ur, "Q"] & (C_- \otimes \varepsilon, (\lambda^i)^* = \alpha_1 B_- \lambda^j \eta_{ji}) \arrow[ul] \\
(C_+ \otimes \delta, (\xi^i)^* = \delta_1 B_+ \xi^j \eta_{ji}) \arrow[r, leftrightarrow, "R"'] \arrow[ur, "S"] & (C_+ \otimes \delta, (\phi^i)^* = \gamma_1 B_- \phi^j \eta_{ji}) \arrow[ul]
\end{tikzcd}$$

5.4 Four-dimensional $\mathcal{N} = 2$ Vector Multiplets and their Lagrangians

The following section will detail $\mathcal{N} = 2$ vector multiplets and their Lagrangians for each four-dimensional signature. These representations of supersymmetry are found by dimensional reduction. This methodology was chosen to emulate the original paper by Cortes and Mohaupt in [20].

This chapter makes use of the work found in Chapter 4 where we derived five-dimensional vector multiplets and Lagrangians in all signatures. It also builds on the standard derivation of four-dimensional multiplets can be found in Section 2.11. This section will summarise the information necessary to construct the Lagrangians.

The theories in this section involve n_V interacting four-dimensional off-shell vector multiplet, which has the field content

$$(X^I, \lambda^{iI}, A^I_\mu, Y^I_{ij}), \quad I = 1, ..., n_V \tag{5.139}$$

where X^I are ϵ-complex scalar field depending on the signature, λ^{iI} are doubled spinors with signature-dependent reality condition, A^I_μ are vector fields and Y^I_{ij} are real SU(2) or SU(1,1)-tensor also depending on signature. Whether the scalar fields are complex or para-complex is a property of the resulting signature, not of the reduction or the starting signature. The pattern alternates, so for t-even we have para-complex scalar fields and t-odd we have complex scalar fields X^I. The conventions of the spinor terms are inherited from the parent theory (up to rewriting quantities in the natural four-dimensional matrices A, B, C as outlined in the previous sections).

The two real fields packaged into the ϵ-complex scalar fields $X^I = \sigma^I + i_\epsilon b^I$ have different origins. σ^I are the real scalar fields inherited from the five-dimensional parent and b^I arise as the dimensionally reduced component of the five-dimensional vector fields.

In five dimensions there are six possible signatures, (t, s) with $t + s = 5$, which have ten different reductions to four-dimensional signatures, (t', s') with $t' + s' = 4$, of which there are five. We find the dimensionally reduced Lagrangians come in four forms (called Type 1, 2, 3 and 4 in the following section) where the coefficients of all terms are the same though the underlying spinor definitions (and those induced onto Y_{ij}) are different due to the different parent theories.

Compared to [20], there are various factors of -1 and i, which arise due to slightly different conventions for dimensionally reducing Clifford algebras and reality conditions employed here. The majority of these are self-explanatory. Additionally, following the Lagrangians there is a short explanation of how to match the results contained here with the standard form in the literature such as in [39].

The five-dimensional coupling matrix $\mathcal{F}_{IJ}$, which was the Hessian of the cubic prepotential $\mathcal{F}(\sigma^I)$, where σ^I are the scalar fields of the five-dimensional theories, gives new coupling matrices in the four-dimensional theories. By extending $\mathcal{F}(\sigma^I)$ to ϵ-complex values $X^I = \sigma^I + i_\epsilon b^I$ we obtain an ϵ-holomorphic prepotential that gives rise to affine special ϵ-Kähler. For t-even the scalar fields live on a special para-Kähler manifold and for t-odd we obtain regular Kähler manifold.

Note that the parameterisation here is in the so-called 'old conventions'. Though it is not done in this thesis, one can change to the 'new conventions' by setting

$$\mathcal{F}^{(new)} = \frac{1}{2i_\epsilon}\mathcal{F}^{(old)}. \tag{5.140}$$

As the Hesse potential is a cubic polynomial, so is any prepotential obtained by dimensional reduction. In four dimensions any ϵ-holomorphic prepotential defines a valid vector multiplet theory provided the scalar and vector coupling matrix $N_{IJ} = Re[F_{IJ}(X)]$

is non-degenerate. In this spirit, though when obtained by dimensional reduction $\mathcal{F}_{IJK}$ is a constant, the Lagrangians will contain $\mathcal{F}_{IJK}$ and $\bar{\mathcal{F}}_{IJK}$ to make the Lagrangians valid for a general ϵ-holomorphic prepotential. Removing the restriction to cubic prepotentials also allows four-fermion terms that contain $\mathcal{F}_{IJKL}$. These will not be included, though they can be derived by considering the supersymmetric variations and the Lagrangian and allowing terms proportional to $\mathcal{F}_{IJKL}$, see the appendix of [20] for further information.

In Euclidean and neutral signature the expressions contain both i and e. In these signatures i arises from the natural complex structure on the spinor module, and e arises from the action of the para-complex tangent bundle of the scalar manifold. In Minkowski signature factors of i can arise from both the spinors and the complex tangent bundle of the scalar manifold.

5.5 Dimensional Reduction

This section goes through the dimensional reduction of each five-dimensional term, demonstrating how the Lagrangians were obtained. In this section μ, ν are the five-dimensional space-time indices and m, n are used for the four-dimensional space-time indices. The dimensionally reduced direction will be called $\#$, which is equal to 0 or 5 depending if the reduction is time-like or space-like respectively.

Recall that the isotropy of the rank-p tensor-valued bilinear form alternates, and the scalar-valued bilinear form is orthogonal i.e.

$$\bar{\lambda}\chi = \bar{\lambda}_+\chi_+ + \bar{\lambda}_-\chi_-, \qquad \bar{\lambda}\gamma^m\chi = \bar{\lambda}_+\gamma^m\chi_- + \bar{\lambda}_-\gamma^m\chi_+, \\ \bar{\lambda}\gamma^{mn}\chi = \bar{\lambda}_+\gamma^{mn}\chi_+ + \bar{\lambda}_-\gamma^{mn}\chi_-. \tag{5.141}$$

We split terms involving bilinears of spinors using these decompositions.

The five-dimensional Lagrangians and supersymmetric variations can be found in 4.8.

5.5.1 Supersymmetry Variations

The vector field A^μ splits into the four-dimensional vector A^m and a scalar field $b = A^\#$. The value of δA^μ is the same in all five-dimensional signatures and so it is in all four-dimensional theories:

$$\delta A^{I\mu} = \frac{1}{2}\bar{\epsilon}\gamma^{\mu}\lambda^{I} \rightarrow \delta A^{Im} = \frac{1}{2}\bar{\epsilon}\gamma^{m}\lambda^{I}, \qquad \delta b^{I} = \delta A^{\#I} = \frac{1}{2}\bar{\epsilon}\gamma^{\#}\lambda^{I}. \tag{5.142}$$

The scalar field is unaffected by dimensional reduction. The generic supersymmetric variation had the form

$$\delta\sigma^{I} = a\bar{\epsilon}\lambda^{I} \tag{5.143}$$

where $a = \frac{1}{2}$ or $\frac{i}{2}$ depending on the signature. This too is unchanged by dimensional reduction. Combining σ^I and b^I into the ϵ-complex scalar field, $X^I = \sigma^I + i_\epsilon b^I$, we find

$$\delta X^{I} = \bar{\epsilon}\big(a + \frac{1}{2}i_{\epsilon}\gamma^{\#}\big)\lambda^{I} = a\bar{\epsilon}\big(1 + \frac{1}{2a}i_{\epsilon}\gamma^{\#}\big)\lambda^{I}. \tag{5.144}$$

This motivates the definition of the chirality matrix $\Gamma_* = \frac{1}{2a}i_\epsilon\gamma^{\#}$ and corresponding projectors $\Gamma_\pm = \frac{1}{2}(1 \pm \Gamma_*)$. $\Gamma_* = e\gamma_*$ for t-even signatures, as defined earlier, but agrees with γ_* when t is odd. From now on all chiral projections are doing using Γ_*, so that $\Gamma_*\lambda_\pm = \pm\lambda_\pm$.

Doing so makes δX entirely chiral:

$$\delta X^{I} = 2a\bar{\epsilon}\left(\frac{1}{2}(1 + \Gamma_*)\right)\lambda^{I} = 2a\bar{\epsilon}_{+}\lambda^{I}_{+}. \tag{5.145}$$

For $\delta\bar{X}^I$, where the bar represents ϵ-complex conjugation, we similarly find

$$\delta\bar{X}^{I} = 2a\bar{\epsilon}_{-}\lambda^{I}_{-}. \tag{5.146}$$

λ^i is unchanged by dimensional reduction, though it can now be split into positive and negative chirality parts $\lambda^i = \lambda^i_+ + \lambda^i_-$ using the projectors $\Gamma_\pm$. The generic form of $\delta\lambda$ variation was

$$\delta\lambda^{Ii} = \beta\gamma\cdot F^{I}\epsilon^{i} + b\not{\partial}\sigma^{I}\epsilon^{i} + uY^{Iij}\epsilon_{j} \tag{5.147}$$

The b here is inherited from the conventions of the five-dimensional chapter and is not to be understood as a scalar field, just a coefficient, with b^I being reserved for scalar fields.

The term proportional to Y^{Iij} is unchanged by the reduction. The $\gamma \cdot F^I$ term reduces according to

$$\gamma \cdot F^I \to \gamma \cdot F^I + 2T\gamma^m\gamma^\# \partial_m b^I = \gamma \cdot F^I - 2aTi_\epsilon \not{\partial} b^I \Gamma_* \tag{5.148}$$

The substitution $\Gamma_* = \Gamma_+ - \Gamma_-$ and some rearrangement was made. $T = +1$ if the reduction is along a space-like direction and $T = -1$ if the reduction was along a time-like direction. The second term is combined with the σ-term to obtain a term in X. This results in

$$\delta\lambda_+^{Ii} = \beta\gamma \cdot F^I \epsilon_+^i + b\not{\partial} X^I \epsilon_-^i + uY^{Iij}\epsilon_{+j}, \tag{5.149}$$

$$\delta\lambda_-^{Ii} = \beta\gamma^{mn} F_{+mn}^I \epsilon_-^i + b\not{\partial}\bar{X}^I \epsilon_+^i + uY^{Iij}\epsilon_{-j}. \tag{5.150}$$

To get the term proportional to ∂X one needs explicit values of a, b, T which always happen to align so that this rewriting is possible (recall that a and b are related by supersymmetry).

Y_{ij}^I, and therefore δY_{ij}^I, is unchanged by the dimensional reduction, though it can now be separated into chiral parts

$$\delta Y_{ij}^I = y\big(\bar{\epsilon}_{+(i}\not{\partial}\lambda_{-j)}^I + \bar{\epsilon}_{-(i}\not{\partial}\lambda_{+j)}^I\big). \tag{5.151}$$

5.5.2 Lagrangian

The kinetic terms for the scalar, spinor and auxiliary field are left unchanged, just removing terms corresponding to the removed dimension:

$$\frac{s_\sigma}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{2}\bar{\lambda}^I\gamma^\mu\partial_\mu\lambda^J + s_Y Y_{ij}^I Y^{ijJ} \qquad \mu = 1,2,3,4,5 \tag{5.152}$$

$$\to \frac{s_\sigma}{2}\partial_m\sigma^I\partial^m\sigma^J - \frac{1}{2}\bar{\lambda}^I\gamma^m\partial_m\lambda^J s_Y Y_{ij}^I Y^{ijJ} \qquad m = 1,2,3,4. \tag{5.153}$$

$s_\lambda = -1$ was filled in as this was chosen in all five-dimensional signatures. Commonly the fermion kinetic term is split into chiral parts:

$$\bar{\lambda}^I\gamma^m\partial_m\lambda^J = \bar{\lambda}_+^I\gamma^m\partial_m\lambda_-^J + \bar{\lambda}_-^I\gamma^m\partial_m\lambda_+^J. \tag{5.154}$$

The vector kinetic term always has the same sign, but the resulting kinetic term for

$b = A^{\#}$ is determined by whether the reduction is space-like or time-like,

$$-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} \to -\frac{1}{4}F^I_{mn}F^{Jmn} - \frac{T}{2}\partial_m b^I \partial^m b^J, \tag{5.155}$$

where again $T = 1$ if reduced along a space-like direction, and $T = -1$ if reduced along a time-like direction. How this interplays with s_σ controls whether the geometry is special Kähler or para-Kähler, and one always finds $T = s_\sigma$ when the number of time-like directions in four-dimensions t' is even and $T = -s_\sigma$ when t' is odd.

By combining σ^I and b^I into a ϵ-complex fields $X^I = \sigma^I + i_\epsilon b^I$ [6] we get the kinetic term for X^I as

$$\frac{s_\sigma}{2}\partial_m X^I \partial^m \bar{X}^J. \tag{5.156}$$

The coupling coefficients are given as the derivatives of the pre-potential $\mathcal{F}(\sigma)$ which is a polynomial in σ^I of maximum degree 3. $\mathcal{F}_{IJ}(\sigma)$ and $\mathcal{F}_{IJK}$ (as $\mathcal{F}(\sigma)$ is cubic this no longer depends on σ) are then defined as

$$\mathcal{F}_{IJ}(\sigma) = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\mathcal{F}(\sigma) \qquad \mathcal{F}_{IJK} = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\frac{\partial}{\partial\sigma^K}\mathcal{F}(\sigma). \tag{5.157}$$

We wish to transform these quantities to depend on X^I; we can expand around σ to find

$$\mathcal{F}_{IJ}(X) = \mathcal{F}_{IJ}(\sigma) + i_\epsilon \mathcal{F}_{IJK} b^K. \tag{5.158}$$

This implies that

$$\bar{\mathcal{F}}_{IJ}(\bar{X}) = \mathcal{F}_{IJ}(\sigma) - i_\epsilon \mathcal{F}_{IJK} b^K, \tag{5.159}$$

which terminates here as $\mathcal{F}_{IJKL} = 0$.

We define $\mathcal{F}_{IJK}(X)$ and $\bar{\mathcal{F}}_{IJK}(\bar{X})$ as the X^K and $\bar{X}^K$ derivatives of $\mathcal{F}_{IJ}(X)$ and

[6] Instead of using para-complex scalar fields (i.e. when $\epsilon = +1$), one can instead work with adapted coordinates, real scalar fields $X^I_\pm = \sigma^I \pm b^I$. This is not done here, but the details and use of these fields can be found in [20].

$\bar{\mathcal{F}}_{IJ}(\bar{X})$ respectively,

$$\mathcal{F}_{IJK}(X) = \frac{\partial}{\partial X^K}\mathcal{F}_{IJ}(X), \qquad \bar{F}_{IJK}(\bar{X}) = \frac{\partial}{\partial \bar{X}^K}\bar{\mathcal{F}}_{IJ}(\bar{X}). \tag{5.160}$$

However we notice that

$$\begin{aligned}\mathcal{F}_{IJK}(X) &= \frac{\partial}{\partial X^K}\mathcal{F}_{IJ}(X) = \frac{\partial}{\partial X^K}(\mathcal{F}_{IJ}(\sigma) + i_\epsilon F_{IJL}b^L) \\ &= \mathcal{F}_{IJL}\frac{\partial \sigma^L}{\partial X^K} + i_\epsilon F_{IJK}\frac{\partial b^L}{\partial X^K} + i_\epsilon b^L \mathcal{F}_{IJLM}\frac{\partial \sigma^M}{\partial X^K} \\ &= \mathcal{F}_{IJL}\frac{\partial}{\partial X^K}(\sigma^L + i_\epsilon b^L) = \mathcal{F}_{IJK}\end{aligned} \tag{5.161}$$

and the same for $\bar{F}_{IJK}(\bar{X}) = \mathcal{F}_{IJK}$. With $\mathcal{F}_{IJK}$ the original 3-σ coefficient of $\mathcal{F}(\sigma)$. However for appearance, the Lagrangians will be written with $\mathcal{F}_{IJK}(X)$ and $\bar{\mathcal{F}}_{IJK}(\bar{X})$ to generalise the Lagrangians to include $\mathcal{F}(X)$ not obtained from dimensional reduction.

The five-dimensional Lagrangians involved $\mathcal{F}_{IJ}$. This is conventionally rewritten as $N_{IJ}(X)$ where $N_{IJ}(X) = Re[\mathcal{F}_{IJ}(X)] = \mathcal{F}_{IJ}(\sigma)$. The kinetic terms for X^I, λ^{iI} and Y^I_{ij} are therefore

$$\left(\frac{s_\sigma}{2}\partial_m X^I \partial^m \bar{X}^J - \frac{1}{2}\bar{\lambda}^I \gamma^m \partial_m \lambda^J s_Y Y^I_{ij} Y^{ijJ}\right) N_{IJ}. \tag{5.162}$$

With that in mind, we turn to the interaction terms, which are going to use the chiral decomposition too.

The first interaction term was chosen to always be of the form

$$\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau}. \tag{5.163}$$

The dimensional reduction of this term results in

$$\frac{1}{4} b^I F^J_{mn} \tilde{F}^{Kmn}, \tag{5.164}$$

with the dual field strength tensor $\tilde{F}^I_{mn} = \frac{1}{2}\varepsilon_{mnpq}F^{Ipq}$. The coefficient is always the same regardless of whether the reduction is over a time-like or space-like direction. This term combines with the kinetic term to allow us to split the kinetic term into self-dual and anti-self dual pieces. Writing

$$F^I_{\pm mn} = \frac{1}{2}\big(F^I_{mn} \pm \frac{1}{i_\epsilon}\tilde{F}^I_{mn}\big) \tag{5.165}$$

we can therefore combine

$$-\frac{1}{4}F^I_{mn}F^{Jmn}N_{IJ} + \frac{1}{4}b^I F^J_{mn}\tilde{F}^{Kmn} = -\frac{1}{4}\big(F^I_{-mn}F^{Jmn}_{-}\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_{+}\bar{\mathcal{F}}_{IJ}(\bar{X})\big) \tag{5.166}$$

The second interaction term was

$$\theta_2\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K\mathcal{F}_{IJK}. \tag{5.167}$$

This reduces to

$$\begin{aligned}&\theta_2\bar{\lambda}^I\gamma^{mn}F^J_{mn}\lambda^K\mathcal{F}_{IJK} + \bar{\lambda}^I\gamma^{m\#}\partial_m b^J\lambda^K\mathcal{F}_{IJK} \\ &= \theta_2\bar{\lambda}^I\gamma^{mn}F^J_{mn}\lambda^K\mathcal{F}_{IJK} + 2\theta_2\bar{\lambda}^I\gamma^m\gamma^\#\partial_m b^J\lambda^K\mathcal{F}_{IJK}.\end{aligned} \tag{5.168}$$

Where # corresponds to the dimensionally reduced direction. The first term is just rewritten in terms of the chiral spinors:

$$\theta_2\bar{\lambda}^I_+\gamma^{mn}F^J_{-mn}\lambda^K_+\mathcal{F}_{IJK} + \theta_2\bar{\lambda}^I_-\gamma^{mn}F^J_{+mn}\lambda^K_-\bar{\mathcal{F}}_{IJK}. \tag{5.169}$$

To write the second term in a totally ϵ-holomorphic form we add an identically zero term, $2i_\epsilon\theta_2\bar{\lambda}^I\not{\partial}\sigma^J\lambda^K\mathcal{F}_{IJK}$, to the second part to get

$$2\theta_2\bar{\lambda}^I_-\not{\partial}\mathcal{F}_{IJ}\lambda^J_+ + 2\theta_2\bar{\lambda}^I_+\not{\partial}\bar{\mathcal{F}}_{IJ}\lambda^J_-. \tag{5.170}$$

Finally, we have the '$\lambda\lambda Y$' interaction term which we just split into chiral pieces

$$\theta_3\bar{\lambda}^{Ii}\lambda^{Jj}Y^K_{ij}\mathcal{F}_{IJK} \to \theta_3\bar{\lambda}^{Ii}_+\lambda^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} + \theta_3\bar{\lambda}^{Ii}_-\lambda^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}. \tag{5.171}$$

5.6 Four-dimensional Lagrangians and Supersymmetry Representations

The supersymmetry variation parameters are doubled spinors denoted ϵ^i that obey the same reality condition as λ^i, which depends on the parent signature and is collected in Table 5.19. In each of these Lagrangians the bilinear form is $C_-\otimes\varepsilon$ and so the $i,j = 1,2$

are raised and lowered using ε^{ij} and ε_{ji}. We use the NW-SE convention such that

$$\lambda^i = \varepsilon^{ij}\lambda_j, \quad \lambda_i = \lambda^j\varepsilon_{ji}, \quad \varepsilon^{ik}\varepsilon_{kj} = -\delta^i{}_j. \tag{5.172}$$

The operation $\bar{\cdot}$ denotes ϵ-complex conjugation for the scalar fields X^I, but Majorana conjugation with C_- for the spinors, $\bar{\lambda}^i = \lambda^{iT}C_-$.

For t-even the Lagrangians are real under simultaneous complex and para-complex conjugation, which acts on both the target space and the spinor module. The scalar fields X^I are para-complex, and the chiral $\lambda^{iI}_\pm$ include the para-complex unit e and the self-dual and anti-self-dual projections of tensors, $F^+_{\mu\nu}$ and $F^-_{\mu\nu}$, are also defined using projections which include a factor e. This is because these fields are vectors on the para-complex target space manifold.

The overall sign of the Lagrangians is fixed so that the kinetic term for the vector field has a negative sign. This choice is so that in Minkowski signature the term gives the correct positive kinetic energy for the vector field. This is not affected by the mostly-plus or mostly-minus convention, see Chapter 4.

The Lagrangians and supersymmetry representations will be provided without comment with discussion to follow.

5.6.1 Type 1: $(1,4) \to (0,4)$, $(1,4) \to (1,3)$ and $(5,0) \to (4,0)$.

Lagrangian

These will be $(\not{1},4)$, $(\not{5},0)$ and $(1,\not{4})$.

$$\begin{aligned} L = &-\frac{1}{4}\big(F^I_{-mn}F^{Jmn}_{-}\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_{+}\bar{\mathcal{F}}_{IJ}(\bar{X})\big) \\ &-\frac{1}{2}\partial_m X^I\partial^m\bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij}Y^J_{ij}N_{IJ}(X,\bar{X}) \\ &-\frac{1}{2}\big(\bar{\lambda}^I_+\not{\partial}\lambda^J_- + \bar{\lambda}^I_-\not{\partial}\lambda^J_+\big)N_{IJ}(X,\bar{X}) \\ &-\frac{1}{4}\big(\bar{\lambda}^I_-\not{\partial}\mathcal{F}_{IJ}(X)\lambda^J_+ + \bar{\lambda}^I_+\not{\partial}\bar{\mathcal{F}}_{IJ}(\bar{X})\lambda^J_-\big) \\ &-\frac{i}{8}\big(\bar{\lambda}^I_+\gamma^{mn}F^J_{-mn}\lambda^K_+\mathcal{F}_{IJK} + \bar{\lambda}^I_-\gamma^{mn}F^J_{+mn}\lambda^K_-\bar{\mathcal{F}}_{IJK}\big) \\ &-\frac{i}{2}\big(\bar{\lambda}^{Ii}_+\lambda^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} + \bar{\lambda}^{Ii}_-\lambda^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\big) \end{aligned} \tag{5.173}$$

$(\not{1},4)$, $(\not{5},0)$ have $X^I = \sigma^I + eb^I$ ergo a para-complex scalar field, composed of the 5D σ^I and the reduced component of the vector field $(A^I)^0 = b^I$. For $(1,\not{4})$ it is a complex scalar field - $X^I = \sigma^I + ib^I$.

This is identical to that found in [20], except for the term $\bar{\lambda}_-^{Ii}\lambda_-^{Jj}Y_{ij}^K\mathcal{F}_{IJK}$ which had an erroneous $\bar{Y}_{ij}^K$ in place of Y_{ij}^K. In Section 5.7 it is demonstrated how to use the map S to rewrite this Lagrangian in terms of Majorana spinors so to align with the common form in the literature.

Representation

$$\begin{aligned}
\delta X^I &= i\bar{\epsilon}_+\lambda_+^I \qquad \delta\bar{X}^I = i\bar{\epsilon}_-\lambda_-^I \\
\delta A_m^I &= \frac{1}{2}\left(\bar{\epsilon}_+\gamma_m\lambda_-^I + \bar{\epsilon}_-\gamma_m\lambda_+^I\right) \\
\delta Y_{ij}^I &= -\frac{1}{2}\left(\bar{\epsilon}_{+(i}\not{\partial}\lambda_{-j)}^I + \bar{\epsilon}_{-(i}\not{\partial}\lambda_{+j)}^I\right) \\
\delta\lambda_+^{Ii} &= -\frac{1}{4}\gamma^{mn}F_{-mn}^I\epsilon_+^i - \frac{i}{2}\not{\partial}X^I\epsilon_-^i - Y^{Iij}\epsilon_{+j} \\
\delta\lambda_-^{Ii} &= -\frac{1}{4}\gamma^{mn}F_{+mn}^I\epsilon_-^i - \frac{i}{2}\not{\partial}\bar{X}^I\epsilon_+^i - Y^{Iij}\epsilon_{-j}
\end{aligned} \tag{5.174}$$

5.6.2 Type 2: $(0,5) \to (0,4)$, $(4,1) \to (3,1)$ and $(4,1) \to (4,0)$.

Lagrangian

The theories include $(\not{4},1)$, $(0,\not{5})$ and $(4,\not{1})$.

$$\begin{aligned}
L = &-\frac{1}{4}\left(F_{-mn}^I F_-^{Jmn}\mathcal{F}_{IJ}(X) + F_{+mn}^I F_+^{Jmn}\bar{\mathcal{F}}_{IJ}(\bar{X})\right) \\
&+ \frac{1}{2}\partial_m X^I\partial^m\bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij}Y_{ij}^J N_{IJ}(X,\bar{X}) \\
&- \frac{1}{2}\left(\bar{\lambda}_+^I\not{\partial}\lambda_-^J + \bar{\lambda}_-^I\not{\partial}\lambda_+^J\right)N_{IJ}(X,\bar{X}) \\
&- \frac{1}{4}\left(\bar{\lambda}_-^I\not{\partial}\mathcal{F}_{IJ}(X)\lambda_+^J + \bar{\lambda}_+^I\not{\partial}\bar{\mathcal{F}}_{IJ}(\bar{X})\lambda_-^J\right) \\
&- \frac{1}{8}\left(\bar{\lambda}_+^I\gamma^{mn}F_{-mn}^J\lambda_+^K\mathcal{F}_{IJK} + \bar{\lambda}_-^I\gamma^{mn}F_{+mn}^J\lambda_-^K\bar{\mathcal{F}}_{IJK}\right) \\
&- \frac{1}{2}\left(\bar{\lambda}_+^{Ii}\lambda_+^{Jj}Y_{ij}^K\mathcal{F}_{IJK} + \bar{\lambda}_-^{Ii}\lambda_-^{Jj}Y_{ij}^K\bar{\mathcal{F}}_{IJK}\right)
\end{aligned} \tag{5.175}$$

For $(0,\not{5})$ and $(4,\not{1})$ X^I is a para-complex scalar field, and for $(\not{4},1)$ it is a complex scalar field.

It should be noted that some of the theories obtained from the reduction of Type 1 and Type 2 theories have the same signature and a different form, but may or may not be isomorphic. These Lagrangians are isomorphic in (0,4) and (4,0) but not in (1,3) and (3,1).

Representation

$$
\begin{aligned}
&\delta X^I = \bar\epsilon_+ \lambda_+^I \qquad \delta \bar X^I = \bar\epsilon_- \lambda_-^I, \quad \delta A_m^I = \frac{1}{2}\big(\bar\epsilon_+ \gamma_m \lambda_-^I + \bar\epsilon_- \gamma_m \lambda_+^I\big) \\
&\delta Y_{ij}^I = -\frac{1}{2}\big(\bar\epsilon_{+(i} \partial\!\!\!/ \lambda_{-j)}^I + \bar\epsilon_{-(i} \partial\!\!\!/ \lambda_{+j)}^I\big) \\
&\delta \lambda_+^{Ii} = -\frac{1}{4}\gamma^{mn} F_{-mn}^I \epsilon_+^i + \frac{1}{2}\partial\!\!\!/ X^I \epsilon_-^i - Y^{Iij}\epsilon_{+j} \\
&\delta \lambda_-^{Ii} = -\frac{1}{4}\gamma^{mn} F_{+mn}^I \epsilon_-^i + \frac{1}{2}\partial\!\!\!/ \bar X^I \epsilon_+^i - Y^{Iij}\epsilon_{-j}
\end{aligned}
\tag{5.176}
$$

5.6.3 Type 3: $(2,3) \to (1,3)$ and $(2,3) \to (2,2)$.

Lagrangian

$(\not{2},3)$ and $(2,\not{3})$. The scalar fields X^I are complex for $(\not{2},3)$, and para-complex for $(2,\not{3})$

$$
\begin{aligned}
L = &-\frac{1}{4}\big(F_{-mn}^I F_-^{Jmn} \mathcal{F}_{IJ}(X) + F_{+mn}^I F_+^{Jmn} \bar{\mathcal{F}}_{IJ}(\bar X)\big) \\
&+ \frac{1}{2}\partial_m X^I \partial^m \bar X^J N_{IJ}(X,\bar X) - Y^{Iij} Y_{ij}^J N_{IJ}(X,\bar X) \\
&- \frac{1}{2}\big(\bar\lambda_+^I \partial\!\!\!/ \lambda_-^J + \bar\lambda_-^I \partial\!\!\!/ \lambda_+^J\big) N_{IJ}(X,\bar X) \\
&- \frac{1}{4}\big(\bar\lambda_-^I \partial\!\!\!/ \mathcal{F}_{IJ}(X) \lambda_+^J + \bar\lambda_+^I \partial\!\!\!/ \bar{\mathcal{F}}_{IJ}(\bar X)\lambda_-^J\big) \\
&- \frac{1}{8}\big(\bar\lambda_+^I \gamma^{mn} F_{-mn}^J \lambda_+^K \mathcal{F}_{IJK} + \bar\lambda_-^I \gamma^{mn} F_{+mn}^J \lambda_-^K \bar{\mathcal{F}}_{IJK}\big) \\
&+ \frac{i}{2}\big(\bar\lambda_+^{Ii} \lambda_+^{Jj} Y_{ij}^K \mathcal{F}_{IJK} + \bar\lambda_-^{Ii} \lambda_-^{Jj} Y_{ij}^K \bar{\mathcal{F}}_{IJK}\big)
\end{aligned}
\tag{5.177}
$$

Representation

$$
\begin{aligned}
\delta X^I &= \bar{\epsilon}_+\lambda_+^I \qquad \delta\bar{X}^I = \bar{\epsilon}_-\lambda_-^I, \delta A_m^I = \frac{1}{2}\big(\bar{\epsilon}_+\gamma_m\lambda_-^I + \bar{\epsilon}_-\gamma_m\lambda_+^I\big)\\
\delta Y_{ij}^I &= -\frac{i}{2}\big(\bar{\epsilon}_{+(i}\not{\partial}\lambda_{-j)}^I + \bar{\epsilon}_{-(i}\not{\partial}\lambda_{+j)}^I\big)\\
\delta\lambda_+^{Ii} &= -\frac{1}{4}\gamma^{mn}F_{-mn}^I\epsilon_+^i + \frac{1}{2}\not{\partial}X^I\epsilon_-^i + iY^{Iij}\epsilon_{+j}\\
\delta\lambda_-^{Ii} &= -\frac{1}{4}\gamma^{mn}F_{+mn}^I\epsilon_-^i + \frac{1}{2}\not{\partial}\bar{X}^I\epsilon_+^i + iY^{Iij}\epsilon_{-j}
\end{aligned}
\tag{5.178}
$$

5.6.4 Type 4: $(3,2)\to(2,2)$ and $(3,2)\to(3,1)$.

$(\cancel{3},2)$ has para-complex scalar fields, and $(3,\cancel{2})$ has complex scalar fields.

Lagrangian

$$
\begin{aligned}
L = &-\frac{1}{4}\big(F_{-mn}^I F_-^{Jmn}\mathcal{F}_{IJ}(X) + F_{+mn}^I F_+^{Jmn}\bar{\mathcal{F}}_{IJ}(\bar{X})\big)\\
&-\frac{1}{2}\partial_m X^I\partial^m\bar{X}^J N_{IJ}(X,\bar{X}) - Y^{Iij}Y_{ij}^J N_{IJ}(X,\bar{X})\\
&-\frac{1}{2}\big(\bar{\lambda}_+^I\not{\partial}\lambda_-^J + \bar{\lambda}_-^I\not{\partial}\lambda_+^J\big)N_{IJ}(X,\bar{X})\\
&-\frac{1}{4}\big(\bar{\lambda}_-^I\not{\partial}\mathcal{F}_{IJ}(X)\lambda_+^J + \bar{\lambda}_+^I\not{\partial}\bar{\mathcal{F}}_{IJ}(\bar{X})\lambda_-^J\big)\\
&-\frac{i}{8}\big(\bar{\lambda}_+^I\gamma^{mn}F_{-mn}^J\lambda_+^K\mathcal{F}_{IJK} + \bar{\lambda}_-^I\gamma^{mn}F_{+mn}^J\lambda_-^K\bar{\mathcal{F}}_{IJK}\big)\\
&-\frac{1}{2}\big(\bar{\lambda}_+^{Ii}\lambda_+^{Jj}Y_{ij}^K\mathcal{F}_{IJK} + \bar{\lambda}_-^{Ii}\lambda_-^{Jj}Y_{ij}^K\bar{\mathcal{F}}_{IJK}\big)
\end{aligned}
\tag{5.179}
$$

Representation

$$
\begin{aligned}
\delta X^I &= i\bar{\epsilon}_+\lambda_+^I \qquad \delta\bar{X}^I = i\bar{\epsilon}_-\lambda_-^I, \quad \delta A_m^I = \frac{1}{2}\big(\bar{\epsilon}_+\gamma_m\lambda_-^I + \bar{\epsilon}_-\gamma_m\lambda_+^I\big)\\
\delta Y_{ij}^I &= -\frac{i}{2}\big(\bar{\epsilon}_{+(i}\not{\partial}\lambda_{-j)}^I + \bar{\epsilon}_{-(i}\not{\partial}\lambda_{+j)}^I\big)\\
\delta\lambda_+^{Ii} &= -\frac{1}{4}\gamma^{mn}F_{-mn}^I\epsilon_+^i - \frac{i}{2}\not{\partial}X^I\epsilon_-^i + iY^{Iij}\epsilon_{+j}\\
\delta\lambda_-^{Ii} &= -\frac{1}{4}\gamma^{mn}F_{+mn}^I\epsilon_-^i - \frac{i}{2}\not{\partial}\bar{X}^I\epsilon_+^i + iY^{Iij}\epsilon_{-j}
\end{aligned}
\tag{5.180}
$$

5.6.5 Discussion

The presented Lagrangians and supersymmetry representations are highly similar, varying only up to factors of ±1 and i. These are due to the definitions of the doubled spinor module, more precisely the signature-dependent reality conditions that change the reality properties of various terms.

The relative signs between the kinetic terms of the scalars $\sigma^I = Re[X^I]$ and $b^I = Im[X^I]$ has already been outlined, and it controls whether the target geometry is complex or para-complex. This aligns with the Abelian factor of the R-symmetry group [20], which were $\mathrm{U}^*(2) \cong \mathrm{SO}(1,1) \times \mathrm{SU}(2)$ in $(0,4)$, $\mathrm{U}(2) \cong \mathrm{U}(1) \cdot \mathrm{SU}(2)$ in $(1,3)$ and $\mathrm{GL}(2,\mathbb{R}) \cong \mathrm{SO}(1,1) \cdot \mathrm{SL}(2,\mathbb{R})$ in $(2,2)$. When the Abelian factor is $\mathrm{U}(1)$ the target space geometry is complex and when it is $\mathrm{SO}(1,1)$ it is para-complex.

The relative sign between the kinetic terms of the various fields depends on the parent signature. In five dimensions the difference between the scalar and vector kinetic terms is a feature mandated by supersymmetry, but in four dimensions this is not always the case. In $(0,4)$ and $(2,2)$ the sign is arbitrary and can be changed through field redefinitions as shown below, arising from the fact there is a single unique $\mathcal{N} = 2$ superalgebra in each signature but multiple possible doubled spinor formulations that lead to a different sign for these terms. In $(1,3)$ the sign is linked to the R-symmetry group (which is linked to the signature of the parent theory). With $\mathrm{U}(2)$ R-symmetry we get the canonical sign attributions, but with $\mathrm{U}(1,1)$ R-symmetry (obtained from the reduction of a $(2,3)$ theory) we get 'ghost' scalar fields, X^I, with negative-definite kinetic energy.

From the classification of $\mathcal{N} = 2$ Poincaré Lie superalgebras in Section 2.8 or equivalently our knowledge of the R-symmetry groups arising from the doubled spinor formulation, we can identify the theories that should be equivalent and provide maps between them. In Section 5.3, we found maps that relate the doubled spinor superalgebras, which will we implement on the Lagrangians and supersymmetry variations. However, they do not often align perfectly and we find that a reparameterisation of the scalar fields is also necessary in $(0,4)$ and $(2,2)$.

5.7 Maps between Equivalent Theories

Previously in Section 5.3 we derived a transformation R that exchanges B_+ and B_- in the reality conditions of the two $(0,4)$ and $(2,2)$ theories together. In this section λ^i will be the spinors with reality condition

$$(\lambda^i)^* = \alpha B_- \lambda^j L_{ji} \tag{5.181}$$

and $\tilde{\lambda}^i$ those in the second theory whose reality condition is

$$(\tilde{\lambda}^i)^* = \beta B_+ \tilde{\lambda}^j L_{ji}. \tag{5.182}$$

$L = \varepsilon$ in $(0,4)$ and η in $(2,2)$. Though other possible reality conditions exist in $(2,2)$ we focus only on those obtained by dimensional reduction.

Under R, λ and $\tilde{\lambda}$ were originally related by

$$\lambda^i \to \frac{1}{\sqrt{2}}(1 - i\gamma_*)\tilde{\lambda}^i. \tag{5.183}$$

However, the vector multiplet theories used $\Gamma_* = e\gamma_*$, so that on the Lagrangian and supersymmetry representations we should implement

$$\lambda^i \to \frac{1}{\sqrt{2}}(1 - ie\Gamma_*)\tilde{\lambda}^i. \tag{5.184}$$

The chiral projections are therefore related by

$$\lambda^i_\pm = \frac{1}{\sqrt{2}}(1 \mp ie)\tilde{\lambda}^i_\pm. \tag{5.185}$$

Let us record the changes this induces in the various bilinears (in all cases the i,j indices are suppressed as they are closed). The scalar bilinear transforms as

$$\begin{aligned}\bar{\epsilon}\lambda \to& \frac{1}{2}\tilde{\epsilon}^T(1 - ie\Gamma_*)^T C_-(1 - ie\Gamma_*)\tilde{\lambda} \\ =& \frac{1}{2}\tilde{\epsilon}^T C_-(1 + (ie)^2 - 2ie\Gamma_*)\tilde{\lambda} \qquad (5.186)\\ =& -ie\bar{\tilde{\epsilon}}\Gamma_*\tilde{\lambda} \\ \implies& \bar{\tilde{\epsilon}}_\pm\lambda_\pm = \mp ie\bar{\tilde{\epsilon}}_\pm\tilde{\lambda}_\pm. \qquad (5.187)\end{aligned}$$

The vector-valued bilinear form is unchanged by this transformation

$$\begin{aligned}\bar{\epsilon}\gamma^\mu\lambda \rightarrow & \frac{1}{2}\tilde{\epsilon}^T(1-ie\Gamma_*)^T C_-\gamma^\mu(1-ie\Gamma_*)\tilde{\lambda}\\ =&\frac{1}{2}\tilde{\epsilon}^T C_-(1-(ie)^2-ie\Gamma_*+ie\Gamma_*)\gamma^\mu\tilde{\lambda}\\ =&\bar{\tilde{\epsilon}}\gamma^\mu\tilde{\lambda}\end{aligned} \tag{5.188}$$

$$\implies \bar{\tilde{\epsilon}}_\pm\gamma^\mu\lambda_\mp = \bar{\tilde{\epsilon}}_\pm\gamma^\mu\tilde{\lambda}_\mp. \tag{5.189}$$

Finally, we have the bilinear with $\gamma^{\mu\nu}$ inserted is

$$\begin{aligned}\bar{\epsilon}\gamma^{\mu\nu}\lambda \rightarrow & \frac{1}{2}\tilde{\epsilon}^T(1-ie\Gamma_*)^T C_-(1-ie\Gamma_*)\gamma^{\mu\nu}\tilde{\lambda}\\ =&\frac{1}{2}\tilde{\epsilon}^T C_-(1+(ie)^2-2ie\Gamma_*)\gamma^{\mu\nu}\tilde{\lambda}\\ =&-ie\bar{\tilde{\epsilon}}\gamma^{\mu\nu}\Gamma_*\tilde{\lambda}\end{aligned} \tag{5.190}$$

$$\implies \bar{\tilde{\epsilon}}_\pm\gamma^{\mu\nu}\lambda_\pm = \mp ie\bar{\tilde{\epsilon}}_\pm\gamma^{\mu\nu}\tilde{\lambda}_\pm. \tag{5.191}$$

The supersymmetric variation of the transformed spinors $\delta\tilde{\lambda}^i_\pm$ is

$$\delta\lambda^{Ii}_+ = -\frac{1}{4}\gamma^{mn}F^I_{-mn}\epsilon^i_+ + \frac{1}{2}\not{\partial}X^I\epsilon^i_- - Y^{Iij}\epsilon_{+j} \tag{5.192}$$

$$\rightarrow = \frac{1}{\sqrt{2}}\big(-\frac{1}{4}\gamma^{mn}F^I_{-mn}(1-ie)\tilde{\epsilon}^i_+ + \frac{1}{2}\not{\partial}X^I(1+ie)\tilde{\epsilon}^i_- - (1-ie)Y^{Iij}\tilde{\epsilon}_{+j}\big).$$

This should be equal to $\frac{1}{\sqrt{2}}(1-ie)\delta\tilde{\lambda}^i_+$. Note that $1+ie = ie(1-ie)$ so we can write

$$= \frac{1}{\sqrt{2}}(1-ie)\big(-\frac{1}{4}\gamma^{mn}F^I_{-mn}\tilde{\epsilon}^i_+ + ie\frac{1}{2}\not{\partial}X^I\tilde{\epsilon}^i_- - Y^{Iij}\tilde{\epsilon}_{+j}\big)$$

Therefore the transformed supersymmetry variation is

$$\delta\tilde{\lambda}^i_+ = -\frac{1}{4}\gamma^{mn}F^I_{-mn}\tilde{\epsilon}^i_+ + ie\frac{1}{2}\not{\partial}X^I\tilde{\epsilon}^i_- - Y^{Iij}\tilde{\epsilon}_{+j}, \tag{5.193}$$

and similarly for $\delta\lambda^i_-$ we find

$$\delta\tilde{\lambda}^{Ii}_- = -\frac{1}{4}\gamma^{mn}F^I_{+mn}\tilde{\epsilon}^i_- - ie\frac{1}{2}\not{\partial}\bar{X}^I\tilde{\epsilon}^i_+ - Y^{Iij}\tilde{\epsilon}_{-j}.$$

Explicit mapping in (0,4)

Using these calculated quantities, we can then relate the two different theories obtained in each signature. As the calculations are mostly the same only $(0,4)$ will be done in-depth, though the features of the transformation for $(2,2)$ are almost identical up to coefficient differences which amount to factors of -1 and i on various terms.

We have two possible parent theories, (1,4) reduced along a time-like direction (for shorthand this is called $(\not{1},4)$) and (0,5) reduced along a space-like direction (similarly called $(0,\not{5})$).

We begin with the $(0,\not{5})$ theory and apply S to obtain

$$\begin{aligned}
L = &-\frac{1}{4}\big(F^I_{-mn}F^{Jmn}_{-}\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_{+}\bar{\mathcal{F}}_{IJ}(\bar{X})\big)\\
&+\frac{1}{2}\partial_m X^I\partial^m\bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij}Y^J_{ij}N_{IJ}(X,\bar{X})\\
&-\frac{1}{2}\big(\bar{\tilde{\lambda}}^I_+\not{\partial}\tilde{\lambda}^J_- + \bar{\tilde{\lambda}}^I_-\not{\partial}\tilde{\lambda}^J_+\big)N_{IJ}(X,\bar{X})\\
&-\frac{1}{4}\big(\bar{\tilde{\lambda}}^I_-\not{\partial}\mathcal{F}_{IJ}(X)\tilde{\lambda}^J_+ + \bar{\tilde{\lambda}}^I_+\not{\partial}\bar{\mathcal{F}}_{IJ}(\bar{X})\tilde{\lambda}^J_-\big)\\
&-\frac{1}{8}\big(-\boldsymbol{ie}\bar{\tilde{\lambda}}^I_+\gamma^{mn}F^J_{-mn}\tilde{\lambda}^K_+\mathcal{F}_{IJK} + \boldsymbol{ie}\bar{\tilde{\lambda}}^I_-\gamma^{mn}F^J_{+mn}\tilde{\lambda}^K_-\bar{\mathcal{F}}_{IJK}\big)\\
&-\frac{1}{2}\big(-\boldsymbol{ie}\bar{\tilde{\lambda}}^{Ii}_+\tilde{\lambda}^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} - \boldsymbol{ie}\bar{\tilde{\lambda}}^{Ii}_-\tilde{\lambda}^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\big).
\end{aligned}\tag{5.194}$$

S transforms the associated supersymmetric variations into

$$\begin{aligned}
\delta X^I &= -\boldsymbol{ie}\bar{\epsilon}_+\tilde{\lambda}^I_+ \qquad \delta\bar{X}^I = \boldsymbol{ie}\bar{\epsilon}_-\tilde{\lambda}^I_-\\
\delta A^I_m &= \frac{1}{2}\big(\bar{\epsilon}_+\gamma_m\tilde{\lambda}^I_- + \bar{\epsilon}_-\gamma_m\tilde{\lambda}^I_+\big)\\
\delta Y^I_{ij} &= -\frac{1}{2}\big(\bar{\epsilon}_{+(i}\not{\partial}\tilde{\lambda}^I_{-j)} + \bar{\epsilon}_{-(i}\not{\partial}\tilde{\lambda}^I_{+j)}\big)\\
\delta\tilde{\lambda}^{Ii}_+ &= -\frac{1}{4}\gamma^{mn}F^I_{-mn}\epsilon^i_+ + \boldsymbol{ie}\frac{1}{2}\not{\partial}X^I\epsilon^i_- - Y^{Iij}\epsilon_{+j}\\
\delta\tilde{\lambda}^{Ii}_- &= -\frac{1}{4}\gamma^{mn}F^I_{+mn}\epsilon^i_- - \boldsymbol{ie}\frac{1}{2}\not{\partial}\bar{X}^I\epsilon^i_+ - Y^{Iij}\epsilon_{-j}
\end{aligned}\tag{5.195}$$

The Lagrangian and supersymmetry representation do not yet look like those in the $(\not{1},4)$ signature theory, though now the spinor bilinear and reality condition are the same. The scalar kinetic terms have different signs, and two interaction terms differ by factors of $\mp e$. In the $(\not{1},4)$ theory we had

$$
\begin{aligned}
&-\frac{1}{8}\big(-ie\bar{\tilde{\lambda}}^I_+\gamma^{mn}F^J_{-mn}\tilde{\lambda}^K_+\mathcal{F}_{IJK}+ie\bar{\tilde{\lambda}}^I_-\gamma^{mn}F^J_{+mn}\tilde{\lambda}^K_-\bar{\mathcal{F}}_{IJK}\big)\\
&-\frac{1}{2}\big(-ie\bar{\tilde{\lambda}}^{Ii}_+\tilde{\lambda}^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK}+ie\bar{\tilde{\lambda}}^{Ii}_-\tilde{\lambda}^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\big).
\end{aligned}
$$

Additionally, the variations for δX^I are not in the same form as the Type 1 variations, they are

$$
\delta X^I=i\bar{\epsilon}_+\tilde{\lambda}^I_+,\qquad \delta\bar{X}^I=i\bar{\epsilon}_-\tilde{\lambda}^I_-
$$

and the $\delta\lambda^i$ terms are also not in the same form, they are supposed to be

$$
\begin{aligned}
\delta\tilde{\lambda}^{Ii}_+&=-\frac{1}{4}\gamma^{mn}F^I_{-mn}\epsilon^i_+-\frac{i}{2}\not{\partial}X^I\epsilon^i_--Y^{Iij}\epsilon_{+j},\\
\delta\tilde{\lambda}^{Ii}_-&=-\frac{1}{4}\gamma^{mn}F^I_{+mn}\epsilon^i_--\frac{i}{2}\not{\partial}\bar{X}^I\epsilon^i_+-Y^{Iij}\epsilon_{-j}.
\end{aligned}
$$

One can see that the terms involving X^I can be fixed by rewriting in terms of $\tilde{X}^I=-eX^I$. This corrects the sign of the kinetic term,

$$
+\frac{1}{2}\partial_mX^I\partial^m\bar{X}^JN_{IJ}\to-\frac{1}{2}\partial_mX^I\partial^m\bar{X}^JN_{IJ}, \tag{5.196}
$$

and fixes the δX and $\delta\tilde{\lambda}^i$ supersymmetry variations.

The prepotential is a function of X^I, so we need to calculate the effect this has on it and its derivatives. The prepotential is a paraholomorphic function and it transforms as a scalar, so that $\tilde{\mathcal{F}}(\tilde{X})=\mathcal{F}(X)$. The Jacobian is

$$
\frac{\partial\tilde{X}^I}{\partial X^J}=-e\delta^I{}_J. \tag{5.197}
$$

The derivatives, therefore, transform as

$$
\begin{aligned}
&\tilde{\mathcal{F}}_I=-e\mathcal{F}_I,\quad \tilde{\mathcal{F}}_{IJ}=\mathcal{F}_{IJ},\quad \tilde{\mathcal{F}}_{IJK}=-e\mathcal{F}_{IJK}\\
&\bar{\tilde{\mathcal{F}}}_I=-e\bar{\mathcal{F}}_I,\quad \bar{\tilde{\mathcal{F}}}_{IJ}=\bar{\mathcal{F}}_{IJ},\quad \bar{\tilde{\mathcal{F}}}_{IJK}=-e\bar{\mathcal{F}}_{IJK}.
\end{aligned}\tag{5.198}
$$

One can see that this corrects all differences between the two Lagrangian and therefore the two Lagrangian obtained from dimensional reduction, written using paracomplex scalar fields are related by using the following maps

$$\lambda^i \to \frac{1}{\sqrt{2}}(1 - ie\Gamma_*)\tilde{\lambda}^i, \qquad \begin{pmatrix} X^I \\ \mathcal{F}_I \end{pmatrix} \to \begin{pmatrix} -eX^I \\ -e\mathcal{F}_I \end{pmatrix}. \tag{5.199}$$

Additionally in the superconformal formulation for supergravity, the Einstein-Hilbert term has the prefactor $-e(X^I\bar{\mathcal{F}}_I - \mathcal{F}_I\bar{X}^I)$. Under the reparameterisation $\tilde{X}^I = -eX^I$ this changes sign, meaning this provides an exact matching of signs to that in [26]. The full treatment of arbitrary signature $\mathcal{N} = 2$ supergravity is future work that the author wishes to pursue one day.

This transformation is superficially similar to that in [25, 54] that acted on the symplectic vector $(X^I, \mathcal{F}_I)$ by e by making a strong-weak duality-like transformation of the field equations..

The duality transformation flips the sign of the vector kinetic term and leaves the sign of the scalar field. Though the transformations differ only by an overall sign, their transformation is non-local and does not include the fermionic terms. The transformation presented here is local and was found as a corollary of an isomorphism between two Euclidean $\mathcal{N} = 2$ superalgebras.

5.7.1 Minkowski Signature in terms of Majorana Spinors

The standard literature ala Van Proeyen/de Wit on $N = 2$ vector multiplets employs the chiral formulation, where chirality is also encoded in the i, j indices. This provides a few differences with our notation. This section will not get to a total matching of the relevant terms, but will provide sufficient detail to show they can be in the same form.

In our formalism an isotropic vector-valued bilinear form $C_+ \otimes \delta$ is written

$$(\gamma^\mu \lambda^i_+)^T C_+ \chi^j_- \delta_{ji} + (\gamma^\mu \lambda^i_-)^T C_+ \chi^j_+ \delta_{ji} \tag{5.200}$$

and in the chiral formulation this is

$$(\gamma^\mu \lambda^i)^T C_+ \chi_i + (\gamma^\mu \lambda^i)^T C_+ \chi_i. \tag{5.201}$$

The standard Majorana spinor formulation corresponds to the doubled spinor description with $C_+ \otimes \delta$ as the bilinear form with a standard Majorana reality condition,

$(\Psi^i)^* = \alpha B_+ \Psi^i$, as expected, though this is slightly hidden. The rank-2 tensor-valued bilinear form is antisymmetric

$$(\Psi^i)^T C_+ \gamma^{\mu\nu} \Psi^j = -(\Psi^j)^T C_+ \gamma^{\mu\nu} \Psi^i \tag{5.202}$$

so that using δ_{ij} to close the indices results in the term vanishing. Therefore we are forced to use

$$(\Psi^i)^T C_+ \gamma^{\mu\nu} \Psi^j \varepsilon_{ji} \tag{5.203}$$

as seen in the literature. This means we cannot have the same bilinear form for all the possible tensor-valued bilinear forms if we wish to use $C_+ \otimes \delta$ to define the superbracket. In this way, it makes the $C_- \otimes \varepsilon$ description 'more natural' as it allows one to consider the same $\mathbb{C}^2$ bilinear form for each possible tensor-valued bilinear form.

We can rewrite the Lagrangian in Section 5.6, which is in terms of symplectic Majorana spinors, in terms of Majorana spinors using the map S. For reference this was

$$\begin{aligned} \lambda^i_+ &= \Psi^i_+ \\ \lambda^i_- &= \Psi^i_- \varepsilon_{ji}. \end{aligned} \tag{5.204}$$

λ^i are symplectic Majorana spinors with $(\lambda^i)^* = \alpha B_- \lambda^j \varepsilon_{ji}$ and Ψ^i are Majorana spinors with $(\Psi^i)^* = \alpha B_+ \Psi^i$.

The vector-valued bilinear form in our formalism is

$$[C_- \otimes \varepsilon](\lambda, \gamma^\mu \chi) = \bar{\lambda}^i_+ \gamma^\mu \chi^j_- \varepsilon_{ji} + \bar{\lambda}^i_- \gamma^\mu \chi^j_+ \varepsilon_{ji}. \tag{5.205}$$

This is mapped to

$$-[C_- \otimes \varepsilon](\lambda, \gamma^\mu \chi) = -\bar{\Psi}^i_+ \gamma^\mu \Omega^j_- \delta_{ji} - \bar{\Psi}^i_- \gamma^\mu \Omega^j_+ \delta_{ji} \tag{5.206}$$

by S. Note the additional minus sign picked up, which arises because to transform C_- to C_+ we must commute γ_* through the γ^μ first. On the kinetic term (which has $\chi = \partial_\mu \lambda$, $\Omega = \partial_\mu \Psi$) we get a sign difference that matches [39].

S transforms the vector-valued bilinear forms into one another, but it does not work like this on the scalar- and tensor-valued bilinear forms. We observe

$$\begin{aligned}
[C_- \otimes \varepsilon](\lambda, \chi) &= (\lambda_+^i)^T C_- \chi_-^j \varepsilon_{ji} + (\lambda_-^i)^T C_- \chi_+^j \varepsilon_{ji} \\
&= (\lambda_+^i)^T C_+ \chi_-^j \varepsilon_{ji} - (\lambda_-^i)^T C_+ \chi_+^j \varepsilon_{ji} \\
&= (\Psi_+^i)^T C_+ \Omega_+^j \varepsilon_{ji} - (\Psi_-^k)^T C_+ \Omega_-^l \varepsilon_{ki} \varepsilon_{lj} \varepsilon_{ji} \\
&= (\Psi_+^i)^T C_+ \Omega_+^j \varepsilon_{ji} - (\Psi_-^i)^T C_+ \Omega_-^j \varepsilon_{ji} \\
&\neq [C_+ \otimes \delta](\Psi, \Omega)
\end{aligned} \tag{5.207}$$

The same holds for $[C_- \otimes \varepsilon](\gamma^{\mu\nu}\lambda, \chi)$ too. Therefore the $\lambda\lambda Y$ term transforms as

$$\begin{aligned}
&(\lambda_+^{iI})^T C_- \lambda_+^{jJ} Y_{ij}^K \mathcal{F}_{IJK} + (\lambda_-^{iI})^T C_- \lambda_-^{jJ} Y_{ij}^K \bar{\mathcal{F}}_{IJK} \\
=&(\Psi_+^{iI})^T C_+ \Psi_+^{jJ} Y_{ij}^K \mathcal{F}_{IJK} - (\Psi_-^{iI})^T C_+ \Psi_-^{jJ} \bar{Y}_{ij}^K \bar{\mathcal{F}}_{IJK}.
\end{aligned} \tag{5.208}$$

And the $\lambda\lambda F$ interaction term becomes

$$\begin{aligned}
&(\lambda_+^{iI})^T C_- \gamma \cdot F_-^J \lambda_+^{jK} \varepsilon_{ji} \mathcal{F}_{IJK} + (\lambda_-^{iI})^T C_- \gamma \cdot F_+^J \lambda_-^{jK} \varepsilon_{ji} \bar{\mathcal{F}}_{IJK} \\
=&(\Psi_+^{iI})^T C_+ \gamma \cdot F_-^J \Psi_+^{jK} \varepsilon_{ji} \mathcal{F}_{IJK} - (\Psi_-^{iI})^T C_+ \gamma \cdot F_+^J \Psi_-^{jK} \varepsilon_{ji} \bar{\mathcal{F}}_{IJK}.
\end{aligned} \tag{5.209}$$

Both of these have a sign difference between the two terms. We can remove this sign difference by taking

$$\begin{pmatrix} X^I \\ \mathcal{F}_I \end{pmatrix} \to \begin{pmatrix} iX^I \\ i\mathcal{F}_I \end{pmatrix}. \tag{5.210}$$

This is highly similar to the reparameterisation we had to perform in $(0,4)$, replacing the natural para-complex unit in that signature with the natural complex unit in $(1,3)$. This transformation keeps the $\mathcal{F}_{IJ}$ and N_{IJ} terms the same, and changes $\mathcal{F}_{IJK} \to -i\mathcal{F}_{IJK}$ and $\bar{\mathcal{F}}_{IJK} \to +i\bar{\mathcal{F}}_{IJK}$. We, therefore, have the terms in the same form given in the literature. In $(1,3)$ X^I are complex scalar fields, and this does not change the sign of their kinetic term.

It is common in the literature to write the kinetic term for the spinors using the covariant derivative. The necessary pieces are already present in our Lagrangian –

$$-\frac{1}{2}\big(\bar{\lambda}_+^I \partial\!\!\!/ \lambda_-^J + \bar{\lambda}_-^I \partial\!\!\!/ \lambda_+^J\big) N_{IJ} - \frac{1}{4}\big(\bar{\lambda}_+^I (\partial\!\!\!/ \bar{\mathcal{F}}_{IJ}) \lambda_-^J + \bar{\lambda}_-^I \partial\!\!\!/ (\mathcal{F}_{IJ}) \lambda_+^J\big). \tag{5.211}$$

The Cristoffel symbol for the Levi-Civita connection for $\mathcal{F}_{IJ}$ is ${\Lambda^I}_{JK} = \frac{1}{2} N^{IL} \mathcal{F}_{JKL}$ for a Kähler manifold such that the second term is equal to

$$
\begin{aligned}
&-\frac{1}{4}\big(\bar{\lambda}^I_+(\not{\partial}\bar{\mathcal{F}}_{IJ})\lambda^J_- + \bar{\lambda}^I_-\not{\partial}(\mathcal{F}_{IJ})\lambda^J_+\big)\\
&= -\frac{1}{4}\big(\bar{\lambda}^I_+(\partial_\mu\bar{X}^K\bar{\mathcal{F}}_{IJK})\gamma^\mu\lambda^J_- + \bar{\lambda}^I_-(\partial_\mu X^K\mathcal{F}_{IJK})\gamma^\mu\lambda^J_+\big)\\
&= -\frac{1}{4}\big(\bar{\lambda}^I_+(\partial_\mu\bar{X}^K N_{IP}N^{PQ}\bar{\mathcal{F}}_{QJK})\gamma^\mu\lambda^J_- + \bar{\lambda}^I_-(\partial_\mu X^K N_{IP}N^{PQ}\mathcal{F}_{QJK})\gamma^\mu\lambda^J_+\big)\\
&= -\frac{1}{4}\big(\bar{\lambda}^I_+(\partial_\mu\bar{X}^K N^{JP}\bar{\mathcal{F}}_{PKL})\gamma^\mu\lambda^K_- + \bar{\lambda}^I_-(\partial_\mu X^K N^{JP}\mathcal{F}_{PKL})\gamma^\mu\lambda^L_+\big)N_{IJ}\\
&= -\frac{1}{2}\big(\bar{\lambda}^I_+(\partial_\mu\bar{X}^K\bar{\Lambda}^J{}_{KL})\gamma^\mu\lambda^L_- + \bar{\lambda}^I_-(\partial_\mu X^K\Lambda^L{}_{KL})\gamma^\mu\lambda^L_+\big)N_{IJ}.
\end{aligned}
\tag{5.212}
$$

Therefore (5.211) becomes

$$
\begin{aligned}
&-\frac{1}{2}\big(\bar{\lambda}^I_+\not{\partial}\lambda^J_- + \frac{1}{2}\bar{\lambda}^I_-\not{\partial}\lambda^J_+ + \bar{\lambda}^I_+(\partial_\mu\bar{X}^K\bar{\Lambda}^J{}_{KL})\gamma^\mu\lambda^L_- + \bar{\lambda}^I_-(\partial_\mu X^K\Lambda^L{}_{KL})\gamma^\mu\lambda^L_+\big)N_{IJ}\\
&= -\frac{1}{2}\big(\bar{\lambda}^I_+\not{D}\lambda^J_- + \frac{1}{2}\bar{\lambda}^I_+\not{D}\lambda^J_-\big)N_{IJ}
\end{aligned}
\tag{5.213}
$$

with $D_\mu\lambda^I_+ = \partial_\mu\lambda^I + \Lambda^I{}_{JK}\partial_\mu X^J\lambda^K$ and $D_\mu\lambda^I_- = \partial_\mu\lambda^I + \bar{\Lambda}^I{}_{JK}\partial_\mu\bar{X}^J\lambda^K$.

This also works after applying S as both terms are vector-valued bilinear forms; they are unchanged by the transformation, up to a sign. Implementing S we obtain

$$
+\frac{1}{2}\big(\bar{\Psi}^I_+\not{D}\Psi^J_- + \bar{\Psi}^I_-\not{D}\Psi^J_+\big)N_{IJ}. \tag{5.214}
$$

Future Directions

This final section will discuss possible future avenues for research and open questions posed by this thesis. These were alluded to in the text but are stated here for easy reference.

First, there are areas where the formalism in Chapter 3 could be expanded. A full classification of supersymmetry algebras up to isomorphism is desirable and could be performed within the scope of this formalism. A few isomorphisms between superalgebras were presented in this thesis, but not in a systematic and exhaustive manner that would be required for a full classification.

Another immediate area for expansion of this formalism could be including BPS/polyvector/central charges. This has been done in [31], which is based on the formalism in [1] that this thesis used heavily for inspiration. The author expects the manner of producing superalgebras presented is readily amenable to this type of extension.

T-duality was explored as a map between supersymmetry algebras of different signatures. String theory with negative branes leads to dynamically changing space-time signature [17] and fully exploring this in our context would be an interesting prospect.

Additionally, while we used the work of [1, 3, 31] heavily the same terminology and formulation were not used. A dictionary between the two formalisms would be desirable. This would involve recasting the analysis performed in Chapter 3 in the style of these papers; i.e. in terms of Schur algebras, analysing the space of super-admissible bilinear forms. One would need to calculate the Schur algebra for an arbitrary number of irreducible spinor modules in arbitrary signature and study the orbit of these larger Schur algebras on the space of superadmissible bilinear forms, as was described in Section 3.10. This would provide a full classification of supersymmetry algebras.

Supergravity theories are commonly made using the ‘superconformal method’ [28,55–58] where a superconformal supergravity theory is derived and gauged to derive a super-Poincaré supergravity theory. Expanding the formalism to include the extra generators found in the superconformal algebra would, therefore, be necessary.

A full supergravity theory involves vector multiplets, hypermultiplets and Weyl multiplets, at least. Having already derived the vector multiplet content (which would also need some correcting to make supersymmetry local), a similar analysis of hypermultiplets and the Weyl multiplets in any signature could be performed. The author has already performed a precursory investigation into hypermultiplets. The obvious starting point would be in five and four dimensions, having already obtained the vector multiplet here. A similar analysis could also be performed in different signatures, for example, this could be a thorough exploration of Type II supergravities that fully demonstrates how supersymmetry causes the sign changes observed in [14] and are detailed in this thesis in Section 2.12.

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