Fuzzy Failure Probability Estimation Applying Intervening Variables

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Abstract

Fuzzy probability offers a framework for taking into account the effects of both aleatoric and epistemic uncertainty on the performance of a system, quantifying its level of safety, for example, in terms of a fuzzy failure probability. However, the practical application of fuzzy probability is often challenging due to increased numerical efforts arising from the need to propagate both types of uncertainties. Hence, this contribution proposes an approach for approximate calculation of fuzzy failure probabilities for a class of problems that involve moderately nonlinear performance functions, where uncertain input parameters of a model are characterized as random variables while their associated distribution parameters (for example, mean and standard deviation) are described as fuzzy variables. The proposed approach is cast as a post-processing step of a standard (yet advanced) reliability analysis. The key issue for performing an approximate calculation of the fuzzy failure probabilities is extracting probability sensitivity information from the reliability analysis stage as well as the introduction of intervening variables that capture – to some extent – the nonlinear relation between distribution parameters and the failure probability. A series of relatively simple illustrative examples demonstrate the capabilities of the proposed approach, highlighting its numerical advantages, as it comprises a single standard reliability analysis plus some additional system analyses.

Keywords: Fuzzy probability, Reliability analysis, Probability sensitivity analysis, Intervening variables

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1. Introduction

Probability theory has been widely accepted by the engineering community as a means to account for the unavoidable effects of uncertainty on the performance of built systems. Hence, the development and application of methods for uncertainty quantification within a probabilistic framework has been the subject of active research, see e.g. [1, 2, 3, 4, 5, 6]. While classical probability theory offers a most appropriate framework for describing aleatoric uncertainty, such may not be the case for those situations where uncertainty arises due to lack of knowledge, vagueness, imprecision, etc. For such cases, non-traditional models for uncertainty quantification may become more suitable, as they may take into account both aleatoric and epistemic uncertainty [7, 8, 9]. Among these non-traditional models, fuzzy probability offers a most convenient framework, as it allows characterizing aleatoric uncertainty through probability distributions while the imprecision on the probabilistic model is described through fuzzy sets. In this way, it is possible to perform an analysis where probabilistic information and imprecision are preserved explicitly, thus providing valuable insight on the behavior of a system, its level of safety and its sensitivity with respect to the imprecision in the specification of the probabilistic model [7]. In other words, the framework provided by fuzzy probability can be seen as a collection of probabilistic models which are indexed by the fuzzy model.

The above discussion clearly indicates that fuzzy probability may convey much more information than a traditional probabilistic analysis. Although this is certainly a most attractive feature, the practical application of fuzzy probabilities can become extremely challenging. In this sense, it should be recalled that traditional probabilistic analysis usually demands considerable numerical efforts, as repeated system analyses are required in order to quantify the effects of uncertainty [3]. Hence, performing fuzzy probability analysis usually becomes even more challenging, as an additional layer (that is, the fuzzy description of the probabilistic model) is included in the analysis as well. The whole problem becomes even more challenging when focusing on the calculation of failure probabilities, that is, probabilities of ocurrence of a certain undesirable event. This is due to the fact that failure probabilities are usually small (e.g. 10^{-3} or less), as they involve events of rare ocurrence. In view of this challenge, several approaches have been developed for coping with problems involving fuzzy probabilities (and in general, imprecise probabilities [7]) and failure probability estimation, including optimization approaches [10, 11], specially devised sampling approaches [12, 13, 14, 15, 16], approximation concepts [17], surrogate models [8, 18], etc. A common feature among these approaches is that they cope (to a certain extent) simultaneously with both aleatoric and epistemic uncertainty.

The purpose of this contribution is proposing an approach for computing failure probabilities within the framework of fuzzy analysis in an approximate way. That is, the objective is characterizing the failure probability associated with a problem in terms of its associated membership function. The proposed approach is cast for the particular case where a probabilistic model describes the uncertainty in the input parameters of a system while the distribution parameters (e.g. mean and standard deviation) of that probabilistic model are characterized by means of fuzzy sets. The novelty of the proposed approach is that it involves a single standard reliability analysis that estimates the probability of failure of a system considering a prescribed probabilistic model plus some additional system analyses. Then, the imprecision due to the fuzzy distribution parameters is captured by retrieving probability sensitivity information from the reliability analysis stage [19] in combination with the application of intervening variables [20, 21]. In this way, numerical efforts associated with the calculation of fuzzy failure probabilities are drastically reduced, as it becomes the byproduct of a standard reliability analysis. The scope of application of the proposed approach comprises systems where the performance function exhibits a moderately nonlinear behavior with respect to the uncertain input parameters of the associated model.

The rest of this paper is organized in the following way. Section 2 describes the specific problem studied in this contribution, that is, calculating the fuzzy failure probability associated with a system. Section 3 presents the proposed framework for approximating fuzzy failure probabilities. The application of this framework is evaluated in Section 4 by means of some relatively simple illustrative examples. The paper closes with conclusions and challenges for future work in Section 5.

2. Problem Statement

2.1. Failure Probability: Precise Distribution Parameters

Assume that there is a certain system of interest whose performance must be quantified. For that purpose, a numerical model of the system is formulated using a suitable technique, for example, the finite element method [22]. During its definition, n_x input variables of this model are identified as uncertain and are characterized as independent random variables $X_i,\ i = 1, \ldots, n_x$ with associated probability density function $f_{X_i}(x_i|\boldsymbol{\theta}_i)$, where $\boldsymbol{\theta}_i$ is a vector of dimension $n_i \times 1$ $(i = 1, \ldots, n_x)$ that contains distribution parameters such as mean, standard deviation, etc. The joint probability density is denoted as $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$, where $\boldsymbol{x} = [x_1, \ldots, x_{n_x}]^T$, $\boldsymbol{\theta} = [\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_{n_x}^T]^T$ and $(\cdot)^T$ denotes transpose; the associated joint cumulative density function is $F_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$.

The above discussion highlights the fact that the performance of the model becomes uncertain due to the uncertainty in its input parameters. In such situation, some particular realizations of the random inputs may cause an undesirable behavior whose chance of occurrence can be quantified in terms of the classical *failure probability* integral p_F, which is equal to:

$$p_F = \int_{g(\boldsymbol{x})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})d\boldsymbol{x} \tag{1}$$

where p_F denotes failure probability and $g(\boldsymbol{x})$ is the so-called performance function [23], which assumes a value equal or smaller than zero whenever a realization $\boldsymbol{x}$ of the random input variables causes the system's response to exceed a prescribed threshold level; in the following, it is assumed that the performance function is twice differentiable. In practical situations, no closed form solutions exist for the failure probability integral. Hence, failure probabilities are often assessed resorting to either approximate techniques [23] or simulation methods [1, 3], which can comprise substantial numerical efforts due to the necessity of repeatedly evaluating the numerical model for different realizations of the uncertain input parameters.

2.2. Failure Probability: Fuzzy Distribution Parameters

The structure of eq. (1) indicates that the value of the failure probability p_F is dependent on the selection of the distribution parameters $\boldsymbol{\theta}$; hence, $p_F = p_F(\boldsymbol{\theta})$. In practical situations, determining the precise values of these distribution parameters may become challenging due to issues such as insufficient knowledge, errors in measurements, lack of data, etc. In such scenario, it may be appropriate to describe these distribution parameters as fuzzy sets. Thus, the fuzzy set $\tilde{\theta}_{l,i}$ associated with the l-th distribution parameter of the i-th random variable is:

$$\tilde{\theta}_{l,i} = \left\{ \left(\theta_{l,i}, \mu_{\tilde{\theta}_{l,i}}(\theta_{l,i})\right) : (\theta_{l,i} \in \Theta_{l,i}) \wedge \left(\mu_{\tilde{\theta}_{l,i}}(\theta_{l,i}) \in [0,1]\right) \right\},$$
$$l = 1, \ldots, n_i,\; i = 1, \ldots, n_x \tag{2}$$

where $\theta_{l,i}$ denotes the value of the l-th distribution parameter associated with the i-th random variable which belongs to the fundamental set $\Theta_{l,i}$ and $\mu_{\tilde{\theta}_{l,i}}(\theta_{l,i})$ is the membership function. Two important issues to be noted from the above characterization are the following. First, the fundamental set $\Theta_{l,i}$ contains all physical values that the distribution parameters $\theta_{l,i}$ may assume, while the fuzzy set $\tilde{\theta}_{l,i}$ associates a membership to each value contained in the fundamental set. Second, in classical set theory, the membership of an element to a set is binary; that is, an element either belongs or not to a set (this is denoted as crisp set). Instead, in fuzzy sets, the membership $\mu_{\tilde{\theta}_{l,i}}(\theta_{l,i})$ represents the degree with which $\theta_{l,i}$ belongs to $\tilde{\theta}_{l,i}$.

It is assumed that the fuzzy sets $\tilde{\theta}_{l,i}$ possess only one element $\theta_{l,i}$ for which $\mu_{\tilde{\theta}_{l,i}}(\theta_{l,i}) = 1$ and that they are convex [7, 9], i.e.:

$$\mu_{\tilde{\theta}_{l,i}}\left(\theta_{l,i}^{C}\right) \geq \min\left(\mu_{\tilde{\theta}_{l,i}}\left(\theta_{l,i}^{L}\right), \mu_{\tilde{\theta}_{l,i}}\left(\theta_{l,i}^{R}\right)\right), \ \forall\ \theta_{l,i}^{L}, \theta_{l,i}^{C}, \theta_{l,i}^{R} \in \Theta_{l,i} \tag{3}$$

such that $\theta_{l,i}^{L} \leq \theta_{l,i}^{C} \leq \theta_{l,i}^{R},\ l = 1, \ldots, n_i,\ i = 1, \ldots, n_x$. A schematic representation of a convex fuzzy set as described above is shown in Figure 1.

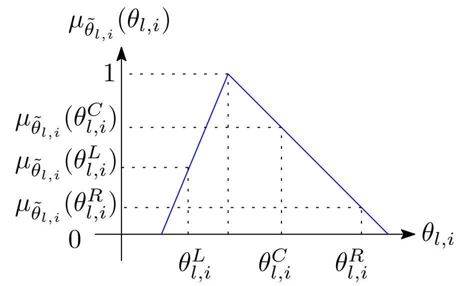

Figure 1: Schematic representation of membership function associated with convex fuzzy set

The fuzziness associated with the distribution parameters propagates to the probabilistic model implying, for example, that there is a *fuzzy set* of cumulative density functions (instead of a single cumulative density function). This idea is represented schematically in Figure 2, where it is assumed for simplicity that $n_i = n_x = 1$. Figure 2(a) illustrates the membership function associated with the distribution parameter while Figure 2(b.1) illustrates the fuzzy cumulative density function. It is important to note from Figure 2(b.1) that there is actually a collection of different cumulative density functions, each of them with an associated membership value μ; Figure 2(b.2) further clarifies this point, by illustrating the membership function of the cumulative density function associated with a specific realization x^* of the uncertain input parameter. Due to the fuzziness in the probabilistic model, the failure probability becomes a fuzzy set as well. This is shown in Figure 2(c). However, the membership function associated with the failure probability is – in general – not known analytically as there is no closed form expression for the failure probability integral in eq. (1). One possibility for determining its membership function in a discrete way is applying the so-called α-level optimization strategy [7, 9], which consists in constructing crisp sets of the distribution parameters by selecting elements from the support of the associated fuzzy set which possess a membership value equal or larger than a certain threshold α, where α denotes the membership level under analysis; clearly, $0 < \alpha \leq 1$. The crisp set associated with the distribution parameter is:

$$\underline{\theta}_{l,i,\alpha_k} = \left\{ \theta_{l,i} \in \Theta_{l,i} : \mu_{\tilde{\theta}_{l,i}}(\theta_{l,i}) \geq \alpha_k \right\}, \\ l = 1, \ldots, n_i, \; i = 1, \ldots, n_x, \; \alpha_k \in (0, 1] \tag{4}$$

where $\alpha_k, \; k = 1, \ldots, N_c$ denotes the α-cut value under consideration and N_c is the number of discrete cuts considered for analysis; $\underline{\theta}_{l,i,\alpha_k}$ denotes the set of possible values of $\theta_{l,i}$ for a given membership value α_k. Figure 2(a) illustrates the crisp set $\underline{\theta}_{\alpha_k}$; recall that in the Figure, it is assumed that $n_i = n_x = 1$ and hence, indexes l and i are omitted. Once the α-cuts of the distribution parameters have been defined as indicated above, the crisp set of the failure probability $\underline{p}_{F,\alpha_k}$ for the specific α-cut value is given by:

$$\underline{p}_{F,\alpha_k} = \{ p_F : \left(\theta_{l,i} \in \underline{\theta}_{l,i,\alpha_k}, \; l = 1, \ldots, n_i, \; i = 1, \ldots, n_x \right) \wedge \\ p_F = p_F(\boldsymbol{\theta}) \} \tag{5}$$

The crisp set $\underline{p}_{F,\alpha_k}$ is represented schematically in Figure 2(c).

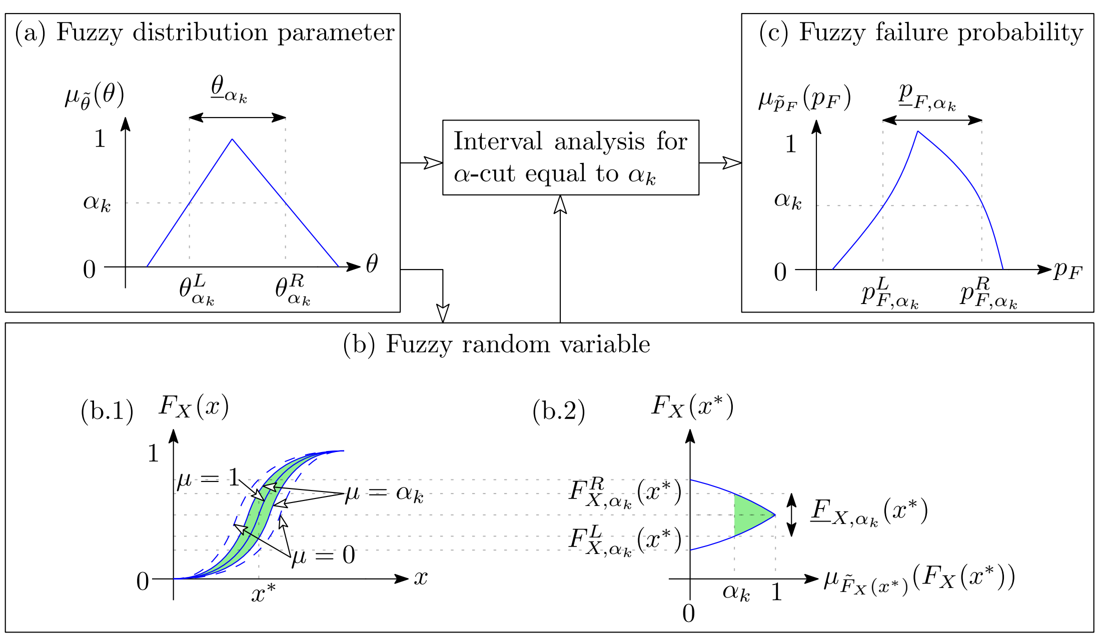

Figure 2: Schematic representation of α-level optimization strategy

Given the assumption that the sets $\underline{\theta}_{l,i,\alpha_k}, \; l = 1, \ldots, n_i, \; i = 1, \ldots, n_x$ are compact and convex, these sets are fully described by their minimum and maximum values, which are denoted with superscripts $(\cdot)^L$ and $(\cdot)^R$, respectively, as shown in Figure 2(a). Moreover, as eq. (1) establishes a continuous mapping between the fuzzy distribution parameters and the failure probability, the crisp set $\underline{p}_{F,\alpha_k}$ is also fully described by its minimum and maximum value, as shown in Figure 2 with superscripts $(\cdot)^L$ and $(\cdot)^R$, respectively. Hence, the description of the crisp set $\underline{p}_{F,\alpha_k}$ involves the solution of the following two optimization problems [9].

$$p^L_{F,\alpha_k} = \min_{\boldsymbol{\theta}} \left(p_F(\boldsymbol{\theta}) \right), \ \theta_{l,i} \in \underline{\theta}_{l,i,\alpha_k}, \ l = 1, \ldots, n_i, \ i = 1, \ldots, n_x \tag{6}$$

$$p^R_{F,\alpha_k} = \max_{\boldsymbol{\theta}} \left(p_F(\boldsymbol{\theta}) \right), \ \theta_{l,i} \in \underline{\theta}_{l,i,\alpha_k}, \ l = 1, \ldots, n_i, \ i = 1, \ldots, n_x \tag{7}$$

Note that the two optimization problems described above correspond actually to performing an interval analysis problem for the α-cut equal to α_k, as indicated in Figure 2. Provided that sufficient α-cut levels are considered, it is possible to generate a discrete approximation of the sought membership function $\mu_{\tilde{p}_F}(p_F)$.

The introduction of fuzziness on the distribution parameters implies that the failure probability becomes fuzzy as well, as already discussed above. The advantage of such formulation is that both probabilistic and imprecise information are kept explicitly and separated. That is, for different membership values, the crisp set associated with the failure probability is known explicitly. Such information is of utmost value for practical analysis, as it may reveal the sensitivity of the failure probability with respect to the level of imprecision in the distribution parameters. Nonetheless, the characterization of the failure probability as a fuzzy variable may become extremely challenging, as the application of the α-level optimization scheme described above demands solving a number of optimization problems involving the failure probability integral as defined in eq. (1), yielding the whole procedure extremely demanding from a numerical viewpoint. In view of this issue, the rest of this contribution proposes a framework for reducing the numerical efforts associated with the calculation of fuzzy probabilities.

3. Approximate Representation of Fuzzy Probability Applying Intervening Variables

3.1. General Remarks

The challenge of applying the α-level optimization scheme as described previously consists in solving the optimization problems in eqs. (6) and (7). A possible means for decreasing the numerical cost associated with this step is approximating the failure probability $p_F(\boldsymbol{\theta})$ with a surrogate model $p_F^S(\boldsymbol{\theta})$ that is an explicit function of the distribution parameters $\boldsymbol{\theta}$. However, the construction of such surrogate model may not be straightforward, as the functional dependence of the failure probability with respect to the distribution parameters may be quite involved. In such situation, the application of *intervening variables* may become helpful. Intervening variables were originally developed in the field of structural optimization, see e.g. [24, 25], and are actually nonlinear functions of the basic variables (in this case, the distribution parameters) and their main characteristic is that the quantity depending upon them (in this case, the failure probability) behaves more linearly with respect to these intervening variables than with respect to the distribution parameters. Hence, intervening variables may have the potential of reducing the nonlinearity of the problem. While intervening variables are used customarily in structural optimization [24, 25], there are few examples of its application for uncertainty quantification, see e.g. [20, 21, 26, 27].

The above discussion highlights the potential benefits of applying intervening variables. However, their practical application demands prescribing their functional form. Such selection has been performed in the past resorting to a physical understanding of a particular system combined with sensitivity analysis, see e.g. [20, 21]. However, for the problem at hand, the selection of the functional form of the intervening variables becomes a problem by itself, as it is not obvious how the distribution parameters $\boldsymbol{\theta}$ affect the failure probability $p_F(\boldsymbol{\theta})$. Thus, in the following, a criterion for selecting the functional form of the intervening variables is discussed, where $\xi_j(\boldsymbol{\theta})$ denotes the j-th intervening variable such that $j = 1, \ldots, n_\xi$. As these intervening variables depend on the distribution parameters, they are termed as *intervening variables associated with the distribution parameters.*

3.2. Approximate Representation of the Performance Function Through Physical Intervening Variables

Before moving into the issue of prescribing a functional form for the intervening variables associated with the distribution parameters, first it is proposed to construct an approximate representation of the performance function $g(\boldsymbol{x})$. Such approximation (denoted in the following as $g^S(\boldsymbol{x})$) will actually allow to formulate later on the sought intervening variables associated with the distribution parameters.

The approximate performance function $g^S(\boldsymbol{x})$ is chosen as linear with respect to the *physical intervening variables* $\eta_i(x_i),\ i = 1, \ldots, n_x$, that is:

$$g(\boldsymbol{x}) \approx g^S(\boldsymbol{x}) = g_0 + \sum_{i=1}^{n_x} g_i \left(\eta_i(x_i) - \eta_i\left(x_i^0\right)\right) \tag{8}$$

where $g_i,\ i = 0, \ldots, n_x$ are real, constant coefficients and $\boldsymbol{x}^0$ is an expansion point associated with the random input parameters of the problem. The expansion point is selected as the expected value of the random input variables assuming that the distribution parameter vector is equal to $\boldsymbol{\theta}^0$, that is $\boldsymbol{x}^0 = \mathbb{E}_{\boldsymbol{X}|\boldsymbol{\theta}^0}[\boldsymbol{x}]$, where $\mathbb{E}_{\boldsymbol{X}|\boldsymbol{\theta}^0}[\cdot]$ denotes expectation with respect to the probability density function associated with the random variable vector $\boldsymbol{X}$ given the distribution parameters $\boldsymbol{\theta}^0$. Details about the selection of $\boldsymbol{\theta}^0$ are discussed in Section 3.4.

The physical intervening variables $\eta_i(x_i),\ i = 1, \ldots, n_x$ are a function of the input parameters of the system and could be regarded as model-based. This is the reason for denoting them as *physical*, in contrast to the intervening variables $\xi_j(\boldsymbol{\theta}),\ j = 1, \ldots, n_\xi$ associated with distribution parameters that affect the probabilistic description. The physical intervening variables are chosen to be of the power type [28, 29], that is:

$$\eta_i(x_i) = x_i^{m_i},\ i = 1, \ldots, n_x \tag{9}$$

where $m_i,\ i = 1, \ldots, n_x$ are real, constant coefficients. Numerical experience in problems of stochastic finite element analysis [20] and fuzzy structural analysis [21] indicates that the intervening variables of power type may be applicable in case of moderately nonlinear performance functions, as encountered in a number of problems of linear static or linear dynamic structural analysis.

The coefficients $g_i,\ i = 0, \ldots, n_x$ and $m_i,\ i = 1, \ldots, n_x$ are chosen by enforcing the following three conditions [20].

(a) The value of the exact and approximate performance functions at the expansion point must be equal, that is, $g(\boldsymbol{x}^0) = g^S(\boldsymbol{x}^0)$.

(b) The first order derivatives of the exact and approximate performance functions at the expansion point are equal, that is, $\partial g(\boldsymbol{x}^0)/\partial x_i = \partial g^S(\boldsymbol{x}^0)/\partial x_i,\ i = 1, \ldots, n_x$.

(c) The second order derivatives of the exact and approximate performance functions at the expansion point are equal, that is, $\partial^2 g(\boldsymbol{x}^0)/\partial x_i^2 = \partial^2 g^S(\boldsymbol{x}^0)/\partial x_i^2, i = 1, \ldots, n_x$. Clearly, this condition excludes second order mixed partial derivatives.

The enforcement of the three conditions described above provides $2n_x + 1$ equations that allow determining the values of the sought coefficients:

$$g_0 = g\left(\boldsymbol{x}^0\right) \tag{10}$$

$$m_i = \begin{cases} 1 & \text{if } \frac{\partial g\left(\boldsymbol{x}^0\right)}{\partial x_i} = 0 \\ 1 + x_i^0 \frac{\partial^2 g\left(\boldsymbol{x}^0\right)/\partial x_i^2}{\partial g(\boldsymbol{x}^0)/\partial x_i} & \text{if } \frac{\partial g\left(\boldsymbol{x}^0\right)}{\partial x_i} \neq 0 \end{cases}, \ i = 1, \ldots, n_x \tag{11}$$

$$g_i = \begin{cases} 0 & \text{if } m_i = 0 \\ \frac{1}{m_i} \frac{\partial g\left(\boldsymbol{x}^0\right)}{\partial x_i} \left(x_i^0\right)^{m_i - 1} & \text{if } m_i \neq 0 \end{cases}, \ i = 1, \ldots, n_x \tag{12}$$

The above equations take into account some ill-conditioned cases that can be found when some of the partial derivatives are equal to zero. For a more detailed description on the criteria for constructing the approximation of the performance function, it is referred to [20, 21].

It is emphasized that the approximate representation of the performance function as proposed in eq. (8) does not play a direct role in evaluating failure probabilities. Instead, it is used as a means for deriving a functional form for the intervening variables associated with the distribution parameters, as discussed in the following.

3.3. Identification of Intervening Variables Associated with Distribution Parameters

The identification of the functional form of the intervening variables associated with the distribution parameters $\xi_j(\boldsymbol{\theta}),\ j = 1, \ldots, n_\xi$ demands gaining some insight on how the distribution parameters $\boldsymbol{\theta}$ affect the failure probability $p_F(\boldsymbol{\theta})$. In order to gain this insight, the so-called first order reliability method (see, e.g. [23]) is applied in the following. It is emphasized that this analysis is not intended for probability estimation but for deducing a functional form for $\xi_j(\boldsymbol{\theta}),\ j = 1, \ldots, n_\xi$ only.

The first step of the analysis is projecting the problem from the space of physical random variables to the standard normal space. Recalling the assumption of independent random variables (see Section 2.1), such projection is accomplished by imposing $F_{X_i}\left(x_i|\boldsymbol{\theta}_i\right) = \Phi(z_i),\ i = 1, \ldots, n_x$, where $\Phi(\cdot)$ is the standard Gaussian cumulative density function, z_i is a realization of the standard Gaussian random variable Z_i and $F_{X_i}(\cdot|\boldsymbol{\theta}_i)$ is the cumulative density function associated with the random variable X_i [23].

Once the reliability problem has been projected into the standard normal space, the failure probability is given by $p_F(\boldsymbol{\theta}) = \Phi(-\beta(\boldsymbol{\theta}))$, where $\beta(\boldsymbol{\theta})$ is the reliability index, that is roughly approximated considering the value of the performance function and its gradient at the origin of the standard normal space ($\boldsymbol{z} = \mathbf{0}$) [23]:

$$\beta(\boldsymbol{\theta}) = \left(\frac{g(\boldsymbol{t}(\boldsymbol{z}|\boldsymbol{\theta}))}{||\nabla_{\boldsymbol{z}} g(\boldsymbol{t}(\boldsymbol{z}|\boldsymbol{\theta}))||} \right) \Bigg|_{\boldsymbol{z}=\mathbf{0}} \tag{13}$$

where $||\cdot||$ denotes Euclidean norm, $\nabla_{\boldsymbol{z}}$ denotes nabla operator, i.e. $\nabla_{\boldsymbol{z}} = [\partial/\partial z_1, \ldots, \partial/\partial z_{n_x}]^T$ and $\boldsymbol{t}(\boldsymbol{z}|\boldsymbol{\theta})$ is the vector-valued transformation function whose i-th component is $x_i = t_i(z_i|\boldsymbol{\theta}_i) = F_{X_i}^{-1}(\Phi(z_i)|\boldsymbol{\theta}_i)$. It is remarked that eq. (13) provides a rough estimate of the actual reliability index; however, it is recalled that this rough estimate is not applied for probability estimation but for visualizing how the distribution parameters affect the failure probability.

By introducing the approximate representation of the performance function $g^S(\boldsymbol{x})$ of eq. (8) (that comprises the physical intervening variables $\eta_i(x_i),\ i = 1, \ldots, n_x$) into eq. (13), it is found that:

$$\beta(\boldsymbol{\theta}) = \sum_{i=1}^{n_x} \left(\frac{g_i\left(\eta_i(t_i(0|\boldsymbol{\theta}_i) - \eta_i\left(x_i^0\right)\right)}{\sqrt{\sum_{i=1}^{n_x}\left(g_i\left(\frac{\partial \eta_i(t_i(z_i|\boldsymbol{\theta}_i))}{\partial z_i}\right)\Big|_{z_i=0}\right)^2}} \right) + \frac{g_0}{\sqrt{\sum_{i=1}^{n_x}\left(g_i\left(\frac{\partial \eta_i(t_i(z_i|\boldsymbol{\theta}_i))}{\partial z_i}\right)\Big|_{z_i=0}\right)^2}} \tag{14}$$

In view of the above discussion, it is proposed to construct the surrogate model for the failure probability as:

$$p_F(\boldsymbol{\theta}) \approx p_F^S(\boldsymbol{\theta}) = \Phi\left(h_0 + \sum_{j=1}^{n_\xi} h_j\left(\xi_j(\boldsymbol{\theta}) - \xi_j\left(\boldsymbol{\theta}^0\right)\right)\right) \tag{15}$$

where $h_j,\ j = 0, \ldots, n_\xi$ are real, constant coefficients (whose calculation is discussed in Section 3.4); $\boldsymbol{\theta}^0$ is a reference value of the distribution parameters (whose selection is discussed in Section 3.4); and $\xi_j(\boldsymbol{\theta}),\ j = 1, \ldots, n_\xi$ denote the intervening variables associated with the distribution parameters, which are selected based on eq. (14) and are equal to:

$$\xi_j(\boldsymbol{\theta}) = \frac{\eta_j(t_j(0|\boldsymbol{\theta}_j))}{\sqrt{\sum_{i=1}^{n_x}\left(g_i\left(\frac{\partial \eta_i(t_i(z_i|\boldsymbol{\theta}_i))}{\partial z_i}\right)\Big|_{z_i=0}\right)^2}},\ j = 1, \ldots, n_x \tag{16}$$

$$\xi_j(\boldsymbol{\theta}) = \frac{1}{\sqrt{\sum_{i=1}^{n_x}\left(g_i\left(\frac{\partial \eta_i(t_i(z_i|\boldsymbol{\theta}_i))}{\partial z_i}\right)\Big|_{z_i=0}\right)^2}},\ j = n_x + 1 \tag{17}$$

where the total number of intervening variables is $n_\xi = n_x + 1$.

It is observed that the approximation proposed in eq. (15) comprises the standard Gaussian cumulative density function, whose argument is a function of the distribution parameters. A similar approximation has been proposed in [30] within the context of reliability-based design optimization. However, when comparing the approximation in eq. (15) with that of [30], it is noted that the major difference is that the one proposed herein includes intervening variables, which help in improving the quality of the approximation. In addition, it is observed that the intervening variables associated with the distribution parameters as proposed in eqs. (16) and (17) possess a functional form that is dependent on both the physical intervening variables and the transformation functions $t_i(z_i|\boldsymbol{\theta}_i),\ i = 1, \ldots, n_x$. As shown in Appendix A, the intervening variables $\xi_j(\boldsymbol{\theta}),\ j = 1, \ldots, n_\xi$ can become nonlinear functions of the distribution parameters and hence, may capture to some extent the nonlinear relation between distribution parameters and the failure probability.

3.4. Construction of Approximate Representation of Fuzzy Probability

Once the intervening variables associated with the distribution parameters have been defined, it is possible to proceed to construct the approximation of the failure probability $p_F^S(\boldsymbol{\theta})$. The first issue to be addressed is the selection of the reference value of the distribution parameters $\boldsymbol{\theta}^0$. A possible criterion is selecting this reference value to be equal to the value of the distribution parameters for which the membership function is equal to 1, that is $\theta^0_{l,i} = \{\theta_{l,i} \in \Theta_{l,i} : \mu_{\tilde{\theta}_{l,i}}(\theta_{l,i}) = 1\}$. Recall that due to the assumptions in Section 2.2, there is a single value for which the membership function is equal to 1. Moreover, other choices for the reference value $\boldsymbol{\theta}^0$ could be devised based on the particular problem at hand.

The second issue for constructing the approximate representation of the failure probability is calculating the coefficients $h_j,\ j = 0, \ldots, n_\xi$ associated with eq. (15). It is proposed to calculate these coefficients based on the information drawn from a single standard reliability analysis carried out considering that the distribution parameters are equal to their reference value $\boldsymbol{\theta}^0$. Such analysis can be performed with any appropriate method (e.g. Monte Carlo simulation, Line Sampling [3], Subset Simulation [1], etc.), providing both the failure probability $p_F(\boldsymbol{\theta}^0)$ and its sensitivity $\partial p_F(\boldsymbol{\theta}^0)/\partial\theta_{l,i},\ l = 1, \ldots, n_i,\ i = 1, \ldots, n_x$. It is important to remark that estimates of the probability sensitivity with respect to distribution parameters can be obtained as a byproduct of a standard reliability analysis, as discussed in [19, 31, 32, 33, 34]. That is, the gradient of the probability with respect to the distribution parameters can be estimated with no additional evaluations of the performance function.

Following the ideas described above, the coefficient h_0 is calculated by enforcing the condition that the failure probability value provided by the standard reliability analysis is equal to the probability provided by the surrogate model at the reference value for the distribution parameters, that is $p_F(\boldsymbol{\theta}^0) = p_F^S(\boldsymbol{\theta}^0)$. This yields:

$$h_0 = \Phi^{-1}\left(p_F\left(\boldsymbol{\theta}^0\right)\right) \tag{18}$$

The rest of the coefficients $h_j,\ j = 1,\ldots,n_\xi$ are calculated by enforcing the condition that the gradient of the failure probability drawn from the standard reliability analysis equals the gradient obtained from the surrogate model at the reference value of the distribution parameters, that is $\partial p_F(\boldsymbol{\theta}^0)/\partial\theta_{l,i} = \partial p_F^S(\boldsymbol{\theta}^0)/\partial\theta_{l,i},\ l = 1,\ldots,n_i,\ i = 1,\ldots,n_x$. This yields the following system of $n_\theta = \sum_{i=1}^{n_x} n_i$ equations:

$$\begin{bmatrix} \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{1,1}} \\ \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{2,1}} \\ \vdots \\ \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{n_i,n_x}} \end{bmatrix} = \phi(h_0) \underbrace{\begin{bmatrix} \frac{\partial\xi_1\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{1,1}} & \cdots & \frac{\partial\xi_{n_\xi}\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{1,1}} \\ \frac{\partial\xi_1\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{2,1}} & \cdots & \frac{\partial\xi_{n_\xi}\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{2,1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial\xi_1\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{n_i,n_x}} & \cdots & \frac{\partial\xi_{n_\xi}\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{n_i,n_x}} \end{bmatrix}}_{\boldsymbol{J}(\boldsymbol{\theta}^0)^T} \begin{bmatrix} h_1 \\ \vdots \\ h_{n_\xi} \end{bmatrix} \tag{19}$$

where $\phi(\cdot)$ denotes the standard Gaussian probability density function and $\boldsymbol{J}(\boldsymbol{\theta}^0)$ is the Jacobian matrix associated with the set of intervening variables $\xi_j(\boldsymbol{\theta}),\ j = 1,\ldots,n_\xi$ evaluated at $\boldsymbol{\theta}^0$. Depending on the particular problem being studied, the above system of equations may not be determined, as it may occur that $n_\theta \neq n_\xi$. Hence, this system of equations is solved by means of the Moore-Penrose inverse, yielding the following expression for the coefficients $h_j,\ j = 1,\ldots,n_\xi$.

$$\begin{bmatrix} h_1 \\ \vdots \\ h_{n_\xi} \end{bmatrix} = \frac{1}{\phi(h_0)} \left(\boldsymbol{J}\left(\boldsymbol{\theta}^0\right)^T\right)^+ \begin{bmatrix} \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{1,1}} \\ \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{2,1}} \\ \vdots \\ \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{n_i,n_x}} \end{bmatrix} \tag{20}$$

In the above equation, $(\cdot)^+$ denotes Moore-Penrose inverse, which can be computed using singular value decomposition [35]. The main advantage of the scheme described above for determining the coefficients $h_j,\ j = 1,\ldots,n_\xi$ is that they are calculated based on the information retrieved from a single standard reliability analysis. Moreover, note that the Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta}^0)$ can be calculated analytically for certain distribution types, as discussed in Appendix A.

3.5. Scope of Application

The proposed approximation for the failure probability as an explicit function of the distribution parameters as presented in eqs. (15), (16), (17), (18) and (20) has been derived based on several approximation concepts. First, the performance function is represented approximately as a linear function in terms of some physical intervening variables (see eq. (8)) without considering interaction between the random variables. Second, the intervening variables associated with the distribution parameters are deduced based on the first order reliability method, introducing a very rough approximation for the reliability index. Finally, the surrogate model for the failure probability is calibrated based on the value of the probability and its sensitivity at a reference value of the distribution parameters. It is clear that all of these assumptions play a determinant role in the quality of the resulting approximation for the failure probability. Hence, it is expected that the proposed approximation is applicable for cases where the performance function is moderately nonlinear with respect to the uncertain input parameters $\boldsymbol{x}$. Such assertion is based on two observations. First, moderately nonlinear performance functions may be conveniently approximated by physical intervening variables, as suggested in [20, 21]. Second, moderately nonlinear performance functions should not exhibit drastic changes in the associated failure probability when the distribution parameters are perturbed.

3.6. Summary of Proposed Approach for Calculating Fuzzy Failure Probabilities

The practical application of the proposed framework for calculating fuzzy failure probabilities comprises the following steps.

(a) **Problem Setting**. Define the problem to be analyzed in terms of its performance function $g(\boldsymbol{x})$, the probability density function $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ describing the uncertainty on the input parameters of the system and the fuzzy description of the distribution parameters $\tilde{\theta}_{l,i},\ l = 1, \ldots, n_i,\ i = 1, \ldots, n_x$. Furthermore, identify a reference value of the distribution parameters $\boldsymbol{\theta}^0$ using, for example, the criterion suggested in Section 3.4.

(b) **Reliability Analysis**. Conduct a reliability analysis for the case where the distribution parameters are equal to their reference value $\boldsymbol{\theta}^0$ using any appropriate approach, for example Monte Carlo simulation, Importance sampling, Line Sampling, Subset Simulation, etc. Retrieve the failure probability $p_F(\boldsymbol{\theta}^0)$ and its sensitivity $\partial p_F(\boldsymbol{\theta}^0)/\partial\theta_{l,i},\ l = 1, \ldots, n_i,\ i = 1, \ldots, n_x$.

(c) **Approximate Representation of the Performance Function**. Calculate the performance function and its first and second order derivatives (excluding cross terms) at a reference point $\boldsymbol{x}^0$ chosen, for example, using the criterion proposed in Section 3.2. For calculating the partial derivatives, apply any appropriate scheme: analytical, semi-analytical or numerical, see e.g. [36]. Compute the coefficients associated with the approximate performance function according to eqs. (10), (11) and (12).

(d) **Surrogate Model of Failure Probability**. Define the intervening variables associated with the distribution parameters following eqs. (16) and (17). Calculate the coefficients $h_j,\ j = 0, \ldots, n_\xi$ associated with the surrogate model of the failure probability applying eqs. (18) and (20).

(e) **Fuzzy Analysis**. Determine the membership function associated with the failure probability by means of the α-level optimization strategy described in Section 2.2. For solving eqs. (6) and (7), consider the surrogate model of the failure probability and apply any appropriate optimization algorithm. As the surrogate is an explicit function of the distribution parameters, the numerical cost associated with this step should be small.

4. Examples

4.1. General Remarks

This section analyzes the application of the proposed approach for calculating fuzzy failure probabilities. In the examples presented, reliability analysis (see step (b) of Section 3.6) is performed by means of Importance Sampling (see, e.g. [3]). A brief overview on this simulation approach can be found in Appendix C. In addition, each of the aforementioned illustrative examples is solved considering two different strategies for approximating the failure probability as an explicit function of the distribution parameters:

(a) The surrogate model proposed in eq. (15) but where the argument of the standard Gaussian cumulative density function is a linear polynomial with respect to the distribution parameters, as discussed in detail in Appendix B. In other words, *direct* variables are considered instead of intervening variables. This approximate representation is termed in the following as *Linear*.

(b) The proposed approach as described in Section 3.6. This is termed in the following as *Proposed*.

The membership functions are represented in a discrete way applying the α-level optimization strategy. The associated optimization problems at each α-cut are solved applying an appropriate global search algorithm.

4.2. Example 1: Linear Performance Function Involving Gaussian Random Variables

This example comprises the analytical performance function $g(\boldsymbol{x}) = 3\sqrt{n_x} - \sum_{i=1}^{n_x} x_i$, where the random variables $X_i,\ i = 1, \ldots, n_x = 2$ are independent and follow Gaussian distributions with mean value μ_i and standard deviation σ_i. The imprecision on the characterization of these distribution parameters is described by means of fuzzy sets whose membership functions are shown in Figure 3.

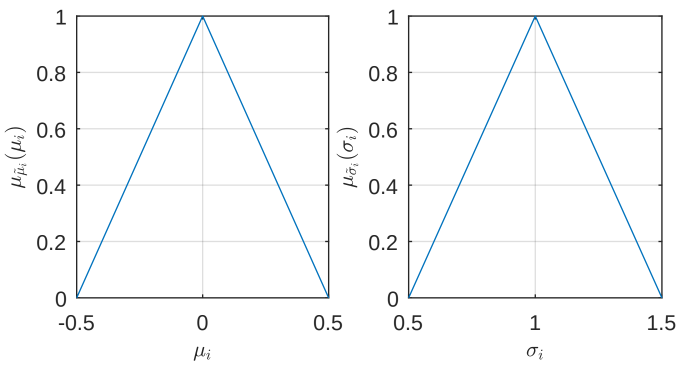

Figure 3: Example 1 – Membership function associated with mean value μ_i and standard deviation σ_i

The reliability problem is solved applying Importance Sampling considering $N = 300$ samples of the uncertain input parameters. This ensures that the failure probability associated with the reference value of the distribution parameters is estimated with a coefficient of variation smaller than 10%. Figure 4 illustrate the failure probabilities calculated in an exact and approximate way for a total of 1000 realizations of the distribution parameters; these realizations are generated at random within the minimum and maximum bounds for the distribution parameters. In the figure, the exact failure probability is calculated by means of a closed form solution available for this particular example while the approximate failure probability is calculated using either the *Linear* approximation or the *Proposed* surrogate model. In addition, the figure includes a line with slope of 45° that serves as a reference. It is seen from Figure 4 that the *Proposed* approximation exhibits good accuracy, as almost all samples lie on the reference line. Such behavior is not surprising, as for the example being studied, it can be shown that the proposed surrogate model for the failure probability can capture perfectly the functional form of the exact failure probability with respect to the distribution parameters. On the contrary, the results provided by the *Linear* approximation exhibit a poor agreement. This highlights the beneficial effect of introducing intervening variables for approximating the failure probability with respect to distribution parameters.

Figure 5 shows the membership function associated with the failure probability in terms of three curves: the *Reference* membership function (which is deduced based on the closed form solution for

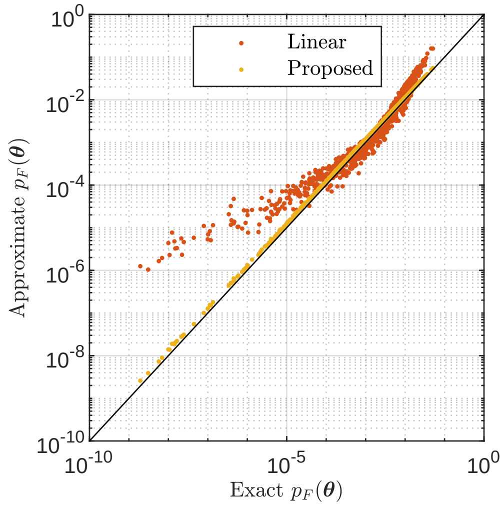

Figure 4: Example 1 – Comparison between exact and approximate failure probability for 1000 realizations of the distribution parameters

the failure probability) as well as the membership functions generated by means of α-level optimization considering the different approximations for the failure probability, that is *Linear* approximation or the *Proposed* approximation. It is noted that the *Proposed* approximation exhibits an almost perfect match with the *Reference*, while the *Linear* approximation deviates considerably from the *Reference*.

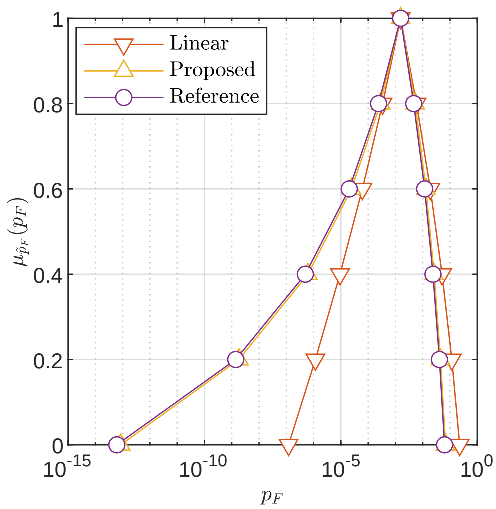

Figure 5: Example 1 – Membership function associated with the failure probability

4.3. Example 2: Shallow Foundation

This example involves estimating the membership function associated with the probability that the vertical displacement below a shallow foundation resting over an elastic soil exceeds a prescribed threshold. Figure 6 illustrates the general layout of the problem. The elastic soil is composed of two layers:

a sand layer of 9 [m] thickness and a gravel layer of 21 [m] thickness; these layers rest over an infinitely stiff rock bed. The Young's moduli of each of the soil layers (denoted as E_1 and E_2, respectively) are modeled as lognormal random variables whose distribution parameters are characterized by means of the fuzzy sets shown in Figure 7. The soil withstands a shallow foundation of 10 [m] width, which applies a distributed load whose intensity q is characterized by means of a lognormal random variable with fuzzy distribution parameters as shown in Figure 7.

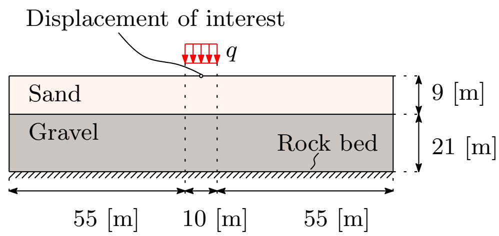

Figure 6: Schematic representation of elastic soil layer subject to loading due to a shallow foundation

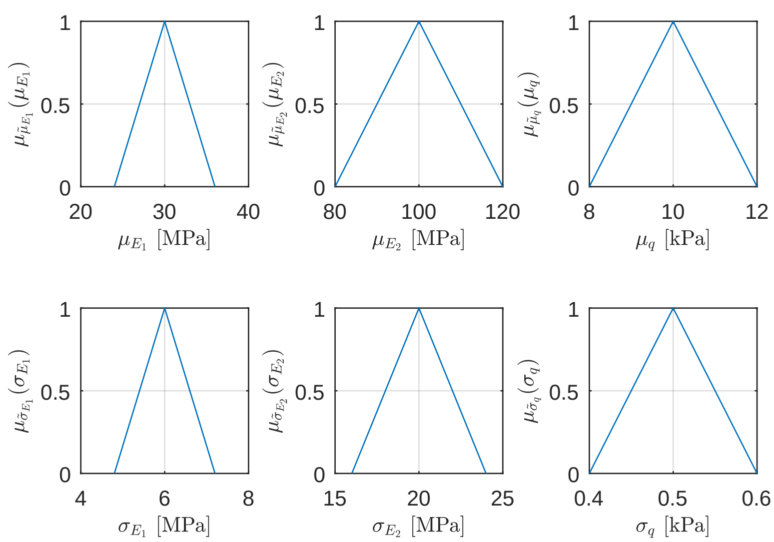

Figure 7: Example 2 – Membership function associated with mean value and standard deviation of: Young's modulus of sand (μ_{E_1} and σ_{E_1}) and gravel (μ_{E_2} and σ_{E_2}); and loading (μ_q and σ_q)

The vertical displacement below the center of the foundation is calculated by means of a small finite element model comprising a total of 160 quadrilateral elements in plain strain, resulting in 320 degrees-of-freedom. The finite element model takes advantage of the symmetry of the problem. The threshold level for the vertical displacement is set equal to 0.07 [m]. It is important to note that given that the response of interest is a displacement, the associated performance function is actually a (moderately) nonlinear function with respect to the Young's moduli.

The failure probability and its sensitivity are estimated applying Importance Sampling considering a total of $N = 300$ samples, thus ensuring that the coefficient of variation of the failure probability (for the reference value of the distribution parameters) is approximately 12%.

Figure 8 compares the failure probabilities calculated with a *Reference* solution and the different approximations of the failure probability for 300 realizations of the distributions parameters. As in the first example, these realizations of the distribution parameters are generated at random within their minimum and maximum possible values. Furthermore, it should be noted that the *Reference* failure probability corresponds to an estimate generated using Importance Sampling with a total of $N = 1000$ random samples; such number of samples ensures a sufficiently small coefficient of variation.

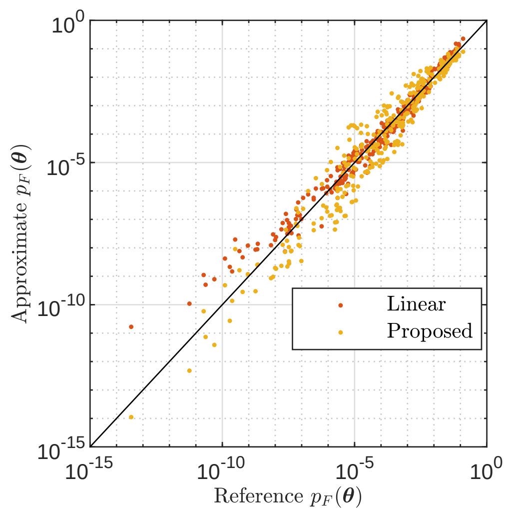

Figure 8: Example 2 – Comparison between reference and approximate failure probability for 300 realizations of the distribution parameters

The results presented in Figure 8 suggest that there is an overall good match between the *Reference* results and those produced with the different approximations. Nonetheless, some discrepancies for the case of the *Linear* approximation are observed for small values of the reference failure probability, where it tends to overestimate the probability.

Figure 9 shows the estimates of the membership function computed using the different approximations of the failure probability (as discussed in Section 4.1) and those produced with a *Reference* solution. In this case, the *Reference* solution is obtained by applying the so-called vertex method (see, e.g. [9]) at each α-cut while the failure probability is calculated by means of Importance Sampling considering $N = 1000$; validation calculations indicate that for this particular example, the vertex method is appropriate for computing the minimum and maximum failure probability for a given α-cut. The results in Figure 9 indicate that the *Proposed* approach is capable of capturing the overall behavior of the membership function, although some differences arise for small membership values. Additionally, it is observed the *Linear* approximation cannot capture the left branch of the membership function; such issue was expected in view of the results presented in Figure 8.

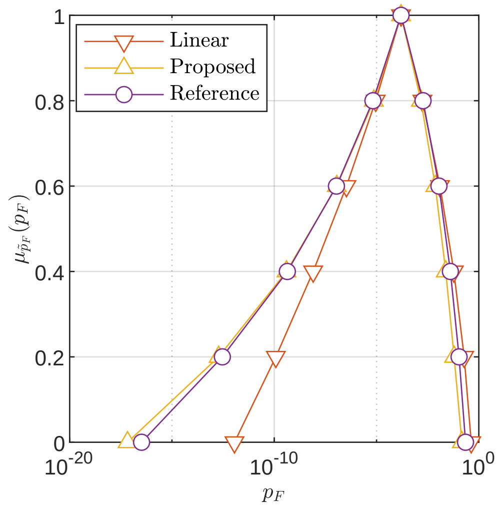

Figure 9: Example 2 – Membership function associated with the failure probability

4.4. *Example 3: Shear Frame Subjected to Horizontal Load*

The third example comprises a 15 degrees-of-freedom shear frame model subjected to horizontal load, as illustrated schematically in Figure 10. Each interstory stiffness $k_i,\ i = 1, \ldots, 15$ is characterized by means of independent lognormal random variables. The mean values and standard deviations of these stiffnesses are modeled as fuzzy, with membership functions as indicated in Figure 11. The objective is determining the fuzzy probability associated with the event where the roof displacement x_{15} exceeds a threshold of 0.075 [m]. In a similar way as discussed for the previous example, it is to be noted that the associated performance function is actually (moderately) nonlinear with respect to the interstory stiffnesses.

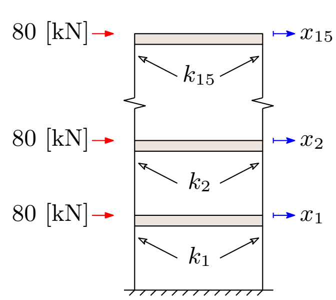

Figure 10: Example 3 – Schematic representation of shear model subjected to horizontal load

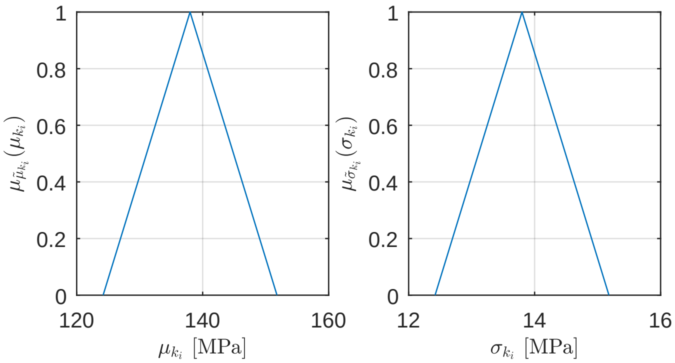

Figure 11: Example 3 – Membership function associated with mean value and standard deviation of interstory stiffness (μ_{k_i} and σ_{k_i}, $i = 1, \ldots, 15$)

The failure probability and its sensitivity are estimated applying Importance Sampling considering a total of $N = 300$ samples, thus ensuring that the coefficient of variation of the failure probability (for the reference value of the distribution parameters) is approximately 11%.

Figure 12 compares the performance of the different approaches for approximating the failure probability as a function of the distributions parameters. In this Figure, the abscissa of each point corresponds to the failure probability calculated by means of a *Reference* solution (that is, Importance Sampling considering $N = 1000$ samples) while its ordinate is the failure probability calculated with one of the approximate approaches. It is observed that the results produced with the *Proposed* approach are closer to the reference line than those produced with the *Linear* approximation.

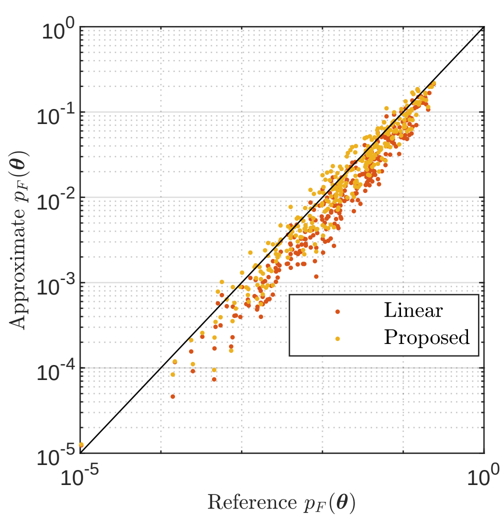

Figure 12: Example 3 – Comparison between reference and approximate failure probability for 300 realizations of the distribution parameters

The membership function associated with the failure probability calculated by means of the different approximate representations of the failure probability and a *Reference* approach are shown in Figure

13. In this case, the *Reference* solution is obtained by applying α-level optimization and Importance Sampling considering $N = 1000$; it is to be noted that the numerical efforts required for producing the reference solution are considerable, as the failure probability must be calculated for several different combinations of the distribution parameters at each α-cut. The results in Figure 13 show that the *Linear* approximation exhibits a good match with the *Reference* in the right branch of the membership function, however it is not that accurate in the left branch. On the contrary, the *Proposed* approach produces an overall good estimate of the membership function.

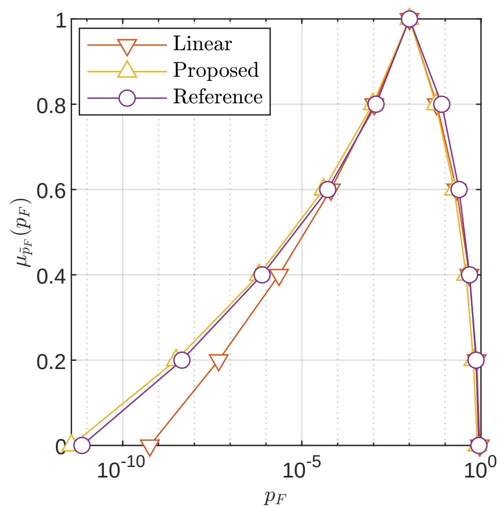

Figure 13: Example 3 – Membership function associated with the failure probability

4.5. Additional Remarks

The results presented in this Section indicate that the failure probability associated with a system can present considerable variations (of several orders of magnitude) due to the effects of imprecision on the distribution parameters of the probabilistic models. This highlights the value of explicit modeling of epistemic uncertainty, as it can reveal valuable information on the sensitivity of a reliability problem.

5. Discussion and Conclusions

The results presented in this contribution suggest that the application of intervening variables associated with the distribution parameters may be of great help for producing an approximate representation of the failure probability. Such finding complements some conclusions drawn in other contributions about the effectiveness of physical intervening variables [20, 21].

The major advantage of the proposed framework for calculating fuzzy probabilities is that it decouples the aleatoric and epistemic steps of analysis. In this way, the proposed framework becomes extremely convenient from a numerical viewpoint, as it comprises a single reliability analysis plus some additional analyses of the system. Moreover, the proposed framework can become quite attractive from a practical point of view, as it may be seen a post-processing step of a standard reliability analysis that conveys additional information (in this case, the membership functions) with little additional effort.

While the above conclusions are encouraging, they cannot be generalized, as the examples analyzed involve: moderately nonlinear performance functions; certain prescribed distribution types (Gaussian and lognormal); and relatively small numerical models of the physical system (which were chosen on purpose in order to carry out validation calculations). Hence, future research efforts aim at revising the aforementioned issues, in order to extend the range of application of the proposed framework. In a similar way, the proposed framework could be extended to more general cases, such as characterization of fuzzy probability density functions applying, for example, the probability density evolution method (PDEM, see e.g. [37]).

6. Acknowledgments

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Appendix A. Intervening Variables Associated with the Distribution Parameters: Gaussian and Lognormal Cases

Consider the case where the random input variables of a problem are characterized as Gaussian, that is $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2),\ i = 1, \ldots, n_x$, where μ_i and σ_i denote the mean and standard deviation and $\mathcal{N}(\cdot, \cdot)$ denotes Gaussian distribution. In such case, the intervening variables associated with the distribution parameters become:

$$\xi_j(\boldsymbol{\theta}) = \frac{\mu_j^{m_j}}{\sqrt{\sum_{i=1}^{n_x}\left(g_i m_i \mu_i^{m_i-1}\sigma_i\right)^2}},\ j = 1,\ldots,n_x \tag{A.1}$$

$$\xi_j(\boldsymbol{\theta}) = \frac{1}{\sqrt{\sum_{i=1}^{n_x}\left(g_i m_i \mu_i^{m_i-1}\sigma_i\right)^2}},\ j = n_x+1 \tag{A.2}$$

In case the random input variables of a problem are characterized as lognormal, that is $X_i \sim \mathcal{LN}(\mu_i, \sigma_i^2),\ i = 1,\ldots,n_x$, where μ_i and σ_i denote the mean and standard deviation and $\mathcal{LN}(\cdot,\cdot)$ denotes lognormal distribution, the intervening variables associated with the distribution parameters become:

$$\xi_j(\boldsymbol{\theta}) = \frac{e^{m_j\mu_j^G}}{\sqrt{\sum_{i=1}^{n_x}\left(g_i m_i \sigma_i^G e^{m_i\mu_i^G}\right)^2}},\ j = 1,\ldots,n_x \tag{A.3}$$

$$\xi_j(\boldsymbol{\theta}) = \frac{1}{\sqrt{\sum_{i=1}^{n_x}\left(g_i m_i \sigma_i^G e^{m_i\mu_i^G}\right)^2}},\ j = n_x+1 \tag{A.4}$$

where $\mu_i^G = \ln\left(\mu_i^2/\sqrt{\mu_i^2+\sigma_i^2}\right)$ and $\sigma_i^G = \sqrt{\ln\left(1+\sigma_i^2/\mu_i^2\right)}$ are the mean and standard deviation of the natural logarithm of the lognormal random variable.

The expressions for the intervening variables as shown in eqs. (A.1), (A.2), (A.3) and (A.4) are explicit functions of the distribution parameters. Hence, the Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ associated with the intervening variables can be calculated analytically for these cases. For the sake of brevity, these analytical expressions are not included here. However, the interested reader can obtain these expressions upon request.

Appendix B. Approximation of the Failure Probability Considering Direct Variables

A possible means for approximating the failure probability as an explicit function of the distribution parameters is resorting to eq. (15) but assuming that the argument of the standard Gaussian cumulative density function is a linear polynomial with respect to the distribution parameters, that is:

$$p_F^S(\boldsymbol{\theta}) = \Phi\left(h_0^L + \sum_{i=1}^{n_x}\sum_{l=1}^{n_i} h_{l,i}^L\left(\theta_{l,i} - \theta_{l,i}^0\right)\right) \tag{B.1}$$

where h_0^L and $h_{l,i}^L,\ l = 1,\ldots,n_i,\ i = 1,\ldots,n_x$ are real, constant coefficients. This approximation is termed as *Linear* in the following and clearly, it does not involve intervening variables; instead, it involves *direct* variables. However, note that the resulting approximation for the failure probability becomes nonlinear with respect to the distribution parameters due to the presence of the standard cumulative density function.

The coefficients required for the linear approximation can be calculated by enforcing that the exact failure probability and its derivatives are equal to their counterparts associated with the linear approximation at the expansion point $\boldsymbol{\theta}^0$. Thus:

$$h_0^L = \Phi^{-1}\left(p_F\left(\boldsymbol{\theta}^0\right)\right) \tag{B.2}$$

and:

$$h_{l,i}^L = \frac{1}{\phi\left(h_0^L\right)} \frac{\partial p_F\left(\boldsymbol{\theta}^0\right)}{\partial \theta_{l,i}},\ l = 1, \ldots, n_i,\ i = 1, \ldots, n_x \tag{B.3}$$

where $\phi(\cdot)$ denotes the standard Gaussian probability density function.

Appendix C. Estimation of Probability and Its Sensitivity Applying Importance Sampling

Importance Sampling (see, e.g. [3]) is a simulation technique that allows estimating failure probabilities. It introduces an importance sampling density function $f_{IS}(\boldsymbol{x}|\boldsymbol{\theta})$ which allows drawing samples of the random input $\boldsymbol{X}$ such that $g(\boldsymbol{x}) \leq 0$ with high frequency. The estimator of the failure probability when applying Importance Sampling is:

$$\begin{aligned} p_F(\boldsymbol{\theta}) &= \int_{g(\boldsymbol{x}) \leq 0} \frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}{f_{IS}(\boldsymbol{x}|\boldsymbol{\theta})} f_{IS}(\boldsymbol{x}|\boldsymbol{\theta}) d\boldsymbol{x} \\ &\approx \frac{1}{N} \sum_{k=1}^{N} I\left(\boldsymbol{x}^{(k)}\right) \frac{f_{\boldsymbol{X}}\left(\boldsymbol{x}^{(k)}|\boldsymbol{\theta}\right)}{f_{IS}\left(\boldsymbol{x}^{(k)}|\boldsymbol{\theta}\right)},\ \boldsymbol{x}^{(k)} \sim f_{IS}\left(\boldsymbol{x}|\boldsymbol{\theta}\right) \end{aligned} \tag{C.1}$$

where $I(\cdot)$ is the indicator function which is equal to one if $g\left(\boldsymbol{x}^{(k)}\right) \leq 0$ and zero otherwise and $\boldsymbol{x}^{(k)},\ k = 1, \ldots, N$ are independent, identically distributed samples drawn according to $f_{IS}\left(\boldsymbol{x}|\boldsymbol{\theta}\right)$. In this contribution, the importance sampling density function is centered at the design point [23], preserving the standard deviation of the reference probability density function. Moreover, as the types of problems considered herein comprise moderately nonlinear performance functions, the design point is identified applying the so-called Hasofer-Lind-Rackwitz-Fiessler algorithm [23].

In a post-processing step, it is possible to estimate the partial derivative of the failure probability with respect to distribution parameters by means of the following expression [19].

$$\begin{aligned}\frac{\partial p_F(\boldsymbol{\theta})}{\partial \theta_{l,i}} &= \int_{g(\boldsymbol{x})\leq 0} \frac{\frac{\partial f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}{\partial \theta_{l,i}}}{f_{IS}(\boldsymbol{x}|\boldsymbol{\theta})} f_{IS}(\boldsymbol{x}|\boldsymbol{\theta}) d\boldsymbol{x} \\ &\approx \frac{1}{N}\sum_{k=1}^{N} I\left(\boldsymbol{x}^{(k)}\right) \frac{\frac{\partial f_{\boldsymbol{X}}\left(\boldsymbol{x}^{(k)}|\boldsymbol{\theta}\right)}{\partial \theta_{l,i}}}{f_{IS}\left(\boldsymbol{x}^{(k)}|\boldsymbol{\theta}\right)},\ \boldsymbol{x}^{(k)} \sim f_{IS}\left(\boldsymbol{x}|\boldsymbol{\theta}\right)\end{aligned} \tag{C.2}$$

Analytical expressions for the derivative of the probability density function with respect to a distribution parameter $\partial f_{\boldsymbol{X}}\left(\boldsymbol{x}|\boldsymbol{\theta}\right)/\partial\theta_{l,i}$ can be found for several types of distributions (see, e.g.[19]).

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