STATISTICAL STABILITY FOR BARGE-MARTIN ATTRACTORS DERIVED FROM TENT MAPS

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ABSTRACT. Let $\{f_t\}_{t\in(1,2]}$ be the family of core tent maps of slopes t. The parameterized Barge-Martin construction yields a family of disk homeomorphisms $\Phi_t\colon D^2 \to D^2$, having transitive global attractors Λ_t on which Φ_t is topologically conjugate to the natural extension of f_t. The unique family of absolutely continuous invariant measures for f_t induces a family of ergodic Φ_t-invariant measures ν_t, supported on the attractors Λ_t.

We show that this family ν_t varies weakly continuously, and that the measures ν_t are physical with respect to a weakly continuously varying family of background Oxtoby-Ulam measures ρ_t.

Similar results are obtained for the family $\chi_t\colon S^2 \to S^2$ of transitive sphere homeomorphisms, constructed in a previous paper of the authors as factors of the natural extensions of f_t.

1. Introduction

The parameterized Barge-Martin construction applied to the family f_t of core tent maps on the interval I yields a family of disk homeomorphisms Φ_t, each of which has a transitive global attractor Λ_t which is homeomorphic to the inverse limit $\varprojlim(I, f_t)$ and on which Φ_t is topologically conjugate to the natural extension $\widehat{f}_t$ [15, 11, 18, 19]. These inverse limits are topologically quite intricate and have been the subject of intense investigation. The main recent focus has been the proof of the Ingram conjecture: for different values of $t \in [\sqrt{2}, 2]$ the inverse limits are not homeomorphic (for references see [13], which contains the final proof for tent maps which are not restricted to their cores). In the case where the parameter t is such that the critical orbit of f_t is dense (a full measure, dense G_δ set of parameters), theorems of Bruin and of Raines imply that the inverse limit $\widehat{I}_t$ is nowhere locally the product of a Cantor set and an interval [21, 25]. Perhaps more striking, Barge, Brooks and Diamond [12] show that there is a dense G_δ set of parameters t for which the inverse limit has a strong self-similarity: every open subset of $\widehat{I}_t$ contains a homeomorphic copy of $\widehat{I}_s$ for every $s \in [\sqrt{2}, 2]$.

A parameterized family of dynamical systems is said to be *statistically stable* if the time averages of observables along typical orbits vary continuously with the parameter: more formally, if there is a family of invariant physical measures which vary continuously in the weak topology. The classic examples of systems statistically stable under perturbation are expanding smooth maps and Axiom A diffeomorphisms restricted to basins of attractors. There have been extensions into the non-uniformly hyperbolic context [10, 2, 26], and a variety of specific, mostly low dimensional, families [4, 3, 7, 9, 6] have been shown to be statistically stable; see [8] for a survey. Of central importance here is a recent result of Alves and Pumariño [5], who prove that the core tent map family $f_t\colon I \to I$ is statistically stable in the stronger sense that the densities of its family of unique absolutely continuous invariant measures (acims) μ_t vary continuously in L^1. These acims μ_t on I for f_t induce a family of ergodic Φ_t-invariant measures ν_t on D^2, supported on the Barge-Martin attractors Λ_t. The main result here, Theorem 4.1, states that this family of Φ_t-invariant measures ν_t is weakly continuous.

The definition of statistical stability also requires that the measures ν_t be physical: the set of regular points of each ν should have positive measure with respect to a background Lebesgue measure. Equivalently,

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if δ_z denotes the Dirac measure at the point z, then when ν is physical

$$\frac{1}{N}\sum_{k=0}^{N-1}\delta_{f^k(x)} \to \nu$$

in the weak topology for a set of initial conditions x of positive Lebesgue probability. This means that the invariant measure ν is physically observable. In the case of planar attractors considered here, the existence of physical (SRB[1]) measures on the attractors have been proved in a variety of cases; see [29, 27] for surveys. These include the classical hyperbolic results and their many non-uniformly hyperbolic extensions as well as the class of attractors defined by Wang and Young [28]. In view of the role of inverse limits of unimodal interval maps as topological models for certain Hénon attractors [11, 14] we note that for the set of Benedicks-Carleson parameters Benedicks and Young [16] have shown the existence of an SRB measure for the Hénon attractors and statistical stability at these parameters is proved in [4].

The Barge-Martin construction rests fundamentally on an inverse limit and there is a natural correspondence between invariant measures for a map and those of its inverse limit. In addition, a key result of Kennedy, Raines and Stockman connects regular points of an invariant measure to those of the corresponding invariant measure in the inverse limit ([23]). However, the background Lebesgue measure on, say, a manifold does not in any natural way produce such a measure on an inverse limit. By using a construction from [19] we can produce a background physical measure of Oxtoby-Ulam type for the Barge-Martin homeomorphisms here. In Section 6.4 we comment on how this can be used with the Homeomorphic Measures Theorem to obtain a background Lebesgue measure.

The first paragraph of the following theorem summarizes relevant results from [18], while the second paragraph contains the main results of this paper.

Theorem 1.1. *There exists a continuously varying family of homeomorphisms $\Phi_t : D^2 \to D^2$, each having a transitive global attractor Λ_t which is homeomorphic to the core tent map inverse limit $\varprojlim(I, f_t)$ and on which Φ_t is topologically conjugate to the natural extension $\widehat{f}_t$. The attractors Λ_t vary Hausdorff continuously.*

The family ν_t of ergodic Φ_t-invariant measures induced by the acims μ_t for f_t is weakly continuous. The measures ν_t are physical with respect to a weakly continuous family of background Oxtoby-Ulam measures ρ_t.

In [19] it is shown how a family $\{\chi_t\}_{t\in(\sqrt{2},2]}$ of transitive sphere homeomorphisms can be constructed from the core tent maps f_t, with each χ_t being a factor of the natural extension $\widehat{f}_t$ by a mild semi-conjugacy. We finish the paper by indicating how the same techniques can be applied to this family.

2. Preliminaries

2.1. Inverse limits

Let X be a compact metric space with metric d, and let $g\colon X \to X$ be continuous and surjective. The *inverse limit* of $g\colon X \to X$ is the space

$$\widehat{X} := \varprojlim(X, g) = \{\mathbf{x} \in X^{\mathbb{N}} \,:\, g(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N}\}.$$

We endow $\widehat{X}$ with a metric, also denoted d, defined by $d(\mathbf{x},\mathbf{y}) = \sum_{n=0}^{\infty} d(x_n, y_n)/2^n$, which induces its natural topology as a subspace of the product $X^{\mathbb{N}}$.

We denote elements of $\widehat{X}$ with angle brackets, $\mathbf{x} = \langle x_0, x_1, x_2, \ldots\rangle$ and refer to them as *threads*. The *projections* are the maps $\pi_n : \widehat{X} \to X$ given by $\pi_n(\langle x_0, x_1, x_2, \ldots\rangle) = x_n$. The *natural extension* of $g\colon X \to X$ is the homeomorphism $\widehat{g}\colon \widehat{X} \to \widehat{X}$ defined by

$$\widehat{g}(\langle x_0, x_1, x_2, \ldots\rangle) = \langle g(x_0), x_0, x_1, x_2, \ldots\rangle\,.$$

Each g-invariant Borel probability measure ν gives rise to a unique $\widehat{g}$-invariant Borel probability measure $\widehat{\nu}$ characterized by $(\pi_n)_*\widehat{\nu} = \nu$ for all n. The measure $\widehat{\nu}$ is ergodic if and only if the measure ν is.

[1]See Remark 5.2 below

2.2. The parameterized Barge-Martin construction

We outline the construction here: for details see [15, 18, 19]. Let $\{f_t\}_{t\in J}$ be the family of core tent maps with slope t, affinely rescaled so they are all defined on the same interval $I=[-1,1]$: that is, f_t has slope t on $[-1,c)$ and slope $-t$ on $(c,1]$, where $c=1-2/t$, so that $f_t(x)=\min(t(x-1)+3,\,t(1-x)-1)$. Here $J=(1,2]$ is the interval of parameters for tent maps with positive topological entropy.

We will use special coordinates for the disk. Let S be the circle of radius 2 centered at the origin of $\mathbb{R}^2$ with angular coordinates y, and let I be the interval $[-1,1]$ in the real axis. Define $G:S\to I$ by $G(y)=\cos(y)$, and let D be a smooth embedding of the mapping cylinder of G. Thus we may view D as the disk of radius 2 whose boundary is the circle S. It is decomposed into a family of smoothly varying, smooth arcs $\eta: S\times[0,1]\to D$ with each $\eta(y,s)$ connecting the point $\eta(y,0)=y\in S$ to the point $\eta(y,1)=G(y)\in I$. The point $\eta(y,s)$ of D is given coordinates (y,s). Thus the interval I consists of all points $(y,1)$ and points in the interior of I have two coordinates. There is a projection $\tau\colon S\to I$ defined $\tau(y)=(y,1)$.

Let $\Upsilon\colon D\to D$ be the near-homeomorphism defined by

$$\Upsilon(y,s)=\begin{cases}(y,2s) & \text{if } s\in[0,1/2],\\ (y,1) & \text{if } s\in[1/2,1].\end{cases}$$

An *unwrapping* of the tent map family $\{f_t\}$ is a continuously varying family of orientation-preserving near-homeomorphisms $\overline{f}_t\colon D\to D$ with the properties that, for each t,

a) $\overline{f}_t$ is injective on I, and $\overline{f}_t(I)\subseteq\{(y,s)\,:\,s\geq 1/2\}$,
b) $\Upsilon\circ\overline{f}_t|_I=f_t$, and
c) for all $y\in S$ and all $s\in[0,1/2]$, the second component of $\overline{f}_t(y,s)$ is s.

Given such an unwrapping, let $H_t=\Upsilon\circ\overline{f}_t\colon D\to D$, which is a near-homeomorphism since both Υ and $\overline{f}_t$ are. By a theorem of Brown ([20]) this implies that the inverse limit $\widehat{D}_t:=\varprojlim(D,H_t)$ is also a topological disk.

Since $H_t=f_t$ on I, there is a copy of $\widehat{I}_t:=\varprojlim(I,f_t)$ (which copy we will also denote $\widehat{I}_t$) canonically embedded in $\widehat{D}_t$, on which the restriction of $\widehat{H}_t$ agrees with $\widehat{f}_t$. Moreover, for every z in the interior of D there is some N such that $H_t^n(z)\in I$ for all $n\geq N$: therefore any $\mathbf{z}$ in the interior of $\widehat{D}_t$ satisfies $\widehat{H}_t^n(\mathbf{z})\to\widehat{I}_t$ as $n\to\infty$, so that $\widehat{I}_t$ is a global attractor for $\widehat{H}_t$.

We work simultaneously with the maps $\{H_t\}_{t\in J}$ by collecting them into a *fat map* $H\colon\Pi:=D\times J\to D$ defined by $H(z,t)=(H_t(z),t)$. The inverse limit $\widehat{\Pi}:=\varprojlim(\Pi,H)$ provides a way of topologizing the disjoint union of the individual inverse limits $\widehat{D}_t$, since for each t the map $\iota_t\colon\mathbf{z}\mapsto\langle(z_0,t),(z_1,t),\ldots\rangle$ is an embedding of $\widehat{D}_t$ into $\widehat{\Pi}$, and $\widehat{\Pi}$ is the disjoint union of the images of these embeddings as t varies through J. We will identify each $\widehat{D}_t$ with its image under this embedding without further comment, thus regarding it as a subset of $\widehat{\Pi}$. Then each $\widehat{D}_t$ is invariant under the natural extension $\widehat{H}$ of H, which acts on it as $\widehat{H}_t$.

A parameterized version [18] of Brown's theorem yields a continuous map $h\colon\widehat{\Pi}\to D^2$, where D^2 is a standard model of the disk, with the property that $h_t:=h|_{\widehat{D}_t}$ is a homeomorphism onto D^2 for each t. The Barge-Martin family of disk homeomorphisms $\{\Phi_t\}_{t\in J}$ is then defined by $\Phi_t=h_t\circ\widehat{H}\circ h_t^{-1}\colon D^2\to D^2$. They have global attractors $\Lambda_t:=h_t(\widehat{I}_t)$, which vary Hausdorff continuously, and on which Φ_t is topologically conjugate to $\widehat{f}_t$.

The ergodic acim μ_t for f_t on I induces an ergodic $\widehat{f}_t$-invariant measure $\widehat{\mu}_t$ on $\widehat{I}_t$. This in turn generates $\nu_t:=(h_t)_*\widehat{\mu}_t$ which is an ergodic Φ_t-invariant measure on D^2 supported on the attractor Λ_t.

2.3. Weak continuity

Let X be a compact metric space. A sequence of Borel probability measures μ_n is said to *converge weakly* to μ_0 if for all $\alpha\in C(X,\mathbb{R})$,

$$\int_X\alpha\,d\mu_n\to\int_X\alpha\,d\mu_0.$$

Since this is the only notion of convergence of measures which we will use, the notation $\mu_n\to\mu_0$ will always denote weak convergence.

The following criteria from pages 16 and 17 of Billingsley [17] will be used. Recall that a Borel set $A \subset X$ is called a *continuity set* for a measure μ if $\mu(\mathrm{Bd}(A)) = 0$.

Theorem 2.1.
(a) $\mu_n \to \mu_0$ *if and only if* $\mu_n(A) \to \mu_0(A)$ *for all* μ_0*-continuity sets* A.
(b) Let $\mathcal{A}$ *be a collection of Borel sets which is closed under finite intersections, with the property that every open subset of* X *is the countable union of sets from* $\mathcal{A}$. *If* $\mu_n(A) \to \mu_0(A)$ *for all* $A \in \mathcal{A}$, *then* $\mu_n \to \mu_0$.

A one-parameter family of measures μ_t is said to be *weakly continuous* if $t_i \to t$ implies $\mu_{t_i} \to \mu_t$. Our starting point in this paper is a recent result of Alves and Pumariño [5], who show that the family of acims μ_t for the core tent maps f_t have densities which vary continuously in L^1, so that, in particular:

Theorem 2.2 (Alves & Pumariño)**.** *The family* μ_t *of acims for the core tent family is weakly continuous.*

3. Weak continuity on the fat inverse limit

In this section we show (Theorem 3.6) that weak continuity of the family μ_t of tent map acims implies weak continuity of the induced measures $\widehat{\mu}_t$ on the inverse limits $\widehat{I}_t$. Since the spaces $\widehat{I}_t$ are varying, this statement needs to be interpreted in the context of fat maps.

Write $P := I \times J$. Given $X \subset P$ and $t \in J$, write $X^{(t)} := X \cap (I \times \{t\})$. For each $t \in J$, let $\sigma_t\colon I \to P$ denote the embedding $x \mapsto (x, t)$, and let $\mu_t^P = (\sigma_t)_* \mu_t$, so that μ_t^P is a measure on P which is supported on $P^{(t)}$.

It is routine to prove weak continuity of the family of measures $\{\mu_t^P\}$ (indeed, if X is any compact metric space and $\{\mu_t\}_{t \in J}$ is a weakly continuously varying family of measures on X, then the corresponding measures μ_t^P on $X \times J$ also vary weakly continuously).

Now let $F\colon P \to P$ be the fat map defined by $F(x, t) = (f_t(x), t)$, and set $\widehat{P} := \varprojlim(P, F)$. As in Section 2.2, we may think of $\widehat{P}$ as the topologized disjoint union of the inverse limits $\widehat{I}_t$, since each map $\iota_t\colon \widehat{I}_t \to \widehat{P}$ defined by $\iota_t(\langle x_0, x_1, \dots\rangle) = \langle (x_0, t), (x_1, t), \dots\rangle$ is a homeomorphism onto its image, which image is called the *t-slice* of $\widehat{P}$.

Recall that $\widehat{\mu}_t$ denotes the $\widehat{f}_t$-invariant measure on $\widehat{I}_t$ induced by the acim μ_t for f_t. For each t, we write $\widehat{\mu_t^P} = (\iota_t)_*(\widehat{\mu}_t)$, a measure on $\widehat{P}$ supported on its t-slice. Our aim in this section is to show that the family of measures $\widehat{\mu_t^P}$ is weakly continuous.

Remark 3.1. *Let* $\pi_n\colon \widehat{P} \to P$ *be the projections on* $\widehat{P}$, *and denote by* $\pi_{n,t}\colon \widehat{I}_t \to I$ *the projections on* $\widehat{I}_t$. *Then* $\sigma_t \circ \pi_{n,t} = \pi_n \circ \iota_t$*: that is,* π_n *acts as* $\pi_{n,t}$ *on the t-slice of* $\widehat{P}$. *It follows that for each* $t \in J$ *we have* $(\pi_n)_* \widehat{\mu_t^P} = (\pi_n \circ \iota_t)_* \widehat{\mu}_t = (\sigma_t \circ \pi_{n,t})_* \widehat{\mu}_t = (\sigma_t)_* \mu_t = \mu_t^P$ *for all* n, *so that* $\widehat{\mu_t^P}$ *is the measure on* $\widehat{P}$ *induced by the measure* μ_t^P *on* P.

To show weak continuity of the family $\widehat{\mu_t^P}$, we will apply Theorem 2.1(b) to the family of subsets of $\widehat{P}$ obtained as finite intersections of sets of the form $\pi_n^{-1}(R)$ for *tilted rectangles* R, which we now define.

Definition 3.2. *A* tilted rectangle *in* $\mathbb{R}^2$ *is obtained by rotating an open (possibly empty) rectangle* $(a_1, a_2) \times (b_1, b_2)$ *by* $\pi/4$. *We write* $\mathcal{R}$ *for the set of all tilted rectangles contained in* P.

Definition 3.3. *A subset* B *of* P *is called* simple *if it satisfies:*
(a) B *is open in* P, *and*
(b) for each $t \in J$, $(\mathrm{Bd}(B))^{(t)}$ *is finite.*

Note that, by (b), a simple subset of P is a continuity set for all of the measures μ_t^P.

Lemma 3.4.
(a) Any finite intersection of simple subsets of P *is simple.*
(b) If $R \in \mathcal{R}$ *and* $n \in \mathbb{N}$, *then* $F^{-n}(R)$ *is simple.*

Proof. (a) is trivial (using $\mathrm{Bd}(B_1 \cap B_2) \subset \mathrm{Bd}(B_1) \cup \mathrm{Bd}(B_2)$). For (b), $F^{-n}(R)$ is clearly open, so we only need show that every $(\mathrm{Bd}(F^{-n}(R)))^{(t)}$ is finite.

If $(x,t) \in P$, then $(x,t) \notin F^{-n}(R)$ if and only if $f_t^n(x)$ is not in the open interval $R^{(t)}$. If this is the case then the same is true for all (x',t') in a neighborhood of (x,t) — and hence $(x,t) \notin \mathrm{Bd}(F^{-n}(R))$ — unless $f_t^n(x) \in \mathrm{Bd}(R^{(t)})$. It follows that $\#(\mathrm{Bd}(F^{-n}(R)))^{(t)}) \leq 2^{n+1}$ for each t. □

Lemma 3.5. *Let $k \geq 1$, $R_1, \dots, R_k \in \mathcal{R}$, and $n_1, \dots, n_k \in \mathbb{N}$ with $n_k = \max(n_1, \dots, n_k)$. Write*

$$A = \pi_{n_1}^{-1}(R_1) \cap \cdots \cap \pi_{n_k}^{-1}(R_k) \subset \widehat{P}.$$

Then $A = \pi_{n_k}^{-1}(B)$ for some simple set B.

Proof. The proof is by induction on k, with trivial base case $k = 1$. For the general case, applying the inductive hypothesis to $\pi_{n_2}^{-1}(R_2) \cap \cdots \cap \pi_{n_k}^{-1}(R_k)$, we can write $A = \pi_{n_1}^{-1}(R_1) \cap \pi_{n_k}^{-1}(B')$ for some simple set B'. Observing that if $n \leq m$ and $B_1, B_2 \subset P$ we have $\pi_n^{-1}(B_1) \cap \pi_m^{-1}(B_2) = \pi_m^{-1}(F^{n-m}(B_1) \cap B_2)$, we can write $A = \pi_{n_k}^{-1}(B)$ with $B = F^{n_1 - n_k}(R_1) \cap B'$, which establishes the result since B is simple by Lemma 3.4. □

Theorem 3.6. *The family of measures $\{\widehat{\mu_t^P}\}$ on $\widehat{P}$ is weakly continuous.*

Proof. Let $\mathcal{A}$ be the collection of all sets A as in the statement of Lemma 3.5: that is, all finite intersections of sets of the form $\pi_n^{-1}(R)$, where $R \in \mathcal{R}$.

If $A \in \mathcal{A}$ then, by Lemma 3.5, we can write $A = \pi_n^{-1}(B)$ for some $n \in \mathbb{N}$ and some simple subset B of P. Then, if $t_i \to t_*$ is any convergent sequence in J, we have

$$\widehat{\mu_{t_i}^P}(A) = \widehat{\mu_{t_i}^P}(\pi_n^{-1}(B)) = \mu_{t_i}^P(B) \to \mu_{t_*}^P(B) = \widehat{\mu_{t_*}^P}(\pi_n^{-1}(B)) = \widehat{\mu_{t_*}^P}(A)$$

by Remark 3.1 and weak continuity of the μ_t^P, since B is a continuity set for $\mu_{t_*}^P$.

The result follows from Theorem 2.1(b), since $\mathcal{A}$ is closed under finite intersections, and any open subset of $\widehat{P}$ is a countable union of sets of $\mathcal{A}$ (the latter is an immediate consequence of the fact that any open subset of $\widehat{P}$ is a countable union of sets of the form $\pi_n^{-1}(U)$, where U is open in P). □

4. Weak continuity of the measures ν_t supported on the attractors

Recall from Section 2.2 that we write $\Pi = D \times J$, $H \colon \Pi \to \Pi$ for the fat version of the near homeomorphisms H_t, and $\widehat{\Pi} := \varprojlim(\Pi, H)$; and that the parameterized version of Brown's theorem yields a continuous $h \colon \widehat{\Pi} \to D^2$ which restricts on each slice to a homeomorphism onto D^2.

Since $I \subset D$ we have $P \subset \Pi$, and $H|_P = F$. Therefore $\widehat{P} \subset \widehat{\Pi}$, and the restriction $h|_{\widehat{P}}$, which we will also denote h, acts on each slice of $\widehat{P}$ as a homeomorphism onto the attractor Λ_t of the Barge-Martin homeomorphism Φ_t. The measures ν_t supported on these attractors, originally defined by $\nu_t = (h_t)_* \widehat{\mu}_t$, can therefore equivalently be defined by $\nu_t = h_* \widehat{\mu_t^P}$. The following, which is the main result of the first part of the paper, is an immediate consequence.

Theorem 4.1. *The Φ_t-invariant measures ν_t on D^2 supported on the attractors Λ_t vary weakly continuously.*

Proof. If $\alpha \in C(D^2, \mathbb{R})$ then $\alpha \circ h \in C(\widehat{P}, \mathbb{R})$ and so by Theorem 3.6, for any $t_i \to t_*$ in J,

$$\int_{\widehat{P}} \alpha \circ h \, d\widehat{\mu_{t_i}^P} \to \int_{\widehat{P}} \alpha \circ h \, d\widehat{\mu_{t_*}^P}.$$

Since $\nu_t = h_* \widehat{\mu_t^P}$ we therefore have $\displaystyle\int_{D^2} \alpha \, d\nu_{t_i} \to \int_{D^2} \alpha \, d\nu_{t_*}$ as required. □

5. Physical measures

5.1. Basic definitions

Let M be a smooth manifold and m a measure on M given by a volume form. It is common to call such a measure "a Lebesgue measure", and we adopt that convention.

Definition 5.1. *Let $f : M \to M$ be continuous with invariant Borel probability measure ν. A point x is* regular *(or* generic*) for ν under f if*

$$\frac{1}{n}\sum_{i=0}^{n-1}\alpha(f^i(x)) \to \int \alpha \, d\nu$$

for all $\alpha \in C(M, \mathbb{R})$. A collection B of regular points is called a basin *for ν. The measure ν is called* physical *if it has a basin with positive Lebesgue measure, $m(B) > 0$.*

Note that the set of regular points for μ under f is completely invariant (i.e. x is regular if and only if $f(x)$ is regular). The acim μ_t of a tent map $f_t\colon I \to I$ is physical, since the pointwise Birkhoff ergodic theorem guarantees that μ_t-almost every (and hence Lebesgue almost every) point is regular for μ_t.

Remark 5.2. *A measure is physical for one Lebesgue measure if and only if it is physical for all Lebesgue measures, justifying the lack of specificity about the background measure.*

There is a substantial variance in terminology surrounding physical measures. Some authors require a basin to have full measure in an open set. Some use SRB measure *as synonymous with physical measure, others regard SRB measures as only existing for smooth systems, where they are defined by a condition on disintegration along unstable manifolds; cf. page 741 of* [29].

5.2. Physical measures for inverse limits

The fundamental relationship between regular points for a base measure and for the induced measure in the inverse limit is ([23]):

Theorem 5.3 (Kennedy, Raines and Stockman)**.** *Let X be a compact metric space, $f\colon X \to X$ be continuous and surjective, and ν be an f-invariant Borel probability measure. Let $\widehat{f}\colon \widehat{X} \to \widehat{X}$ be the natural extension of f, and $\widehat{\nu}$ be the induced $\widehat{f}$-invariant measure.*

(a) A point $x \in X$ is regular for ν if and only if one (and hence every) point of $\pi_0^{-1}(x)$ is regular for $\widehat{\nu}$.

(b) A set B is a basin for the measure ν if and only if $\pi_0^{-1}(B)$ is a basin for $\widehat{\nu}$.

Remark 5.4. *If $\mathbf{x}, \mathbf{x}' \in \pi_0^{-1}(x)$, then $d(\widehat{f}^i(\mathbf{x}), \widehat{f}^i(\mathbf{x}')) = d(\mathbf{x}, \mathbf{x}')/2^i$. It is therefore immediate that if one point of $\pi_0^{-1}(x)$ is regular, then every point of $\pi_0^{-1}(x)$ is regular.*

One would like to be able to show that if ν is a physical measure for (M, f), then $\widehat{\nu}$ is a physical measure for $(\varprojlim(M, f), \widehat{f})$. The problem, as already mentioned, is that $\varprojlim(M, f)$ is usually not a manifold and so there is no natural way to define a background Lebesgue measure. One way forward is via the following reasonable definition from [23]:

Definition 5.5. *Let X be compact with a Lebesgue measure m, and $f : X \to X$ be continuous and surjective. An invariant measure $\widehat{\nu}$ for the natural extension $\widehat{f}\colon \widehat{X} \to \widehat{X}$ is called an* inverse limit physical measure *if it has a basin $\widehat{B}$ which satisfies $m(\pi_0(\widehat{B})) > 0$.*

The following is then immediate from Theorem 5.3:

Corollary 5.6 (Kennedy, Raines and Stockman)**.** *Let X be compact with a Lebesgue measure m, $f\colon X \to X$ be continuous and surjective, and ν be an f-invariant Borel probability measure. Then ν is physical if and only if the induced $\widehat{f}$-invariant measure $\widehat{\nu}$ is inverse limit physical.*

5.3. Inverse limit physical measures for tent maps

Recall from Section 2.2 that in the Barge-Martin construction we regard the interval I as being contained in the disk D, and construct near-homeomorphisms $H_t\colon D \to D$ which agree with f_t on I. We will now regard the acim μ_t for f_t as a measure on D, supported on I, and maintain its name. In this section we show that it is a physical measure for H_t with respect to the (y, s)-coordinate measure — denoted m and called "Lebesgue" — on D. It then follows by Corollary 5.6 that $\widehat{\mu}_t$, regarded as a measure on $\widehat{D}_t$ supported on $\widehat{I}_t$, is an inverse limit physical measure for $\widehat{H}_t$.

The Barge-Martin construction depends on the choice of unwrapping $\{\overline{f}_t\}$ of the tent family $\{f_t\}$. From now on we will assume that our unwrapping satisfies the additional condition that $\overline{f}_t$ restricts to the identity on the annulus $S \times [0, 3/4] \subset D$ for all t. (Such an unwrapping could be constructed from an arbitrary unwrapping $\{\overline{g}_t\}$ by rescaling $\overline{g}_t$ to be defined on $S \times [7/8, 1]$ and interpolating between the identity and $\overline{g}_t|_{S\times\{0\}}$ in $S \times [3/4, 7/8]$.)

Theorem 5.7. *For each t, the measure $\widehat{\mu}_t$ on $\widehat{D}_t$ is an inverse limit physical measure for $\widehat{H}_t$.*

Proof. Recall that $\tau\colon S \to I$ is the projection $y \mapsto (y, 1)$ from the boundary circle of D onto the interval. For each t, denote by X_t the set of regular points for μ_t under f_t, and let $Y_t = \tau^{-1}(X_t)$. Since μ_t is ergodic and absolutely continuous with respect to Lebesgue, X_t has full Lebesgue measure in I; since τ is smooth and at most two to one, Y_t has full angular Lebesgue measure 2π in S.

Let $Z_t = Y_t \times [1/2, 3/4] \subset D$. If $(y, s) \in Z_t$ then $H_t(y, s) = \tau(y) \in X_t$, which is regular for μ_t under H_t since $H_t|_I = f_t$: therefore, by complete invariance of the set of regular points, every point of Z_t is regular for μ_t under H_t. Since $m(Z_t) > 0$, μ_t is physical for H_t, and the result follows from Corollary 5.6. □

6. A BACKGROUND MEASURE FOR THE INVERSE LIMIT

The Barge-Martin construction of Section 2.2 produces a family of homeomorphisms $\Phi_t\colon D^2 \to D^2$ having global attractors Λ_t, varying Hausdorff continuously, with $\Phi_t|_{\Lambda_t}$ topologically conjugate to $\widehat{f}_t\colon \widehat{I}_t \to \widehat{I}_t$. Theorem 4.1 states that the Φ_t-invariant measures ν_t on D^2 induced from $\widehat{\mu}_t$ vary weakly continuously.

In order to complete the proof of Theorem 1.1, we need to show that the ν_t are physical measures. While there is no natural connection between Lebesgue measure on D^2 and any structure on the inverse limits, we will now use a modification of a construction from [19] to obtain a weakly continuous family of Oxtoby-Ulam measures ρ_t on D^2, with respect to which the ν_t are physical.

6.1. HOMEOMORPHISMS ONTO THE COMPLEMENT OF THE ATTRACTORS

Let A be the half-open annulus $A = S \times [0, \infty)$. In this section we will define a slice-preserving map $\Psi\colon A \times J \to \widehat{\Pi}$, whose image is the complement of the union of the attractors $\widehat{I}_t$, and which is a homeomorphism onto its image. This homeomorphism will be used to transfer Lebesgue measure on each slice $A \times \{t\}$ to an Oxtoby-Ulam measure on $\widehat{D}_t$.

We start by defining Ψ on each slice.

Definition 6.1. *For each $t \in J$, define $\Psi_t\colon A \to \widehat{D}_t \setminus \widehat{I}_t$ by*

$$\Psi_t(y,s) = \begin{cases} \langle (y,s), (y,s/2), (y,s/4), \dots \rangle & \text{if } s \in [0,1), \\ \left\langle f_t^{\lfloor s \rfloor - 1}(H_t(y,v)), \dots, f_t(H_t(y,v)), H_t(y,v), (y,v), (y,v/2), \dots \right\rangle & \text{otherwise,} \end{cases}$$

where, in the second case, $v = (u+1)/2$, with $u = s - \lfloor s \rfloor$ the fractional part of s.

Lemma 6.2. *Each Ψ_t is a bijection*

Proof. Let $\mathbf{z} = \langle z_0, z_1, \dots \rangle \in \widehat{D}_t \setminus \widehat{I}_t$, so that each $z_n \in D$ and $H_t(z_{n+1}) = z_n$ for each n. Note that any point $(y, s) \in D \setminus I$ (i.e. with $s < 1$) has unique H_t-preimage $(y, s/2)$. Therefore if $z_0 = (y, s) \notin I$, then $\mathbf{z} = \Psi_t(y, s)$. On the other hand, if $z_0 \in I$ then, since $\mathbf{z} \notin \widehat{I}_t$, there is some least $k > 0$ with $z_k \notin I$. Writing $z_k = (y, v)$ with $v \in [1/2, 1)$ we have $\mathbf{z} = \Psi_t(y, k + 2v - 1)$. Therefore Ψ_t is surjective.

Given $\mathbf{z} = \Psi_t(y, s)$, we can determine the integer part of s as the first k with $z_k \notin I$; and y and the fractional part of s from z_k. Therefore Ψ_t is injective. □

Recall from Section 2.2 the embeddings $\iota_t\colon \widehat{D}_t \to \widehat{\Pi}$ defined by $\mathbf{z} \mapsto \langle (z_0, t), (z_1, t), \dots \rangle$. Write $\widehat{\Pi}_C = \widehat{\Pi} \setminus \bigsqcup_{t\in J} \iota_t(\widehat{I}_t)$, the complement of the union of the attractors, and define $\Psi\colon A \times J \to \widehat{\Pi}_C$ by $\Psi((y, s), t) = \iota_t(\Psi_t(y, s))$.

Theorem 6.3. Ψ *is a slice-preserving homeomorphism. In particular, each Ψ_t is a homeomorphism.*

Proof. That Ψ is a slice-preserving bijection is immediate from its definition and from Lemma 6.2. The restriction of Ψ to $S \times [0, 1) \times J$ is given by

$$\Psi((y,s),t) = \langle ((y,s),t), ((y,s/2),t), \dots \rangle ,$$

which is evidently a homeomorphism onto its image. Let $G\colon A\times J\to A\times J$ be the homeomorphism defined by

$$G((y,s),t)=\begin{cases}((y,2s),t) & \text{if } s\in[0,1],\\ ((y,s+1),t) & \text{if } s\in[1,\infty).\end{cases}$$

It follows immediately from the definitions that $\widehat{H}\circ\Psi=\Psi\circ G\colon A\times J\to\widehat{\Pi}_C$ (the three cases $s\in[0,1/2)$, $s\in[1/2,1)$, and $s\in[1,\infty)$ need to be checked separately). Therefore, for each $N\geq 1$, since G^{-N} maps $S\times[0,N+1)\times J$ onto $S\times[0,1)\times J$, we have

$$\Psi|_{S\times[0,N+1)\times J}=\widehat{H}^N\circ\Psi|_{S\times[0,1)\times J}\circ G^{-N}|_{S\times[0,N+1)\times J},$$

so that Ψ is a homeomorphism on each $S\times[0,N+1)\times J$, and hence is a homeomorphism. □

6.2. The background measure on $\widehat{D}_t$

Let m be the probability measure defined on A by $dm=\frac{1}{K}\arctan(s)\,dyds$, where K is the normalization constant. For each $t\in J$, let ρ'_t be the measure on $\widehat{D}_t$ obtained by pushing forward m with Ψ_t, and assigning $\rho'_t(\widehat{I}_t)=0$.

Recall (see for example [1, 22]) that a Borel probability measure on a manifold M is called *Oxtoby-Ulam (OU)* or *good* if it is non-atomic, positive on open sets, and assigns zero measure to the boundary of M (if it exists).

Theorem 6.4. *For each t, ρ'_t is an OU measure on $\widehat{D}_t$ with respect to which the set of points which are regular for $\widehat{\mu}_t$ under $\widehat{H}_t$ has positive measure.*

Proof. Because Ψ_t is a homeomorphism, $(\Psi_t)_*(m)$ is an OU measure on $\widehat{D}_t\setminus\widehat{I}_t$. Since $\widehat{I}_t$ is closed and nowhere dense in $\widehat{D}_t$, the extension ρ'_t of this measure is also OU.

Recall from the proof of Theorem 5.7 that every point of $Z_t=Y_t\times[1/2,3/4]\subset D$ is regular for μ_t under H_t. It follows by Theorem 5.3 that every point of $R'_t:=\pi_0^{-1}(Z_t)$ is regular for $\widehat{\mu}_t$ under $\widehat{H}_t$. However, the points of R'_t are precisely threads $\langle(y,s),(y,s/2),\ldots\rangle$ with $y\in Y_t$ and $s\in[1/2,3/4]$: that is, $R'_t=\Psi_t(Y_t\times[1/2,3/4])$. Since Y_t has positive one-dimensional Lebesgue measure we have $m(Y_t\times[1/2,3/4])>0$, and the result follows. □

6.3. The background measures on D^2

Recall from Section 2.2 that the parameterized Barge-Martin construction yields a continuous map $h\colon\widehat{\Pi}\to D^2$ which restricts on each slice to a homeomorphism $h_t\colon\widehat{D}_t\to D^2$; that the disk homeomorphisms Φ_t are defined by $\Phi_t=h_t\circ\widehat{H}_t\circ h_t^{-1}$; and that the measure ν_t on D^2 is defined by $\nu_t=(h_t)_*\widehat{\mu}_t$.

Theorem 4.1 states that the measures ν_t vary weakly continuously. We can now complete the proof of Theorem 1.1:

Lemma 6.5. *For each t, the measure ν_t is physical with respect to the OU measure $\rho_t:=(h_t)_*\rho'_t$. The family of measures ρ_t itself varies weakly continuously.*

Proof. Recall from the proof of Theorem 6.4 that $R'_t=\pi_0^{-1}(Z_t)$ consists of regular points for $\widehat{\mu}_t$ under $\widehat{H}_t$, and is assigned positive measure by the OU measure ρ'_t. Since h_t is a homeomorphism which conjugates $\widehat{H}_t$ and Φ_t, it is immediate that $R_t:=h_t(R'_t)$ consists of regular points for ν_t under Φ_t, and is assigned positive measure by the OU measure ρ_t.

It therefore only remains to show weak continuity of the family $\{\rho_t\}$, so let $t_i\to t_*$ in J. By Theorem 6.3 we have that $h_{t_i}\circ\Psi_{t_i}\to h_{t_*}\circ\Psi_{t_*}$ pointwise.

Let $\alpha\in C(D^2,\mathbb{R})$. Since ρ_t is supported on $h_t\circ\Psi_t(A)$, we have

$$\int_{D^2}\alpha\,d\rho_{t_i}=\int_{h_{t_i}\circ\Psi_{t_i}(A)}\alpha\,d(h_{t_i}\circ\Psi_{t_i})_*m=\int_A\alpha\circ h_{t_i}\circ\Psi_{t_i}\,dm\to\int_A\alpha\circ h_{t_0}\circ\Psi_{t_0}\,dm=\int_{D^2}\alpha\,d\rho_{t_0}$$

as required, by the bounded convergence theorem. □

6.4. Using the homeomorphic Measures Theorem

For a measure to be physical with respect to a background OU measure is not entirely satisfactory, since positive measure sets for an OU measure may have zero Lebesgue measure.

The Homeomorphic Measures Theorem due to Oxtoby and Ulam, and also to von Neumann, states that, for any OU measure ρ on a manifold M, there is a homeomorphism $g\colon M \to M$ such that $g_*\rho$ is Lebesgue measure [24, 1]. In particular, if λ is a fixed Lebesgue measure on D^2 we can find disk homeomorphisms Θ_t with $(\Theta_t)_*\rho_t = \lambda$. Then the disk homeomorphisms $\Omega_t = \Theta_t^{-1} \circ \Phi_t \circ \Theta_t$ have ergodic invariant measures $\xi_t = (\Theta_t)_*\nu_t$, supported on the global attractors $\Theta_t(\Lambda_t)$; and the measures ξ_t will be physical with respect to the fixed Lebesgue measure λ.

A natural question is whether, since the ρ_t vary weakly continuously, the homeomorphisms Θ_t can be chosen to vary continuously with t: without this, there is no reason to suppose that the Ω_t will vary continuously, or the ξ_t weakly continuously. This is an open question posed by Fathi [22], who gives some partial answers: there are also relevant results due to Peck and Prasad (personal communication). However these results don't apply in the present context, unless one could succeed in imposing very particular properties on the family of homeomorphisms h_t.

7. Application to a family of transitive sphere homeomorphisms

In [19] it is shown how a family of transitive sphere homeomorphisms $\chi_t\colon S^2 \to S^2$ can be constructed from the core tent maps f_t for $t > \sqrt{2}$. These sphere homeomorphisms are factors of the natural extensions $\widehat{f}_t\colon \widehat{I}_t \to \widehat{I}_t$ by mild semi-conjugacies. The following result is a combination of Theorems 5.19 and 5.32 of [19].

Theorem 7.1. *Let $J = (\sqrt{2}, 2]$. There is a continuously varying family $\{\chi_t\}_{t\in J}$ of self-homeomorphisms of S^2, with each χ_t being a factor of $\widehat{f}_t\colon \widehat{I}_t \to \widehat{I}_t$ by a semi-conjugacy $g_t\colon \widehat{I}_t \to S^2$, all of whose fibers except perhaps one has at most 3 points, and whose exceptional fiber carries no topological entropy.*

Each χ_t is topologically transitive, has dense periodic points, and has topological entropy $\log t$. Moreover, $\eta_t = (g_t)_*(\widehat{\mu}_t)$ is an ergodic invariant OU measure of maximal entropy.

The techniques used in this paper can also be applied to show that the measures η_t vary weakly continuously:

Theorem 7.2. *The measures η_t on S^2 from the statement of Theorem 7.1 vary weakly continuously.*

Sketch proof. The details used in this sketch are contained in Section 5.4 of [19]. In that paper, the Barge-Martin construction is carried out not in the disk D, but in the sphere T obtained by collapsing the boundary of D, giving rise to Barge-Martin sphere homeomorphisms $\widehat{H}_t\colon \widehat{T}_t \to \widehat{T}_t$ having global attractors $\widehat{I}_t$.

The inverse limit of the fat family of Barge-Martin near homeomorphisms, here denoted $\widehat{\Pi}$, is there denoted $\widehat{T}_*$; and the commutative diagram of Figure 15 of [19] includes a slice-preserving map $k = K\circ\pi\colon \widehat{T}_* \to S^2 \times J$, whose restriction $k_t\colon \widehat{T}_t \to S^2$ to the t-slice satisfies $k_t|_{\widehat{I}_t} = g_t$, the semi-conjugacy of Theorem 7.1. That is, the semi-conjugacies g_t can be gathered into a single continuous map $g\colon \widehat{P} \to S^2$, with the measures η_t given by $\eta_t = g_*\widehat{\mu_t^P}$. That these measures vary weakly continuously then follows exactly as in the proof of Theorem 4.1.

□

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