Sensitivity to Small Delays of Mean Square Stability for Stochastic Neutral Evolution Equations

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Abstract: In this work, we are concerned about the mean square exponential stability property for a class of stochastic neutral functional differential equations with small delay parameters. Both distributed and point delays under the neutral term are considered. Sufficient conditions are given to capture the exponential stability in mean square of the stochastic system under consideration. As an illustration, we present some practical systems to show their exponential stability which is not sensitive to small delays in the mean square sense.

Keywords: Stability sensitive to small delay; Exponential stability; Stochastic neutral functional differential equation.

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1. Introduction

We begin our statement with the consideration of a simple one dimensional deterministic time delay linear differential equation in $\mathbb{R}$,

$$\begin{cases} dy(t) = \alpha y(t)dt + \beta y(t-r)dt, \quad t \geq 0, \\ y(0) = \phi_0, \; y(\theta) = \phi_1(\theta), \; \theta \in [-r, 0], \; (\phi_0, \phi_1) \in \mathbb{R} \times L^2([-r, 0]; \mathbb{R}), \end{cases} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $r \geq 0$. From Bátkai and Piazzera [1], it is known that under the condition $\alpha + \beta < 0$, there exists a number $r_0 > 0$ such that the null solution of (1.1) is exponentially stable for all $r \in [0, r_0)$, while for $r > r_0$, the null solution is exponentially unstable. That is, the exponential stability of (1.1) is not sensitive to small delays in this situation.

Now we consider a similar problem for stochastic systems. More precisely, let us study a stochastic version of (1.1) in the following form

$$\begin{cases} dy(t) = \alpha y(t)dt + \beta y(t-r)dt + \sigma y(t)dw(t), \quad t \geq 0, \\ y(0) = \phi_0, \; y(\theta) = \phi_1(\theta), \; \theta \in [-r, 0], \; (\phi_0, \phi_1) \in \mathbb{R} \times L^2([-r, 0]; \mathbb{R}), \end{cases} \tag{1.2}$$

where $r \geq 0$, α, β, $\sigma \in \mathbb{R}$ and $w(t)$, $t \geq 0$, is a standard real Brownian motion. If $r = 0$, it is a well known fact that if $\alpha + \beta < \sigma^2/2$, then the null solution of (1.2) is exponentially stable in the almost sure sense, i.e., the Lyapunov exponent of (1.2) satisfies

$$\limsup_{t \to \infty} \frac{\log |y(t)|}{t} = \alpha + \beta - \frac{\sigma^2}{2} < 0 \qquad a.s.$$

In addition to the condition $\alpha + \beta < \sigma^2/2$, it was shown by Bierkens [2] that the null solution of (1.2) is exponentially stable in the almost sure sense if the delay parameter $r > 0$ is sufficiently small. Also, Appleby and Mao [3] obtained some similar results for some nonlinear stochastic systems.

When we turn to consider an infinite dimensional system, the situation becomes complicated. It was observed by Datko et al. [4] that small delays may destroy stability for a partial differential equation. On the other hand, it is noticed that if some natural conditions are imposed, a similar result to those as above in finite dimensional spaces could be true. To state this, let us denote by $\mathscr{L}(X, Y)$ the space of all bounded, linear operators from X into Y where X and Y are arbitrary Banach spaces with their respective norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. If $X = Y$, we simply write $\mathscr{L}(X)$ for $\mathscr{L}(X, X)$. Consider the following deterministic differential equation in a Hilbert space H,

$$\begin{cases} dy(t) = Ay(t)dt + A_0 y(t-r)dt, \quad t \geq 0, \\ y(0) = \phi_0 \in H, \; y(\theta) = \phi_1(\theta), \; \theta \in [-r, 0], \; \phi_1 \in L^2([-r, 0]; H), \end{cases} \tag{1.3}$$

where A generates a C_0-semigroup e^{tA}, $t \geq 0$, on H and $A_0 \in \mathscr{L}(H)$. It was shown in Theorem 7.5, [1] that if e^{tA}, $t \geq 0$, is a norm continuous C_0-semigroup, i.e., the mapping

$t \to e^{tA}$ is continuous from $(0,\infty)$ to $\mathscr{L}(H)$, and the C_0-semigroup generated by $A + A_0$ is exponentially stable, then there exists $r_0 > 0$ such that the null solution of (1.3) is exponentially stable for all $r \in [0, r_0)$. In other words, the exponential stability property is not sensitive to small delays in this situation. In particular, consider a linear partial differential equation in $H = L^2(0,\pi)$,

$$\begin{cases} dy(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} y(t,\xi)dt + \alpha y(t-r,\xi)dt, \quad t \geq 0, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \ t \geq 0, \ \ y(0,\cdot) = \phi_0(\cdot) \in L^2(0,\pi), \\ y(\theta) = \phi_1(\theta) \in L^2([-r,0]; L^2(0,\pi)), \end{cases} \tag{1.4}$$

where $r \geq 0$ and $\alpha \in \mathbb{R}$. Let $A = \partial^2/\partial\xi^2$, which generates a norm continuous C_0-semigroup on H and $A_0 = \alpha I$, where I is the identity operator on H. If $r = 0$, it is well known that when $\alpha < 1$, the trivial solution of (1.4) is exponentially stable. If $r \neq 0$ and $\alpha < 1$, we have as above that the trivial solution of (1.4) is exponentially stable when $r \in (0, r_0)$ for some $r_0 > 0$.

We turn our attention to stochastic systems and consider their sensitivity problem of exponential stability to small delays. As a motivation example, let $r \geq 0$ and consider a stochastic version of the delay partial differential equation (1.4),

$$\begin{cases} dy(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} y(t,\xi)dt + \alpha y(t-r,\xi)dt + \sigma \displaystyle\int_{-r}^{0} y(t+\theta,\xi)d\theta dw(t), \ t \geq 0, \ \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \ t \geq 0, \ \ y(0) = \phi_0 \in L^2(0,\pi), \\ y_0(\cdot) = \phi_1(\cdot) \in L^2([-r,0]; L^2(0,\pi)), \end{cases} \tag{1.5}$$

where α, $\sigma \in \mathbb{R}$ and w is a standard real Brownian motion. If $r > 0$, it was shown in [5] that under the condition $\alpha - 1 < \sigma^2/2$, the pathwise exponential stability of the trivial solution to (1.5) is not sensitive to small delays $r > 0$.

Now let us consider a time delay version of (1.5) of neutral type in the following form

$$\begin{cases} d(y(t,\xi) - \gamma y(t-r,\xi)) = \dfrac{\partial^2}{\partial \xi^2}(y(t,\xi) - \gamma y(t-r,\xi))dt \\ \qquad\qquad + \sigma \displaystyle\int_{-r}^{0} y(t+\theta,\xi)d\theta dw(t), \quad t \geq 0, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \ t \geq 0, \ \ y(0) = \phi_0 \in L^2(0,\pi), \\ y_0(\cdot) = \phi_1(\cdot) \in L^2([-r,0]; L^2(0,\pi)), \end{cases} \tag{1.6}$$

where $\gamma \in \mathbb{R}$. In comparison with (1.5), the novelty of system (1.6) is that a delay term appears under the differentiation and the second order derivative, i.e., Laplacian, operator of (1.6) on one hand, and a time delay appears in the diffusion term on the other. Here we want to know whether, in addition to some natural conditions on σ, γ, the trivial solution of equation (1.6) can still secure its exponential stability, at least for sufficiently small delay parameter $r > 0$. In [6], the answer was shown to be affirmative for the pathwise exponential stability for a similar system to (1.6).

In this work, we shall consider the sensitivity problem to small delays of exponential stability in the mean square sense for such stochastic neutral functional differential equations as (1.6). The organization of this work is as follows. In Section 2, we first develop a C_0-semigroup theory of functional differential equations, which enables us to lift up stochastic time delay systems into non time delay ones in the subsequent section. To justify the stochastic stability for our systems, it is important to know when the associated "lift-up" solution semigroups are exponentially stable. To this end, we distinguish in Section 3 between two kinds of the most popular delays, i.e., distributed and point delays, and treat the corresponding systems separately. Last, we apply the results established in this work to various examples to illustrate our theory.

2. Deterministic Neutral Systems

In the sequel, we shall focus on a class of norm continuous semigroups e^{tA}, $t \geq 0$, formulated by a variational approach. Precisely, let V be a separable Hilbert space and $a : V \times V \to \mathbb{R}$ a bilinear form satisfying the so-called Gårding's inequalities

$$|a(x, y)| \leq \beta \|x\|_V \|y\|_V, \quad a(x, x) \leq -\alpha \|x\|_V^2, \qquad \forall\, x,\, y \in V, \tag{2.1}$$

for some constants $\beta > 0$, $\alpha > 0$. In association with the form $a(\cdot, \cdot)$, let A be a linear operator defined by

$$a(x, y) = \langle x, Ay \rangle_{V,V^*}, \quad x,\ y \in V, \tag{2.2}$$

where V^* is the dual space of V and $\langle \cdot, \cdot \rangle_{V,V^*}$ is the dual pairing between V and V^*. Then $A \in \mathscr{L}(V, V^*)$. Moreover, it can be shown (see, e.g., [9]) that A generates a bounded, analytic semigroup e^{tA}, $t \geq 0$, on V^* such that $e^{tA} : V^* \to V$ for each $t > 0$ and for some constant $M > 0$,

$$\|e^{tA}\|_{\mathscr{L}(V^*)} \leq M \qquad \text{for all} \quad t \geq 0.$$

We also introduce the standard interpolation Hilbert space $H = (V, V^*)_{1/2,2}$, which is described by

$$H = \Big\{ x \in V^* : \int_0^\infty \|Ae^{tA}x\|_{V^*}^2 dt < \infty \Big\}$$

with inner product

$$\langle x, y \rangle_H = \langle x, y \rangle_{V^*} + \int_0^\infty \langle Ae^{tA}x, Ae^{tA}y \rangle_{V^*} dt, \quad x,\ y \in V^*.$$

We identity the dual H^* of H with H, then it is easy to see that

$$V \hookrightarrow H = H^* \hookrightarrow V^* \tag{2.3}$$

where the imbedding $\hookrightarrow$ is dense and continuous with

$$\|x\|_H^2 \leq \nu \|x\|_V^2, \quad \forall\, x \in V \quad \text{for some constant } \nu > 0. \tag{2.4}$$

Hence, $\langle x, Ay\rangle_H = \langle x, Ay\rangle_{V,V^*}$ for all $x \in V$ and $y \in V$ with $Ay \in H$. Moreover, for any $T \geq 0$ it is well known that

$$L^2([0,T],V) \cap W^{1,2}([0,T],V^*) \subset C([0,T],H)$$

where $W^{1,2}([0,T],V^*)$ is the Sobolev space consisting of all functions $y:\ [0,T] \to V^*$ such that y and its first order distributional derivative are in $L^2([0,T],V^*)$ and $C([0,T],H)$ is the space of all continuous functions from $[0,T]$ into H, respectively.

Let $r > 0$ and $T \geq 0$. For $x \in L^2([-r,T],V)$, we always write $x_t(\theta) := x(t+\theta)$ for any $t \geq 0$ and $\theta \in [-r,0]$ in this work. Suppose that $D_1 \in \mathscr{L}(V)$, $D_2 \in \mathscr{L}(L^2([-r,0],V),V)$, $F_1 \in \mathscr{L}(V,V^*)$ and $F_2 \in \mathscr{L}(L^2([-r,0],V),V^*)$. We introduce two linear mappings D and F on $C([-r,T],V)$, respectively, by

$$Dx_t = D_1x(t-r) + D_2x_t, \qquad t \in [0,T], \qquad \forall\, x(\cdot) \in C([-r,T],V),$$

and

$$Fx_t = F_1x(t-r) + F_2x_t, \qquad t \in [0,T], \qquad \forall\, x(\cdot) \in C([-r,T],V).$$

Both the mappings D and F have a bounded, linear extension to $L^2([-r,T],V)$ such that for any $x \in L^2([-r,T],V)$,

$$\int_0^T \|Dx_t\|_V^2 dt \leq C_1 \int_{-r}^T \|x(t)\|_V^2 dt, \qquad C_1 > 0, \tag{2.5}$$

and

$$\int_0^T \|Fx_t\|_{V^*}^2 dt \leq C_2 \int_{-r}^T \|x(t)\|_V^2 dt, \qquad C_2 > 0. \tag{2.6}$$

Let $\mathcal{H} = H \times L^2([-r,0],V)$ and consider the following deterministic functional differential equation of neutral type in V^*,

$$\begin{cases} d(x(t) - Dx_t) = A(x(t) - Dx_t)dt + Fx_t dt, \qquad t \geq 0, \\ x_0 = \phi_1,\ \phi = (\phi_0, \phi_1) \in \mathcal{H}, \end{cases} \tag{2.7}$$

or its integral form

$$\begin{cases} x(t) - Dx_t = e^{tA}\phi_0 + \displaystyle\int_0^t e^{(t-s)A} F x_s ds, \qquad t \geq 0, \\ x_0 = \phi_1,\ \phi = (\phi_0, \phi_1) \in \mathcal{H}. \end{cases} \tag{2.8}$$

It is known (see [8]) that there is a unique solution x of (2.8) in $[0,T]$ such that

$$x \in L^2([0,T],V) \cap W^{1,2}([0,T],V^*)$$

and the equation (2.8) is satisfied almost everywhere in $[0,T]$, $T \geq 0$.

Let $x(t)$, $t \geq -r$, denote the unique solution of system (2.8) with $x_0 = \phi_1$, $\phi = (\phi_0, \phi_1) \in \mathcal{H}$. We define a family of operators $\mathcal{S}(t) : \mathcal{H} \to \mathcal{H}$, $t \geq 0$, by

$$\mathcal{S}(t)\phi = (x(t) - Dx_t, x_t) \quad \text{for any} \quad \phi \in \mathcal{H}. \tag{2.9}$$

The family $t \to \mathcal{S}(t)$ is a strongly continuous semigroup on $\mathcal{H}$. Moreover, we have (see [8]) the following result which completely describes the generator $\mathcal{A}$ of semigroup $\mathcal{S}(t)$ or $e^{t\mathcal{A}}$, $t \geq 0$.

Theorem 2.1. *The generator $\mathcal{A}$ of the strongly continuous semigroup $\mathcal{S}(t)$, $t \geq 0$, is given by*

$$\mathscr{D}(\mathcal{A}) = \Big\{(\phi_0, \phi_1) \in \mathcal{H} : \phi_1 \in W^{1,2}([-r, 0], V),\ \phi_0 = \phi_1(0) - D\phi_1 \in V,\ A\phi_0 + F\phi_1 \in H\Big\}$$

and for each $\phi = (\phi_0, \phi_1) \in \mathscr{D}(\mathcal{A})$,

$$\mathcal{A}\phi = (A\phi_0 + F\phi_1, \phi_1') \in \mathcal{H}.$$

3. Stochastic Neutral Differential Equations

In this work, we are concerned about a class of infinite dimensional stochastic systems, especially stochastic functional differential equations of neutral type. To this end, assume that $\{\Omega, \mathscr{F}, \mathbb{P}\}$ is a complete probability space equipped with some filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and K is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_K$. Let $W_Q(t), t \geq 0$, denote an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted, Q-Wiener process in K, defined on $\{\Omega, \mathscr{F}, \mathbb{P}\}$ with covariance operator Q satisfying that

$$\mathbb{E}\langle W_Q(t), u\rangle_K \langle W_Q(s), v\rangle_K = (t \wedge s)\langle Q(u), v\rangle_K \quad \text{for all} \quad u, v \in K, \tag{3.1}$$

where Q is a positive, self-adjoint and trace class operator on K. We introduce the subspace $K_Q = Ran(Q^{1/2})$, the range of $Q^{1/2}$, of K and let $\mathscr{L}_2 = \mathscr{L}_2(K_Q, H)$ denote the space of all Hilbert-Schmidt operators from K_Q into H.

To proceed further, we split this section into two parts. The first part concentrates on the distributed delay case and the other concentrates on the point delay one.

3.1. Distributed Delays under Neutral Terms

Let A be a linear operator given as in (2.2). Consider the following retarded differential equation of neutral type in space V^*,

$$\begin{cases} d\Big(y(t) - \displaystyle\int_{-r}^0 G_1(\theta)y(t+\theta)d\theta\Big) = A\Big(y(t) - \displaystyle\int_{-r}^0 G_1(\theta)y(t+\theta)d\theta\Big)dt \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{-r}^0 A_1(\theta)y(t+\theta)d\theta dt + By(t)dW_Q(t), \quad t \geq 0, \\ y(0) = \phi_0,\ y(\theta) = \phi_1(\theta),\ \theta \in [-r, 0],\ (\phi_0, \phi_1) \in \mathcal{H} = H \times L^2([-r, 0]; V), \end{cases} \tag{3.2}$$

where $G_1(\cdot) \in L^2([-r,0];\mathscr{L}(V))$, $A_1(\cdot) \in L^2([-r,0];\mathscr{L}(V,V^*))$, $B \in \mathscr{L}(H,\mathscr{L}_2)$ and $W_Q(\cdot)$ is a Q-Wiener process satisfying (3.1).

In association with (3.2), it is immediate by Theorem 2.1 that

$$\mathcal{A}\phi = \Big(A\phi_0 + \int_{-r}^0 A_1(\theta)\phi_1(\theta)d\theta, \frac{d\phi_1(\theta)}{d\theta}\Big), \qquad \phi \in \mathscr{D}(\mathcal{A}),$$

and we further define a bounded, linear operator $\mathcal{B}: \mathcal{H} \to \mathscr{L}_2(K_Q, \mathcal{H})$ by

$$\mathcal{B}\phi(k) = \Big(B\Big(\phi_0 + \int_{-r}^0 G_1(\theta)\phi_1(\theta)d\theta\Big)(k), 0\Big), \qquad \phi \in \mathcal{H}, \qquad k \in K_Q.$$

By exploiting the standard lift-up method (see, e.g., Liu [8]), we can rewrite (3.2) as an equivalent stochastic evolution equation without delay,

$$\begin{cases} dY(t) = \mathcal{A}Y(t)dt + \mathcal{B}Y(t)dW_Q(t), \quad t \geq 0, \\ Y(0) = \phi \in \mathcal{H}, \end{cases} \tag{3.3}$$

where

$$Y(t) = \Big(y(t) - \int_{-r}^0 G_1(\theta)y(t+\theta)d\theta,\ y(t+\cdot)\Big),\ t \geq 0$$

is the lift-up process of $y(t)$, $t \geq -r$. It can be easily shown that the stochastic exponential stability of the null solution of (3.3) is equivalent to the corresponding exponential stability of (3.2).

The following proposition which is taken from Theorem 2.2.2 in Liu [7] is important in establishing the main results in this work.

Proposition 3.1. *Suppose that $\varLambda : \mathscr{D}(\varLambda) \subset H \to H$ generates a C_0-semigroup $e^{t\varLambda}$, $t \geq 0$, on the Hilbert space H. For the linear stochastic evolution equation,*

$$\begin{cases} dy(t) = \varLambda y(t)dt + \Sigma y(t)dW_Q(t), \quad t \geq 0, \\ y(t) = y_0 \in H, \end{cases} \tag{3.4}$$

where $\Sigma \in \mathscr{L}(H,\mathscr{L}_2)$ and $W_Q(t)$, $t \geq 0$, is a Q-Wiener process in K. If there exists a constant $\gamma > 0$ such that

$$\|e^{t\varLambda}\| \leq e^{-\gamma t}, \quad t \geq 0, \tag{3.5}$$

and

$$\Big\| \int_0^\infty e^{t\varLambda^*}\Delta(I)e^{t\varLambda}dt \Big\| < 1, \quad t \geq 0, \tag{3.6}$$

then there exist positive constants $M \geq 1, \mu > 0$ such that

$$\mathbb{E}\|y(t,t_0)\|^2 \leq M\|y_0\|^2 e^{-\mu t}, \quad t \geq 0,$$

where $\Delta(I) \in \mathscr{L}(H)$ is the unique operator defined by the form

$$\langle x, \Delta(I)y\rangle_H := Tr\{\Sigma(x)Q^{1/2}(\Sigma(y)Q^{1/2})^*\}, \qquad x,\ y \in H.$$

Here $Tr\{\cdot\}$ means the trace of operators.

Now we are in a position to state one of the main results in this work.

Theorem 3.1. *Suppose that* $\langle x, Ax\rangle_{V,V^*} \le -\alpha\|x\|_V^2$ *for all* $x \in V$ *and some constant* $\alpha > 0$. *Assume that the delay parameter* $r \in [0,1]$ *and for some* $0 < \lambda < \alpha$,

$$\int_{-r}^{0} e^{-2\lambda\nu^{-1}\theta}\Big[\|A_1(\theta)\|^2 + 2(\alpha-\lambda)^2\|G_1(\theta)\|^2 + 2(\alpha-\lambda)\|A_1(\theta)\|\|G_1(\theta)\|\Big]d\theta < (\alpha-\lambda)^2, \tag{3.7}$$

then we have the relation

$$\|e^{t\mathcal{A}}\| \le e^{-\lambda\nu^{-1}t}, \quad t \ge 0. \tag{3.8}$$

Proof. We intend to find an equivalent inner product $(\cdot,\cdot)_{\mathcal{H}}$ to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H} = H \times L^2([-r,0];V)$ such that

$$(\mathcal{A}\phi,\phi)_{\mathcal{H}} \le -\lambda\nu^{-1}\|\phi\|_{\mathcal{H}} \quad \text{for all} \quad \phi \in \mathscr{D}(\mathcal{A}).$$

To this end, define $(\cdot,\cdot)_{\mathcal{H}}$ by

$$(\phi,\psi)_{\mathcal{H}} = \langle\phi_0,\psi_0\rangle_H + \int_{-r}^{0}\gamma(\theta)\langle\phi_1(\theta),\psi_1(\theta)\rangle_V d\theta,\ \phi,\ \psi \in \mathcal{H},$$

where $\gamma(\cdot)$ is given by

$$\gamma(\theta) = e^{2\lambda\nu^{-1}\theta} \cdot \left[\alpha - \lambda - \frac{\int_{\theta}^{0} e^{-2\lambda\nu^{-1}\tau}\big[\|A_1(\tau)\|^2 + 2(\alpha-\lambda)^2\|G_1(\tau)\|^2 + 2(\alpha-\lambda)\|A_1(\tau)\|\|G_1(\tau)\|\big]d\tau}{\alpha-\lambda}\right],$$
$$\theta \in [-r,0].$$

From (3.7), it is immediate that $\gamma(-r) > 0$. Hence, it is easy to see that $(\cdot,\cdot)_{\mathcal{H}}$ satisfy all the conditions of a valid inner product due to the fact that $\gamma(\theta) \ge \gamma(-r) > 0$ for $\theta \in [-r,0]$.

Since $\langle x, Ax\rangle_{V,V^*} \le -\alpha\|x\|_V^2$ for any $x \in V$, it follows for any $\phi \in \mathscr{D}(\mathcal{A})$ that

$$\begin{aligned}
&(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}}\\
&= \Big\langle\phi_0, A\phi_0 + \lambda\nu^{-1}\phi_0 + \int_{-r}^{0}A_1(\theta)\phi_1(\theta)d\theta\Big\rangle_H\\
&\quad + \int_{-r}^{0}\gamma(\theta)\langle\phi_1(\theta),\dot{\phi}_1(\theta) + \lambda\nu^{-1}\phi_1(\theta)\rangle_V d\theta\\
&\le \Big\langle\phi_0, A\phi_0 + \int_{-r}^{0}A_1(\theta)\phi_1(\theta)d\theta\Big\rangle_{V,V^*} + \int_{-r}^{0}\gamma(\theta)\langle\phi_1(\theta),\dot{\phi}_1(\theta)\rangle_V d\theta\\
&\quad + \lambda\|\phi_0\|_V^2 + \lambda\nu^{-1}\int_{-r}^{0}\gamma(\theta)\|\phi_1(\theta)\|_V^2 d\theta\\
&\le (\lambda-\alpha)\|\phi_0\|_V^2 + \|\phi_0\|_V\int_{-r}^{0}\|A_1(\theta)\|\cdot\|\phi_1(\theta)\|_V d\theta + \frac{1}{2}\int_{-r}^{0}\gamma(\theta)d\|\phi_1(\theta)\|_V^2\\
&\quad + \lambda\nu^{-1}\int_{-r}^{0}\gamma(\theta)\|\phi_1(\theta)\|_V^2 d\theta.
\end{aligned} \tag{3.9}$$

By using integration by parts, one can easily obtain for $\phi \in \mathscr{D}(\mathcal{A})$ that

$$\begin{aligned}
&\frac{1}{2}\int_{-r}^{0}\gamma(\theta)d\|\phi_1(\theta)\|_V^2 \\
&= \frac{1}{2}\gamma(0)\|\phi_1(0)\|_V^2 - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 - \frac{1}{2}\int_{-r}^{0}\|\phi_1(\theta)\|_V^2 d\gamma(\theta) \\
&= \frac{1}{2}\gamma(0)\|\phi_1(0)\|_V^2 - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 - \lambda\nu^{-1}\int_{-r}^{0}\|\phi_1(\theta)\|_V^2\gamma(\theta)d\theta \\
&\quad - \frac{\int_{-r}^{0}\|\phi_1(\theta)\|_V^2\big[\|A_1(\theta)\|^2 + 2(\alpha-\lambda)^2\|G_1(\theta)\|^2 + 2(\alpha-\lambda)\|A_1(\theta)\|\cdot\|G_1(\theta)\|\big]d\theta}{2(\alpha-\lambda)}.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9), we thus get for $\phi \in \mathscr{D}(\mathcal{A})$ that

$$\begin{aligned}
&(\phi, (\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} \\
&\le (\lambda-\alpha)\|\phi_0\|_V^2 + \|\phi_0\|_V\int_{-r}^{0}\|A_1(\theta)\|\cdot\|\phi_1(\theta)\|_V d\theta \\
&\quad + \frac{1}{2}(\alpha-\lambda)\Big\|\phi_0 + \int_{-r}^{0}G_1(\theta)\phi_1(\theta)\mathrm{d}\theta\Big\|_V^2 - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 \\
&\quad - \frac{\int_{-r}^{0}\|\phi_1(\theta)\|_V^2\big[\|A_1(\theta)\|^2 + 2(\alpha-\lambda)^2\|G_1(\theta)\|^2 + 2(\alpha-\lambda)\|A_1(\theta)\|\cdot\|G_1(\theta)\|\big]d\theta}{2(\alpha-\lambda)}.
\end{aligned} \tag{3.11}$$

On the other hand, it is easy to see by the well-known Hölder's inequality and assumption $r \in [0,1]$ that

$$\begin{aligned}
&\Big\|\phi_0 + \int_{-r}^{0}G_1(\theta)\phi_1(\theta)d\theta\Big\|_V^2 \\
&\le \|\phi_0\|_V^2 + \Big\|\int_{-r}^{0}G_1(\theta)\phi_1(\theta)d\theta\Big\|_V^2 + 2\|\phi_0\|_V\int_{-r}^{0}\|G_1(\theta)\|\|\phi_1(\theta)\|_V d\theta \\
&\le \|\phi_0\|_V^2 + \int_{-r}^{0}\|G_1(\theta)\|^2\|\phi_1(\theta)\|_V^2 d\theta + 2\|\phi_0\|_V\int_{-r}^{0}\|G_1(\theta)\|\|\phi_1(\theta)\|_V d\theta.
\end{aligned} \tag{3.12}$$

By substituting (3.12) into (3.11), it follows immediately that

$$\begin{aligned}
&(\phi, (\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} \\
&\le -\frac{1}{2}(\alpha-\lambda)\|\phi_0\|_V^2 + \|\phi_0\|_V\int_{-r}^{0}\|\phi_1(\theta)\|_V\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]d\theta \\
&\quad - \frac{\int_{-r}^{0}\|\phi_1(\theta)\|_V^2\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]^2 d\theta}{2(\alpha-\lambda)}.
\end{aligned} \tag{3.13}$$

If $\|\phi_0\|_V = 0$, it follows from (3.13) that

$$(\phi, (\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} \le 0 \quad \text{for all} \quad \phi \in \mathscr{D}(\mathcal{A}).$$

If $\|\phi_0\|_V \neq 0$, we have from (3.13) and the assumption $r \in [0,1]$ that

$$
\begin{aligned}
&(\phi, (\mathcal{A} + \lambda\nu^{-1})\phi)_{\mathcal{H}} \\
&\le -\frac{\|\phi_0\|_V^2}{2}\int_{-r}^0 \Big\{\frac{(\alpha-\lambda)}{r} - \frac{2\|\phi_1(\theta)\|_V\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]}{\|\phi_0\|} \\
&\quad + \frac{\|\phi_1(\theta)\|_V^2\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]^2}{(\alpha-\lambda)\|\phi_0\|_V^2}\Big\}d\theta \\
&\le -\frac{\|\phi_0\|_V^2}{2}\int_{-r}^0 \Big\{(\alpha-\lambda) - \frac{2\|\phi_1(\theta)\|_V\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]}{\|\phi_0\|_V} \\
&\quad + \frac{\|\phi_1(\theta)\|_V^2\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]^2}{(\alpha-\lambda)\|\phi_0\|_V^2}\Big\}d\theta \\
&= -\frac{\|\phi_0\|_V^2}{2(\alpha-\lambda)}\int_{-r}^0 \Big((\alpha-\lambda) - \frac{\|\phi_1(\theta)\|_V\big[\|A_1(\theta)\| + (\alpha-\lambda)\|G_1(\theta)\|\big]}{\|\phi_0\|_V}\Big)^2 d\theta \\
&\le 0.
\end{aligned}
\tag{3.14}
$$

Hence, we get the relation

$$(\phi, \mathcal{A}\phi)_{\mathcal{H}} \le -\lambda\nu^{-1}\|\phi\|_{\mathcal{H}} \ \text{ for all } \ \phi \in \mathscr{D}(\mathcal{A}).$$

Further, it follows from Proposition 2.1.4 in Liu [5] that the C_0-semigroup $e^{t\mathcal{A}}$, $t \ge 0$ is exponentially stable. □

Theorem 3.2. *Suppose that all the conditions in Theorem 3.1 are satisfied and further $B \in \mathscr{L}(H, \mathscr{L}_2(K_Q, H))$ in (3.2) with $\|B\| := \|B\|_{\mathscr{L}(H,\mathscr{L}_2(K_Q,H))}$. Assume that the following relation holds:*

$$\lambda > \frac{\|B\|^2\nu}{2}\Big(1 + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\Big),$$

then the null solution of (3.2) is exponentially stable in the mean square sense.

Proof. From Theorem 3.1, we know that

$$\|e^{t\mathcal{A}}\| \le e^{-\lambda\nu^{-1}t}, \qquad t \ge 0. \tag{3.15}$$

That is, (3.5) is satisfied for the system (3.3). Let $\mathcal{I}$ denote the identity operator in $\mathcal{H}$ and define a linear operator $\Delta(\mathcal{I}) \in \mathscr{L}(\mathcal{H})$ by the relation

$$\langle \phi, \Delta(\mathcal{I})\phi\rangle_{\mathcal{H}} = Tr\{\mathcal{B}(\phi)Q^{1/2}(\mathcal{B}(\phi)Q^{1/2})^*\}, \quad \phi \in \mathcal{H}.$$

Then, by virtue of (3.15), we have

$$
\begin{aligned}
\Big\|\int_0^\infty e^{t\mathcal{A}^*}\Delta(\mathcal{I})e^{t\mathcal{A}}dt\Big\| &\le \|\Delta(\mathcal{I})\|\int_0^\infty \|e^{t\mathcal{A}}\|^2 dt \\
&\le \frac{\|\mathcal{B}\|^2_{\mathscr{L}(\mathcal{H},\mathscr{L}_2(K_Q,\mathcal{H}))}\cdot\nu}{2\lambda}.
\end{aligned}
\tag{3.16}
$$

On the other hand, for any $\phi \in \mathcal{H}$, we easily get by Hölder's inequality that

$$
\begin{aligned}
&\|\mathcal{B}(\phi)\|^2_{\mathscr{L}_2(K_Q,\mathcal{H})} \\
&= \Big\| B\Big(\phi_0 + \int_{-r}^0 G_1(\theta)\phi_1(\theta)d\theta\Big)\Big\|^2_{\mathscr{L}_2(K_Q,H)} \\
&\le \|B\|^2\Big[\|\phi_0\|^2_H + \Big\|\int_{-r}^0 G_1(\theta)\phi_1(\theta)d\theta\Big\|^2_H + 2\|\phi_0\|_H\nu^{1/2}\int_{-r}^0 \|G_1(\theta)\phi_1(\theta)\|_V d\theta\Big] \\
&\le \|B\|^2\Big[\|\phi_0\|^2_H + \nu\Big\|\int_{-r}^0 G_1(\theta)\phi_1(\theta)d\theta\Big\|^2_V + 2\|\phi_0\|_H\nu^{\frac{1}{2}}\int_{-r}^0 \|G_1(\theta)\|\|\phi_1(\theta)\|_V d\theta\Big] \\
&\le \|B\|^2\Big[\|\phi_0\|^2_H + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\cdot\int_{-r}^0 \|\phi_1(\theta)\|^2_V d\theta \\
&\quad + 2\nu^{\frac{1}{2}}\|\phi_0\|_H\Big(\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_{-r}^0 \|\phi_1(\theta)\|^2_V d\theta\Big)^{\frac{1}{2}}\Big] \qquad (3.17)\\
&\le \|B\|^2\Big[\|\phi_0\|^2_H + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\cdot\int_{-r}^0 \|\phi_1(\theta)\|^2_V d\theta \\
&\quad + \|\phi_0\|^2_H\nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta + \int_{-r}^0 \|\phi_1(\theta)\|^2_V d\theta\Big] \\
&= \|B\|^2\Big(1 + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\Big)\Big(\|\phi_0\|^2_H + \int_{-r}^0 \|\phi_1(\theta)\|^2_V d\theta\Big) \\
&= \|B\|^2\Big(1 + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\Big)\|\phi\|^2_{\mathcal{H}}.
\end{aligned}
$$

Therefore, it follows that

$$
\|\mathcal{B}\|_{\mathscr{L}(\mathcal{H},\mathscr{L}_2(K_Q,\mathcal{H}))} \le \|B\|\sqrt{1 + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta}. \qquad (3.18)
$$

Substituting (3.18) into (3.16) and using the condition

$$
\lambda > \frac{\|B\|^2\nu}{2}\Big(1 + \nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\Big),
$$

we obtain the desired relation

$$
\Big\|\int_0^\infty e^{t\mathcal{A}^*}\Delta(\mathcal{I})e^{t\mathcal{A}}dt\Big\| < 1.
$$

Therefore, the conditions (3.5) and (3.6) in Proposition 3.1 are satisfied and the null solution of (3.2) is exponential stability in mean square. The proof is thus complete. □

Corollary 3.1. *Let* $K = \mathbb{R}$, $W_Q(t) = w(t)$, *a standard real Brownian motion,* $B \in \mathscr{L}(H)$

and assume that the delay parameter $r \in [0,1]$. Further, suppose that for some $\lambda \in (0,\alpha)$,

$$\|e^{tA}\| \le e^{-\alpha t}, \quad \frac{\|B\|^2_{\mathscr{L}(H)} \cdot \nu}{2}\Big(1+\nu\int_{-r}^0 \|G_1(\theta)\|^2 d\theta\Big) < \lambda,$$
$$\int_{-r}^0 e^{-2\lambda\nu^{-1}\theta}\big[\|A_1(\theta)\|^2 + 2(\alpha-\lambda)^2\|G_1(\theta)\|^2 + 2(\alpha-\lambda)\|A_1(\theta)\|\|G_1(\theta)\|\big]d\theta$$
$$< (\alpha-\lambda)^2, \tag{3.19}$$

then the null solution of (3.2) is exponentially stable in the mean squarer sense.

Example 3.1. Consider the following stochastic functional evolution equation of neutral type,

$$\begin{cases} d\Big(y(t,\xi) - \int_{-r}^0 \gamma y(t+\theta,\xi)d\theta\Big) = \Delta\Big(y(t,\xi) - \int_{-r}^0 \gamma y(t+\theta,\xi)d\theta\Big)dt \\ \qquad\qquad +\sigma y(t,\xi)dw(t), \quad t\ge 0,\ \xi\in[0,\pi], \\ y(t,0) = y(t,\pi) = 0, \\ y(0,\cdot) = \phi_0(\cdot) \in L^2(0,\pi),\ y_0(\cdot,\cdot) = \phi_1(\cdot,\cdot) \in L^2([-r,0];L^2(0,\pi)), \end{cases} \tag{3.20}$$

where $r \le 1$, γ, $\sigma \in \mathbb{R}$ with $\sigma \ne 0$ and $\Delta = \partial^2/\partial\xi^2$ is the Laplace operator.

Let $V = H_0^1(0,\pi) \cap H^2(0,\pi)$ and $H = L^2(0,\pi)$. If $\gamma = 0$, it is known (see, e.g., Liu [7]) that whenever

$$\frac{\sigma^2}{2} < 1, \tag{3.21}$$

the null solution of (3.20) is exponentially stable in mean square.

If $\gamma \ne 0$, by virtue of Corollary 3.1, we have that if there exists $0 < \lambda < 1$ ($\alpha = 1$, on this occasion) such that

$$\frac{\sigma^2}{2}\Big(1 + \int_{-r}^0 \gamma^2 d\theta\Big) \le \frac{\sigma^2}{2}(1+\gamma^2) < \lambda < 1,$$
$$\frac{(e^{2\lambda r}-1)[2(1-\lambda)^2\gamma^2]}{2\lambda} < (1-\lambda)^2, \tag{3.22}$$

then the null solution of (3.20) is exponentially stable in the mean square sense. Now suppose that (3.21) is true and let γ^2, $\varepsilon > 0$ be so small that

$$\frac{\sigma^2}{2}(1+\gamma^2) + \varepsilon < 1,$$

then it may be verified that

$$\lambda := \frac{\sigma^2}{2}(1+\gamma^2) + \varepsilon < 1$$

with sufficiently small $\varepsilon > 0$ satisfies the first condition of (3.22). In this case, the second inequality in (3.22) reduces, by letting $\varepsilon \to 0$, to

$$r < \frac{1}{\sigma^2(1+\gamma^2)} \ln\Big[1 + \frac{\sigma^2(1+\gamma^2)}{2\gamma^2}\Big]. \tag{3.23}$$

In other words, if $\sigma^2(1+\gamma^2)/2 < 1$, i.e.,

$$|\gamma| < \frac{\sqrt{2}}{|\sigma|}\sqrt{1 - \frac{\sigma^2}{2}},$$

and $r \in [0, 1]$ is so small, satisfying (3.23), then the null solution of (3.20) is exponentially stable in the mean square sense.

3.2. Point Delays under Neutral Terms

In this subsection, we shall consider the following stochastic functional differential equation of neutral type in space V^*,

$$\begin{cases} d(y(t) - G_2 y(t-r)) = A(y(t) - G_2 y(t-r))dt + A_2 y(t-r)dt \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{-r}^{0} By(t+\theta)d\theta dW_Q(t), \quad t \geq 0, \\ y(0) = \phi_0,\ y(\theta) = \phi_1(\theta),\ \theta \in [-r, 0],\ (\phi_0, \phi_1) \in \mathcal{H} = H \times L^2([-r, 0]; V). \end{cases} \tag{3.24}$$

where $A_2 \in \mathscr{L}(V, V^*)$, $G_2 \in \mathscr{L}(V)$ and $B \in \mathscr{L}(V, \mathscr{L}_2(K_Q, H))$ with $B(V) \subset V$.

To this end, let

$$\mathcal{A}\phi = \Big(A\phi_0 + A_2\phi_1(-r), \frac{d\phi_1(\theta)}{d\theta}\Big), \qquad \phi \in \mathscr{D}(\mathcal{A}),$$

and meanwhile we define

$$\mathcal{B}\phi(k) = \Big(\int_{-r}^{0} B\phi_1(\theta)d\theta(k), 0\Big), \qquad \phi \in \mathcal{H}, \quad k \in K_Q.$$

We can rewrite (3.24) as an equivalent stochastic evolution equation without delay,

$$\begin{cases} dY(t) = \mathcal{A}Y(t)dt + \mathcal{B}Y(t)dW_Q(t), \quad t \geq 0, \\ Y(0) = \phi \in \mathcal{H}, \end{cases} \tag{3.25}$$

where $Y(t) = (y(t) - G_2 y(t-r),\ y(t+\cdot)),\ t \geq 0$ is the lift-up process of $y(t)$.

Theorem 3.3. *Suppose that $\langle x, Ax\rangle_{V,V^*} \leq -\alpha\|x\|_V^2$ for all $x \in V$ and some $\alpha > 0$. Assume further that for some $0 < \lambda < \alpha$,*

$$e^{2\lambda\nu^{-1}r}\big[\|A_2\|^2 + 2\|G_2\|(\alpha-\lambda) + 2\|G_2\|^2(\alpha-\lambda)^2\big] < (\alpha-\lambda)^2, \tag{3.26}$$

then

$$\|e^{t\mathcal{A}}\| \leq e^{-\lambda\nu^{-1}t}, \quad t \geq 0. \tag{3.27}$$

Proof. As before, we intend to find an equivalent inner product $(\cdot,\cdot)_{\mathcal{H}}$ to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H} = H \times L^2([-r,0];V)$ such that

$$(\phi, \mathcal{A}\phi)_{\mathcal{H}} \le -\lambda\nu^{-1}\|\phi\|_{\mathcal{H}} \quad \text{for all} \quad \phi \in \mathscr{D}(\mathcal{A}).$$

Define $(\cdot,\cdot)_{\mathcal{H}}$ by

$$(\phi,\psi)_{\mathcal{H}} := \langle\phi_0,\psi_0\rangle_H + \int_{-r}^0 \gamma(\theta)\langle\phi_1(\theta),\psi_1(\theta)\rangle_V d\theta, \quad \phi, \quad \psi \in \mathcal{H},$$

where $\gamma(\theta)$ is given by

$$\gamma(\theta) = e^{2\lambda\nu^{-1}\theta}(\alpha-\lambda), \;\; \theta\in[-r,0]. \tag{3.28}$$

It can be verified that $(\cdot,\cdot)_{\mathcal{H}}$ satisfies the definition of a valid inner product due to the fact that $\gamma(\theta) \ge \gamma(-r) > 0$ for $\theta \in [-r,0]$. For any $\phi \in \mathscr{D}(\mathcal{A})$, it follows that

$$\begin{aligned}
&(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}}\\
&= \langle\phi_0, A\phi_0 + \lambda\nu^{-1}\phi_0 + A_2\phi_1(-r)\rangle_H + \int_{-r}^0 \gamma(\theta)\langle\phi_1(\theta),\dot\phi_1(\theta)+\lambda\nu^{-1}\phi_1(\theta)\rangle_V d\theta\\
&\le \langle\phi_0, A\phi_0 + A_2\phi_1(-r)\rangle_{V,V^*} + \int_{-r}^0 \gamma(\theta)\langle\phi_1(\theta),\dot\phi_1(\theta)\rangle_V d\theta\\
&\quad + \lambda\|\phi_0\|_V^2 + \lambda\nu^{-1}\int_{-r}^0 \gamma(\theta)\|\phi_1(\theta)\|_V^2 d\theta\\
&\le (\lambda-\alpha)\|\phi_0\|_V^2 + \|\phi_0\|_V\|\phi_1(-r)\|_V\|A_2\| + \frac{1}{2}\int_{-r}^0 \gamma(\theta) d\|\phi_1(\theta)\|_V^2\\
&\quad + \lambda\nu^{-1}\int_{-r}^0 \gamma(\theta)\|\phi_1(\theta)\|_V^2 d\theta.
\end{aligned} \tag{3.29}$$

where $\|A_2\| := \|A_2\|_{\mathscr{L}(V,V^*)}$. Using integration by parts, one can derive for $\phi \in \mathscr{D}(\mathcal{A})$ that

$$\begin{aligned}
&\frac{1}{2}\int_{-r}^0 \gamma(\theta) d\|\phi_1(\theta)\|_V^2\\
&= \frac{1}{2}\gamma(0)\|\phi_1(0)\|_V^2 - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 - \frac{1}{2}\int_{-r}^0 \|\phi_1(\theta)\|_V^2 d\gamma(\theta)\\
&= \frac{1}{2}\gamma(0)\|\phi_1(0)\|_V^2 - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 - \lambda\nu^{-1}\int_{-r}^0 \|\phi_1(\theta)\|_V^2 \gamma(\theta) d\theta.
\end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29), we immediately obtain

$$\begin{aligned}
&(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}}\\
&\le (\lambda-\alpha)\|\phi_0\|_V^2 + \|\phi_0\|_V\|\phi_1(-r)\|_V\|A_2\| + \frac{1}{2}(\alpha-\lambda)\|\phi_0 + G_2\phi_1(-r)\|_V^2\\
&\quad - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2, \quad \phi \in \mathscr{D}(\mathcal{A}).
\end{aligned} \tag{3.31}$$

On the other hand, it is immediate that

$$\|\phi_0 + G_2\phi_1(-r)\|_V^2 \leq \|\phi_0\|_V^2 + \|G_2\|^2\|\phi_1(-r)\|_V^2 + 2\|\phi_0\|_V\|G_2\|\|\phi_1(-r)\|_V. \tag{3.32}$$

Hence, from (3.32) and (3.31) we have for any $\phi \in \mathscr{D}(\mathcal{A})$ that

$$\begin{aligned}(\phi, (\mathcal{A} + \lambda\nu^{-1})\phi)_{\mathcal{H}} &\leq -\frac{1}{2}(\alpha-\lambda)\|\phi_0\|_V^2 + \|\phi_0\|_V\|\phi_1(-r)\|_V\left[\|A_2\| + (\alpha-\lambda)\|G_2\|\right] \\ &\quad + \frac{1}{2}(\alpha-\lambda)\|G_2\|^2\|\phi_1(-r)\|_V^2 - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 \\ &= -\frac{1}{2}(\alpha-\lambda)\|\phi_0\|_V^2 + \|\phi_0\|_V\|\phi_1(-r)\|_V\left[\|A_2\| + (\alpha-\lambda)\|G_2\|\right] \\ &\quad - \frac{1}{2}\|\phi_1(-r)\|_V^2\left[\gamma(-r) - (\alpha-\lambda)\|G_2\|^2\right].\end{aligned} \tag{3.33}$$

If $\|\phi_0\|_V = 0$, it follows from (3.26) that

$$\gamma(-r) - (\alpha-\lambda)\|G_2\|^2 > \frac{\|A_2\|^2 + 2\|G_2\|(\alpha-\lambda) + \|G_2\|^2(\alpha-\lambda)^2}{\alpha-\lambda} \geq 0. \tag{3.34}$$

Combining (3.33) and (3.34), we thus have

$$(\phi, (\mathcal{A} + \lambda\nu^{-1})\phi)_{\mathcal{H}} \leq 0 \quad \text{for all} \quad \phi \in \mathscr{D}(\mathcal{A}).$$

If $\|\phi_0\|_V \neq 0$, we have from (3.33) and (3.34) that for any $\phi \in \mathscr{D}(\mathcal{A})$,

$$\begin{aligned}(\phi, (\mathcal{A} + \lambda\nu^{-1})\phi)_{\mathcal{H}} &= -\frac{\|\phi_0\|_V^2}{2}\Bigg[(\alpha-\lambda) - \frac{2\|\phi_1(-r)\|_V\left[\|A_2\| + (\alpha-\lambda)\|G_2\|\right]}{\|\phi_0\|_V} \\ &\quad + \frac{\|\phi_1(-r)\|_V^2\left[\gamma(-r) - (\alpha-\lambda)\|G_2\|^2\right]}{\|\phi_0\|_V^2}\Bigg] \\ &\leq -\frac{\|\phi_0\|_V^2}{2}\Bigg[(\alpha-\lambda) - \frac{2\|\phi_1(-r)\|_V\left[\|A_2\| + (\alpha-\lambda)\|G_2\|\right]}{\|\phi_0\|_V} \\ &\quad + \frac{\|\phi_1(-r)\|_V^2\left[\|A_2\| + (\alpha-\lambda)\|G_2\|\right]^2}{\|\phi_0\|_V^2(\alpha-\lambda)}\Bigg] \\ &= -\frac{\|\phi_0\|_V^2}{2(\alpha-\lambda)}\left((\alpha-\lambda) - \frac{\|\phi_1(-r)\|_V\left[\|A_2\| + (\alpha-\lambda)\|G_2\|\right]}{\|\phi_0\|_V}\right)^2 \\ &\leq 0.\end{aligned} \tag{3.35}$$

Hence, we obtain that

$$(\phi, \mathcal{A}\phi)_{\mathcal{H}} \leq -\lambda\nu^{-1}\|\phi\|_{\mathcal{H}} \quad \text{for all} \quad \phi \in \mathscr{D}(\mathcal{A}).$$

Thus, the C_0-semigroup $e^{t\mathcal{A}},\ t \geq 0$, is exponentially stable. □

By using a similar method to that in Theorem 3.2, we obtain sufficient conditions to guarantee the exponential stability for the stochastic system (3.24).

Theorem 3.4. *Suppose that all the conditions in Theorem 3.3 are satisfied. Further, if*

$$\lambda > \frac{\|B\|^2 \cdot \nu^2}{2},$$

where $\|B\| := \|B\|_{\mathscr{L}(V,\mathscr{L}_2(K_Q,H))}$, *then the solution of (3.24) is mean square exponential stable.*

Proof. From Theorem 3.3, we know $\|e^{t\mathcal{A}}\| \le e^{-\lambda\nu^{-1}t}$, $t \ge 0$. That is, the condition (3.5) is satisfied for the system (3.25). Define a linear operator $\Delta(\mathcal{I}) \in \mathscr{L}(\mathcal{H})$ by the relation

$$\langle \phi, \Delta(\mathcal{I})\phi\rangle_{\mathcal{H}} = Tr\{\mathcal{B}(\phi)Q^{1/2}(\mathcal{B}(\phi)Q^{1/2})^*\}, \quad \phi \in \mathcal{H}.$$

From (3.27), it is easy to see that

$$\Big\|\int_0^\infty e^{t\mathcal{A}^*}\Delta(\mathcal{I})e^{t\mathcal{A}}dt\Big\| \le \|\Delta(\mathcal{I})\|\int_0^\infty \|e^{t\mathcal{A}}\|^2 dt \le \frac{\|\mathcal{B}\|^2\cdot\nu}{2\lambda}. \tag{3.36}$$

On the other hand, for any $\phi \in \mathcal{H}$, we easily obtain

$$\begin{aligned}\|\mathcal{B}(\phi)\|^2_{\mathscr{L}_2(K_Q,\mathcal{H})} &= \int_{-r}^0 \|B\phi_1(\theta)\|^2_{\mathscr{L}_2(K_Q,H)}d\theta\\ &\le \|B\|^2\int_{-r}^0 \|\phi_1(\theta)\|^2_H d\theta\\ &\le \|B\|^2\nu\Big(\|\phi_0\|^2_H + \int_{-r}^0 \|\phi_1(\theta)\|^2_V d\theta\Big)\\ &= \|B\|^2\nu\|\phi\|^2_{\mathcal{H}}.\end{aligned} \tag{3.37}$$

Thus, it is immediate that

$$\|\mathcal{B}\| \le \|B\|\sqrt{\nu}. \tag{3.38}$$

Substituting (3.38) into (3.36) and using the condition $\lambda > \|B\|^2\cdot\nu^2/2$, we get immediately that

$$\Big\|\int_0^\infty e^{t\mathcal{A}^*}\Delta(\mathcal{I})e^{t\mathcal{A}}dt\Big\| < 1.$$

Therefore, the two conditions in Proposition 3.1 are satisfied for the stochastic system (3.24) and the mean square exponential stability is thus obtained. The proof is complete now. □

As an immediate consequence of Theorem 3.4, we have the following result.

Corollary 3.2. *Let* $K = \mathbb{R}$, $W_Q(t) = w(t)$ *and* $B \in \mathscr{L}(H)$. *If*

$$\begin{aligned}&\|e^{tA}\| \le e^{-\alpha t}, \quad \frac{\|B\|^2\cdot\nu^2}{2} < \lambda < \alpha,\\ &0 < r < \frac{1}{2\lambda\nu^{-1}}\ln\frac{(\alpha-\lambda)^2}{\|A_2\|^2 + 2\|G_2\|(\alpha-\lambda) + 2\|G_2\|^2(\alpha-\lambda)^2},\end{aligned} \tag{3.39}$$

then the mild solution Y *is exponentially stable in the mean square sense.*

Example 3.2. Consider the following linear neutral type stochastic delay partial differential equation,

$$\begin{cases} d(y(t,\xi)-\gamma y(t-r,\xi)) = \dfrac{\partial^2}{\partial \xi^2}(y(t,\xi)-\gamma y(t-r,\xi))dt \\ \qquad\qquad\qquad\qquad\qquad + \sigma \displaystyle\int_{-r}^{0} y(t+\theta,\xi)d\theta dw(t), \quad t \geq 0,\ \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \quad t \geq 0, \\ y(0,\cdot) = \phi_0(\cdot) \in L^2(0,\pi), y_0(\cdot,\cdot) = \phi_1(\cdot,\cdot) \in L^2([-r,0];L^2(0,\pi)). \end{cases} \tag{3.40}$$

where γ, $\sigma \in \mathbb{R}$ with $\sigma \neq 0$, $V = H_0^1(0,\pi) \cap H^2(0,\pi)$ and $H = L^2(0,\pi)$.

If $\gamma = 0$, it can be shown, similarly to Corollary 4.3.1 in Liu [7], that whenever

$$\frac{\sigma^2}{2} < 1, \tag{3.41}$$

the null solution of (3.40) is exponentially stable in mean square.

If $\gamma \neq 0$, we impose the following condition on σ, γ,

$$\sigma^2 < 2, \qquad |\gamma| < \frac{\sqrt{1+2(1-\frac{\sigma^2}{2})^2}-1}{2-\sigma^2}. \tag{3.42}$$

Then it is easy to see from (3.42) that

$$2|\gamma| + 2|\gamma|^2\Big(1-\frac{\sigma^2}{2}\Big) < 1-\frac{\sigma^2}{2}.$$

which immediately yields

$$\frac{1-\frac{\sigma^2}{2}}{2|\gamma|+2|\gamma|^2(1-\frac{\sigma^2}{2})} > 1.$$

On this occasion, it is easy to see that α in Corollary 3.2 is equal to 1. Hence, by virtue of (3.39), if there exists $\lambda > 0$ such that

$$\frac{\sigma^2}{2} < \lambda < 1, \quad 0 < r < \frac{1}{2\lambda}\ln\frac{1-\lambda}{2|\gamma|+2|\gamma|^2(1-\lambda)}, \tag{3.43}$$

then the null solution of (3.40) is exponentially stable in the mean square sense. It may be verified that $\lambda := \frac{\sigma^2}{2}+\varepsilon$ with $\varepsilon > 0$ sufficient small satisfies the first condition of (3.43). The second condition in (3.43), by letting $\varepsilon \to 0$, reduces to

$$0 < r < \frac{1}{2\sigma^2}\ln\frac{1-\frac{\sigma^2}{2}}{2|\gamma|+2|\gamma|^2(1-\frac{\sigma^2}{2})}. \tag{3.44}$$

In summary, in the case $\gamma \neq 0$ and $\sigma^2 < 2$, we have that whenever

$$|\gamma| < \frac{\sqrt{1 + 2(1 - \frac{\sigma^2}{2})^2} - 1}{2 - \sigma^2}$$

and r is so small, satisfying (3.44), the null solution of (3.40) is exponentially stable in the mean square sense. In other words, the mean square exponential stability of (3.40) is not sensitive to small delays in this situation.

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