A relaxed quadratic function negative-determination lemma and its application to time-delay systems ⋆

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Abstract

The quadratic function with respect to the time-varying delay has often been introduced for the analysis of systems with time-varying delays. To determine the negative definiteness of such function, this paper develops a parameter-adjustable-based lemma, which contains the lemma popularly used in literature as a special case and has potential to reduce the conservatism without requiring extra decision variables. A stability criterion for a linear time-delay system is established by using the proposed lemma, whose advantage is demonstrated via a numerical example, and the criterion is finally applied to analyze the stability of load frequency control scheme for a singe-area power system.

Key words: Time-delay system, time-varying delay, a relaxed quadratic function lemma, stability

1 Introduction

As the common phenomenon in networked control systems, a time delay has become an important factor to be considered due to its potential harm to system stability [1]. The delay in practical systems is usually a time-varying function with assessable bounds. Thus, among the methodologies for stability analysis of time-delay systems, the one based on Lyapunov-Krasovskii functional (LKF) and linear matrix inequality (LMI) is the most popular due to its adaptability to the time-varying delay. Under this framework, how to reduce the conservatism of the criteria has attracted considerable attention over the past decades [2]. For deriving stability criteria with conservatism as small as possible, many techniques have been developed, for example, different LKFs (see e.g., augmented LKF [3], delay-partition-based LKF [4], multiple-integral based LKF [5], discretized LKF [6], delay-product-type LKF [7], matrix-refined-function-based LKF [8], etc.), different methods for estimating integral terms (see e.g., free-weighting-matrix approach [9], Jensen inequality [10], Wirtinger based inequality [11], auxiliary-based inequality [12], Bessel-Legendre-based inequality [13], free-matrix-based inequality [14], etc.), and different methods of handling the reciprocal convexity for time-varying-delay systems (see e.g., reciprocally convex combination lemma [15], relaxed reciprocally convex matrix inequalities [16–19], generalized reciprocally convex combination lemmas [20], etc.).

Among the above methods, Bessel inequality, together with suitable augmented LKFs, provides an effective way to reduce conservatism, especially, Bessel inequality with enough high order has potential to derive criteria without conservatism for systems with constant delays [13]. However, for the more common case that the system has a time-varying delay, there exists an issue needing further investigation during applying the high-order Bessel inequality to reduce conservatism [21–23]. Specifically, consider a linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)),\ t \geq 0 \\ x(t) = \phi(t),\ t \in [h_2, 0] \end{cases} \tag{1}$$

where $x(t)$ is the system state, A and A_d are known real constant matrices, $\phi(t)$ is the initial condition, and $d(t)$ is the time-varying delay satisfying

$$0 = h_0 \leq h_1 \leq d(t) \leq h_2 \tag{2}$$

with h_1 and h_2 being constants. Let $h_{12} = h_2 - h_1$. During the developing of stability criteria, it needs to find

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the condition that guarantees the negative definiteness of the derivative of the LKF. When using high-order Bessel inequality to estimate the derivative, the original version of negative requirement depends on the following quadratic function with respect to the time-varying delay [21]:

$$f(y) = a_2 y^2 + a_1 y + a_0 \tag{3}$$

where $a_i \in \mathcal{R}$, $i = 0, 1, 2$ and $h_1 \leq y = d(t) \leq h_2$. It is important issue to find negativity conditions of this quadratic function for obtaining tractable LMI-based stability criteria. So far, a few work on the negative-determination of $f(y)$ has been reported. In [24], a simple condition, $f(h_i) < 0, i = 1, 2$, was given to guarantee $f(y) < 0$, while this condition is only suitable for the case of $a_2 \geq 0$. A sufficient condition reported in [21] is commonly used in literature and summarized as follows.

Lemma 1 *For a quadratic function $f(y)$ defined in (3), $f(y) < 0$ holds for $h_1 \leq y \leq h_2$ if the following holds*

$$\mathcal{L}_{1,i} = f(h_i) < 0, i = 1, 2 \tag{4}$$
$$\mathcal{L}_{1,3} = -h_{12}^2 a_2 + f(h_1) < 0 \tag{5}$$

Although the requirement of $a_2 \geq 0$ is removed in Lemma 1, it is still conservative to require (4) and (5) for guaranteeing $f(y) < 0$. For example, such requirement may limit the potential advantage of a tighter integral inequality (see example studies in Section 4 for details). It motivates the current research to develop a relaxed requirement of $f(y) < 0$.

This paper develops a new quadratic function negative-determination lemma, in which an adjustable parameter introduced makes it cover Lemma 1 and also provides potential to reduce the conservatism. Then, the proposed lemma, together with the generalized reciprocally convex combination lemma, is applied to develop a stability criterion for a linear time-delay system. The advantage of the proposed lemma is demonstrated through a numerical example and the application of the proposed stability criterion is studied for a practical example.

Notations: Throughout this paper, $\mathcal{R}^n$ refers to the n-dimensional Euclidean space; $\|\cdot\|$ means the Euclidean vector norm; the superscripts T and -1 stand for the transpose and the inverse of a matrix, respectively; $col\{y_1, y_2, \ldots, y_n\} = \left[y_1^T, y_2^T, \ldots, y_n^T\right]^T$; $X > 0$ (≥ 0) represents that X is a positive-definite (semi-positive-definite) and symmetric matrix; $Sym\{X\} = X + X^T$; $diag\{\cdot\}$ refers to a block-diagonal matrix; and the notation $*$ represents the symmetric term in a symmetric matrix.

2 A relaxed lemma

A relaxed quadratic function negative-determination lemma is developed as follows.

Lemma 2 *For a quadratic function $f(y)$ defined in (3), $f(y) < 0$ holds for $h_1 \leq y \leq h_2$ if the following holds for any given β within $[0, 1]$:*

$$\mathcal{L}_{2,i} = f(h_i) < 0, i = 1, 2 \tag{6}$$
$$\mathcal{L}_{2,3} = -\beta^2 h_{12}^2 a_2 + f(h_1) < 0 \tag{7}$$
$$\mathcal{L}_{2,4} = -(1-\beta)^2 h_{12}^2 a_2 + f(h_2) < 0 \tag{8}$$

Proof. For the case of $a_2 \geq 0$, $f(y)$ is convex in $[h_1, h_2]$. Thus, $f(y) < 0$ for $h_1 \leq y \leq h_2$ is guaranteed if (6) holds. For the case of $a_2 < 0$, $f(y)$ is concave in $[h_1, h_2]$. By letting y_0 be any constant, $f(y)$ is rewritten as:

$$\begin{aligned} f(y) &= (2a_2 y_0 + a_1)y - a_2 y_0^2 + a_0 + a_2(y - y_0)^2 \\ &\leq (2a_2 y_0 + a_1)y - a_2 y_0^2 + a_0 \qquad (9) \\ &:= g(y) \qquad (10) \end{aligned}$$

Since $g(y)$ is a linear function with respect to y, $f(y) \leq g(y) < 0$ holds for $h_1 \leq y \leq h_2$ if the following holds

$$g(h_1) = f(h_1) - a_2(h_1 - y_0)^2 < 0 \tag{11}$$
$$g(h_2) = f(h_2) - a_2(h_2 - y_0)^2 < 0 \tag{12}$$

Let $y_0 = (1-\beta)h_1 + \beta h_2$ with β being any constant within $[0, 1]$. (11) and (12) respectively lead to (7) and (8). Thus, (7) and (8) lead to $f(y) < 0$ with $a_2 < 0$ for $h_1 \leq y \leq h_2$. This completes the proof. ■

Remark 1 *If set $\beta = 1$, then (6)-(8) of Lemma 2 reduce to (4) and (5) of Lemma 1. Thus, the conditions (4) and (5) of Lemma 1 are special cases of the conditions (6)-(8) of Lemma 2, which means that the stability criterion obtained by Lemma 2 is at least not more conservative than that obtained by Lemma 1.*

Remark 2 *For the case of $a_2 \geq 0$, it can be found that conditions of Lemma 1 and Lemma 2 are all simplified as (4) due to $\mathcal{L}_{1,3} \leq \mathcal{L}_{1,1}$, $\mathcal{L}_{2,3} \leq \mathcal{L}_{2,1} = \mathcal{L}_{1,1}$, and $\mathcal{L}_{2,4} \leq \mathcal{L}_{2,2} = \mathcal{L}_{1,2}$. For the case of $a_2 < 0$, it follows from (5), (7), and (8) that*

$$\mathcal{G}_1 = \mathcal{L}_{1,3} - \mathcal{L}_{2,3} = (\beta^2 - 1)h_{12}^2 a_2 \tag{13}$$
$$\mathcal{G}_2 = \mathcal{L}_{1,3} - \mathcal{L}_{2,4} = ((\beta-1)^2 - 1)h_{12}^2 a_2 - f(h_2) + f(h_1) \tag{14}$$

Obviously, $\beta \in [0, 1]$ and $a_2 < 0$ imply $\mathcal{G}_1 > 0$, i.e., $\mathcal{L}_{2,3} < \mathcal{L}_{1,3}$, which means that (7) is relaxed than (5); Similarly, by choosing suitable β, one can obtain $\mathcal{G}_2 > 0$ such that (8) is also relaxed than (5). Thus, the conditions (6)-(8) of Lemma 2 with a suitably selected β are relaxed than the conditions (4) and (5) of Lemma 1, which means that the conservatism of stability criterion obtained by Lemma 1 can be reduced by using Lemma 2.

Remark 3 *The contribution to reduce conservatism via Lemma 2 benefits from the free selection of β within $[0, 1]$. For Lemma 2, there is no requirement that the conditions*

(6)-(8), for all β within $[0,1]$, are always relaxed than the conditions (4) and (5). In fact, (8) with few values of β may be strict than (5), which means that the stability criterion obtained by Lemma 2, if β is not suitably preset, is more conservative than that obtained by Lemma 1 (See example study for details).

3 A stability criterion

Before developing the stability criterion, the following lemmas are given at first.

Lemma 3 *[12] For a matrix $R>0$, scalars a and b with $b>a$, and a vector x such that the integrations concerned are well defined, the following inequality holds:*

$$(b-a)\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \geq \sum_{i=1}^{3}(2i-1)\chi_i^T R\chi_i \tag{15}$$

where $\chi_1 = x(b)-x(a)$, $\chi_2 = x(b)+x(a)-2\int_a^b \frac{x(s)}{b-a}ds$, and $\chi_3 = x(b)-x(a)+6\int_a^b \frac{x(s)}{b-a}ds-12\int_a^b\int_\theta^b \frac{x(s)}{(b-a)^2}dsd\theta$.

Lemma 4 *For a scalar $\alpha\in(0,1)$, a matrix $R\in\mathcal{R}^{m\times m}$ and $R>0$, a matrix $\Gamma\in\mathcal{R}^{2m\times l}$ with $rank(\Gamma)=2m$ and $2m\leq l$, and any matrices $N_1\in\mathcal{R}^{l\times m}$ and $N_2\in\mathcal{R}^{l\times m}$, the following inequality holds:*

$$\Gamma^T\hat{R}(\alpha)\Gamma \geq \Gamma^T\bar{R}(\alpha)\Gamma + Sym\left\{\Gamma^T\begin{bmatrix}(1-\alpha)N_1^T\\ \alpha N_2^T\end{bmatrix}\right\} -\alpha N_1R^{-1}N_1^T-(1-\alpha)N_2R^{-1}N_2^T \tag{16}$$

where

$$\hat{R}(\alpha)=\begin{bmatrix}\frac{1}{\alpha}R & 0\\ 0 & \frac{1}{1-\alpha}R\end{bmatrix},\quad \bar{R}(\alpha)=\begin{bmatrix}(2-\alpha)R & 0\\ 0 & (1+\alpha)R\end{bmatrix}$$

Proof. The above statement can be found in the proof of Lemma 2 in [20]. ■

The following stability criterion is developed based on the proposed lemma.

Theorem 1 *For a fixed β freely selected within $[0,1]$ and given $h_i, 1=1,2$, system (1) with the delay satisfying (2) is asymptotically stable if there exist $P>0$, $Q_i>0$ and $R_i>0$, i=1,2, any matrices L_1, L_2, N_1 and N_2, such that the following holds*

$$\Theta_i=\begin{bmatrix}\Upsilon(h_1)-\delta_i^2h_{12}^2\Upsilon_0 & N_2\\ * & -\hat{R}_2\end{bmatrix}<0,\quad i=1,2 \tag{17}$$

$$\Theta_i=\begin{bmatrix}\Upsilon(h_2)-\delta_i^2h_{12}^2\Upsilon_0 & N_1\\ * & -\hat{R}_2\end{bmatrix}<0,\quad i=3,4 \tag{18}$$

where $\delta_1=\delta_3=0$, $\delta_2=\beta$, $\delta_4=1-\beta$ and

$$\begin{aligned}
\Upsilon_0 &= Sym\{\Pi_0^TP\Pi_2\}\\
\Pi_0 &= col\{0,0,0,0,e_9+e_{10}\}\\
\Upsilon(d(t)) &= \Upsilon_1(d(t))+\Upsilon_2+\Upsilon_3-\Upsilon_4(d(t))+\Upsilon_5(d(t))\\
\Upsilon_1(d(t)) &= Sym\Big\{\Pi_1^T(d(t))P\Pi_2\Big\}\\
\Pi_1(d(t)) &= col\{e_1,h_1e_5,e_{11}+e_{12},h_1^2e_8,E_a\}\\
E_a &= (d(t)-h_1)^2e_9+(h_2-d(t))^2e_{10}+(h_2-d(t))e_{11}\\
\Pi_2 &= col\{e_s,e_1-e_2,e_2-e_4,h_1(e_1-e_5),E_b\}\\
E_b &= h_{12}e_2-e_{11}-e_{12}\\
\Upsilon_2 &= e_1^TQ_1e_1-e_2^T(Q_1-Q_2)e_2-e_4^TQ_2e_4\\
\Upsilon_3 &= e_s^T(h_1^2R_1+h_{12}^2R_2)e_s-E_1^T\hat{R}_1E_1\\
\Upsilon_4(d(t)) &= \begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\frac{2h_2-d(t)-h_1}{h_{12}}\hat{R}_2 & 0\\ 0 & \frac{h_2+d(t)-2h_1}{h_{12}}\hat{R}_2\end{bmatrix}\begin{bmatrix}E_2\\ E_3\end{bmatrix}\\
&\quad +Sym\left\{\begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\frac{h_2-d(t)}{h_{12}}N_1^T\\ \frac{d(t)-h_1}{h_{12}}N_2^T\end{bmatrix}\right\}\\
\Upsilon_5(d(t)) &= Sym\{[e_6^T,e_{11}^T]L_1[(d(t)-h_1)e_6-e_{11}]\}\\
&\quad +Sym\{[e_7^T,e_{12}^T]L_2[(h_2-d(t))e_7-e_{12}]\}\\
E_i &= col\{e_i-e_{i+1},e_i+e_{i+1}-2e_{i+4},\\
&\qquad e_i-e_{i+1}+6e_{i+4}-12e_{i+7}\},i=1,2,3\\
\hat{R}_i &= diag\{R_i,3R_i,5R_i\},i=1,2\\
e_i &= [0_{n\times(i-1)n},I,0_{n\times(12-i)n}],i=1,2,\ldots,12\\
e_s &= Ae_1+A_de_3
\end{aligned}$$

Proof. Consider the following LKF candidate:

$$V(t,x_t,\dot{x}_t)=V_1(t,x_t)+V_2(t,x_t)+V_3(t,\dot{x}_t) \tag{19}$$

where

$$\begin{aligned}
V_1(t,x_t) &= \varsigma^T(t)P\varsigma(t)\\
V_2(t,x_t) &= \int_{t-h_1}^{t}x^T(s)Q_1x(s)ds+\int_{t-h_2}^{t-h_1}x^T(s)Q_2x(s)ds\\
V_3(t,\dot{x}_t) &= \sum_{i=1}^{2}(h_i-h_{i-1})\int_{-h_i}^{-h_{i-1}}\int_{t+\theta}^{t}\dot{x}^T(s)R_i\dot{x}(s)dsd\theta
\end{aligned}$$

and $\varsigma(t) = col\{x(t),h_1w(h_1,h_0,t),h_{12}w(h_2,h_1,t),h_1^2v(h_1,h_0,t),h_{12}^2v(h_2,h_1,t)\}$ with

$$w(a,b,t)=\int_{t-a}^{t-b}\frac{x(s)}{a-b}ds,\ v(a,b,t)=\int_{t-a}^{t-b}\int_{\theta}^{t-b}\frac{x(s)}{(a-b)^2}dsd\theta$$

and $P>0$, $Q_i>0$, and $R_i>0$, $i=1,2$, which shows $V(t,x_t,\dot{x}_t)\geq-\epsilon\|x(t)\|^2$ for a sufficient small $\varepsilon>0$.

Calculating the derivative of the $V_1(t,x_t)$ along the solution of (1), using $h_{12}^2v(h_2,h_1,t)=E_a\xi(t)$ and $\frac{d}{dt}[h_{12}^2v(h_2,h_1,t)]=E_b\xi(t)$, and following the similar calculations in [12] yield:

$$\dot{V}_1(t,x_t)=2\varsigma^T(t)P\dot{\varsigma}(t)=\xi^T(t)\Upsilon_1(d(t))\xi(t) \tag{20}$$

where $\xi(t) = col\{x(t),\ x(t-h_1),\ x(t-d(t)),\ x(t-h_2),\ w(h_1,h_0,t),\ \ w(d(t),h_1,t),\ \ w(h_2,d(t),t),\ \ v(h_1,h_0,t),\ v(d(t),h_1,t),\ v(h_2,d(t),t),\ (d(t)-h_1)w(d(t),h_1,t),\ (h_2-d(t))w(h_2,d(t),t)\}$.

Calculating the derivative of the $V_2(t,x_t)$ and $V_3(t,\dot{x}_t)$ along the solution of (1) yields [17]:

$$\begin{aligned}\dot{V}_2(t,x_t) &= x^T(t)Q_1x(t) - x^T(t-h_1)(Q_1-Q_2)x(t-h_1)\\ &\quad -x^T(t-h_2)Q_2x(t-h_2)\\ &= \xi^T(t)\Upsilon_2\xi(t) \qquad (21)\\ \dot{V}_3(t,\dot{x}_t) &= \dot{x}^T(t)\big(h_1^2R_1 + h_{12}^2R_2\big)\dot{x}(t) - J_1 - J_2 \qquad (22)\end{aligned}$$

where

$$J_1 = h_1\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds$$
$$J_2 = h_{12}\int_{t-d(t)}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds + h_{12}\int_{t-h_2}^{t-d(t)}\dot{x}^T(s)R_2\dot{x}(s)ds$$

Based on (15), J_1 with $R_1 > 0$ and J_2 with $R_2 > 0$ are respectively estimated as [12]:

$$\begin{aligned}J_1 &\geq \xi^T(t)E_1^T\hat{R}_1E_1\xi(t) \qquad (23)\\ J_2 &\geq \xi^T(t)\left(\frac{h_{12}E_2^T\hat{R}_2E_2}{d(t)-h_1} + \frac{h_{12}E_3^T\hat{R}_2E_3}{h_2-d(t)}\right)\xi(t)\end{aligned}$$

For any matrices N_1 and N_2, J_2 is further estimated, based on (16) with $\Gamma^T = [E_2^T, E_3^T]$ and $\alpha = \frac{d(t)-h_1}{h_{12}}$, as

$$J_2 \geq \xi^T(t)(\Upsilon_4(d(t)) - \bar{\Upsilon}_4(d(t)))\xi(t) \qquad (24)$$

where

$$\bar{\Upsilon}_4(d(t)) = \frac{d(t)-h_1}{h_{12}}N_1\hat{R}_2^{-1}N_1^T + \frac{h_2-d(t)}{h_{12}}N_2\hat{R}_2^{-1}N_2^T$$

For any matrices L_1 and L_2, the following holds

$$\begin{aligned}g_1 &= [w^T(d(t),h_1,t),(d(t)-h_1)w^T(d(t),h_1,t)]L_1\\ &\quad\times\left[(d(t)-h_1)\int_{t-d(t)}^{t-h_1}\frac{x(s)}{d(t)-h_1}ds - \int_{t-d(t)}^{t-h_1}x(s)ds\right] = 0\\ g_2 &= [w^T(h_2,d(t),t),(h_2-d(t))w^T(h_2,d(t),t)]L_2\\ &\quad\times\left[(h_2-d(t))\int_{t-h_2}^{t-d(t)}\frac{x(s)}{h_2-d(t)}ds - \int_{t-h_2}^{t-d(t)}x(s)ds\right] = 0\end{aligned}$$

which imply

$$2g_1 + 2g_2 = \xi^T(t)\Upsilon_5(d(t))\xi(t) = 0 \qquad (25)$$

It follows from (19)-(25) that

$$\begin{aligned}\dot{V}(t,x_t,\dot{x}_t) &= \dot{V}_1(t,x_t) + \dot{V}_2(t,x_t) + \dot{V}_3(t,\dot{x}_t) + 2g_1 + 2g_2\\ &\leq \xi^T(t)[\Upsilon(d(t)) + \bar{\Upsilon}_4(d(t))]\xi(t) \qquad (26)\end{aligned}$$

It is found that $\xi^T(t)[\Upsilon(d(t)) + \bar{\Upsilon}_4(d(t))]\xi(t)$ satisfies the quadratic function defined in (3) with $y = d(t)$ and $a_2 = \xi^T(t)\Upsilon_0\xi(t)$. Thus, based on Lemma 2, the following inequality

$$\xi^T(t)[\Upsilon(d(t)) + \bar{\Upsilon}_4(d(t))]\xi(t) < 0 \qquad (27)$$

holds if the following holds for any given $\beta \in [0,1]$:

$$\begin{aligned}\Upsilon(h_1) + \bar{\Upsilon}_4(h_1) &< 0 \qquad (28)\\ -\beta^2h_{12}^2\Upsilon_0 + \Upsilon(h_1) + \bar{\Upsilon}_4(h_1) &< 0 \qquad (29)\\ \Upsilon(h_2) + \bar{\Upsilon}_4(h_2) &< 0 \qquad (30)\\ -(1-\beta)^2h_{12}^2\Upsilon_0 + \Upsilon(h_2) + \bar{\Upsilon}_4(h_2) &< 0 \qquad (31)\end{aligned}$$

It follows from Schur complement that $\Theta_1 < 0 \Longrightarrow (28)$, $\Theta_2 < 0 \Longrightarrow (29)$, $\Theta_3 < 0 \Longrightarrow (30)$, and $\Theta_4 < 0 \Longrightarrow (31)$. Therefore, if LMIs (17) and (18) hold, then $\dot{V}(t,x_t,\dot{x}_t) \leq -\epsilon\|x(t)\|^2$ for a sufficient small $\varepsilon > 0$.

Based on the above discussion, system (1) is stable if $P > 0$, $Q_i > 0$, $R_i > 0$, $i = 1,2$, and LMIs (17) and (18) hold. This completes the proof. ■

If (27) is handled by using Lemma 1, then the following stability criterion is easily obtained.

Corollary 1 *For given h_1 and h_2, system (1) with the delay satisfying (2) is asymptotically stable if there exist $P > 0$, $Q_i > 0$ and $R_i > 0$, i=1,2, any matrices L_1, L_2, N_1, and N_2, such that*

$$\bar{\Theta}_i = \begin{bmatrix}\Upsilon(h_1) - \bar{\delta}_ih_{12}^2\Upsilon_0 & N_2\\ * & -\hat{R}_2\end{bmatrix} < 0,\quad i = 1,2 \qquad (32)$$

$$\bar{\Theta}_3 = \begin{bmatrix}\Upsilon(h_2) & N_1\\ * & -\hat{R}_2\end{bmatrix} < 0 \qquad (33)$$

where $\bar{\delta}_1 = 0$, $\bar{\delta}_2 = 1$, and the other notations are defined in Theorem 1.

Remark 4 *On the one hand, based on Remarks 1-3, Theorem 1 with a suitably preset β has less conservative in comparison to Corollary 1. On the other hand, compared with Corollary 1, Theorem 1 does not require any extra decision variable since β is preset and Theorem 1 only adds one condition to be checked. It means that the conservatism-reduction via Theorem 1 does not increase too much complexity.*

4 Examples

Example 1 *Consider system (1) with*

$$A = \begin{bmatrix}0 & 1\\ -10 & -1\end{bmatrix},\quad A_d = \begin{bmatrix}0 & 0.1\\ 0.1 & 0.2\end{bmatrix} \qquad (34)$$

For different given h_1, the allowably maximal h_2 can be obtained via Theorem 1 and Corollary 1 (one can refer to [26] for the algorithm). The results provided by Theorem 1 and Corollary 1 and the ones reported in literature are listed in Table 1, where the values of β_o are respectively 0.38 ($h_1 = 0$), 0.48 ($h_1 = 0.3$), 0.53 ($h_1 = 0.7$), and 0.55 ($h_1 = 1.0$) (they are obtained by increasing β from 0 to 1 with the step of 0.01 and selecting the one that makes Theorem 1 provide least conservative results).

The following observations are summarized based on the results listed in Table 1:

- The drawback of Lemma 1 is found from the results provided by Corollary 1 and reported in [20]. Compared with Theorem 1(vi) of [20], Corollary 1 was derived by using a tighter inequality and a more general LKF, namely, (15) is tighter than the inequality used in [20] and LKF (19) contains the one used in [20]. However, Table 1 shows that Corollary 1 does not always lead to less conservative results (for example, the cases of $h_1 \in \{0, 0.3\}$). That is, using Lemma 1 to handle $d^2(t)$-dependent (27) leads to extra conservatism and limits the potential advantages of the tighter inequality and the more general LKF. It shows the necessity of developing a relaxed lemma to handle $d^2(t)$-dependent term.
- Theorem 1 with $\beta = \beta_o$ provides less conservative results than the others reported in literature. Especially, Theorem 1 with $\beta = \beta_o$ provides less conservative results in comparison to Theorem 1(vi) of [20], which means that the contributions of the tighter inequality and the more general LKF to reduce conservatism are well reflected when using Lemma 2 to handle $d^2(t)$-dependent (27). It shows the contribution and advantage of the proposed lemma.
- Compared with Corollary 1, Theorem 1 successfully reduces the conservatism by choosing suitable value of β. Theorem 1 with different values of β leads to the results with different levels of conservatism, and one can select the ones with least conservatism (the ones for $\beta = \beta_o$). It verifies the statements of Remark 2.
- Compared with Corollary 1, Theorem 1 with a specific value of β leads to more conservative result (for example, the case that $h_1 = 1.0$ and $\beta = 0$), which verifies the statements of Remark 3.

Example 2 *Consider the load frequency control scheme of single power system [26] modeled as system (1) with*

$$x(t) = \begin{bmatrix} \Delta f & \Delta P_m & \Delta P_v & \int ACE\ ds \end{bmatrix}^T$$

$$A = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{T_t} & \frac{1}{T_t} & 0 \\ -\frac{1}{RT_g} & 0 & -\frac{1}{T_g} & 0 \\ \bar{\beta} & 0 & 0 & 0 \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{K_p\bar{\beta}}{T_g} & 0 & 0 & -\frac{K_i}{T_g} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where Δf, ΔP_m, and ΔP_v are respectively the deviations of frequency, generator mechanical output, and valve position, ACE is the area control error, D is the generator damping coefficient, M is the moment of inertia of the generator, T_g and T_t are the time constants of the governor and the turbine, respectively, R is the speed drop, $\bar{\beta}$ is the frequency bias factor, and K_p and K_i are the gains of PI controller (One can refer to [26] for more details).

Table 1
The allowably maximal h_2 for different h_1 (Example 1).

Methods	h_1				β
	0	0.3	0.7	1.0	
[11]	1.59	2.01	2.41	2.62	
[12]	1.64	2.13	2.70	2.96	
[25]	1.80	2.19	2.58	2.79	
Th.1(vi) [20]	1.862	2.288	2.695	2.895	
Corollary 1	1.748	2.240	2.849	3.118	
Theorem 1	1.977	2.561	2.992	3.213	β_o
Theorem 1	1.862	2.380	2.870	3.113	0.0
Theorem 1	1.939	2.465	2.908	3.137	0.2
Theorem 1	1.975	2.545	2.966	3.185	0.4
Theorem 1	1.880	2.504	2.980	3.207	0.6
Theorem 1	1.783	2.290	2.886	3.151	0.8
Theorem 1	1.748	2.240	2.849	3.118	1.0

Let $T_t = 0.3$, $T_g = 0.1$, $R = 0.05$, $D = 1.0$, $\bar{\beta} = 21$, $M = 10$, $K_p = 0.1$, and $K_i \in \{0.05, 0.10, 0.15\}$. The allowably maximal values of h_2 for $h_1 = 2$ calculated via Theorem 1 and Corollary 1 are listed in Table 2. It is found that, compared with Corollary 1, Theorem 1 provides less conservative results, which consequently means that Lemma 2 is more effective than Lemma 1. It shows the advantage of the proposed method.

Table 2
The allowably maximal h_2 for different K_i (Example 2).

Methods	K_i		
	0.05	0.10	0.15
Corollary 1	13.774	10.980	8.581
Theorem 1	13.900	11.091	8.619

Simulation tests are carried out for 200 sets of randomly chosen cases that delays satisfy $d(t) \in [2, 11.091]$ and initial frequency deviations satisfy $\Delta f \in [-0.02, 0.02]$, and Fig. 1 shows the responses of system state for those cases. It is observed that system is asymptotically stable.

5 Conclusions

In order to handle the $d^2(t)$-dependent quadratic function often arising in consideration of systems with time-varying delays, this paper has developed a relaxed

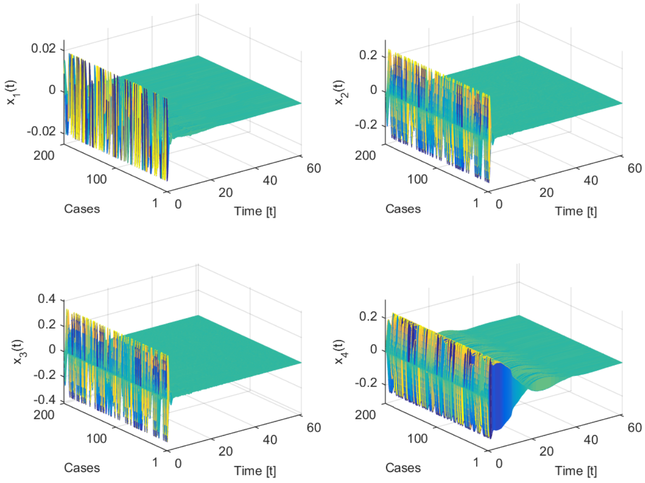

Fig. 1. Responses of system states.

quadratic function negative-determination lemma. This lemma has introduced an adjustable parameter to reduce the conservatism, it reduces to the popular lemma used currently by fixing such parameter as a special value, and its advantages has been shown based on a numerical example. For a linear system with a time-varying delay, a new stability criterion has been established via the developed lemma, together with generalized reciprocally convex combination, and it has been applied to analyze the load frequency control scheme of power systems.

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