TYPICAL PATH COMPONENTS IN TENT MAP INVERSE LIMITS

PHILIP BOYLAND, ANDRÉ DE CARVALHO, AND TOBY HALL

ABSTRACT. In the inverse limit $\hat{I}_s$ of a tent map f_s restricted to its core, the set $\mathcal{GR}$ of points whose path components are bi-infinite and bi-dense has full measure with respect to the measure induced on $\hat{I}_s$ by the unique absolutely continuous invariant measure of f_s. With respect to topology, there is a dichotomy. When the parameter s is such that the critical orbit of f_s is not dense, $\mathcal{GR}$ contains a dense G_δ set. In contrast, when the critical orbit of f_s is dense, the complement of $\mathcal{GR}$ contains a dense G_δ set.

1. Introduction

The inverse limits $\hat{I}_s$ of tent maps f_s restricted to their core intervals have been the subject of intense investigation in dynamics and topology. In dynamics they are models for attractors [18, 10, 6, 15], while in topology they are studied for their intrinsic topological complexity. The main recent focus of topological investigations has been the proof of the Ingram conjecture: for different values of $s \in [\sqrt{2}, 2]$ the inverse limits are not homeomorphic (for references see [5], which contains the final proof for tent maps which are not restricted to their cores). Many distinguishing properties have been discovered. For example, when the parameter s is such that critical orbit of f_s is dense (a full measure, dense G_δ set of parameters), theorems of Bruin and of Raines imply that the inverse limit $\hat{I}_s$ is nowhere locally the product of a Cantor set and an interval [10, 20]. Perhaps more striking, for a dense, G_δ set of parameters s, Barge, Brooks and Diamond [4] show that the inverse limit has a strong self-similarity: every open subset of $\hat{I}_s$ contains a homeomorphic copy of $\hat{I}_t$ for every $t \in [\sqrt{2}, 2]$.

In this paper we take an alternative point of view and study the abundance of tame behavior. A point $\underline{x} \in \hat{I}_s$ is called *globally leaf regular* if its path component is intrinsically homeomorphic to $\mathbb{R}$ (that is, if it is a continuous injective image of $\mathbb{R}$), and this path component is dense and metrically infinite in both directions. For certain values of s the set $\mathcal{GR}$

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of globally regular points is well understood. For example, if the critical point is n-periodic then $\mathcal{GR}$ consists of the entire inverse limit except for n components which are intrinsically homeomorphic to $[0,\infty)$ [4, 11]. Here we show that, for all s, the globally leaf regular points are typical in the inverse limit with respect to a natural measure. In contrast, globally leaf regular points are topologically typical only for those s for which the critical orbit of f_s is not dense.

Theorem 1.1. *Let $s \in (\sqrt{2}, 2]$, and let $\hat{I}_s$ be the inverse limit of the tent map f_s restricted to its core I_s.*

(a) For all s, the set of globally leaf regular points has full measure with respect to the measure induced on $\hat{I}_s$ by the unique absolutely continuous invariant measure for f_s.
(b) If the critical orbit of f_s is not dense in I_s, then the set of globally leaf regular points of $\hat{I}_s$ contains a dense G_δ set.
(c) If the critical orbit of f_s is dense in I_s, then the complement of the set of globally leaf regular points of $\hat{I}_s$ contains a dense G_δ set. Indeed, there is a dense G_δ set of points whose path components are either points or are locally homeomorphic to $[0,1)$, and whose $\hat{f}_s$-orbits are dense in $\hat{I}_s$.

Parts (b) and (c) of this result are built upon the foundation of well-known topological properties of tent map inverse limits and, while new, are expected from the existing literature (see, in particular, [10]). Part (a), on the other hand, requires the introduction of new measure theoretic ideas: in particular, a measure α_x defined on almost every fiber $\pi_0^{-1}(x)$, where $\pi_0\colon \hat{I}_s \to I_s$ is the projection. This measure is defined by an explicit formula in Definition 6.2: its crucial property, given by Theorem 7.1, is its holonomy invariance on appropriate collections of arcs in $\hat{I}_s$ which connect two fibers. We also show (Theorem 8.1) that the collection of measures $\{\alpha_x\}$ is a prescribed scalar multiple of the disintegration onto the fibers of the measure induced on $\hat{I}_s$ by the unique absolutely continuous invariant measure for f_s. These properties, coupled with the topological structure of $\hat{I}_s$ and the ergodicity of the natural extension, yield part (a) of the theorem.

An additional motivation for this work is the discovery in [8] that the natural extensions of tent maps are semi-conjugate to sphere homeomorphisms, by semi-conjugacies for which all fibers except perhaps one are finite. For some parameters s these sphere homeomorphisms are pseudo-Anosov maps, and for some they are generalized pseudo-Anosovs (as defined in [12]). Our results here show that for the other parameters there is still an analog of an invariant unstable foliation, which carries a holonomy invariant transverse measure. It is important to note that these foliations have to be understood in a suitable sense. There are always bi-infinite leaves but there need not be even measurable foliations: we will show in a subsequent paper that when the critical orbit of f_s is dense there is no measurably foliated chart in the neighborhood of any point.

In [22], Su shows that in the inverse limit of a rational map of the Riemann sphere the typical path component (with respect to a natural measure) has the affine structure of the complex plane. The measure theoretic results presented here can be seen as analogs of this result, as we show that, with respect to a natural measure, the typical path component is intrinsically isometric to the real line. The results here were in part inspired by Su's paper and we borrowed several ideas from it, most prominently the use of boxes, and the main ideas in the proofs of our Lemma 8.3 and Theorem 1.1(a). Note also that Lyubich and Minksy give a deep study of the inverse limits of rational maps from a somewhat different point of view in [17].

2. Basic topology and notation

We consider a fixed tent map $f_s : I \to I$ with slope $s \in (\sqrt{2}, 2]$ and critical point c, restricted to its core $I_s = [f_s^2(c), f_s(c)]$. Since we consider a fixed map, we suppress the dependence on s and rescale so that the core is $[0, 1]$. As a result, although there are no subscripts s in the remainder of the paper, $f\colon I \to I$ will always denote a core tent map of slope s defined on $I = [0, 1]$ (so that $f(c) = 1$ and $f^2(c) = 0$), and $\hat{I} := \varprojlim(f, I)$ will always denote the inverse limit of this core tent map.

Points (also called *threads*) in $\hat{I}$ are denoted $\underline{x} = \langle x_0, x_1, x_2, \ldots\rangle$, with $f(x_{i+1}) = x_i$ for all $i \geq 0$. The standard metric on $\hat{I}$ is given by

$$d(\underline{x}, \underline{x}') = \sum_{i=0}^{\infty} \frac{|x_i - x_i'|}{2^i}\,.$$

The *projections* are the maps $\pi_n : \hat{I} \to I$ given by $\pi_n(\underline{x}) = x_n$, for $n \geq 0$. The *shift* or *natural extension of* f is the homeomorphism $\hat{f} : \hat{I} \to \hat{I}$ given by $\hat{f}(\underline{x}) = \langle f(x_0), f(x_1), f(x_2), \ldots\rangle = \langle f(x_0), x_0, x_1, \ldots\rangle$. Fundamental relations are $f \circ \pi_n = \pi_n \circ \hat{f}$ and $\pi_{m+n} \circ \hat{f}^n = \pi_m$. If $K \subseteq \hat{I}$, we write $K_n := \pi_n(K)$.

Until the end of Section 5, where we complete the proofs of parts (b) and (c) of Theorem 1.1, we will assume that $s \in (\sqrt{2}, 2)$, in order to avoid complicating some statements with exceptions. When $s = 2$ it is straightforward to show that there is a dense G_δ set of globally regular points.

Convention 2.1. For brevity, we use the term *interval* exclusively to mean a non-trivial subinterval of I (open, half-open, or closed); the term *arc* to mean a subset of $\hat{I}$ which is intrinsically homeomorphic to such an interval (and so may be open, half-open, or closed); and the term *continuum* to mean a subcontinuum of $\hat{I}$ (which, as usual, must contain more than one point).

The following fact about the dynamics of tent maps is well known.

Lemma 2.2. *Let J be an interval. If $f^2(J) \neq I$, then $|f^2(J)| \geq s^2|J|/2$. In particular, if $N \geq -2\log(|J|)/\log(s^2/2)$ then $f^N(J) = I$.*

The analysis of path components has been a central part of the study of tent map inverse limits (see, for example, [7, 9, 11]). The starting point is this well-known basic characterization [15].

Theorem 2.3. *The inverse limit $\hat{I}$ contains no subset homeomorphic to a circle or to the letter "Y". Therefore every path component of $\hat{I}$ is either a point or an arc.*

Definition 2.4 (Locally leaf regular, terminal, and solitary points)**.** A point $\underline{x} \in \hat{I}$ is called *locally leaf regular* if its local path component is homeomorphic to $(0, 1)$, a *terminal point* if its local path component is homeomorphic to $[0, 1)$ and a *solitary point* if its local path component is just itself.

Definition 2.5 (End continuum)**.** A continuum K is an *end continuum* if, whenever A and B are continua with $K \subseteq A$ and $K \subseteq B$, then either $A \subseteq B$ or $B \subseteq A$.

The following result is Lemma 7 of [4].

Theorem 2.6 (Barge, Brucks, & Diamond)**.** *If K is a continuum with $0 \in K_n$ for infinitely many n, then K is an end continuum.*

Remark 2.7. Since $f^{-1}(0) = \{1\}$ and $f^{-1}(1) = \{c\}$, it follows that $0 \in K_n$ for infinitely many n if and only if $1 \in K_n$ for infinitely many n if and only if $c \in K_n$ for infinitely many n.

3. Countable 0-flat decomposition of arcs

3.1. **Flat arcs and interval threads.** The next definition formalizes and names a standard tool in the theory of inverse limits of interval maps.

Definition 3.1 (Interval thread)**.** Let $J_0, J_1, J_2, \ldots$ be a sequence of intervals, with $f(J_{i+1}) = J_i$ for each i. We write

$$\underline{J} = \langle J_0, J_1, J_2, \ldots \rangle := \{\underline{x} \in \hat{I} \,:\, x_i \in J_i \text{ for each } i\} \subseteq \hat{I},$$

and call $\underline{J}$ an *interval thread*. Equivalently, we can write $\underline{J} = \varprojlim(J_i, f_{|J_i})$.

Every continuum K is an interval thread (on a sequence of closed intervals), since each K_n is a closed interval and $K = \langle K_0, K_1, K_2, \ldots \rangle$.

Definition 3.2 (Flat arc and flat interval thread)**.** An arc γ is *m-flat (over the interval J)* if $\pi_{m|\gamma}$ is a homeomorphism onto its image $\pi_m(\gamma) = J$. An arc is called *flat* if it is flat for some $m \geq 0$.

An interval thread $\langle J_0, J_1, J_2, \ldots \rangle$ is *m-flat* if f sends J_{i+1} homeomorphically onto J_i for all $i \geq m$, or, equivalently, if $c \notin \operatorname{Int} J_i$ for all $i > m$. An interval thread is called *flat* if it is flat for some $m \geq 0$.

An arc γ (respectively an interval thread $\underline{J}$) is m-flat if and only if $\hat{f}^{-m}(\gamma)$ (respectively $\hat{f}^{-m}(\underline{J})$) is 0-flat. Therefore many proofs of properties of flat arcs and interval threads reduce to the 0-flat case.

Any flat arc γ is equal to the interval thread $\langle \gamma_0, \gamma_1, \gamma_2, \ldots \rangle$ and any flat interval thread is an arc. Moreover, it is easy to check that an arc is m-flat if and only if it is an m-flat interval thread. Because of this equivalence we will go back and forth freely between the terminology and notation of flat arcs and flat interval threads.

0-flat arcs are closely related to the *basic arcs* defined symbolically in [10, 2], and elsewhere: a 0-flat arc is a (non-degenerate) subarc of a basic arc.

By Theorem 2.6 and Remark 2.7, if K is a continuum but not an end continuum, then $c \in K_n$ for only finitely many n. We therefore have the following corollary:

Corollary 3.3 (Brucks and Bruin [9])**.** *If K is a continuum but not an end continuum, then it is a flat closed arc.*

3.2. **The 0-flat decomposition.**

Definition 3.4 (0-flat decomposition, node)**.** A 0-*flat decomposition* of an arc γ is a countable collection of 0-flat arcs $\gamma^{(i)}$ such that

(1) $\gamma = \cup \gamma^{(i)}$,
(2) $\gamma^{(i+1)} \cap \gamma^{(i)}$ is a single point, called a *node of the decomposition* and denoted $\underline{z}^{(i)}$, and
(3) $\gamma^{(i)} \cap \gamma^{(j)} = \emptyset$ when $|i - j| > 1$.

The 0-flat decomposition is called *efficient* if whenever $\gamma' \subseteq \gamma$ is 0-flat, we have $\gamma' \subseteq \gamma^{(i)}$ for some i.

If an arc has an efficient 0-flat decomposition, then this decomposition is unique and is determined by its nodes. In this case we refer to these nodes as *the nodes of the arc.*

The next lemma is certainly known to experts but does not seem to be stated in the literature in the form we need; it is implicit in [10] and [2]. Rather than introduce the symbolic machinery used in those papers we maintain a strictly topological perspective for brevity of exposition and self-sufficiency.

Lemma 3.5.

(a) Every flat arc has a finite efficient 0*-flat decomposition.*
(b) Every closed arc which is contained in an open arc is flat.
(c) Every open arc has an efficient 0*-flat decomposition.*

Proof. For (a), let $\gamma = \langle \gamma_0, \gamma_1, \gamma_2, \ldots \rangle$ be an m-flat arc. Let the closed intervals of monotonicity of $f^m_{|\gamma_m}$ be $I^{(1)}, \ldots, I^{(N)} \subseteq \gamma_m$, ordered from left to right; and define $z^{(i)}$ by $I^{(i)} \cap I^{(i+1)} = \{z^{(i)}\}$ for $1 \le i \le N-1$.

By assumption, $f_{|\gamma_\ell}$ is a homeomorphism for all $\ell > m$. Therefore, if $1 \le i \le N$ and $k > 0$, there is a unique interval $I^{(i)}_{m+k} \subseteq \gamma_{m+k}$ for which $f^k \colon I^{(i)}_{m+k} \to I^{(i)}$ is a homeomorphism. So for each such i there is a 0-flat interval thread

$$\underline{I}^{(i)} = \left\langle f^m(I^{(i)}), \ldots, f(I^{(i)}), I^{(i)}, I^{(i)}_{m+1}, I^{(i)}_{m+2}, \ldots \right\rangle \subseteq \gamma.$$

Similarly, if $1 \le i < N$ and $k > 0$, there is a unique $z^{(i)}_{m+k} \in \gamma_{m+k}$ for which $f^k(z^{(i)}_{m+k}) = z^{(i)}$, giving threads

$$\underline{z}^{(i)} = \left\langle f^m(z^{(i)}), \ldots, f(z^{(i)}), z^{(i)}, z^{(i)}_{m+1}, z^{(i)}_{m+2}, \ldots \right\rangle \in \gamma.$$

It is straightforward to check that the collection of arcs $\underline{I}^{(i)}$ $(1 \le i \le N)$ is a 0-flat decomposition of γ with nodes $\underline{z}^{(i)}$. Since the nodes are exactly the points $\underline{x} \in \gamma$ satisfying $x_j = c$ for some $1 \le j \le m$, no 0-flat subarc of γ can contain a node in its interior. The decomposition is therefore efficient.

(b) follows immediately from Corollary 3.3, since a closed arc which is contained in an open arc cannot be an end continuum.

For (c), let γ be an open arc, and write γ as an increasing union $\gamma = \bigcup \gamma^{(n)}$ of closed (and therefore flat) arcs. Then the union of the nodes of the arcs $\gamma^{(n)}$ determines an efficient 0-flat decomposition of γ. □

4. Global leaf regularity

4.1. The metric on arcs.

Definition 4.1 (The metric ρ on an open or flat arc γ)**.** Let γ be an open arc or a flat arc, and let $\underline{x}$ and $\underline{x}'$ be distinct elements of γ. We define

$$\rho(\underline{x},\underline{x}') = \sum_{i=0}^{N-1} \left| z_0^{(i)} - z_0^{(i+1)} \right|,$$

where $\underline{z}^{(0)} = \underline{x}$, $\underline{z}^{(N)} = \underline{x}'$, and $\underline{z}^{(1)},\ldots,\underline{z}^{(N-1)}$ are the nodes of the efficient 0-flat decomposition of the (flat) closed subarc of γ with endpoints $\underline{x}$ and $\underline{x}'$.

Remark 4.2. A more standard metric on γ is the intrinsic metric: choose a parameterization $\sigma : [0,1] \to \gamma$ of the subarc with endpoints $\underline{x}$ and $\underline{x}'$, and set

$$\beta(\underline{x},\underline{x}') = \sup \left\{ \sum_{i=0}^{n-1} d(\sigma(t_i),\sigma(t_{i+1})), \right\}$$

where the supremum is over all subdivisions $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$. We will show that $\beta(\underline{x},\underline{x}') = \frac{2s}{2s-1}\,\rho(\underline{x},\underline{x}')$ for all $\underline{x},\underline{x}' \in \gamma$, so that the two metrics are just scaled versions of one another. The use of ρ makes some calculations cleaner.

It is enough to show this in the case where γ is 0-flat, since it is immediate from the definitions that (using the notation of Definition 4.1) we have $\rho(\underline{x},\underline{x}') = \sum_{i=0}^{N-1} \rho(\underline{z}^{(i)},\underline{z}^{(i+1)})$ and $\beta(\underline{x},\underline{x}') = \sum_{i=0}^{N-1} \beta(\underline{z}^{(i)},\underline{z}^{(i+1)})$.

Assume, then, that γ is 0-flat, so that $\rho(\underline{x},\underline{x}') = |x_0 - x_0'|$. Since $f^n_{|\gamma_n}$ is a homeomorphism with derivative $\pm 1/s^n$ for each $n > 0$, we have that $|x_n - x_n'| = |x_0 - x_0'|/s^n$, from which it follows that $d(\underline{x},\underline{x}') = \frac{2s}{2s-1}\,\rho(\underline{x},\underline{x}')$. On the other hand, again using that $f^n_{|\gamma_n}$ is a homeomorphism for all n, if $\underline{x}''$ lies on the subarc of γ with endpoints $\underline{x}$ and $\underline{x}'$, then x_n'' lies between x_n and x_n' for all n, so that $d(\underline{x},\underline{x}') = d(\underline{x},\underline{x}'') + d(\underline{x}'',\underline{x}')$. It follows that $\beta(\underline{x},\underline{x}') = d(\underline{x},\underline{x}') = \frac{2s}{2s-1}\,\rho(\underline{x},\underline{x}')$ as required.

Given a flat closed arc γ with endpoints $\underline{x}$ and $\underline{x}'$, we write $\rho(\gamma) := \rho(\underline{x},\underline{x}')$.

Lemma 4.3. *If γ is a flat closed arc, then $\rho(\hat{f}(\gamma)) = s\rho(\gamma)$.*

Proof. As in Remark 4.2, it suffices to show this when $\gamma = \langle \gamma_0, \gamma_1, \gamma_2, \ldots \rangle$ is 0-flat. If $c \notin \operatorname{Int}\gamma_0$ then $\hat{f}(\gamma)$ is also 0-flat, and the result follows since $|\hat{f}(\underline{x})_0 - \hat{f}(\underline{x}')_0| = |f(x_0) - f(x_0')| = s|x_0 - x_0'|$ (where $\underline{x}$ and $\underline{x}'$ are the endpoints of γ).

On the other hand, if $c \in \operatorname{Int}\gamma_0$, let $\underline{x}'' \in \gamma$ be the point with $x_0'' = c$: then the efficient 0-flat decomposition of $\hat{f}(\gamma)$ has node $\hat{f}(\underline{x}'')$, and hence

$\rho(\hat{f}(\underline{x}), \hat{f}(\underline{x}')) = \rho(\hat{f}(\underline{x}), \hat{f}(\underline{x}'')) + \rho(\hat{f}(\underline{x}''), \hat{f}(\underline{x}')) = s\rho(\underline{x}, \underline{x}') + s\rho(\underline{x}'', \underline{x}') = s\rho(\underline{x}, \underline{x}')$ as required. □

Corollary 4.4. *If γ is a flat closed arc with $\gamma_\ell = I$ for some $\ell \geq 0$, then $\rho(\gamma) \geq s^\ell$.*

Proof. We have $(\hat{f}^{-\ell}(\gamma))_0 = I$, so that $\rho(\hat{f}^{-\ell}(\gamma)) \geq 1$ by Definition 4.1. □

4.2. **Density.** Recall that a subset K of $\hat{I}$ is *ϵ-dense* in $\hat{I}$ if $d(\underline{x}, K) < \epsilon$ for all $\underline{x} \in \hat{I}$.

Lemma 4.5. *Let K be a continuum.*

(a) If $K_\ell = I$ for some $\ell > 0$, then K is $2^{-\ell}$-dense in $\hat{I}$.
(b) Let $|\pi_0(K)| = \delta > 0$. If $N \geq -2\log(\delta)/\log(s^2/2)$, then for all $j > 0$, $\hat{f}^{N+j}(K)$ is 2^{-j}-dense in $\hat{I}$.

Proof. Part (a) is obvious, and (b) follows from Lemma 2.2 and (a). □

Definition 4.6 (Metrically infinite)**.** A path-connected subset S of $\hat{I}$ is *metrically infinite* if, for all $N \geq 0$, there is a flat closed arc $\gamma \subseteq S$ with $\rho(\gamma) > N$.

Lemma 4.7. *A path connected subset S of $\hat{I}$ is dense in $\hat{I}$ if and only if for every $\ell \geq 0$ there is a flat closed arc $\gamma \subseteq S$ with $\gamma_\ell = I$. In this case, S is also metrically infinite.*

Proof. A flat closed arc γ with $\gamma_\ell = I$ is $2^{-\ell}$-dense in $\hat{I}$ by Lemma 4.5 (a), which establishes sufficiency of the condition. Such an arc has $\rho(\gamma) \geq s^\ell$ by Corollary 4.4, so the condition also implies that S is metrically infinite.

For the converse, suppose that S is dense in $\hat{I}$, so that S is either an open arc or a half-open arc. Removing an endpoint in the half-open case, we can assume that S is an open arc, so that there is a continuous bijection $\sigma\colon (-1, 1) \to S$. For each $k \geq 2$, let $\gamma^{(k)} = \sigma([-1 + 1/k, 1 - 1/k]) \subseteq S$, a flat closed arc. We show that for every $\ell \geq 0$ there is some k with $\gamma^{(k)}_\ell = I$, which will establish the result.

Suppose for a contradiction that there is some fixed ℓ such that $\gamma^{(k)}_\ell \neq I$ for all k. By Lemma 2.2 we have $|\gamma^{(k)}_{\ell+2}| < 2/s^2 < 1$ for all k. Since $\gamma^{(k)}_{\ell+2}$ is an increasing sequence of intervals, there is an open interval J which is disjoint from all of the $\gamma^{(k)}_{\ell+2}$. Thus $\pi^{-1}_{\ell+2}(J)$ is disjoint from the dense set S, which is the required contradiction. □

The converse of the last statement in the lemma is not true in general: there may be metrically infinite path connected sets which are not dense.

Definition 4.8 (Metrically bi-infinite and bi-dense open arcs)**.** Let γ be an open arc. We say that γ is *metrically bi-infinite* (respectively *bi-dense*) if for some (and hence for all) $p \in \gamma$, both components of $\gamma \setminus \{p\}$ are metrically infinite (respectively dense in $\hat{I}$).

Lemma 4.9. *If an open arc γ is bi-dense then it is metrically bi-infinite, and is a path component of $\hat{I}$.*

Proof. That γ is metrically bi-infinite follows from Lemma 4.7. To see that it is a path component of $\hat{I}$, suppose to the contrary that there is some $q \notin \gamma$ which is in the path component of γ. Let $p \in \gamma$. By Theorem 2.3 there is a unique closed arc Γ in $\hat{I}$ with endpoints p and q. Then Γ contains one of the two components of $\gamma \setminus \{p\}$, contradicting the fact that these rays are both dense in $\hat{I}$. □

4.3. **Global leaf regularity.** A point $\underline{x} \in \hat{I}$ is called *globally leaf regular* if its path component is a bi-dense (and hence metrically bi-infinite) open arc. Let $\mathcal{GR}$ denote the collection of globally leaf regular points,

Lemma 4.10. *Let $\underline{x} \in \hat{I}$. The following two conditions each imply that $\underline{x} \in \mathcal{GR}$.*

(a) There exists $\epsilon > 0$ such that, for arbitrarily large n, there is a flat closed arc γ with (i) $\underline{x} \in \operatorname{Int}(\hat{f}^n(\gamma))$; and (ii) each component T of $\gamma \setminus \{\hat{f}^{-n}(\underline{x})\}$ satisfies $|\pi_0(T)| \geq \epsilon$.

(b) There exists $\delta > 0$ such that $|x_n - c| \geq \delta$ for all n.

Proof. Suppose that the condition in (a) holds, so that in particular $\underline{x}$ is locally leaf regular. Let C be the path component of $\underline{x}$, and let S be the union of $\underline{x}$ with either one of the path components of $C \setminus \{\underline{x}\}$. We will show that S is dense in $\hat{I}$: it follows that S is a half-open arc, and hence that C is a bi-dense open arc as required.

Let $N > -2\log(\epsilon)/\log(s^2/2)$. Given any $\ell \geq 0$, pick $n \geq \ell + N$ for which a flat closed arc γ as in (a) exists. Let Γ be the flat closed arc given by the union of $\hat{f}^{-n}(\underline{x})$ and the component of $\gamma \setminus \{\hat{f}^{-n}(\underline{x})\}$ which ensures $\hat{f}^n(\Gamma) \subseteq S$. Then

$$\pi_{n-N}(\hat{f}^n(\Gamma)) = \pi_0(\hat{f}^N(\Gamma)) = f^N(\pi_0(\Gamma)) = I$$

by Lemma 2.2, since $|\pi_0(\Gamma)| \geq \epsilon$. Since $n - N \geq \ell$, we have $\pi_\ell(\hat{f}^n(\Gamma)) = I$, and hence S is dense in $\hat{I}$ by Lemma 4.7, as required.

For (b), take any $n \geq 0$ and define intervals $\gamma_i = [x_{n+i} - \delta/s^i, x_{n+i} + \delta/s^i]$. Then $\gamma = \langle \gamma_0, \gamma_1, \gamma_2, \ldots \rangle$ is a 0-flat closed arc which satisfies the conditions of (a) for $\epsilon = 2\delta$. □

There are many cases in which one can check directly that particular points are globally leaf regular using these criteria. In the statement below we use the following notation:

Notation 4.11 ($\hat{X}$)**.** If X is a compact subset of I with $f(X) = X$, then we write $\hat{X} := \varprojlim(f_{|X}, X)$.

Notice that $\hat{f}(\hat{X}) = \hat{X}$. Examples of such invariant compact subsets are provided by the orbit closures $\overline{\text{orb}(x)}$ of recurrent points $x \in I$.

Corollary 4.12. *If $X \subseteq I$ is compact, with $f(X) = X$ and $c \notin X$, then each $\underline{x} \in \hat{X}$ is globally leaf regular. In particular, if x is recurrent and $c \notin \overline{\text{orb}(x)}$, then each $\underline{x} \in \widehat{\overline{\text{orb}(x)}}$ is globally leaf regular.*

Proof. Immediate from Lemma 4.10 (b) with $\delta = d(X, c) > 0$. □

The simplest examples which satisfy the criterion of Corollary 4.12 are periodic points $\underline{x}$ of $\hat{f}$ with $x_n \neq c$ for all n. The collection of such periodic points is dense in $\hat{I}$.

5. Typical in topology

5.1. Boxes.

Definition 5.1 (Boxes)**.** An open (respectively closed) m-box B is a union of open (respectively closed) arcs, all of which are m-flat over the same open (respectively closed) interval J. Thus an m-box may be written as a union

$$B = \bigcup \gamma^\eta$$

where each γ^η is an m-flat arc with $\gamma^\eta_m = J$.

The *maximal m-box* B over an interval J is the union of all arcs which are m-flat over J.

Remarks 5.2.

(a) Open and closed boxes need not be open and closed subsets of $\hat{I}$.

(b) A subset B of $\hat{I}$ is an m-box over J if and only if $\hat{f}^{-m}(B)$ is a 0-box over J.

(c) The arcs γ^η of an open m-box are mutually disjoint, whereas those of a closed m-box may intersect at their endpoints.

(d) Let B be an open 0-box over J. For each $a \in J$, write $B_a = \pi_0^{-1}(a) \cap B$. Let $\underline{x}^{a,\eta}$ denote the intersection point of B_a and γ^η: thus $x_i^{a,\eta} \in \gamma_i^\eta$ for each i. For each N there is some $\epsilon > 0$ such that if $d(\underline{x}^{a,\eta}, \underline{x}^{a,\eta'}) < \epsilon$ then $x_i^{a,\eta'} \in \gamma_i^\eta$ for $0 \leq i \leq N$, so that $\gamma_i^\eta = \gamma_i^{\eta'}$ for $0 \leq i \leq N$. It follows that, for each $b \in J$, the function $\psi_{a,b}\colon B_a \to B_b$ defined by $\psi_{a,b}(\underline{x}^{a,\eta}) = \underline{x}^{b,\eta}$ is a homeomorphism, and hence that the function $\underline{x}^{x,\eta} \mapsto (x, \underline{x}^{a,\eta})$ is a homeomorphism $B \to J \times B_a$.

Lemma 5.3. *The closure in $\hat{I}$ of a box is a closed box. In particular, the maximal box over a closed interval is closed in $\hat{I}$.*

Proof. By Remark 5.2 (b) it suffices to consider the case where $B = \bigcup \gamma^\eta$ is a 0-box over an interval J. Moreover, we can assume without loss of generality that J is closed, for if not then $\bigcup \mathrm{Cl}(\gamma^\eta) \subseteq \mathrm{Cl}(B)$ is a 0-box over $\mathrm{Cl}(J)$.

A 0-flat arc $\gamma \subseteq J \times I^\infty$ over J is the graph of the function $F\colon J \to I^\infty$ defined by $F(x_0) = (x_1, x_2, \dots)$, where $\langle x_0, x_1, x_2, \dots\rangle \in \gamma$: in other words, $F = \hat{f}^{-1} \circ \pi_0|_\gamma^{-1}$. The function F is Lipschitz, since if $x_0, x_0' \in J$ with $\pi_0|_\gamma^{-1}(x_0) = \underline{x}$ and $\pi_0|_\gamma^{-1}(x_0') = \underline{x}'$, then, as in Remark 4.2,

$$d(F(x_0), F(x_0')) = \sum_{i=1}^\infty \frac{|x_i - x_i'|}{2^{i-1}} = \sum_{i=1}^\infty \frac{|x_0 - x_0'|}{2^{i-1}\, s^i} = \frac{2}{2s-1}\,|x_0 - x_0'|.$$

Therefore the 0-box B is the union of a collection of graphs of uniformly Lipschitz functions. Conversely, the graph of any function $J \to I^\infty$ is a 0-flat arc over J, provided that it is contained in $\hat{I}$, which is guaranteed if it is contained in $\mathrm{Cl}(B)$. Now if X and Y are compact metric spaces, then, by Arzelà–Ascoli, the closure in $X \times Y$ of any union of graphs of uniformly Lipschitz functions $X \to Y$ is a union of graphs of functions $X \to Y$. The result follows. ☐

5.2. **Proof of Theorem 1.1 (b) and (c).** For (b), suppose that the critical orbit of f is not dense in I, so that $Y := I \setminus \overline{\mathrm{orb}(c)} \neq \emptyset$. Let $J = (a, b)$ be a component of Y. If $n \geq 1$ then $f^{-n}(J)$ is a union of components of Y, to each of which f^n restricts to a homeomorphism onto J. Therefore $B = \pi_0^{-1}(J)$ is a union of 0-flat arcs over J, i.e. an open 0-box.

Since f is transitive, so also is $\hat{f}$, and hence $\hat{f}^{-1}$. By a theorem of Birkhoff (see Theorem 5.8 in [23]), there is a dense G_δ subset Z of $\hat{I}$ consisting of points whose $\hat{f}^{-1}$-orbits are dense. We will establish (b) by showing that $Z \subseteq \mathcal{GR}$.

Let $\epsilon = (b-a)/4$ and set $J' = (a+\epsilon, b-\epsilon)$ and $B' = \pi_0^{-1}(J') \subseteq B$. Let $\underline{x} \in Z$. Then, since B' is open in $\hat{I}$, there are arbitrarily large integers n with $\hat{f}^{-n}(\underline{x}) \in B'$. For each such n, the arc γ of $\mathrm{Cl}(B)$ to which $\hat{f}^{-n}(\underline{x})$ belongs satisfies the conditions of Lemma 4.10 (a). Therefore $\underline{x} \in \mathcal{GR}$ as required.

For (c), suppose that $\overline{\mathrm{orb}(c)} = I$. In this case, $\hat{I}$ is nowhere locally the product of a zero-dimensional set and an interval (see Proposition 1 of [10] or

Theorem 6.4 of [20]), so that no box contains an open subset of $\hat{I}$. Let $\{U_j\}$ be a collection of open intervals which form a countable base for the topology of I, and for each $m \geq 0$ let $B_{m,j}$ be the maximal m-box over $\mathrm{Cl}(U_j)$. Then each $B_{m,j}$ is closed in $\hat{I}$ by Lemma 5.3, and so is nowhere dense. Therefore, by Baire's theorem, the complement Z of $\bigcup B_{m,j}$ is dense G_δ.

By Lemma 3.5 (b), every locally leaf regular point is contained in some $B_{m,j}$, so that Z consists entirely of terminal and solitary points. Since the set of points whose $\hat{f}$-orbits are dense is also dense G_δ, the result follows. □

Remark 5.4. It follows from the proof of Theorem 1.1 (c) that, when the critical orbit is dense, the (disjoint) union of the set of solitary points and the set of terminal points is dense G_δ. The interesting question of which one of these is dense G_δ remains open. Proposition 4.29 in [1] shows that for s in the dense G_δ set of parameters identified in [4] the solitary points and terminal points are each dense in the core inverse limit $\hat{I}_s$. The situation under the weaker hypothesis of a dense critical orbit is unclear and, more generally, Problem 9 in [1] asks for conditions on the orbit of the critical point that ensure the existence of solitary points.

6. Measure preliminaries

6.1. **Cylinder sets and fibers.** Let $J_0, \ldots, J_n$ be intervals with $f(J_{i+1}) = J_i$ for each i. The associated *interval cylinder set* is

$$[J_0, J_1, \ldots, J_n] = \{\underline{x} \in \hat{I} \colon x_i \in J_i \text{ for } 0 \leq i \leq n\}.$$

Since $[J_0, J_1, \ldots, J_n] = \pi_n^{-1}(J_n)$, it is open in $\hat{I}$ if J_n is open. The collection of interval cylinder sets for all n, with J_n open in I (that is, the collection of all $\pi_n^{-1}(J_n)$) generates both the topology and the Borel σ-algebra of $\hat{I}$.

The set $\pi_n^{-1}(x)$ is called the π_n*-fiber* over x. A π_0-fiber is sometimes just called a *fiber*. A *point cylinder set* in the fiber over y_0 is

$$[y_0, y_1, \ldots, y_n] = \{\underline{x} \in \hat{I} \colon x_i = y_i \text{ for } 0 \leq i \leq n\}.$$

Note that $[y_0, y_1, \ldots, y_n] = \pi_n^{-1}(y_n) \subseteq \pi_0^{-1}(y_0) = \pi_0^{-1}(f^n(y_n))$. The point cylinder set $[y_0, y_1, \ldots, y_n]$ is open in $\pi_0^{-1}(y_0)$.

6.2. **Invariant measures.** We now summarize some basic results about the "physical" measure for tent maps. This summary includes contributions of several authors, and has been extended in a variety of directions [16, 13, 14, 21, 3]. As before, $f \colon [0,1] \to [0,1]$ denotes a tent map of fixed slope $s \in (\sqrt{2}, 2]$, restricted to its core.

Theorem 6.1. *f has a unique invariant Borel probability measure μ which is absolutely continuous with respect to Lebesgue measure m, and $d\mu = \varphi dm$ with $\varphi \in L^1(m)$ defined on $[0,1] \setminus \operatorname{orb}(c)$. The function φ can be chosen in its L^1-class to be strictly positive, of bounded variation, and*

$$\varphi(x) = \sum_{f^n(y)=x} \frac{\varphi(y)}{s^n} \qquad \text{(all } x \notin \operatorname{orb}(c) \text{ and } n \geq 0\text{).} \tag{6.1}$$

Finally, μ is ergodic.

Note that if $x \notin \operatorname{orb}(c)$, so that $\varphi(x)$ is defined, then $\varphi(y)$ is also defined whenever $f^n(y) = x$. In particular, given a thread $\langle x_0, x_1, x_2, \ldots \rangle$, if $\varphi(x_0)$ is defined, then so is each $\varphi(x_i)$. The measure μ is conventionally called the unique *acim* (absolutely continuous invariant measure) for f. The symbol φ will always denote the density of this measure.

If ν is an f-invariant Borel probability measure on I then there is a unique $\hat{f}$-invariant Borel probability measure $\hat{\nu}$ on $\hat{I}$ with the property that $(\pi_n)_*\hat{\nu} = \nu$ for all n [19]. The measure $\hat{\nu}$ is sometimes called the *inverse limit* or *natural extension* of the f-invariant measure ν. The measure $\hat{\nu}$ is $\hat{f}$-ergodic if and only if ν is f-ergodic. We will be exclusively concerned with the $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{I}$ derived from the acim μ on I.

6.3. **A measure on fibers.** The formalities of Borel measures on fibers are very similar to those on symbolic subshifts, which is one reason for adopting the language of cylinder sets. We next define explicitly a measure on fibers, which turns out to be $\varphi(x)$ times the disintegration of $\hat{\mu}$ onto fibers (see Theorem 8.1).

Definition 6.2 (The measures α_x)**.** For each $x \in I \setminus \operatorname{orb}(c)$ and each point cylinder set in the fiber $\pi_0^{-1}(x)$, define

$$\alpha_x([x, x_1, \ldots, x_n]) = \frac{\varphi(x_n)}{s^n}. \tag{6.2}$$

By (6.1), α_x is finitely additive on the semi-algebra of point cylinder sets. Exactly as in the case of symbolic subshifts (see §0.2 of [23]), α_x extends to the σ-algebra generated by the cylinder sets, namely the Borel σ-algebra of $\pi_0^{-1}(x)$. We regard each α_x as a measure on $\hat{I}$ supported on $\pi_0^{-1}(x)$, so that if E is a Borel subset of $\hat{I}$ we have $\alpha_x(E) = \alpha_x(E \cap \pi_0^{-1}(x))$.

7. Holonomy invariance of α_x in 0-boxes

Theorem 7.1. *If B is a 0-box over J then, for all $a, b \in J \setminus \operatorname{orb}(c)$,*

$$\alpha_a(B) = \alpha_b(B).$$

Proof. Write $B = \bigcup \gamma^\eta$, where each $\gamma^\eta = \langle J, \gamma_1^\eta, \gamma_2^\eta, \ldots\rangle$ is 0-flat over J. For each $n \geq 1$, let $[J, J_{1,n}^{(i)}, J_{2,n}^{(i)}, \ldots, J_{n,n}^{(i)}]$ $(1 \leq i \leq N(n))$ be the interval cylinder sets which are realized by the first $n+1$ entries of some γ^η. That is, for each i there is some η with $J_{j,n}^{(i)} = \gamma_j^\eta$ for $1 \leq j \leq n$, and each η arises in this way. Then, for each n,

$$B \subseteq \bigcup_{i=1}^{N(n)} \left[J, J_{1,n}^{(i)}, J_{2,n}^{(i)}, \ldots, J_{n,n}^{(i)}\right],$$

and the sets in this union are mutually disjoint except perhaps along the fibers of endpoints of J, if those endpoints lie in $\operatorname{orb}(c)$. Moreover,

$$B = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N(n)} \left[J, J_{1,n}^{(i)}, J_{2,n}^{(i)}, \ldots, J_{n,n}^{(i)}\right].$$

Now let $a, b \in J\backslash\operatorname{orb}(c)$. Since $f^j\colon J_{j,n}^{(i)} \to J$ is a homeomorphism for each i, j, and n, there is a unique point $a_{j,n}^{(i)} \in J_{j,n}^{(i)}$ with $f^j(a_{j,n}^{(i)}) = a$. Therefore

$$B \cap \pi_0^{-1}(a) = \bigcap_{n=1}^{\infty} \bigsqcup_{i=1}^{N(n)} \left[a, a_{1,n}^{(i)}, a_{2,n}^{(i)}, \ldots, a_{n,n}^{(i)}\right].$$

Since α_a is a regular measure, (6.2) gives

$$\alpha_a(B) = \lim_{n\to\infty} \sum_{i=1}^{N(n)} \frac{\varphi(a_{n,n}^{(i)})}{s^n} \quad \text{and analogously} \quad \alpha_b(B) = \lim_{n\to\infty} \sum_{i=1}^{N(n)} \frac{\varphi(b_{n,n}^{(i)})}{s^n}.$$

For each n and i, the points $a_{n,n}^{(i)}$ and $b_{n,n}^{(i)}$ are both in the interval $J_{n,n}^{(i)}$. As i varies, the intervals $J_{n,n}^{(i)}$ are disjoint except perhaps at their endpoints. Recalling that φ is of bounded variation, let $V < \infty$ be its total variation. Then

$$\begin{aligned} |\alpha_a(B) - \alpha_b(B)| &\leq \lim_{n\to\infty} \sum_{i=1}^{N(n)} \frac{|\varphi(a_{n,n}^{(i)}) - \varphi(b_{n,n}^{(i)})|}{s^n} \\ &\leq \lim_{n\to\infty} \frac{V}{s^n} = 0. \end{aligned}$$

□

8. Typical in measure

8.1. **Disintegration of the measure $\hat{\mu}$.** The fibers $\{\pi_0^{-1}(x)\}$ provide a measurable partition of $\hat{I}$. Thus, by Rokhlin's disintegration theorem, there is a family of probability measures $\{\hat{\mu}_x\}$, defined for μ-a.e. $x \in I$, with $\hat{\mu}_x$ supported on the fiber $\pi_0^{-1}(x)$, having the property that for any Borel subset

E of $\hat{I}$,

$$\hat{\mu}(E) = \int_I \hat{\mu}_x(E)\, d\mu(x). \tag{8.1}$$

Note that $\hat{\mu}_x(E) = \hat{\mu}_x(E \cap \pi_0^{-1}(x))$, since each $\hat{\mu}_x$ is supported on the fiber $\pi_0^{-1}(x)$. The measures $\hat{\mu}_x$ are called the disintegrations of $\hat{\mu}$ onto fibers, or alternatively the conditional measures of $\hat{\mu}$ on fibers. We next show that these conditional measures are simple multiples of the measures α_x. In this statement, and in the remainder of the paper, "almost every" means with respect to μ or, equivalently, with respect to Lebesgue measure m.

Theorem 8.1. *$d\alpha_x = \varphi(x)\, d\hat{\mu}_x$ for a.e. $x \in I$. In particular, for any Borel subset E of $\hat{I}$,*

$$\hat{\mu}(E) = \int_I \alpha_x(E)\, dm(x).$$

Proof. It suffices to show that for a.e. $x \in I$ we have

$$\alpha_x([x, x_1, \ldots, x_n]) = \varphi(x)\ \hat{\mu}_x([x, x_1, \ldots, x_n])$$

for each point cylinder set $[x, x_1, \ldots, x_n]$ in $\pi_0^{-1}(x)$. Since $\mathrm{orb}(c)$ is countable, we can assume that $x \notin \mathrm{orb}(c)$, so that $x_i \neq c$ for all i. There is therefore some ϵ_0 with the property that, for all $\epsilon < \epsilon_0$, the restriction of f^n to $J_\epsilon = [x_n - \epsilon, x_n + \epsilon]$ is a homeomorphism onto its image $I_\epsilon = [x - s^n\epsilon, x + s^n\epsilon]$.

Write $K_\epsilon = \pi_n^{-1}(J_\epsilon)$, so that $\pi_0(K_\epsilon) = I_\epsilon$. By (8.1),

$$\hat{\mu}(K_\epsilon) = \int_{I_\epsilon} \hat{\mu}_y(K_\epsilon)\, d\mu(y) = \int_{I_\epsilon} \hat{\mu}_y(K_{\epsilon_0})\, d\mu(y)$$

since $\hat{\mu}_y(K_\epsilon) = \hat{\mu}_y(K_{\epsilon_0})$ for $y \in I_\epsilon$. By the Lebesgue differentiation theorem, for a.e. $x \in I$,

$$\lim_{\epsilon \to 0} \frac{\hat{\mu}(K_\epsilon)}{\mu(I_\epsilon)} = \hat{\mu}_x(K_{\epsilon_0}) = \hat{\mu}_x(K_{\epsilon_0} \cap \pi_0^{-1}(x)) = \hat{\mu}_x([x, x_1, \ldots, x_n]).$$

Since $d\mu = \varphi\, dm$ we have $\lim_{\epsilon\to 0} \mu(I_\epsilon)/m(I_\epsilon) = \varphi(x)$ for a.e. $x \in I$, so that

$$\varphi(x)\, \hat{\mu}_x([x, x_1, \ldots, x_n]) = \lim_{\epsilon \to 0} \frac{\hat{\mu}(K_\epsilon)}{m(I_\epsilon)} \quad \text{for a.e. } x \in I,$$

and it only remains to show that $\lim_{\epsilon\to 0} \hat{\mu}(K_\epsilon)/m(I_\epsilon) = \alpha_x([x, x_1, \ldots, x_n])$ for a.e. $x \in I$.

To show this, let $g = (f^n_{|J_\epsilon})^{-1} \colon I_\epsilon \to J_\epsilon$ (so that g has constant slope $\pm 1/s^n$). Observing that $\hat{\mu}(K_\epsilon) = \mu(J_\epsilon)$ (since $\mu = (\pi_n)_*\hat{\mu}$), we have that for a.e. $x \in I$,

$$\begin{aligned}\lim_{\epsilon\to 0}\hat{\mu}(K_\epsilon)/m(I_\epsilon) &= \lim_{\epsilon\to 0}\mu(J_\epsilon)/m(I_\epsilon)\\ &= \lim_{\epsilon\to 0}\frac{1}{m(I_\epsilon)}\int_{J_\epsilon}\varphi(y)\,dm(y)\\ &= \lim_{\epsilon\to 0}\frac{1}{m(I_\epsilon)}\int_{I_\epsilon}\varphi(g(u))\,|g'(u)|\,dm(u)\\ &= \frac{\varphi(x_n)}{s^n} = \alpha_x([x, x_1, \ldots, x_n])\end{aligned}$$

as required, using (6.2) and the Lebesgue differentiation theorem. □

The important consequence of this result, together with Theorem 7.1, for what follows is that the restriction of $\hat{\mu}$ to an open 0-box is a product:

Corollary 8.2. *Let B be an open 0-box over an interval J and take any $a \in J \setminus \mathrm{orb}(c)$. Under the homeomorphism $B \to J \times (\pi_0^{-1}(a) \cap B)$ defined in Remark 5.2 (d), the restriction of $\hat{\mu}$ to B pushes forward to $m \times \alpha_a$. In particular, $\hat{\mu}(B) = m(J)\alpha_a(B)$.*

8.2. **Positive measure boxes.**

Lemma 8.3. *Let $M = \sup\{\varphi(x) \colon x \in I \setminus \mathrm{orb}(c)\}$. For all $N > 1$ there exists an open 0-box B over an interval J such that, for all $x \in J \setminus \mathrm{orb}(c)$,*

$$\alpha_x(B) \geq M\left(1 - \frac{1}{s^{N-1}}\right), \tag{8.2}$$

and in particular

$$\hat{\mu}(B) \geq M\left(1 - \frac{1}{s^{N-1}}\right)m(J) > 0. \tag{8.3}$$

Proof. Fix $N > 1$, and let J be a component of $I \setminus \{c, f(c), \ldots, f^N(c)\}$ with $\sup\{\varphi(x) : x \in J \setminus \mathrm{orb}(c)\} = M$.

Let $S = \{m \in \mathbb{N} : f^m(c) \in J\}$. For each $m \in S$, let K_m be the component of $f^{-m}(J)$ containing c, and define

$$C_m = [f^m(K_m), \ldots, K_m] \subset \pi_0^{-1}(J).$$

Now suppose that $\underline{x} = \langle x_0, x_1, x_2, \ldots\rangle \in \pi_0^{-1}(J) \setminus \bigcup_{m\in S} C_m$. Define intervals $J_m \subset f^{-m}(J)$ for $m \geq 0$ inductively by $J_0 = J$, and J_{m+1} is the component of $f^{-1}(J_m)$ which contains x_{m+1}. Then $c \notin J_m$ for all m, for if $c \in J_m$ we would have $J_m \subset K_m$ and hence $\underline{x} \in C_m$. It follows that $0 \notin J_m$ for all m, and hence $f(J_{m+1}) = J_m$ for all m. Therefore $\underline{J} = \langle J, J_1, J_2, \ldots\rangle$ is a 0-flat arc over J which contains $\underline{x}$. Hence

$$B = \pi_0^{-1}(J_0) \setminus \bigcup_{m\in S} C_m$$

is the maximal 0-box over J.

If $c \in f^i(K_m)$ for some $m \in S$ and $1 \le i < m$, then $m - i \in S$, $f^i(K_m) \subset K_{m-i}$, and $C_m \subset C_{m-i}$. Defining

$$T = \{m \in S \,:\, c \notin f^i(K_m) \text{ for } 1 \le i < m\},$$

we therefore have $B = \pi_0^{-1}(J_0) \setminus \bigcup_{m \in T} C_m$.

Let $x \in J \setminus \mathrm{orb}(c)$, and write $T_x = \{m \in T \,:\, x \in f^m(K_m)\} \subset T$. For each $m \in T_x$, and each $1 \le i < m$, let $x_i^{(m)}$ be the unique point of $f^{m-i}(K_m)$ with $f^i(x_i^{(m)}) = x$. Then $\pi_0^{-1}(x) \cap C_m \subset [x, x_1^{(m)}, \dots, x_{m-1}^{(m)}]$, and hence

$$\pi_0^{-1}(x) \cap B \supset \pi_0^{-1}(x) \setminus \bigcup_{m \in T_x} \left[x, x_1^{(m)}, \dots, x_{m-1}^{(m)}\right].$$

Since $\alpha_x([x, x_1^{(m)}, \dots, x_{m-1}^{(m)}]) = \varphi(x_{m-1}^{(m)})/s^{m-1}$, and $m > N$ for all $m \in T_x$ by choice of J, we have

$$\begin{aligned}\alpha_x(B) &\ge \alpha_x(\pi_0^{-1}(x)) - \sum_{m \in T_x} \frac{\varphi(x_{m-1}^{(m)})}{s^{m-1}} \\ &\ge \varphi(x) - \frac{M}{s^{N-1}}.\end{aligned}$$

By Theorem 7.1, $\alpha_x(B)$ is independent of $x \in J \setminus \mathrm{orb}(c)$. (8.2) therefore holds since J was chosen so that $M = \sup\{\varphi(x) : x \in J \setminus \mathrm{orb}(c)\}$, and (8.3) follows by Corollary 8.2. ☐

8.3. **Proof of Theorem 1.1 (a).** The proof is almost identical to that of Theorem 1.1 (b), using ergodicity rather than transitivity of $\hat{f}^{-1}$.

By Lemma 8.3, there is a 0-box B over an interval $J = (a, b)$ with $\hat{\mu}(B) > 0$. Let $\epsilon = (b-a)/4$, and set $J' = (a+\epsilon, b-\epsilon)$ and $B' = \pi_0^{-1}(J') \cap B$. By Corollary 8.2, $\hat{\mu}(B') = \hat{\mu}(B)/2 > 0$.

Since $\hat{f}^{-1}$ is ergodic with respect to $\hat{\mu}$, there is a full $\hat{\mu}$-measure subset Z of $\hat{I}$ with the property that, for each $\underline{x} \in Z$, there are arbitrarily large integers n with $\hat{f}^{-n}(\underline{x}) \in B'$. For each such n, the arc γ of $\mathrm{Cl}(B)$ to which $\hat{f}^{-n}(\underline{x})$ belongs satisfies the conditions of Lemma 4.10 (a). Therefore $Z \subseteq \mathcal{GR}$. ☐

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Department of Mathematics, University of Florida, 372 Little Hall, Gainesville, FL 32611-8105, USA
Email address: boyland@ufl.edu

Departamento de Matemática Aplicada, IME-USP, Rua do Matão 1010, Cidade Universitária, 05508-090 São Paulo SP, Brazil
Email address: andre@ime.usp.br

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, UK
Email address: tobyhall@liv.ac.uk